EXTREMALS FOR THE SOBOLEV INEQUALITY ON THE SEVEN DIMENSIONAL QUATERNIONIC HEISENBERG GROUP AND THE QUATERNIONIC CONTACT YAMABE PROBLEM

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Abstract. A complete solution to the quaternionic contact Yamabe problem on the seven dimensional sphere is given. Extremals for the Sobolev inequality on the seven dimensional Hesenberg group are explicitly described and the best constant in the L^2 Folland-Stein embedding theorem is determined.

Contents

1. Introduction

A quaternionic contact (qc) structure, introduced in [Biq1, Biq2], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. A qc structure on a real (4n+3)-dimensional manifold M is a codimension three distribution H locally given as the kernel of 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the three 2-forms $d\eta_i|_H$ are the fundamental 2-forms of a quaternionic structure on H, i.e., there exists a Riemannian metric g on H and three local almost complex structures I_i on H satisfying the commutation relations of the imaginary quaternions, $I_1 I_2 I_3 = -1$, such that, $d\eta_i|_H = 2g(I_i., .)$. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of Riemannian metrics and a 2-sphere bundle of almost complex structures, the quaternionic bundle $\mathbb{Q}$. The 2-sphere bundle of one forms determines uniquely the associated metric and a conformal change of the metric is equivalent to a conformal change of the one forms. To every metric in the fixed conformal class one can associate a linear connection preserving the qc structure, see [Biq1], which we shall call the Biquard connection.

If the first Pontrijagin class of M vanishes then the 2-sphere bundle of $\mathbb{R}^3$-valued 1-forms is trivial [AK], i.e. there is a globally defined 3-contact form η that anihilates H, we denote the corresponding

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QC manifold (M,η). In this case the 2-sphere of associated almost complex structures is also globally defined on H.

Examples of QC manifolds are given in [Biq1, Biq2, IMV, D1]. In particular, any totally umbilic hypersurface of a quaternionic Kähler or hyperKähler manifold carries such a structure [IMV]. A basic example is provided by any 3-Sasakian manifold which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyperKähler manifold. It was shown in [IMV] that the torsion endomorphism of the Biquard connection is the obstruction for a given qc-structure to be locally 3-Sasakian, up to a multiplication with a constant factor and a $SO(3)$-matrix.

For a fixed metric in the conformal class one associates the scalar curvature of the associated Biquard connection, called the qc-scalar curvature. The quaternionic contact Yamabe problem is: *given a compact QC manifold (M,η), find a conformal 3-contact form for which the qc-scalar curvature is constant.*

In this paper, following the geometrical approach developed in [IMV], we provide a solution of this problem on the seven dimensional sphere equipped with its natural quaternionic contact structure. The question reduces to the solvability of the Yamabe equation (3.4). Taking the conformal factor in the form $\bar\eta = u^{1/(n+1)}\eta$ turns (3.4) into the equation

$$\mathfrak{L}u \;\equiv\; 4\frac{n+2}{n+1}\,\triangle u - \;u\,Scal \;=\; -\,u^{2^*-1}\,\overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle h \;=\; tr^g(\nabla dh)$, $Scal$ and $\overline{Scal}$ are the qc-scalar curvatures correspondingly of $(M,\,\eta)$ and $(M,\,\bar\eta)$, and $2^* = \frac{2Q}{Q-2}$, with $Q = 4n+6$–the homogeneous dimension. On a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the Yamabe equation characterizes the non-negative extremals of the Yamabe functional defined by

$$\Upsilon(u) \;=\; \int_M 4\frac{n+2}{n+1}\,|\nabla u|^2 \;+\; \mathrm{Scal}\,u^2\,dv_g, \qquad \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0.$$

The Yamabe constant is defined as the infimum

$$\lambda(M) = \inf\{\Upsilon(u) : \int_M u^{2^*}\,dv_g \;=\; 1,\; u>0\}.$$

When the Yamabe constant $\lambda(M)$ is less than that of the quaternionic sphere the existence of solutions to the quaternionic contact Yamabe problem is shown in [W], see also [JL2]. We consider the Yamabe problem on the standard unit $(4n+3)$-dimensional quaternionic sphere. The standard 3-Sasaki structure on the sphere is a qc-Einstein structure $\tilde\eta$ having constant qc-scalar curvature $\widetilde{\mathrm{Scal}} = 16n(n+2)$. Its images under conformal quaternionic contact automorphism have again constant qc-scalar curvature. In [IMV] we conjectured that these are the only solutions to the Yamabe problem on the quaternionic sphere. The purpose of this paper is to confirm this conjecture when the dimension is equal to seven, i.e., $n=1$. We prove the following theorem.

Theorem 1.1. *Let $\tilde\eta = \frac{1}{2h}\eta$ be a conformal deformation of the standard qc-structure $\tilde\eta$ on the quaternionic unit sphere S^7. If η has constant qc-scalar curvature, then up to a multiplicative constant η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism. Furthermore, $\lambda(S^7) = \Upsilon(\tilde\eta) = 48\,(4\pi)^{1/5}$ and this minimum value is achieved only by $\tilde\eta$ and its images under conformal quaternionic contact automorphisms.*

In [IMV] a weaker result was proven, namely the conclusion holds (in all dimensions) provided the vertical space of η is integrable.

Another motivation for studying the Yamabe equation comes from its connection with the determination of the norm and extremals in a relevant Sobolev-type embedding on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, [GV1] and [Va1] and [Va]. As it is well known, the Yamabe equation is essentially the Euler-Lagrange equation of the extremals for the L^2 case of such embedding results. In the considered setting we have the following Theorem due to Folland and Stein [FSt].

Theorem 1.2 (Folland and Stein). *Let $\Omega \subset \boldsymbol{G}$ be an open set in a Carnot group $\boldsymbol{G}$ of homogeneous dimension Q. For any $1 < p < Q$ there exists $S_p = S_p(\boldsymbol{G}) > 0$ such that for $u \in C_o^\infty(\Omega)$*

$$\left(\int_\Omega |u|^{p^*}\, dH(g)\right)^{1/p^*} \leq S_p \left(\int_\Omega |Xu|^p\, dH(g)\right)^{1/p}, \tag{1.1}$$

where $|Xu| = \sum_{j=1}^m |X_j u|^2$ with $X_1, \dots, X_m$ denoting a basis of the first layer of $\boldsymbol{G}$.

Let S_p be the best constant in the Folland-Stein inequality, i.e., the smallest constant for which (1.1) holds. The second result of this paper is the following Theorem, which determines the extremals and the best constant in Theorem 1.2 when $p = 2$ for the case of the seven dimensional quaternionic Heisenberg group. Our result confirms the Conjecture made after [GV1, Theorem 1.1] and shows that the assumption of partial-symmetry in [GV1, Theorem 1.6] is superfluous in the case of the first quaternionic Heisenberg group.

Theorem 1.3. *Let $\boldsymbol{G}(\mathbb{H}) = \mathbb{H} \times Im\mathbb{H}$ be the seven dimensional quaternionic Heisenberg group of Heisenebrg type. The best constant in the L^2 Folland-Stein embedding theorem is*

$$S_2 = \frac{15^{1/10}}{\pi^{2/5}\, 2\sqrt{2}}.$$

An extremal is given by the function

$$F(q,\omega) = \gamma \left[(1+|q|^2)^2 + 16|\omega|^2)\right]^{-2}, \ (q,\omega) \in \boldsymbol{G}(\mathbb{H})$$

where

$$\gamma = 32\, \pi^{-17/50}\, 2^{1/5}\, 15^{2/5}.$$

Any other non-negative extremal is obtained from F by translations (5.2) and dilations (5.3).

