tt^* GEOMETRY, FROBENIUS MANIFOLDS, THEIR CONNECTIONS, AND THE CONSTRUCTION FOR SINGULARITIES

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ABSTRACT. The base space of a semiuniversal unfolding of a hypersurface singularity carries a rich geometry. By work of K. Saito and M. Saito it can be equipped with the structure of a Frobenius manifold. By work of Cecotti and Vafa it can be equipped with tt^* geometry if the singularity is quasihomogeneous. tt^* geometry generalizes the notion of variation of Hodge structures.

In the second part of this paper (chapters 6–8) Frobenius manifolds and tt^* geometry are constructed for any hypersurface singularity, using essentially oscillating integrals; and the intimate relationship between polarized mixed Hodge structures and this tt^* geometry is worked out.

It builds on the first part (chapters 2–5). There tt^* geometry and Frobenius manifolds and their relations are studied in general. To both of them flat connections with poles are associated, with distinctive common and different properties. A frame for a simultaneous construction is given.

CONTENTS

1. Introduction

10 years ago Cecotti and Vafa [CV1][CV2] considered moduli spaces of $N = 2$ supersymmetric quantum field theories and introduced a geometry on them which is governed by the tt^*-equations. This tt^* geometry generalizes variations of Hodge structures. Its most important ingredients are a real structure, given by a fiberwise $\mathbb{C}$-antilinear involution κ, a hermitian metric h, and a Higgs field C.

A distinguished class of these moduli spaces is associated to quasihomogeneous singularities. They are the base spaces of semiuniversal unfoldings of such singularities.

Already 10 years earlier K. Saito [SK2][SK3] and M. Saito [SM2] introduced another geometry on these moduli spaces of singularities, the structure of Frobenius manifolds. This is a purely holomorphic structure, but it shares many properties with the tt^* geometry. One can combine them to a structure which is called CDV-structure in this paper because of the papers [CV1][Du2][CV2]. It is defined below in definition 1.2.

One purpose of this paper is to construct without physics CDV-structures on the base spaces of semiuniversal unfoldings of all hypersurface singularities, not only quasihomogeneous ones, and to study their properties. This is done in chapters 6–8. It requires a careful analysis of the common and different properties of tt^* geometry and Frobenius manifolds and of the connections with poles which belong to them. This is carried out in chapters 2–5.

Both structures are studied first on abstract holomorphic vector bundles $K \to M$, and only later on the holomorphic tangent bundle TM. The tt^* geometry on an abstract bundle is called CV-structure, the part of a Frobenius manifold structure which makes sense is called a Frobenius type structure. CV-structures make the contact to the work of Simpson [Si1]–[Si4]. They are special cases of Simpson's harmonic bundles. In a harmonic bundle one has a hermitian metric, a Higgs field and the tt^*-equations, but not the real structure. Simpson defined them as a generalization of complex variations of Hodge structures and used them for his nonabelian Hodge theorem [Si3]. So his manifolds are not germs or small representatives of germs, but compact Kähler manifolds.

A common feature of CV-structures and Frobenius type structures is that there are two ways to describe them: in terms of geometry on a bundle $K \to M$ or in terms of a flat connection with poles and a pairing on the lifted tangent bundle π^*K, where $\pi : \mathbb{P}^1 \times M \to M$ is the projection.

The second point of view is most important. One can say that these connections with poles present the true geometry in CV-structures and Frobenius type structures. Several correspondences between connections with poles on π^*K and geometry on K are given in chapters 2–5. They are quite intricate because the structures on both sides are so rich. The correspondence for CV-structures is richer than, but closely related to Simpson's correspondence between harmonic bundles and variations of twistor structures [Si5][Sab6].

The similarity of CV-structures and Frobenius type structures is visible in the restriction of the connections to $\pi^*K|_{\mathbb{C}\times M}$, the difference in the extension to $\{\infty\}\times M$. On $\pi^*K|_{\mathbb{C}\times M}$ one has in both cases a flat connection with a pole of Poincaré rank 1 along $\{0\}\times M$ and a flat pairing P on $\pi^*K|_{\mathbb{C}^*\times M}$ with certain properties. In the case of a CV-structure the bundle $\pi^*K|_{\mathbb{C}^*\times M}$ has a real flat subbundle. This structure is called a $(TERP)$-structure, here 'T' stands for Twistor, 'E' for Extension of the connection in z-direction, so 'TE' for a pole of Poincaré rank 1, 'R' for Real, 'P' for Pairing.

Along $\{\infty\}\times M$ the meromorphic connection of a Frobenius type structure has a logarithmic pole, the connection for a CV-structure has an 'antiholomorphic twin' of a pole of Poincaré rank 1 (see definition 2.6 and theorem 2.19). There is freedom in the choice of the logarithmic pole, but the antiholomorphic twin at $\{\infty\}\times M$ of a pole of Poincaré rank 1 is obtained uniquely with the real structure from the pole at $\{0\}\times M$.

So $(TERP)$-structures are the common ferry to CV-structures and to Frobenius type structures. In singularity theory the Frobenius manifolds arise from $(TERP)$-structures which come essentially from oscillating integrals. The unique antiholomorphic extension of these $(TERP)$-structures thus induces a unique tt^* geometry on these Frobenius manifolds.

One may hope that $(TERP)$-structures and thus tt^* geometry also exist in quantum cohomology and in the Barannikov–Kontsevich construction of Frobenius manifolds.

In definition 1.2 CDV-structures are defined using the connection with poles. First, the definition of Frobenius manifolds is recalled.

Definition 1.1. A *Frobenius manifold* is a complex manifold M with a commutative and associative multiplication $\circ$ on the holomorphic tangent bundle TM, with a unit field e, an *Euler field* E, and a holomorphic metric g such that the following holds. Let $C:\mathcal{T}_M\to\Omega^1_M\otimes\mathcal{T}_M$ be the Higgs field with $C_XY:=-X\circ Y$. Let ∇^g be the Levi-Civita connection of the metric g.

Then ∇^g is flat; the potentiality condition $\nabla^g(C)=0$ holds; e is flat; the Euler field satisfies $\mathrm{Lie}_E(\circ)=\circ$ and $\mathrm{Lie}_E(g)=(2-d)\cdot g$ for some $d\in\mathbb{C}$; and

$$g(C_XY,Z)=g(Y,C_XZ)\quad\text{for } X,Y,Z\in\mathcal{T}_M\ , \tag{1.1}$$

that means, g is multiplication invariant.

A meromorphic connection is associated to a Frobenius manifold as follows. Define the endomorphisms $\mathcal{U}$ and $\mathcal{V}$ of the holomorphic tangent bundle TM,

$$\mathcal{U} := E\circ\ , \tag{1.2}$$

$$\mathcal{V} := \nabla^g_{\bullet}E-\frac{2-d}{2}\,\mathrm{id}=\nabla^g_E-\mathrm{Lie}_E-\frac{2-d}{2}\,\mathrm{id}\ , \tag{1.3}$$

and lift $\nabla^g,C,\mathcal{U},\mathcal{V}$ canonically to π^*TM, where $\pi:\mathbb{P}^1\times M\to M$. Then it is well known (e.g. lemma 5.11) that

$$\nabla:=\nabla^g+\frac{1}{z}C+(\frac{1}{z}\mathcal{U}-\mathcal{V})\frac{\mathrm{d}z}{z} \tag{1.4}$$

is a flat connection on $\pi^*TM|_{\mathbb{C}^*\times M}$. It is called (first) structure connection of the Frobenius manifold. The flatness of ∇^g encodes many properties in the definition of a Frobenius manifold.

Definition 1.2. A CDV-structure is a Frobenius manifold $(M, \circ, e, E, g)$ together with a real structure on TM given by a fiberwise $\mathbb{C}$-antilinear involution $\kappa : T_tM \to T_tM$ such that the following holds.

κ is an automorphism of TM as a real analytic real vector bundle. The form $h := g(\cdot, \kappa\cdot)$ is hermitian (but not necessarily positive definitive) and satisfies

$$\begin{aligned} h(C_X Y, Z) &= h(Y, \kappa C_X \kappa Z) \quad \text{for } X, Y, Z \in \mathcal{T}_M \ , && (1.5)\\ \mathrm{Lie}_e(h) &= 0 \ , && (1.6)\\ \mathrm{Lie}_{E-\overline{E}}(h) &= 0 \ . && (1.7)\end{aligned}$$

The metric connection D on TM for h respects κ, i.e. $D(h) = 0$ and $D(\kappa) = 0$. The number d with $\mathrm{Lie}_E(g) = (2 - d) \cdot g$ is real. Define an endomorphism $\mathcal{Q}$ of TM as real analytic complex vector bundle by

$$\mathcal{Q} := D_E - \mathrm{Lie}_E - \frac{2-d}{2}\,\mathrm{id} \ . \tag{1.8}$$

Lift D and $\mathcal{Q}$ canonically to π^*TM. Then

$$\nabla^{CV} := D + \frac{1}{z}C + z\kappa C\kappa + (\frac{1}{z}\mathcal{U} - \mathcal{Q} - z\kappa\mathcal{U}\kappa)\frac{\mathrm{d}z}{z} \tag{1.9}$$

is a flat non holomorphic connection on $\pi^*TM|_{\mathbb{C}^*\times M}$.

In [He4] Frobenius manifold structures on the base spaces M of semiuniversal unfoldings F of hypersurface singularities $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ were constructed with the Gauß–Manin connection, following K. Saito and M. Saito. In section 8.1 they are reconstructed using oscillating integrals and $(TERP)$-structures. A result of chapter 8 is the following.

Theorem 1.3. *Each of these Frobenius manifolds carries a canonical CDV-structure outside of a real analytic subvariety $R \subset M$. The set R is invariant under the flow of $E - \overline{E}$.*

A CDV-structure with positive definite hermitian metric gives on the subvariety $\{t \in M \mid \mathcal{U}|_t = 0\}$ a variation of polarized Hodge structures (chapter 3). In the singularity case this subvariety parametrizes the quasihomogeneous singularities. The μ-constant stratum is the larger set $\{t \in M \mid \mathcal{U}|_t \text{ is nilpotent }\}$. Here polarized mixed Hodge structures turn up. In chapter 7 the relations between CV-structures and polarized mixed Hodges structures are studied in general. In chapter 8 this is used to prove a part of the following conjecture.

Conjecture 1.4. *Let M be the base space of an unfolding F of a singularity with a CDV-structure from chapter 8. Let $t \in M$ be any point of M.*

If one starts at t and goes sufficiently far along the flow of the real vector field $E+\overline{E}$ then one does not meet R anymore, the metric h is positive definite and the eigenvalues of $\mathcal{Q}$ tend to the set $\mathrm{Exp}(F_t) - \frac{n+1}{2}$.

This conjecture is motivated by work in [CV1][CV2]. There the behaviour of a CV-structure with respect to the renormalization group flow is an important aspect. That flow corresponds to the vector field $E + \overline{E}$. The set $\mathrm{Exp}(F_t)$ is the union of the exponents (the spectral numbers $+1$) of the singularities of F_t; that is a tuple of μ numbers in $(0, n + 1) \cap \mathbb{Q}$ which are symmetric around $\frac{n+1}{2}$ and which come from Hodge filtrations and monodromies.

The endomorphism $\mathcal{Q}$ is perhaps the most fascinating piece of a CDV-structure. It is the 'new supersymmetric index' from [CFIV]. If h is positive definite then $\mathcal{Q}$ is a hermitian endomorphism, it is semisimple with real eigenvalues symmetric around 0. The eigenvalues vary real analytically on M. But on the set $\{t \in M \mid \mathcal{U}|_t = 0\}$ they are constant. There $e^{2\pi i \mathcal{Q}}$ is an automorphism of the variation of Hodge structures, and the eigenspaces of $\mathcal{Q}$ are subspaces of the Hodge decomposition. In the singularity case there the set of eigenvalues of $\mathcal{Q}$ is $\mathrm{Exp}(F_t) - \frac{n+1}{2}$. That fits to the conjecture, because the set $\{t \in M \mid \mathcal{U}|_t = 0\}$ is the fixed point set of E. The following is the main result of chapter 8.

Theorem 1.5. *Conjecture 1.4 is true for points $t \in M$ such that either $\mathcal{U}|_t$ is semisimple with μ different eigenvalues or $\mathcal{U}|_t$ is nilpotent.*

The semisimple case is a direct consequence of a result of Dubrovin [Du2][CV2] (cf. chapter 6). The nilpotent case uses polarized mixed Hodge structures and their relations to nilpotent orbits (theorem 7.5) and to Euler field orbits of $(TERP)$-structures (theorem 7.20). Associated to the $E + \overline{E}$ orbit of a point t in the μ-constant stratum is an orbit $e^{rN} F^\bullet(t)$ of Hodge like filtrations. Because the filtration $F(t)$ at t is part of a polarized mixed Hodge structure they become Hodge filtrations of polarized pure Hodge structures if one goes sufficiently far along $E + \overline{E}$. This implies with some rather long technical arguments (theorem 7.20) that there h is positive definite.

This paper grew out of the attempt to understand the physicists' results in [CV1][CV2] and other papers about singularities and to recover them. This is only a first step, much more is in these papers, and many wishes and questions are open.

The first part of the paper benefitted a lot from Dubrovin's paper [Du2]. The philosophy to generalize variations of Hodge structures by considering connections with poles of Poincaré rank 1 is also present in the work of Barannikov, the real structure is treated in [Ba2, Remark 5.3].

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2. tt^* geometry on abstract bundles

In section 2.2 some basic notions in the work of Simpson are recalled, Higgs bundles, λ-connections, and (T)-structures, which are families of λ-connections. In section 2.3 $(\widetilde{T})$-structures, the 'antiholomorphic twins' of (T)-structures, are defined. This allows to give in section 2.4 Simpson's notion of variation of twistor structures and to recover in lemma 2.11 the correspondence between them and harmonic bundles without metric ($(DC\widetilde{C})$-structures). In section 2.5 and section 2.6 the structures on both sides of the correspondence are enriched. The notion of $(TERP)$-structures is central in the whole paper. Finally, in section 2.7 the correspondence between these enriched structures is presented.

