Mirror Symmetry and Arnold's Duality

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Abstract

We introduce the K3 mirror surfaces and establish a link between this symmetry and the duality of Arnold on 14 exceptional singularities of modality one. The combinatorial description of non-linear change of variables is presented to the study of birational geometry in the mirror symmetry context for manifolds with vanishing first Chern class.

Introduction

This note will be devoted to discussing the topological properties of mirror symmetry of complex manifolds with trivial first Chern class. We shall study the mirror manifolds by orbifold construction. The combinatorial nature of the cohomology (with the coefficient in complex numbers) will be our main concern here, especially those properties related to the corresponding N=2 conformal field theory [22]. The mirror symmetry has been studied on a large class of $c_1 = 0$ Kahler 3–folds, i.e. Calabi-Yau spaces [2, 5, 8, 15, 22]. The topological aspect of this symmetry principle essentially lies in the context of toric geometry [15], hence the same treatment works also for manifolds of any dimension [3, 16]. A remarkable fact in dimension 2 and 3 about this approach appears in the non-singular structure of the canonical model in this construction [10, 14], even though the same property is expected to hold in general. It has been known that the orbifold construction of mirror manifolds is closely related to the strange duality of Arnold for the 14 exceptional singularities of modality one [1, 18, 11, 19, 21]. But the mathematical link between these two subjects has not been thoroughly explored yet. It is the aim of this paper is to extract this relation from the mirror structure of K3 surfaces. At this moment, it is not clear how general the orbifold technique and the method of non-linear change of variables [9, 22] (or fractional transformation in [19]) could be for constructing Calabi-Yau mirror manifolds, even though they are extremely useful in most known examples. However the deeper understanding of the relation between mirror symmetry and Arnold's duality should provide further informations for the general mirror construction of $c_1 = 0$ manifolds.

The mirror symmetry means a pair of $c_1 = 0$ m-folds having the same cohomology by interchanging $H^{1,1}$ and $H^{m-1,1}$. For $m = 2$, these manifolds are K3 surfaces, hence with the same $H^{1,1} = H^{m-1,1}$. Nevertheless, one can still introduce a refined structure on $H^{1,1}$ for those K3 in this context, and again speak of the mirror symmetry as higher dimensional cases. The space $H^{1,1}$ is now expressed by a sum of certain subspaces, which depends on a birational model of the K3 as a hypersurface in weighted 3–space. The description will be given in Sect. 3. Having the mirror K3 surfaces, we can make contact between Arnold's duality and this mirror structure in Sect. 4. In Sect. 1, we briefly review the main results in [10, 15, 16] which will be needed for the discussion of this paper. The non-linear change of variables in [9, 11, 19] is quite similar to the birational technique studied by T. Shioda [17], but some extra consideration is required for the $c_1 = 0$ structure. In Sect. 2, we shall derive the mathematical structure of non-linear change of variables using the combinatorial description of Sect. 1. For the purpose of illustration, most of the discussions in this paper is followed by some specific examples.

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Section 1. Mirror Mainifolds

As a preparation to the discussion of this paper, we begin with a general framework on mirror manifolds through the orbifold construction.

Let $\mathcal{Z}$ be a degree d quasi-smooth hypersurface in the weighted projective space $WP^{n-1}_{(n_1,\ldots,n_n)}$ defined by a quasi-homogeneous polynomial $f(Z)$ $(= f(Z_1,\ldots,Z_n))$. Assume $n > 3$, and

$$d = \sum_{j=1}^{n} n_j \ , \ \gcd(n_j | j \neq i) = 1 \ \forall \ i \, . \tag{1}$$

Let G be a diagonal subgroup of

$$SD := \{\text{dia}[\alpha_1,\ldots,\alpha_n] \in SL_n(\mathbf{C}) |\ f(\alpha_1 Z_1,\ldots,\alpha_n Z_n) = f(Z_1,\ldots,Z_n)\}. \tag{2}$$

Then the quotient

$$\mathcal{X} = \mathcal{Z}/G \, .$$

is a V-manifold with the trivial canonical sheaf. The cohomology of $\mathcal{X}$ can be described by the Jacobian ring of $f(Z)$,

$$\mathfrak{J} = \mathbf{C}[Z_1,\ldots,Z_n] / \left(\frac{\partial f}{\partial Z_i} \right) ,$$

in the following procedure. Denote

$$q_i = \frac{n_i}{d} \ \text{ for } i = 1,\ldots,n \, ,$$
$$Z^k = Z_1^{k_1} \cdots Z_n^{k_n} \ \text{, for } k = (k_1,\ldots,k_n) \in \mathbf{Z}^n \, ,$$

$$\mathfrak{J}_I = \text{ the Jacobian ring of } f(Z|Z_i = 0 \text{ for } i \in I), \;\; I \subset \{1, \dots, n\}.$$

The linear action of G induces one on $\mathbf{C}[Z]$, hence on $\mathfrak{J}$, $\mathfrak{J}_I$. For a non-negative integer m, let

$$\begin{aligned} (\mathfrak{J}_I)^{G,m} = &\text{ the subspace of } (\mathfrak{J}_I)^G \text{ generated by } \left[Z^k\right] \text{ with} \\ & k = (k_1, \dots, k_n)\,, \;\; k_i = 0 \text{ for } i \in I, \;\; \sum_{j \notin I} q_j(k_j + 1) = m + 1. \\ & \mathfrak{J}^{G,m} = (\mathfrak{J}_\emptyset)^{G,m} \;. \end{aligned} \tag{3}$$

By $\sum_i q_i = 1$,

$$\mathfrak{J}^{G,m} = \text{the subspace of } \mathfrak{J}^G \text{ generated by } \left[Z^k\right] \text{ with } \sum_{i=1}^{n} q_i k_i = m.$$

Then the following is a well-known fact for Hodge theory of $\mathcal{X}$ [20] :

$$H^{n-3,1}(\mathcal{X})_0 \simeq \mathfrak{J}^{G,1} \;\;, \tag{4}$$

(Here the subscript in $H^{n-3,1}(\mathcal{X})_0$ refers the primitive part of the cohomology; in the case $n > 4$, this subscript can be dropped.)

Consider the special marginal deformation of the Fermat hypersurface in $\mathbf{WP}^{n-1}_{(n_1,\dots,n_n)}$:

$$\mathcal{Z} : \; f(Z) = \sum_{i=1}^{n} \frac{1}{d_i} Z_i^{d_i} - t Z_1 \cdots Z_n = 0 \;\;, \;\; t \in \mathbf{C} - \{0\}. \tag{5}$$

Now the space (4) has a canonical basis expressed by a combinatorial data depending only on $q_i \left(= \frac{1}{d_i}\right)$. It is defined by the method of toric geometry as follows. Given a $n-$dimensional lattice N and its dual $M := Hom_{\mathbf{Z}}(N, \mathbf{Z})$, denote $N_{\mathbf{Q}} = N \otimes_{\mathbf{Z}} \mathbf{Q}$, $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$, $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$, and the pairing of $N_{\mathbf{R}}$ and $M_{\mathbf{R}}$ by

$$<,>: \; N_{\mathbf{R}} \times M_{\mathbf{R}} \to \mathbf{R}, \; (x, y) \rightsquigarrow < x, y > \;.$$

Consider a $n-$dimensional rational simplicial cone C in $N_{\mathbf{R}}$ with its dual cone

$$\check{C} := \{ y \in M_{\mathbf{R}} | \; < x, y > \geq 0 \text{ for } x \in C \} \,.$$

For a given data $(N, \mathcal{C}, q, v)$ with $q \in N \cap \mathrm{Int}(\mathcal{C})$, $v \in M \cap \mathrm{Int}\left(\check{\mathcal{C}}\right)$ and $< q, v >= 1$, define

$$E(N, \mathcal{C}, q, v) = \{x \in N \cap \mathcal{C} | < x, v >= 1\} \bigcap \bigcup_{1<i<(n-1)} \mathrm{Int}(i\text{-dim face of } \mathcal{C}),$$

$$e^1, \ldots, e^n : \text{ the primitive elements in } N \text{ with } \mathcal{C} = \sum_{i=1}^{n} \mathbf{R}_{\geq 0} e^i,$$

$$\check{e}_1, \ldots, \check{e}_n : \text{ the primitive elements in } M \text{ with } \check{\mathcal{C}} = \sum_{i=1}^{n} \mathbf{R}_{\geq 0} \check{e}_i.$$