In [IMV] we proved Theorem 1.1 with the 'extra' assumption of the integrability of the vertical distribution using the classification of all qc-Einstein structures conformal to the standard. In the present paper we remove the 'extra' integrability assumption in dimension seven. This goal is achieved with the help of a suitable divergence formula, Theorem 4.4, see [JL3] for the CR case and [Ob], [LP] for the Riemannian case. Once it is known that the qc-structure conformal to the standard 3-Sasakian qc-structure on S^7 is again qc-Einstein, which under the assumption of constant scalar curvature is equivalent to the integrability of the vertical space, we apply the result of [IMV] to conclude the proof of Theorem 1.1.

Organization of the paper. The paper relies heavily on [IMV]. In order to make the present paper self-contained, in Section 2 we give a review of the notion of a quaternionic contact structure and collect formulas and results from [IMV] that will be used in the subsequent sections.

Section 3 and 4 are of technical nature. In the former we find some transformations formulas for relevant tensors, while in the latter we prove certain divergence formulas. Of course, the main result is Theorem 4.4. In the last Section we prove the main Theorems.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [Biq1] and [IMV].

For the purposes of this paper, a quaternionic contact (QC) manifold $(M, g, \mathbb{Q})$ is a $4n+3$ dimensional manifold M with a codimension three distribution H equipped with a metric g and an Sp(n)Sp(1) structure, i.e., we have

i) a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures, such that, we have $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 : \ a^2+b^2+c^2 = 1\}$, where the almost complex structures $I_s : H \to H, \quad I_s^2 = -1, \quad s = 1,\ 2,\ 3,$ satisfy the commutation relations of the imaginary quaternions $I_1I_2 = -I_2I_1 = I_3$;
ii) H is the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds

$$2g(I_sX, Y) \ = \ d\eta_s(X, Y), \quad s = 1, 2, 3, \quad X, Y \in H.$$

Correspondingly, given a quaternionic contact manifold we shall denote with η any associated contact form. The associated contact form is determined up to an $SO(3)$-action, namely if $\Psi \in SO(3)$ then $\Psi\eta$ is again a contact form satisfying the above compatibility condition (rotating also the almost complex structures) . On the other hand, if we consider the conformal class $[g]$, the associated contact forms are determined up to a multiplication with a positive function μ and an $SO(3)$-action, namely if $\Psi \in SO(3)$ then $\mu\Psi\eta$ is a contact form associated with a metric in the conformal class $[g]$.

We shall denote with (M, η) a QC manifold with a fixed globally defined contact form. A special phenomena here, noted by Biquard [Biq1], is that the 3-contact form η determines the quaternionic structure and the metric on the horizontal bundle in a unique way.

A QC manifold $(M, \bar{g}, \mathbb{Q})$ is called conformal to $(M, g, \mathbb{Q})$ if $\bar{g} \in [g]$. In that case, if $\bar{\eta}$ is a corresponding associated one-form with complex structures $\bar{I}_s$, $s = 1, 2, 3$, we have $\bar{\eta} \ = \ \mu\, \Psi\, \eta$ for some $\Psi \in SO(3)$ and a positive function μ. In particular, starting with a QC manifold (M, η) and defining $\bar{\eta} \ = \ \mu\, \eta$ we obtain a QC manifold $(M, \bar{\eta})$ conformal to the original one.

Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into $Sp(n)$-invariant parts as follows $\quad \Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}, \quad$ where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger \ = \ I_1 \otimes I_1 \ + \ I_2 \otimes I_2 \ + \ I_3 \otimes I_3 \tag{2.1}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = \frac{|\Psi|^2}{4} Id_{|H}$.

On a quaternionic contact manifold there exists a canonical connection defined in [Biq1] when the dimension $(4n+3) > 7$, and by D. Duchemin [D] in the 7-dimensional case.

Theorem 2.1. [Biq1] *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n+3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

i) ∇ preserves the decomposition $H \oplus V$ and the metric g;
ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;
iii) ∇ preserves the $Sp(n)Sp(1)$-structure on H, i.e., $\nabla g \ = \ 0$ and $\nabla\mathbb{Q} \subset \mathbb{Q}$;
iv) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset so(4n)$;
v) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.

We shall call the above connection *the Biquard connection*. Biquard [Biq1] also described the supplementary subspace V explicitly, namely, locally V is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\begin{aligned} \eta_s(\xi_k)=\delta_{sk}, \qquad (\xi_s\lrcorner d\eta_s)_{|H}=0,\\ (\xi_s\lrcorner d\eta_k)_{|H}=-(\xi_k\lrcorner d\eta_s)_{|H}. \end{aligned} \tag{2.2}$$

The vector fields ξ_1,ξ_2,ξ_3 are called Reeb vector fields or fundamental vector fields.

If the dimension of M is seven, the conditions (2.2) do not always hold. Duchemin shows in [D] that if we assume, in addition, the existence of Reeb vector fields as in (2.2), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.2).

Notice that equations (2.2) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1,\xi_2,\xi_3\}=V\perp H$ and $g(\xi_s,\xi_k)=\delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on $TM, \nabla g=0$. We shall also extend the quternionic structure by setting $I_{s|V}=0$.

The fundamental 2-forms $\omega_i, i=1,2,3$ of the quaternionic structure Q are defined by

$$2\omega_{i|H} \;=\; d\eta_{i|H}, \qquad \xi\lrcorner\omega_i=0, \quad \xi\in V. \tag{2.3}$$

Due to (2.3), the torsion restricted to H has the form

$$T(X,Y)=-[X,Y]_{|V}=2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s, \qquad X,Y\in H. \tag{2.4}$$

The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi=T(\xi,.):H\rightarrow H, \quad \xi\in V$. We decompose the endomorphism $T_\xi\in(sp(n)+sp(1))^\perp$ into symmetric part T^0_ξ and skew-symmetric part $b_\xi, \quad T_\xi=T^0_\xi+b_\xi$. We summarize the description of the torsion due to O. Biquard in the following Proposition.

Proposition 2.2. [Biq1] *The torsion T_ξ is completely trace-free,*

$$tr\,T_\xi=g\left(T_\xi(e_a),e_a\right)=0, \quad tr\,T_\xi\circ I=g\left(T_\xi(e_a),Ie_a\right)=0, \quad I\in Q,$$

where $e_1\dots e_{4n}$ is an orthonormal basis of H. The symmetric part of the torsion has the properties:

$$T^0_{\xi_i}I_i=-I_iT^0_{\xi_i}, \qquad i=1,2,3;$$

$$I_2(T^0_{\xi_2})^{+--}=I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-}=I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+}=I_3(T^0_{\xi_3})^{+--}.$$

The skew-symmetric part can be represented in the following way

$$b_{\xi_i}=I_iu, \quad i=1,2,3,$$

where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1,I_2,I_3.

If $n=1$ then the tensor u vanishes identically, $u=0$ and the torsion is a symmetric tensor, $T_\xi=T^0_\xi$.

The covariant derivative of the quaternionic contact structure with respect to the Biquard connection and the covariant derivative of the distribution V are given by

$$\nabla I_i=-\alpha_j\otimes I_k+\alpha_k\otimes I_j, \qquad \nabla\xi_i=-\alpha_j\otimes\xi_k+\alpha_k\otimes\xi_j, \tag{2.5}$$

where the $sp(1)$-connection 1-forms α_s on H are given by [Biq1]

$$\alpha_i(X)=d\eta_k(\xi_j,X)=-d\eta_j(\xi_k,X), \quad X\in H, \quad \xi_i\in V \tag{2.6}$$

while the $sp(1)$-connection 1-forms α_s on the vertical space V are calculated in [IMV]

$$\alpha_i(\xi_s) \;=\; d\eta_s(\xi_j,\xi_k)-\,\delta_{is}\left(\frac{Scal}{16n(n+2)}\;+\;\frac{1}{2}\left(d\eta_1(\xi_2,\xi_3)\;+\;d\eta_2(\xi_3,\xi_1)\;+\;d\eta_3(\xi_1,\xi_2)\right)\right), \tag{2.7}$$

$s=1,2,3$ and (i,j,k) is any cyclic permutation of $(1,2,3)$.

