# RESOLVING MIXED HODGE MODULES ON CONFIGURATION SPACES

**E. Getzler**

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

Department of Mathematics
Northwestern University
Evanston
IL 60208-2730

USA

MPI 96-144

# RESOLVING MIXED HODGE MODULES ON CONFIGURATION SPACES

E. GETZLER

If $X/S$ is a separated scheme over a base $S$ and $n$ is a natural number, let $X^n/S$ be the $n$th fibred power of $X$ with itself, and let $\mathsf{F}(X/S,n)$ be the configuration space, whose fibre $\mathsf{F}(X/S,n)_s$ over a point $s \in S$ is a configuration of $n$ distinct points in the fibre $X_s$, or equivalently, the complement of the $\binom{n}{2}$ diagonals in $X^n/S$. Let $j(n) : \mathsf{F}(X/S,n) \hookrightarrow X^n/S$ be the natural open immersion.

Given a sheaf $\mathcal{F}$ of abelian groups on $X^n/S$, we introduce in this paper a natural resolution $\mathcal{L}^\bullet(X/S,\mathcal{F},n)$ of the sheaf $j(n)_!j(n)^*\mathcal{F}$, whose underlying graded sheaf is a sum of terms of the form $i(J)_!i(J)^*\mathcal{F}$, where $i(J)$ is the closed immersion of a diagonal in $X^n/S$. This resolution has the property that if $\mathcal{F}$ is an $\mathbb{S}_n$-equivariant sheaf (where the symmetric group $\mathbb{S}_n$ acts on $X^n/S$ by permuting the factors in the fibred product), the resolution is $\mathbb{S}_n$-equivariant as well.

For example, if $n = 2$, we have the exact sequence of sheaves

$$(0.1) \qquad 0 \longrightarrow j(2)_!j(2)^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \longrightarrow 0,$$

where $i : X \hookrightarrow X^2/S$ is the immersion of the diagonal.

Let $\pi(n) : \mathsf{F}(X/S,n) \to S$ and $\pi(n) : X^n/S \to S$ be the projections to $S$. (We denote them by the same symbol, since confusion is hardly likely to arise.) The objects $\pi(n)_!j(n)^*\mathcal{F}$ and $\pi(n)_!\mathcal{L}^\bullet(X/S,\mathcal{F},n)$ are isomorphic in the derived category of sheaves on $S$. We use this isomorphism to calculate the $\mathbb{S}_n$-equivariant Euler characteristic of $\pi(n)_!j(n)^*\mathcal{F}$.

When $X/S$ is a quasi-projective morphism of varieties over $\mathbb{C}$, it is not hard to extend our resolution to mixed Hodge modules: in this way, we obtain a new proof of the formula of [10] for the Serre polynomial of the configuration space $\mathsf{F}(X,n)$. (The Serre polynomial is the Euler characteristic of $H^\bullet_c(\mathsf{F}(X,n),\mathbb{Q})$ in the Grothendieck group of mixed Hodge structures.) The virtue of this new proof is that it applies with no modification to the relative case.

A similar spectral sequence has been obtained by Totaro [27], in the case where $S = \operatorname{Spec}(\mathbb{C})$. Since he works with cohomology, and not cohomology with compact support, his results depend on the dimension of $X$ and require $X$ to be smooth; however, when $X$ is smooth, our spectral sequence is equivalent to his.

To extend our resolution from sheaves to mixed Hodge modules, we have to modify it, since $i^*$ is not a $t$-exact functor of mixed Hodge modules (or of perverse sheaves, which underly mixed Hodge modules), even when $i$ is a closed immersion; already for $n = 2$, it is well-known that (0.1) must be replaced by an exact triangle. This difficulty is overcome by introducing Čech resolutions for the sheaves $i(J)_!i(J)^*\mathcal{F}$, which are constructed using the property of mixed Hodge modules that $f_!$ is $t$-exact for open affine immersions.

If we apply our resolution to the universal elliptic curve, we obtain a formula for the relative $\mathbb{S}_n$-equivariant Serre polynomial of $\mathcal{M}_{1,n}/\mathcal{M}_{1,1}$. Eichler-Shimura theory, which calculates the cohomology of polynomials of the Hodge local system on $\mathcal{M}_{1,1}$, then leads to a formula for the $\mathbb{S}_n$-equivariant Serre polynomial of $\mathcal{M}_{1,n}$.

The first value of $n$ for which $\mathcal{M}_{1,n}$ has a contribution from the cusp forms, and hence has non-Tate cohomology, is $n = 11$, for which the (non-equivariant) Serre polynomial may be calculated by (5.6) to be

$$\begin{aligned}\mathsf{e}(\mathcal{M}_{1,11}) = \mathsf{L}^{11} - 330\,\mathsf{L}^9 + 4575\,\mathsf{L}^8 - 30657\,\mathsf{L}^7 + 124992\,\mathsf{L}^6 - \mathsf{S}_{12} \\ - 336820\,\mathsf{L}^5 + 584550\,\mathsf{L}^8 - 406769\,\mathsf{L}^3 - 865316\,\mathsf{L}^2 + 2437776\,\mathsf{L} - 1814400;\end{aligned}$$

here, $\mathsf{L}$ denotes the mixed Hodge structure $\mathbb{Q}(-1)$, and $\mathsf{S}_{12}$ is a two-dimensional Hodge structure of weight 11, associated to the discriminant cusp form $\Delta$. (We do not reproduce the equivariant Serre-Hodge polynomial $\mathsf{e}^{\mathbb{S}_{11}}(\mathcal{M}_{1,11})$ for lack of space: there are 56 irreducible representations of $\mathbb{S}_{11}$, although not all of these occur.) In particular, $\mathcal{M}_{1,11}$ has Euler characteristic $-302400$.

The virtual Euler characteristic of the orbifold $\mathcal{M}_{1,1}$ equals $-1/12$. (This is a special case of the formula of Harer and Zagier [16], but is easy to prove directly, using the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane.) It follows by induction on $n$, using the fibrations $\mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$, that the virtual Euler characteristic of $\mathcal{M}_{1,n}$ equals $(-1)^n(n-1)!/12$. For $n \geq 5$, $\mathcal{M}_{1,n}$ is a fine moduli space (that is, no automorphism of an elliptic curve fixes 5 points), and thus its virtual Euler characteristic equals its Euler characteristic. The agreement between the resulting formula for $\chi(\mathcal{M}_{1,11})$ and the value which we have calculated provides a (modest) consistency check between our work and that of Harer and Zagier. We show in Proposition (5.7) that our formula for the Serre polynomial of $\mathcal{M}_{1,n}$ does give the correct value of $\chi(\mathcal{M}_{1,n})$, for all $n \geq 5$.

In a sequel to this paper [11], we show how to sum the Serre polynomials of the strata of $\overline{\mathcal{M}}_{1,n}$ to obtain a formula for its Hodge polynomial. For example,

$$\begin{aligned}\mathsf{e}(\overline{\mathcal{M}}_{1,11}) = \mathsf{L}^{11} + 2037\,\mathsf{L}^{10} + 213677\,\mathsf{L}^9 + 4577630\,\mathsf{L}^8 + 30215924\,\mathsf{L}^7 + 74269967\,\mathsf{L}^6 \\ - \mathsf{S}_{12} + 74269967\,\mathsf{L}^5 + 30215924\,\mathsf{L}^4 + 4577630\,\mathsf{L}^3 + 213677\,\mathsf{L}^2 + 2037\,\mathsf{L} + 1.\end{aligned}$$

**Outline of the paper.** In Section 1, we explain the relationship between Arnold's calculation of the cohomology of the configuration spaces $\mathsf{F}(\mathbb{C}, n)$ and the theory of Stirling numbers of the first and second kinds.

Section 2 is devoted to the construction of the resolution in the simpler case of sheaves of abelian group. This is generalized in Section 3 to the cases of perverse sheaves and of mixed Hodge modules.

In Section 4, we apply the associated spectral sequence to generalize the formula of [10] for the $\mathbb{S}_n$-equivariant Serre polynomial of $\mathsf{F}(X, n)$ to the relative case.

In Section 5, we apply the formulas of Section 4 to calculate the $\mathbb{S}_n$-equivariant Serre polynomial of the moduli space $\mathcal{M}_{1,n}$.

**Acknowledgments.** I wish to thank the Department of Mathematics at the Université de Paris-VII and the Max-Planck-Institut für Mathematik in Bonn for their hospitality during the inception and completion of this paper, respectively. I am grateful to E. Looijenga for introducing me to the Eichler-Shimura theory, and to D. Zagier for his help with the proof of Proposition (5.7).

The author is partially supported by a research grant of the NSF and a fellowship of the A.P. Sloan Foundation.

## 1. The combinatorics of partitions and Stirling numbers

In this section, we recall the cohomology of the configuration space $\mathsf{F}(\mathbb{C}, n)$ and its relationship with the Stirling numbers. We give more detail on the theory of Stirling numbers than is necessary, since it illuminates the combinatorics which we will apply to construct our resolution.

(1.1). **Partitions.** A partition $J$ of $n$ is a decomposition of the set $\{1,\dots,n\}$ into disjoint non-empty subsets: for example, the partitions of $\{1,2,3,4\}$ are

$$\{1,2,3,4\} \quad \{12,3,4\} \quad \{13,2,4\} \quad \{14,2,3\} \quad \{23,1,4\} \quad \{24,1,3\} \quad \{34,1,2\}$$
$$\{123,4\} \quad \{124,3\} \quad \{134,2\} \quad \{234,1\} \quad \{12,34\} \quad \{13,24\} \quad \{14,23\} \quad \{1234\},$$

where we abbreviate the subset $\{i_1,\dots,i_\ell\}$ to $i_1\dots i_\ell$. We denote the subsets of $J$ by $\{J_1,\dots,J_k\}$, in no particular order. Denote by $\mathsf{S}(n,k)$ the set of partitions of $n$ into $k$ non-empty subsets.

Associated to a partition $J$ of $n$ is an equivalence relation on $\{1,\dots,n\}$, such that $i \sim_J j$ iff $i$ and $j$ lie in the same part of $J$. The set of all partitions of $n$ is a poset: if $J$ and $K$ are partitions, $J \prec K$ iff $i \sim_J j$ implies that $i \sim_K j$, that is, iff $K$ is coarser than $J$.

If $\mathbf{a} = (a_n \mid n \geq 1)$ is a sequence of natural numbers, let $|\mathbf{a}| = \sum_{n=1}^{\infty} n a_n$.

**Lemma (1.2).** *The exponential generating function of the number $p(\mathbf{a})$ of partitions of $|\mathbf{a}|$ into $a_j$ subsets of size $j$, $j \geq 1$, is*

$$B(\mathbf{t}) = \sum_{\mathbf{a}} p(\mathbf{a}) \frac{\mathbf{t}^{\mathbf{a}}}{|\mathbf{a}|!} = \exp\Big(\sum_{j=1}^{\infty} \frac{t_j}{j!}\Big).$$

*Proof.* Indeed, $p(\mathbf{a})$ is the number of automorphisms of the set with $|\mathbf{a}|$ elements divided by the number of automorphisms of such a partition, namely

$$p(\mathbf{a}) = |\mathbf{a}|! \Big/ \prod_{j=1}^{\infty} j!^{a_j} a_j!,$$

from which the lemma follows. □

Let $f$ be a power series

$$f(t) = \sum_{k=1}^{\infty} \frac{f_k t^k}{k!}.$$

Define the partial Bell polynomials $B_{n,k}$ by the generating function

$$\exp(xf(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} B_{n,k}(f_1,\dots,f_n) x^k.$$

Setting $t_j = xt^j f_j$ in the generating function $B(\mathbf{t})$ of Lemma (1.2), we obtain the explicit formula

$$B_{n,k}(f_1,\dots,f_n) = \sum_{J \in \mathsf{S}(n,k)} \prod_{i=1}^{k} f_{|J_i|}. \tag{1.3}$$

(See Ex. 2.11 of Macdonald [19].) In particular, the partial Bell polynomials have positive integral coefficients.

