Simple mass formulas

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Abstract

We give a unified formulation of geometric mass for abelian varieties with additional structures and show that it equals a weighted class number of a reductive $\mathbb{Q}$-group G relative to an open compact subgroup $U \subset G(\mathbb{A}_f)$, or simply called an *arithmetic mass*. The proof replies on the results of Zarhin, Faltings and de Jong on endomorphisms of abelian varieties and those of their attached ℓ-divisible groups. We also classify the special objects for which our formulation remains valid over algebraically closed fields.

§1. Introduction

The study of some class numbers using elliptic curves might go back to Kronecker or even to Gauss. The celebrated Eichler-Deuring mass formula says

$$\sum_{E \in \Lambda_p} \frac{1}{\# \operatorname{Aut}(E)} = \frac{p-1}{24}, \tag{1}$$

where Λ_p is the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$. It is well-known that the set Λ_p is in bijection correspondence with the double coset space attached to the definite quaternion algebra $\mathbf{B} := B_{p,\infty}$ over $\mathbb{Q}$ of discriminant p relative to a maximal order $O_{\mathbf{B}}$. More precisely, let G' be the group scheme over $\mathbb{Z}$ attached to the multiplicative group $O_{\mathbf{B}}^{\times}$; then one has the following natural bijection:

$$\Lambda_p \simeq G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/G'(\hat{\mathbb{Z}}). \tag{2}$$

We write $\operatorname{mass}(\Lambda_p)$ for the left hand side of (1), and call it the mass of Λ_p. It also equals an arithmetically defined mass for G' relative to the open compact subgroup $G'(\hat{\mathbb{Z}})$ now defined as follows. For an $\mathbb{R}$-anisotropic reductive $\mathbb{Q}$-group G, and an open compact subgroup U of $G(\mathbb{A}_f)$, the mass for G relative to U is defined to be

$$\operatorname{mass}(G, U) := \sum_{c} \frac{1}{\# \Gamma_c}, \tag{3}$$

where c runs through a complete set of representatives for the double coset space $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/U$, and $\Gamma_c := G(\mathbb{Q}) \cap cUc^{-1}$.

The analogous result for Siegel moduli spaces was generalized by Ekedahl [2] (also see [17]). A similar bijection (2) holds as well where Λ_p is replaced by the set $\Lambda_{g,p}$ of the isomorphism classes of g-dimensional *superspecial* principally polarized abelian varieties over

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$\overline{\mathbb{F}}_p$, and G' is replaced by the group scheme G'_g over $\mathbb{Z}$ obtained from $M_g(O_{\mathbf{B}})$ with equations given by $g^*g = 1$, where $g \mapsto g^*$ is the standard involution. Namely, one has

$$\Lambda_{g,p} \simeq G'_g(\mathbb{Q})\backslash G'_g(\mathbb{A}_f)/G'_g(\hat{\mathbb{Z}}) \quad \text{and} \quad \mathrm{mass}(\Lambda_{g,p}) = \mathrm{mass}(G'_g, G'_g(\hat{\mathbb{Z}})), \tag{4}$$

where $\mathrm{mass}(\Lambda_{g,p}) := \sum_{(A,\lambda)\in\Lambda_{g,p}} \frac{1}{\#\operatorname{Aut}(A,\lambda)}$. Applying the Hashimoto-Ibukiyama formula for $\mathrm{mass}(G'_g, G'_g(\hat{\mathbb{Z}}))$ [9] to the second formula of (4), Ekedahl obtained the geometric mass formula

$$\sum_{(A,\lambda)\in\Lambda_{g,p}} \frac{1}{\#\operatorname{Aut}(A,\lambda)} = \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \cdot \prod_{k=1}^{g} \left\{ (p^k + (-1)^k \right\}. \tag{5}$$

In [19] the correspondence in (4) is generalized to supersingular polarized abelian varieties of Hilbert-Siegel type. The geometric mass formulas are explicitly calculated for superspecial points of Hilbert-Blumenthal type (see Corollary 2.3, 2.5 and Theorem 3.7 of loc. cit.). The latter relies on Shimura's arithmetic mass formula for quaternion unitary groups [16] and local indices computation.

In this paper we give a uniform formulation of the geometric mass $\mathrm{mass}(\Lambda)$ for arbitrary abelian varieties with additional structures over arbitrary (finitely generated) fields, and show that it equals an arithmetic mass defined by some (G, U); see Section 2 for precise statements. The description, though being surprisingly simple, replies on the deep results of Zarhin, Faltings and de Jong on the endomorphisms of abelian varieties, Tate modules, and p-divisible groups; see [21], [3] (cf. [4]) and [1, Theorem 2.6]. We call the formula established in Theorem 2.2 *simple mass formula*. The simple mass formula connects a geometrically defined mass and an arithmetically defined mass; but it provides no clue of computing either side explicitly. It is useful to prove a geometric mass formula from a known arithmetic mass formula and vice versa, or to verify an arithmetic mass formula by a geometric method and vice versa. Ekedahl's formula above is the simplest example. A worth note is that a geometric mass then becomes to have good properties as an arithmetic mass does. For example, it has a simple relation between different levels and the calculation can be reduced to local volume computation.

In the second part of this paper, we study certain special abelian varieties in question (called *of arithmetic type*, see Definitions 3.1 and 3.10). For those the hidden Galois structure required in the formula is superfluous, thus the description can be extended in the geometric setting. This explains why a good formulation of the mass for supersingular elliptic curves or supersingular abelian varieties is possible. We remark that the parallel description for CM abelian varieties in characteristic zero is well-known and this has been playing the important role on explicit reciprocity laws in class field theory, known as the main theorem of complex multiplication. Our description could be used to create new explicit reciprocity laws.

In the last part of this paper, we classify the abelian varieties of arithmetic type in question. In the case of characteristic zero, the possibility occurs only when the semi-simple involuted algebra $(B, *)$ is componentwise of second kind; and then every abelian variety of arithmetic type is essentially a product of a simple CM abelian variety. In the case of characteristic p, we show that an object $\underline{A}$ is of arithmetic type if and only if it is basic in the sense of Kottwitz [10].

Finally we mention that the function field analog of the geometric mass formulas can be also considered where supersingular Drinfeld modules take place the role of supersingular elliptic curves. This was obtained by Gekeler [5, 6, 7, 8] for the cases (a) rank $r = 2$ and any global function and (b) the rational function fields and any rank r, recently obtained by Jing Yu and the author in general [20].