The vanishing of the $sp(1)$-connection 1-forms on H is equivalent to the vanishing of the torsion endomorphism of the Biquard connection, see [IMV].

2.1. **The qc-Einstein condition and Bianchi identities.** We explain briefly the consequences of the Bianchi identities and the notion of qc-Einstein manifold introduced in [IMV] since it plays a crucial role in solving the Yamabe equation in the quaternionic seven dimensional sphere. For more details see [IMV].

Let $R=[\nabla,\nabla]-\nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. The Ricci tensor and the scalar curvature $Scal$ of the Biquard connection, called *qc-Ricci tensor* and *qc-sacalr curvature*, respectively, are defined by

$$Ric(X,Y)=g(R(e_a,X)Y,e_a),\quad X,Y\in H,\qquad Scal=Ric(e_a,e_a)=g(R(e_b,e_a)e_a,e_b).$$

Here and everywhere in the paper we adoped the
Convention 1: In what follows, $\{e_1,\dots,e_{4n}\}$ will denote an orthonormal basis of H and the summation convention over repeated vectors from the basis $\{e_1,\dots,e_{4n}\}$ is always assumed. For example, the formula for the qc-scalar curvature means

$$Scal=\sum_{a=1}^{4n}Ric(e_a,e_a)=\sum_{a,b=1}^{4n}g(R(e_b,e_a)e_a,e_b).$$

Convention 2. The triple (i,j,k) is always a cyclic permutation of $(1,2,3)$.

According to [Biq1] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc-Ricci tensor is zero we call the quaternionic structure *a qc-Einstein manifold* [IMV].
It is shown in [IMV] that the qc-Ricci tensor is completely determined by the components of the torsion. First, recall the notion of the $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors T^0,U on H introduced in [IMV] by

$$T^0(X,Y)\overset{def}{=}g((T^0_{\xi_1}I_1+T^0_{\xi_2}I_2+T^0_{\xi_3}I_3)X,Y),\quad U(X,Y)\overset{def}{=}g(uX,Y),\quad X,Y\in H.$$

The tensor T^0 belongs to [-1]-eigenspace while U is in the [3]-eigenspace of the operator $\dagger$ given by (2.1), i.e., they have the properties:

$$\begin{aligned}T^0(X,Y)+T^0(I_1X,I_1Y)+T^0(I_2X,I_2Y)+T^0(I_3X,I_3Y)=0,\\ 3U(X,Y)-U(I_1X,I_1Y)-U(I_2X,I_2Y)-U(I_3X,I_3Y)=0.\end{aligned}$$

Theorem 1.3, Theorem 3.12 and Corollary 3.14 in [IMV] give

Theorem 2.3. [IMV] *Let $(M^{4n+3},g,\mathbb{Q})$ be a quaternionic contact $(4n+3)$-dimensional manifold, $n>1$. For any $X,Y\in H$ the qc-Ricci tensor and the qc-scalar curvature satisfy*

$$\begin{aligned}Ric(X,Y)&=(2n+2)T^0(X,Y)+(4n+10)U(X,Y)+\frac{Scal}{n}g(X,Y)\\ Scal&=-8n(n+2)g(T(\xi_1,\xi_2),\xi_3)\end{aligned}$$

For $n=1$ the above formulas hold with $U=0$.

In particular, the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. If $Scal\neq 0$ the latter holds exactly when the qc-structure is 3-sasakian up to a multiplication by a constant and an $SO(3)$-matrix with smooth entries.

We recall that the Ricci 2-forms $\rho_s,\quad s=1,2,3$ are defined by

$$4n\ \rho_s(B,C)=g(R(B,C)e_a,I_se_a),\ B,C\in\Gamma(TM).$$

For ease of reference, in the following Theorem we summarize the properties of the Ricci 2-forms, the scalar curvature and the torsion evaluated on the vertical space established in Lemma 3.11, Corollary 3.14 Proposition 4.3 and Proposition 4.4 of [IMV] .

Theorem 2.4. [IMV] *The Ricci 2-forms satisfy*

$$\begin{aligned} \rho_1(X,Y) &= 2g((T^0_{\xi_2})^{--+}I_3X,Y) - 2g(I_1uX,Y) - \frac{Scal}{8n(n+2)}\omega_1(X,Y),\\ \rho_2(X,Y) &= 2g((T^0_{\xi_3})^{+--}I_1X,Y) - 2g(I_2uX,Y) - \frac{Scal}{8n(n+2)}\omega_2(X,Y),\\ \rho_3(X,Y) &= 2g((T^0_{\xi_1})^{-+-}I_2X,Y) - 2g(I_3uX,Y) - \frac{Scal}{8n(n+2)}\omega_3(X,Y).\end{aligned} \tag{2.8}$$

$$\begin{aligned} &\rho_i(X,\xi_i) = -\frac{X(Scal)}{32n(n+2)} + \frac12(\omega_i([\xi_j,\xi_k],X) - \omega_j([\xi_k,\xi_i],X) - \omega_k([\xi_i,\xi_j],X)),\\ &\rho_i(X,\xi_j) = \omega_j([\xi_j,\xi_k],X), \quad \rho_i(X,\xi_k) = \omega_k([\xi_j,\xi_k],X),\\ &\rho_i(I_kX,\xi_j) = -\rho_i(I_jX,\xi_k) = g(T(\xi_j,\xi_k),I_iX) = \omega_i([\xi_j,\xi_k],X),\end{aligned} \tag{2.9}$$

$$\rho_i(\xi_i,\xi_j) + \rho_k(\xi_k,\xi_j) = \frac{1}{8n(n+2)}\xi_j(Scal). \tag{2.10}$$

The torsion of the Biquard connection restricted to V satisfies the equality

$$T(\xi_i,\xi_j) = -\frac{Scal}{8n(n+2)}\xi_k - [\xi_i,\xi_j]_H, \tag{2.11}$$

where $[\xi_i,\xi_j]_H$ denotes the projection on H parallel to the vertical space V.

We also recall the definition of the vector field $\mathbb{A}$, which appeared naturally in the Bianchi identities in [IMV]

$$\mathbb{A} \;=\; I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2].$$

We shall denote with the same letter the corresponding horizontal one-form, i.e.,

$$\mathbb{A}(X) \;=\; g(I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2],X).$$

The horizontal divergence ∇^*P of a (0,2)-tensor field P on M with respect to Biquard connection is defined to be the (0,1)-tensor field

$$\nabla^*P(.) = (\nabla_{e_a}P)(e_a,.).$$

Then we conclude from [IMV, Theorem 4.8], that

Theorem 2.5. [IMV] *On any QC manifold of dimension $(4n+3)$ we have the formulas*

$$\nabla^*T^0 = (n+2)\mathbb{A}, \qquad \nabla^*U = \frac{1-n}{2}\mathbb{A} \tag{2.12}$$

3. Conformal transformations

Note that a conformal quaternionic contact transformation between two quaternionic contact manifold is a diffeomorphism Φ which satisfies

$$\Phi^*\eta = \mu\ \Psi\cdot\eta,$$

for some positive smooth function μ and some matrix $\Psi\in SO(3)$ with smooth functions as entries and $\eta=(\eta_1,\eta_2,\eta_3)^t$ is considered as an element of $\mathbb{R}^3$. The Biquard connection does not change under rotations, i.e., the Biquard connection of $\Psi\cdot\eta$ and η coincides. Hence, studying conformal transformations we may consider only transformations $\Phi^*\eta \;=\; \mu\,\eta$.

Let h be a positive smooth function on a QC manifold (M,η). Let $\bar\eta = \frac{1}{2h}\eta$ be a conformal deformation of the QC structure η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus, $d\bar\eta = -\frac{1}{2h^2}\,dh\wedge\eta + \frac{1}{2h}d\eta$ and $\bar g = \frac{1}{2h}g$. The new triple $\{\bar\xi_1,\bar\xi_2,\bar\xi_3\}$ is determined by the conditions defining the Reeb vector fields. We have

$$\bar\xi_s \;=\; 2h\,\xi_s \;+\; I_s\nabla h, \quad s=1,2,3, \tag{3.1}$$

where ∇h is the horizontal gradient defined by $g(\nabla h,X) = dh(X), \quad X\in H$.