**Proposition (1.4).** *If $g$ is the inverse power series to $f$ (that is, $g(f(t)) = t$), then the matrices $F_{nk} = B_{n,k}(f_1,\dots,f_n)$ and $G_{nk} = B_{n,k}(g_1,\dots,g_n)$ are inverse to each other.*

*Proof.* The matrix $F$ is the transition matrix between the bases $(t^k/k! \mid k \geq 0)$ and $(f(t)^k/k! \mid k \geq 0)$ of $\mathbb{Q}[t]$. Its inverse $F^{-1}$ is thus the transition matrix between the bases $(f(t)^k/k! \mid k \geq 0)$ and $(t^k/k! \mid k \geq 0)$. Changing variables from $t$ to $g(t)$, the result follows. □

(1.5). **Stirling numbers of the first kind.** The Stirling number of the first kind $s(n,k)$ may be defined as $(-1)^{n-k}$ times the number of permutations on $n$ letters with $k$ cycles. A permutation of the set $\{1,\dots,n\}$ is the same thing as a partition of $n$, together with a cyclic order on each part of the partition. Since a set of cardinality $i$ has $(i-1)!$ cyclic orders, we see that

$$s(n,k) = \sum_{J\in \mathsf{S}(n,k)} \prod_{i=1}^{n} (-1)^{|J_i|-1}(|J_i|-1)!. \tag{1.6}$$

Applying (1.3) with $f_k = (-1)^{k-1}(k-1)!$ (or $f(t) = \log(1+t)$), we see that

$$1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} s(n,k)\frac{t^n x^k}{n!} = (1+t)^x. \tag{1.7}$$

In particular, for $n \geq 1$,

$$\sum_{k=1}^{n} s(n,k)x^k = x(x-1)\dots(x-n+1). \tag{1.8}$$

(1.9). **Stirling numbers of the second kind.** The number $S(n,k)$ of partitions of $n$ with $k$ parts (i.e. the cardinality of $\mathsf{S}(n,k)$) is called a Stirling number of the second kind. The special case of (1.3) with $f(t) = e^t - 1$ (and hence $f_k = 1$ for all $k$) shows that the Stirling numbers of the second kind have generating function

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} S(n,k)\frac{t^n x^k}{n!} = e^{x(e^t-1)} - 1. \tag{1.10}$$

For the reader's edification, we display the first few rows of the matrices of first and second Stirling numbers:

$$s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \ddots \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \ddots \end{bmatrix}.$$

Applying Proposition (1.4) to the functions $f(t) = e^t - 1$ and $g(t) = \log(1+t)$, we see that the matrices $s$ and $S$ formed from the numbers $s(n,k)$ and $S(n,k)$ are inverse to each other.

**Proposition (1.11).**

$$\sum_{n=1}^{\infty} s(j,n)S(n,k) = \delta(j,k)$$

We may rewrite (1.8) in the form

$$\sum_{j=1}^{n} s(j,n)x^{-n} = x^{-j}(1-x)\dots(1-(j-1)x).$$

From Proposition (1.11), it now follows easily that

$$\sum_{n=k}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)},$$

or equivalently,

$$S(n,k) = \sum_{1\le i_1\le\dots\le i_k\le n} i_1\dots i_k.$$

This is also not difficult to prove directly, by induction on $n$.

(1.12). **The cohomology of the configuration spaces** $\mathsf{F}(\mathbb{C},n)$. Let $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ be the cohomology of the configuration space of the complex line. Given distinct $j$ and $k$ in $\{1,\dots,n\}$, let $\omega_{jk}\in H^1(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ be the integral cohomology class represented by the closed differential form

$$\Omega_{jk} = \frac{1}{2\pi i}\frac{d(z_j-z_k)}{z_j-z_k}.$$

By induction on $n$, Arnold shows in [1] that the cohomology ring $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ is generated by the classes $\omega_{jk}$, subject to the relations $\omega_{jk}=\omega_{kj}$ and

$$\omega_{ij}\omega_{jk}+\omega_{jk}\omega_{ki}+\omega_{ki}\omega_{ij}=0. \tag{1.13}$$

The action of the group $\mathbb{S}_n$ on the configuration space $\mathsf{F}(\mathbb{C},n)$ induces an action on the cohomology ring $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$, which permutes the generators, by the formula

$$\sigma\cdot\omega_{ij}=\omega_{\sigma(i)\sigma(j)}.$$

Using the above presentation of $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$, Arnold shows that $H^{n-k}(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ is a free abelian group of rank $(-1)^{n-k}s(n,k)$. This motivates the definition of a graded $\mathbb{S}_n$-module $\mathsf{s}(n,k)$, with

$$\mathsf{s}(n,k)^i = \begin{cases} H^i(\mathsf{F}(\mathbb{C},n),\mathbb{Z}), & i=n-k,\\ 0. & \text{otherwise,}\end{cases}$$

We may think of $\mathsf{s}(n,k)$ as a lift of the Stirling number $s(n,k)$ to the category of graded $\mathbb{S}_n$-modules.

Denote by $\mathsf{L}(n)$ the graded $\mathbb{S}_n$-module $\mathsf{s}(n,1)$. More generally, if $\mathbf{n}$ is a finite set of cardinality $n$, let $\mathsf{L}(\mathbf{n})$ be the graded $\mathrm{Aut}(\mathbf{n})$-module defined in the same way as $\mathsf{L}(n)$ but with the set $\{1,\dots,n\}$ replaced by $\mathbf{n}$. It is isomorphic to $\mathsf{L}(n)$, but to obtain an isomorphism, we must choose a total order on $\mathbf{n}$.

The following theorem is proved by Orlik and Solomon for general hyperplane arrangements [21] (see Theorem 4.21 of Orlik [20]). See also Lehrer-Solomon [18] for the special case which we consider.

**Theorem (1.14).** *There is a natural decomposition*

$$\mathsf{s}(n,k) = \bigoplus_{J\in\mathsf{S}(n,k)} \mathsf{s}(n,J), \tag{1.15}$$

*together with natural isomorphisms*

$$\mathsf{s}(n,J) \cong \bigotimes_{i=1}^{k} \mathsf{L}(J_i).$$

*Proof.* The graded $\mathbb{S}_n$-module $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ is spanned by monomials in the generators $\omega_{ij}$. To such a monomial, we associate a forest (a graph each of whose components is a tree) with vertices the set $\{1,\dots,n\}$, and with an edge between vertices $i<j$ if and only if the generator $\omega_{ij}$ occurs in the monomial. Such a forest determines a partition of the set $\{1,\dots,n\}$. Let $\mathsf{s}(n,J)$ be the span of the monomials associated to the partition $J$.

Since all of the forests related by application of one of Arnold's relations (1.13) give rise to the same partition, we see that $\mathsf{s}(n,J)$ is well-defined. If $J$ has $k$ parts, its associated forest has $n-k$ edges, and hence $\mathsf{s}(n,J)$ is a subgroup of $H^{n-k}(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$. In particular, if $J$ is the unique partition in $\mathsf{S}(n,1)$, we see that $\mathsf{s}(n,J)$ is generated by all trees with $n$ labelled vertices, modulo the Arnold relations. From this, it is easy to see that

$$\mathsf{s}(n,J) \cong \bigotimes_{i=1}^{k} \mathsf{s}(J_i,1). \quad \square$$

The characters of the $\mathbb{S}_n$-modules $\mathsf{L}(n)$ have been calculated by Hanlon [13] and Stanley [24]. From their formula, one may calculate the characters of $\mathsf{s}(n,k)$ for all $k$.

**Lemma (1.16).** *The equivariant Euler characteristic of the graded $\mathbb{S}_n$-module $\mathsf{L}(n)$, evaluated at $\sigma \in \mathbb{S}_n$, is given by the formula*

$$\chi_\sigma(\mathsf{L}(n)) = \begin{cases} -\dfrac{\mu(d)}{n}(-d)^{n/d}(n/d)!, & \text{if } \sigma \text{ has } n/d \text{ cycles of length } d, \\ 0, & \text{otherwise.} \end{cases}$$

(1.17). **A differential on $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$.** We now study the differential

$$\partial : H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z}) \longrightarrow H^{\bullet-1}(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$$

of the algebra $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$ associated to the diagonal action of the multiplicative group $\mathbb{C}^\times$ on $\mathsf{F}(\mathbb{C},n)$; it is given by capping with the fundamental class of the circle $\mathrm{U}(1) \subset \mathbb{C}^\times$. It follows from the definition of $\omega_{ij}$ that $\partial\omega_{ij}=1$. One can easily check that $\partial$ is well-defined, by showing that the differential of the relation (1.13) vanishes:

$$\partial\big(\omega_{ij}\omega_{jk}+\omega_{jk}\omega_{ki}+\omega_{ki}\omega_{ij}\big) = \big(\omega_{jk}-\omega_{ij}\big)+\big(\omega_{ki}-\omega_{jk}\big)+\big(\omega_{ij}-\omega_{ki}\big)=0.$$

The following lemma reflects the fact that the action of $\mathbb{C}^\times$ on $\mathsf{F}(\mathbb{C},n)$ is free if $n>1$, and that the resulting principal fibration is trivial.

**Lemma (1.18).** *If $n>1$, the complex $(H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z}),\partial)$ is acyclic.*

*Proof.* Let $H$ denote the operator of multiplication by $\omega_{12}$. Since $\partial\omega_{12}=1$, we see that $\partial\cdot H + H\cdot\partial$ equals the identity operator, proving acyclicity. $\square$

If $J \in \mathsf{S}(n,j)$ and $K \in \mathsf{S}(n,k)$ are partitions of $\{1,\dots,n\}$, denote by $\partial_{JK}$ the component of $\partial$ mapping from $\mathsf{s}(n,J)$ to $\mathsf{s}(n,K)$; thus, $\partial_{JK}$ vanishes unless $k=j+1$.

**Lemma (1.19).** *The differential $\partial_{JK}$ vanishes unless $K \prec J$.*

*Proof.* Let $\alpha$ be a monomial in the generators $\omega_{ij}$ of $H^\bullet(\mathsf{F}(\mathbb{C},n),\mathbb{Z})$. By the definition of $\partial$, $\partial\alpha$ is a sum of terms, in each of which one of the factors $\omega_{ij}$ occurring in $\alpha$ is omitted. Such a term corresponds to partition of $\{1,\dots,n\}$ in which $i$ and $j$ are no longer equivalent: in other words, the partition associated to the new monomial is a refinement of the partition associated to $\alpha$. □

## 2. Resolving sheaves on configuration spaces

Before turning to the construction of resolutions of sheaves over configuration spaces, we explain by an informal argument why one expects Stirling numbers to arise in the construction.

Let $X/S$ be a separated scheme over a base $S$, and let $X^n/S$ be the $n$th fibred power of $X$ with itself, defined inductively by $X^0/S = S$ and

$$X^{n+1}/S = (X^n/S)\times_S X.$$

Denote by $\pi(n) : X^n/S \to S$ the projection to $S$.

The scheme $X^n/S$ has a stratification, with strata indexed by the poset of partitions $J$ of $\{1,\dots,n\}$: the stratum associated to a partition $J$ is given by

$$\mathsf{F}(X/S,J) = \{(x_1,\dots,x_n)\in X^n/S \mid x_i = x_j \text{ iff } i\sim_J j\}.$$

A stratum $\mathsf{F}(X/S,K)$ lies in the closure of $\mathsf{F}(X/S,J)$ if and only if $J\prec K$; the closure of $\mathsf{F}(X/S,J)$ is the diagonal

$$X^J/S = \{(x_1,\dots,x_n)\in X^n/S \mid x_i = x_j \text{ if } i\sim_J j\}.$$

If $J\in\mathsf{S}(n,k)$, denote by $i(J) : X^J/S \hookrightarrow X^n/S$ the diagonal immersion. If $\mathcal{F}$ is a sheaf on $X^n/S$, denote by $\mathcal{F}(J)$ the sheaf $i(J)_! i(J)^*\mathcal{F}$ on $X^n/S$.

If $J\in\mathsf{S}(n,k)$, $\mathsf{F}(X/S,J)$ is isomorphic to $\mathsf{F}(X/S,k)$; thus, we may represent the above stratification of $X^n/S$ (in)formally as

$$X^n/S = \coprod_{k=1}^{n} S(n,k)\cdot\mathsf{F}(X/S,k).$$

Proposition (1.11) leads us to expect that there is a "virtual stratification" of $\mathsf{F}(X/S,n)$, of the form

$$\mathsf{F}(X/S,n) = \coprod_{k=1}^{n} s(n,k)\cdot X^k/S. \tag{2.1}$$

Rewritten in terms of generating functions, this becomes

$$\sum_{n=0}^{\infty}\frac{t^n[\mathsf{F}(X/S,n)]}{n!} = \sum_{k=0}^{\infty}\frac{\log(1+t)^n[X^n/S]}{n!} = (1+t)^{[X/S]},$$

where we think of the symbol $[X^n/S]$ as the $n$th power of $[X/S]$, as indeed it is in the Grothendieck group of motivic sheaves on $S$.