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§2. Main theorem

(2.1) Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with a positive involution $*$, and O_B an order of B stable under $*$.

A *polarized abelian O_B-variety* is a triple $\underline{A} = (A, \lambda, \iota)$ where (A, λ) is a polarized abelian variety and $\iota : O_B \to \mathrm{End}(A)$ is a ring monomorphism such that $\lambda\iota(a^*) = \iota(a)^t\lambda$ for all $a \in O_B$. For any $\underline{A}$ and any prime ℓ (not necessarily invertible in the ground field), we write $\underline{A}(\ell)$ for the associated ℓ-divisible group with additional structures $(A[\ell^\infty], \lambda_\ell, \iota_\ell)$, where λ_ℓ is the induced quasi-polarization from $A[\ell^\infty]$ to $A^t[\ell^\infty] = A[\ell^\infty]^t$ (the Serre dual), and $\iota_\ell : O_B \otimes \mathbb{Z}_\ell \to \mathrm{End}(A[\ell^\infty])$ the induced ring monomorphism.

For any two $\underline{A}_1$ and $\underline{A}_2$ over a field k, denote by

- $\text{Q-isom}_k(\underline{A}_1, \underline{A}_2)$ (resp. $\mathrm{Isom}_k(\underline{A}_1, \underline{A}_2)$) the set of O_B-linear quasi-isogenies (resp. isomorphisms) $\varphi : A_1 \to A_2$ over k such that $\varphi^*\lambda_2 = \lambda_1$; and
- $\text{Q-isom}_k(\underline{A}_1(\ell), \underline{A}_2(\ell))$ (resp. $\mathrm{Isom}_k(\underline{A}_1(\ell), \underline{A}_2(\ell))$) the set of $O_B \otimes \mathbb{Z}_\ell$-linear quasi-isogenies (resp. isomorphisms) $\varphi : A_1[\ell^\infty] \to A_2[\ell^\infty]$ such that $\varphi^*\lambda_2 = \lambda_1$.

Let $x := \underline{A}_0 = (A_0, \lambda_0, \iota_0)$ be a polarized abelian O_B-variety. Choose a finitely generated extension field k over its prime field so that the object $\underline{A}_0$ and all endomorphisms of A_0 are defined over k. Denote by $\Lambda_x(k)$ the set of isomorphisms classes of polarized abelian O_B-varieties $\underline{A}$ over k such that

(i) $\mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell)) \neq \emptyset$ for all ℓ, and

(ii) $\text{Q-isom}_k(\underline{A}_0, \underline{A}) \neq \emptyset$.

Let G_x be the automorphism group scheme over $\mathbb{Z}$ associated to $\underline{A}_0$; for any commutative ring R, its group of R-points is

$$G_x(R) = \{g \in \mathrm{End}_{O_B}(A_{0/k}) \otimes R \,|\, g'g = 1\}, \tag{6}$$

where $g \mapsto g'$ is the Rosati involution induced by λ_0. Note that $G_x(\mathbb{Q}) = \text{Q-isom}_k(\underline{A}_0, \underline{A}_0)$. By the theorems of Zarhin, Faltings, and de Jong on the endomorphisms of abelian varieties, Tate modules and p-divisible groups (see [21], [3] (cf. [4]) and [1, Theorem 2.6]), we have the natural isomorphisms

$$G_x(\mathbb{Z}_\ell) = \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}_0(\ell)) \quad \text{and} \quad G_x(\mathbb{Q}_\ell) = \text{Q-isom}_k(\underline{A}_0(\ell), \underline{A}_0(\ell)) \tag{7}$$

for all ℓ.

(2.2) Theorem

(1) *There is a natural bijection between the following two pointed sets*

$$\Lambda_x(k) \simeq G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}}).$$

In particular, $\Lambda_x(k)$ is finite.

(2) *Define*

$$\mathrm{mass}[\Lambda_x(k)] := \sum_{\underline{A} \in \Lambda_x(k)} \frac{1}{\# \mathrm{Aut}_k(\underline{A})}.$$

Then one has $\mathrm{mass}[\Lambda_x(k)] = \mathrm{mass}[G_x, G_x(\hat{\mathbb{Z}})]$.

PROOF. (1) Given an element $\underline{A} \in \Lambda_x(k)$, consider the natural map

$$m(\underline{A}) : \text{Q-isom}(\underline{A}, \underline{A}_0) \times \prod_\ell \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell)) \to \prod_\ell{}' \text{Q-isom}_k(\underline{A}_0(\ell), \underline{A}_0(\ell)) = G_x(\mathbb{A}_f) \tag{8}$$

which sends $(\phi, (\alpha_\ell)_\ell)$ to $(\phi\alpha_\ell)_\ell$. Clearly if c is an element in the image $c(\underline{A})$ of $m(\underline{A})$, then $c(\underline{A})$ equals the double coset $G_x(\mathbb{Q})\, c\, G_x(\hat{\mathbb{Z}})$. Thus, $c(\underline{A})$ defines an element in $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$.

Let $\underline{A}, \underline{A}' \in \Lambda_x(k)$ such that $c(\underline{A}) = c(\underline{A}')$. Write $c(\underline{A}) = [(\phi\alpha_\ell)_\ell]$ and $c(\underline{A}') = [(\phi'\alpha'_\ell)_\ell]$. Then there exist $b \in G_x(\mathbb{Q})$ and $k_\ell \in G_x(\mathbb{Z}_\ell)$ for all ℓ such that $b\phi\alpha_\ell k_\ell = \phi'\alpha'_\ell$. Then

$$(b\phi)^{-1}\phi' = \alpha_\ell k_\ell(\alpha'_\ell)^{-1} \in \text{Q-isom}_k(\underline{A}', \underline{A}) \cap \prod_\ell \mathrm{Isom}_k(\underline{A}'(\ell), \underline{A}(\ell)) = \mathrm{Isom}_k(\underline{A}', \underline{A}).$$

Thus $\underline{A}' \simeq \underline{A}$ and this shows the injectivity of c.