The components of the torsion tensor transform according to the following formulas from [IMV, Section 5]

$$\overline{T}^0(X,Y) = T^0(X,Y) + h^{-1}[\nabla dh]_{[sym][-1]}, \tag{3.2}$$

$$\bar{U}(X,Y) = U(X,Y) + (2h)^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}, \tag{3.3}$$

where the symmetric part is given by

$$[\nabla dh]_{[sym]}(X,Y) = \nabla dh(X,Y) + \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X,Y).$$

In addition, the qc-scalar curvature changes according to the formula [Biq1]

$$\overline{\text{Scal}} = 2h\,(\text{Scal}) - 8(n+2)^2\,h^{-1}|\nabla h|^2 + 8(n+2)\,\triangle h. \tag{3.4}$$

The following vectors will be important for our considerations,

$$A_i = I_i[\xi_j,\xi_k], \qquad \text{hence} \quad A = A_1 + A_2 + A_3. \tag{3.5}$$

Lemma 3.1. *Let h be a positive smooth function on a QC manifold $(M,g,\mathbb{Q})$ with constant qc-scalar curvature $Scal = 16n(n+2)$ and $\bar\eta = \frac{1}{2h}\eta$ a conformal deformation of the qc structure η. Suppose $\bar\eta$ is a 3-Sasakian structure Then we have the formulas*

$$\begin{aligned} A_1(X) &= -\frac12 h^{-2}dh(X) - \frac12 h^{-3}|\nabla h|^2 dh(X)\\ &- \frac12 h^{-1}\Big(\nabla dh(I_2X,\xi_2) + \nabla dh(I_3X,\xi_3)\Big) + \frac12 h^{-1}\Big(dh(\xi_2)\,dh(I_2X) + dh(\xi_3)\,dh(I_3X)\Big)\\ &+ \frac14 h^{-1}\Big(\nabla dh(I_2X,I_2\nabla h) + \nabla dh(I_3X,I_3\nabla h)\Big). \end{aligned} \tag{3.6}$$

The expressions for A_2 and A_3 can be obtained from the above formula by a cyclic permutation of $(1,2,3)$. *Thus, we have also*

$$\begin{aligned} A(X) &= -\frac32 h^{-2}dh(X) - \frac32 h^{-3}|\nabla h|^2 dh(X)\\ &- h^{-1}\sum_{s=1}^{3}\nabla dh(I_sX,\xi_s) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)\,dh(I_sX) + \frac12 h^{-2}\sum_{s=1}^{3}\nabla dh(I_sX,I_s\nabla h) \end{aligned}$$

Proof. First we calculate the $sp(1)$-connection 1-forms of the Biquard connection ∇. For a 3-Sasaki structure we have $d\bar\eta_i(\bar\xi_j,\bar\xi_k) = 2$, $\bar\xi_i\lrcorner d\bar\eta_i = 0$, the non-zero $sp(1)$-connection 1-forms are $\alpha_i(\xi_i) = -2, i = 1,2,3$, and the qc-scalar curvature $\overline{S}cal = 16n(n+2)$ (see [Example 4.12,[IMV]]). Then (3.1), (2.6), and (2.7) yield

$$\begin{aligned} 2d\eta_i(\xi_j,\xi_k) &= 2h^{-1} + h^{-2}\|dh\|^2, & \alpha_i(X) &= -h^{-1}dh(I_iX),\\ \alpha_i(\xi_j) &= -h^{-1}dh(\xi_k) = -\alpha_j(\xi_i), & 4\alpha_i(\xi_i) &= -4 - 2h^{-1} - h^{-2}\|dh\|^2. \end{aligned} \tag{3.7}$$

From the 3-Sasakian assumption the commutators are $[\bar\xi_i,\bar\xi_j] = -2\bar\xi_k$. Thus, for $X\in H$ taking also into account (3.1) we have

$$g([\bar\xi_i,\bar\xi_j],\,I_3X) = -2\,g(\bar\xi_3,\,I_3X) = -2\,g(2\,h\,\xi_3 + I_3\nabla h,\,I_3\nabla h) = -2\,dh\,(X).$$

Therefore, using again (3.1), we obtain

$$\begin{aligned} -2\,dh(X) &= g([\bar\xi_1,\bar\xi_2],\,I_3X) = g\big([2h\xi_1 + I_1\nabla h, 2h\xi_2 + I_2\nabla h],\,I_3X\big)\\ &= -4h^2A_3(X) + 2hg([\xi_1,\,I_2\nabla h],\,I_3X) + 2hg([I_1\nabla h,\xi_2],\,I_3X) + g([I_1\nabla h,\,I_2\nabla h],I_3X) \end{aligned} \tag{3.8}$$

The last three terms are transformed as follows. The first equals

$$\begin{aligned} g([\xi_1,\,I_2\nabla h],I_3X) &= g((\nabla_{\xi_1}I_2)\,dh + I_2\nabla_{\xi_1}dh,\,I_3X) - g(T(\xi_1,\,I_2dh),\,I_3X)\\ &= -\alpha_3(\xi_1)\,dh(I_2X) + \alpha_1(\xi_1)\,dh(X) - \nabla dh\,(\xi_1,I_1X) - g(T(\xi_1,\,I_2dh),\,I_3X), \end{aligned}$$

where we use (2.5) and the fact that ∇ preserves the splitting $H \oplus V$, the second and the tird equal

$$\begin{aligned} g\,([I_1\nabla h, \xi_2], I_3\,X) \;&=\; \alpha_2(\xi_2)\,dh(X) \;+\; \alpha_3(\xi_2)\,dh(I_1X) - \;\nabla dh\,(\xi_2, I_2X) \;-\; g\,(T(I_1dh,\,\xi_2),\; I_3X),\\ g\,([I_1\nabla h, I_2\nabla h], I_3\,X) \;&=\; -\alpha_3(I_1dh)\,dh(I_2X) \;+\; \alpha_1(I_1dh)\,dh(X) \;-\; \nabla dh\,(I_1\nabla h, I_1X)\\ &\quad +\; \alpha_2(I_2dh)\,dh(X) \;+\; \alpha_3(I_2dh)\,dh(I_1X) \;-\; \nabla dh\,(I_2\nabla h, I_2X). \end{aligned}$$

Next we apply (3.7) to the last three equalities, then substitute their sum into (3.8), after which we use the commutation relations

$$\begin{aligned} \nabla dh\,(X,Y) - \nabla dh\,(Y,X) \;&=\; -dh(T(X,Y)) = \;-2\sum_{s=1}^{3}\omega_s(X,Y)\,dh(\xi_s),\\ \nabla dh\,(X,\xi) - \nabla dh\,(\xi,X) \;&=\; -dh(T(X,\xi)), \qquad X,\,Y\in H,\quad \xi\in V. \end{aligned} \tag{3.9}$$

The result is the following identity

$$\begin{aligned} 4h^2A_3(X) \;&=\; (-4h \;+\; h^{-1}\|\nabla h\|^2)\,dh(X)\\ &- 2h\left[\nabla dh\,(I_1X,\xi_1) \;+\; \nabla dh\,(I_2X,\xi_2)\right] - \;\left[\nabla dh\,(I_1X, I_1\nabla h)) \;+\; \nabla dh\,(I_2X, I_2\nabla h)\right]\\ &+\; 2\left[dh(\xi_1)\,dh(I_1X) \;+\; dh(\xi_2)\,dh(I_2X) \;+\; 2\,dh(\xi_3)\,dh(I_3X)\right]\\ &+\; 2h\left[T(\xi_1, I_1X, dh) \;+\; T(\xi_2, I_2X, dh) \;-\; T(\xi_1, I_2\nabla h, I_3dh) \;+\; T(\xi_2, I_1\nabla h, I_3dh)\right]. \end{aligned} \tag{3.10}$$