2.1. **Some notations.** Throughout the whole paper, M will denote a complex manifold of dimension m. The sheaf of holomorphic k-forms is Ω^k_M, the sheaf of C^∞ (p,q)-forms is $\mathcal{A}^{p,q}_M$, and $\mathcal{A}^k_M := \bigoplus_{p+q=k} \mathcal{A}^{p,q}_M$. Especially $\mathcal{O}_M = \Omega^0_M$ and $C^\infty_M = \mathcal{A}^0_M$. The sheaf of holomorphic vector fields is $\mathcal{T}_M$, the sheaf of C^∞-vector fields is $\mathcal{T}^{\mathbb{C}}_M = \mathcal{T}^{1,0}_M \oplus \mathcal{T}^{0,1}_M$, with $\mathcal{T}^{1,0}_M = \mathcal{T}_M \otimes_{\mathcal{O}_M} C^\infty_M$, and $\mathcal{T}^{0,1}_M = \overline{\mathcal{T}^{1,0}_M}$.

Vector fields denoted by latin letters (X,Y,Z) will always mean vector fields in $\mathcal{T}^{1,0}_M$, sometimes even in $\mathcal{T}_M$. For vector fields not necessarily of type $(1,0)$ greek letters will be used, $\xi, \xi_1, \xi_2 \in \mathcal{T}^{\mathbb{C}}_M$.

If not said otherwise, vector bundles will be complex C^∞-bundles, sometimes with additional structure (real analytic, holomorphic, antiholomorphic).

Let $K \to M$ be a vector bundle with sheaf $C^\infty(K)$ of C^∞-sections. A connection is a map $D : C^\infty(K) \to \mathcal{A}^1_M \otimes C^\infty(K)$ which satisfies the Leibniz rule, $D(a\cdot s) = \mathrm{d}a \otimes s + a \cdot Ds$ for $a \in C^\infty_M,\ s \in C^\infty(K)$. With the extension

$$D : \mathcal{A}^1_M \otimes C^\infty(K) \to \mathcal{A}^2_M \otimes C^\infty(K),\ \omega \otimes s \mapsto \mathrm{d}\omega \otimes s - \omega \wedge Ds$$

the curvature of D is the map $D^2 : C^\infty(K) \to \mathcal{A}^2_M \otimes C^\infty(K)$. The connection is flat if $D^2 = 0$.

A $(1,0)$-connection is a map $D' : C^\infty(K) \to \mathcal{A}^{1,0}_M \otimes C^\infty(K)$ which satisfies the Leibniz rule for $(1,0)$-vector fields, $D'(a\cdot s) = \partial a \otimes s + a \cdot D's$. It is extended to

$$D' : \mathcal{A}^1_M \otimes C^\infty(K) \to \mathcal{A}^2_M \otimes C^\infty(K),\ \omega \otimes s \mapsto \partial\omega \otimes s - \omega \wedge D's\ .$$

It is called flat if $D'^2 = 0$. A $(0,1)$-connection is defined analogously. A connection D is a sum $D = D^{(1,0)} + D^{(0,1)}$ of a $(1,0)$-connection and a $(0,1)$-connection.

Giving a holomorphic structure on $K \to M$ is equivalent to giving a flat $(0,1)$-connection: If a holomorphic structure with sheaf $\mathcal{O}(K)$ of holomorphic sections is given one extends the map $D'' : \mathcal{O}(K) \to 0$ with the Leibniz rule to a flat $(0,1)$-connection. If a flat $(0,1)$-connection D'' is given then, because of the Newlander-Nirenberg theorem, $\ker D''$ is the sheaf of holomorphic sections of a holomorphic structure on $K \to M$.

A C^∞_M-linear map $C : C^\infty(K) \to \mathcal{A}^1_M \otimes C^\infty(K)$ has the canonical extension

$$C : \mathcal{A}^1_M \otimes C^\infty(K) \to \mathcal{A}^2_M \otimes C^\infty(K),\ \omega \otimes s \mapsto -\omega \wedge Cs$$

For operators D, D', D'', C as above,the compositions $D^2,\ D'^2,\ D''^2,\ D'D''+D''D',\ D(C) := DC+CD,\ D'(C) := D'C+CD',\ D''(C) := D''C+CD''$ and C^2 are all maps from $C^\infty(K)$ to $\mathcal{A}^2_M \otimes C^\infty(K)$. Compatibility conditions will be written subsequently as vanishing of such conditions. Therefore it is useful to rewrite these compositions after inserting vector fields $X, Y \in \mathcal{T}^{1,0}_M$ or $\xi_1, \xi_2 \in \mathcal{T}^{\mathbb{C}}_M$. One easily sees

$$
\begin{aligned}
D^2(\xi_1,\xi_2) &= [D_{\xi_1}, D_{\xi_2}] - D_{[\xi_1,\xi_2]}\ , && (2.1)\\
D'^2(X,Y) &= [D'_X, D'_Y] - D'_{[X,Y]}\ , && (2.2)\\
D''^2(\overline{X},\overline{Y}) &= [D''_{\overline{X}}, D''_{\overline{Y}}] - D''_{[\overline{X},\overline{Y}]}\ , && (2.3)\\
(D'D''+D''D')(X,\overline{Y}) &= [D'_X, D''_{\overline{Y}}] - (D'+D'')_{[X,\overline{Y}]}\ , && (2.4)\\
D(C)(\xi_1,\xi_2) &= D_{\xi_1}(C_{\xi_2}) - D_{\xi_1}(C_{\xi_2}) - C_{[\xi_1,\xi_2]}\ , && (2.5)\\
D'(C)(X,Y) &= D'_X(C_Y) - D'_Y(C_X) - C_{[X,Y]}\ , && (2.6)\\
D''(C)(\overline{X},Y) &= D''_{\overline{X}}(C_Y) \quad \text{for } X,Y \in \mathcal{T}_M\ , && (2.7)\\
C^2(\xi_1,\xi_2) &= C_{\xi_1}C_{\xi_2} - C_{\xi_2}C_{\xi_1}\ , && (2.8)
\end{aligned}
$$

and if C and $\widetilde{C}$ are two C^∞_M-linear maps as above then

$$
(C\widetilde{C} + \widetilde{C}C)(\xi_1,\xi_2) = [C_{\xi_1}, \widetilde{C}_{\xi_2}] - [C_{\xi_2}, \widetilde{C}_{\xi_1}]\ . \qquad (2.9)
$$

2.2. Higgs bundles, λ-connections and (T)-structures.

Definition 2.1. [Si4] A *Higgs bundle* is a holomorphic vector bundle $K \to M$ together with an $\mathcal{O}_M$-linear map $C : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K)$ which satisfies

$$
C_X C_Y = C_Y C_X \quad \text{for } X, Y \in \mathcal{T}_M\ , \qquad (2.10)
$$

that is, $C^2 = 0$. The map C is called a *Higgs field.*

Equivalently, a Higgs bundle is a triple (K, C, D'') where $K \to M$ is a complex C^∞-vector bundle, C is a C^∞_M-linear map $C : C^\infty(K) \to \mathcal{A}^{1,0}_M \otimes C^\infty(K)$, and D'' is a $(0,1)$-connection, which satisfy $(D''+C)^2 = 0$, that is,

$$
D''^2 = 0,\ D''(C) = 0,\ C^2 = 0\ . \qquad (2.11)
$$

D'' gives a holomorphic structure on $K \to M$ with sheaf of holomorphic sections $\mathcal{O}(K)$, and $D''(C) = 0$ says that $C_X : \mathcal{O}(K) \to \mathcal{O}(K)$ for $X \in \mathcal{T}_M$.

The notion of a λ-connection generalizes the notions of a holomorphic connection and of a Higgs bundle. It is due to Deligne [Si3].

Definition 2.2. Fix $\lambda \in \mathbb{C}$ and a holomorphic vector bundle $K \to M$. A *λ-connection* is a map $D : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K)$ which satisfies $D(a\cdot s) = \lambda \mathrm{d}a \otimes s + a \cdot Ds$. It has the natural extension

$$
D : \Omega^1_M \otimes \mathcal{O}(K) \to \Omega^2_M \otimes \mathcal{O}(K),\ \omega \otimes s \mapsto \lambda \mathrm{d}\omega \otimes s - \omega \wedge Ds\ . \qquad (2.12)
$$

A λ-connection is *flat* if $D^2 = 0$.

For $\lambda \neq 0$, D is a (flat) λ-connection if and only if $\frac{1}{\lambda}D$ is a (flat) holomorphic connection. A Higgs bundle is a flat 0-connection.

A holomorphic family of flat λ-connections, $\lambda \in \mathbb{C}$, is equivalent to the following structure. It will be a basic playing character and will be equipped with additional structure in section 2.5.

Definition 2.3. Let $H \to \mathbb{C} \times M$ be a holomorphic vector bundle. A (T)-structure is a pair (H, ∇). Here ∇ is a map

$$\nabla : \mathcal{O}(H) \to \frac{1}{z}\mathcal{O}_{\mathbb{C}\times M} \cdot \Omega^1_M \otimes \mathcal{O}(H) \tag{2.13}$$

such that for any $z \in \mathbb{C}^*$ the restriction of D to $H|_{\{z\}\times M}$ is a flat connection ('T' as in 'Twistor' [Si5] or in 'Topological' [CV1]).

Lemma 2.4. *Let (H, ∇) be a (T)-structure. Define $K := H|_{\{0\}\times M}$. The map $z \cdot \nabla : \mathcal{O}(H) \to \mathcal{O}_{\mathbb{C}\times M} \cdot \Omega^1_M \otimes \mathcal{O}(H)$ restricts to a flat z-connection $\nabla^{(z)}$ on $H|_{\{z\}\times M}$ for any $z \in \mathbb{C}$ (including $z = 0$). The holomorphic bundle K together with $C := \nabla^{(0)}$ is a Higgs bundle, the Higgs bundle of the (T)-structure. In view of $\mathcal{O}(K) = \mathcal{O}(H)/z\mathcal{O}(H)|_{\{0\}\times M}$ one has for $X \in \mathcal{T}_M,\ a \in \mathcal{O}(H), [a] \in \mathcal{O}(K)$*

$$C_X([a]) = (z \cdot \nabla_X(a)) \mod z\mathcal{O}(H)\ . \tag{2.14}$$

Proof. trivial. □

Remarks 2.5. (i) The support of a Higgs bundle $(K \to M, C)$ is the set

$$\bigcup_{t\in M} \{\lambda \in T^*_t M \mid \forall\ X \in T_t M\ \ \ker(C_X - \lambda(X)\,\mathrm{id} : T_t M \to T_t M) \neq 0\}$$

of simultaneous eigenvalues of all C_X at all $t \in M$. It is a subvariety of dimension m in the holomorphic cotangent bundle T^*M, and it is finite over M.

(ii) The Higgs bundle of a (T)-structure has the special property that its support is Lagrange in T^*M. This follows from [Sab6, Proposition 1.25] and the fact that in the notation of [Sab6] a (T)-structure is a locally free $\mathcal{O}_{\mathcal{X}}$-module and a strict holonomic $\mathcal{R}_{\mathcal{X}}$-module.

(iii) Usually a (T)-structure turns up as part of a meromorphic connection ∇ on H with a pole of Poincaré rank 1 along $\{0\} \times M$, see definition 2.12.

(iv) In (2.13) Ω^1_M is written for $\pi^{-1}\Omega^1_M$, where $\pi : \mathbb{C} \times M \to M$. Similar notations will be used in the formulas (2.15), (2.16), (2.29). Because of these formulas we write $\mathcal{O}_{\mathbb{C}\times M} \cdot \Omega^1_M$ instead of $\pi^*\Omega^1_M$ or $\Omega^1_{\mathbb{C}\times M/M}$.

2.3. **C^∞-considerations.** In order to define an antiholomorphic twin of a (T)-structure, one has to start with a mixture of C^∞ and holomorphic data.

For $U \subset \mathbb{P}^1$ open let $\mathcal{O}_U C^\infty_M$ be the sheaf of C^∞-functions on $U \times M$ which are holomorphic in z-direction (z is the coordinate on $\mathbb{C}$). A complex C^∞-vector bundle $H \to U \times M$ is said to carry a $C^{h\infty}$ structure if in the sheaf $C^\infty(H)$ of all C^∞-sections a subsheaf $C^{h\infty}(H)$ is fixed which is a locally free $\mathcal{O}_U C^\infty_M$-module of rank $\mathrm{rk}\, H$. Such a bundle H can be seen as a C^∞-family over M of holomorphic vector bundles on U.

It will also be useful to consider everything with C^∞ replaced by real analytic; we will simply replace C^∞ and $C^{h\infty}$ by C^ω and $C^{h\omega}$.

Now a (T)-structure is a $C^{h\infty}$-vector bundle $H \to \mathbb{C} \times M$ with a map

$$\nabla : C^{h\infty}(H) \to (\frac{1}{z}\mathcal{O}_{\mathbb{C}}C_M^\infty \cdot \mathcal{A}_M^{1,0} + \mathcal{O}_{\mathbb{C}}C_M^\infty \cdot \mathcal{A}_M^{0,1}) \otimes C^{h\infty}(H) \tag{2.15}$$

such that for any $z \in \mathbb{C}^*$ the restriction of ∇ to $H|_{\{z\}\times M}$ is a flat connection. The $(0,1)$-part of ∇ together with the $C^{h\infty}$-structure on H induces a holomorphic structure on H. An antiholomorphic twin of a (T)-structure is defined as follows.

Definition 2.6. A $(\widetilde{T})$-*structure* is a pair $(\widetilde{H}, \widetilde{\nabla})$. Here $\widetilde{H} \to (\mathbb{P}^1 - \{0\}) \times M$ is a $C^{h\infty}$-vector bundle and $\widetilde{\nabla}$ is a map

$$\widetilde{\nabla} : C^{h\infty}(H) \to (\mathcal{O}_{\mathbb{P}^1-\{0\}}C_M^\infty \cdot \mathcal{A}_M^{1,0} + z \cdot \mathcal{O}_{\mathbb{P}^1-\{0\}}C_M^\infty \cdot \mathcal{A}_M^{0,1}) \otimes C^{h\infty}(H) \tag{2.16}$$

such that for any $z \in \mathbb{C}^*$ the restriction of $\widetilde{\nabla}$ to $H|_{\{z\}\times M}$ is a flat connection.