Given $[(\phi_\ell)_\ell]$ in $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$, choose an positive integer N such that $f_\ell := N\phi_\ell^{-1}$ is an isogeny for all ℓ. Let H be the product of the kernels of $N\phi_\ell^{-1}$; it is a finite subgroup scheme over k invariant under the O_B-action. Take $A := A_0/H$ and let $\pi : A_0 \to A$ be the natural projection; A is defined over k and it is equipped with a natural action by O_B so that π is O_B-linear. Let $\lambda \in \mathrm{Hom}(A, A^t) \otimes \mathbb{Q}$ be the fractional polarization on A such that $(N^{-1}\pi)^*\lambda = \lambda_0$; it is O_B-linear as π is so. As π_ℓ and f_ℓ have the same kernel, there is an element $\alpha_\ell \in \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell))$ such that $\alpha_\ell f_\ell = \pi_\ell$. This shows $\lambda \in \mathrm{Hom}_{k,O_B}(A, A^t)$ and one obtains $\underline{A} \in \Lambda_x(k)$. Put $\phi := (N^{-1}\pi)^{-1} \in \text{Q-isom}_k(\underline{A}, \underline{A}_0)$. One checks

$$\phi\alpha_\ell = N\pi_\ell^{-1}\alpha_\ell = Nf_\ell^{-1} = \phi_\ell.$$

This shows $c(\underline{A}) = [(\phi_\ell)_\ell]$ and the surjectivity of c.

(2) It suffices to show that if $x' = \underline{A} \in \Lambda_x(k)$ and c any representative for the double coset $c(\underline{A})$, then $\mathrm{Aut}_k(\underline{A}) \simeq \Gamma_c$. Write $G_{x'}$ for the group scheme over $\mathbb{Z}$ associated to $\underline{A}$ defined as (6) in (2.1). Choose $\phi \in \text{Q-isom}_k(\underline{A}_0, \underline{A})$ such that $\phi c_\ell \in \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell))$ for all ℓ. Note that $\alpha \in \mathrm{Aut}_k(\underline{A})$ if and only if $\alpha \in G_{x'}(\mathbb{Q})$ and $\alpha_\ell \in \mathrm{Aut}_k(\underline{A}(\ell))$ for all ℓ.

The map ϕ gives an isomorphism $G_x(\mathbb{Q}) \to G_{x'}(\mathbb{Q})$ which sends β to $\phi\beta\phi^{-1} =: \alpha$. Note that $\alpha \in G_{x'}(\hat{\mathbb{Z}})$ if and only if $(\phi c)^{-1}\alpha(\phi c) \in G_x(\hat{\mathbb{Z}})$. The latter is equivalent to $c^{-1}\beta c \in G_x(\hat{\mathbb{Z}})$. Therefore, the above isomorphism gives $\Gamma_c \simeq \mathrm{Aut}_k(\underline{A})$. This completes the proof. ∎

(2.3) Let N be any positive integer and U_N be the kernel of the reduction map $G_x(\hat{\mathbb{Z}}) \to G_x(\hat{\mathbb{Z}}/N\hat{\mathbb{Z}})$. Let $\underline{A}$ be a polarized abelian O_B-variety. By an $(\underline{A}_0, U_N)$*-level structure on* $\underline{A}$ we mean a non-empty U_N-orbit $\bar{\eta}$ of isomorphisms η in $\prod_\ell \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell))$. The existence of such $\bar{\eta}$ implies that the first condition for objects lying in $\Lambda_x(k)$ is satisfied. Let $\bar{\eta}_0$ be the U_N-orbit of the identity in $\prod_\ell \mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}_0(\ell))$. Now we change our notation a bit in the remaining of this section. We write $\underline{A}_0$ for $(A_0, \lambda_0, \iota_0, \bar{\eta}_0)$ and $\underline{A}$ for $(A, \lambda, \iota, \bar{\eta})$ in brief.

For any two $\underline{A}_1$ and $\underline{A}_2$ over a field k, denote by $\text{Q-isom}_k(\underline{A}_1, \underline{A}_2)$ and $\text{Q-isom}_k(\underline{A}_1(\ell), \underline{A}_2(\ell))$ the sets which have the same meaning as in (2.1); denote by $\mathrm{Isom}_k(\underline{A}_1, \underline{A}_2)$ the set of elements φ in $\mathrm{Isom}_k((A_1, \lambda_1, \iota_1), (A_2, \lambda_2, \iota_2))$ satisfying $\varphi_*\bar{\eta}_1 = \bar{\eta}_2$; and denote by $\mathrm{Isom}_k(\underline{A}_1(\ell), \underline{A}_2(\ell))$ the set of elements φ in $\mathrm{Isom}_k((A_1, \lambda_1, \iota_1)(\ell), (A_2, \lambda_2, \iota_2)(\ell))$ satisfying $\varphi_*\bar{\eta}_{1,\ell} = \bar{\eta}_{2,\ell}$.

Let $\Lambda_{x,N}(k)$ denote the set of isomorphism classes of polarized abelian O_B-varieties with an $(\underline{A}_0, U_N)$-level structure $(A, \lambda, \iota, \bar{\eta})$ over k such that $\text{Q-isom}_k(\underline{A}_0, \underline{A}) \neq \emptyset$. The same proof of Theorem 2.2 without modification gives the following variant.

(2.4) Theorem *There is a natural bijection*

$$\Lambda_{x,N}(k) \simeq G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/U_N.$$

Furthermore, one has $\mathrm{mass}[\Lambda_{x.N}(k)] = \mathrm{mass}(G_x, U_N)$.

(2.5) Lemma *If* $N \geq 3$, *then* $\mathrm{Aut}_k(\underline{A})$ *is trivial for any object* $\underline{A} = (A, \lambda, \iota, \bar{\eta})$ *in* $\Lambda_{x,N}(k)$.

Proof. An element $g \in \mathrm{Aut}_k(A, \lambda, \iota) = G_{(A,\lambda,\iota)}(\mathbb{Z})$ preserves $\bar{\eta}$ if and only if its image $\bar{g}$ in $G_{(A,\lambda,\iota)}(\mathbb{Z}/N\mathbb{Z})$ is trivial. Choose a faithful and integral representation ρ of $\mathrm{Aut}_k(\underline{A})$ on a finite free $\mathbb{Z}$-module $V_{\mathbb{Z}}$. Then every matrix $\rho(g)$ in the image satisfies the property $\rho(g)^m = I$ for some m and $\rho(g) \equiv I \mod N$, thus $\mathrm{Aut}_k(\underline{A})$ is trivial. ∎

(2.6) Remark The assumption in Theorems 2.2 and 2.4 that all endomorphisms of A are defined over k is superfluous.

§3. Geometric Setting

Abelian varieties of CM type and supersingular abelian varieties have rich arithmetic properties so that the mass formula as (1) can be formulated over an algebraically closed field. This leads us to the following definition.