With the help of Proposition 2.2 the sum of the torsion terms can be seen to equal $2T^{0^{--+}}(X,\,\nabla h) - 4\,U(X,\nabla h)$ which allows us to rewrite (3.10) in the form

$$\begin{aligned} 4A_3(X) \;&=\; (-4h^{-1} \;+\; h^{-3}\|\nabla h\|^2)\,dh(X) - \;2\,h^{-1}\left[\nabla dh\,(I_1X,\xi_1) \;+\; \nabla dh\,(I_2X,\xi_2)\right]\\ &+\; 2\,h^{-2}\left[dh(\xi_1)\,dh(I_1X) \;+ dh(\xi_2)\,dh(I_2X) \;+\; 2\,dh(\xi_3)\,dh(I_3X)\right]\\ &-\; h^{-2}\left[\nabla dh\,(I_1X, I_1\nabla h)) \;+\; \nabla dh\,(I_2X, I_2\nabla h)\right] \;+\; 4h^{-1}\left[(T^{0^{--+}}(\nabla h, X) \;-\; 2U(\nabla h, X)\right]. \end{aligned} \tag{3.11}$$

Using (3.2) the $T^{0^{--+}}$ component of the torsion can be expressed by h as follows

$$\begin{aligned} 4T^{0^{--+}}(\nabla h, X) \;&=\; T^0(\nabla h, X) \;-\; T^0(I_1\nabla h, I_1X) \;-\; T^0(I_2\nabla h, I_2X) \;+\; T^0(I_3\nabla h, I_3X) =\\ -\,h^{-1}&\Big\{[\nabla dh]_{[-1]}(\nabla h, X) - [\nabla dh]_{[-1]}(I_1\nabla h, I_1X) - [\nabla dh]_{[-1]}(I_2\nabla h, I_2X) + [\nabla dh]_{[-1]}(I_3\nabla h, I_3X)\Big\}\\ &- h^{-1}\sum_{s=1}^{3}\Big\{dh(\xi_s)\,\Big[g(I_s\nabla h, X) \;-\; g(I_sI_1\nabla h, I_1X) - \;g(I_sI_2\nabla h, I_2X) \;+\; g(I_sI_3\nabla h, I_3X)\Big]\Big\}\\ &=\; -h^{-1}\,\Big\{\nabla dh\,(\nabla h, X) \;-\; \nabla dh\,(I_1\nabla h, I_1X) \;-\; \nabla dh\,(I_2\nabla h, I_2X) \;+\; \nabla dh\,(I_3\nabla h, I_3X)\Big\}\\ &\qquad -\; 4h^{-1}\,dh(\xi_3)\,dh(I_3X). \end{aligned}$$

On the other hand, (3.3) and the Yamabe equation (3.4) give

$$
\begin{aligned}
(3.12)\quad 8\,U(\nabla h, X) \;&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_sX)\\
&\qquad -\,2h^{-1}\|\nabla h\|^2\,dh(X) \;-\; \frac{\triangle h}{n}\,dh(X) \;+\; 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_sX)\Big\}\\
&\quad -\; h^{-1}\Big\{-2h^{-1}\|\nabla h\|^2\,dh(X) \;-\; \frac{2n-4nh+(n+2)h^{-1}\|\nabla h\|^2}{n}\,dh(X) \;+\; 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_sX)\Big\} -\; h^{-1}\big(-3h^{-1}\|\nabla h\|^2 \;-\; 2 \;+\; 4h\big)dh(X).
\end{aligned}
$$

Substituting the last two formulas in (3.11) gives A_3 in the form (3.6). □

4. Divergence formulas

We shall need the divergences of various vector/forms through the almost complex structures, so we start with a general formula valid for any horizontal vector/form A. Let $\{e_1, \dots, e_{4n}\}$ be an orthonormal basis of H.

$$\nabla^*(I_1A) \;\equiv\; (\nabla_{e_a}(I_1A))(e_a) \;=\; -(\nabla_{e_a}A)(I_1e_a) \;-\; A((\nabla_{e_a}I_1)e_a),$$

recalling $I_1A(X) = -A(I_1X)$.

We say that an orthonormal frame

$$\{e_1, e_2 = I_1e_1, e_3 = I_2e_1, e_4 = I_3e_1 \dots, e_{4n} = I_3e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

is a qc-normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 4.5 in [IMV] asserts that a qc-normal frame exists at each point of a QC manifold. With respect to a qc-normal frame the above divergence reduces to

$$\nabla^*(I_1A) \;\;=\;\; -(\nabla_{e_a}A)(I_1e_a).$$

Lemma 4.1. *Suppose $(M, \eta, \mathbb{Q})$ is a quaternionic contact manifold with constant qc-scalar curvature. For any function h we have the following formulas*

$$
\begin{aligned}
div\Big(\sum_{s=1}^{3} dh(\xi_s)I_sA_s\Big) \;&=\; \sum_{s=1}^{3}\;\nabla dh\,(I_se_a, \xi_s)A_s(e_a)\\
div\Big(\sum_{s=1}^{3} dh(\xi_s)I_sA\Big) \;&=\; \sum_{s=1}^{3}\;\nabla dh\,(I_se_a, \xi_s)A(e_a).
\end{aligned}
$$

Proof. We note that the vectors $\sum_{s=1}^{3} dh(\xi_s)I_sA_s$ and $\sum_{s=1}^{3} dh(\xi_s)I_sA$ are $Sp(n)Sp(1)$ invariant vectors so it is sufficient to compute their divergences in a qc-normal frame. To avoid the introduction of new variables, in this proof, we shall assume that $\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$ is a qc-normal frame.

We apply (2.11). Using that the Biquard connection preserves the splitting of TM, we find

$$
\begin{aligned}
\nabla^*[\xi_1, \xi_2] \;&=\; -g(\nabla_{e_a}\big(T(\xi_1, \xi_2)\big), e_a)\\
&=\; -\Big[g((\nabla_{e_a}T)\,(\xi_1, \xi_2), e_a) \;+\; g(T(\nabla_{e_a}\xi_1, \xi_2), e_a) \;+\; g(T(\xi_1, \nabla_{e_a}\xi_2), e_a)\Big].
\end{aligned}
$$

From Bianchi's identity we have ($\sigma_{A,B,C}$ means a cyclic sum over (A,B,C))

$$\begin{aligned} g((\nabla_{e_a}T)(\xi_1,\xi_2),e_a) \;&=\; -g((\nabla_{\xi_1}T)(\xi_2,e_a),e_a) \;-\; g((\nabla_{\xi_2}T)(e_a,\xi_1),e_a) \\ &\quad -\; \sigma_{e_a,\xi_1,\xi_2}\Big\{T(T(e_a,\xi_1),\xi_2,e_a)\Big\} \;+\sigma_{e_a,\xi_1,\xi_2}\Big\{R(e_a,\xi_1,\xi_2,e_a)\Big\} \\ &= -\Big[T(T(e_a,\xi_1),\xi_2,e_a) \;+\; T(T(\xi_1,\xi_2),e_a,e_a) \;+\; T(T(\xi_2,e_a),\xi_1,e_a)\Big] \\ &= \Big[g(T(T_{\xi_1}e_a,\xi_2),e_a) \;-\; g(T(T_{\xi_2}e_a,\xi_1),e_a) \;-\; g(T(T(\xi_1,\xi_2),e_a),e_a)\Big], \end{aligned}$$

after using the properties of the curvature tensor to obtain the next to last line. The last term is equal to zero as

$$\begin{aligned} g(T(T(\xi_1,\xi_2),e_a),e_a) \;&=\; g\big(T(-\frac{\mathrm{Scal}}{n}\xi_3 \;-\; [\xi_1,\xi_2]_H,e_a),e_a\big) \\ &= \;-\frac{\mathrm{Scal}}{n}\,g(T(\xi_3,e_a),e_a) \;=0, \end{aligned}$$

taking into account that the torsion T_{ξ_3} is traceless and $T([\xi_1,\xi_2]_H,e_a)$ is a vertical vector. On the other hand,

$$\begin{aligned} &\Big[g(T(T_{\xi_1}e_a,\xi_2),e_a) \;-\; g(T(T_{\xi_2}e_a,\xi_1),e_a)] \\ &\qquad = \;-\sum_{a,b=1}^{4n}[T(e_b,\xi_2,e_a)\,T(\xi_1,e_a,e_b) \;-\; T(e_b,\xi_1,e_a)\,T(\xi_2,e_a,e_b)\Big] \\ &\qquad = \sum_{a,b=1}^{4n}\Big[T(\xi_2,e_b,e_a)\,T(\xi_1,e_a,e_b) \;-\; T(\xi_1,e_b,e_a)\,T(\xi_2,e_a,e_b)\Big] \;=\; 0. \end{aligned}$$