There is a natural correspondence between (T)-structures and $(\widetilde{T})$-structures. One needs two operations, the $\mathbb{C}$-antilinear map

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1,\ z \mapsto \frac{1}{\overline{z}}\ , \tag{2.17}$$

and complex conjugation of vector bundles. The complex conjugate $\overline{H} \to U \times M$ of a vector bundle $H \to U \times M$ is the same as H except that the fibers are equipped with the complex conjugate $\mathbb{C}$-linear structure.

Lemma 2.7. *If (H, ∇) is a (T)-structure (respectively a $(\widetilde{T})$-structure) then $\overline{\gamma^* H} =: \widetilde{H}$ carries a canonical $(\widetilde{T})$-structure (respectively a (T)-structure). This gives a one-to-one correspondence between (T)-structures and $(\widetilde{T})$-structures.*

Proof. The equation $\overline{\gamma^*(\frac{1}{z}\mathcal{O}_{\mathbb{C}}C_M^\infty)} = z\mathcal{O}_{\mathbb{P}^1-\{0\}}C_M^\infty$ shows that $\widetilde{H}$ is a $C^{h\infty}$-vector bundle with $C^{h\infty}(\widetilde{H}) = \overline{\gamma^*(C^{h\infty}(H))}$ and that the induced map $\widetilde{\nabla}$ on $C^{h\infty}(\widetilde{H})$ satisfies (2.16). □

2.4. **Variation of twistor structures.** Simpson [Si5, ch. 3] defined the notion of a variation of twistor structures. Sabbah called a variation of twistor structures a smooth twistor structure [Sab6, 2.2.a] and greatly generalized this notion.

Definition 2.8. (a) A $(T\widetilde{T})$-*structure* is a $C^{h\infty}$-vector bundle $H \to \mathbb{P}^1 \times M$ and a map ∇ such that $(H, \nabla)|_{\mathbb{C}\times M}$ is a (T)-structure and $(H, \nabla)|_{(\mathbb{P}^1-\{0\})\times M}$ is a $(\widetilde{T})$ structure.

(b) [Si5, ch. 3][Sab6, 2.2.a] A *variation of twistor structures* is a $(T\widetilde{T})$-structure such that for each $t \in M$ the restriction $H|_{\mathbb{P}^1\times\{t\}}$ is a trivial bundle. We call it also a $(trT\widetilde{T})$-structure ('tr' for trivial).

A (T)-structure (H, ∇) equips $K := H|_{\{0\}\times M}$ with the structure of a Higgs bundle, that is, a flat $(0,1)$-connection D'' and a C_M^∞-linear map

$$C : C^\infty(K) \to \mathcal{A}_M^{1,0} \otimes C^\infty(K) \tag{2.18}$$

with $(D''+C)^2 = 0$. A $(\widetilde{T})$-structure (H, ∇) equips $H|_{\{\infty\}\times M}$ with the antiholomorphic twin of a Higgs bundle, which is by definition a flat $(1,0)$-connection D' and a C^∞_M-linear map

$$\widetilde{C} : C^\infty(H|_{\{\infty\}\times M}) \to \mathcal{A}^{0,1}_M \otimes C^\infty(H|_{\{\infty\}\times M}) \tag{2.19}$$

with $(D' + \widetilde{C})^2 = 0$. Here $\widetilde{C} := \lim_{z\to\infty}(\frac{1}{z}\nabla)|_{\{z\}\times M}$. In the case of a $(trT\widetilde{T})$-structure K and $H|_{\{\infty\}\times M}$ are canonically isomorphic and thus all four operators D'', C, D', $\widetilde{C}$ live on K. Their properties are stated in definition 2.9 and lemma 2.11.

Definition 2.9. A $(DC\widetilde{C})$-structure is a C^∞ vector bundle $K \to M$ together with a connection D on it and two C^∞_M-linear maps

$$C : C^\infty(K) \to \mathcal{A}^{1,0}_M \otimes C^\infty(K) \tag{2.20}$$
$$\widetilde{C} : C^\infty(K) \to \mathcal{A}^{0,1}_M \otimes C^\infty(K) \tag{2.21}$$

with the following properties. Let D' and D'' be the $(1,0)$-part and the $(0,1)$-part of D. Then

$$(D''+C)^2 = 0, \quad (D'+\widetilde{C})^2 = 0, \tag{2.22}$$
$$D'(C) = 0, \quad D''(\widetilde{C}) = 0, \tag{2.23}$$
$$D'D'' + D''D' = -(C\widetilde{C} + \widetilde{C}C)\ . \tag{2.24}$$

Remarks 2.10. (i) In formula (2.11) it was already stated that $(D''+C)^2 = 0$ decomposes into $D''^2 = 0$, $D''(C) = 0$, $C^2 = 0$. An analogous statement holds for $(D' + \widetilde{C})^2 = 0$.

(ii) The equations (2.23) and (2.24) are called tt^*-equations in [CV1]. The equation $D'(C) = 0$ says (cf. (2.6))

$$D'_X(C_Y) - D'_Y(C_X) = C_{[X,Y]} \quad \text{for } X, Y \in \mathcal{T}^{1,0}_M\ . \tag{2.25}$$

We call $D'(C) = 0$ potentiality condition, because it turns up in the case of a Frobenius manifold as one formulation of potentiality. The curvature condition (2.24) can be written as (cf. (2.4) and(2.9))

$$[D'_X, D''_{\overline{Y}}] - (D' + D'')_{[X,\overline{Y}]} = -[C_X, \widetilde{C}_{\overline{Y}}] \quad \text{for } X, Y \in \mathcal{T}^{1,0}_M\ . \tag{2.26}$$

Lemma 2.11. [Si5, Lemma 3.1][Sab6, Lemma 2.2.2] *There is a one-to-one correspondence between variations of twistor structures (i.e. $(trT\widetilde{T})$-structures) and $(DC\widetilde{C})$-structures. It is given by the steps in (a) and (b). They are inverse to one another.*

(a) Let $(K \to M, D, C, \widetilde{C})$ be a $(DC\widetilde{C})$-structure. Let $\pi : \mathbb{P}^1 \times M \to M$ be the projection. The lifted bundle $H := \pi^*K$ on $\mathbb{P}^1 \times M$ is a $C^{h\infty}$-bundle. The operators $D, C, \widetilde{C}$ are lifted canonically to H. Then the bundle H and the operator

$$\nabla := D + \frac{1}{z}C + z\widetilde{C} \tag{2.27}$$

form a variation of twistor structure.

(b) In the case of a variation of twistor structures (H,∇) the four operators $D'',C,D',\widetilde{C}$ on $K:=H|_{\{0\}\times M}$ defined before definition 2.9 yield a $(DC\widetilde{C})$-structure.

Proof. (a) The only thing to be shown is $\nabla^2=0$. This is equivalent to (2.22)–(2.24).

(b) The operators $D'',C,D',\widetilde{C}$ are lifted canonically to H, using the canonical isomorphism $\pi^*K\to H$, respectively the isomorphism $C^\infty(K)\cong \pi_*C^{h\infty}(H)$. One has to show

$$\nabla = D'+D''+\frac{1}{z}C+z\widetilde{C}\ . \tag{2.28}$$

Consider the operator $\nabla_X-(D'_X+D''_X+\frac{1}{z}C_X+z\widetilde{C}_X)=\nabla_X-D'_X-\frac{1}{z}C_X$ for $X\in\mathcal{T}^{1,0}_M$. It maps sections in $\pi_*C^{h\infty}(H)$ to $C^{h\infty}$-sections which have no pole along $\{0\}\times M$ because of the definition of C and which vanish along $\{\infty\}\times M$ because of the definition of D'. Therefore they vanish globally. An analogous statement holds for $\overline{X}$. Therefore (2.28) holds. The flatness $\nabla^2=0$ gives the conditions (2.22)–(2.24). □

2.5. $(TERP)$**-structures.** We will now equip both sides in the correspondence in lemma 2.11 with additional structures. In definition 2.12 a (T)-structure will be enriched to a $(TERP(w))$-structure. Here ‘E’ stands for Extension of ∇ in z-direction, ‘R’ stands for Real structure, ‘P’ stands for Pairing, and w will be an integer.

Such a structure induces a $(T\widetilde{T})$-structure and additional geometry on $K:=H|_{\{0\}\times M}$ (lemma 2.14). If the $(T\widetilde{T})$-structure is a $(trT\widetilde{T})$-structure then one obtains a rich geometric structure on K, a CV-structure (definition 2.16). Theorem 2.19 extends lemma 2.11 to an equivalence between CV-structures and $(trTERP(w))$-structures.

They are really two sides of the same coin. But the formulation of the properties of real structure and pairings on both sides is rather lengthy, and passing from one side to the other side is nontrivial.

$(TERP(w))$-structures turn up in [CV1][CFIV][CV2] and in singularity theory (see chapter 8). The discussion here and the equivalence in theorem 2.19 are obtained by unifying considerations in [CV1][CFIV][Du2][Si4][Si5].

Definition 2.12. A $(TERP(w))$-structure is a tuple $(H,\nabla,H_{\mathbb{R}},P)$ with the following properties.

(a) The space H is a holomorphic vector bundle $H\to\mathbb{C}\times M$.

(b) It is equipped with a flat holomorphic connection ∇ on $H|_{\mathbb{C}^*\times M}$ with a pole of Poincaré rank 1 along $\{0\}\times M$, that is,

$$\begin{aligned}\nabla:\mathcal{O}(H) &\to (\frac{1}{z}\mathcal{O}_{\mathbb{C}\times M}\cdot\Omega^1_M+\frac{1}{z}\mathcal{O}_{\mathbb{C}\times M}\frac{\mathrm{d}z}{z})\otimes\mathcal{O}(H) \\ &= \frac{1}{z}\Omega^1_{\mathbb{C}\times M}(\log(\{0\}\times M))\otimes\mathcal{O}(H)\end{aligned} \tag{2.29}$$

and $\nabla^2=0$.

(c) The bundle $H|_{\mathbb{C}^*\times M}$ contains a real bundle $H_{\mathbb{R}}\to\mathbb{C}^*\times M$ which is ∇-flat and satisfies $H_{(z,t)}=(H_{\mathbb{R}})_{(z,t)}\oplus i(H_{\mathbb{R}})_{(z,t)}$.

(d) $w \in \mathbb{Z}$. There is a $\mathbb{C}$-bilinear pairing

$$P : H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C} \quad \text{for any } (z,t) \in \mathbb{C}^* \times M \tag{2.30}$$

with the following properties. It is nondegenerate and $(-1)^w$-symmetric (symmetric for even w, antisymmetric for odd w). It extends for an open subset $U_1 \times U_2 \subset \mathbb{C} \times M$ to a nondegenerate pairing

$$\begin{aligned} P : \mathcal{O}(H)(U_1 \times U_2) \times \mathcal{O}(H)((-U_1) \times U_2) &\to z^w \mathcal{O}_{\mathbb{C}\times M}(U_1 \times U_2) \\ (a,b) &\mapsto ((z,t) \mapsto P(a(z,t), b(-z,t))) \ . \end{aligned} \tag{2.31}$$

Thus, P is $\mathcal{O}_M$-bilinear, but z-sesquilinear,

$$f(z,t)P(a,b) = P(f(z,t)a,b) = P(a, f(-z,t)b) \quad \text{for } f \in \mathcal{O}_{\mathbb{C}\times M} \ . \tag{2.32}$$

It is ∇-flat, that is, $\nabla(P) = 0$, or explicitely for $X \in \mathcal{T}_M(U_2),\ a \in \mathcal{O}(H)(U_1 \times U_2),\ b \in \mathcal{O}(H)((-U_1) \times U_2)$,

$$X\, P(a,b) = P(\nabla_X a, b) + P(a, \nabla_X b) \ , \tag{2.33}$$

$$z\partial_z\, P(a,b) = P(\nabla_{z\partial_z} a, b) + P(a, \nabla_{z\partial_z} b) \ . \tag{2.34}$$

The pairing P takes values in $i^w\mathbb{R}$ on $H_{\mathbb{R}}$,

$$P : (H_{\mathbb{R}})_{(z,t)} \times (H_{\mathbb{R}})_{(-z,t)} \to i^w\mathbb{R} \ . \tag{2.35}$$

Remarks 2.13. (i) The definition of weaker structures is obvious, for example
(TE): (H,∇) with (a) and (b),
(TER): $(H,\nabla,H_{\mathbb{R}})$ with (a), (b), and (c),
$(TEP(w)$: (H,∇,P) with (a), (b), (d) except for (2.35),
$(TP(w))$: a (T)-structure (H,∇) with a pairing P as in (d) except for (2.34), (2.35).

In [Si5] and [Sab6] (T)-structures with hermitian pairings $H_{(z,t)} \times H_{(-\gamma(z),t)} \to \mathbb{C}$ for $(z,t) \in \mathbb{C}^* \times M$ are considered ($\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ was defined in (2.17)). Here such a pairing arises as $P(\cdot, \tau\cdot)$ for τ as in lemma 2.14 (d).

(ii) In chapter 8 $(TERP(w))$-structures will be discussed which come from oscillating integrals for isolated singularities. $(TERP(w))$-structures turn also up in [CV1]. The first structure connections of any holomorphic Frobenius manifold (without Euler field) contain $(TEP(w))$-structures (respectively $(TP(w))$-structures), see section 5.3.

Establishing such structures often is the first and most difficult part in the construction of Frobenius manifolds. In quantum cohomology and in the Barannikov–Kontsevich construction in most cases only formal versions of $(TEP(w))$-structures are established. But one may expect that they are convergent and carry real structures as in (c).

