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On prime decompositions of knotted graphs and orbifolds

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Abstract

The famous Kneser-Milnor Theorem states that every compact orientable 3-manifold can be presented as a connected sum of prime factors, which are unique up to order. Is a similar prime decomposition theorem true for 3-orbifolds? In 1994 Carlo Petronio posed this question and answered it positively for 3-orbifolds containing neither bad nor nonseparating 2-suborbifolds. The question also makes sense for knotted graphs in 3-manifolds. We show that in general the answer is negative in both cases. Moreover, the set of all possible counterexamples admits an acceptable description. We conjecture that, in a certain sense, it is generated by a finite number of basic counterexamples.

1 Introduction

We begin by considering knotted graphs in 3-manifolds. Transition from knotted graphs to orbifolds is straightforward. We describe it in the last section.

Definition 1. *A knotted graph is a pair (M, G), where M is a closed oriented 3-manifold and G an arbitrary graph (compact one-dimensional polyhedron) in M. Two knotted graphs are equivalent if they are homeomorphic as pairs.*

Definition 2. *A 2-sphere $S \subset (M, G)$ is called compressible if there is a disc $D \subset M$ such that $D \cap S = \partial D$, $D \cap G = \emptyset$, and each of the two discs bounded by ∂D on S intersects G. Otherwise S is incompressible.*

By analogy with connected sums of 3-manifolds we define three types of connected sums of knotted graphs, see [2, 5]. Let $(M_i, G_i), i = 1, 2$, be knotted graphs and $B_i \subset M_i$ 3-balls such that both pairs $(B_i, B_i \cap G_i)$ are homeomorphic to $(\mathrm{Con}\, S^2, \mathrm{Con}\, X)$, where a subset X of S^2 consists of 0, 2, or 3 points and Con is the cone. Additionally we assume that both spheres $S_i = \partial B_i$ are incompressible. Choose a homeomorphism $h\colon (S_1, S_1 \cap G_1) \to (S_2, S_2 \cap G_2)$ which reverses the induced orientations of S_1, S_2.

Definition 3. *The pair (M, G) obtained by gluing together the manifolds $M_i \setminus$ Int B_i, $i = 1, 2$, by h is called a connected sum of (M_i, G_i) and denoted $(M_1, G_1)\#(M_2, G_2)$.*

Altogether there are 1, 2, or 6 different homeomorphisms for $k = 0, 2$, and 3 respectively. Suppose that one of the pairs (M_i, G_i) (say (M_1, G_1) is trivial, i.e., homeomorphic to the suspension (double cone) over a 2-sphere S with $k = 0, 2$, or 3 selected points. Moreover, we assume that S is the boundary of the ball B_1 used for the summation. Then the sum coincides with the other pair. Such a summation is called *trivial.*

Definition 4. *A knotted graph (M, G) is called prime if it cannot be presented as a nontrivial connected sum.*

2 Existence of prime decompositions

It is convenient to have an independent description of operations inverse to connected summations.

Definition 5. *Let (M, G) be a knotted graph and $S \subset (M, G)$ a general position sphere in (M, G). Then S is called admissible if S is incompressible, separates M, and $X = S \cap G$ consists of* 0, 2, *or* 3 *points. An admissible sphere is called trivial if it bounds a ball $B \subset M$ such that $(B, B \cap G)$ is homeomorphic to* $(\mathrm{Con}\ S^2, \mathrm{Con}\ X)$.

Definition 6. *Let (M, G) be a knotted graph and $S \subset (M, G)$ an admissible sphere. Then the reduction of (M, G) along S consists in cutting (M, G) along S and taking cones over $(S_\pm, S_\pm \cap G)$, where $S_\pm$ are two copies of S appearing under the cut. The resulting knotted graphs will be denoted by $(M_S, G_S), (M'_S, G'_S)$. The reduction along S is nontrivial if so is S.*

The following example shows that our additional assumption that spheres used for constructing connected sums are incompressible is essential.