The equalities $\nabla^*(I_1A_1) = \nabla^*(I_2A_2) = 0$ with respect to a qc-normal frame can be obtained similarly. Hence, the first formula in Lemma 4.1 follows.

We are left with proving the second divergence formula. Since the scalar curvature is constant (2.9) implies

$$A(X) = -2\sum_{s=1}^{3}\rho_s(X,\xi_s). \tag{4.1}$$

Fix an $s\in\{1,2,3\}$. Working again in a qc-normal frame we have

$$(\nabla_{e_a}A),I_se_a) \;=\; -2\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_se_a,\xi_t).$$

A calculation involving the expressions (2.8) and the properties of the torsion shows that

$$\mathrm{tr}\,(\rho_t\circ I_s) \;=\; -4\,\delta_{st}\,\mathrm{Scal}. \tag{4.2}$$

The second Bianchi identity

$$\begin{aligned} 0 \;=\; &(\nabla_{e_a}R)(I_se_a,\xi_t,e_b,I_te_b) \;+\; (\nabla_{I_se_a}R)(\xi_t,e_a,e_b,I_te_b) \;+\; (\nabla_{\xi_t}R)(e_a,I_se_a,e_b,I_te_b) \\ &+\; R(T(e_a,I_se_a),\xi_t,e_b,I_te_b) \;+\; R(T(I_se_a,\xi_t),e_a,e_b,I_te_b) \;+\; R(T(\xi_t,e_a),I_se_a,e_b,I_te_b), \end{aligned}$$

together with the constancy of the qc-scalar curvature and (4.2) show that the third term on the right is zero and thus

$$\sum_{t=1}^{3}\Big\{2\,(\nabla_{e_a}\rho_t)(I_se_a,\xi_t) \;-\; 2\,\rho_t(T(\xi_t,I_se_a),e_a) \;+\; \rho_t(T(e_a,I_se_a),\xi_t)\Big\} \;=\; 0.$$

Substituting (2.4) in the above equality we come to the equation

$$\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_se_a,\xi_t) \;=\; \sum_{t=1}^{3}\rho_t(T(\xi_t,I_se_a),e_a)-4n\sum_{t=1}^{3}\rho_t(\xi_s,\xi_t)=0, \tag{4.3}$$

where the vanishing of the second term follows from (2.10), while the vanishing of the first term is seen as follows. The definition of $T^0_{\xi_s}$, the formulas in Theorem 2.4 and Proposition 2.2 imply

$$\begin{aligned}
&\sum_{s=1}^{3}\rho_s(T(\xi_s,I_1e_a),e_a)\\
&\quad= g(\rho_1,T^0_{\xi_1}I_1)\;+\;g(\rho_2,T^0_{\xi_2}I_1)\;+\;g(\rho_3,T^0_{\xi_3}I_1)\;-\;g(\rho_1,u)\;-\;g(\rho_2,I_3u)\;+\;g(\rho_3,I_2u)\\
&\quad= g(\rho_1,T^0_{\xi_1}I_1)\;+\;g(\rho_2,T^0_{\xi_2}I_1)\;+\;g(\rho_3,T^0_{\xi_3}I_1)\\
&\quad= g(2(T^0_{\xi_2})^{--+}I_3\;-\;2I_1u\;-\;\frac{Scal}{n}I_1,\,T^0_{\xi_1}I_1)+\\
&g(2(T^0_{\xi_3})^{+--}I_1\;-\;2I_2u\;-\;\frac{Scal}{n}I_2,\,T^0_{\xi_2}I_1)+\;g(2(T^0_{\xi_1})^{-+-}I_2\;-\;2I_3u\;-\;\frac{Scal}{n}I_3,\,T^0_{\xi_3}I_1)\\
&\quad= -2g((T^0_{\xi_2})^{--+}I_2,T^0_{\xi_1})\;+\;2g((T^0_{\xi_3})^{+--},T^0_{\xi_2})\;+\;2g((T^0_{\xi_1})^{-+-}I_3,T^0_{\xi_3})\\
&\quad= 2g((T^0_{\xi_3})^{+--},(T^0_{\xi_2})^{+--})\;+\;2g((T^0_{\xi_1})^{-+-},I_3(T^0_{\xi_3})^{+--})\\
&\quad= 2g(I_2(T^0_{\xi_3})^{+--},I_2(T^0_{\xi_2})^{+--})\;-\;2g(I_1(T^0_{\xi_1})^{-+-},I_2(T^0_{\xi_3})^{+--})\;=\;0.
\end{aligned}$$

Renaming the almost complex structures shows that the same conclusion is true when we replace I_1 with I_2 or I_3 in the above calculation.

Finally, the second formula in Lemma 4.1 follows from (4.1) and (4.3). ∎

We shall also need the following one-forms

$$\begin{aligned}
D_1(X) \;&=\; -h^{-1}T^{0^{+--}}(X,\nabla h)\\
D_2(X) \;&=\; -h^{-1}T^{0^{-+-}}(X,\nabla h)\\
D_1(X) \;&=\; -h^{-1}T^{0^{--+}}(X,\nabla h)
\end{aligned} \tag{4.4}$$

For simplicity, using the musical isomorphism, we will denote with D_1, D_2, D_3 the corresponding (horizontal) vector fields, for example $\quad g(D_1,X)=D_1(X) \qquad \forall X\in H.\quad$ Finally, we set

$$D\;=\;D_1\;+\;D_2\;+\;D_3\;=\;-h^{-1}\,T^0(X,\nabla h).$$

Lemma 4.2. *Suppose (M,η) is a quaternionic contact manifold with constant qc-scalar curvature $Scal=16n(n+2)$. Suppose $\bar\eta=\frac{1}{2h}\eta$ has vanishing $[-1]$-torsion component $\overline{T}^0=0$. We have*

$$D(X)\;=\;\frac{1}{4}h^{-2}\Big(3\,\nabla dh(X,\nabla h)\;-\;\sum_{s=1}^{3}\,\nabla dh(I_sX,I_s\nabla h)\Big)\;+\;h^{-2}\sum_{s=1}^{3}dh(\xi_s)\,dh(I_sX).$$

and the divergence of D satisfies

$$div\,D\;=\;|T^0|^2\;-h^{-1}g(dh,D)\;-\;h^{-1}(n+2)\,g(dh,A).$$

Proof. a) The formula for D follows immediately from (3.2).