Lemma 2.14. *Let $(H,\nabla,H_{\mathbb{R}},P)$ be a $(TERP(w))$-structure. Then $K := H|_{\{0\}\times M}$ together with the tensor C in lemma 2.4 is a Higgs bundle. There is additional structure with the following properties.*

(a) There is a symmetric nondegenerate holomorphic pairing g on K,

$$\begin{aligned} g &: K_t \times K_t \to \mathbb{C} \quad \textit{for } t \in M \ , \\ g &: \mathcal{O}(K) \times \mathcal{O}(K) \to \mathcal{O}_M \quad \mathcal{O}_M\textit{-bilinear} \ , \end{aligned} \tag{2.36}$$

defined by

$$g([a],[b]) := (z^{-w}P(a,b)) \mod z\mathcal{O}(H) \tag{2.37}$$

for $[a],[b]\in\mathcal{O}(K)$ and lifts $a,b\in\mathcal{O}(H)$ (in suitable open sets). It satisfies

$$g(C_X[a],[b]) = g([a],C_X[b]) \quad \text{for } X\in\mathcal{T}_M\ . \tag{2.38}$$

(b) There is a holomorphic endomorphism $\mathcal{U}$ of K,

$$\begin{aligned} \mathcal{U} &: K_t\to K_t \quad \text{for } t\in\mathcal{U}\ ,\\ \mathcal{U} &: \mathcal{O}(K)\to\mathcal{O}(K)\quad \mathcal{O}_M\text{-linear}\ , \end{aligned} \tag{2.39}$$

defined by

$$\mathcal{U}([a]) := (z^2\nabla_{\partial_z}(a)) \mod z\mathcal{O}(H) \tag{2.40}$$

for $[a]\in\mathcal{O}(K),\ a\in\mathcal{O}(H)$. It satisfies

$$\mathcal{U}C_X = C_X\mathcal{U} \quad \text{for } X\in\mathcal{T}_M\ , \tag{2.41}$$

$$g(\mathcal{U}[a],[b]) = g([a],\mathcal{U}[b]) \quad \text{for } [a],[b]\in\mathcal{O}(K)\ . \tag{2.42}$$

(c) Let $\kappa_H : H|_{\mathbb{C}^*\times M}\to H|_{\mathbb{C}^*\times M}$ be the fiberwise $\mathbb{C}$-antilinear involution with $\kappa_H|_{H_\mathbb{R}}=\mathrm{id}$. The points $z\in\mathbb{C}^*$ and $\gamma(z):=\frac{1}{\overline{z}}$ (cf. (2.17)) are contained in the real halfline $\{\zeta\in\mathbb{C}^* \mid \arg\zeta=\arg z\}$. For $(z,t)\in\mathbb{C}^*\times M$ let $\gamma_\nabla : H_{(z,t)}\to H_{(\gamma(z),t)}$ be the isomorphism which one obtains from the ∇-flat shift along this half line. Then the map

$$\tau_{real} := \kappa_H\circ\gamma_\nabla = \gamma_\nabla\circ\kappa_H : H_{(z,t)}\to H_{(\gamma(z),t)} \quad \text{for } (z,t)\in\mathbb{C}^*\times M \tag{2.43}$$

is $\mathbb{C}$-antilinear and ∇-flat and satisfies $\tau_{real}^2=\mathrm{id}$ and $\tau_{real}(z\cdot a)=\frac{1}{\overline{z}}\tau_{real}(a)$ (this last equation has to be understood with z as a function on $\mathbb{C}^*$, not pointwise).

(d) The map

$$\begin{aligned} \tau : H_{(z,t)} &\to H_{(\gamma(z),t)} \quad \text{for } (z,t)\in\mathbb{C}^*\times M\\ a &\mapsto \tau_{real}(z^{-w}a) \end{aligned} \tag{2.44}$$

is $\mathbb{C}$-antilinear and satisfies $\tau^2=\mathrm{id}$, $\tau(z\cdot a)=\frac{1}{\overline{z}}\tau(a)$, and

$$\nabla_\xi\circ\tau = \tau\circ\nabla_{\overline{\xi}} \text{ and } \nabla_{z\partial_z}\circ\tau = \tau\circ(\nabla_{-z\partial_z}+w\cdot\mathrm{id})\ . \tag{2.45}$$

It is an isomorphism

$$\tau : H|_{\mathbb{C}^*\times M}\to\overline{(\gamma^*H)|_{\mathbb{C}^*\times M}}\ , \tag{2.46}$$

which respects the $C^{h\infty}$-structures and the restrictions of the flat connections to the slices $\{z\}\times M$ for $z\in\mathbb{C}^*$.

(e) The bundles H and $\overline{\gamma^*H}$ are glued with τ to a bundle $\hat{H}$. Then $(\hat{H},\nabla)$ is a $(T\widetilde{T})$-structure.

Proof. (a) The pairing g is symmetric because P is $(-1)^w$-symmetric and z-sesquilinear. The pairing g is nondegenerate and holomorphic because of (2.31). It satisfies (2.38), because for $X\in\mathcal{T}_M,\ a,b,\in\mathcal{O}(H),\ [a],[b]\in\mathcal{O}(K)$ one has

$$\begin{aligned} 0 &= zX\, z^{-w}P(a,b) \mod z\mathcal{O}_{\mathbb{C}\times M}\\ &= z^{-w}P(z\nabla_X a,b) - z^{-w}P(a,z\nabla_X b) \mod z\mathcal{O}_{\mathbb{C}\times M}\\ &= g(C_X[a],[b]) - g([a],C_X[b])\ . \end{aligned} \tag{2.47}$$

(b) One has for $X \in \mathcal{T}_M,\ a, b, \in \mathcal{O}(H),\ [a], [b] \in \mathcal{O}(K)$

$$\begin{aligned} 0 &= ([\nabla_{z^2\partial_z}, \nabla_{zX}] - \nabla_{z^2X})a \\ &= [\nabla_{z^2\partial_z}, \nabla_{zX}]a \mod z\mathcal{O}(H) \\ &= (\mathcal{U}C_X - C_X\mathcal{U})[a] \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} 0 &= z^2\partial_z\, z^{-w}P(a,b) \mod z\mathcal{O}_{\mathbb{C}\times M} \\ &= (-w)z\cdot z^{-w}P(a,b) + z^{-w}P(\nabla_{z^2\partial_z}a, b) - z^{-w}P(a, \nabla_{z^2\partial_z}b) \\ &\quad \mod z\mathcal{O}_{\mathbb{C}\times M} \\ &= g(\mathcal{U}[a],[b]) - g([a],\mathcal{U}[b])\ . \end{aligned} \tag{2.49}$$

(c) is obvious.

(d) Because of $\overline{\gamma}(z) = \frac{1}{z}$ and $\overline{\gamma}^*(\frac{dz}{z}) = -\frac{dz}{z}$ the flatness $\nabla(\tau_{real}) = 0$ means $\nabla_\xi \circ \tau_{real} = \tau_{real} \circ \nabla_{\overline{\xi}}$ and $\nabla_{z\partial_z} \circ \tau_{real} = \tau_{real} \circ \nabla_{-z\partial_z}$. The rest is obvious.

(e) Use lemma 2.7. □

Definition 2.15. A $(trTERP(w))$-structure is a $(TERP(w))$-structure such that the $(T\widetilde{T})$-structure defined in lemma 2.14 (e) is a $(trT\widetilde{T})$-structure.

A $(trTERP(w))$-structure gives rise to a harmonic bundle with additional structure. This structure is discussed in the next section, the correspondence is presented in section 2.7.

2.6. **CV-structures.**

The structures in [CV1] motivated the following definition.

Definition 2.16. A CV-structure is a tuple $(K \to M, D, C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ such that $(K \to M, D, C, \widetilde{C})$ is a $(DC\widetilde{C})$-structure (definition 2.9) and the other objects have the following properties.

(a) κ is a fiberwise $\mathbb{C}$-antilinear automorphism of K as C^∞-bundle with

$$\kappa^2 = \mathrm{id}\ , \tag{2.50}$$

$$D(\kappa) = 0\ , \tag{2.51}$$

$$\kappa C \kappa = \widetilde{C}\ . \tag{2.52}$$

(b) h is a hermitian pseudo-metric on K; that means, it is linear on the left, semilinear on the right, nondegenerate and satisfies $h(b,a) = \overline{h(a,b)}$. It also has the three properties:
it takes real values on the real subbundle $K_{\mathbb{R}} := \ker(\kappa - \mathrm{id}) \subset K$;

$$D(h) = 0\ ; \tag{2.53}$$

$$h(C_Xa, b) = h(a, \widetilde{C}_{\overline{X}}b) \tag{2.54}$$

for $a, b \in C^\infty(K),\ X \in \mathcal{T}_M^{1,0}$.

(c) $\mathcal{U}$ and $\mathcal{Q}$ are C_M^∞-linear endomorphisms of K with

$$[C,\mathcal{U}] = 0 \ , \tag{2.55}$$
$$D'(\mathcal{U}) - [C,\mathcal{Q}] + C = 0 \ , \tag{2.56}$$
$$D''(\mathcal{U}) = 0 \ , \tag{2.57}$$
$$D'(\mathcal{Q}) + [C, \kappa\mathcal{U}\kappa] = 0 \ , \tag{2.58}$$
$$\mathcal{Q} + \kappa\mathcal{Q}\kappa = 0 \ , \tag{2.59}$$
$$h(\mathcal{U}a,b) = h(a,\kappa\mathcal{U}\kappa b) \ , \tag{2.60}$$
$$h(\mathcal{Q}a,b) = h(a,\mathcal{Q}b) \ . \tag{2.61}$$

Remarks 2.17. (i) If the hermitian pseudo-metric h is positive definite, we call the CV-structure a CV$\oplus$-structure.

(ii) A CV-structure on $K \to M$ is given by the data $(DC\widetilde{C}\kappa h\mathcal{U}\mathcal{Q})$. One can easily define weaker structures, for example with the data $(DC\widetilde{C}\kappa)$, $(DC\widetilde{C}h)$, $(DC\widetilde{C}\kappa h)$, $(DC\widetilde{C}\kappa\mathcal{U}\mathcal{Q})$ and with those properties which are formulated in these data. Then Simpson's notion of a harmonic bundle [Si4] is a $(DC\widetilde{C}h)$-structure with positive definite h.

(iii) κ yields a real structure on K, a real subbundle $K_\mathbb{R} := \ker(\kappa - \mathrm{id})$ with

$$K = K_\mathbb{R} \oplus iK_\mathbb{R} \quad \text{and} \quad iK_\mathbb{R} = \ker(\kappa + \mathrm{id}) \ . \tag{2.62}$$

(iv) κ and h induce a C_M^∞-bilinear nondegenerate pairing $g := h(\cdot, \kappa\cdot)$ on K. It satisfies $D(g) = 0$. It is symmetric because h takes real values on $K_\mathbb{R}$. These two properties are equivalent.

(v) The pairing g satisfies

$$\begin{aligned}
g(C_X a, b) &= g(a, C_X b) \quad \text{for } X \in \mathcal{T}_M^{1,0} \ , & (2.63)\\
g(\widetilde{C}_{\overline{X}} a, b) &= g(a, \widetilde{C}_{\overline{X}} b) \quad \text{for } X \in \mathcal{T}_M^{1,0} \ , & (2.64)\\
g(\mathcal{U}a, b) &= g(a, \mathcal{U}b) \ , & (2.65)\\
g(\kappa\mathcal{U}\kappa a, b) &= g(a, \kappa\mathcal{U}\kappa b) \ , & (2.66)\\
g(\mathcal{Q}a, b) &= -g(a, \mathcal{Q}b) \ . & (2.67)
\end{aligned}$$

Two of the three properties (2.52), (2.54), (2.63) imply the third. Two of the three properties (2.59), (2.61), (2.67) imply the third.

(vi) With (2.52) one recovers $\widetilde{C}$ from C and κ, with (2.54) one recovers $\widetilde{C}$ from C and h. With (2.53) one recovers D as the metric connection from h and from the holomorphic structure $\mathcal{O}(K)$ defined by D''.

(vii) Let C_M^ω be the sheaf of real analytic functions on M and define the real analytic structure on K with sheaf of sections $C^\omega(K) := \mathcal{O}(K) \otimes_{\mathcal{O}_M} C_M^\omega$. It turns out that the whole CV-structure is automatically compatible with this real analytic structure, see lemma 2.18 (a). But regarding the weaker structures in (ii), this property is only clear for $(DC\widetilde{C}\kappa\mathcal{U}\mathcal{Q})$. Therefore definition 2.16 was not given rightaway in the real analytic category.

(viii) The endomorphism $\mathcal{Q}$ turns up in [CFIV] as a 'new supersymmetric index'. In the singularity case its real analytically varying eigenvalues are generalizations of the (shifted) spectral numbers. This will be discussed in chapter 8. In general a CV-structure is a grand generalization of a variation of

Hodge structures, see chapter 3, and the eigenspaces of $\mathcal{Q}$ are generalizations of the subspaces of the Hodge decomposition.

(ix) Often there exists a global vector field $E \in \mathcal{T}_M$ with $\mathcal{U} = -C_E$. Then (2.52), (2.58) and (2.59) show

$$D'_E(\mathcal{Q}) = D''_{\overline{E}}(\mathcal{Q}) = [\mathcal{U}, \kappa\mathcal{U}\kappa] \text{ and } D_{E-\overline{E}}(\mathcal{Q}) = 0 . \tag{2.68}$$

When a CV-structure comes together with a Frobenius manifold (see section 5.4) the Euler field has this property. Then the behaviour of $\mathcal{Q}$ along $E+\overline{E}$ is very interesting.

Lemma 2.18. *Consider a CV-structure.*

(a) The tensors $\widetilde{C}, \kappa, h, \mathcal{Q}$ and the connection D respect the real analytic structure $C^\omega(K)$. The other tensors $C, g, \mathcal{U}$ respect the holomorphic structure $\mathcal{O}(K)$. Therefore a CV-structure is a real analytic structure.