We may make sense of (2.1) in the following way. Let $j(n) : \mathsf{F}(X/S,n)\hookrightarrow X^n/S$ be the open immersion of the configuration space in $X^n/S$. There is a natural resolution $\mathcal{L}^\bullet(X/S,n,\mathcal{F})$ of $j(n)_! j(n)^*\mathcal{F}$, whose underlying graded sheaf has the form

$$\mathcal{L}^{n-k}(X/S,n,\mathcal{F}) = \bigoplus_{J\in\mathsf{S}(n,k)}\operatorname{Hom}(\mathsf{s}(n,J),\mathcal{F}(J)). \tag{2.2}$$

For example, if $n=2$, we recover (0.1) while if $n=3$, we obtain the resolution

$$0 \to j(3)_! j(3)^* \mathcal{F} \to \mathcal{F}(1,2,3) \to \mathcal{F}(12,3) \oplus \mathcal{F}(13,2) \oplus \mathcal{F}(23,1) \to \mathcal{F}(123) \oplus \mathcal{F}(123) \to 0.$$

In the special case that $\mathcal{F}$ is a constant sheaf, we may interpret $\mathcal{F}(J)$ as a copy of the diagonal $X^J$, which is isomorphic to $X^k$ when $J \in \mathsf{S}(n,k)$; replacing the $\mathbb{S}_n$-module $\mathsf{s}(n,J)$ by its Euler characteristic and bearing in mind (1.15), or its numerical version (1.6), we are led to (2.1).

(2.3). **Construction of the resolution.** If $J \prec K$ are partitions of $\{1,\dots,n\}$, denote by $i(J,K)$ the inclusion $X^K/S \hookrightarrow X^J/S$, and by $i(J,K)^* : \mathcal{F}(J) \to \mathcal{F}(K)$ the induced map of sheaves. Let $\mathcal{L}^\bullet(X/S,n,\mathcal{F})$ be the complex of sheaves (2.2), with differential

$$d = \sum_{J \prec K} \partial^*_{KJ} \otimes i(J,K)^*. \tag{2.4}$$

For example, $\mathcal{L}^0(X/S,n,\mathcal{F}) \cong \mathcal{F}$, while $\mathcal{L}^1(X/S,n,\mathcal{F})$ is the direct sum

$$\mathcal{L}^1(X/S,n,\mathcal{F}) = \bigoplus_{1\le k<l\le n} \mathcal{F}(kl,1,\dots,\widehat{k},\dots,\widehat{l},\dots,n),$$

since $\dim \mathsf{s}(n,J) = 1$ for all $J \in \mathsf{S}(n,n-1)$. In particular, $j^*\mathcal{L}^1(X/S,n,\mathcal{F}) = 0$.

Denote by $\eta : j(n)_! j(n)^* \Rightarrow \mathrm{Id}$ the unit of the adjunction between $j(n)_!$ and $j(n)^*$; it induces a map, also denoted by $\eta$, from $j(n)_! j(n)^* \mathcal{F}$ to $\mathcal{F} = \mathcal{L}^0(X/S,n,\mathcal{F})$. The composition of arrows

$$j(n)_! j(n)^* \mathcal{F} \xrightarrow{\eta} \mathcal{L}^0(X/S,n,\mathcal{F}) \xrightarrow{d} \mathcal{L}^1(X/S,n,\mathcal{F})$$

is zero, showing that $\eta : j(n)_! j(n)^* \mathcal{F} \to \mathcal{L}^\bullet(X/S,n,\mathcal{F})$ is a morphism of complexes.

**Theorem (2.5).** *The morphism $\eta : j(n)_! j(n)^* \mathcal{F} \to \mathcal{L}(X/S,n,\mathcal{F})$ is a quasi-isomorphism.*

*Proof.* We apply the following lemma.

**Lemma (2.6).** *Let $X$ be a stratified space with strata $\{X_J\}$, and let $j(J)$ be the locally closed immersion of the stratum $X_J$ in $X$. Then a map of complexes of sheaves $\eta : \mathcal{F}_1 \to \mathcal{F}_2$ on $X$ is a quasi-isomorphism if and only if $\eta : j(J)_! j(J)^* \mathcal{F}_1 \to j(J)_! j(J)^* \mathcal{F}_2$ is a quasi-isomorphism for all strata.*

*Proof.* If there is only one stratum, the lemma is a tautology. We now argue by induction on the number of strata. Let $X_J$ be an open stratum of $X$, and let $Z$ be its complement in $X$, with closed immersion $i : Z \hookrightarrow X$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j(J)_! j(J)^* \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & i_! i^* \mathcal{F}_1 & \longrightarrow & 0 \\
 & & \Big\downarrow \eta & & \Big\downarrow \eta & & \Big\downarrow \eta & & \\
0 & \longrightarrow & j(J)_! j(J)^* \mathcal{F}_2 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & i_! i^* \mathcal{F}_1 & \longrightarrow & 0
\end{array}.$$

Since the rows are exact, we conclude by the five-lemma that $\eta : \mathcal{F}_1 \to \mathcal{F}_2$ is a quasi-isomorphism if and only if $\eta : j(J)_! j(J)^* \mathcal{F}_1 \to j(J)_! j(J)^* \mathcal{F}_2$ and $\eta : i_! i^* \mathcal{F}_1 \to i_! i^* \mathcal{F}_2$ are. By the induction hypothesis, $\eta : i_! i^* \mathcal{F}_1 \to i_! i^* \mathcal{F}_2$ is a quasi-isomorphism if and only if $\eta : j(K)_! j(K)^* \mathcal{F}_1 \to j(K)_! j(K)^* \mathcal{F}_2$ are for all $K \neq J$; this proves the induction step. □

If $J$ is a partition of $\{1,\dots,n\}$, let $j(J) : \mathsf{F}(X/S,J) \hookrightarrow X^n/S$ be the inclusion of the locally closed subscheme $\mathsf{F}(X/S,J)$. By the base change theorem,

$$j(J)_! j(J)^* j(n)_! j(n)^* \mathcal{F} \cong \begin{cases} j(n)_! j(n)^* \mathcal{F}, & \text{if } J \text{ is the unique partition in } \mathsf{S}(n,n), \\ 0, & \text{otherwise.} \end{cases}$$

Let the parts of $J$ be $\{J_1,\dots,J_k\}$, in no particular order, and let $n_i$ be the cardinality of $J_i$. Applying Theorem (1.14), we see that

$$\begin{aligned} j(J)_! j(J)^* \mathcal{L}^\bullet(X/S,n,\mathcal{F}) &\cong \operatorname{Hom}\Big(\bigoplus_{K \prec J} \mathsf{s}(n,K), j(J)_! j(J)^* \mathcal{F}\Big) \\ &\cong \operatorname{Hom}\Big(\bigotimes_{i=1}^{k} H^{n_i - \bullet}(\mathsf{F}(\mathbb{C},n_i),\mathbb{Z}), j(J)_! j(J)^* \mathcal{F}\Big). \end{aligned}$$

The differential on this complex of sheaves is induced by the differentials on the factors $H^{n_i-\bullet}(\mathsf{F}(\mathbb{C},n_i),\mathbb{Z})$, and hence by Lemma (1.18) is acyclic if $n_i > 1$ for any $i$.

We see that the hypotheses of Lemma (2.6) are fulfilled: if $J$ is a partition of $\{1,\dots,n\}$, $\eta : j(J)_! j(J)^* \mathcal{F} \to j(J)_! j(J)^* \mathcal{L}(X/S,n,\mathcal{F})$ is a quasi-isomorphism, since the two complexes are equal if $J \in \mathsf{S}(n,n)$, while they are both acyclic otherwise. □

## 3. Mackey 2-functors

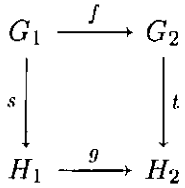
In this section, we axiomatize those properties of the 2-functor associating to a scheme its derived category of mixed Hodge modules which will be used in constructing the analogue of the resolution $\mathcal{L}^\bullet(X/S,n,\mathcal{F})$. It turns out that these axioms define the natural analogue for 2-functors of Dress's Mackey functors [8].

Impatient readers may skip to Section (3.5): all they need to know about the Mackey 2-functor underlying the theory of mixed Hodge modules is that the usual properties of the functors $f_!$ and $f^*$ for locally closed immersions hold, such as the base change theorem (in particular, we make no use of Verdier duality). In Section (3.5), we impose sufficient additional hypotheses on these functors to allow us to construct Čech-type resolutions of $j_! j^* \mathcal{F}$ when $j$ is an open immersion and of $i_! i^* \mathcal{F}$ when $i$ is a closed immersion. The analogue of $\mathcal{L}(X/S,n,\mathcal{F})$ for mixed Hodge modules is defined by replacing the sheaf $i(J)_! i(J)^* \mathcal{F}$ by this Čech complex.

(3.1). **Mackey functors.** Mackey functors were introduced by Dress [8] as an axiomatization of induction in the theory of group representations. The motivating example is the functor $G \mapsto R(G)$ on the category of finite groups, which assigns to a group $G$ its virtual representation ring. Given a morphism $f : G \to H$ of finite groups, there is a contravariant map $f^\bullet : R(H) \to R(G)$, pull-back along $f$, and a covariant map $f_\bullet : R(G) \to R(H)$ generalizing induction:

$$f_\bullet V = (\mathbb{C}[G] \otimes V)^H.$$

These functors satisfy the Mackey double coset formula, which says that given a Cartesian square of finite groups



we have the equality $g^\bullet t_\bullet = s_\bullet f^\bullet$.

(3.2). **Mackey 2-functors.** The group of virtual representations $R(G)$ is the Grothendieck group of the category $\mathsf{Proj}(G)$ of finite-dimensional projective representations. The 2-functor $G \mapsto \mathsf{Proj}(G)$ satisfies axioms which are the natural analogue for 2-functors of the definition of a Mackey functor; we call a 2-functor satisfying these axioms a Mackey 2-functor.

This concept is not new: it was introduced by Deligne in Exposé XVII of SGA 4 [2] (though not under this name).

**Definition (3.3).** A Mackey 2-functor from a category $\mathcal{C}$ to a 2-category $\mathbb{T}$ consists of a pair of 2-functors $\mathsf{D}^\bullet : \mathcal{C}^\circ \to \mathbb{T}$ (where $\mathcal{C}^\circ$ is the opposite of $\mathcal{C}$) and $\mathsf{D}_\bullet : \mathcal{C} \to \mathbb{T}$, such that

(i) if $X$ is an object of $\mathcal{C}$, the objects $\mathsf{D}^\bullet(X)$ and $\mathsf{D}_\bullet(X)$ are identical — we denote this object by $\mathsf{D}(X)$, and if $f : X \to Y$ is a morphism of $\mathcal{C}$, we denote the 1-morphism $\mathsf{D}^\bullet(f) : \mathsf{D}(Y) \to \mathsf{D}(X)$ by $f^\bullet$ and the 1-morphism $\mathsf{D}_\bullet(f) : \mathsf{D}(X) \to \mathsf{D}(Y)$ by $f_\bullet$;

(ii) (additivity) there are 2-morphisms of bifunctors $\alpha : \mathsf{D}^\bullet(X \coprod Y) \Rightarrow \mathsf{D}^\bullet(X) \oplus \mathsf{D}^\bullet(Y)$ and $\beta : \mathsf{D}_\bullet(X \coprod Y) \Rightarrow \mathsf{D}_\bullet(X) \oplus \mathsf{D}_\bullet(Y)$, such that

$$\alpha_{X,Y} = \beta_{X,Y} : \mathsf{D}(X \textstyle\coprod Y) \Longrightarrow \mathsf{D}(X) \oplus \mathsf{D}(Y);$$

(iii) (base change) to each Cartesian square

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ Y_1 & \xrightarrow{g} & Y_2 \end{array}$$

in $\mathcal{C}$ is associated a natural 2-morphism $\phi : g^\bullet t_\bullet \Rightarrow s_\bullet f^\bullet$, such that given a diagram each square of which is Cartesian

$$\begin{array}{ccccc} X_1 & \xrightarrow{f} & X_2 & \xrightarrow{f'} & X_3 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} \\ Y_1 & \xrightarrow{g} & Y_2 & \xrightarrow{g'} & Y_3 \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle t'} & & \\ Z_1 & \xrightarrow{h} & Z_2 & & \end{array}$$

the 2-morphism $\phi$ associated to the top (resp. left) pair of squares is the 2-composition of the 2-morphisms associated to the squares from which it is formed.