Assume the following conditions hold:

$$< e^i, v >=< q, \check{e}_i >= 1 \quad \text{for all } i\,. \tag{6}$$

Then we have

$$\begin{aligned} q = \sum_{i=1}^{n} q_i e^i = \sum_{i=1}^{n} \check{e^i}, \quad v = \sum_{i=1}^{n} q_i \check{e}_i = \sum_{i=1}^{n} e_i\ , \\ N(\mathcal{C}) := \bigoplus_{i=1}^{n} \mathbf{Z} e^i \subset N \subset \bigoplus_{i=1}^{n} \mathbf{Z} \check{e^i}\ , \\ \bigoplus_{i=1}^{n} \mathbf{Z} \check{e}_i \subset M \subset N(\mathcal{C})^* := \bigoplus_{i=1}^{n} \mathbf{Z} e_i\ , \end{aligned} \tag{7}$$

here $\{e_i\}_{i=1}^n$ $\left(\left\{\check{e^i}\right\}_{i=1}^n\right)$ is the dual basis of $\{e^i\}_{i=1}^n$ ($\{\check{e}_i\}_{i=1}^n$ resp.) in $M_{\mathbf{Q}}$ ($N_{\mathbf{Q}}$ resp.), and $q_i = \frac{1}{d_i}$ for some positive integer d_i. It follows

$$\check{e}_i = d_i e_i \quad \text{for all } i\,. \tag{8}$$

Regard $\mathbf{WP}^{n-1}_{(n_i)}$ as a compactification of the $(n-1)$-torus T with

$$Hom_{\mathrm{alg.\,gp}}(\mathsf{T}, \mathbf{C}^*) = N(\mathcal{C})/(N(\mathcal{C}) \cap \mathbf{Q}q)\quad . \tag{9}$$

Then the homogeneous coordinate Z_i of $\mathbf{WP}^{n-1}_{(n_i)}$ can be identified with $e_i \in N(\mathcal{C})^*$. Hence we have the correspondence:

$$\begin{aligned} \check{e}_i &\leftrightarrow Z_i^{d_i} \quad \text{for all } i\,, \\ v &\leftrightarrow Z_1 \cdots Z_n \quad , \end{aligned}$$

and these are the monomials appeared in the equation (5). One can identify the quotient $\oplus_{i=1}^n \mathbf{Z}\check{e^i} \,/\, \oplus_{i=1}^n \mathbf{Z} e^i$ with the diagonal group

$$D_q := \left\{ \mathrm{dia}[\alpha_1, \ldots, \alpha_n] \in GL_n(\mathbf{C}) |\ \alpha_i^{d_i} = 1\ \forall\, i \right\}.$$

It acts on $\mathbf{C}^n$, hence on $\mathbf{C}[Z_1, \ldots, Z_n]$. Now the subgroup of D_q,

$$SD_q = D_q \cap SL_n(\mathbf{C}) \quad ,$$

is just the group SD of (2) for the polynomial $f(Z)$ defined by (5). It contains the cyclic group

$$Q =< \mathrm{dia}\left[e^{2\pi i q_1}, \ldots, e^{2\pi i q_n}\right] > \quad . \qquad (10)$$

The lattice N corresponds the subgroup

$$G(N) = N / \oplus_{i=1}^{n} \mathbf{Z} e^i \qquad (11)$$

with

$$Q \subset G(N) \subset SD_q \ . \qquad (12)$$

For $G = G(N)$, we shall also denote the lattices N, M by N_G, M_G respectively. Regard the lattice $N_G/(N_G \cap \mathbf{Q}q)$ as the group of 1–parameter subgroups of the $(n-1)$-torus T/G:

$$Hom_{\mathrm{alg.\,gp}}(\mathrm{T}/G, \ \mathbf{C}^*) = N_G/(N_G \cap \mathbf{Q}q) \quad ,$$

here T is the torus in (9). The quotient $\mathbf{WP}^{n-1}_{(n_i)}/G$ is considered as the T/G-compactification associated to the cone $\mathcal{C}$, and we have the Calabi-Yau hypersurface

$$\mathcal{X} = \mathcal{Z}/G \qquad (13)$$

with $\mathcal{Z}$ defined by (5). From now on we will usually write

$$\overline{[Z_1, \ldots, Z_n]} = \text{the orbit of } [Z_1, \ldots, Z_n] \text{ in } \mathcal{X} \text{ for } [Z_1, \ldots, Z_n] \in \mathcal{Z} \ .$$

Consider the dual data $\left(M, \check{\mathcal{C}}, v, q\right)$ of $(N, \mathcal{C}, q, v)$. Through the identification

$$D_q = \bigoplus_{i=1}^{n} \mathbf{Z}\check{e}^i / \bigoplus_{i=1}^{n} \mathbf{Z}e^i = \bigoplus_{i=1}^{n} \mathbf{Z}e_i / \bigoplus_{i=1}^{n} \mathbf{Z}\check{e}_i \ ,$$

the subgroup $G(M)$ also satisfies the condition (12). We shall call $G(N)$, $G(M)$ the mirror subgroups. From this definition, the characterization of mirror subgroups G, G' of SD_q is given by one of the following equivalent conditions:

$$\begin{cases} N_G = M_{G'} \\ M_G = N_{G'} \end{cases} .$$

In particular, Q and SD_q are mirrors of each other. The subset $E\left(M_G, \check{\mathcal{C}}, v, q\right)$ in the lattice M_G encodes the essential part of the cohomology of $\mathcal{X}$, and the following proposition provides a combinatorial description of $H^{n-3,1}(\mathcal{X})_0$. For its argument, one could refer to [16].

Proposition 1.

$$H^{n-3,1}(\mathcal{X})_0 \simeq \mathfrak{J}^{G,1} \ ,$$

and the elements

$$\left[Z^k\right], \ k \in E\left(M_G, \check{\mathcal{C}}, v, q\right) \cup \{v\} ,$$

form a basis of $\mathfrak{J}^{G,1}$.

For convenience, in the following we shall simply write $E(N), E(M)$ for $E(N, \mathcal{C}, q, v)$, $E\left(M, \check{\mathcal{C}}, v, q\right)$ if there is no danger of confusion. Now the simplicial cones $\mathcal{C}$, $\check{\mathcal{C}}$ are the first quadrant cones with respective to the bases $\{e^i\}_{i=1}^n, \{\check{e}_i\}_{i=1}^n$ respectively. For a subset I of $\{1, \ldots, n\}$, by the $I-$face of the cone, we mean those elements having zero value for all the $j-$th coordinates with $j \notin I$. Define

$$\begin{aligned} E(N; I) &= E(N) \cap (I - \text{face of } \mathcal{C}), \\ E(N; I)^* &= E(N; I) - \bigcup_{J \subsetneq I} E(N; J), \end{aligned}$$

and similarly for $E(M; I)$, $E(M; I)^*$. Then we have

$$E(N) = \bigcup_{|I|=2}^{n-2} E(N; I)^*, \ E(M) = \bigcup_{|I|=2}^{n-2} E(M; I)^* \ . \qquad (14)$$

For a group G satisfying (12), there is a birational model $\widehat{\mathcal{Z}/G}$ for $\mathcal{Z}/G$ such that the exceptional divisors in $\widehat{\mathcal{Z}/G}$ are essentially described by $E(N_G)$. Moreover, for $n = 4$ and 5, these birational models $\widehat{\mathcal{Z}/G}$ are non-singular projective manifolds. Using Proposition 1 and a description of the deformation of $\widehat{\mathcal{Z}/G}$, one can establish a correspondence between cohomology $H^{1,1}, H^{n-3,1}$ of $\widehat{\mathcal{Z}/G}$ and $\widehat{\mathcal{Z}/G'}$ for a mirror pair G, G'. The detail can be found in [16]. In particular, for $n = 5$, $\widehat{\mathcal{Z}/G}$ and $\widehat{\mathcal{Z}/G'}$ are Calabi-Yau spaces with

$$\begin{aligned} H^{2,1}\left(\widehat{\mathcal{Z}/G}\right) &\simeq H^{1,1}\left(\widehat{\mathcal{Z}/G'}\right), \\ H^{1,1}\left(\widehat{\mathcal{Z}/G}\right) &\simeq H^{2,1}\left(\widehat{\mathcal{Z}/G'}\right). \end{aligned}$$

For $n = 4$, we have $H^{n-3,1} = H^{1,1}$. However, the mirror property for K3 surfaces $\widehat{\mathcal{Z}/G}$, $\widehat{\mathcal{Z}/G'}$ appears in a more refined structure of $H^{1,1}$ which will be discussed in Sect. 3.

