# Holomorphic Automorphisms of Quadrics II

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# HOLOMORPHIC AUTOMORPHISMS OF QUADRICS II

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ABSTRACT. This paper is the continuation of [2]. We consider the automorphisms of 4 different types of quadrics in $\mathbb{C}^6$.

## 1. INTRODUCTION

Let $(z, w = u + iv)$ be coordinates in $\mathbb{C}^3 \times \mathbb{C}^3$ with $w = u + iv$, $u, v \in \mathbb{R}^3$. We consider quadrics $Q$ of the form:

$$\begin{aligned} v^1 &= \langle z,z\rangle^1 = \sum H^1_{ij} z^i \bar{z}^j \\ v^2 &= \langle z,z\rangle^2 = \sum H^2_{ij} z^i \bar{z}^j \\ v^3 &= \langle z,z\rangle^3 = \sum H^3_{ij} z^i \bar{z}^j, \end{aligned} \tag{1}$$

where $H^k_{ij} = \overline{H^k_{ji}}$.

The quadrics $Q$ are presumed to be nondegenerate, i.e.

*i.)* $\langle z, b\rangle^j = 0$ for all $z$ implies $b = 0$

*ii.)* $\langle z, z\rangle^j$ are linearly independent $j = 1, \ldots, k$.

We are interested in finding the isotropy groups, i.e. the groups of holomorphic automorphisms preserving the origin.

It follows from the results of Henkin, Tumanov and Forstnerič [3, 4, 5] that any local CR diffeomorhism of $Q$ extends to a birational map of $\mathbb{C}^6$.

Beloshapka proved, that quadrics of the form (1) in general position are rigid, i.e. their isotropy group consists of the trivial automorphisms

$$\begin{aligned} z &\mapsto cz \\ w &\mapsto |c|^2 w \end{aligned}$$

for some $c \in \mathbb{C}$

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But he found a quadric of form (1) with a 19 dimensional isotropy group. That is the maximally possible dimension. This quadric has the following defining equation

$$\begin{aligned} v^1 &= |z^1|^2 \\ v^2 &= z^1\bar{z}^2 + z^2\bar{z}^1 \\ v^3 &= z^1\bar{z}^3 + z^3\bar{z}^1, \end{aligned} \tag{2}$$

It is called nullquadric, because the characteristic polynomial

$$\det(t_1 H^1 + t_2 H^2 + t_3 H^3)$$

vanishes identically.

The goal of the present paper is to calculate the explicit isotropy group of the nullquadric.

In the case of quadrics of codimension 2 in $\mathbb{C}^4$ the authors obtained the isotropy groups by means of some matrix substitution in the well known formulas of sphere automorphisms (see [2]).

We present a matrix substitution leading to the isotropy group of the nullquadric and three other types quadrics (1).

Remark. As mentioned in the previous paper [2] quadrics are related to Siegel domains of second kind. However the quadrics being considered here are strongly 1-concave, and therefore do not realize Siegel domains. In the case $n = k = 2$ being considered in [2] only the hyperbolic quadric realizes a Siegel domain: the direct product of two balls.

## 2. The matrix substitution

For $\epsilon, \delta \in \mathbb{R}$, we consider the algebras $\mathfrak{A}_{\epsilon\delta}$ consisting of matrices

$$Z = \begin{pmatrix} z^1 & \epsilon\delta z^3 & \epsilon\delta z^2 \\ z^2 & z^1 & \epsilon z^3 \\ z^3 & \delta z^2 & z^1 \end{pmatrix}$$

with conjugation

$$\bar{Z} = \begin{pmatrix} \bar{z}^1 & \epsilon\delta\bar{z}^3 & \epsilon\delta\bar{z}^2 \\ \bar{z}^2 & \bar{z}^1 & \epsilon\bar{z}^3 \\ \bar{z}^3 & \delta\bar{z}^2 & \bar{z}^1 \end{pmatrix}.$$

These algebras are commutative. Let $\sigma$ be the lifting

$$\sigma : \mathbb{C}^3 \longrightarrow \mathfrak{A}_{\delta\epsilon}$$

given by the formula above.

The equation of the nullquadric takes the form

$$\operatorname{Im} W = Z\bar{Z}$$

where $W = \sigma(w)$, $Z = \sigma(z)$ for $\epsilon = \delta = 0$.

For $\epsilon\delta > 0$, $\epsilon\delta < 0$, and $\epsilon = 0$, $\delta \neq 0$ other pairwise nonequivalent typs of quadrics appear.