(3.1) Definition Let $\underline{A} = (A, \lambda, \iota)$ be a polarized abelian O_B-variety a field k finitely generated over its prime field for which all endomorphisms of A are defined (this will be assumed in the rest). Let $\rho_\ell : \mathcal{G}_k := \mathrm{Gal}(k_s/k) \to \mathrm{GAut}_{k_s}(\underline{A}(\ell))$ the associated ℓ-adic Galois representation for $\ell \neq \mathrm{char}\, k$, where k_s denotes a separable closure of k. We call $\underline{A}$ over k is *of arithmetic type* if the image $\rho_\ell(\mathcal{G}_k)$ is contained in the center of $\mathrm{GAut}_{k_s}(\underline{A}(\ell))$ for all $\ell \neq \mathrm{char}\, k$.

Although it is not necessary to assume below, we are only interested in these cases.

(3.2) Assumption. Let g be the dimension of A. For the remaining of this paper, we assume that the datum $(B, *, g)$ satisfies the condition that there is a g-dimensional polarized abelian O_B-variety over a field *of characteristic zero*.

This condition says that there exists a non-degenerate $\mathbb{Q}$-valued skew-Hermitian B-space (V, ψ) such that $2g = \dim_{\mathbb{Q}} V$.

This assumption will exclude, for example, the case where A is a supersingular elliptic curve ($g = 1$) and $B = \mathrm{End}^0(A)$.

(3.3) Basic properties for abelian varieties of arithmetic type:

(3.3.1) The definition of arithmetic type is related with the endowed endomorphism structure ι. It is possible that an object $\underline{A} = (A, \lambda, \iota)$ is of arithmetic type, while its underlying polarized abelian variety $f(\underline{A}) := (A, \lambda)$ is not. Clearly if $f(\underline{A})$ is of arithmetic type, then so as $\underline{A}$. When $\mathrm{char}\, k = 0$, any polarized abelian variety cannot be of arithmetic type. Indeed, if the image of the Galois group lies in the center of $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$, which consists of the scalar matrices, then one has $\dim \mathrm{End}^0(A) = 4g^2$. This contradicts with the fact that $\dim \mathrm{End}^0(A) \le 2g^2$. So we show that (A, λ) is of arithmetic type if and only if $\mathrm{char}\, k = p$ and A is supersingular.

(3.3.2) Any polarized CM-abelian variety by O_L (the ring of integers of a CM algebra L) is of arithmetic type as $\mathrm{End}_{O_L}(A)$ is already commutative.

(3.3.3) If $\underline{A}$ is of arithmetic type and $\underline{A}'$ is another polarized abelian O_B-variety such that $\text{Q-isom}_k(\underline{A}, \underline{A}') \neq \emptyset$, then clearly $\underline{A}'$ is also of arithmetic type. Hence being of arithmetic type is an isogenous property.

(3.3.4) We write $E := \mathrm{End}_k^0(A)$, $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $B_\ell := B \otimes \mathbb{Q}_\ell$, $T_\ell := T_\ell(A_{k_s})$ and $V_\ell = T_\ell \otimes \mathbb{Q}_\ell$. Let $G_\ell := \rho_\ell(\mathcal{G}_k)$ and G_ℓ^{alg} be the algebraic envelope of G_ℓ. Then $\mathrm{End}_{k_s}(A[\ell^\infty]) = \mathrm{End}(T_\ell)$. Suppose that the polarization λ is also E-linear, we have

$$G_\ell \subset \mathrm{GAut}_{E_\ell}(T_\ell) \subset \mathrm{GAut}_{B\otimes\mathbb{Q}_\ell}(T_\ell) \subset \mathrm{GAut}(T_\ell),$$

$$Z(\mathrm{GAut}(T_\ell)) \subset Z(\mathrm{GAut}_{B\otimes\mathbb{Q}_\ell}(T_\ell)) \subset Z(\mathrm{GAut}_{E_\ell}(T_\ell)).$$

Let $\iota_E : \mathrm{End}(A) \to \mathrm{End}(A)$ be the identity and put $\underline{A}_E := (A, \lambda, \iota_E)$. Clearly if $\underline{A}$ is of arithmetic type, then so as $\underline{A}_E$.

(3.4) It is known that any abelian O_B-variety admits an O_B-linear polarization [11, Section 9]. We will see that the polarization structure will not play a role in the definition of arithmetic type. Therefore, the notion of arithmetic type tests a special property of abelian variety up to isogeny endowed with a B-linear action.

Let $\underline{A} = (A, \lambda, \iota)$ is a polarized abelian O_B-variety of arithmetic type (3.1). Write the semi-simple algebra B into simple factors $\oplus_{i=1}^r M_{n_i}(D_i)$, where D_i is a division algebra over $\mathbb{Q}$ with a positive involution $*_i$. According this decomposition the abelian variety A is isogenous to $\prod A_i^{n_i}$; one has ring monomorphism $D_i \to \mathrm{End}^0(A_i)$. Write V_i for $T_\ell(A_i) \otimes \mathbb{Q}_\ell$ and one has

$$\mathrm{End}_B(V_\ell) = \oplus_{i=1}^r \mathrm{End}_{D_i}(V_i).$$

Let $g \mapsto g'$ be the adjoint with respect to the alternating pairing $\langle\, ,\rangle$ on V_ℓ. Then $\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle)$ consists of elements $g = (g_i) \in \prod \mathrm{End}_{D_i}(V_i)$ such that $g_1'g_1 = g_2'g_2 = \cdots = g_r'g_r \in \mathbb{Q}_\ell^\times$.

We have projections $p_i : \mathrm{GAut}_B(V_\ell, \langle\, ,\rangle) \to \mathrm{GAut}_{D_i}(V_i, \langle\, ,\rangle_i)$ and these induce $p_i : Z(\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle)) \to Z(\mathrm{GAut}_{D_i}(V_i, \langle\, ,\rangle_i))$. If $\rho_{i,\ell}$ the ℓ-adic Galois representation attached to $\underline{A}_i$, then one has $p_i \circ \rho_\ell = \rho_{i,\ell}$. This shows the if $\underline{A}$ is of arithmetic type, then each $\underline{A}_i$ is of arithmetic type. The converse is also true as $Z(\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle)) \hookrightarrow \prod_i Z(\mathrm{GAut}_{D_i}(V_i, \langle\, ,\rangle_i))$. We have proved

(3.4.1) Notation as above. Then $\underline{A}$ is of arithmetic type if and only if each $\underline{A}_i$ is of arithmetic type.

We now compute $Z(\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle))$. We may assume that B is a division algebra.