Section 2. Non – linear Change of Variables

In this section, the non-linear change of variables will be derived using the combinatorial data introduced in the previous section. Let $(N, \mathcal{C}, q, v)$, $\left(M, \check{\mathcal{C}}, v, q\right)$, $\{e^i\}_{i=1}^n$, $\left\{\check{e}^i\right\}_{j=1}^n$, $\{e_i\}_{i=1}^n$, $\{\check{e}_i\}_{i=1}^n$ be the same as before, and they satisfy the conditions (6) (7) (8). Then $N = N_G$ with $G = G(N)$ defined by (11). Consider a n−dimensional rational simplicial cone $\mathcal{F}$ in $N_{\mathbf{R}}$ generated by n elements $\mathfrak{f}^1, \ldots, \mathfrak{f}^n$ of $\mathcal{C} \cap N$ with

$$q \in \mathrm{Int}(\mathcal{F}), \ < \mathfrak{f}^j, v >= 1 \ \forall j \ . \qquad (15)$$

The dual basis $\{\mathfrak{f}_j\}_{j=1}^n$ of $\left\{\mathfrak{f}^j\right\}_{j=1}^n$ generates the dual cone $\check{\mathcal{F}}$, and we have

$$\mathcal{F} \subset \mathcal{C}, \ \check{\mathcal{F}} \supset \check{\mathcal{C}} \quad .$$

Write

$$\mathfrak{f}^j = \sum_{i=1}^n f_i^j e^i \quad , 1 \leq j \leq n,$$

then

$$f_i^j \in \mathbf{Q}_{\geq 0} \ , \ \sum_{i=1}^n f_i^j = 1 \ . \qquad (16)$$

From $\sum_{j=1}^n \mathbf{Z}\mathfrak{f}^j \subset N$, it follows that

$$M \subset \sum_{j=1}^n \mathbf{Z}\mathfrak{f}_j \ ,$$

and

$$\begin{pmatrix} \check{e}_1 \\ \vdots \\ \check{e}_n \end{pmatrix} = \left(m_i^j\right)_{1 \leq i,j \leq n} \begin{pmatrix} \mathfrak{f}_1 \\ \vdots \\ \mathfrak{f}_n \end{pmatrix} \qquad (17)$$

for $m_i^j \in \mathbf{Z}_{\geq 0}$. In fact, we have

$$m_i^j = d_i f_i^j \ , \ \text{ for } 1 \leq i, j \leq n \ ,$$

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = (f_j^i)_{1 \leq i,j \leq n} \begin{pmatrix} \mathfrak{f}_1 \\ \vdots \\ \mathfrak{f}_n \end{pmatrix} . \qquad (18)$$

By (15), one has the expression

$$q = \sum_{j=1}^{n} q'_j \mathfrak{f}^j \quad \text{with} \quad q'_j = \frac{n'_j}{d'} \ , \tag{19}$$

n'_j, d' satisfying (1). And the relation (16) gives

$$v = \sum_{i=1}^{n} \mathfrak{f}_i \quad . \tag{20}$$

Let $\mathsf{T}^\sharp$ be the $(n-1)$−torus which has the 1-parameter subgroups defined by

$$Hom_{\text{alg. gp}}\left(\mathsf{T}^\sharp, \mathsf{C}^*\right) = (\oplus_{j=1}^{n} \mathsf{Z}\mathfrak{f}^j)/((\oplus_{j=1}^{n} \mathsf{Z}\mathfrak{f}^j) \cap \mathsf{Q}q) \quad . \tag{21}$$

Then $\mathsf{WP}^{n-1}_{(n'_i)}$ is the compactification of $\mathsf{T}^\sharp$ associated to the cone $\mathcal{F}$, and has the homogeneous coordinate $[Y_1, \ldots, Y_n]$ through the identification:

$$\mathfrak{f}_i \leftrightarrow Y_i \quad \text{for all } i \, . \tag{22}$$

By (17) (20), we have the correspondence:

$$\breve{e}_i \leftrightarrow \prod_{j=1}^{n} Y_j^{m_i^j} \quad \text{for all } i \, ,$$
$$v \leftrightarrow Y_1 \cdots Y_n \quad .$$

Now the relation (18) defines a non-linear change of variables from $\mathsf{WP}^{n-1}_{(n'_i)}$ to $\mathsf{WP}^{n-1}_{(n_i)}$:

$$X_i = \prod_{j=1}^{n} Y_j^{f_i^j} \quad \text{for } i = 1, \ldots, n \, . \tag{23}$$

From the above relation, the equation (5) is transformed into

$$g(Y) = \sum_{i=1}^{n} \frac{1}{d_i} \prod_j Y_j^{m_i^j} - tY_1 \cdots Y_n \ , \tag{24}$$

and denote

$$\mathcal{Y} = \left\{ [Y] \in \mathsf{WP}^{n-1}_{(n'_i)} | \ g(Y) = 0 \right\} \quad . \tag{25}$$

We may regard

$$G^\sharp := N / \left(\sum_{j=1}^{n} \mathsf{Z}\mathfrak{f}^j \right) \, , \tag{26}$$

as a diagonal group acting on $[Y_1, \ldots, Y_n]$ which leaves the polynomial $g(Y)$ invariant. The quotient $\mathbf{WP}^{n-1}_{(n'_i)}/G^\sharp$ is the compactification of $(n-1)$–torus $\mathsf{T}^\sharp/G^\sharp$, which satisfies

$$Hom_{\mathrm{alg.\,gp}}\left(\mathsf{T}^\sharp/G^\sharp,\, \mathbf{C}^*\right) = N/(N \cap \mathbf{Q}q) \quad .$$

As $\mathsf{T}^\sharp/G^\sharp$ and T/G have the same 1-parameter subgroups, the relation (23) defines an isomorphism:

$$\mathsf{T}^\sharp/G^\sharp \simeq \mathsf{T}/G\,,$$

hence birational maps

$$\mathbf{WP}^{n-1}_{(n'_i)}/G^\sharp \;\leftarrow \cdots \rightarrow\; \mathbf{WP}^{n-1}_{(n_i)}/G\,,$$

$$\mathcal{Y}/G^\sharp \quad \leftarrow \cdots \rightarrow \quad \mathcal{X}\;, \tag{27}$$

with $\mathcal{X}$, $\mathcal{Y}/G^\sharp$ defined by (13) (25) (26). Its inverse is given by

$$Y_i = \prod_{j=1}^{n} X_j^{h^i_j} \quad \text{for } i = 1, \ldots, n\,, \tag{28}$$

with $\left(h^i_j\right)_{1\le i,j\le n} = \left(f^j_i\right)^{-1}_{1\le i,j\le n}$.

Notice that one can also reverse the above construction by starting from the non-linear change of variables (23), which is the procedure given in [9, 11, 19]. We now illustrate the above discussion by the following examples of Calabi-Yau spaces.

Example 1. Consider the hypersurface in $\mathbf{WP}^4_{(2,2,2,1,1)}$:

$$\mathcal{Z}\;: f(Z) = \frac{1}{4}Z_1^4 + \frac{1}{4}Z_2^4 + \frac{1}{4}Z_3^4 + \frac{1}{8}Z_4^8 + \frac{1}{8}Z_5^8 \; - tZ_1Z_2Z_3Z_4Z_5 \;\; = 0\,.$$

Now the group SD_q is isomorphic to $(\mathbf{Z}/3\mathbf{Z})^3 \times (\mathbf{Z}/8\mathbf{Z})$ and Q is generated by $\mathrm{dia}\left[e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{8}}, e^{\frac{2\pi i}{8}}\right]$. Let G be the subgroup of SD_q defined by

$$G =< \mathrm{dia}\left[e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{8}}, e^{\frac{2\pi i}{8}}\right], \mathrm{dia}\left[1, 1, 1, e^{\frac{2\pi i}{8}}, e^{-\frac{2\pi i}{8}}\right] >,$$

and

$$\mathcal{X} = \mathcal{Z}/G\,,$$
$$N = N_G.$$

By (6) (7), we have

$$e^1, \dots, e^5 = \text{the standard base in } \mathbf{R}^5 ,$$

$$e_1, \dots, e_5 = \text{the standard base in } \mathbf{R}^{5*} ,$$

$$\check{e^i} = \frac{1}{4} e^i, \ \ \check{e}_i = 4e_i \ \ (i = 1,2,3) \ ; \ \ \check{e^j} = \frac{1}{8} e^j, \ \ \check{e}_j = 4e_j \ \ (j = 4,5.)$$

$$\mathcal{C} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} e^i \ , \ \ \check{\mathcal{C}} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} e_i \ ,$$

$$q = \frac{1}{4} \sum_{i=1}^{3} e^i + \frac{1}{8}(e^4 + e^5) \ , \ \ v = \sum_{i=1}^{5} e_i \ .$$

Consider the rational cone

$$\mathcal{F} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} \mathfrak{f}^i$$

where

$$(\ \mathfrak{f}^1 \ \ \mathfrak{f}^2 \ \ \mathfrak{f}^3 \ \ \mathfrak{f}^4 \ \ \mathfrak{f}^5\ ) = (\ e^1 \ \ e^2 \ \ e^3 \ \ e^4 \ \ e^5\ ) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & \frac{7}{8} \end{pmatrix} ,$$

or

$$(\ e^1 \ \ e^2 \ \ e^3 \ \ e^4 \ \ e^5\ ) = (\ \mathfrak{f}^1 \ \ \mathfrak{f}^2 \ \ \mathfrak{f}^3 \ \ \mathfrak{f}^4 \ \ \mathfrak{f}^5\ ) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{7} \\ 0 & 0 & 0 & 0 & \frac{8}{7} \end{pmatrix} .$$