Example 1. Let a knot K in $M = S^2 \times S^1$ be obtained from the knot $K_0 = \{*\} \times S^1$ by tying a local trefoil t. See Fig. 1. Note that K is in fact isotopic to K_0. Indeed, by deforming some little arc of t all the way across $S^2 \times \{*\}$, we can change an overcrossing to an undercrossing so that K comes undone. Let $S \subset M$ be a sphere surrounding t, see Fig. 1. Note that S is compressible. If we were allowed to use compressible spheres for constructing connected sums and reductions, we would have $(M, K) = (M, K_0)\#(S^3, t)$. Therefore there would be no hope for existence of a prime decomposition for (M, K_0), since the above splitting could be iterated *ad infinitum.*

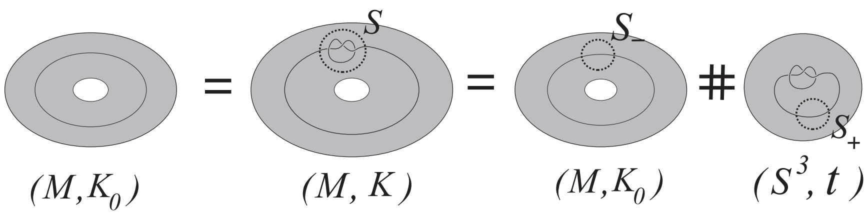

Figure 1:

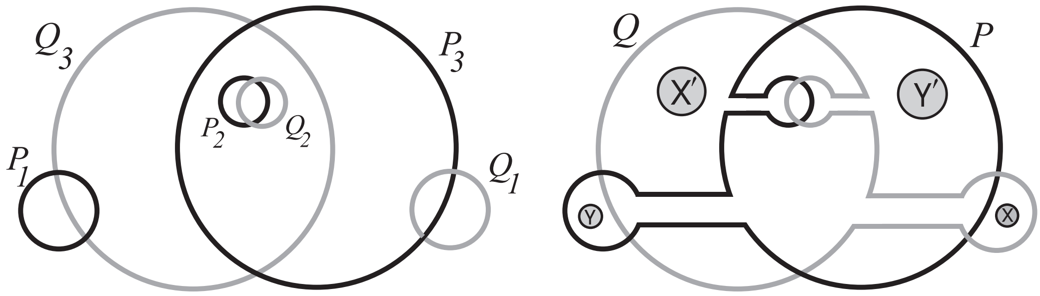

Figure 2:

On the other hand, the incompressibility assumption guaranties existence of prime decompositions.

Proposition 1. *Every nontrivial knotted graph can be presented as a connected sum of prime factors.*

Proof. Let us apply to a given knotted graph (M, G) nontrivial spherical reductions as long as possible. It follows from a version of the H. Kneser Lemma proved in [2] (Lemma 5) that the number of those reductions is bounded by a constant depending only on (M, G). So we stop. ∎

3 A counterexample to uniqueness

Let $P_i, Q_i, 1 \leq i \leq 3$, be six 2-spheres in S^3 such that the following conditions hold:

1. $P_i \cap Q_j$ is a circle for $(i, j) = (1, 3), (3, 1), (2, 2), (3, 3)$ and empty for all other pairs (i, j).

2. P_3 separates P_1 from P_2, and Q_3 separates Q_1 from Q_2. See Fig. 2 (left).

Let us join P_3 with P_1 and P_2 by thin tubes and denote by P the resulting 2-sphere. Similarly, we join Q_3 with Q_1, Q_2 by tubes and get a 2-sphere Q.