The Grothendieck group $K_0(\mathsf{D})$ of a triangulated category $\mathsf{D}$ is the abelian group generated by the isomorphism classes of objects of $\mathsf{D}$ (where we assume that these form a set); we impose the relation $[V] \sim [U]+[W]$ for all exact triangles $(U, V, W)$ in $\mathsf{D}$. For example, if $\mathsf{D}$ is the derived category $\mathsf{D}^b(\mathsf{Ab})$ of bounded complexes in an abelian category $\mathsf{Ab}$, then $K_0(\mathsf{D})$ may be identified with the Grothendieck group $K_0(\mathsf{Ab})$ of the abelian category $\mathsf{Ab}$, which is the abelian group generated by the isomorphism classes of objects of $\mathsf{Ab}$, with the relation $[V] \sim [U]+[W]$ whenever $V$ is an extension of $W$ by $U$. Although we will not need it, the following result goes some way towards justifying our introduction of Mackey 2-functors.

**Proposition (3.4).** *The composition of a Mackey 2-functor* $\mathsf{D}$ *with the functor* $K_0$ *is a Mackey functor* $\mathsf{K}$ *on* $C$.

(3.5). **Exact Mackey 2-functors.** Let $\mathsf{Var}$ be the category of quasi-projective varieties over $\mathbb{C}$, with morphisms the locally closed immersions. Let $\mathbb{T}$ be the 2-category defined as follows:

**objects:** $t$-categories (Beilinson-Bernstein-Deligne [4]);
**1-morphisms:** right $t$-exact functors possessing a right adjoint;
**2-morphisms:** natural isomorphisms.

**Definition (3.6).** An exact Mackey 2-functor on $\mathsf{Var}$ is a Mackey 2-functor with values in $\mathbb{T}$ such that

(i) for closed immersions $i : Z \hookrightarrow X$, $i^\bullet$ has right adjoint $i_\bullet$, and $i_\bullet$ is fully faithful;
(ii) for open immersions $j : U \hookrightarrow X$, $j_\bullet$ is fully faithful, and has right adjoint is $j^\bullet$;
(iii) if $i : Z \hookrightarrow X$ is a closed immersion, and $j : U \hookrightarrow X$ is the open immersion of the complement $U = X \setminus Z$, there is an exact triangle $(j_\bullet j^\bullet V, V, i_\bullet i^\bullet V)$ for each object $V$ of $\mathsf{D}(X)$.
(iv) if $j$ is an affine open immersion, the functor $j_\bullet$ is $t$-exact.

We have in mind three examples of exact Mackey 2-functors.

**Example (3.7).** The 2-functors $X \longmapsto (D^b(X), f_\bullet = f_!, f^\bullet = f^*)$ assigning to $X$ the derived category of sheaves of abelian groups with the usual $t$-structure.

**Example (3.8).** The 2-functors $X \longmapsto ({}^pD^b_c(X), f_\bullet = f_!, f^\bullet = f^*)$ assigning to $X$ the derived category of bounded complexes of sheaves with constructible cohomology, together with the $t$-structure associated to the *middle* perversity $p$ (axiom (iv) is Corollaire 4.1.3 of [4]).

**Example (3.9).** The 2-functors $X \longmapsto (D^b(\mathsf{MHM}(X)), f_\bullet = f_!, f^\bullet = f^*)$ assigning to $X$ the derived category of mixed Hodge modules with the natural $t$-structure.

(3.10). **Čech complexes.** Let $j : U \hookrightarrow X$ be an open immersion and let $\mathcal{U} = \{U_i\}_{1\le i\le d}$ be a finite cover of $U$ by affine open immersions $j_i : U_i \hookrightarrow X$. For example, we might take the $U_i$ to be complements of Cartier divisors in $X$. Using these data, we now define a Čech-type resolution of $j_\bullet j^\bullet \mathcal{F}$, where $\mathsf{D}$ is an exact Mackey 2-functor. (See also Proposition 2.19 of Saito [22] and Section 3.4 of Beilinson [3].)

**Definition (3.11).** The Čech-complex $C_\bullet(X,\mathcal{U},\mathcal{F})$ is the graded object of $\mathsf{D}(X)$

$$C_k(X,\mathcal{U},\mathcal{F}) = \bigoplus_{i_0<\cdots<i_k} (j_{i_0\dots i_k})_\bullet (j_{i_0\dots i_k})^\bullet \mathcal{F},$$

where $j_{i_0\dots i_k}$ is the open immersion of $U_{i_0\dots i_k} = \bigcap_{\ell=0}^k U_i$ in $X$. Its differential is the sum of maps

$$\partial = \sum_{\ell=0}^k (-1)^\ell \partial_\ell : C_k(X,\mathcal{U},\mathcal{F}) \longrightarrow C_{k-1}(X,\mathcal{U},\mathcal{F}).$$

Here, $\partial_\ell : (j_{i_0\dots i_k})_\bullet (j_{i_0\dots i_k})^\bullet \mathcal{F} \to (j_{i_0\dots\widehat{i_\ell}\dots i_k})_\bullet (j_{i_0\dots\widehat{i_\ell}\dots i_k})^\bullet \mathcal{F}$ is induced by the adjunction $q_\bullet q^\bullet \Rightarrow \mathrm{Id}$ associated to the open immersion $q : U_{i_0\dots i_k} \hookrightarrow U_{i_0\dots\widehat{i_\ell}\dots i_k}$.

If $\mathsf{D}$ is a $t$-category, let $H^0(\mathsf{D})$ be its heart. Recall from Section 3.1.9 of [4] the realization functor

$$\mathrm{real} : D^b(H^0(\mathsf{D})) \longrightarrow \mathsf{D}^b;$$

this is an exact functor mapping bounded complexes $\mathcal{C}_\bullet(H^0(\mathsf{D}))$ to objects $\mathrm{real}(\mathcal{C}_\bullet)$ in $\mathsf{D}^b$.

**Proposition (3.12).** *Let $\mathsf{D}$ be an exact Mackey 2-functor, let $j : U \hookrightarrow X$ be an open immersion, let $i : Z \hookrightarrow X$ be the closed immersion of the complement $Z = X \setminus U$, and let $\mathcal{U} = \{U_i\}_{0\le i\le d}$ be a cover of $U$ by affine open immersions $j_i : U_i \hookrightarrow X$. Then*

(i) $\mathrm{real}\big(\mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F})\big) \in \mathrm{Ob}\,\mathsf{D}(X)^{[-d,0]}$ *is isomorphic to* $j_\bullet j^\bullet\mathcal{F}$;

(ii) $\mathrm{real}\big(\mathrm{cone}(\mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F}) \to \mathcal{F})\big) \in \mathrm{Ob}\,\mathsf{D}(X)^{[-d-1,0]}$ *is isomorphic to* $i_\bullet i^\bullet\mathcal{F}$.

*Proof.* Note that (ii) is implied by (i) and the five-lemma: in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & i_\bullet i^\bullet\mathcal{F} & \longrightarrow & j_\bullet j^\bullet\mathcal{F}[1] & \longrightarrow & 0 \\ & & \simeq\downarrow & & \downarrow & & \simeq\downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathrm{cone}\big(\mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{F}\big) & \longrightarrow & \mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F})[1] & \longrightarrow & 0 \end{array}$$

the top row is exact by axiom (iii) for exact Mackey 2-functors, and the bottom row is obviously exact.

We now prove (i) by induction on $d$: for $d = 0$, $\mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F}) \cong j_\bullet j^\bullet\mathcal{F}$, and the proposition is a tautology.

The open subset $U^0 = \bigcup_{i=1}^d U_i$ of $X$ has cover $\mathcal{U}^0 = \{U_i\}_{1\le i\le d}$. Let $j^0$ and $j_0$ be the open immersions of $U^0$ and $U_0$ in $X$, and define $\mathcal{F}^0 = (j^0)_\bullet(j^0)^\bullet\mathcal{F}$ and $\mathcal{F}_0 = (j_0)_\bullet(j_0)^\bullet\mathcal{F}$. Let $p$ be the locally closed immersion of $U \setminus U^0 = U_0 \setminus U^0$ in $X$. We now form the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^0 & \longrightarrow & j_\bullet j^\bullet\mathcal{F} & \longrightarrow & p_\bullet p^\bullet\mathcal{F} & \longrightarrow & 0 \\ & & \simeq\downarrow & & \downarrow & & \simeq\downarrow & & \\ 0 & \longrightarrow & \mathcal{C}_\bullet(X,\mathcal{U}^0,\mathcal{F}^0) & \to & \mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F}) & \to & \mathrm{cone}(\mathcal{C}_\bullet(X,\mathcal{U}^0\cap U_0,\mathcal{F}_0) \to \mathcal{F}_0) & \longrightarrow & 0 \end{array}$$

The lower row is defined by dividing the summands $(j_{i_0\dots i_k})_\bullet(j_{i_0\dots i_k})^\bullet\mathcal{F}$ of $\mathcal{C}(X,\mathcal{U},\mathcal{F})$ into two classes:

(i) if $i_0 > 0$, the term $(j_{i_0\dots i_k})_\bullet(j_{i_0\dots i_k})^\bullet\mathcal{F}$ is a summand of $\mathcal{C}_\bullet(\mathcal{U}^0, (j^0)_\bullet(j^0)^\bullet\mathcal{F})$;

(ii) if $i_0 = 0$, the term $(j_{0i_1\dots i_k})_\bullet(j_{0i_1\dots i_k})^\bullet\mathcal{F}$ is a summand of $\mathrm{cone}_k(\mathcal{C}_\bullet(\mathcal{U}^0\cap U_0,\mathcal{F}_0) \to \mathcal{F}_0)$.

In particular, the bottom row is exact.

The top row is exact by axiom (iii) for exact Mackey 2-functors, applied to the closed immersion of $U \setminus U^0$ in $U$, while the outer vertical arrows are quasi-isomorphisms by the induction hypothesis. The proposition now follows by the five-lemma. □

(3.13). **The resolution $\mathcal{L}^\bullet_{\mathcal{U}}(X/S,n,\mathcal{F})$ for mixed Hodge modules.** Using the Čech-complexes $\mathcal{C}_\bullet(X,\mathcal{U},\mathcal{F})$, we now construct a resolution of $j_!j^*\mathcal{F}$, where $\mathcal{F}$ is a mixed Hodge module. The functor $i^*$ is not $t$-exact on mixed Hodge modules for general closed immersions $i$; for this reason, our construction depends on the choice of an auxiliary cover $\mathcal{U}$ of $\mathsf{F}(X/S,2)$ by affine open immersions $j_i : U_i \hookrightarrow X$. (If $X/S$ is a smooth family of curves, we may take the cover to have one element $\mathcal{U} = \{\mathsf{F}(X/S,2)\}$, since in that case, the diagonal in $X^2/S$ is a Cartier divisor.) We will actually construct the resolution in the more general setting of exact Mackey 2-functors.

For $k, \mathsf{L} \in \{1,\dots,n\}$ with $k \ne l$, let $\pi_{kl} : X^n/S \to X^2/S$ be the morphism which projects onto the $k$th and $l$th factors. Define a cover $\mathcal{U}(J)$ of the complement of the diagonal $i(J) : X^J/S \hookrightarrow X^n/S$ by

$$\mathcal{U}(J) = \{\pi_{kl}^{-1}(U_i) \mid k \sim_J l \text{ and } U_i \in \mathcal{U}\}.$$

The open immersions $\pi_{kl}^{-1}(U_i) \hookrightarrow X^n/S$ are affine, since affine open immersions are preserved under base change (EGA II, 1.6.2 [12]).

By Proposition (3.12), the realization of the complex $\operatorname{cone}(\mathcal{C}_\bullet(X^n/S, \mathcal{U}(J), \mathcal{F}) \to \mathcal{F})$ is quasi-isomorphic to $i(J)_\bullet i(J)^\bullet \mathcal{F}$. If $J \prec K$, the morphism

$$i(J,K)^* : i(J)_\bullet i(J)^\bullet \mathcal{F} \longrightarrow i(K)_\bullet i(K)^\bullet \mathcal{F}$$

is induced by an inclusion of complexes

$$i(J,K)^* : \mathcal{C}_\bullet(X^n/S, \mathcal{U}(J), \mathcal{F}) \longrightarrow \mathcal{C}_\bullet(X^n/S, \mathcal{U}(K), \mathcal{F}),$$

which exists because the cover $\mathcal{U}(K)$ contains the open cover $\mathcal{U}(J)$.