All these quadrics have a 9 dimensional subgroup of automorphisms $\Phi$ with the property

$$d\Phi|_{T_0^{\mathbb{C}}Q} = \mathrm{id}\,. \tag{3}$$

We obtain these automorphisms by inserting matrices of $\mathfrak{A}_{\epsilon\delta}$ into the Poincaré formula (cp. [2]) for the sphere $\operatorname{Im} w = |z|^2$ in $\mathbb{C}^2$:

$$\begin{aligned} Z &\mapsto (Z + AW)(\mathrm{id} - 2i\bar{A}Z - (R + iA\bar{A})W)^{-1} \\ W &\mapsto W(\mathrm{id} - 2i\bar{A}Z - (R + iA\bar{A})W)^{-1}, \end{aligned}$$

where $A, R \in \mathfrak{A}_{\epsilon\delta}$ and $R = \bar{R}$.

Adding to these groups the linear automorphisms

$$\begin{aligned} z &\mapsto Cz \\ w &\mapsto \rho z, \end{aligned}$$

with $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$, we get the whole automorphism groups of dimension 19 in the case of the nullquadric, and of dimension 15,15,17, respectively, in the other cases. This follows from a uniqueness theorem of Beloshapka [1].

We are now able to define the chains, analogous to the Chern Moser chains in the case of a hyperquadric. They are 3 dimensional real analytic surfaces which can be obtained as images of the standard chain $z = 0, v = 0$ via some automorphism. They have the form

$$\begin{aligned} Z &= (\mathrm{id} - iA\bar{A}U)^{-1}AU \\ W &= (\mathrm{id} - iA\bar{A}U)^{-1}U, \end{aligned}$$

where $U$ is the parameter and $A$ is fixed with $A, U \in \mathfrak{A}_{\epsilon\delta}$ and $U = \bar{U}$.

The linear automorphisms can be obtained similarly as in the case $n = k = 2$. Solving a system of linear equations one get the corresponding Lie algebras. The images of the Lie algebras under the exponential map are the desired groups.

In the case of the nullquadric $Q_{00}$ we have

$$C = a \begin{pmatrix} 1 & 0 & 0 \\ b & \alpha & \beta \\ c & \gamma & \xi \end{pmatrix},$$

with $a, b, a \in \mathbb{C}$ and $\alpha, \beta, \gamma, \xi \in \mathbb{R}$.

In the case of the quadric $Q_{10}$ we have

$$C = a \begin{pmatrix} 1 & 0 & 0 \\ b & \alpha & 0 \\ c & \beta & \alpha^2 \end{pmatrix},$$

with $a, b, c \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.

In the case of $Q_{1-1}$ the linear groups are:

$$C = \begin{pmatrix} a & \epsilon c & \epsilon b \\ b & a & \epsilon c \\ c & b & a \end{pmatrix},$$

with $a, b, c \in \mathbb{C}$.

Remark 1. Three types of quadrics with $n = k = 3$ in $\mathbb{C}^6$ with nontrivial isotopy group can be obtained as direct products of a sphere $S^3 \in \mathbb{C}^2$ with the three types of quadrics with $n = k = 2$.

The direct products of the sphere with the hyperbolic, elliptic and parabolic quadrics have isotropy groups of dimension 15,15,16 respectively.

It is easy to verify that $S^3 \times Q_{-1}$ and $S^3 \times Q_1$ are not equivalent to $Q_{1-1}$ and $Q_{11}$: In fact, we consider the set of isotropic vectors $\{z \in \mathbb{C}^3 | \langle z, z \rangle = 0\}$. In the case of $S^3 \times Q_{-1}$ it consists of the nullvector, in the case $S^3 \times Q_1$ it consists of a single complex ray $\{(0, a, -a) | a \in \mathbb{C}\}$, and in the cases of $Q_{11}$ and $Q_{1-1}$ it contains a continuum of complex rays.

It is more difficult to show that $Q_{11}$ and $Q_{1-1}$ are not equivalent.

Suppose for a moment, they were equivalent and the linear transformation $z \mapsto Cz, w \mapsto \rho w$ maps $Q_{11}$ to $Q_{1-1}$.

Since chains of $Q_{11}$ transform to chains of $Q_{1-1}$ a linear isomorphism of the spaces of matrix lines in $\mathfrak{A}^2_{11}$ and $\mathfrak{A}^2_{1-1}$ occurs.

We consider the matrix line $w = z$ in $\mathfrak{A}^2_{11}$. Without loss of generality we may assume that its image is $w = z$ in $\mathfrak{A}^2_{1-1}$, because linear automorphisms of $Q_{1-1}$ act transitively at the matrix line space.

This means that $(w, w) \mapsto (w^*, w^*) = (Cw, \rho w)$ for any $w$. It follows $C = \rho$.

Furthermore, let $z = aw$ be any matrix line. It will be sent to $z^* = a^* w^*$, where $a^* = Ca\rho^{-1} = \rho a \rho^{-1}$.

Hence, the algebras $\mathfrak{A}_{11}$ and $\mathfrak{A}_{1-1}$ would be adjoint. We show that this is impossible. The contradiction proves that the two mentioned types are not equivalent.