b) We work in a qc-normal frame. Since the scalar curvature is assumed to be constant we use (2.12) to find

$$\begin{aligned}
\nabla^*D\;&=\;-h^{-1}\,dh(e_a)D(e_a)\;-\;h^{-1}\nabla^*T^0(\nabla h)\;-\;h^{-1}T^0(e_a,e_b)\,\nabla dh(e_a,e_b)\\
&=\;-h^{-1}\,dh(e_a)D(e_a)\;-\;h^{-1}(n+2)\,dh(e_a)A(e_a)\;-\;g(T^0,\,h^{-1}\,\nabla dh)\\
&=\;|T^0|^2\;-h^{-1}dh(e_a)D(e_a)\;-\;h^{-1}(n+2)\,dh(e_a)A(e_a),
\end{aligned}$$

using (3.2) in the last equality. ∎

Let us also consider the following one-forms (and corresponding vectors)

$$F_s(X) \;=\; -h^{-1}\, T^0(X, I_s\nabla h), \quad s=1,2,3.$$

From the definition of F_1 and (4.4) we find

$$\begin{aligned} F_1(X) \;&=\; -h^{-1}T^0(X, I_1\nabla h)\\ &=\; -h^{-1}T^{0+--}(X, I_1\nabla h) \;-\; h^{-1}T^{0-+-}(X, I_1\nabla h) \;-h^{-1}T^{0--+}(X, I_1\nabla h)\\ &=\; h^{-1}T^{0+--}(I_1X, \nabla h) \;-\; h^{-1}T^{0-+-}(I_1X, \nabla h) \;-h^{-1}T^{0--+}(I_1X, \nabla h)\\ &=\; -D_1(I_1X) \;+\; D_2(I_1X) \;+\; D_3(I_1X). \end{aligned}$$

Thus, the forms F_s can be expressed by the forms D_s as follows

$$\begin{aligned} F_1(X) \;&=\; -D_1(I_1X) \;+\; D_2(I_1X) \;+\; D_3(I_1X)\\ F_2(X) \;&=\; D_1(I_2X) \;-\; D_2(I_2X) \;+\; D_3(I_2X) \qquad (4.5)\\ F_3(X) \;&=\; D_1(I_3X) \;+\; D_2(I_3X) \;-\; D_3(I_3X). \end{aligned}$$

Lemma 4.3. *Suppose (M,η) is a quaternionic contact manifold with constant qc-scalar curvature $Scal = 16n(n+2)$. Suppose $\bar\eta = \frac{1}{2h}\eta$ has vanishing $[-1]$-torsion component, $\overline{T}^0 = 0$. We have*

$$\begin{aligned} div\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^3\Big[\,\nabla dh\,(I_se_a,\xi_s)F_s(I_se_a)\Big]\\ &\quad+\; h^{-1}\sum_{s=1}^3\Big[dh(\xi_s)dh(I_se_a)D(e_a) \;+(n+2)\,dh(\xi_s)dh(I_se_a)\,A(e_a)\Big]. \end{aligned}$$

Proof. We note that the vector $\sum_{s=1}^3 dh(\xi_s)F_s$ is an $Sp(n)Sp(1)$ invariant vector, hence, we may assume that $\{e_1,\dots,e_{4n},\xi_1,\xi_2,\xi_3\}$ is a qc-normal frame. Since the scalar curvature is assumed to be constant we can apply Theorem 2.5, thus $\nabla^*T^0 \;=\; (n+2)A$. Turning to the divergence, we compute

$$\begin{aligned} \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^3\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big] \;-\; \sum_{s=1}^3 h^{-1}\,dh(\xi_s)\,\nabla^*T^0(I_s\nabla h)\\ &\quad+\sum_{s=1}^3\sum_{a,\,b=1}^{4n}\Big[h^{-2}\,dh(\xi_s)\,dh(e_a)T^0(e_a,I_se_b)\,dh(e_b) \;-\; h^{-1}\,dh(\xi_s)\,T^0(e_a,I_se_b)\,\nabla dh\,(e_a,e_b)\Big]\\ &=\; \sum_{s=1}^3\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big] \;-\; \sum_{s=1}^3 h^{-1}\,dh(\xi_s)\,\nabla^*T^0(I_s\nabla h)\\ &\quad+\sum_{s=1}^3\Big[\,h^{-1}\,dh(\xi_s)\,dh(I_se_a)\,D(e_a) \;+\; h^{-1}(n+2)\,dh(\xi_s)\,dh(I_se_a)\,A(e_a)\Big]\\ &=\; \sum_{s=1}^3\Big[\,\nabla dh\,(e_a,\xi_s)F_s(e_a) \;+\; h^{-1}\,dh(\xi_s)\,dh(I_se_a)\,D(e_a) \;+\; h^{-1}(n+2)\,dh(\xi_s)\,dh(I_se_a)\,A(e_a)\Big], \end{aligned} \tag{4.6}$$

using the symmetry of T^0 in the next to last equality, and the fact

$$T^0(e_a,I_1e_b)\;\;\nabla dh\,(e_a,e_b) \;=\; 0,$$

which is a consequence of (3.2). Switching to the basis $\{I_se_a : a=1,\dots,4n\}$ in the first term of the right-hand-side of (4.6) completes the proof. ∎

At this point we restrict our considerations to the 7-dimensional case. Following is our main technical result.

Theorem 4.4. *Suppose (M^7,η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^7,\bar\eta)$, $\tilde\eta = \frac{1}{2h}\eta$. If $Scal_\eta = Scal_{\tilde\eta} = 16n(n+2)$, then with f given by*

$$f = \frac{1}{2} + h + \frac{1}{4}h^{-2}|\nabla h|^2,$$

the following identity holds

$$div\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\, F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\, I_s A\Big) = f|T^0|^2 + h\,\langle QV, V\rangle.$$

Here, Q is a positive definite matrix and $V = (D_1, D_2, D_3, A_1, A_2, A_3)$ with A_s, D_s defined, correspondingly, in (3.5) and (4.4).

Proof. From Lemmas 4.1, 4.3 and 4.2 it follows

$$\begin{aligned}
(4.7)\quad & \operatorname{div}\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\, F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\, I_s A\Big)\\
&= \sum_{a=1}^{4}\Big(dh(e_a) - \frac{1}{2}h^{-3}dh(e_a)|\nabla h|^2 + \frac{1}{2}h^{-2}\ \nabla dh(e_a,\nabla h)\Big)\, D(e_a)\\
&+ f\,\Big(|T^0|^2 - h^{-1}\sum_{a=1}^{4} dh(e_a) D(e_a) - h^{-1}(n+2)\sum_{a=1}^{4} dh(e_a) A(e_a)\Big)\\
&+ \sum_{s=1}^{3}\sum_{a=1}^{4}\ \nabla dh(I_s e_a,\xi_s) F_s(I_s e_a)\\
&+ h^{-1}\sum_{s=1}^{3}\sum_{a=1}^{4}\Big[dh(\xi_s)\, dh(I_s e_a)\, D(e_a) + h^{-1}(n+2)\, dh(\xi_s)\, dh(I_s e_a)\, A(e_a)\Big]\\
&+ 4\sum_{s=1}^{3}\sum_{a=1}^{4} \nabla dh(I_s e_a,\xi_s)\, A_s(e_a) - \frac{10}{3}\sum_{s=1}^{3}\sum_{a=1}^{4}\ \nabla dh(I_s e_a,\xi_s)\, A(e_a)
\end{aligned}$$

Since the dimension of M is seven we have $U = \bar U = [\,\nabla dh - 2h^{-1} dh\otimes dh]_{[3][0]} = 0$. The last equation together with the Yamabe equation (3.4) yield the formula, cf. (3.12),

$$(4.8)\quad \nabla dh\,(\nabla h, X) + \sum_{s=1}^{3} \nabla dh\,(I_s\nabla h, I_s X) + \big(-2 + 4h - 3h^{-1}\|\nabla h\|^2\big)\, dh(X) = 0.$$