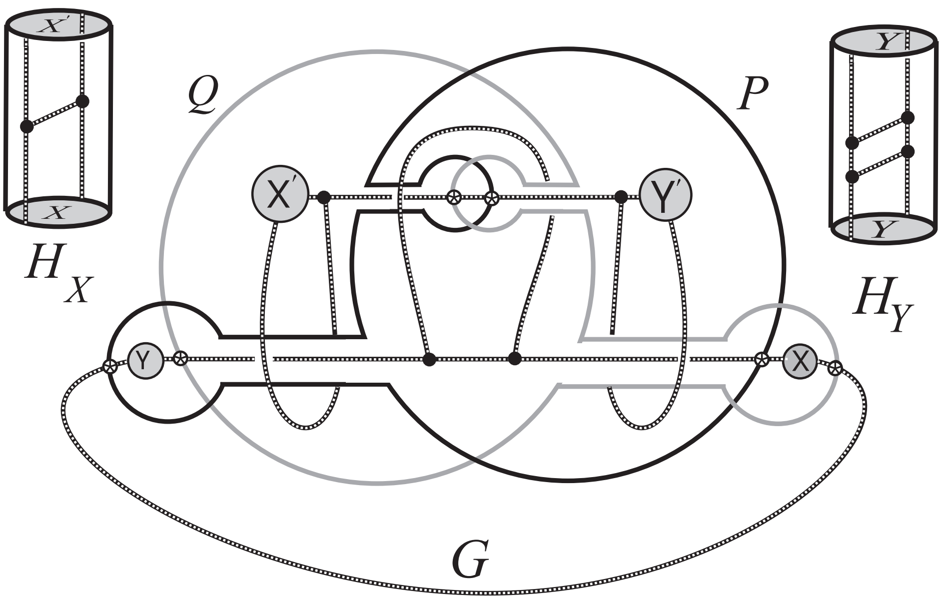

Figure 3:

Note that $P \cap Q$ consists of four circles, which decompose each sphere into an annulus, a twice punctured disc, and three discs. Choose 3-balls X, X', Y, Y' in the complement of $P \cup Q$ such that X, X' are outside P but inside Q, and Y, Y' are inside Q but outside P. See Fig. 2 (right).

Let us cut off those balls and attach to the resulting four times punctured sphere S^3_* two 3-dimensional handles $H_X, H_Y \approx S^2 \times I$ such that they join ∂X with $\partial X'$ and ∂Y with $\partial Y'$ respectively. We get a 3-manifold $M \approx (S^2 \times S^1)\#(S^2 \times S^1)$ with spheres P, Q inside.

Let us describe a knotted graph $G \subset M$. It has only 3-valent vertices. Two of them lie in H_X, four in H_Y, and the remaining four in S^3_*. The edges inside the handles serve as decorations for distinguishing them. G intersects all 6 disc components of $(P \cup Q) \setminus (P \cap Q)$, each at exactly one point. See Fig. 3.

Since P is admissible, we may reduce (M, G) along it. We get two knotted graphs G_P, G'_P in two exemplars of $S^2 \times S^1$. See Fig. 4. Since any separating 2-sphere in $S^2 \times S^1$ bounds a ball, one can easily prove that both knotted graphs are prime. Note that G_P has eight vertices (four of them lie in H_Y) while G'_P has four vertices (two in H_X). The reduction of (M, G) along Q also gives two knotted graphs G_Q, G'_Q similar to G_P, G'_P. The only difference is that G_Q and G'_Q have six vertices each. Those numbers show that decompositions $(M, G) = (S^2 \times S^1, G_P)\#(S^2 \times S^1, G'_P)$ and $(M, G) = (S^2 \times S^1, G_Q)\#(S^2 \times S^1, G'_Q)$ are distinct.

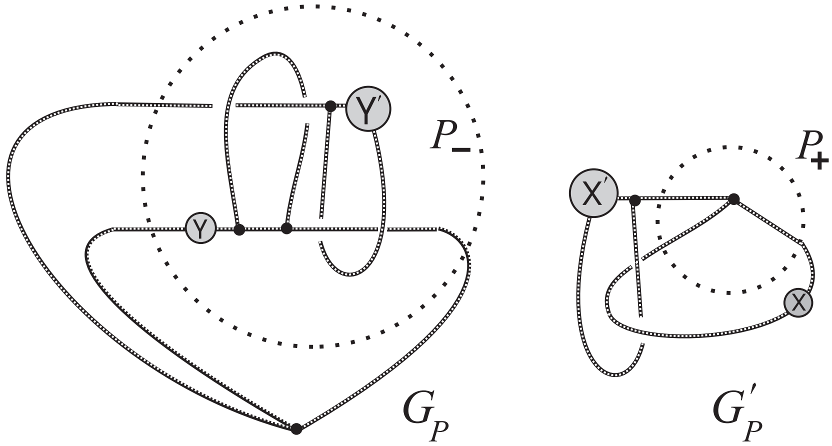

Figure 4:

4 General structure of counterexamples

Let us divide the set $\mathcal{G}$ of all knotted graphs into two subsets $\mathcal{C}$ and $\mathcal{R}$. The first one consists of counterexamples to the prime decomposition theorem, i.e., of knotted graphs having different prime decompositions. Each knotted graph from $\mathcal{R}$ has only one prime decomposition. We call such graphs *regular*.

Let (M,G) be a knotted graph. Note that if $(M,G)=(M_1,G_1)\#(M_2,G_2)$ and one of the factors, say, (M_1,G_1), is a counterexample, then so is (M,G). One may say that (M,G) is induced by (M_1,G_1).

Definition 7. *A knotted graph $(M,G)\in\mathcal{C}$ is called basic if it is not induced by a graph from $\mathcal{C}$ or, equivalently, if all of its factors are regular. The set of all basic counterexamples will be denoted by $\mathcal{B}$.*

It follows from the definition that if $(M,G)\in\mathcal{B}$, then the following statements hold:

1. For any nontrivial admissible sphere $S\subset(M,G)$ each of the graphs $(M_S,G_S),(M'_S,G'_S)$ obtained by the reduction along S has a unique prime decomposition.

2. There is a pair of admissible spheres $P,Q\subset(M,G)$ such that the union of prime factors of $(M_P,G_P),(M'_P,G'_P)$ is different from the one of $(M_Q,G_Q),(M'_Q,G'_Q)$.

We shall say that the pair of spheres $P,Q\subset(M,G)$ as above is *significant*.

Definition 8. *Let $P_i, Q_i \subset (M_i, G_i), i = 1, 2,$ be significant pairs of spheres and $N_i = N(P_i \cup Q_i)$ regular neighborhoods of their unions. Then the pairs P_i, Q_i are called equivalent if there is a homeomorphism $h\colon N_1 \to N_2$ taking $P_1 \cup Q_1$ to $P_2 \cup Q_2$.*

Conjecture 1. *Any basic counterexample $(M, G) \in \mathcal{C}$ contains a significant pair P_i, Q_i of spheres such that the intersection of the spheres is transversal and consists of no more than six circles.*

Conjecture 2. *The number of equivalence classes of significant pairs of spheres is finite.*

One can easily see that the first conjecture implies the second. Let us describe a few arguments in favor of Conjecture 1. The main idea is to decrease the number $\#(P \cap Q)$ of circles in $P \cap Q$ by several surgery tricks similar to ones used in [2, 3]. Suppose that there is a disc $D \subset P$ such that $D \cap Q = \partial D$ and $D \cap G = \emptyset$. Then the surgery of Q along D transforms Q into two spheres S_1, S_2. See Fig. 5. Since Q is incompressible, at least one of them (say S_1) does not intersect G.

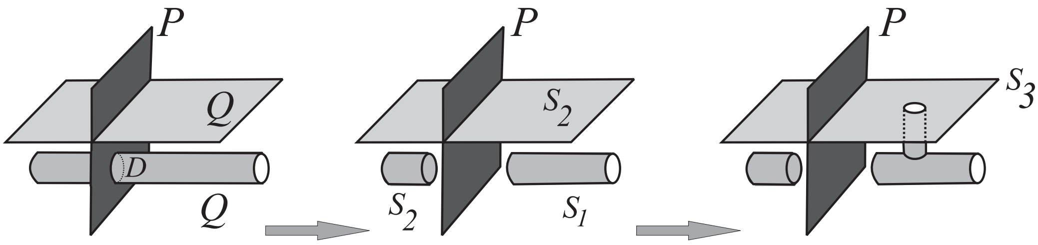

Figure 5:

Case 1. Suppose that S_1 is separating. If it is nontrivial, then at least one of the pairs P, S_1 and Q, S_1 is significant. By construction, $\#(P \cap S_1)$ and $\#(Q \cap S_1)$ are less than $\#(P \cap Q)$ (after a small isotopy of S_1). If S_1 is trivial, then $\#(P \cap Q)$ can be decreased by an isotopy of Q invariant on G.