As in the case of sheaves, our resolution of $j_\bullet j^\bullet \mathcal{F}$ is a sum over partitions (2.2); unlike that case, the result is a double complex, and we must take the realization of its total complex to obtain an object of $\mathsf{D}(X^n/S)$. Let

$$\mathcal{L}_{\mathcal{U}}^{n-k,-j}(X/S,n,\mathcal{F}) = \bigoplus_{J \in \mathsf{S}(n,k)} \operatorname{Hom}(\mathsf{s}(n,J), \operatorname{cone}_j(\mathcal{C}_\bullet(X^n/S, \mathcal{U}(J), \mathcal{F}) \longrightarrow \mathcal{F})).$$

There are two differentials: the analogue of differential (2.4),

$$d = \sum_{J \prec K} \partial_{KJ}^* \otimes i(J,K)^*,$$

and the Čech-differential $\mathcal{L}_{\mathcal{U}}^{n-k,-j}(X/S,n,\mathcal{F}) \to \mathcal{L}_{\mathcal{U}}^{n-k,1-j}(X/S,n,\mathcal{F})$.

We may identify $\mathcal{L}_{\mathcal{U}}^{0,\bullet}(X/S,n,\mathcal{F})$ with $\mathcal{F}$; thus, there is a natural coaugmentation

$$\eta : j_\bullet j^\bullet \mathcal{F} \longrightarrow \mathcal{L}_{\mathcal{U}}^{0,\bullet}(X/S,n,\mathcal{F}).$$

If $\sigma$ is a permutation, the cover $\mathcal{U}(J)$ is carried into $\mathcal{U}(\sigma \cdot J)$ by the action of $\sigma$, so that $\sigma$ maps $\mathcal{C}_\bullet(X^n/S, \mathcal{U}(J), \mathcal{F})$ isomorphically to $\mathcal{C}_\bullet(X^n/S, \mathcal{U}(\sigma \cdot J), \mathcal{F})$. The differential of $\mathcal{L}_{\mathcal{U}}^{\bullet,\bullet}(X/S,n,\mathcal{F})$ is invariant under the action of $\sigma$, showing that $\mathcal{L}_{\mathcal{U}}^{\bullet,\bullet}(X/S,n,\mathcal{F})$ carries an action of $\mathbb{S}_n$. It is clear that the coaugmentation $\eta$ is $\mathbb{S}_n$-equivariant.

We now come to the main result of this paper; we omit the proof, since it is essentially identical to that of Theorem (2.5).

**Theorem (3.14).** *Let* $\mathsf{D}$ *be an exact Mackey* 2*-functor. If* $\pi : X \to S$ *is a morphism of quasi-projective varieties over* $\mathbb{C}$, $\mathcal{F}$ *is an object of* $\mathsf{D}(X)$ *and* $\mathcal{U}$ *is a cover of* $\mathsf{F}(X/S,2)$ *by affine open immersions, the coaugmentation* $\eta : j_\bullet j^\bullet \mathcal{F} \to \mathcal{L}_{\mathcal{U}}^{\bullet,\bullet}(X/S,n,\mathcal{F})$ *induces an* $\mathbb{S}_n$*-equivariant quasi-isomorphism in* $\mathsf{D}(X^n/S)$ *between* $j_\bullet j^\bullet \mathcal{F}$ *and* $\operatorname{real}(\operatorname{Tot} \mathcal{L}_{\mathcal{U}}^{\bullet,\bullet}(X/S,n,\mathcal{F}))$.

## 4. The $\mathbb{S}_n$-equivariant relative Serre polynomial of $j(n)_\bullet j(n)^\bullet \mathcal{E}^{\boxtimes n}$

(4.1). **The relative Serre polynomial.** Let $\mathsf{D}$ be an exact Mackey 2-functor, and let $\pi : X \to S$ be a morphism of quasi-projective varieties over $\mathbb{C}$. If $\mathcal{F}$ is an object of $\mathsf{D}(X)$, the *relative Serre polynomial* $\mathsf{e}_S(X,\mathcal{F})$ of $\mathcal{F}$ is the class of $\pi_\bullet \mathcal{F}$ in $\mathsf{K}(S)$.

This terminology is motivated by the special case in which $\mathsf{D} = D^b(\mathsf{MHM})$ and $S = \operatorname{Spec}(\mathbb{C})$. If we apply to $\mathsf{e}(X,\mathcal{F})$ the augmentation $\varepsilon : \mathsf{K}(\mathsf{MHM}(\operatorname{Spec}(\mathbb{C}))) \to \mathbb{Z}[t]$ defined by

$$\varepsilon(V) = \sum_{i,k} (-1)^i \dim \operatorname{gr}_k^W V^i t^k,$$

we obtain the Serre polynomial of $\mathcal{F}$,

$$\mathsf{e}(X,\mathcal{F}) = \sum_{i,k} (-1)^i H_c^i(X,\mathcal{F}) t^k.$$

Now suppose a finite group $\Gamma$ acts on $X$ and $Y$, and the morphism $\pi$ is $\Gamma$-equivariant. If $\mathcal{F}$ is an object of $\mathsf{D}^\Gamma(X)$, the equivariant relative Serre polynomial $\mathsf{e}_S^\Gamma(X,\mathcal{F})$ is the class of $\pi_\bullet\mathcal{F}$ in $\mathsf{K}^\Gamma(S)$, where $\mathsf{K}^\Gamma(X)$ is the Grothendieck group of $\Gamma$-equivariant objects in $\mathsf{D}(X)$.

When $\mathsf{D}$ is defined over a field $\mathsf{k}$ of characteristic 0 and the categories $\mathsf{D}(X)$ have tensor products, the associated Grothendieck groups $\mathsf{K}^\Gamma(X)$ are $\lambda$-rings, by the arguments of [10]. If in addition the action of $\Gamma$ on $X$ is trivial, the Peter-Weyl theorem (see Theorem 3.2 of [10]) implies the isomorphism of $\lambda$-rings

$$\mathsf{K}^\Gamma(X) \cong R(\Gamma) \otimes \mathsf{K}(X),$$

where $R(\Gamma)$ is the virtual representation ring of $\Gamma$ (the Grothendieck group of finite-dimensional $\mathsf{k}[\Gamma]$-modules).

In this section, we calculate $\mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet\mathcal{E}^{\boxtimes n})$. We obtain a generalization of Theorem 5.6 of [10], which is the special case new result where $\mathsf{D} = \mathsf{MHM}$, $S = \operatorname{Spec}(\mathbb{C})$ and $\mathcal{E} = \mathbb{1}$ is the unit of $\mathsf{D}(X)$.

(4.2). **Green 2-functors.** The reader interested only in the case of mixed Hodge modules may omit this section, whose rôle is to axiomatize the projection axiom.

If $G$ is a finite group, $R(G)$ is not only an abelian group, but also a commutative ring: furthermore, the functor $f^\bullet$ preserves this product, and we have the projection axiom, which says that if $f : G \to H$ is a morphism of finite groups, there is a commutative diagram

$$\begin{array}{ccccc} & & R(G)\otimes R(H) & & \\ & \swarrow^{1\otimes f^\bullet} & & \searrow^{f_\bullet\otimes 1} & \\ R(G)\otimes R(G) & & & & R(H)\otimes R(H) \\ \downarrow & & & & \downarrow \\ R(G) & & \xrightarrow[f_\bullet]{\qquad\qquad} & & R(H) \end{array}$$

where the vertical arrows are multiplication in $R(G)$ and $R(H)$. Dress calls a Mackey functor with these additional structures a Green functor.

As in [10], a rring is a symmetric monoidal category $\mathcal{R}$ with coproducts, denoted $A \oplus B$, such that there are natural isomorphisms

$$(A\oplus B)\otimes C \cong (A\otimes C)\oplus(B\otimes C) \quad \text{and} \quad A\otimes 0 \cong 0$$

satisfying the coherence axioms of Laplaza [17]. Let RING denote the 2-category whose objects are all rrings, whose 1-morphisms are all symmetric monoidal functors, and whose 2-morphisms are all natural isomorphisms of these.

**Definition (4.3).** A Green 2-functor from a 2-category $\mathcal{C}$ to the 2-category of rrings RING is a Mackey 2-functor on $\mathcal{C}$ such that:

(i) the 2-functor $\mathsf{D}^\bullet$ has a lift to a 2-functor $\mathsf{D}^\bullet : \mathcal{C}^\circ \to$ RING, and the 2-isomorphism of bifunctors $\alpha : \mathsf{D}^\bullet(X \coprod Y) \Rightarrow \mathsf{D}^\bullet(X) \oplus \mathsf{D}^\bullet(Y)$ is a 2-isomorphism of rrings.

(ii) (the projection axiom) given a morphism $f : X \to Y$ in $\mathcal{C}$, there is a natural 2-morphism

$$\begin{array}{ccc} & \mathsf{D}(X)\otimes\mathsf{D}(Y) & \\ {\scriptstyle 1\otimes f^\bullet}\swarrow & & \searrow{\scriptstyle f_\bullet\otimes 1} \\ \mathsf{D}(X)\otimes\mathsf{D}(X) & \overset{\psi_f}{\Longrightarrow} & \mathsf{D}(Y)\otimes\mathsf{D}(Y) \\ \downarrow & & \downarrow \\ \mathsf{D}(X) & \xrightarrow[f_\bullet]{\hspace{6em}} & \mathsf{D}(Y) \end{array}$$

This must satisfy the condition that for any pair of composable arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, the 2-morphism $\psi_{gf}$ equals the 2-pasting of the following diagram:

$$\begin{array}{ccccc} & & \mathsf{D}(X)\otimes\mathsf{D}(Z) & & \\ & {\scriptstyle 1\otimes g^\bullet}\swarrow & & \searrow{\scriptstyle 1\otimes f_\bullet} & \\ & \mathsf{D}(X)\otimes\mathsf{D}(Y) & = & \mathsf{D}(Y)\otimes\mathsf{D}(Z) & \\ {\scriptstyle 1\otimes f^\bullet}\swarrow & & \searrow{\scriptstyle f_\bullet\otimes 1}\quad{\scriptstyle 1\otimes g^\bullet}\swarrow & & \searrow{\scriptstyle g_\bullet\otimes 1} \\ \mathsf{D}(X)\otimes\mathsf{D}(X) & \overset{\psi_f}{\Longrightarrow} & \mathsf{D}(Y)\otimes\mathsf{D}(Y) & \overset{\psi_g}{\Longrightarrow} & \mathsf{D}(Z)\otimes\mathsf{D}(Z) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{D}(X) & \xrightarrow[f_\bullet]{\hspace{6em}} & \mathsf{D}(Y) & \xrightarrow[g_\bullet]{\hspace{6em}} & \mathsf{D}(Z) \end{array}$$

**Definition (4.4).** An exact Green 2-functor is a Green 2-functor whose underlying Mackey 2-functor is an exact Mackey 2-functor.

All three examples of exact Mackey 2-functors which we gave above are exact Green 2-functors, with respect to the usual tensor product.

(4.5). **A formula for $\mathsf{e}_S^{\mathbf{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet\mathcal{E}^{\boxtimes n})$.** Let $\mathsf{D}$ be an exact Green 2-functor. If $\pi : X \to S$ is a morphism of quasi-projective varieties and $\mathcal{E}$ is an object of $\mathsf{D}(X)$, we define

$$\mathcal{E}^{\boxtimes n} = \pi_1^\bullet\mathcal{E} \otimes \ldots \otimes \pi_n^\bullet\mathcal{E} \in \operatorname{Ob}\mathsf{D}^{\mathbf{S}_n}(X^n/S),$$

where $\pi_i : X^n/S \to X$ is the $i$th projection.

In calculating $\mathsf{e}_S^{\mathbf{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet\mathcal{E}^{\boxtimes n})$, we make free use of the results of [10], in particular the relationship between the ring $\Lambda$ of symmetric functions and representations of symmetric groups. Recall from loc. cit. that if $R$ is a complete $\lambda$-ring, with decreasing filtration $F_iR$, $i \geq 0$, there is an operation

$$\operatorname{Exp}(x) = \sum_{n=0}^{\infty} \sigma_n(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n}\psi_n(x)\Big) : F_1R \longrightarrow 1 + F_1R$$

an analogue of the exponential, with inverse

$$\operatorname{Log}(x) = \sum_{n=0}^{\infty} \frac{\mu(n)}{n} \log(\psi_n(x)) : 1 + F_1R \longrightarrow F_1R.$$

Here, $\sigma_n$ is the $n$th $\sigma$-operation on $R$, and $\psi_n$ is the $n$th Adams operation.