Then we have

$$N = \sum_{i=1}^{5} \mathbf{Z} \mathfrak{f}^i + \mathbf{Z} e^1 + \mathbf{Z} q \ ,$$

$$q = \frac{1}{4}(\mathfrak{f}^1 + \mathfrak{f}^2 + \mathfrak{f}^3) + \frac{3}{28} \mathfrak{f}^4 + \frac{1}{7} \mathfrak{f}^5 \ ,$$

hence (24) (25) (26) are now

$$G^{\natural} = < \mathrm{dia}\left[e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{5\pi i}{28}}, e^{\frac{2\pi i}{7}} \right] >,$$

$$\mathbf{WP}^4_{(n_i)} = \mathbf{WP}^4_{(2,2,2,1,1)}, \ \mathbf{WP}^4_{(n_i')} = \mathbf{WP}^4_{(7,7,7,3,4)},$$

$$\mathcal{Y} : g(Y) = \frac{1}{5} \left(\sum_{i=1}^{3} Y_i^4 + Y_4^8 Y_5 + Y_5^7 \right) - t Y_1 Y_2 Y_3 Y_4 Y_5 = 0 \ \text{ in } \ \mathbf{WP}^4_{(7,7,7,3,4)} ,$$

$$\mathcal{Y} = \mathcal{Y} / G^{\natural} \ .$$

The corresponding non-linear changes of variables (23) (28) are expressed by

$$\begin{cases} Z_i = Y_i \quad (i = 1, 2, 3), \\ Z_4 = Y_4 Y_5^{\frac{1}{8}} \\ Z_5 = Y_5^{\frac{7}{8}} \end{cases} , \qquad \begin{cases} Y_i = Z_i \quad (i = 1, 2, 3), \\ Y_4 = Z_4 Z_5^{-\frac{1}{7}} \\ Y_5 = Z_5^{\frac{8}{7}} \end{cases} , \tag{29}$$

which define the biraional map $\mathcal{Y} \leftarrow \cdots \rightarrow \mathcal{X}$. q.e.d.

Example 2. Consider the quintic in projective 4–space:

$$\mathcal{Z} \ : f(Z) = \frac{1}{5} \sum_{i=1}^{5} Z_i^5 \ - t Z_1 Z_2 Z_3 Z_4 Z_5 \ = 0 \quad \text{ in } \mathbf{P}^4.$$

Let

$$\begin{aligned} G &= SD_q, \\ \mathcal{X} &= \mathcal{Z} / SD_q \,, \\ N &= N_{SD_q} \,. \end{aligned}$$

By (6) (7), we have

$$\begin{gathered} e^1, \ldots, e^5 = \text{the standard base in } \mathbf{R}^5 \,, \\ e_1, \ldots, e_5 = \text{the standard base in } \mathbf{R}^{5*} \,, \\ \check{e}^i = \frac{1}{5} e^i \ , \ \check{e}_i = 5 e_i \quad \forall i \,, \\ \mathcal{C} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} e^i \ , \ \check{\mathcal{C}} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} e_i \ , \\ q = \frac{1}{5} \sum_{i=1}^{5} e^i \ , \ v = \sum_{i=1}^{5} e_i \ . \end{gathered}$$

Consider the rational cone

$$\mathcal{F} = \sum_{i=1}^{5} \mathbf{R}_{\geq 0} \mathfrak{f}^i$$

where

$$(\mathfrak{f}^1 \quad \mathfrak{f}^2 \quad \mathfrak{f}^3 \quad \mathfrak{f}^4 \quad \mathfrak{f}^5) = (e^1 \quad e^2 \quad e^3 \quad e^4 \quad e^5) \begin{pmatrix} \frac{4}{5} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & 0 & 0 & \frac{4}{5} \end{pmatrix},$$

or

$$
(\,e^1 \quad e^2 \quad e^3 \quad e^4 \quad e^5\,) = (\,\mathfrak{f}^1 \quad \mathfrak{f}^2 \quad \mathfrak{f}^3 \quad \mathfrak{f}^4 \quad \mathfrak{f}^5\,)\begin{pmatrix} \frac{256}{205} & \frac{-64}{205} & \frac{16}{205} & \frac{-4}{205} & \frac{1}{205} \\ \frac{1}{205} & \frac{256}{205} & \frac{-64}{205} & \frac{16}{205} & \frac{-4}{205} \\ \frac{-4}{205} & \frac{1}{205} & \frac{256}{205} & \frac{-64}{205} & \frac{16}{205} \\ \frac{16}{205} & \frac{-4}{205} & \frac{1}{205} & \frac{256}{205} & \frac{-64}{205} \\ \frac{-64}{205} & \frac{16}{205} & \frac{-4}{205} & \frac{1}{205} & \frac{256}{205} \end{pmatrix}.
$$

Then we have

$$
N = \sum_{i=1}^{5} \mathbf{Z}\mathfrak{f}^i + \mathbf{Z}e^1 + \mathbf{Z}q \ ,
$$

$$
q = \frac{1}{5}\sum_{i=1}^{5} e^i = \frac{1}{5}\sum_{i=1}^{5} \mathfrak{f}^i \quad ,
$$

hence

$$
G^{\natural} = < e^{\frac{2\pi i}{5}} id. \, , \mathrm{dia}\left[e^{\frac{51}{205}2\pi i}, e^{\frac{1}{205}2\pi i}, e^{\frac{-4}{205}2\pi i}, e^{\frac{16}{205}2\pi i}, e^{\frac{-64}{205}2\pi i}\right] >,
$$

$$
\mathbf{WP}^4_{(n_i)} = \mathbf{WP}^4_{(n_i')} = \mathbf{P}^4,
$$

$$
\mathcal{Y}: \ g(Y) = \frac{1}{5}\left(\sum_{i=1}^{4} Y_i^4 Y_{i+1} + Y_5^4 Y_1\right) - tY_1Y_2Y_3Y_4Y_5 = 0 \ \text{ in } \ \mathbf{P}^4 .
$$

The corresponding non-linear change of variables for the biraional map $\mathcal{Y}/G^{\natural} \leftarrow \cdots \rightarrow \mathcal{X}$ is given by

$$
\begin{cases} Z_1 = Y_1^{\frac{4}{5}} Y_2^{\frac{1}{5}} \\ Z_2 = Y_2^{\frac{4}{5}} Y_3^{\frac{1}{5}} \\ Z_3 = Y_3^{\frac{4}{5}} Y_4^{\frac{1}{5}} \\ Z_4 = Y_4^{\frac{4}{5}} Y_5^{\frac{1}{5}} \\ Z_5 = Y_5^{\frac{4}{5}} Y_1^{\frac{1}{5}} \end{cases} ,
$$

$$
\begin{cases} Y_1 = Z_1^{\frac{256}{205}} Z_2^{\frac{1}{205}} Z_3^{\frac{-4}{205}} Z_4^{\frac{16}{205}} Z_5^{\frac{-64}{205}} \\ Y_2 = Z_1^{\frac{-64}{205}} Z_2^{\frac{256}{205}} Z_3^{\frac{1}{205}} Z_4^{\frac{-4}{205}} Z_5^{\frac{16}{205}} \\ Y_3 = Z_1^{\frac{16}{205}} Z_2^{\frac{-64}{205}} Z_3^{\frac{256}{205}} Z_4^{\frac{1}{205}} Z_5^{\frac{-4}{205}} \\ Y_4 = Z_1^{\frac{-4}{205}} Z_2^{\frac{16}{205}} Z_3^{\frac{-64}{205}} Z_4^{\frac{256}{205}} Z_5^{\frac{1}{205}} \\ Y_5 = Z_1^{\frac{1}{205}} Z_2^{\frac{-4}{205}} Z_3^{\frac{16}{205}} Z_4^{\frac{-64}{205}} Z_5^{\frac{256}{205}} \end{cases} ,
$$

q.e.d.

For the case of interest to us, we shall focus only on those polynomials $g(Y)$ of (24) whose zeros have exactly one critical point in $\mathbf{C}^n$ at the origin. In this situation, both $\mathcal{Y}/G^\sharp$, $\mathcal{X}$ have only abelian quotient singularities, where the method of toroidal compactification can be applied for resolving the singularities. For $n = 4$ and 5, the "minimal" toroidal resolution $\widehat{\mathcal{Y}/G^\sharp}$, $\widehat{\mathcal{X}}$ are non-singular projective manifolds with the trivial canonical bundle. From all the examples we have studied, there should exist a biregular isomorphism between $\widehat{\mathcal{Y}/G^\sharp}$ and $\widehat{\mathcal{X}}$ in a general context such that it is compactible with the map (27). Here we shall illustrate this phenomenon through an example of Calabi-Yau space. For $n = 4$, more examples related to Arnold's strange duality will be discussed in Sect. 4.