Case 2. Suppose that S_1 is nonseparating. Then we connect S_1, S_2 with a thin tube so as to obtain a new admissible sphere S_3, see Fig. 5 to the right. Since Q is nontrivial, so is S_3. As above, at least one of the pairs P, S_3 and Q, S_3 is significant and $\#(P \cap S_3)$, $\#(Q \cap S_3)$ are less than $\#(P \cap Q)$.

Further on we may assume that every disc component of $(P \cup Q) \setminus (P \cap Q)$ intersects G. Since P, Q are admissible, each of them contains either two or three disc components. All other components are annuli, except one disc with two holes in the case of three disc components.

Suppose that $\#(P \cap Q) > 6$. Then one can find two neighboring annular components $A_1, A_2 \subset P$ such that for $i = 1, 2$ we have $A_i \cap Q = \partial A_i$ and

$A_i \cap G = \emptyset$. This was the motivation behind the restriction on the number of intersection circles in Conjecture 1. Let us perform two surgeries of Q. The first surgery along A_1 transforms Q into into a sphere S_1 and a torus T. The second surgery along a parallel copy of A_2 transforms $S_1 \cup T$ into either a sphere S_2 or the union of a sphere and a torus. See Fig. 6. If one of the spheres S_1, S_2 together with P form a significant pair, we are done. The obstacle is that this may be not the case. However, in all examples I have considered the described surgery tricks turned to be sufficient for decreasing the number of intersection circles to no more than six.

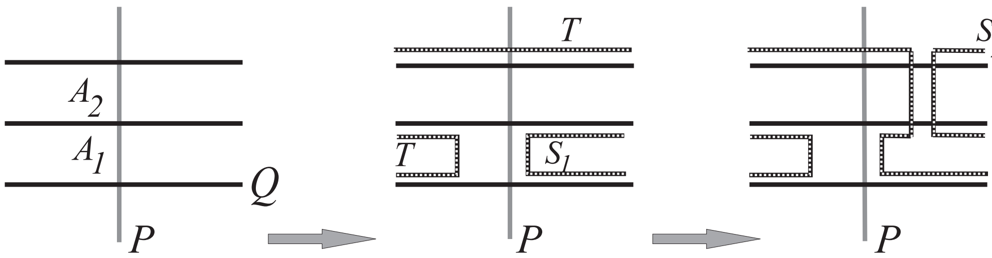

Figure 6:

Another argument in favor of Conjectures 1,2 is that for global knots (i.e., for knots in 3-manifolds) they are true. Indeed, one can extract from the prime decomposition theorem of K. Miyazaki [4] that for knots there is actually only one basic counterexample. The intersection of corresponding significant spheres consists of 3 circles.

5 From knotted graphs to orbifolds

It is well-known that any closed orientable 3-orbifold can be presented as a pair (M, G^*), where M is an orientable 3-manifold and G^* a collection of disjoint circles and trivalent graphs in M such that each circle and each edge of G^* carries an order in $\{n \in \mathbb{N} \colon n \geq 2\}$. The edges outgoing any vertex should have orders $(2, 2, n)$ for $n \geq 2$, or $(2, 3, n)$ for $n \in \{3, 4, 5\}$. See [1, 5]. Let G be a knotted graph in a 3-manifold M such that G is a disjoint union of circles and trivalent graphs. In order to transform (M, G) into a 3-orbifold it suffices to equip all circles and edges of G with appropriate orders. The simplest way to do that is to take all orders equal to 2. So the counterexample in section 3 can be easily converted into a counterexample for the prime decomposition theorem for 3-orbifolds.

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