(3.5) Definition Keep the notation in (2.1) and assume that B is a division algebra. A polarized abelian O_B-variety is said to be of type (D & 0-dim) if B is of type (III) in the Albert classification and $2 \dim A = [B : \mathbb{Q}]$.

Recall that B is of type (III) if B is a totally definite quaternion algebra over a totally real number field F and the main involution $*$ is the unique positive involution.

(3.6) Lemma *Suppose $\underline{A}$ is of type* (D & 0-dim)*.*

(1) *T_ℓ is a free $O_F \otimes \mathbb{Z}_\ell$-module of rank* 4*.*

(2) *V_ℓ is a free B_ℓ-module of rank* 1*.*

(3) $\mathrm{GAut}_{B_\ell}(V_\ell, \langle\, , \rangle)$ *is an extension of a normal commutative subgroup by a finite 2-torsion group.*

(4) *The center $Z(\mathrm{GAut}_{B_\ell}(V_\ell, \langle\, , \rangle))$ consists of elements a in $F_\ell := F \otimes \mathbb{Q}_\ell$ with $a^2 \in \mathbb{Q}_\ell^\times$.*

PROOF. The statement (1) follows from the fact that $\mathrm{Tr}(a; V_\ell/\mathbb{Q}_\ell) = 4\,\mathrm{Tr}_{F/\mathbb{Q}}(a)$ for all $a \in O_F$. The statement (2) follows from (1). To show the statement (3), we regard $G := \mathrm{GAut}_B(V_\ell, \langle\, , \rangle)$ as an algebraic group over $\mathbb{Q}_\ell$ and show that its neutral component G^0 is a torus.

Let $V_\ell = B_\ell$ as a left B_ℓ-module. Let $(\, ,) : B_\ell \times B_\ell \to B_\ell$ be the lifting of $\langle\, , \rangle$. One has $\langle x, y\rangle = \mathrm{Trd}_{B_\ell/\mathbb{Q}_\ell}(x\alpha y^*)$. where $\alpha := (1,1)$ with $\alpha^* = -\alpha$. Any element in $\mathrm{End}_{B_\ell}(V_\ell)$ is a right translation ρ_g for a $g \in B_\ell$. The condition $\langle xg, yg\rangle = c(g)\langle x, y\rangle$ gives $\mathrm{Trd}_{B_\ell/\mathbb{Q}_\ell}(xg\alpha g^* y^*) = \mathrm{Trd}_{B_\ell/\mathbb{Q}_\ell}(x\, c(g)\alpha y^*)$. Therefore, the group G is the subgroup of $B_\ell^{\mathrm{opp},\times}$ defined by the relation $g\alpha g^* = c(g)\alpha$ for some $c(g) \in \mathbb{Q}_\ell^\times$. Choose the isomorphism $B_\ell^{\mathrm{opp}} \simeq B_\ell$ which sends $g \mapsto g^{-1}$; the group G is identified with the subgroup of $B_\ell^\times$ defined by the same relation.

For each $\sigma \in \Sigma := \mathrm{Hom}(F_\ell, \overline{\mathbb{Q}}_\ell)$, put $B_\sigma = B_\ell \otimes_{F_\ell,\sigma} \overline{\mathbb{Q}}_\ell \simeq M_2(\overline{\mathbb{Q}}_\ell)$. Let $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $g \in B_\sigma$, one computes

$$jg^*j^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = g^t.$$

Write $\alpha = \beta j$, then $\beta^t = j\beta^* j^{-1} = -\alpha^* j^{-1} = \beta$ and the relation defining G becomes $g\beta g^t = c(g)\beta$ for some $c(g)$. We proved

$$G_{\overline{\mathbb{Q}}_\ell} \simeq \{(g_\sigma) \in \mathrm{GL}_2^\Sigma \ ; \ g_\sigma g_\sigma^t = c \ \text{ for some } c \in \overline{\mathbb{Q}}_\ell^\times \text{ (independent of } \sigma), \forall\, \sigma \in \Sigma\}, \quad \text{and}$$

$$G^0_{\overline{\mathbb{Q}}_\ell} \simeq \left\{ \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \in \mathrm{GL}_2^d \ ; \ a_i^2 + b_i^2 = c \ \text{ for some } c \in \overline{\mathbb{Q}}_\ell^\times, \forall\, 1 \le i \le d \right\}.$$

This shows that G^0 is a torus.

(4) This follows directly from the computation in (3). ∎

(3.7) Lemma *The center $Z(\mathrm{GAut}_B(V_\ell, \langle\, , \rangle))$ consists of elements $a \in Z(B) \otimes \mathbb{Q}_\ell$ such that $a^* a \in \mathbb{Q}_\ell^\times$.*

PROOF. One may first reduce to the case where B is a division algebra. The case of type (D & 0-dim) has been treated in Lemma 3.6. Now suppose that $\underline{A}$ is not of type (D & 0-dim).

For an algebra E and subset G, write $Z(E,G) := \{x \in E;\ gx = xg\ \forall\, g \in G\}$. Write $E := \mathrm{End}_B(V_\ell)$ and G for the algebraic group over $\mathbb{Q}_\ell$ defined by $\underline{V}_\ell$; we have particularly $G(\mathbb{Q}_\ell) = \mathrm{GAut}_B(V_\ell, \langle\,,\rangle)$. It suffices to show that $Z(B) \otimes \mathbb{Q}_\ell = Z(E, G(\mathbb{Q}_\ell))$. We have $Z(B) \otimes \mathbb{Q}_\ell \subset Z(E, G(\mathbb{Q}_\ell))$ and hence need to show $\dim_{\mathbb{Q}_\ell} Z(B) \otimes \mathbb{Q}_\ell = \dim_{\mathbb{Q}_\ell} Z(E, G(\mathbb{Q}_\ell))$. Since $\dim_{\mathbb{Q}_\ell} Z(B) \otimes \mathbb{Q}_\ell = \dim_{\overline{\mathbb{Q}}_\ell} Z(B) \otimes \overline{\mathbb{Q}}_\ell$ and $\dim_{\mathbb{Q}_\ell} Z(E, G(\mathbb{Q}_\ell)) \ge \dim_{\overline{\mathbb{Q}}_\ell} Z(E \otimes \overline{\mathbb{Q}}_\ell, G(\overline{\mathbb{Q}}_\ell))$ (equality holds if $G(\mathbb{Q}_\ell)$ is Zariski dense in G), it suffices to show that $\dim_{\overline{\mathbb{Q}}_\ell} Z(B) \otimes \overline{\mathbb{Q}}_\ell = \dim_{\overline{\mathbb{Q}}_\ell} Z(E \otimes \overline{\mathbb{Q}}_\ell, G(\overline{\mathbb{Q}}_\ell))$. Decomposing into simple factors, we have three cases:

(a) $E = M_n(\overline{\mathbb{Q}}_\ell) \times M_n(\overline{\mathbb{Q}}_\ell)$, $* : (A, B) \mapsto (B^t, A^t)$ and $G = GU_n$.