(b) Suppose that h is positive definite. Then $\mathcal{Q}$ is a hermitian endomorphism, it is semisimple with (real analytically varying) real eigenvalues which are distributed symmetrically around 0. Define $K_{t,\lambda} := \ker(\mathcal{Q} - \lambda \operatorname{id} : K_t \to K_t)$ for $\lambda \in \mathbb{R}$. Then

$$\kappa : K_{t,\lambda} \to K_{t,-\lambda} , \tag{2.69}$$
$$h(K_{t,\lambda}, K_{t,\lambda'}) = 0 \quad \text{for } \lambda \neq \lambda' , \tag{2.70}$$
$$g(K_{t,\lambda}, K_{t,\lambda'}) = 0 \quad \text{for } \lambda + \lambda' \neq 0 . \tag{2.71}$$

Proof. (a) The statement on $C, g, \mathcal{U}$ follows from $D''(C) = 0$, $D''(\mathcal{U}) = 0$, $D(g) = 0$. The statement on $\widetilde{C}, \kappa, h, \mathcal{Q}$ and D will follow from the proof of theorem 2.19, see remark 2.20 (iii).

(b) This follows from (2.59), (2.61), (2.67). □

2.7. A correspondence.

Theorem 2.19. *Fix $w \in \mathbb{Z}$. There is a one-to-one correspondence between $(trTERP(w))$-structures and CV-structures. It extends that in lemma 2.11. It is given by the steps in (a) and (b). They are inverse to one another.*

(a) Let $(K \to M, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ be a CV-structure. Let $\pi : \mathbb{P}^1 \times M \to M$ be the projection. Define $\hat{H} := \pi^* K$ and $H := \hat{H}|_{\mathbb{C}\times M}$. Extend $C, \widetilde{C}, \kappa, \mathcal{U}, \mathcal{Q}$ canonically to $\hat{H}$. Extend D to a connection on $\hat{H}$ such that D_{∂_z} and $D_{\overline{\partial_z}}$ vanish on sections in $\pi^{-1}C^\infty(K)$.

(α) Define

$$\nabla := D + \frac{1}{z}C + z\widetilde{C} + (\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\operatorname{id} - z\kappa\mathcal{U}\kappa)\frac{\mathrm{d}z}{z} . \tag{2.72}$$

Then ∇ is a flat connection on $H|_{\mathbb{C}^*\times M}$, the $(0,1)$-part $\nabla^{(0,1)} = D^{(0,1)} + z\widetilde{C}$ defines a holomorphic structure $\mathcal{O}_\nabla(H)$ on H, the connection ∇ is compatible with it and has a pole of Poincaré rank 1 along $\{0\} \times M$.

(β) Define $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ and $\gamma_\nabla : H_{(z,t)} \to H_{(\gamma(z),t)}$ for $z \in \mathbb{C}^*$ as in lemma 2.14 (c). Let $\psi_z : \hat{H}_{(z,t)} \to K_t$ for $z \in \mathbb{P}^1$ be the canonical projection. Define a $\mathbb{C}$-antilinear map

$$\tau : \hat{H}_{(z,t)} \to \hat{H}_{(\gamma(z),t)} , \ \tau(a) := \psi^{-1}_{\gamma(z)} \circ \kappa \circ \psi_z(a) \ \text{ for } z \in \mathbb{P}^1 . \tag{2.73}$$

Then $\tau^2 = \mathrm{id}$ *and* $\tau(za) = \frac{1}{z}\tau(a)$ *(this has to be understood with* z *as function on* $\mathbb{C}^*$*, not pointwise). For* $(z,t) \in \mathbb{C}^* \times M$ *define the maps*

$$\tau_{real} : H_{(z,t)} \to H_{(\gamma(z),t)} \ , \ \tau_{real}(a) = \tau(z^w a) \ , \tag{2.74}$$
$$\kappa_H := \gamma_\nabla \circ \tau_{real} : H_{(z,t)} \to H_{(z,t)} \ . \tag{2.75}$$

Then τ_{real} *and* κ_H *are* ∇*-flat, fiberwise* $\mathbb{C}$*-antilinear and satisfy* $\tau_{real}^2 = \mathrm{id}$ *and* $\kappa_H^2 = \mathrm{id}$*. Thus* κ_H *defines a* ∇*-flat real subbundle* $H_{\mathbb{R}} := \ker(\kappa_H - \mathrm{id}) \subset H|_{\mathbb{C}^*\times M}$*.*

(γ) *Define a pairing*

$$P : H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C} \quad \text{for } (z,t) \in \mathbb{C}^* \times M \tag{2.76}$$
$$(a,b) \mapsto z^w \cdot g(\psi_z a, \psi_{-z} b) \ . \tag{2.77}$$

It satisfies all properties in definition 2.12 (d). The tuple $(H,\nabla,H_{\mathbb{R}},P)$ *is a* $(trTERP(w))$*-structure.*

(b) Let $(H,\nabla,H_{\mathbb{R}},P)$ *be a* $(trTERP(w))$*-structure. Let* $K := H|_{\{0\}\times M}$ *and* $D'', C, D', \widetilde{C}$ *be the* $(DC\widetilde{C})$*-structure in lemma 2.11 (b). Define* g*,* $\mathcal{U}$*,* τ *and* $\hat{H}$ *as in lemma 2.14. Because* $\hat{H}|_{\mathbb{P}^1\times\{t\}}$ *is a trivial bundle for any* $t \in M$*, there is a canonical projection* $\psi : \hat{H} \to K$*. It restricts to canonical isomorphisms* $\psi_z : H_{(z,t)} \to K_t$ *for* $(z,t) \in \mathbb{P}^1 \times M$*. Then the map*

$$\kappa : K_t \to K_t \ , \ \kappa(a) := \psi_{\gamma(z)} \circ \tau \circ \psi_z^{-1}(a) \text{ for some } z \in \mathbb{C}^* \tag{2.78}$$

is independent of the choice of z *and is a* $\mathbb{C}$*-antilinear involution. Define* $h := g(\cdot, \kappa\cdot)$ *on* K*. Finally, lift* $D, C, \widetilde{C}, \kappa, \mathcal{U}$ *canonically to* $\hat{H}$*, using* ψ*. Then there exists a unique endomorphism* $\mathcal{Q}$ *of* K *such that* (2.72) *holds. The tuple* $(K \to M, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ *is a CV-structure.*

Proof. (a) (α) ∇^2 maps sections in $\pi_* C^{h\infty}(\hat{H})$ to sections in

$$\Big(\frac{1}{z^2}\mathcal{A}_M^{2,0} + \frac{1}{z}\mathcal{A}_M^{2,0} + \frac{1}{z}\mathcal{A}_M^{1,1} + \mathcal{A}_M^{2} + z\mathcal{A}_M^{1,1} + z\mathcal{A}_M^{0,2} + z^2\mathcal{A}_M^{0,2} \tag{2.79}$$
$$+\big(\frac{1}{z^2}\mathcal{A}_M^{1,0} + \frac{1}{z}\mathcal{A}_M^{1} + \mathcal{A}_M^{1} + z\mathcal{A}_M^{1} + z^2\mathcal{A}_M^{0,1}\big) \wedge \frac{\mathrm{d}z}{z}\Big) \otimes \pi_* C^{h\infty}(\hat{H}) \ .$$

$\nabla^2 = 0$ holds because it splits into (2.22)–(2.24), (2.55)–(2.58), and the four identities

$$[\widetilde{C}, \kappa\mathcal{U}\kappa] = 0 \ , \tag{2.80}$$
$$D''(\kappa\mathcal{U}\kappa) + [\widetilde{C}, \mathcal{Q}] + \widetilde{C} = 0 \ , \tag{2.81}$$
$$D'(\kappa\mathcal{U}\kappa) = 0 \ , \tag{2.82}$$
$$D''(\mathcal{Q}) - [\widetilde{C}, \mathcal{U}] = 0 \ , \tag{2.83}$$

which follow from (2.55)–(2.59) and (2.50)–(2.52). For the summand C in (2.56) remark that D in (2.72) contains a covariant derivative D_{∂_z}.

(β) The only nontrivial claim is the flatness $\nabla(\tau_{real}) = 0$. It is equivalent to

$$\nabla_\xi \circ \tau = \tau \circ \nabla_{\overline{\xi}} \text{ and } \nabla_{z\partial_z} \circ \tau = \tau \circ (\nabla_{-z\partial_z} + w \cdot \mathrm{id}) \ . \tag{2.84}$$

Both τ and κ act on the sheaf $\pi_* C^{h\infty}(\hat{H})$ of fiberwise global and holomorphic C^∞-sections, and these actions coincide. Therefore for $a \in \pi_* C^{h\infty}(\hat{H})$

$$\begin{aligned} & (\nabla_\xi \circ \tau - \tau \circ \nabla_{\overline{\xi}})(a) && (2.85) \\ = \;& (D_\xi + \frac{1}{z} C_\xi + z\widetilde{C}_\xi)\tau(a) - \tau(D_{\overline{\xi}} + \frac{1}{z} C_{\overline{\xi}} + z\widetilde{C}_{\overline{\xi}})(a) \\ = \;& (D_\xi \circ \kappa - \kappa \circ D_{\overline{\xi}})(a) + \frac{1}{z}(C_\xi \circ \kappa - \kappa \circ \widetilde{C}_{\overline{\xi}})(a) + z(\widetilde{C}_\xi \circ \kappa - \kappa \circ C_{\overline{\xi}})(a) \\ = \;& 0 \end{aligned}$$

and

$$\begin{aligned} & (\nabla_{z\partial_z} \circ \tau - \tau \circ \nabla_{-z\partial_z} - w \,\mathrm{id})(a) && (2.86) \\ = \;& (\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa)\tau(a) + \tau(\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa)(a) - w \cdot a \\ = \;& -(\mathcal{Q} + \kappa\mathcal{Q}\kappa)\kappa(a) = 0\ . \end{aligned}$$

(γ) The pairing P is $(-1)^w$-symmetric because g is symmetric. The only nontrivial claims are $\nabla(P) = 0$ and that P takes values in $i^w\mathbb{R}$ on $H_\mathbb{R}$. Consider $a, b \in C^\infty(K)$ and their canonical lifts $\widetilde{a}, \widetilde{b} \in \pi_* C^{h\infty}(\hat{H})$ to $\hat{H}$. The following calculation shows $\nabla(P) = 0$. It uses the z-sesquilinearity of P and in the last step $D(g) = 0$ and (2.63)–(2.67).

$$\begin{aligned} & \nabla(P)(\widetilde{a}, \widetilde{b}) && (2.87) \\ = \;& P(\nabla\widetilde{a}, \widetilde{b}) + P(\widetilde{a}, \nabla\widetilde{b}) - \mathrm{d}(P(\widetilde{a}, \widetilde{b})) \\ = \;& z^w g((D + \frac{1}{z} C + z\widetilde{C})a, b) + z^w g(a, (D - \frac{1}{z} C - z\widetilde{C})b) \\ & + z^w g((\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa)a, b)\frac{\mathrm{d}z}{z} \\ & + z^w g(a, (-\frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} + z\kappa\mathcal{U}\kappa)b)\frac{\mathrm{d}z}{z} \\ & - z^w \mathrm{d}(g(a,b)) - w z^w g(a,b)\frac{\mathrm{d}z}{z} \\ = \;& 0\ . \end{aligned}$$

Because $H_\mathbb{R}$ and P are ∇-flat it is now sufficient to show $P((H_\mathbb{R})_{(1,t)} \times (H_\mathbb{R})_{(-1,t)}) \subset i^w\mathbb{R}$. But for $a, b \in K_t$ and $\psi_{\pm 1}^{-1}(a) \in H_{(\pm 1,t)}$

$$\kappa_H(\psi_{\pm 1}^{-1}(a)) = \tau_{real}(\psi_{\pm 1}^{-1}(a)) = (\pm 1)^w \tau(\psi_{\pm 1}^{-1}(a)) = (\pm 1)^w \psi_{\pm 1}^{-1}\kappa(a)\ . \quad (2.88)$$

Therefore

$$(H_\mathbb{R})_{(\pm 1,t)} = \psi_{\pm 1}^{-1}(\ker(\kappa - (\pm 1)^w\,\mathrm{id}) : K_t \to K_t)\ , \quad (2.89)$$

and with (2.62)

$$(H_\mathbb{R})_{(1,t)} = \psi_1^{-1}(K_{\mathbb{R},t}) \text{ and } (H_\mathbb{R})_{(-1,t)} = \psi_{-1}^{-1}(i^w K_{\mathbb{R},t})\ , \quad (2.90)$$

and for $a \in K_{\mathbb{R},t}$, $b \in i^w K_{\mathbb{R},t}$

$$P(\psi_1^{-1}(a), \psi_{-1}^{-1}(b)) = g(a,b) = (-1)^w h(a,b) \in i^w\mathbb{R}\ . \quad (2.91)$$

(b) The map τ from lemma 2.14 acts on $\hat{H}$ by definition of $\hat{H}$ and thus also on $\pi_* C^{h\infty}(\hat{H}) \cong C^\infty(K)$. Therefore κ in (2.78) is independent of the choice of z.

The lifts with the projection $\psi : \hat{H} \to K$ of $D, C, \widetilde{C}, \kappa, \mathcal{U}$ to $\hat{H}$ act on $\pi_* C^{h\infty}(\hat{H}) \cong C^\infty(K)$, and by definition of κ

$$\kappa = \tau \quad \text{on } \pi_* C^{h\infty}(\hat{H}) \ . \tag{2.92}$$

Define on $C^{h\infty}(H|_{\mathbb{C}^*\times M})$ the map

$$\mathcal{Q} := -\nabla_{z\partial_z} + D_{z\partial_z} + \frac{1}{z}\mathcal{U} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa \ . \tag{2.93}$$

Then $\tau(zb) = \frac{1}{z}\tau(b)$, (2.45) and (2.92) show for $a \in \pi_* C^{h\infty}(\hat{H})$

$$\mathcal{Q}a = -\tau\mathcal{Q}\tau a \ . \tag{2.94}$$

$\mathcal{Q}a$ extends as a $C^{h\infty}$-section to $\{0\} \times M$ because of the definition of $\mathcal{U}$ and to $\{\infty\} \times M$ because of (2.94). Therefore $\mathcal{Q}a \in \pi_* C^{h\infty}(\hat{H})$, and $\mathcal{Q}$ descends to a C^∞-linear endomorphism of K. Together with the proof of lemma 2.11 this shows (2.72). Now (2.94) implies $\mathcal{Q} = -\kappa\mathcal{Q}\kappa$. The calculation (2.85) and (2.45) show $\widetilde{C} = \kappa C \kappa$ and $D(\kappa) = 0$. By definition $\kappa^2 = \mathrm{id}$.