For later use, we describe the connection between the non-linear change of variables and birational relations of cyclic quotient singularity in surface cases. The results should be well-known to specialists. But the required formulation could not be found in literature, so we just derive it here. For a positive integer d, we denote

$$A_{d-1} = \mathbf{C}^2/\mathrm{dia}\left[e^{\frac{2\pi i}{d}}, e^{\frac{-2\pi i}{d}}\right],$$
$$\overline{(z_1, z_2)} = \text{the element in } A_{d-1} \text{ determined by } (z_1, z_2) \in \mathbf{C}^2,$$
$$\widehat{A_{d-1}} = \text{the minimal resolution of } A_{d-1}.$$

<u>Lemma 1</u> For positive integers d, h with $d > h$, the map

$$\iota : A_{h-1} \to A_{d-1}$$

defined by the relation

$$\overline{(z_1, z_2)} = \overline{\left(y_1 y_2^{\frac{d-h}{d}}, y_2^{\frac{h}{d}}\right)} \quad \text{for } \overline{(y_1, y_2)} \in A_{h-1}, \ \overline{(z_1, z_2)} \in A_{d-1}, \qquad (30)$$

is an injective morphism with the image $A_{d-1} - \left\{\overline{(z_1, 0)} \mid z_1 \neq 0\right\}$. Furthermore, the morphism ι induces the embedding between their minimal resolutions,

$$\hat{\iota} : \widehat{A_{h-1}} \to \widehat{A_{d-1}}.$$

Proof. It is obvious that the morphism ι defines an isomorphism between A_{h-1} and $A_{d-1} - \left\{\overline{(z_1, 0)} \mid z_1 \neq 0\right\}$, and its inverse is given by

$$\overline{(y_1, y_2)} = \overline{\left(z_1 z_2^{-\frac{d-h}{h}}, z_2^{\frac{d}{h}}\right)} \quad \text{for } \overline{(y_1, y_2)} \in A_{h-1}, \ \overline{(z_1, z_2)} \in A_{d-1}.$$

The expression of local coordinates of $\widehat{A_{d-1}}$ is well-known. It can be obtained by the method of toroidal compactification using the combinatorial data from the intersection of first quadrant cone with the lattice

$$\mathbf{Z}\begin{pmatrix}1\\0\end{pmatrix} + \mathbf{Z}\begin{pmatrix}0\\1\end{pmatrix} + \mathbf{Z}\begin{pmatrix}\frac{1}{d}\\ \frac{d-1}{d}\end{pmatrix}.$$

In this set, all the elements with 1 as the sum of its components are given by

$$\rho^i = \begin{pmatrix} \frac{i}{d} \\ \frac{d-i}{d} \end{pmatrix}, \quad i = 0, \ldots, d.$$

For each ρ^i, there associates a divisor D_{ρ^i} in $\widehat{A_{d-1}}$. Then D_{ρ^i} $(1 \leq i \leq d-1)$ are the exceptional divisors, and D_{ρ^0}, D_{ρ^d} the proper transform of $\{z_1 = 0\}$, $\{z_2 = 0\}$ respectively. For each k, (ρ^{k-1}, ρ^k) forms a basis of the above lattice and its dual basis defines a local affine coordinate of $\widehat{A_{d-1}}$, denoted by (s_k, t_k). The local equations of $D_{\rho^{k-1}}$, D_{ρ^k} are $\{s_k = 0\}$, $\{t_k = 0\}$ respectively, and the relation between (s_k, t_k) and (z_1, z_2) is given by

$$\left\{ \begin{array}{l} s_k = z_1^k z_2^{k-d} \\ t_k = z_1^{1-k} z_2^{d-k+1} \end{array} \right. , \text{ for } k = 1, \ldots, d. \tag{31}$$

Similarly, $\widehat{A_{h-1}}$ has the divisors $D_{(\rho^i)'}$, $0 \leq i \leq h$, and coordinate systems (s'_k, t'_k), $1 \leq k \leq h$,

$$\left\{ \begin{array}{l} s'_k = y_1^k y_2^{k-h} \\ t'_k = y_1^{1-k} y_2^{h-k+1} \end{array} \right. , \text{ for } k = 1, \ldots, h. \tag{32}$$

By (30) (31) (32), one has

$$\left\{ \begin{array}{l} s_k = s'_k \\ t_k = t'_k \end{array} \right. \qquad \text{for } k = 1, \ldots, h.$$

Therefore the morphism $\hat{\iota}$ is an embedding under which

$$\hat{\iota} : \widehat{A_{h-1}} \simeq \widehat{A_{d-1}} - \bigcup_{j=h+1}^{d} D_{\rho^j}.$$

q.e.d.

Example 3. Let $\mathcal{X}$, $\mathcal{Y}$ be the Calabi-Yau orbifolds in Example 1. Their singularities are described by

$$\text{Sing}(\mathcal{Y}) = \{Y_4 = Y_5 = 0\} \cup \{[0,0,0,1,0]\},$$
$$\text{Sing}(\mathcal{X}) = \overline{\{X_4 = X_5 = 0\}}.$$

The exceptional divisors in their Calabi-Yau resolutions

$$\sigma_{\mathcal{Y}} : \widehat{\mathcal{Y}} \to \mathcal{Y},$$
$$\sigma_{\mathcal{X}} : \widehat{\mathcal{X}} \to \mathcal{X}$$

are given by

$$\sigma_{\mathcal{Y}}^{-1}(\{Y_4 = Y_5 = 0\}) = \bigcup_{k=1}^{6} E_k \ , \text{ a fiber bundle with fiber } \widehat{A_6},$$
$$\sigma_{\mathcal{Y}}^{-1}(\{[0,0,0,1,0]\}) = E \simeq \mathrm{P}^2 \text{ with } \mathcal{O}_{\mathrm{P}^2}(-3) \text{ normal bundle},$$
$$\sigma_{\mathcal{X}}^{-1}\left(\overline{\{X_4 = X_5 = 0\}}\right) = \bigcup_{k=1}^{7} D_k \ , \text{ a fiber bundle with fiber } \widehat{A_7}.$$

The non-linear change of variables (23) defines a birational morphism

$$\varphi : \mathcal{Y} - \{[0,0,0,1,0]\} \to \mathcal{X}$$

with $\varphi(\{Y_5 = 0\}) = \mathrm{Sing}(\mathcal{X})$. In fact, φ defines the biregular morphism

$$\varphi : \mathcal{Y} - \{Y_5 = 0\} \simeq \mathcal{X} - \overline{\{Z_5 = 0\}} \, .$$

The above isomorphism can be extended to the one between $\widehat{\mathcal{Y}}$ and $\widehat{\mathcal{X}}$ with

$$E \leftrightarrow \text{ the proper transform of } \overline{\{Z_5 = 0\}} \text{ in } \widehat{\mathcal{X}} \ ,$$
$$E_i \ \leftrightarrow \ D_i \quad \text{for } i = 1, \ldots, 6,$$

such that the fiber $\widehat{A_6}$ of $\bigcup_{i=1}^{6} E_i$ embeds in $\widehat{A_7}$ of $\bigcup_{i=1}^{7} D_i$ as in Lemma 1. By identifying $\widehat{\mathcal{Y}}$ with $\widehat{\mathcal{X}}$, $\mathcal{Y}$ is obtained by blowing down $\bigcup_{i=1}^{6} E_i$ to a curve, E to a point; while $\mathcal{X}$ by blowing down $\bigcup_{i=1}^{7} D_i$ to a curve, (see Figure 1). q.e.d.

Remark. By (19) (20), the data $(\mathcal{F}, N, q, v)$ discussed in this section satisfies only half of the conditions in (6) (7). In particular, $< q, \check{f}_i > \neq 1$ for some i, which means the equation (24) of $\mathcal{Y}$ is not of Fermat-type. The combinatorial description of Sect. 1 does not apply to $\widehat{\mathcal{Y}}$. The question of a canonical representative in the lattice N for the cohomology of $\widehat{\mathcal{Y}}$ is rather intriguing. It should shed some light on some further principles of the mirror symmetry.