For example, if $R = \mathbb{Z}[\![q]\!]$ has its standard $\lambda$-ring structure, then

$$\operatorname{Exp}(a_1 q + a_2 q^2 + a_3 q^3 + \dots) = (1-q)^{-a_1}(1-q^2)^{-a_2}(1-q^3)^{-a_3}\dots.$$

The ring of symmetric functions $\Lambda$ may be identified with the free $\lambda$-ring on one generator $h_1$, such that $\sigma_n(h_1) = h_n$ is the $n$th complete symmetric function, and $\psi_n(h_1) = p_n$ is the $n$th power sum. It is a complete $\lambda$-ring. If $V$ is an $\mathbb{S}_n$-module, denote by $\operatorname{ch}(V) \in \Lambda$ its Frobenius characteristic; this is the degree $n$ symmetric function given by the explicit expression

$$\operatorname{ch}(V) = \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} \operatorname{Tr}_V(\sigma) p_\sigma,$$

where $p_\sigma$ is the monomial in the power sums obtained by taking one factor $p_k$ for each cycle of $\sigma$ of length $k$.

If $R$ is a $\lambda$-ring, denote by $\Lambda\hat{\otimes}R$ the complete tensor product of $\Lambda$ with $R$.

**Theorem (4.6).** *Let $\mathsf{D}$ be an exact Green 2-functor, and let $\pi : X \to S$ be a morphism of quasi-projective varieties over $\mathbb{C}$. If $\mathcal{E}$ is an object of $\mathsf{D}(X)$, the following equality holds in $\Lambda\hat{\otimes}\,\mathsf{K}(S)$:*

$$\sum_{n=0}^\infty \mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet \mathcal{E}^{\boxtimes n}) = \operatorname{Exp}\Big(\sum_{n=1}^\infty \frac{\mu(n)}{n}\,\mathsf{e}\big(X/S, \log(1 + p_n\otimes\mathcal{E}^{\otimes n})\big)\Big).$$

*Proof.* By Theorem (3.14), we know that

$$\begin{aligned}\mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet\mathcal{E}^{\boxtimes n}) &= \mathsf{e}_S^{\mathbb{S}_n}(X^n/S, j(n)_\bullet j(n)^\bullet \mathcal{E}^{\boxtimes n}) \\ &= \sum_{k=1}^n \mathsf{e}_S^{\mathbb{S}_n}\Big(X^n/S, \bigoplus_{J\in\mathsf{S}(n,k)} \operatorname{Hom}(\mathsf{s}(n,J), \mathcal{E}^{\boxtimes n}(J))\Big) \\ &= \sum_{k=1}^n \bigoplus_{J\in\mathsf{S}(n,k)} \operatorname{ch}(\mathsf{s}(n,J)^\vee)\otimes \mathsf{e}_S(X^n/S, \mathcal{E}^{\boxtimes n}(J)) \in \Lambda_n\otimes\mathsf{K}(S).\end{aligned}$$

We may replace $\mathsf{s}(n,k)^\vee$ by $\mathsf{s}(n,k)$, since any $\mathbb{S}_n$-module is isomorphic to its dual. Applying Theorem (1.14), we obtain

$$\sum_{k=1}^n \bigoplus_{J\in\mathsf{S}(n,k)} \prod_{i=1}^k \operatorname{ch}(\mathsf{L}(J_i))\otimes \mathsf{e}_S(X, \mathcal{E}^{\otimes|J_i|}),$$

where the product is taken in the ring $\Lambda\hat{\otimes}\,\mathsf{K}(S)$. Summing over $n$ gives

$$\sum_{n=0}^\infty \mathsf{e}_S^{\mathbb{S}_n}(X^n/S, j(n)_\bullet j(n)^\bullet\mathcal{E}^{\boxtimes n}) = \operatorname{Exp}\Big(\sum_{n=1}^\infty \operatorname{ch}(\mathsf{L}(n))\otimes \mathsf{e}_S(X,\mathcal{E}^{\otimes n})\Big).$$

The theorem now follows from the character formula of Lemma (1.16), which may be rewritten as

$$\operatorname{ch}(\mathsf{L}(n)) = \frac{1}{n}\sum_{d|n}(-1)^{n/d-1}\mu(d)p_d^{n/d}.$$

We see that

$$\begin{aligned}\sum_{n=1}^{\infty} \mathsf{L}(n) \otimes \mathsf{e}_S(X, \mathcal{E}^{\otimes n})) &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} (-1)^{n/d-1} \mu(d) p_d^{n/d} \, \mathsf{e}_S(X, \mathcal{E}^{\otimes n}) \\ &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{e=1}^{\infty} \frac{(-1)^{e-1}}{e} p_d^e \, \mathsf{e}_S(X, \mathcal{E}^{\otimes de}) \\ &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \, \mathsf{e}_S\big(X, \log(1 + p_d \otimes \mathcal{E}^{\otimes d})\big). \quad \square\end{aligned}$$

**Remark (4.7).** Rewriting Exp in terms of Adams operations, Theorem (4.6) becomes

$$\sum_{n=0}^{\infty} \mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet \mathcal{E}^{\boxtimes n}) = \exp\Big( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} p_n^\ell \sum_{d|n} \mu(n/d) \psi_d\big(\mathsf{e}_S\big(X, \mathcal{E}^{\otimes \ell n/d}\big)\big) \Big).$$

For example, with the notation $\mathsf{e}(n) = \mathsf{e}_S(X, \mathcal{E}^{\otimes n})$, we have

$$\mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n), j(n)^\bullet \mathcal{E}^{\boxtimes n}) = \sum_{\lambda \vdash n} s_\lambda \otimes \Phi_\lambda,$$

where $\Phi_{1^n} = \sigma_{1^n}(\mathsf{e}(1))$, while the other $\Phi_\lambda$ are as follows for $|\lambda| \leq 4$:

$$\begin{aligned}
\Phi_2 &= \sigma_2(\mathsf{e}(1)) - \mathsf{e}(2), \\
\Phi_3 &= \sigma_3(\mathsf{e}(1)) - \mathsf{e}(1)\,\mathsf{e}(2), \quad \Phi_{21} = \sigma_{21}(\mathsf{e}(1)) - \mathsf{e}(1)\,\mathsf{e}(2) + \mathsf{e}(3), \\
\Phi_4 &= \sigma_4(\mathsf{e}(1)) - \sigma_2(\mathsf{e}(1))\,\mathsf{e}(2) + \sigma_{1^2}(\mathsf{e}(2)), \\
\Phi_{31} &= \sigma_{31}(\mathsf{e}(1)) - \sigma_2(\mathsf{e}(1))\,\mathsf{e}(2) - \sigma_{11}(\mathsf{e}(1))\,\mathsf{e}(2) + \mathsf{e}(1)\,\mathsf{e}(3) + \sigma_2(\mathsf{e}(2)) - \mathsf{e}(4), \\
\Phi_{2^2} &= \sigma_{2^2}(\mathsf{e}(1)) - \sigma_2(\mathsf{e}(1))\,\mathsf{e}(2) + \mathsf{e}(1)\,\mathsf{e}(3) + \sigma_{1^2}(\mathsf{e}(2)), \\
\Phi_{21^2} &= \sigma_{21^2}(\mathsf{e}(1)) - \sigma_{1^2}(\mathsf{e}(1))\,\mathsf{e}(2) + \mathsf{e}(1)\,\mathsf{e}(3) - \mathsf{e}(4).
\end{aligned}$$

Note that the operations $\Phi_\lambda$ of [10] are the specializations of these polynomials obtained on setting $\mathsf{e}(n) = \mathsf{e}(1)$ for all $n \geq 1$.

Applying Theorem (4.6) with $\mathcal{E}$ equal to the unit object $\mathbb{1}$ of $\mathsf{D}(X)$, and using that $\mathbb{1}^{\otimes n} = \mathbb{1}$ for all $n$, we obtain the following corollary. Here, we abbreviate $\mathsf{e}_S(X, \mathbb{1})$ to $\mathsf{e}_S(X)$.

**Corollary (4.8).** $\sum_{n=0}^{\infty} \mathsf{e}_S^{\mathbb{S}_n}(\mathsf{F}(X/S,n)) = \operatorname{Exp}\big(\operatorname{Log}(1+p_1)\,\mathsf{e}_S(X)\big)$

Theorem (4.6), and its corollary, generalize immediately to the equivariant situation, in which a finite group $\Gamma$ acts on $X$ and $S$, and the morphism $\pi : X \to S$ and $\mathcal{E}$ are $G$-equivariant. The calculations now take place in the complete $\lambda$-ring $\Lambda \hat{\otimes} \mathsf{K}^\Gamma(S)$, and the formulas do not change.

(4.9). **The configuration spaces of group schemes.** If $\mathbb{G}$ is a group scheme over $S$ and $n > 0$, the scheme $\mathsf{F}(\mathbb{G}/S, n)$ is an $\mathbb{S}_n$-equivariant $\mathbb{G}$-torsor, and we may consider the quotient scheme $\mathbb{G}\backslash\mathsf{F}(\mathbb{G}/S,n)$. Imitating the above proof, we now calculate its $\mathbb{S}_n$-equivariant relative Euler characteristic. There is also a $\Gamma$-equivariant generalization, when a finite group $\Gamma$ acts on all of the data; however, it is formally identical, so we simplify notation by only treating the case $\Gamma = 1$.

**Theorem (4.10).** *If in the setting of Theorem (4.6) $X = \mathbb{G}$ is a group scheme, then*

$$\sum_{n=1}^{\infty} \mathsf{e}_S^{\mathbb{S}_n}(\mathbb{G}\backslash \mathsf{F}(\mathbb{G}/S,n)) = \frac{\operatorname{Exp}\big(\operatorname{Log}(1+p_1)\,\mathsf{e}_S(\mathbb{G})\big)-1}{\mathsf{e}_S(\mathbb{G})}.$$

*Proof.* We must first choose a $\mathbb{G}$-equivariant cover $\mathcal{U}$ of $\mathsf{F}(\mathbb{G}/S,2)$ by affine open immersions. Observe that the automorphism $(g,h)\mapsto(g,g^{-1}h)$ of $\mathbb{G}^2/S$ identifies $\mathsf{F}(\mathbb{G}/S,2)$ with $\mathbb{G}\times_S\mathbb{G}_0$, where $\mathbb{G}_0$ is the complement of the identity section of $\mathbb{G}$. Under this identification, the action of $\mathbb{G}$ on $\mathsf{F}(\mathbb{G}/S,2)$ corresponds to left translation in the first factor of $\mathbb{G}\times_S\mathbb{G}_0$. We now choose a cover $\{U_i\}$ of $\mathbb{G}_0$ by affine open immersions $j_i : U_i \hookrightarrow \mathbb{G}_0$; the cover $\mathcal{U}$ of $\mathsf{F}(\mathbb{G}/S,2)$ is the pullback of this cover by the projection from $\mathsf{F}(\mathbb{G}/S,2)\cong\mathbb{G}\times_S\mathbb{G}_0$ to $\mathbb{G}_0$. (Here, we use that to be an affine open immersion is preserved under base change.)

If $n>0$, the morphism $j(n) : \mathsf{F}(\mathbb{G}/S,n)\hookrightarrow\mathbb{G}^n/S$ is an $\mathbb{S}_n$-equivariant immersion of $\mathbb{G}$-torsors. On quotienting by the action of the group scheme $\mathbb{G}$, we obtain an $\mathbb{S}_n$-equivariant immersion

$$\bar{\jmath}(n) : \mathbb{G}\backslash\mathsf{F}(\mathbb{G}/S,n) \longhookrightarrow \mathbb{G}\backslash(\mathbb{G}^n/S).$$

(Note that $\mathbb{G}\backslash\mathbb{G}^n/S$ is isomorphic to $\mathbb{G}^{n-1}/S$; however, this isomorphism obscures the action of the symmetric group $\mathbb{S}_n$.) Since the cover $\mathcal{U}$ of $\mathsf{F}(\mathbb{G}/S,2)$ is $\mathbb{G}$-equivariant, the resolution $\mathcal{L}^\bullet_{\mathcal{U}}(\mathbb{G}/S,\mathbb{1},n)$ is $\mathbb{S}_n\times\mathbb{G}$-equivariant, so descends to an $\mathbb{S}_n$-equivariant resolution $\mathbb{G}\backslash\mathcal{L}^\bullet_{\mathcal{U}}(\mathbb{G}/S,\mathbb{1},n)$ of $\bar{\jmath}(n)_\bullet\bar{\jmath}(n)^\bullet\mathbb{1}$. The theorem now follows by a proof which is entirely parallel to that of Theorem (4.6) (in the special case that $\mathcal{E}=\mathbb{1}$), provided we observe that

$$\begin{aligned}\mathsf{e}_S^{\mathbb{S}_n}(\mathbb{G}\backslash(\mathbb{G}^n/S),\mathbb{G}\backslash\mathcal{L}^{n-k}_{\mathcal{U}}(\mathbb{G}/S,\mathbb{1},n)) &= \sum_{k=1}^{n}\operatorname{ch}(\mathsf{s}(n,k))\,\mathsf{e}_S(\mathbb{G})^{k-1}\\ &= \frac{1}{\mathsf{e}_S(\mathbb{G})}\,\mathsf{e}_S^{\mathbb{S}_n}(\mathbb{G}^n/S,\mathcal{L}^{n-k}(\mathbb{G}/S,\mathbb{1},n)). \qquad\square\end{aligned}$$

Note that if $\mathbb{G}$ is a family of elliptic curves, the proof of Theorem (4.10) simplifies, since we may take for the cover $\mathcal{U}$ of $\mathsf{F}(\mathbb{G}/S,2)$ the canonical choice $\{\mathsf{F}(\mathbb{G}/S,2)\}$. In fact, this is the case of Theorem (4.10) which we apply in the next section, to the universal family $E(N)$ of elliptic curves over the modular curve $Y(N)$.