Splitting $\nabla^2 = 0$ into pieces with (2.72) and (2.79) gives (2.22)–(2.24) and (2.55)–(2.58).

Next we have to show

$$P(a, b) = z^w g(\psi_z a, \psi_{-z} b) \ . \tag{2.95}$$

Consider a neighborhood $U_1 \times U_2 \subset \mathbb{P}^1 \times M$ of a point in $\{0\} \times M$ such that U_1 is invariant with respect to $z \mapsto -z$ and consider sections $a, b \in \mathcal{O}(\hat{H})(U_1 \times U_2)$. Then $P(a, b) \in z^w \mathcal{O}_{U_1\times U_2}$ by (2.31), and the definitions of τ_{real} and τ show

$$P(\tau_{real}(a), \tau_{real}(b)) \in \overline{\gamma}(z)^w \mathcal{O}_{\gamma(U_1)} C^\infty_M \ , \tag{2.96}$$

$$P(\tau(a), \tau(b)) \in z^w \mathcal{O}_{\gamma(U_1)} C^\infty_M \ . \tag{2.97}$$

Therefore

$$P(a, b) \in z^w \cdot C^\infty_M \quad \text{for} \ \ a, b \in \pi_* C^{h\infty}(\hat{H}) \tag{2.98}$$

and (2.95) holds.

Now one can repeat the calculation in (2.87). Together with $\nabla(P) = 0$ it shows $D(g) = 0$ and (2.63)–(2.67). By definition of h these imply (2.60) and (2.61). The pairing P takes values in $i^w\mathbb{R}$ on $H_\mathbb{R}$. Define $K_\mathbb{R} := \ker(\kappa - \mathrm{id})$. Now (2.92) shows (2.88)–(2.90). Reading (2.91) backwards shows that h takes real values on $K_\mathbb{R}$. This and the symmetry of g imply that h is hermitian. $D(h) = 0$ follows from $D(\kappa) = 0$ and $D(g) = 0$. □

Remarks 2.20. (i) One can formulate and show many intermediate correspondences for structures stronger than those in lemma 2.11 and weaker than those in theorem 2.19. For example in [Si5, Lemma 3.1] and [Sab6, Lemma 2.2.2] a correspondence is given between $(DC\widetilde{C}h)$-structures and $(trT\widetilde{T})$-structures with a sesquilinear pairing, which turns up as $P(\cdot, \tau\cdot)$ in the case of a CV-structure.

(ii) Harmonic bundles [Si4] are $(DC\widetilde{C}h)$-structures with a positive definite hermitian metric h. In Simpson's nonabelian Hodge theorem [Si3] real structures are not considered. But in [Si5] (T)-structures with real structures with properties as τ_{real} for $w = 0$ are discussed under the name (variation of) 'circular real structures'. But the $(trTERP(w))$-structures and CV-structures here are more motivated by the work of Cecotti and Vafa and by singularity theory than by Simpson's work.

(iii) The proof of lemma 2.18 (a): The bundle H in lemma 2.14 carries a $C^{h\omega}$-structure with sheaf $C^{h\omega}(H)$ of real analytic sections which are holomorphic in z-direction. The map τ in lemma 2.14 respects this structure and the induced structure on $\hat{H}$. Therefore κ in theorem 2.19 (b) respects the real analytic structure of K. Because all other nonholomorphic operators in the CV-structure are obtained from κ and holomorphic operators, the whole CV-structure is real analytic. This shows lemma 2.18 (a).

3. Variation of Hodge structures

Simpson [Si1]–[Si4] defined harmonic bundles as a generalization of variations of polarized Hodge structures. But from the harmonic bundle of a variation of Hodge structures one cannot recover the Hodge filtrations without additional information. The tensors $\mathcal{U}$ and $\mathcal{Q}$ of a CV-structure provide such information. They allow to formulate the correspondence in theorem 3.1. Of course, most of it is due to Simpson. Also in [CV1][CV2] it is stated that the structures there generalize variations of Hodge structures.

Theorem 3.1. *Fiz an integer w. There is a natural correspondence between the following structures.*
(α) CV$\oplus$-structures with $\mathcal{U}=0$.
(β) Sums of two variations of polarized Hodge structures; one is of weight w, the other is of weight $w-1$ and is equipped with an automorphism with eigenvalues $\neq 1$.

In the first section the notion of a variation of Hodge structures is recalled. Subsequently the correspondence is made precise and is proved. The last section contains some remarks about special Kähler manifolds.

3.1. **Variation of polarized Hodge structures.** A *variation of Hodge structures of weight* w is a holomorphic vector bundle $H\to M$ with the additional structure:
(α) a flat holomophic connection $\nabla:\mathcal{O}(H)\to\Omega^1_M\otimes\mathcal{O}(H)$,
(β) a ∇-flat real subbundle $H_\mathbb{R}\to M$ with $H_t=H_{\mathbb{R},t}\oplus iH_{\mathbb{R},t}$ for $t\in M$; the corresponding complex conjugation is denoted by $\bar{\cdot}$ or by κ.
(γ) an exhaustive decreasing filtration $F^\bullet$ by holomorphic subbundles $F^p\subset H$ such that

$$H_t=F^p_t\oplus\overline{F^{w+1-p}_t}\quad\text{for any } t\in M \tag{3.1}$$

and (Griffiths transversality)

$$\nabla:\mathcal{O}(F^p)\to\Omega^1_M\otimes\mathcal{O}(F^{p-1})\ . \tag{3.2}$$

The Hodge subbundles are the complex C^∞-subbundles $H^{p,w-p}$ (in fact, real analytic) with

$$H^{p,w-p}_t=F^p_t\cap\overline{F^{w-p}_t}\quad\text{for } t\in M\ . \tag{3.3}$$

They satisfy $\overline{H^{p,w-p}}=H^{w-p,p}$. Property (3.1) is equivalent to the Hodge decomposition

$$F^p_t=\oplus_{q\geq p}H^{q,w-q}_t\quad\text{for } t\in M\ . \tag{3.4}$$

A *polarization* of such a variation of Hodge structures is a ∇-flat $\mathcal{O}_M$-bilinear pairing S on H which takes real values on $H_\mathbb{R}$, is nondegenerate and $(-1)^w$-symmetric and satisfies

$$S(F^p_t,F^q_t)=0\quad\text{for } p+q>w\ , \tag{3.5}$$

$$i^{p-(w-p)}S(v,\overline{v})>0\quad\text{for } v\in H^{p,w-p}_t-\{0\}\ . \tag{3.6}$$

(3.5) is equivalent to

$$S(H_t^{p,w-p}, H_t^{q,w-q}) = 0 \quad \text{for } p+q \neq w\ . \tag{3.7}$$

(3.6) motivates the definition of a sesquilinear form

$$\begin{aligned} h &: H_t \times H_t \to \mathbb{C} \\ h &: H_t^{p,w-p} \times H_t^{q,w-q} \to 0 \quad \text{for } p \neq q\ , & (3.8) \\ h(a,b) &:= \frac{(-1)^p}{(2\pi i)^w} S(a,\overline{b}) \quad \text{for } a,b \in H_t^{p,w-p}\ . & (3.9) \end{aligned}$$

The factor $1/(2\pi)^w$ is inserted in order to make h and S consistent with the singularity case, see section 8.1. The form h is hermitian, positive definite, and takes real values on $H_{\mathbb{R}}$.

An automorphism of a variation of polarized Hodge structures is a ∇-flat bundle automorphism which respects $H_{\mathbb{R}}, F^\bullet, S$, and h. Because h is positive definite, the automorphism is semisimple with eigenvalues on the unit circle.

3.2. **Correspondence with special CV-structures.**

Lemma 3.2. *Let $(H \to M, \nabla, H_{\mathbb{R}}, S, A)$ be a variation of polarized Hodge structures of weight w with an automorphism A with eigenvalues $\neq (-1)^{w+1}$.*

Define $\kappa = \bar{\ }$ and h as above. There exist unique maps $D', D'', C, \widetilde{C} : C^\infty(H) \to \mathcal{A}^1_M \otimes C^\infty(H)$ with

$$\begin{aligned} D' : C^\infty(H^{p,w-p}) &\to \mathcal{A}^{1,0}_M \otimes C^\infty(H^{p,w-p})\ , & (3.10) \\ D'' : C^\infty(H^{p,w-p}) &\to \mathcal{A}^{0,1}_M \otimes C^\infty(H^{p,w-p})\ , & (3.11) \\ C : C^\infty(H^{p,w-p}) &\to \mathcal{A}^{1,0}_M \otimes C^\infty(H^{p-1,w+1-p})\ , & (3.12) \\ \widetilde{C} : C^\infty(H^{p,w-p}) &\to \mathcal{A}^{0,1}_M \otimes C^\infty(H^{p+1,w-1-p})\ , & (3.13) \end{aligned}$$

and

$$\nabla = D' + D'' + C + \widetilde{C}\ . \tag{3.14}$$

Define $\mathcal{U} := 0$ and define the endomorphism $\mathcal{Q} : H_t \to H_t$ by

$$\mathcal{Q}|\ (H_t^{p,w-p} \cap \ker(A - \lambda\,\mathrm{id})) := \alpha\,\mathrm{id} \tag{3.15}$$

$$\text{for } \alpha \in \mathbb{R} \text{ with } e^{2\pi i \alpha} = \lambda,\ \left[\alpha + \frac{w+1}{2}\right] = p\ . \tag{3.16}$$

Then $e^{2\pi i \mathcal{Q}} = A$, and $(H \to M, D', D'', C, \widetilde{C}, \kappa, h, \mathcal{U}, \mathcal{Q})$ is a CV⊕-structure.

Proof. Griffiths transversality, holomorphicity of the subbundles F^p and ∇-flatness of $H_{\mathbb{R}}$ show

$$\nabla : C^\infty(F^p) \to \mathcal{A}^{1,0}_M \otimes C^\infty(F^{p-1}) + \mathcal{A}^{0,1}_M \otimes C^\infty(F^p) \quad , \tag{3.17}$$

$$\nabla : C^\infty(\overline{F^p}) \to \mathcal{A}^{0,1}_M \otimes C^\infty(\overline{F^{p-1}}) + \mathcal{A}^{1,0}_M \otimes C^\infty(\overline{F^p}) \quad , \tag{3.18}$$

$$\begin{aligned} \nabla : C^\infty(H^{p,w-p}) &\to \mathcal{A}^{1,0}_M \otimes C^\infty(H^{p,w-p}) + \mathcal{A}^{0,1}_M \otimes C^\infty(H^{p,w-p}) \\ &+ \mathcal{A}^{1,0}_M \otimes C^\infty(H^{p-1,w+1-p}) + \mathcal{A}^{0,1}_M \otimes C^\infty(H^{p+1,w-1-p}) \quad . \end{aligned} \tag{3.19}$$

The maps $D', D'', C, \widetilde{C}$ with (3.10)–(3.14) are well defined. $D := D' + D''$ is a connection, C and $\widetilde{C}$ are C^∞_M-linear. Splitting ∇^2 into pieces according to the pieces $\mathcal{A}_M^{r,2-r}\otimes C^\infty(H^{p,w-p})$ of $\mathcal{A}_M^2\otimes C^\infty(H)$ shows that $(H\to M,D,C,\widetilde{C})$ is a $(DC\widetilde{C})$-structure (definition 2.9).

The complex conjugation respects the Hodge decomposition. Therefore the flatness $\nabla(\kappa)=0$ of $H_\mathbb{R}$ splits into $D(\kappa)=0$ and $(C+\widetilde{C})(\kappa)=0$. The last equation means $\kappa C\kappa=\widetilde{C}$. Because of (3.7) the flatness $\nabla(S)=0$ of the pairing S splits into $D(S)=0$ and

$$S(C_Xa,b)+S(a,C_Xb) \;=\; 0\;, \tag{3.20}$$
$$S(\widetilde{C}_{\overline{X}}a,b)+S(a,\widetilde{C}_{\overline{X}}b) \;=\; 0\;. \tag{3.21}$$

The definition of h, $D(S)=0$, $D(\kappa)=0$, and (3.10)+(3.11) show $D(h)=0$ and

$$\begin{aligned} h(C_Xa,\overline{b}) &= \frac{(-1)^{p-1}}{(2\pi i)^w}S(C_Xa,\overline{b})=\frac{(-1)^p}{(2\pi i)^w}S(a,C_X\overline{b})\\ &= h(a,\kappa C_X\kappa b)=h(a,\widetilde{C}_{\overline{X}}b) \end{aligned}\tag{3.22}$$

for $a\in C^\infty(H^{p,w-p})$. It rests to show the equations for $\mathcal{U}$ and $\mathcal{Q}$ in a CV-structure. The automorphism A is $A=e^{2\pi i\mathcal{Q}}$ and commutes with C_X, because C_X comes from the variation of Hodge structures. The definition of $\mathcal{Q}$ shows $D(\mathcal{Q})=0$ and

$$C_X:\ker(\mathcal{Q}-\alpha\,\mathrm{id})\to\ker(\mathcal{Q}-(\alpha-1)\,\mathrm{id})\;. \tag{3.23}$$

Therefore $[C,\mathcal{Q}]-C=0$. The automorphism A respects the real structure. Therefore

$$\kappa(H^{p,w-p}\cap\ker(A-\lambda\,\mathrm{id}))=H^{w-p,p}\cap\ker(A-\overline{\lambda}\,\mathrm{id}). \tag{3.24}$$

This together with the identity for $\alpha\in\mathbb{R}-(\frac{w+1}{2}+\mathbb{Z})$

$$\left[-\alpha+\frac{w+1}{2}\right]-(w-p)=-(\left[\alpha+\frac{w+1}{2}\right]-p) \tag{3.25}$$

shows $\kappa\mathcal{Q}\kappa=-\mathcal{Q}$. Finally, h is invariant under A, so

$$h:\big(H_t^{p,w-p}\cap\ker(A-\lambda_1\,\mathrm{id})\big)\times\big(H_t^{q,w-q}\cap\ker(A-\lambda_2\,\mathrm{id})\big)\to 0 \quad \text{for } (p,\lambda_1)\neq(q,\lambda_2)\;, \tag{3.26}$$

and $h(\mathcal{Q}a,b)=h(a,\mathcal{Q}b)$. □

Remark 3.3. If one starts just with a variation of polarized Hodge structures of weight w, one can define an automorphism $A:=(-1)^w\,\mathrm{id}$ and one obtains a CV$\oplus$-structure. If one has already an automorphism, but with $(-1)^{w+1}$ as eigenvalue. one can change A to the sum of A on the subbundle $\bigoplus_{\lambda\neq(-1)^{w+1}}\ker(A-\lambda\,\mathrm{id})$ and to $-A$ on the subbundle $\ker(A-(-1)^{w+1}\,\mathrm{id})$ and then proceed.