(b) $E = M_{2n}(\overline{\mathbb{Q}}_\ell)$, $*$ is the standard symplectic involution, and $G = GSp_{2n}$.

(c) $E = M_{2n}(\overline{\mathbb{Q}}_\ell)$, $* : A \mapsto A^t$ and $G = GO_{2n}$ $(n \ge 2)$.

Then we have the cases (a) $Z(E, G) = \{(aI_n, bI_n); a, b \in \overline{\mathbb{Q}}_\ell\}$; (b) $Z(E, G) = \{aI_{2n}; a \in \overline{\mathbb{Q}}_\ell\}$; (c) $Z(E, G) = \{aI_{2n}; a \in \overline{\mathbb{Q}}_\ell\}$. From this one sees that $\dim_{\overline{\mathbb{Q}}_\ell} Z(B) \otimes \overline{\mathbb{Q}}_\ell = \dim_{\overline{\mathbb{Q}}_\ell} Z(E \otimes \overline{\mathbb{Q}}_\ell, G(\overline{\mathbb{Q}}_\ell))$. This finishes the proof. ∎

(3.8) Lemma *Let $\underline{A} = (A, \lambda, \iota)$ be a polarized abelian O_B-variety and λ' be another O_B-linear polarization. Then (A, λ', ι) is of arithmetic type if and only if $\underline{A}$ is of arithmetic type.*

Proof. By Lemma 3.7, the center of $\mathrm{Aut}_B(V_\ell, \langle\,,\rangle)$ is independent of the polarization. Therefore, the assertion is proved. ∎

(3.9) Lemma *Let k_0 be a field of finite type over its prime field and k be an extension of k of finite type. Let $\underline{A}$ be a polarized abelian variety over k_0. If $\underline{A}$ is of arithmetic type over k_0 then so as $\underline{A}$ over k. Conversely, if $\underline{A}$ is of arithmetic type over k, then so as $\underline{A}$ over a finite extension of k_0.*

Proof. Let k_s be a separable closure of k and $k_{0,s}$ the algebraic closure of k_0 in k_s. Let k_1 be the algebraic closure of k_0 in k. The restriction gives a surjective homomorphism $r : \mathcal{G}_k \to \mathcal{G}_{k_1}$ of Galois groups. We also have Galois equivariant isomorphism $s : A[\ell^n](k_{0,s}) \simeq A[\ell^n](k_s)$ in the sense that $r(\sigma)x = \sigma(s(x)$ for $x \in A[\ell^n](k_{0,s})$ and $\sigma \in \mathcal{G}_k$. This gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{G}_k & \xrightarrow{\rho_{A_k}} & \mathrm{Aut}(T_\ell(A_k)) \\ {\scriptstyle r}\downarrow & & \simeq\downarrow \\ \mathcal{G}_{k_1} & \xrightarrow{\rho_{A_{k_0}}} & \mathrm{Aut}(T_\ell(A_{k_0})), \end{array}$$

and one has $\rho_{A_k}(\mathcal{G}_k) = \rho_{A_{k_0}}(\mathcal{G}_{k_1})$. It follows that $\underline{A}$ over k is of arithmetic if and only if $\underline{A}$ over k_1 is so, and clearly if $\underline{A}$ over k_0 is of arithmetic type then so as $\underline{A}$ over k_1. This proves the lemma. ∎

Combining (3.3.3) and Lemma 3.8 and 3.9, we should make the notion of arithmetic type more precisely.

(3.10) Definition Let $(B,*)$ be as in (2.1) and (A,ι) be an abelian B-variety up to isogeny over a field k of finite type over its prime field. The pair (A,ι) is said to be *of B-arithmetic type* or simply *of arithmetic type* if there is a finite extension k'/k such that $\rho_\ell(\mathcal{G}_{k'})$ lies in the center of $\mathrm{GAut}_B(V_\ell)$ for a B-linear polarization λ and for one $\ell \neq \mathrm{char}\, k$. An abelian B-variety up to isogeny is said to be *of arithmetic type* if it is so over a field of finite type over its prime field.

(3.11) Lemma *Let $\underline{A}=(A,\iota)$ be an abelian B-variety up to isogeny. If $\underline{A}$ is of arithmetic type, then A is of CM type.*

Proof. Since $\underline{A}$ is of arithmetic type, G_ℓ is commutative. Let $\mathbb{Q}_\ell[\pi]$ be the subalgebra of $\mathrm{End}(V_\ell)$ generated by G_ℓ. By the semi-simplicity of Tate modules due to Faltings and Zarhin [3, 21], $\mathbb{Q}_\ell[\pi]$ is a (commutative) semi-simple subalgebra. Let L be a maximal semi-simple commutative subalgebra in $\mathrm{End}^0(A)$, then $L\otimes\mathbb{Q}_\ell$ is a maximal commutative semi-simple algebra in $\mathrm{End}^0(A)\otimes\mathbb{Q}_\ell$. By the theorem of Faltings and Zarhin on Tate's conjecture loc. cit., we have $\mathrm{End}^0(A)\otimes\mathbb{Q}_\ell=\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(V_\ell)$. Hence $L\otimes\mathbb{Q}_\ell$ becomes a maximal semi-simple commutative subalgebra in $\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(V_\ell)$. Since $\mathbb{Q}_\ell[\pi]$ is commutative and semi-simple, any maximal semi-simple commutative subalgebra in $\mathrm{End}_{\mathbb{Q}_\ell[\pi]}(V_\ell)$ has degree $2g$ over $\mathbb{Q}_\ell$. This shows $[L:\mathbb{Q}]=2g$ and the proof is complete. ∎

(3.12) Proposition *Let (A,ι) be an abelian B-variety of arithmetic type over a field k finitely generated over its prime field. Then G_ℓ^{alg} is independent of ℓ for all $\ell\neq\mathrm{char}\,k$. That is, there is a $\mathbb{Q}$-subgroup G of GL_{2g} such that $G\otimes\mathbb{Q}_\ell\simeq G_\ell^{\mathrm{alg}}$ for all $\ell\neq\mathrm{char}\,k$.*

Proof. By Lemma 3.11, A is of CM-type. The semi-simple part of G_ℓ^{alg} is trivial. By Bogomolov's theorem, $(G_\ell^{\mathrm{alg}})^0$ is independent of ℓ (see [15, 2.2.5] also see the remark in 2.3 of loc. cit. for the function field case). By a theorem of Serre [15] that the component group $G_\ell^{\mathrm{alg}}/(G_\ell^{\mathrm{alg}})^0$ is independent of ℓ, one shows G_ℓ^{alg} is independent of ℓ. ∎

(3.13) Remark In (3.3)–(3.12) we have shown that $\underline{A}$ is of arithmetic type in the sense of (3.1) if and only if its underlying abelian O_B-variety is of arithmetic type in the sense of (3.10).