## 5. The $\mathbb{S}_n$-equivariant Serre polynomial of the moduli space $\mathcal{M}_{1,n}$

Let $\mathcal{M}_{1,n}(N)$ be the fine moduli space of smooth elliptic curves of level $N\geq 3$ with $n$ marked points; it is a smooth quasi-projective variety. The finite group $\mathrm{SL}(2,\mathbb{Z}/N)$ acts on $\mathcal{M}_{1,n}(N)$, with quotient $\mathcal{M}_{1,n}$ the coarse moduli space of smooth elliptic curves.

Let $Y(N)$ be the modular curve $\mathcal{M}_{1,1}(N)$, and let $E(N)\to Y(N)$ be the universal elliptic curve of level $N$. The relative configuration space $\mathsf{F}(E(N)/Y(N),n)$ is an $\mathrm{SL}(2,\mathbb{Z}/N)$-equivariant $E(N)$-torsor with base $Y(N)$.

Denote by $\mathsf{H}$ the mixed Hodge module $\mathsf{R}^1 f_!\mathbb{Q}$ on $Y(N)$; it is of course an $\mathrm{SL}(2,\mathbb{Z}/N)$-equivariant local system of rank 2, known as the Hodge local system. The sub-$\lambda$-ring which it generates in $\mathsf{K}^{\mathrm{SL}(2,\mathbb{Z}/N)}(\mathsf{MHM}(Y(N)))$ is isomorphic to the Grothendieck group of polynomial representations of the algebraic group $\mathrm{GL}(2)$; this is the polynomial ring $\mathbb{Z}[\mathsf{H},\mathsf{L}]$, with $\sigma_t(\mathsf{H})=(1-t\mathsf{H}+t^2\mathsf{L})^{-1}$ and $\sigma_t(\mathsf{L})=(1-t\mathsf{L})^{-1}$. In this notation, we have

$$\mathsf{e}_{Y(N)}^{\mathrm{SL}(2,\mathbb{Z}/N)}(E(N)) = 1-\mathsf{H}+\mathsf{L}.$$

We may now apply the $\mathrm{SL}(2,\mathbb{Z}/N)$-equivariant version of Theorem (4.10), obtaining the following formula.

**Proposition (5.1).**

$$\sum_{n=1}^{\infty} \mathsf{e}^{\mathrm{SL}(2,\mathbb{Z}/N)\times\mathbb{S}_n}_{Y(N)}(\mathcal{M}_{1,n}(N)) = \frac{\left\{\prod_{k=1}^{\infty}(1+p_k)^{\frac{1}{k}\sum_{d|k}\mu(k/d)(1-\psi_d(\mathsf{H})+\mathsf{L}^d)}\right\}-1}{1-\mathsf{H}+\mathsf{L}}$$

Denote the $n$th symmetric power of $\mathsf{H}$ by $\mathsf{H}_n$; it is a rank $(n+1)$ $\mathrm{SL}(2,\mathbb{Z}/N)$-equivariant local system on $Y(N)$, given by the Chebyshev polynomial of the second kind*

$$\mathsf{H}_n = U_n(\mathsf{H}/2).$$

The following table gives $\mathsf{e}^{\mathrm{SL}(2,\mathbb{Z}/N)\times\mathbb{S}_n}_{Y(N)}(\mathcal{M}_{1,n}(N))$ for $n \le 5$. This table was calculated using J. Stembridge's symmetric function package SF [25] for `maple`.

| $n$ | $\mathsf{e}^{\mathrm{SL}(2,\mathbb{Z}/N)\times\mathbb{S}_n}_{Y(N)}(\mathcal{M}_{1,n}(N))$ |
|---|---|
| 1 | $\mathsf{H}_0$ |
| 2 | $\mathsf{H}_0\mathsf{L}s_2 - \mathsf{H}_1 s_{1^2}$ |
| 3 | $\mathsf{H}_0\mathsf{L}^2 s_3 - \mathsf{H}_1(\mathsf{L}s_{21} - s_3) + \mathsf{H}_2 s_{1^3}$ |
| 4 | $\mathsf{H}_0(\mathsf{L}^3-\mathsf{L})s_4 - \mathsf{H}_1(\mathsf{L}^2 s_{31} - \mathsf{L}(s_4+s_{31}) + s_{2^2}) + \mathsf{H}_2(\mathsf{L}s_{21^2} - s_{31}) - \mathsf{H}_3 s_{1^4}$ |
| 5 | $\mathsf{H}_0(\mathsf{L}^4 s_5 - \mathsf{L}^2(s_5+s_{41}) + \mathsf{L}s_{32})$ <br> $- \mathsf{H}_1(\mathsf{L}^3 s_{41} - \mathsf{L}^2(s_5+s_{41}+s_{32}) - \mathsf{L}(s_{32}+s_{2^21}) + s_{31^2})$ <br> $+ \mathsf{H}_2(\mathsf{L}^2 s_{31^2} - \mathsf{L}(s_{41}-s_{32}-s_{31^2}) + (s_5+s_{32}+s_{2^21})) - \mathsf{H}_3(\mathsf{L}s_{21^3} - s_{31^2}) + \mathsf{H}_4 s_{1^5}$ |

(5.2). **The Eichler-Shimura isomorphism.** Let $S_\ell(N)$ be the spaces of cusp forms of weight $\ell$ for the congruence group $\Gamma(N) = \ker(\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N))$. It is an $\mathrm{SL}(2,\mathbb{Z}/N)$-module, and its invariant subspace $S_\ell = S_\ell(1)$ is the space of cusp forms of level 1.

Let $E_\ell(N)$ be the space of Eisenstein series of weight $\ell$. If $\ell > 2$, this is isomorphic as a $\mathrm{SL}(2,\mathbb{Z}/N)$-module to the induced representation

$$\Sigma_\ell(N) = \mathrm{Ind}_{P(N)}^{\mathrm{SL}(2,\mathbb{Z}/N)} \chi_\ell,$$

where $P(N) \subset \mathrm{SL}(2,\mathbb{Z}/N)$ is the parabolic subgroup of upper triangular matrices, with generators $T = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $-I$, and $\chi_\ell$ is the character of $P(N)$ which equals 1 on $T$ and $(-1)^\ell$ on $-I$.

The space $E_2(N)$ is smaller than $\Sigma_2(N)$: it is isomorphic to $H_0(\mathrm{SL}(2,\mathbb{Z}/N),\Sigma_2(N))$.

If $\ell$ is even, $\Sigma_\ell(N)$ is the permutation representation of $\mathrm{SL}(2,\mathbb{Z}/N)$ on the set of cusps, and the $\mathrm{SL}(2,\mathbb{Z}/N)$-invariant subspace is one-dimensional; the corresponding subspace of $E_\ell(N)$ is spanned by the level 1 Eisenstein series $E_\ell$. If $\ell$ is odd, there are no $\mathrm{SL}(2,\mathbb{Z}/N)$-invariant elements of $E_\ell(N)$, reflecting the fact that there are no level 1 Eisenstein series of odd weight. In all cases, $\Sigma_\ell(N)$ has dimension

$$[\mathrm{SL}(2,\mathbb{Z}/N) : P(N)] = \frac{k^2}{2}\prod_{p|k}(1-p^{-2})$$

equal to the number of cusps of the congruence subgroup $\Gamma(N)$.

*These polynomials have generating function $\sum_{n=0}^{\infty} t^n U_n(x) = (1-2xt+t^2)^{-1}$.

Eichler and Shimura have calculated the cohomology of the sheaves $\mathsf{H}_n$. This calculation is explained in Verdier [26] and Shimura [23]. The mixed Hodge structure on this cohomology may be calculated by the same technique that Deligne uses in [5] to calculate the action of the Frobenius operator on the étale cohomology groups. Define the Hodge structure $\mathsf{S}_{n+2}(N)$ to be $\operatorname{gr}^W_{n+1} H^1_c(Y(N),\mathsf{H}_n)$.

**Theorem (5.3).** *The vector spaces* $\operatorname{gr}^W_k H^i_c(Y(N),\mathsf{H}_n)$ *associated to the weight filtration on the cohomology groups* $H^\bullet_c(Y(N),\mathsf{H}_n)$ *vanish, with the exception of*

$$\begin{aligned}\operatorname{gr}^W_0 H^1_c(Y(N),\mathsf{H}_n) &\cong E_{n+2}(N),\\ \operatorname{gr}^W_{n+1} H^1_c(Y(N),\mathsf{H}_n) &\cong \mathsf{S}_{n+2}(N), \quad \textit{and}\\ \operatorname{gr}^W_2 H^2_c(Y(N),\mathsf{H}_0) &= \mathsf{L}.\end{aligned}$$

*The Hodge filtration of* $\mathsf{S}_{n+2}(N)$ *has two steps:* $0 \subset F^0\mathsf{S}_{n+2}(N) \subset \mathsf{S}_{n+2}(N)$, *and the vector space* $F^0\mathsf{S}_{n+2}(N)$ *is naturally isomorphic to* $S_{n+2}(N)$.

**Corollary (5.4).** *The equivariant Serre polynomial*

$$\mathsf{e}^{\mathrm{SL}(2,\mathbf{Z}/N)\times\mathbb{S}_n}(\mathcal{M}_{1,n}(N)) \in \mathsf{K}^{\mathrm{SL}(2,\mathbf{Z}/N)}(\mathsf{MHM}(\operatorname{Spec}(\mathbb{C}))) \otimes \Lambda_n$$

*is obtained from the equivariant Serre polynomial*

$$\mathsf{e}^{\mathrm{SL}(2,\mathbf{Z}/N)\times\mathbb{S}_n}_{Y(N)}(\mathcal{M}_{1,n}(N)) \in \mathsf{K}^{\mathrm{SL}(2,\mathbf{Z}/N)}(\mathsf{MHM}(Y(N))) \otimes \Lambda_n$$

*by the substitution* $\mathsf{H}_n \mapsto \delta_{n,0}(\mathsf{L}+1) - \Sigma_{n+2}(N) - \mathsf{S}_{n+2}(N)$.