The inverse of lemma 3.2 is straightforward. We formulate it.

Lemma 3.4. *Let $(H\to M,D,C,\widetilde{C},\kappa,h,\mathcal{U},\mathcal{Q})$ be a CV$\oplus$-structure with $\mathcal{U}=0$ and such that $\mathcal{Q}$ has no eigenvalues in $\frac{w+1}{2}+\mathbb{Z}$.*

Define a connection $\nabla := D + C + \widetilde{C}$ *and define*

$$
\begin{aligned}
H_t^{p,w-p} &:= \bigoplus_{\alpha:\ [\alpha+\frac{w+1}{2}]=p} \ker(\mathcal{Q} - \alpha\,\mathrm{id} : H_t \to H_t)\ , && (3.27)\\
F_t^p &:= \bigoplus_{q\geq p} H_t^{q,w-q}\ , && (3.28)\\
S &: \ H_t \times H_t \to \mathbb{C} \quad \textit{with} && (3.29)\\
S(a,b) &:= (2\pi i)^w (-1)^p h(a,\overline{b}) \quad \textit{for } a \in H_t^{p,w-p},\ b\in H_t\ , \\
A &:= e^{2\pi i \mathcal{Q}}\ . && (3.30)
\end{aligned}
$$

Then $(H \to M, \nabla, H_\mathbb{R}, S, F^\bullet, A)$ *is a variation of polarized Hodge structures of weight* w *with an automorphism* A.

Proof. ∇ is flat because of lemma 2.11 (a). $\mathcal{Q}$ is semisimple with real eigenvalues by lemma 2.18 (b). It satisfies the equations $D(\mathcal{Q}) = 0$, $[C,\mathcal{Q}] - C = 0$ and $[\widetilde{C},\mathcal{Q}] + \widetilde{C} = 0$. They show $[C,A] = 0 = [\widetilde{C},A]$, $\nabla(A) = 0$, and (3.10)–(3.13). Therefore Griffiths transversality holds and the bundles F^p are holomorphic subbundles with respect to the holomorphic structure on H defined by $\nabla^{(0,1)}$. The equation $\kappa\mathcal{Q}\kappa = -\mathcal{Q}$ shows $\kappa A\kappa = A$ and $\kappa\ker(\mathcal{Q}-\alpha\,\mathrm{id}) = \ker(\mathcal{Q}+\alpha\,\mathrm{id})$. With (3.25) this gives $\overline{H^{p,w-p}} = H^{w-p,p}$, (3.3) and (3.1).

The pairing S satisfies (3.5), (3.6) and $D(S) = 0$ by definition. It is $(-1)^w$-symmetric and takes real values on $H_\mathbb{R}$ because h is hermitian and takes real values on $H_\mathbb{R}$. The equation $h(C_X a, b) = h(a, \widetilde{C}_{\overline{X}} b)$ shows (3.20) and (3.21). With $D(S) = 0$ this implies $\nabla(S) = 0$. Finally, because of $h(\mathcal{Q}a,b) = h(a,\mathcal{Q}b)$ the pairings h and S are invariant under the automorphism A. □

Proof of theorem 3.1. From (α) to (β): On the subbundle $\bigoplus_{\alpha\neq(-1)^w}\ker(\mathcal{Q}-\alpha\,\mathrm{id})$ one applies lemma 3.4 with $w-1$ instead of w and equips the resulting variation of polarized Hodge structures of weight $w-1$ with the automorphism $(-1)^w A$. On the subbundle $\ker(\mathcal{Q}-(-1)^w\,\mathrm{id})$ one applies lemma 3.4 with w and obtains a variation of polarized Hodge structures of weight w.

From (β) to (α): One equips the variation of Hodge structures of weight w with the automorphism $A := (-1)^w\,\mathrm{id}$, and the one with weight $w-1$ with the automorphism $A := (-1)^w \cdot$ (*the given automorphism*). Then one applies lemma 3.2, with $w-1$ instead of w in the case of the variation of Hodge structures of weight $w-1$. One adds the resulting CV⊕-structures.

It is clear that these two procedures are inverse to one another. □

Remarks 3.5. (i) A CV⊕-structure obtained as in lemma 3.2 decomposes into CV⊕-structures on the subbundles $\ker(A-\lambda\,\mathrm{id}) + \ker(A-\overline{\lambda}\,\mathrm{id})$. For $\lambda\neq\pm1$ one can apply lemma 3.4 with $w-1$ instead of w and one gets a variation of Hodge structures of weight $w-1$ on the subbundle, having started with one of weight w. But for $\lambda = \pm 1$ this is not possible; the smallest shift of the weight of Hodge structures on $\ker(A-(\pm1)\,\mathrm{id})$ is 2, using the Tate twist.

(ii) The formulation and the proof of theorem 3.1 fit to the case of families $f_t, t\in M$, of quasihomogeneous singularities $f_t : \mathbb{C}^w \to \mathbb{C}$. There one has a flat bundle $H \to M$ of middle cohomologies of Milnor fibers with a semisimple monodromy $(-1)^w A$; on the subbundle $\ker((-1)^w A - \mathrm{id}) \subset H$ one has a variation of Hodge structures of weight w and on the subbundle $\bigoplus_{\lambda\neq 1} \ker((-1)^w A - \lambda\,\mathrm{id}) \subset H$ a variation of Hodge structures of weight $w-1$. See step 4 in section 8.1.

(iii) The point of view of $(TERP)$-structures for variations of Hodge structures will be discussed in section 7.4.

3.3. **Special geometry.** In [CV1] and [Du2] it is mentioned that CV$\oplus$-structures are generalizations of special geometry. One can consider 'special geometry' roughly as a synonym for variations of polarized Hodge structures. Then theorem 3.1 makes this remark in [CV1] and [Du2] precise. But one can also give 'special geometry' a more precise and narrower meaning.

In [Fr] and [ACD] two versions are considered: affine special Kähler manifolds and projective special Kähler manifolds. The notion of projective special Kähler manifolds is the richer one. It axiomatizes the geometry on moduli spaces of Calabi–Yau threefolds, which is induced from the corresponding variations of Hodge structures of weight 3 on the middle cohomologies [BrG][Co][Fr][BCOV]. In fact, it is equivalent to such a variation of Hodge structures. We refer to these references for a discussion of this.

But in the definition in [Fr] and [ACD] of an affine special Kähler manifold such a description in terms of variations of Hodge structures is not given. The only point in this section is proposition 3.7, which gives such a description.

By definition in [ACD], an *affine special complex manifold* (M, J, ∇) is a complex manifold (M, J), where $J : T^{\mathbb{R}}M \to T^{\mathbb{R}}M$ with $J^2 = -\mathrm{id}$ gives the complex structure, together with a torsion free flat connection ∇ on $T^{\mathbb{R}}M$ with

$$(\nabla_{\xi_1} J)(\xi_2) = (\nabla_{\xi_2} J)(\xi_1) \quad \text{for } \xi_1, \xi_2 \in \mathcal{T}_M^{\mathbb{C}} \ . \tag{3.31}$$

By definition in [ACD] and [Fr], an *affine special Kähler manifold* (M, J, ∇, ω) is an affine special complex manifold (M, J, ∇) with a ∇-flat symplectic form ω on $T^{\mathbb{R}}M$ which is J-invariant and such that the pseudo Kähler metric $h := \omega(J\cdot, \bar{\cdot})$ is positive definite.

Each fiber of the complex tangent bundle $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ of a complex manifold carries a natural Hodge structure of weight 1 with

$$0 = F_t^2 \subset F_t^1 = T_t^{1,0}M \subset F_t^0 = T_t^{\mathbb{C}}M \ . \tag{3.32}$$

Lemma 3.6. *Let ∇ be a flat torsion free connection on the complex tangent bundle $T^{\mathbb{C}}M$ of a complex manifold M. Then $T^{1,0}M = F^1$ is a holomorphic subbundle of $T^{\mathbb{C}}M$ with the holomorphic structure defined by ∇ if and only if* (3.31) *holds.*

Proof. The first condition is equivalent to

$$\nabla_{\overline{Y}} X = 0 \quad \text{for all } X, Y \in \mathcal{T}_M \ . \tag{3.33}$$

The second condition is equivalent to

$$(\nabla_X J)(\overline{Y}) = (\nabla_{\overline{Y}} J)(X) \quad \text{for all } X, Y \in \mathcal{T}_M \ . \tag{3.34}$$

Consider $X, Y \in \mathcal{T}_M$. Then $\mathrm{Lie}_X(J) = 0$ and

$$\begin{aligned} 0 &= [X, J(\overline{Y})] - J([X, \overline{Y}]) \\ &= \nabla_X J(\overline{Y}) - \nabla_{J(\overline{Y})} X - J\nabla_X \overline{Y} + J\nabla_{\overline{Y}} X \\ &= (\nabla_X J)(\overline{Y}) - (\nabla_{\overline{Y}} J)(X) + \nabla_{\overline{Y}} J(X) - \nabla_{J(\overline{Y})} X \\ &= \big((\nabla_X J)(\overline{Y}) - (\nabla_{\overline{Y}} J)(X)\big) + 2i(\nabla_{\overline{Y}} X)\ . \end{aligned} \tag{3.35}$$

This shows the lemma. □

Proposition 3.7. *Let (M, J) be a complex manifold and $F^\bullet$ as in* (3.32).

(a) (M, J, ∇) is an affine special complex manifold if and only if ∇ is a torsion free connection on $T^{\mathbb{R}}M$ and ∇ and $F^\bullet$ give a variation of Hodge structures on $T^{\mathbb{C}}M$.

(b) (M, J, ∇, ω) is an affine special Kähler manifold if and only if ∇ is a torsion free connection on $T^{\mathbb{R}}M$ and ∇, $F^\bullet$ and $S := 2\pi\omega$ give a variation of polarized Hodge structures of weight 1 *on $T^{\mathbb{C}}M$.*

Proof. (a) Lemma 3.6

(b) Start with a polarizing form S; define $\omega := \frac{1}{2\pi} S$ and $h := \omega(J\cdot, \bar{\cdot})$. Then h and S are related as in (3.9), and h is positive definite. □

So indeed, affine special complex manifolds and affine special Kähler manifolds are very natural objects. By Lemma 3.2 they give $(DC\widetilde{C}\kappa\mathcal{U}\mathcal{Q})$-structures (defined in remark 2.17 (ii)) respectively CV⊕-structures on the complex tangent bundle.

4. tt^* GEOMETRY ON THE TANGENT BUNDLE

Suppose that a CV-structure on a vector bundle $K \to M$ is given such that $\operatorname{rk} K = \dim M$ and $\mathcal{O}(K)$ is a free $\mathcal{T}_M$-module of rank 1. Let C be the Higgs field on K. Choose a generator $\zeta \in \mathcal{O}(K)$ of $\mathcal{O}(K)$ as a $\mathcal{T}_M$-module. Then one can shift the whole CV-structure with the isomorphism $C_\bullet\zeta : \mathcal{T}_M \to \mathcal{O}(K)$ to the holomorphic tangent bundle TM. Most of the induced structure depends on the choice of ζ, but not all.

In section 4.1 some induced structure, which does not depend on the choice of ζ, is studied, a multiplication on the tangent bundle (lemma 4.1), a unit field e and an Euler field E. They form an F-manifold (definition 4.2 and lemma 4.3).

In section 4.2 it is discussed how the induced CV-structure on TM behaves with respect to the flows of unit field e, Euler field E, and their complex conjugates $\overline{e}$ and $\overline{E}$. Theorem 4.5 leads to some wishes about the CV-structures on TM. In section 5.4 it will be shown how a careful choice of ζ satisfies these wishes and gives a Frobenius manifold structure.

4.1. **Multiplication on the tangent bundle.** Let M be a complex manifold of dimension $m \geq 1$. A multiplication $\circ$ on the holomorphic tangent bundle TM is an $\mathcal{O}_M$-bilinear commutative and associative map $\circ : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{T}_M$. A global holomorphic vector field e is called a unit field if $e\circ = \mathrm{id}$.

A multiplication provides TM with the structure of a Higgs bundle with the Higgs field $C : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ defined by

$$C_X Y := -X \circ Y\ . \tag{4.1}$$

The minus sign is chosen only for the sake of the discussion of the singularity case in section 8.1 (cf. also [Sab4, 0.13.d]).

A multiplication can arise from an abstract Higgs bundle in the following way.