From Lemma 4.2 and 4.4 we have

$$(4.9)\quad \begin{aligned}
D_1(X) &= h^{-2}\, dh(\xi_1)\, dh(I_1 X) + \frac{1}{4}h^{-2}\big[\,\nabla dh\,(X,\nabla h) + \nabla dh\,(I_1 X, I_1\nabla h)\\
&\qquad - \nabla dh\,(I_2 X, I_2\nabla h) - \nabla dh\,(I_3 X, I_3\nabla h)\big],\\
D_2(X) &= h^{-2}\, dh(\xi_2)\, dh(I_2 X) + \frac{1}{4}h^{-2}\big[\nabla dh\,(X,\nabla h) - \nabla dh\,(I_1 X, I_1\nabla h)\\
&\qquad + \nabla dh\,(I_2 X, I_2\nabla h) - \nabla dh\,(I_3 X, I_3\nabla h)\big],\\
D_3(X) &= h^{-2}\, dh(\xi_3)\, dh(I_3 X) + \frac{1}{4}h^{-2}\big[\nabla^2 h(X,\nabla h) - \nabla dh\,(I_1 X, I_1\nabla h)\\
&\qquad - \nabla dh\,(I_2 X, I_2\nabla h) + \nabla dh\,(I_3 X, I_3\nabla h)\big].
\end{aligned}$$

Lemma 3.1, (4.8), (4.9) and (4.5) allow us to rewrite the divergence (4.7) in the form

$$\begin{aligned}\operatorname{div}\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big)\\ = f|T^0|^2 + h\,\sigma_{1,2,3}\Big\{ g\,(D_1,\, 3A_1 - A_2 - A_3 + 2\,D_1)\\ + g\,(A_1,\, \frac{22}{3}A_1 - \frac{2}{3}A_2 - \frac{2}{3}A_3 + \frac{11}{3}D_1 - \frac{1}{3}D_2 - \frac{1}{3}D_3)\Big\},\end{aligned}$$

where $\sigma_{1,2,3}$ denotes the sum over all positive permutations of $(1,2,3)$, Let Q equal to

$$Q := \begin{bmatrix} 2 & 0 & 0 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3}\\ 0 & 2 & 0 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3}\\ 0 & 0 & 2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3}\\ \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3}\\ -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

so that

$$\operatorname{div}\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4dh(\xi_s)\,I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big) = f\,|T^0|^2 + h\,\langle QV, V\rangle,$$

with $V = (D_1, D_2, D_3, A_1, A_2, A_3)$. It is not hard to see that the eigenvalues of Q are given by

$$\{0, \quad 0, \quad 2\,(2+\sqrt{2}), \quad 2\,(2-\sqrt{2}), \quad 10, \quad 10\},$$

which shows that Q is a non-negative matrix. □

5. Proofs of the main theorems

The proofs rely on Theorem 4.4. Consider first the case of the (seven dimensional) sphere. Integrating the divergence formula of Theorem 4.4 we see that according to divergence theorem established in [IMV, Proposition 8.1] the integral of the left-hand side is zero. Thus, the right-hand side vanishes, as well, which shows that the new structure has vanishing torsion, i.e., it is also qc-Einstein and has integrable vertical distribution. An application of [IMV, Theorem 1.2] proves the first claim of Theorem 1.1. It follows from [GV2], [Va1, Va] and [IMV] that the minimum $\lambda(S^{4n+3})$ is achieved and every minimizer is a smooth 3-contact form (of constant qc-scalar curvature). This shows the second claim of Theorem 1.1.

The proof of Theorem 1.3 follows as well invoking the Cayley transform, which is a quaternionic contact conformal diffeomorphism, see [IMV, Section 8.3] for details. To be precise, we should say that in [IMV] we used left invariant orthonormal basis in which the quaternionic Heisenberg group is not a group of Heisenberg type, but this does not affect the results with the exception of the look of some formulas. We finish the section with the formulas one obtains when working in the basis and metric from [IMV, Section 5.2].

As a manifold the quaternionic Heisenberg group of dimension 7 is $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}\times \operatorname{Im}\mathbb{H}$. The group law is given by $(q', \omega') = (q_o, \omega_o)\circ(q, \omega) = (q_o + q, \omega + \omega_o + 2\,\operatorname{Im}\, q_o\,\bar{q})$, where $q,\ q_o \in \mathbb{H}$ and $\omega, \omega_o \in \operatorname{Im}\mathbb{H}$. The standard qc structure is determined by the left-invariant 3-contact form $\tilde\Theta = (\tilde\Theta_1,\ \tilde\Theta_2,\ \tilde\Theta_3) = \frac{1}{2}\,(d\omega - q'\cdot d\bar q' + dq'\cdot \bar q')$. The left-invariant orthonormal basis is

$$\begin{aligned} T_1 &= \frac{\partial}{\partial t_1} + 2x^1\frac{\partial}{\partial x} + 2y^1\frac{\partial}{\partial y} + 2z^1\frac{\partial}{\partial z} & X_1 &= \frac{\partial}{\partial x_1} - 2t^1\frac{\partial}{\partial x} - 2z^1\frac{\partial}{\partial y} + 2y^1\frac{\partial}{\partial z}\\ Y_1 &= \frac{\partial}{\partial y_1} + 2z^1\frac{\partial}{\partial x} - 2t^1\frac{\partial}{\partial y} - 2x^1\frac{\partial}{\partial z} & Z_1 &= \frac{\partial}{\partial z_1} - 2y^1\frac{\partial}{\partial x} + 2x^1\frac{\partial}{\partial y} - 2t^1\frac{\partial}{\partial z}\end{aligned}$$

using $q = t_1 + i\,x_1 + j\,y_1 + k\,z_1$ and $\omega = i\,x + j\,y + k\,z$. The central (vertical) vector fields ξ_1, ξ_2, ξ_3 are described as follows

$$\xi_1 = 2\frac{\partial}{\partial x} \qquad \xi_2 = 2\frac{\partial}{\partial y} \qquad \xi_3 = 2\frac{\partial}{\partial z}\,.$$

The group $\boldsymbol{G}\,(\mathbb{H})$ can be identified with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H} \ :\ \Re\, p' = |q'|^2\},$$

by using the map $(q', \omega') \mapsto (q', |q'|^2 - \omega')$. The Cayley transform is the map $\mathcal{C} : S \mapsto \Sigma$ from the sphere $S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$ minus a point to the Heisenberg group Σ, with $\mathcal{C}$ defined by

$$(q', p') = \mathcal{C}\,\Big((q, p)\Big), \qquad q' = (1+p)^{-1}\, q, \quad p' = (1+p)^{-1}\,(1-p).$$

We have

$$\lambda\ \cdot (\mathcal{C}^{-1})^*\,\tilde\eta\ \cdot \bar\lambda = \frac{8}{|1+p'\,|^2}\,\tilde\Theta,$$

where $\lambda = \frac{1+p'}{|1+p'\,|}$ is a unit quaternion and $\tilde\eta$ is the standard quaternionic contact form on the sphere,

$$\tilde\eta = dq\cdot \bar q + dp\cdot \bar p - q\cdot d\bar q - p\cdot d\bar p.$$

With the above notation, let $\Theta = \frac{1}{2h}\tilde\Theta$ be a conformal deformation of the standard qc-structure $\tilde\Theta$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. Our result is that if $n = 1$ and Θ has constant scalar curvature, then up to a left translation the function h is given by

$$h = c\,\Big[\big(1 + \nu\,|q|^2\big)^2 + \nu^2\,(x^2 + y^2 + z^2)\Big],$$

where c and ν are any positive constants. A small calculation shows that this is equivalent to the fact that if we set

$$F = 1024\,[(1+|q|^2)^2 + |\omega|^2]^{-2},$$

then F satisfies the Yamabe equation

$$\big(T_1^2 + X_1^2 + Y_1^2 + Z_1^2\big)\,F = -F^{2^*-1} \tag{5.1}$$

and all other solution of (5.1) are obtained from F by translations and dilations,

$$\tau_{(q_o,\omega_o)}F\ (q,\omega) \overset{def}{=} F(q_o+q, \omega+\omega_o), \tag{5.2}$$

$$F_\lambda\,(q) \overset{def}{=} \lambda^4\ F(\lambda q, \lambda^2\omega), \qquad \lambda > 0. \tag{5.3}$$

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