Section 3. Mirror Symmetry of K3 Surfaces

In this section the case of $n = 4$ is discussed. Then the only I's involved in (14) are $|I| = 2$ and we have $E(N)_I = E(N)_I^*$. In this case, the resolution of $\mathcal{X}$,

$$\sigma : \widehat{\mathcal{X}} \to \mathcal{X} = \mathcal{Z}/G \quad ,$$

is a K3 surface. The singular set of $\mathcal{X}$ is a union of $\mathcal{X}_I \left(:= \ \mathcal{X} \cap \overline{\{Z_i = 0, \ i \in I\}}\right)$. In fact,

$$\mathcal{X}_I \subset \mathrm{Sing}(X) \ \Leftrightarrow \ E(N_G)_I \neq \emptyset \, .$$

In which case $\mathcal{X}_I$ consists of only a finite elements, and $\sigma^{-1}(x)$ is a union of exceptional divisors parametrized by $E(N_G)_I$ for each $x \in \mathcal{X}$. Let $\delta_{I'}$ be the 0–cocycle of $\mathcal{X}_I$ with the value one for each of its elements. Then the complement of $\delta_{I'}$ in $H^0(\mathcal{X}_I)$ is $H^0(\mathcal{X}_I)_0$ having a basis with indices $E(M_G)_{I'}$. As always, we identify

$$(E(N_G)_I \times \delta_{I'}) \coprod (E(N_G)_I \times E(M_G)_{I'}) \ ,$$

with a basis of

$$\bigoplus \left\{ H^0(D) | D : \text{ exceptional divisor in } \widehat{\mathcal{X}} \text{ over } \mathcal{X}_I \right\} ,$$

which is contained in $H^{1,1}\left(\widehat{\mathcal{X}}\right)$ via the isomorphism:

$$H^{1,1}\left(\widehat{\mathcal{X}}\right) \simeq H^{1,1}(\mathcal{X}) \bigoplus \oplus \left\{ H^0(D) | D : \text{ exceptional divisor in } \widehat{\mathcal{X}} \right\},$$
$$H^{1,1}(\mathcal{X}) \simeq \mathbf{C}(\text{Fubini}) \oplus H^{1,1}(\mathcal{X})_0 \ .$$

(Here (Fubini) denotes the class of the Fubini metric of $\mathbf{WP}^3_{(n_i)}$). Define

$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_c := H^{1,1}(\mathcal{X})_0 \ ,$$
$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_d := \mathbf{C}(\text{Fubini}) \oplus \bigoplus_I \{ \mathbf{C}\beta | \ \beta \in E(N_G)_I \times \delta_{I'} \}, \qquad (33)$$
$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_n := \bigoplus_I \{ \mathbf{C}\beta | \ \beta \in E(N_G)_I \times E(M_G)_{I'} \},$$

then we have

$$H^{1,1}\left(\widehat{\mathcal{X}}\right) \simeq H^{1,1}\left(\widehat{\mathcal{X}}\right)_c \bigoplus H^{1,1}\left(\widehat{\mathcal{X}}\right)_n \bigoplus H^{1,1}\left(\widehat{\mathcal{X}}\right)_d \ . \qquad (34)$$

Now, using Proposition 1 and the relation (33), from Hodge theory and the structure of exceptional divisors of $\widehat{\mathcal{X}}$, one can derive the following mirror property of K3 orbifolds by the same argument as Theorem 4 of [15]:

Proposition 2 Let G, G' be a pair of mirror subgroups of SD_q and $\mathcal{X} = \mathcal{Z}/G, \ \mathcal{X}' = \mathcal{Z}/G'$. Then

$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_c \simeq H^{1,1}\left(\widehat{\mathcal{X}'}\right)_d ,$$
$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_d \simeq H^{1,1}\left(\widehat{\mathcal{X}'}\right)_c ,$$
$$H^{1,1}\left(\widehat{\mathcal{X}}\right)_n \simeq H^{1,1}\left(\widehat{\mathcal{X}'}\right)_n \ .$$

Section 4. Strange Duality of Arnold

Among the 14 exceptional singularities of modality one, there is a duality of Arnold with the Dolgachev and Gabrielov numbers exchanged [1, 18]. Each of these 14 families is given by 3 indices k_1, k_2, k_3 of homogeneity and an integer $d(= |\text{Coxter number}|)$ with the relation $d = k_1 + k_2 + k_3 + 1$, hence corresponds to an one-parameter family of quasi-smooth hypersurfaces in $WP^3_{(k_1,k_2,k_3,1)}$. Through this relation, these 14 exceptional singularities have been explained by K3 surfaces in [6, 12]. In this section, we shall connect the Arnold's duality with mirror K3 surfaces introduced in the previous section. At this moment, only part of these 14 exceptional singularities are found to have link with the mirror K3 surfaces of Fermat-type in weighted 3–sapces. Here is the list (with notations in [1]) of which these relations have been obtained in this paper:

Notation,	Homogeneity,	$d =$|Coxter number|
U_{12}	$4, 4, 3$	12
W_{12}	$10, 5, 4$	20
E_{14}	$12, 3, 8$	24
Q_{10}	$6, 9, 8$	24
Q_{11}	$6, 4, 7$	18
Z_{13}	$3, 5, 9$	18

(35)

For each d in the above table, there exist integers n_1, n_2, n_3 such that

$$\begin{aligned} & n_i \geq 2, \quad d_i = \frac{d}{n_i} \in \mathbf{Z}\,, \ \text{ for } i = 1, 2, 3, \\ & d = n_1 + n_2 + n_3 + 1, \\ & n_1 \text{ is divisible by } n_2\,. \end{aligned} \tag{36}$$

The solutions of n_1, n_2, n_3 are as follows:

d	n_1, n_2, n_3
12	$4, 4, 3$
20	$10, 5, 4$
24	$12, 3, 8$
18	$6, 2, 9$

(37)

For the above integers d, n_1, n_2, n_3, we consider the following two families of hypersurfaces in $WP^3_{(n_1,n_2,n_3,1)}$:

$$\mathcal{Z} \ : \ f(Z) = Z_1^{d_1} + Z_2^{d_2} + Z_3^{d_3} + Z_4^d - tZ_1Z_2Z_3Z_4 \ = 0\,,$$

$$\mathcal{Z}^* : f^*(Z) = \sum_{\substack{1 \leq i \neq j \leq 3 \\ n_j | n_i}} Z_i Z_j^{\frac{d-n_i}{n_j}} + \sum_{\substack{1 \leq j \leq 3 \\ n_j \text{ not facctor of other } n_i}} Z_j^{d_j} + Z_4^d - tZ_1Z_2Z_3Z_4 = 0.$$

By the list of weighted K3 hypersurfaces [7, 13], the integers d, n_1, n_2, n_3 which satisfy the condition (36) with quasi-smooth families $\mathcal{Z}, \mathcal{Z}^*$ are given by (37) plus one more solution

$$(d;\ n_1, n_2, n_3) = (10;\ 2, 2, 5) \quad . \qquad (38)$$

Denote SD_q, SD^* the groups (2) for $f(Z)$, $f^*(Z)$ respectively. The subgroup Q of (10) is now given by $q_i = \frac{1}{d_i}$ $(i = 1, 2, 3)$, $q_4 = \frac{1}{d}$. For all cases in (37) and (38), we have

$$|SD_q| = d \cdot d_1 \ .$$

In fact, SD_q is generated by the subgroup Q and the element $\mathrm{dia}\left[e^{\frac{2\pi i}{d_1}}, e^{-\frac{2\pi i}{d_1}}, 1, 1\right]$. Then

$$d = d_2 d_3 \ , \ (n_1, n_2, n_3) = \left(\frac{d_2}{d_1} d_3, d_3, d_2\right), \ \gcd(d_2, d_3) = 1. \qquad (39)$$

For SD^*, we have

$$SD^* = \begin{cases} Q & \text{for } d = 12, 20, 24, \\ < Q, \mathrm{dia}[1, -1, -1, 1] > & \text{for } d = 18, \\ < Q, \mathrm{dia}\left[e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, 1, 1\right] > & \text{for } d = 10 \ . \end{cases}$$

Hence

$$\overline{SD^*} := SD^*/Q \simeq \begin{cases} 0 & \text{for } d = 12, 20, 24, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } d = 18, \\ \mathbb{Z}/3\mathbb{Z} & \text{for } d = 10 \ . \end{cases} \qquad (40)$$

$$\mathcal{Z}^*/SD^* = \mathcal{Z}^*/\overline{SD^*} \quad .$$

As the singularities of $\mathcal{Z}$, $\mathcal{Z}^*$ have the same structure, the same decomposition (34) of the cohomology of $\widehat{\mathcal{Z}}$ applies also to $\widehat{\mathcal{Z}^*}$. Hence the K3 surfaces $\widehat{\mathcal{X}}$, $\widehat{\mathcal{Z}^*}$ form a mirror pair in the sense that Proposition 2 holds. Through the non-linear change of coordinates of $\mathcal{X}$, we are going to describe a quasi-smooth hypersurface $\mathcal{Y}$ in some $WP^3_{(n'_1, n'_2, n'_3, 1)}$ with $d = 1 + \sum_{i=1}^{3} n'_i$ for each d, such that $\mathcal{Y}/\overline{SD^*}$ is birational to $\mathcal{X}$. Let $\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3, \mathfrak{f}^4$ be the elements in N_{SD_q} defined by

$$(\mathfrak{f}^1, \mathfrak{f}^2) = \begin{cases} (e^1, e^2)\begin{pmatrix} \frac{1}{d_1} & 1 \\ \frac{d_1 - 1}{d_1} & 0 \end{pmatrix} & \text{if } n_1 \neq n_2, \\ (e^1, e^2)\begin{pmatrix} \frac{d_1 - 1}{d_1} & \frac{1}{d_1} \\ \frac{1}{d_1} & \frac{d_1 - 1}{d_1} \end{pmatrix} & \text{if } n_1 = n_2, \end{cases} \qquad (41)$$

$$\mathfrak{f}^3 = e^3 \ , \ \mathfrak{f}^4 = e^4 \ .$$

Then $\{f^i\}_{i=1}^4$ generates a simplicial cone $\mathcal{F}$ in $(N_{SD_q})_{\mathbf{R}}$ satisfying (15). We have

$$N_{SD_q} = \sum_{i=1}^{4} \mathbf{Z} f^i + \mathbf{Z} q \quad \text{for } d = 12, 20, 24,$$
$$N_{SD_q} = \sum_{i=1}^{4} \mathbf{Z} f^i + \mathbf{Z} q + \mathbf{Z}\, e^2 \quad \text{for } d = 18, 10.$$