We may now descend to level 1 by applying the augmentation

$$\varepsilon : \mathsf{K}^{\mathrm{SL}(2,\mathbf{Z}/N)}(\mathsf{MHM}(\operatorname{Spec}(\mathbb{C}))) \longrightarrow \mathsf{K}(\mathsf{MHM}(\operatorname{Spec}(\mathbb{C}))),$$

given explicitly by $\varepsilon(\mathsf{S}_\ell(N)) = \mathsf{S}_\ell$ and

$$\varepsilon(\Sigma_\ell(N)) = \begin{cases} \mathbb{Q}, & \ell \text{ even},\\ 0, & \ell \text{ odd}.\end{cases}$$

The following table gives the $\mathbb{S}_n$-equivariant Serre polynomial $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n})$ for $n \le 5$, together with the underlying Serre polynomial $\mathsf{e}(\mathcal{M}_{1,n})$ and Euler characteristic.

| $n$ | $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n})$ | $\mathsf{e}(\mathcal{M}_{1,n})$ | $\chi(\mathcal{M}_{1,n})$ |
|---|---|---|---|
| 1 | $\mathsf{L}s_1$ | $\mathsf{L}$ | 1 |
| 2 | $s_2\mathsf{L}^2$ | $\mathsf{L}^2$ | 1 |
| 3 | $s_3\mathsf{L}^3 - s_{1^3}$ | $\mathsf{L}^3 - 1$ | 0 |
| 4 | $s_4\mathsf{L}^4 - s_4\mathsf{L}^2 - s_{21^2}\mathsf{L} + s_{31}$ | $\mathsf{L}^4 - \mathsf{L}^2 - 3\,\mathsf{L} + 3$ | 0 |
| 5 | $s_5\mathsf{L}^5 - (s_5 + s_{41})\mathsf{L}^3 + (s_{32} - s_{31^2})\mathsf{L}^2$ | $\mathsf{L}^5 - 5\,\mathsf{L}^3 - \mathsf{L}^2 + 15\,\mathsf{L} - 12$ | $-2$ |
| | $+ (s_{41} + s_{32} + s_{31^2})\mathsf{L}$ | | |
| | $- (s_5 + s_{32} + s_{2^21} + s_{1^5})$ | | |

Corollary (5.4) may be expressed in closed form:

$$(5.5)\quad \sum_{n=1}^{\infty} \mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n}) = \operatorname{res}_0\Biggl[\Biggl(\frac{\prod_{n=1}^{\infty}(1+p_n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)(1-\omega^d-\mathsf{L}^d/\omega^d+\mathsf{L}^d)}-1}{1-\omega-\mathsf{L}/\omega+\mathsf{L}}\Biggr) \times \Biggl(\sum_{k=1}^{\infty}\Bigl(\frac{\mathsf{S}_{2k+2}+1}{\mathsf{L}^{2k+1}}\Bigr)\omega^{2k}-1\Biggr)(\omega-\mathsf{L}/\omega)\,d\omega\Biggr],$$

where $\operatorname{res}_0$ is the residue of the differential form at the origin. This is an easy consequence of the Weyl integration formula for SU(2), in the form

$$-\frac{1}{2}\operatorname{res}_0\Biggl[\Biggl(\frac{\omega^{k+1}-(\mathsf{L}/\omega)^{k+1}}{\omega-\mathsf{L}/\omega}\Biggr)\Biggl(\frac{\omega^{\ell+1}-(\mathsf{L}/\omega)^{\ell+1}}{\omega-\mathsf{L}/\omega}\Biggr)(\omega-\mathsf{L}/\omega)^2\frac{d\omega}{\omega}\Biggr] = \mathsf{L}^{k+1}\delta_{k\ell}.$$

To obtain a formula for the non-equivariant Serre polynomials, we replace $p_n$, $n > 1$, by 0, and expand in $p_1$, which gives

$$(5.6)\quad \frac{\mathsf{e}(\mathcal{M}_{1,n+1})}{n!} = \operatorname{res}_0\Biggl[\binom{\mathsf{L}-\omega-\mathsf{L}/\omega}{n}\Biggl(\sum_{k=1}^{\infty}\Bigl(\frac{\mathsf{S}_{2k+2}+1}{\mathsf{L}^{2k+1}}\Bigr)\omega^{2k}-1\Biggr)(\omega-\mathsf{L}/\omega)\,d\omega\Biggr].$$

From (5.6), we can calculate the Euler characteristic $\chi(\mathcal{M}_{1,n})$ directly. The following proof was shown to the author by D. Zagier.

**Proposition (5.7).** *If $n \geq 5$, $\chi(\mathcal{M}_{1,n}) = (-1)^n(n-1)!/12$.*

*Proof.* If in (5.6), we replace $\mathsf{L} = 1$ and $\mathsf{S}_{k+2}$ by $2\dim(S_{k+2})$, we see that

$$\frac{\chi(\mathcal{M}_{1,n+1})}{n!} = \operatorname{res}_0\Biggl[\binom{1-\omega-\omega^{-1}}{n}\frac{(1-\omega^2-2\omega^4-\omega^6+\omega^8)}{(1+\omega^2)(1-\omega^6)}\frac{d\omega}{\omega}\Biggr].$$

The poles of this differential form are all simple, and are located at $\omega = 0$ and $\omega = \infty$, and at values of $\omega$ such that $\omega+\omega^{-1}$ is an integer in the interval $[-2,2]$ (the latter poles are on the unit circle). Since it is invariant under $\omega \mapsto \omega^{-1}$, its residues at 0 and $\infty$ are equal. By the residue theorem, it follows that

$$\frac{\chi(\mathcal{M}_{1,n+1})}{n!} = -\frac{1}{2}\sum_{z\in\{\pm1,\pm i,\pm\rho,\pm\rho^2\}}\operatorname{res}_z\Biggl[\binom{1-\omega-\omega^{-1}}{n}\frac{(1-\omega^2-2\omega^4-\omega^6+\omega^8)}{(1+\omega^2)(1-\omega^6)}\frac{d\omega}{\omega}\Biggr],$$

where $\rho$ is a primitive sixth root of unity.

The residues of this differential form on the unit circle are as follows:

$$(5.8)\quad \operatorname{res}_z\Biggl[\frac{(1-\omega^2-2\omega^4-\omega^6+\omega^8)}{(1+\omega^2)(1-\omega^6)}\Biggr] = \begin{cases} 1/6, & |z+z^{-1}| = 2, \\ -1/3, & |z+z^{-1}| = 1, \\ -1/2, & |z+z^{-1}| = 0. \end{cases}$$

At each of these poles except $\omega = 1$, the binomial coefficient $\binom{1-\omega-\omega^{-1}}{n}$ vanishes for $n \geq 4$. This leaves the residue at 1, which equals $(-1)^{n+1}/12$. □

We close the paper with a calculation of the Serre polynomials of the spaces $\mathcal{M}_{1,n}/\mathbb{S}_n$. If we substitute $x^n$ for $p_n$ in (5.5), we obtain the generating function for the $\mathbb{S}_n$-invariant parts of the local systems $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n}/\mathcal{M}_{1,1})$. By Corollary (5.7) of [10], we have

$$\begin{aligned}\sum_{n=1}^\infty H^0(\mathbb{S}_n, \mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n}/\mathcal{M}_{1,1}))x^n &= \frac{\prod_{n=1}^\infty (1+x^n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)(1-\omega^d-\mathsf{L}^d/\omega^d+\mathsf{L}^d)} - 1}{1-\omega-\mathsf{L}/\omega+\mathsf{L}} \\ &= \frac{1}{1-\omega-\mathsf{L}/\omega+\mathsf{L}}\left\{\frac{(1-\omega x)(1-\mathsf{L}x/\omega)(1-x^2)(1-\mathsf{L}x^2)}{(1-x)(1-\mathsf{L}x)(1-\omega x^2)(1-\mathsf{L}x^2/\omega)} - 1\right\} \\ &= x\left(\frac{1-\mathsf{L}x^3}{1-\mathsf{L}x}\right)\frac{1}{1-(\omega+\mathsf{L}/\omega)x^2+\mathsf{L}x^4} \\ &= x\left(\frac{1-\mathsf{L}x^3}{1-\mathsf{L}x}\right)\sum_{k=0}^\infty \mathsf{H}_k x^{2k}.\end{aligned}$$

Applying the functor $H_c^\bullet(\mathcal{M}_{1,1}, -)$, we obtain the following result.

**Proposition (5.9).**

$$\sum_{n=1}^\infty \mathsf{e}(\mathcal{M}_{1,n}/\mathbb{S}_n)x^n = x\left(\frac{1-\mathsf{L}x^3}{1-\mathsf{L}x}\right)\sum_{k=0}^\infty \mathsf{e}(\mathcal{M}_{1,1}, \mathsf{H}_{2k})x^{4k}$$

Applying the augmentation $\varepsilon : K_0(\mathsf{MHM}) \to \mathbb{Z}$, we obtain the following corollary:

$$\begin{aligned}\sum_{n=1}^\infty \chi(\mathcal{M}_{1,n}/\mathbb{S}_n)x^n &= (x+x^2+x^3)\sum_{n=0}^\infty \chi(\mathcal{M}_{1,1}, \mathsf{H}_n)x^{4n} \\ &= (x+x^2+x^3)\frac{(1-x^4-2x^8-x^{12}+x^{16})}{(1-x^8)(1-x^{12})}.\end{aligned}$$

The corresponding formulas in genus 0 are $\mathsf{e}(\mathcal{M}_{0,n}/\mathbb{S}_n) = \mathsf{L}^{n-3}$ and $\chi(\mathcal{M}_{0,n}/\mathbb{S}_n) = 1$, for all $n \geq 3$.

## References

[1] V.I. Arnold, *The cohomology ring of the colored braid group*, Mat. Zametki **5** (1969), 227–231.

[2] M. Artin et al., "Théorie de topos et cohomologie étale des schémas, Tome 3 (SGA4)," Lecture Notes in Math., vol. 305, Springer, Berlin-Heidelberg-New York, 1973.

[3] A.A. Beilinson, *On the derived category of perverse sheaves*, in "$K$-theory, arithmetic and geometry (Moscow, 1984–1986)," Lecture Notes in Math., vol. 1289, Springer, Berlin-Heidelberg-New York, 1987, 27–41.

[4] A.A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982), 5–171.

[5] P. Deligne, *Formes modulaires et représentations $\ell$-adiques*, Sémin. Bourbaki 1968/69, no. 347, Lecture Notes in Math., vol. 179, Springer, Berlin-Heidelberg-New York, 1971, 139–172.

[6] P. Deligne, *Théorie de Hodge III*, Publ. Math. IHES **44** (1974), 5–78.

[7] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52** (1980), 137–252.

[8] A. Dress, *Contributions to the theory of induced representations*, in "Algebraic $K$-theory II," Lecture Notes in Math., vol. 342, Springer, Berlin-Heidelberg-New York, 1973, 183–240.

[9] G. Faltings, C.L. Chai, "Degeneration of Abelian varieties," Ergeb. Math. Grenzgeb. (3) vol. 22, Springer, Berlin-Heidelberg-New York, 1990.

[10] E. Getzler, *Mixed Hodge structures of configuration spaces*, Max-Planck-Institut preprint MPI-96-61, `alg-geom/9510018`.

[11] E. Getzler, *The semi-classical approximation for modular operads*, Max-Planck-Institut preprint MPI-96-145, `alg-geom/9611003`.

[12] A. Grothendieck, *Eléments de Géometrie Algebrique*, Publ. Math. IHES **8** (1961).

[13] P. Hanlon, *The fixed-point partition lattice*, Pacific J. Math. **96** (1981), 319–341.

[14] P. Hanlon, *A proof of a conjecture of Stanley concerning partitions of a set,* European J. Combin. **4** (1983), 137–141.

[15] P. Hanlon, *The action of $S_n$ on the components of the Hodge decomposition of Hochschild homology,* Michigan Math. J. **37** (1990), 105–124.

[16] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves,* Invent. Math. **85** (1986), 457–485.

[17] M.L. Laplaza, *Coherence for distributivity,* in "Coherence in categories," Lecture Notes in Math., vol. 281, Springer, Berlin-Heidelberg-New York, 1972, 29–65.

[18] G.I. Lehrer and L. Solomon, *On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes,* J. Algebra **104** (1986), 410–424.

[19] I.G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon Press, Oxford, 1979.

[20] P. Orlik, *Introduction to arrangements,* CBMS Regional Conf. Ser. Math. **72** (1989).

[21] P. Orlik and L. Solomon, *Combinatorics and topology of complements of hyperplanes,* Invent. Math. **56** (1980), 167–189.

[22] M. Saito, *Mixed Hodge modules,* Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.

[23] G. Shimura, "Introduction to the arithmetic theory of automorphic functions," (Publications of the Mathematical Society of Japan, vol. 11) Princeton University Press, Princeton, N.J., 1971.

[24] R.P. Stanley, *Some aspects of groups acting on finite posets,* J. Combin. Theory **A32** (1982), 132–161.

[25] J. Stembridge, *A Maple package for symmetric functions.* `ftp://ftp.math.lsa.umich.edu/pub/jrs/software/SF.tar.Z.`

[26] J.L. Verdier, *Sur les integrales attacheées aux formes automorphes (d'après Goro Shimura),* Sémin. Bourbaki 1960/61, no. 216.

[27] B. Totaro, *Configuration spaces of algebraic varieties,* Topology **35** (1996), 1057–1068.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN-STR. 26, D-53225 BONN, GERMANY
*Current address*: Department of Mathematics, Northwestern University, Evanston, IL 60208-2730, USA
*E-mail address*: `getzler@math.nwu.edu`