(3.14) Theorem *Let k be an algebraically closed field and let $x=\underline{A}$ be a polarized abelian O_B-variety over k. Suppose that $\underline{A}$ is of arithmetic type.*

(1) $G_x(\mathbb{Z}_\ell)=\mathrm{Isom}_k(\underline{A}(\ell))$ *for all ℓ.*

(2) *There is a natural bijection*

$$\Lambda_{x,N}(k)\simeq G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/U_N.$$

(3) $\mathrm{mass}[\Lambda_{x.N}(k)] = \mathrm{mass}(G_x, U_N)$.

PROOF. The statements (2) and (3) follow from the statement (1) and Theorems 2.2 and 2.4. We now prove (1). Let k_0 be a finitely generated field for which $\underline{A}$ is defined and we may assume that k is an algebraic closure of k_0.

If $\mathrm{char}\, k = 0$, then $G_\ell := \rho_\ell(\mathcal{G}_{k_0})$ is in $Z(\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle))$ for all ℓ. By Faltings' theorem [3], one has

$$G_x(\mathbb{Z}_\ell) = \mathrm{Aut}_{k_0}(\underline{A}(\ell)) = Z(\mathrm{Aut}_k(\underline{A}(\ell)), G_\ell).$$

Since G_ℓ is in $Z(\mathrm{GAut}_B(V_\ell, \langle\, ,\rangle))$, the latter is simply $\mathrm{Aut}_k(\underline{A}(\ell))$. This proves the case of characteristic zero.

If $\mathrm{char}\, k = p > 0$, then replacing $\underline{A}$ by an isogeny we may assume that k_0 is a finite field, as A is of CM-type (Lemma 3.11 and a theorem of Grothendieck [12, p. 220]). Using (3.4.1), we may also assume that B is a division algebra. Let Frob be the geometric Frobenius in $\mathcal{G}_{k_0}$ and π_A the relative Frobenius endomorphism on A, one has $\rho(\mathrm{Frob}) = \pi_A$ in $\mathrm{End}_{k_0}(A[p^\infty])$. By the p-adic version of Tate's theorem on endomorphisms over finite fields, it then suffices to show that π_A lies in $Z(B)\otimes\mathbb{Q}_p$, which is the center of $\mathrm{End}^0_B(A[p^\infty])$. Since A is of arithmetic type, $\pi_A = \rho_\ell(\mathrm{Frob}) \in Z(B)\otimes\mathbb{Q}_\ell$ (Lemma 3.7). Consider $\mathbb{Q}[\pi_A]$ and $Z(B)$ as linear subspaces of $\mathrm{End}^0(A)$; then $\mathbb{Q}[\pi_A] = \mathbb{Q}_\ell[\pi_A] \cap \mathrm{End}^0(A) \subset Z(B)\otimes\mathbb{Q}_\ell \cap \mathrm{End}^0(A) = Z(B)$, and thus $\pi_A \in Z(B)$. ∎

§4. Classification

In this section we classify abelian B-varieties of arithmetic type up to isogeny. Due to Lemma 3.11, it suffices to classify the objects which are defined either over a number field or a finite field. We may also assume, without loss of generosity due to (3.4.1), that B is a division algebra.

Let $\mathbb{P}$ be a prime field, k be an algebraic closure of $\mathbb{P}$ and k_0 be a finite extension of $\mathbb{P}$ in k. Let $\underline{A}$ be an abelian O_B-variety over k_0.

(4.1) Lemma *If the positive involution $*$ on B is of first kind, then $\underline{A}$ is of arithmetic type if and only if* $\mathrm{char}\, k = p > 0$ *and A is supersingular.*

PROOF. If $\underline{A}$ is of arithmetic type, then by Lemma 3.7 G_ℓ is contained in $\mathbb{Q}_\ell^\times$ after replacing k_0 by a finite extension. Then $\mathrm{End}^0(A)$ has dimension $4g^2$ by Tate's theorem. This implies $\mathrm{char}\, k = p > 0$ and A is supersingular. The other implication is obvious. ∎

(4.2) Lemma 4.1 classifies the abelian varieties of B-arithmetic type in the case of first kind. Thereafter, we suppose that $*$ is of second kind. Let K be the center of B and F be the maximal totally real subfield of K.

(4.2.1) Let $\iota_0 : O_K \to \mathrm{End}(A)$ be the restriction of ι. Then (A, ι) is of B-arithmetic type if and only if (A, ι_0) is of K-arithmetic type. Indeed, it follows from Lemma 3.7 that the centers $Z(\mathrm{GAut}_B(V_\ell, \langle\, , \rangle))$ and $Z(\mathrm{GAut}_K(V_\ell, \langle\, , \rangle))$ are the same. Therefore, the classification is reduced to the case where B is a CM field K.

(4.2.2) Write A isogenous to $\prod_{j=1}^r A_j^{n_j}$, denoted by $A \sim \prod_{j=1}^r A_j^{n_j}$, where each A_j is a simple abelian variety and A_i is not isogenous A_j for $i \neq j$. If A is of K-arithmetic type, then we have

$$\mathrm{End}_K^0(A) \otimes \mathbb{Q}_\ell \simeq \mathrm{End}_{K_\ell}(V_\ell).$$

Note that V_ℓ is a free K_ℓ-module. The latter is isomorphic to $M_n(K_\ell)$ and has dimension $n^2 d$, where $[K : \mathbb{Q}] = d$ and $2g = dn$. Put $B_j := A_j^{n_j}$, $b_j = \dim B_j$ and let $2b_j = dm_j$. We have

$$\dim_{\mathbb{Q}} \mathrm{End}_K^0(B_j) \leq \dim_{\mathbb{Q}_\ell} \mathrm{End}_{K_\ell}(V_\ell(B_j)) = dm_j^2.$$

We also have $\mathrm{End}_K^0(A) = \prod_j \mathrm{End}_K^0(B_j)$. From the dimensions of the abelian varieties and those of the endomorphism algebras, we have

$$\sum_j m_j = n, \quad n^2 \leq \sum_{j=1}^r m_j^2.$$

This shows $r = 1$. We showed that if A is of arithmetic type then it is isogenous to a product of a simple factor.