The non-linear change of variables associated to $\{f^i\}_{i=1}^4$ will determine a degree d quasi-smooth hypersurface $\mathcal{Y}$ in $\mathbf{WP}^3_{(n'_1,n'_2,n'_3,1)}$ and a diagonal group $G^\sharp$ such that $\mathcal{X}$ is birational to $\mathcal{Y}/G^\sharp$. In fact, the $\mathbf{WP}^3_{(n'_1,n'_2,n'_3,1)}$, $\mathcal{Y}, \mathcal{Y}/G^\sharp$ are as follows:

$$d = 12, 20, \quad (n'_1, n'_2, n'_3) = (n_1, n_2, n_3), \quad \mathcal{Y}/G^\sharp = \mathcal{Y} = \mathcal{Z}^* \;;$$

$$\begin{array}{ll} d = 24, & (n'_1, n'_2, n'_3) = (6, 9, 8), \qquad \mathcal{Y}/G^\sharp = \mathcal{Y}, \\ & \mathcal{Y} = \left\{ Y_1 Y_2^2 + Y_1^4 + Y_3^2 + Y_4^{24} - t \prod_{j=1}^{4} Y_j = 0 \right\}; \\ d = 18, & (n'_1, n'_2, n'_3) = (3, 5, 9), \quad \mathcal{Y}/G^\sharp = \mathcal{Y}/ < \mathrm{dia}[-1, -1, 1, 1] >, \\ & \mathcal{Y} = \left\{ Y_1 Y_2^3 + Y_1^6 + Y_3^2 + Y_4^{18} - t \prod_{j=1}^{4} Y_j = 0 \right\}; \\ d = 10, & (n'_1, n'_2, n'_3) = (2, 2, 5), \quad \mathcal{Y} = \mathcal{Z}^*, \quad \mathcal{Y}/G^\sharp = \mathcal{Z}^*/\overline{SD^*}\ . \end{array} \tag{42}$$

In the above discussion, we have seen that the mirror pair $(\mathcal{X}, \mathcal{Z}^*)$ leads to a pair $(\mathcal{Y}, \mathcal{Z}^*/\overline{SD^*})$ of K3 orbifolds with the hypersurface $\mathcal{Y}$. Using (40) and the technique of non-linear change of variables, one can show that $\mathcal{Z}^*/\overline{SD^*}$ is birational to the following hypersurfaces in weighted 3–space:

$$d = 12, 20, 24 \qquad \mathcal{Z}^*/\overline{SD^*} = \mathcal{Z}^* \;;$$

$$d = 18, \qquad \mathcal{Z}^*/\overline{SD^*} \leftarrow \cdots \rightarrow \text{hypersurface in } \mathbf{WP}^3_{(6,4,7,1)} \text{ defined by}$$
$$W_1^3 + W_1 W_2^3 + W_2 W_3^2 + W_4^{18} - t \prod_{j=1}^{4} W_j = 0;$$
$$d = 10, \qquad \mathcal{Z}^*/\overline{SD^*} \leftarrow \cdots \rightarrow \text{hypersurface in } \mathbf{WP}^3_{(2,2,5,1)} \text{ defined by}$$
$$W_1^3 W_2^2 + W_1^2 W_2^3 + W_3^2 + W_4^{10} - t \prod_{j=1}^{4} W_j = 0\,.$$

These hypersurfaces are all quasi-smooth except $d = 10$. For $d \neq 10$, i.e. d in (37), the pairing of $\mathcal{Y}$ with the above hypersurface is the ones of Arnold's duality in (35).

We now describe the birational equivalence between $\mathcal{X}$ and $\mathcal{Y}/G^\sharp$ from the identified K3 surface $\widehat{\mathcal{X}} = \widehat{\mathcal{Y}/G^\sharp}$. By (39), one can show that

$$\begin{aligned}
\mathcal{Z} \cap \{Z_3 = Z_4 = 0\} &= \{[\alpha, 1, 0, 0] | \alpha^{d_1} = -1\}, \\
\mathcal{Z} \cap \{Z_1 = Z_4 = 0\} &= \left\{ \left[0, 1, e^{\frac{\pi i}{d_3}}, 0\right] \right\}, \\
\mathcal{Z} \cap \{Z_2 = Z_4 = 0\} &= \left\{ \left[1, 0, e^{\frac{\pi i}{d_1}}, 0\right] \right\}, \\
\mathcal{Z} \cap \{Z_1 = Z_2 = 0\} &= \{[0, 0, \beta, 1] | \beta^{d_3} = -1\}.
\end{aligned}$$

Therefore the singular set of $\mathcal{X}\ (:= \mathcal{Z}/SD_q)$ consists of $3 + d_3$ elements, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \ldots, \mathbf{b}_{d_3}$, where

$$\mathbf{a}_1 = \overline{\left[0, 1, e^{\frac{\pi i}{d_3}}, 0\right]}, \ \mathbf{a}_2 = \overline{\left[1, 0, e^{\frac{\pi i}{d_1}}, 0\right]}, \ \mathbf{a}_3 = \overline{\left[e^{\frac{\pi i}{d_1}}, 1, 0, 0\right]},$$
$$\mathbf{b}_k = \overline{\left[0, 0, e^{\frac{\pi i}{d_3}(1+2k)}, 1\right]} \quad , \ k = 1, \ldots, d_3 ,$$

with the singularity-type

$$\mathbf{b}_k: \ A_{d_1-1} \quad (1 \leq k \leq d_3),$$
$$\mathbf{a}_1 : A_{d_1-1} , \quad \mathbf{a}_2 : A_{\gcd(n_1,n_3)\cdot d_1-1} , \quad \mathbf{a}_3 : A_{n_2-1} .$$

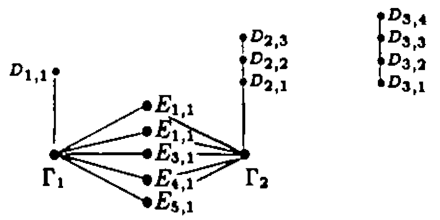
Let $\sigma : \widehat{\mathcal{X}} \to \mathcal{X}$ be the K3 resolution. Consider the following $\mathbf{P}^1$ curves in $\widehat{\mathcal{X}}$:

$$\begin{aligned}
&\sigma^{-1}(\mathbf{b}_k) = E_{k,1} \cup \cdots \cup E_{k,d_1-1} \ , 1 \leq k \leq d_3 , \\
&\sigma^{-1}(\mathbf{a}_1) = D_{1,1} \cup \cdots \cup D_{1,d_1-1} \ , \\
&\sigma^{-1}(\mathbf{a}_2) = D_{2,1} \cup \cdots \cup D_{2,\gcd(n_1,n_3)d_1-1} , \\
&\sigma^{-1}(\mathbf{a}_3) = D_{3,1} \cup \cdots \cup D_{3,n_2-1} \ , \\
&\Gamma_i = \text{the proper transform of } \overline{\{Z_i = 0\}} \text{ in } \mathcal{X}, \ i = 1, 2,
\end{aligned}$$

such that the only intersection among these curves are given by:

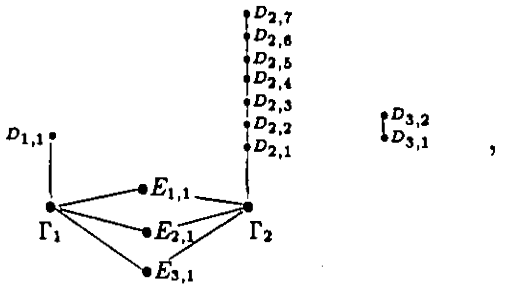
$$\begin{aligned}
&E_{k,j} \cdot E_{k,j+1} = 1 \ , \quad D_{i,j} \cdot D_{i,j+1} = 1, \\
&\Gamma_1 \cdot D_{1,1} = 1, \quad \Gamma_2 \cdot D_{2,\gcd(n_1,n_3)d_1-1} = 1, \\
&\Gamma_1 \cdot E_{1,1} = 1, \quad \Gamma_2 \cdot E_{k,d_1-1} = 1 .
\end{aligned}$$