Let $a = \sigma(a_1, a_2, a_3)$ and $a^* = \sigma(a_1^*, a_2^*, a_3^*)$. Then the characteristic polynomials $\det(a - \lambda \,\mathrm{id})$ and $\det(a^* - \lambda \,\mathrm{id})$ are identical. It follows immediately that $a_1 = a_1^*$, $a_2 a_3 = a_2^* a_3^*$, and $a_2^3 + a_3^3 = (a_2^*)^3 + (a_3^*)^3$. These equations lead to $a_2 = a_2^*$, and $a_3 = a_3^*$. Thus, $\rho = C$ has to be the unit matrix.

Remark 2. In the case of the nullquadric the subgroup of automorphisms with (3) can be obtained in another way. Therefore, let $Q_0$ be the parabolic quadric of codimension 2 in $\mathbb{C}^4$, i.e. the quadric given by

$$\begin{aligned} v^1 &= |z^1|^2 \\ v^2 &= z^1 \bar{z}^2 + z^2 \bar{z}^1. \end{aligned} \tag{4}$$

Furthermore, let $\pi$ be the projetion to the sphere $v = |z|^2$ in $\mathbb{C}^2$ defined by

$$(z^1, z^2, w^1, w^2) \mapsto (z^1, w^1) = (z, w).$$

Now the nullquadric is the fibred product of two copies of $Q_0$ over the sphere. Since the $(z^1, w^1)$ components of the automorphisms of $Q_0$ depend only on the $(z^1, w^1)$ variables, there is a canonical projection of the isotropy group of $Q_0$ onto the isotropy group of the sphere (see [2]). Therefore a subgroup of the isotropy group of the nullquadric can be obtained as fibred product of two copies of the isotropy group of $Q_0$ over the isotropy group of the sphere. This subgroup has only dimension 17. However, it contains all automorphisms with (3).

Remark 3. Analogously to the case of 2 quadrics in $\mathbb{C}^4$ there exists a linear representation of the automorphism groups in $\mathbb{C}^9$, namely

Let $\mathfrak{A}^3$ be the $\mathfrak{A}$ module with $\mathfrak{A} = \mathfrak{A}_{\epsilon\delta}$ (for $\epsilon, \delta = 1, 1, -1$) of triples $(\Theta_0, \Theta_1, \Theta_2)$ with $\Theta_i \in \mathfrak{A}$. By $\mathfrak{A}^*$ we denote the group of invertible elements of $\mathfrak{A}$ and by $\hat{\mathfrak{A}}^3$ the factor space under the natural action of $\mathfrak{A}^*$. $\hat{\mathfrak{A}}^3$ is a compact manifold which can be considered as a compactification of $\mathbb{C}^6 = \mathfrak{A}^2_{\epsilon\delta}$ by the embedding

$$(Z, W) \mapsto (\mathrm{id}, Z, W),$$

where $Z, W$ are $\sigma(z), \sigma(w)$.

Now, any automorphism of $Q_{\epsilon\delta}$ can be represented as a linear transformation of $\mathbb{C}^9$ in the following way:

Let $Q_{\epsilon\delta}$ be given in the form $\operatorname{Im} W = Z\bar{Z}$. Then the automorphisms can be written as a composition of

$$\begin{aligned} Z &\mapsto (Z + AW)(\mathrm{id} - 2i\bar{A}z - (R + iA\bar{A})W)^{-1} \\ W &\mapsto W(\mathrm{id} - 2i\bar{A}z - (R + iA\bar{A})W)^{-1}, \end{aligned}$$

where $A, R \in \mathfrak{A}$, with $R = \bar{R}$, and a linear $(C, \rho)$ transformation.

The first map induces the following linear transformation in $\mathfrak{A}^3$:

$$\begin{aligned}\Theta_0 &\mapsto \Theta_0 - 2i\bar{A}\Theta_1 - (R + iA\bar{A})\Theta_2\\ \Theta_1 &\mapsto \Theta_1 + A\Theta_2\\ \Theta_2 &\mapsto \Theta_2.\end{aligned}$$

Let $\theta_i$ for $i = 0, 1, 2$ be the projections of $\mathfrak{A}$ to resp. $\mathbb{C}^3$, such that $\sigma(\theta_i) = \Theta_i$. Then

$$\begin{aligned}\theta_0 &\mapsto \theta_0 - 2i\bar{A}\theta_1 - (R + iA\bar{A})\theta_2,\\ \theta_1 &\mapsto \theta_1 + A\theta_2,\\ \theta_2 &\mapsto \theta_2.\end{aligned}$$

Together with the linear transformation $C, \rho$ we obtain

$$\begin{aligned}\theta_0 &\mapsto \theta_0 - 2i\bar{A}\theta_1 - (R + iA\bar{A})\theta_2,\\ \theta_1 &\mapsto C\theta_1 + CA\theta_2,\\ \theta_2 &\mapsto \rho\theta_2.\end{aligned}$$

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