(4.3) Proposition *If* $\mathrm{char}\, k = 0$, *then* A *is of* K*-arithmetic type if and only if* $A \sim A_1^n$, A_1 *is simple abelian variety with CM by* K_1 *and the image of the homomorphism* $\iota : K \to \mathrm{End}^0(A) = M_n(K_1)$ *contains the center* K_1.

Proof. $(\Rightarrow)$ If A is of K-arithmetic type, the first and second statements are proved in (4.2.2). We regard K as a subfield of $\mathrm{End}^0(A)$ via ι. Let $\widetilde{K}$ the composite of K and K_1. It suffices to show that $K = \widetilde{K}$. The centralizer of K in $M_n(K_1)$, same as that of $\widetilde{K}$, has dimension $[\widetilde{K} : \mathbb{Q}](\dim V/[\widetilde{K} : \mathbb{Q}])^2 = 4g^2/[\widetilde{K} : \mathbb{Q}]$. While $\mathrm{End}_{K_\ell}(V_\ell)$ has dimension $4g^2/[K_\ell : \mathbb{Q}_\ell]$. It follows that $[\widetilde{K} : \mathbb{Q}] = [K : \mathbb{Q}]$, hence K contains K_1.

$(\Leftarrow)$ It suffices to show that $\iota(K_\ell) \supset \mathrm{End}_{\mathrm{End}^0(A)}(V_\ell)$, as G_ℓ is contained in the latter. As $n[K_1 : \mathbb{Q}] = \dim_{\mathbb{Q}_\ell} V_\ell$, the commudant $\mathrm{End}_{\mathrm{End}^0(A)}(V_\ell)$ is $K_{1,\ell}$. And it is contained in $\iota(K_\ell)$ as the assumption. This completes the proof. ∎

We recall the definition of basic abelian varieties with additional structures in the sense of Kottwitz ([10], [14, p. 291, 6.25]). Thereafter, the characteristic of k will be $p > 0$.

(4.4) Definition Let W be the ring of Witt vectors over k and L be the fractional field of W. Let $(B, *)$ remain as in (2.1).

(1) Let (V_p, ψ_p) be a $\mathbb{Q}_p$-valued non-degenerate skew-Hermitian B_p-module, where $B_p := B \otimes \mathbb{Q}_p$. A polarized abelian O_B-variety $\underline{A}$ over k is said to be *related to* (V_p, ψ_p) if there is

a $B_p \otimes L$-linear isomorphism $\alpha : M(\underline{A}) \otimes_W L \simeq (V_p, \psi_p) \otimes L$ which preserves the pairings for a suitable identification $L(1) \simeq L$, where $M(\underline{A})$ is the covariant Dieudonné module with additional structures associated to $\underline{A}$.

Let $G' := \mathrm{GAut}_{B_p}(V_p, \psi_p)$ be the algebraic group of B_p-linear similitudes. A choice α gives rise to an element $b \in G'(L)$ so that one has an isomorphism of isocrystals with additional structures $M(\underline{A}) \otimes L \simeq (V_p \otimes L, \psi_p, b(\mathrm{id} \otimes \sigma))$. The decomposition of $V_p \otimes L$ into isoclinic components induces a $\mathbb{Q}$-graded structure, and thus defines a (slope) homomorphism $\nu_{[b]} : \mathbf{D} \to G'$ over some finite extension $\mathbb{Q}_{p^s}$ of $\mathbb{Q}_p$, where $\mathbf{D}$ is the pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$.

(2) A polarized abelian O_B-variety $\underline{A}$ is called *basic with respect to* (V_p, ψ_p) if
   (i) $\underline{A}$ is related to (V_p, ψ_p), and
   (ii) the slope homomorphism ν is central.

(3) $\underline{A}$ is called *basic* if it is basic with respect to (V_p, ψ_p) for some skew-Hermitian space (V_p, ψ_p).

(4.5) Lemma *Let $\underline{A}$ be a polarized abelian O_B-variety over k. The following statements are equivalent.*

(a) $\underline{A}$ is basic.

(b) Let Z be the center of B and $Z_p = Z \otimes \mathbb{Q}_p = \prod_{\mathbf{p}|p} Z_{\mathbf{p}}$ be the decomposition as a product of local fields. Let $N = M(\underline{A}) \otimes_W L$ be the isocrystals with additional structures associated to $\underline{A}$ and $N = \oplus_{\mathbf{p}|p} N_{\mathbf{p}}$ be the decomposition with respect to the Z_p-action. Then each component $N_{\mathbf{p}}$ is isoclinic.

Proof. See a proof in 6.25 of [14]. ∎

Using Lemma 4.5, one can check a given abelian variety with additional structures to be basic by the statement (b). Note that the statement (b) only depends on the underlying structure of B-action, not on polarizations. This is also a property of those of arithmetic type; see Lemma 3.8. Indeed, we have

(4.6) Proposition *An abelian O_B-variety $\underline{A} = (A, \iota)$ over k is of arithmetic type if and only if it is basic.*

Proof. Using the notation $A \sim \prod_j A_j^{n_j}$, one can show using (b) of Lemma 4.5 that A is basic if and only if each $A_j^{n_j}$ is basic. Therefore, we may assume that B is a division algebra.

If $(B, *)$ is of first kind, then by Lemma 4.5 $\underline{A}$ is basic if and only if $\underline{A}$ is supersingular. Then this follows from Lemma 4.1.

Suppose that $(B, *)$ is of second kind. By Lemma 6.28 of Rapoport-Zink [14], $\underline{A}$ is basic if and only there is a finite field k_0 such that the relative Frobenius morphism π_{A/k_0} lies in the center K of B. The latter statement is equivalent to that the Galois representation ρ_ℓ factors through the center $Z(\mathrm{GAut}_B(V_\ell, \langle\, , \rangle))$; see the proof in Theorem 3.14. This completes the proof. ∎

(4.7) Remark The statement of Proposition 4.6 remains valid when k is an arbitrary algebraically closed field of characteristic p.

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