The above relations can be realized in their dual graphs:
$d = 20$,



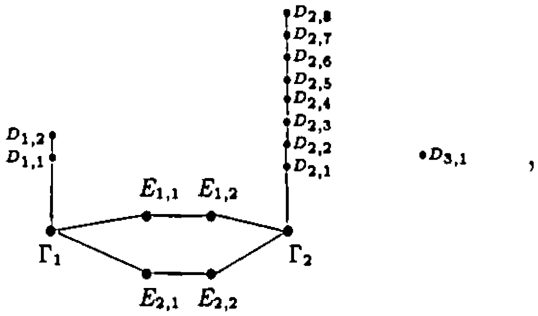
,

$d = 24,$



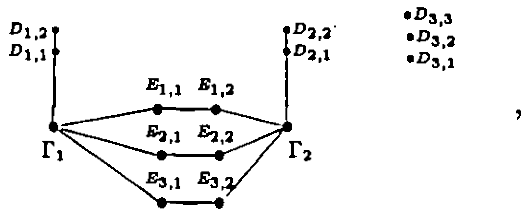
,

$d = 18,$



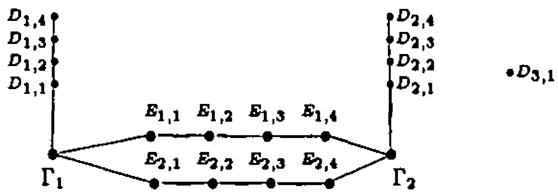
,

$d = 12,$



,

$d = 10,$



.

By (41), $\mathcal{X}$ is birational to $\mathcal{Y}/G^\natural$ by the change of variables

$$Z_3 = Y_3$$

$$Z_4 = Y_4$$

$$\begin{cases} Z_1 = & Y_1^{\frac{1}{d_1}} Y_2 \\ Z_2 = & Y_1^{\frac{d_1-1}{d_1}} \end{cases} \qquad \begin{cases} Y_1 = & Z_2^{\frac{d_1}{d_1-1}} \\ Y_2 = & Z_1 Z_2^{\frac{-1}{d_1-1}} \end{cases} \quad \text{for } d = 20, 24, 18,$$

$$\begin{cases} Z_1 = & Y_1^{\frac{d_1-1}{d_1}} Y_2^{\frac{1}{d_1}} \\ Z_2 = & Y_1^{\frac{1}{d_1}} Y_2^{\frac{d_1-1}{d_1}} \end{cases} \qquad \begin{cases} Y_1 = & Z_1^{\frac{d_1-1}{d_1-2}} Z_2^{\frac{-1}{d_1-2}} \\ Y_2 = & Z_1^{\frac{-1}{d_1-2}} Z_2^{\frac{d_1-1}{d_1-2}} \end{cases} \quad \text{for } d = 12, 10.$$

Identify $\widehat{\mathcal{X}}$ with $\widehat{\mathcal{Y}/G^\natural}$ through Lemma 1. Then $\mathcal{Y}/G^\natural$ is obtained by blowing down the following curves:

$$\begin{cases} \bigcup_{j=1}^{d_1-1} D_{1,j} & \mapsto \overline{\left[1,0,e^{\frac{\pi i}{d_3}},0\right]} \\ \Gamma_2 \cup \bigcup_{j=1}^{\gcd(n_1,n_3)d_1-1} D_{2,j} & \mapsto \overline{[0,1,0,0]} \\ \bigcup_{j=1}^{n_2-1} D_{3,j} & \mapsto \overline{[1,-1,0,0]} \end{cases} \quad \text{for } d = 20, 24, 18,$$

$$\begin{cases} \Gamma_1 \cup \bigcup_{j=1}^{d_1-1} D_{1,j} & \mapsto \overline{[0,1,0,0]} \\ \Gamma_2 \cup \bigcup_{j=1}^{d_1-1} D_{2,j} & \mapsto \overline{[1,0,0,0]} \\ \bigcup_{j=1}^{n_2-1} D_{3,j} & \mapsto \overline{[1,-1,0,0]} \end{cases} \quad \text{for } d = 12, 10.$$

The above data determine the Dolgachev numbers of $\widehat{\mathcal{Y}}$ for the families $d = 12, 20, 24$ in (35). We can also obtain the Dolgachev number of $\widehat{\mathcal{Z}^*}$ from the dual diagram of its exceptional curves, then compare it with the corresponding $H^{1,1}(\mathcal{Y})_0$. This indicates the coincidence between the Dolgachev and Gabrielov numbers in the Arnold's duality. For $d = 18$, the diagrams for resolution of $\mathcal{Y}$, $\mathcal{Z}^*/\overline{SD^*}$ can also be obtained, and one can find their relation with the ones for $\mathcal{Y}/G^\natural$, $\mathcal{Z}^*$. We shall not give the detail here.

Remark. All the hypersurfaces discussed in this section are defined by a quais-homogenous polynomial $f(Z_1, Z_2, Z_3, Z_4)$ in some $\mathbf{WP}^3_{(\ell_1,\ell_2,\ell_3,1)}$ with the degree $d = \ell_1 + \ell_2 + \ell_3 + 1$. Any such surface has the same cohomology as the Milnor fiber $V := \{(z_1, z_2, z_3) \in \mathbf{C}^3 | F(z_1, z_2, z_3) = 1\}$, where

$$F(Z_1, Z_2, Z_3) := f(Z_1, Z_2, Z_3, 0).$$

In fact, the hypersurface is the compactification $\overline{V}$ of V. Since the curve $\left\{[Z_1, Z_2, Z_3] \in \mathbf{WP}^2_{(n_1,n_2,n_3)} | F(Z_1, Z_2, Z_3) = 0\right\}$ is of genus 0, the Hodge theory of Milnor fiber V implies

$$H^2\left(\overline{V}\right)_0 \simeq \mathbf{C}[z_1, z_2, z_3] / \left(\frac{\partial F}{\partial z_i}\right)$$

[20]. Therefore the conclusion obtained in this section can also be formulated in terms of Jacobian ring of the polynomial $F(Z_1, Z_2, Z_3)$.

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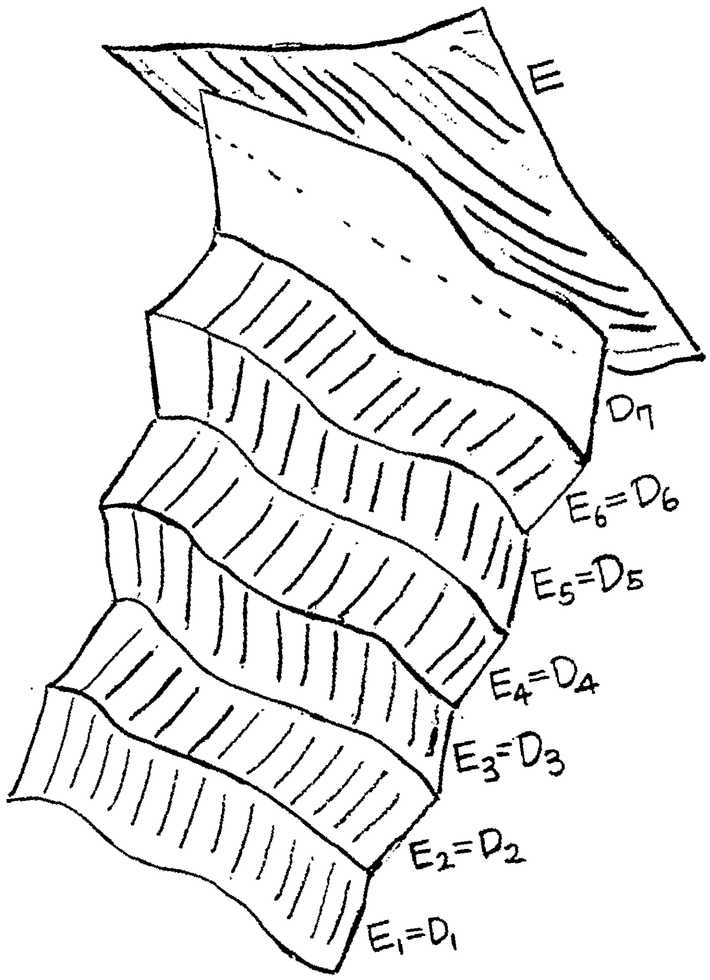

Figure 1