Lemma 4.1. *Let $(K \to M, C)$ be a Higgs bundle with $\operatorname{rk} K = \dim M$ such that $\mathcal{O}(K)$ is a locally free $\mathcal{T}_M$-module of rank 1. Then there is a unique multiplication $\circ$ on M with $C_X C_Y = -C_{X\circ Y}$ and there is a unique unit field e. It satisfies $C_e = -\mathrm{id}$.*

Proof. Choose a (local) generator $\zeta \in \mathcal{O}(K)$ of $\mathcal{O}(K)$ as a $\mathcal{T}_M$-module. The map $C_\bullet\zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism. Define $e \in \mathcal{T}_M$ by $C_e\zeta = -\zeta$ and define $X \circ Y$ for $X, Y \in \mathcal{T}_M$ by $C_X C_Y \zeta = -C_{X\circ Y}\zeta$. Then $(M, \circ, e)$ is a manifold with commutative and associative multiplication $\circ$ and unit field e. Because of $C_e C_Z \zeta = C_Z C_e \zeta = -C_Z\zeta$ and $C_X C_Y C_Z \zeta = ... = -C_{X\circ Y} C_Z \zeta$ the unit field e and the multiplication are independent of the choice of the generator ζ and satisfy $C_e = -\mathrm{id}$ and $C_X C_Y = -C_{X\circ Y}$. □

If a Higgs bundle as in lemma 4.1 is part of a richer structure, a $(DC\widetilde{C})$-structure or even a CV-structure, one can choose a generator ζ of $\mathcal{O}(K)$ as $\mathcal{T}_M$-module and pull the richer structure to the tangent bundle TM. Lemma 4.3 says that then the multiplication and the Lie bracket for vector fields satisfy the compatibility condition (4.2).

Definition 4.2. [HM][Man, I.5]

(a) A manifold M with multiplication $\circ$ on the holomorphic tangent bundle and unit field e is an *F-manifold* if the multiplication satisfies

$$\mathrm{Lie}_{X\circ Y}(\circ) = X\circ \mathrm{Lie}_Y(\circ) + Y\circ \mathrm{Lie}_X(\circ)\ . \tag{4.2}$$

(b) Let $(M,\circ,e)$ be an F-manifold. A vector field $E\in\mathcal{T}_M$ is an *Euler field* if it satisfies

$$\mathrm{Lie}_E(\circ) = \circ\ . \tag{4.3}$$

F-manifolds with Euler fields are studied in [He4, part 1]. It seems that all interesting manifolds with multiplication on the tangent bundle are F-manifolds.

Lemma 4.3. *Let $(M,\circ,e)$ be a manifold with multiplication and unit field. Define a Higgs field C by (4.1). Let D' be a $(1,0)$-connection on TM as C^∞-bundle with $D'(C)=0$.*

(a) Then $(M,\circ,e)$ is an F-manifold.

(b) Let $E\in\mathcal{T}_M$ be a holomorphic vector field and $\mathcal{Q}$ a C^∞-endomorphism of TM with

$$D'(-C_E) - [C,\mathcal{Q}] + C = 0\ . \tag{4.4}$$

Then E is an Euler field of the F-manifold, that is, $\mathrm{Lie}_E(\circ)=\circ$.

Proof. (a) Formula (2.6) shows that $D'(C)=0$ serves as a replacement for torsion freeness of D',

$$[X,Y]\circ Z = D'_X(Y\circ)(Z) - D'_Y(X\circ)(Z)\ . \tag{4.5}$$

Using this, one simply calculates for $X,Y,Z,W\in\mathcal{T}_M^{1,0}$ (or $\mathcal{T}_M$)

$$\begin{aligned}
& (\mathrm{Lie}_{X\circ Y}(\circ) - X\circ\mathrm{Lie}_Y(\circ) - Y\circ\mathrm{Lie}_X)(Z,W) \qquad (4.6)\\
=\ & [X\circ Y, Z\circ W] - [X\circ Y,Z]\circ W - Z\circ[X\circ Y,W]\\
& -X\circ[Y,Z\circ W] + X\circ[Y,Z]\circ W + X\circ Z\circ[Y,W]\\
& -[X,Z\circ W]\circ Y + [X,Z]\circ Y\circ W + Z\circ[X,W]\circ Y\\
=\ & D_{X\circ Y}(Z\circ W\circ)(e) - D_{Z\circ W}(X\circ Y\circ)(e)\\
& -D_{X\circ Y}(Z\circ)(W) + D_Z(X\circ Y\circ)(W)\\
& -Z\circ D_{X\circ Y}(W\circ)(e) + Z\circ D_W(X\circ Y\circ)(e)\\
-\ & X\circ D_Y(Z\circ W\circ)(e) + X\circ D_{Z\circ W}(Y\circ)(e)\\
& +X\circ D_Y(Z\circ)(W) - X\circ D_Z(Y\circ)(W)\\
& +X\circ Z\circ D_Y(W\circ)(e) - X\circ Z\circ D_W(Y\circ)(e)\\
-\ & D_X(Z\circ W\circ)(Y) + D_{Z\circ W}(X\circ)(Y)\\
& +D_X(Z\circ)(Y\circ W) - D_Z(X\circ)(Y\circ W)\\
& +Z\circ D_X(W\circ)(Y) - Z\circ D_W(X\circ)(Y)\\
=\ & 0\ .
\end{aligned}$$

(b) One calculates with (2.6) in the second and third step and (4.4) in the fifth step

$$\begin{aligned}
& C_{\mathrm{Lie}_E(\circ)(X,Y)-X\circ Y} && (4.7)\\
= \;& C_{[E,X\circ Y]} - C_{X\circ[E,Y]} - C_{[E,X]\circ Y} - C_{X\circ Y} \\
= \;& D'_E(C_{X\circ Y}) - D'_{X\circ Y}(C_E) + C_X C_{[E,Y]} + C_{[E,X]}C_Y + C_X C_Y \\
= \;& -D'_E(C_X C_Y) - D'_{X\circ Y}(C_E) + C_X(D'_E(C_Y) - D'_Y(C_E)) \\
& +(D'_E(C_X) - D'_X(C_E))C_Y + C_X C_Y \\
= \;& -D'_{X\circ Y}(C_E) - C_X D'_Y(C_E) - D'_X(C_E)C_Y + C_X C_Y \\
= \;& [C_{X\circ Y}, \mathcal{Q}] + C_X[C_Y,\mathcal{Q}] + [C_X,\mathcal{Q}]C_Y \\
= \;& 0\ .
\end{aligned}$$

□

4.2. **CV-structures on the tangent bundle.** In the situation assumed at the beginning of the chapter, one obtains an induced CV-structure on TM and the structure of an F-manifold $(M,\circ,e,E)$ with unit field e and Euler field E. On the tangent bundle one has additionally to the induced connection D the Lie derivative of vector fields. In theorem 4.5 the actions of $D_e, \mathrm{Lie}_e, D_E, \mathrm{Lie}_E$ and also $D_{\overline{e}}, \mathrm{Lie}_{\overline{e}}, D_{\overline{E}}, \mathrm{Lie}_{\overline{E}}$ on the CV-structure on TM will be discussed.

Remarks 4.4. (i) An $\mathcal{O}_M$-linear map $\mathcal{T}_M^{\otimes k} \to \mathcal{T}_M^{\otimes l}$ is called a holomorphic (k,l)-tensor; a C_M^∞-linear map $(\mathcal{T}_M^{1,0})^{\otimes k} \to (\mathcal{T}_M^{1,0})^{\otimes l}$ is called a C^∞ (k,l)-tensor. Vector fields in $\mathcal{T}_M^{1,0}$ can be identified with C^∞ $(0,1)$-tensors by $C_M^\infty \to \mathcal{T}_M^{1,0}$, $a\mapsto a\cdot X$, for $X\in\mathcal{T}_M^{1,0}$. We will also consider non C^∞ tensors; for example a sesquilinear pairing on TM is a C_M^∞-sesquilinear $(2,0)$-tensor.

(ii) Let D be a C^∞-connection on TM with $D^{(0,1)}\mathcal{T}_M = 0$. Then one has for $X\in\mathcal{T}_M$

$$\mathrm{Lie}_{\overline{X}}(T) = D_{\overline{X}}(T) \quad \text{for } T \text{ a } C^\infty \text{ tensor}\ , \tag{4.8}$$
$$\mathrm{Lie}_{\overline{X}}(T) = 0 = D_{\overline{X}}(T) \quad \text{for } T \text{ a holomorphic tensor.} \tag{4.9}$$

But $D_{\overline{X}} - \mathrm{Lie}_{\overline{X}}$ acts nontrivially on non C^∞ tensors.

Theorem 4.5. *Let $(TM, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ be a CV-structure on TM such that $\mathcal{T}_M$ is a locally free module over itself with respect to the Higgs field C.*

(a) There is a unique multiplication $\circ$ with $C_X C_Y = -C_{X\circ Y}$, and there is a unique unit field e. Define $E := \mathcal{U}(e)$. Then $(M,\circ,e,E)$ is an F-manifold with Euler field E. Furthermore

$$\mathcal{U} = E\circ\ , \tag{4.10}$$
$$\mathrm{Lie}_e(\circ) = 0\ , \tag{4.11}$$
$$\mathrm{Lie}_{\overline{e}}(\mathcal{Q}) = D_{\overline{e}}(\mathcal{Q}) = D_e(\mathcal{Q}) = 0\ . \tag{4.12}$$

(b) We have the equivalences

$$
\begin{aligned}
& D_e - \mathrm{Lie}_e = 0 \quad \textit{on } C^\infty \textit{ tensors} && (4.13)\\
\iff\ & D_e e = 0 && (4.14)\\
\iff\ & \mathrm{Lie}_e(h) = 0 \iff \mathrm{Lie}_{\overline{e}}(h) = 0 && (4.15)\\
\iff\ & \mathrm{Lie}_e(\kappa) = 0 \iff \mathrm{Lie}_{\overline{e}}(\kappa) = 0\ . && (4.16)
\end{aligned}
$$

If the six statements in the equivalences are true then

$$\mathrm{Lie}_e(g) = 0 \quad \textit{and} \quad \mathrm{Lie}_e(\mathcal{Q}) = 0 \tag{4.17}$$

and then all the tensors $\circ, e, \kappa, g, h, \mathcal{Q}$ *are invariant under the flows of* e *and* $\overline{e}$.

(c) Fix some $d \in \mathbb{R}$ *and define the map*

$$\widetilde{\mathcal{Q}} := D'_E - \mathrm{Lie}_E - \frac{2-d}{2}\,\mathrm{id} : \mathcal{T}_M^{1,0} \to \mathcal{T}_M^{1,0}\ . \tag{4.18}$$

Then

$$D'(\mathcal{U}) - [C, \widetilde{\mathcal{Q}}] + C = 0\ , \tag{4.19}$$

$$\mathrm{Lie}_E(\widetilde{\mathcal{Q}}) = D_E(\widetilde{\mathcal{Q}})\ , \quad \mathrm{Lie}_{\overline{E}}(\widetilde{\mathcal{Q}}) = D_{\overline{E}}(\widetilde{\mathcal{Q}})\ , \tag{4.20}$$

and the two equivalences

$$\mathrm{Lie}_{E-\overline{E}}(h) = 0 \iff h(\widetilde{\mathcal{Q}}X, Y) = h(X, \widetilde{\mathcal{Q}}Y) \textit{ for } X, Y \in \mathcal{T}_M^{1,0}\ , \tag{4.21}$$

$$\mathrm{Lie}_E(g) = (2-d)g \iff g(\widetilde{\mathcal{Q}}X, Y) = -g(X, \widetilde{\mathcal{Q}}Y) \textit{ for } X, Y \in \mathcal{T}_M^{1,0} \tag{4.22}$$

hold. If $\mathrm{Lie}_{E-\overline{E}}(h) = 0$ *and* $\mathrm{Lie}_E(g) = (2-d)g$ *are true then* $(TM, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \widetilde{\mathcal{Q}})$ *is a CV-structure and the following equations hold,*

$$\mathrm{Lie}_{E-\overline{E}}(\widetilde{\mathcal{Q}}) = D_{E-\overline{E}}(\widetilde{\mathcal{Q}}) = 0 = D_{E-\overline{E}}(\mathcal{Q})\quad , \tag{4.23}$$

$$
\begin{aligned}
\mathrm{Lie}_E(\widetilde{\mathcal{Q}}) = D_E(\widetilde{\mathcal{Q}}) &= \frac{1}{2}\mathrm{Lie}_{E+\overline{E}}(\widetilde{\mathcal{Q}}) = \frac{1}{2}D_{E+\overline{E}}(\widetilde{\mathcal{Q}}) && (4.24)\\
&= [\mathcal{U}, \kappa\mathcal{U}\kappa] = D_E(\mathcal{Q}) = \frac{1}{2}D_{E+\overline{E}}(\mathcal{Q})\quad .
\end{aligned}
$$

Especially, then the tensors h *and* $\widetilde{\mathcal{Q}}$ *are invariant under the flow of* $E - \overline{E}$.

Proof. (a) Lemma 4.1 and lemma 4.3 show everything except for (4.10)–(4.12). $\mathrm{Lie}_e(\circ) = 0$ follows from (4.2) for $X = Y = e$. The calculation

$$\mathcal{U}(X) = -\mathcal{U}C_X(e) = -C_X\mathcal{U}(e) = X \circ E$$

shows $\mathcal{U} = E\circ$. The identity (4.8) together with (2.59), (2.51) , (2.58) gives

$$\mathrm{Lie}_{\overline{e}}(\mathcal{Q}) = D''_{\overline{e}}(\mathcal{Q}) = -\kappa D'_e(\mathcal{Q})\kappa = \kappa[C_e, \kappa\mathcal{U}\kappa]\kappa = 0\ .$$

(b) Fix a holomorphic vector field $X \in \mathcal{T}_M$. Then for $Y, Z \in \mathcal{T}_M^{1,0}$

$$
\begin{aligned}
\mathrm{Lie}_X(h)(Y,Z) &= X\,h(Y,Z) - h(\mathrm{Lie}_X Y, Z) - h(Y, \mathrm{Lie}_{\overline{X}} Z)\\
&= h((D_X - \mathrm{Lie}_X)Y, Z)\ , && (4.25)
\end{aligned}
$$

using $D(h)=0$ and $\mathrm{Lie}_{\overline{X}}Z = D_{\overline{X}}Z$. Similarly

$$\begin{aligned}
\mathrm{Lie}_{\overline{X}}(h)(Y,Z) &= h(Y,(D_X-\mathrm{Lie}_X)Z)\ , && (4.26)\\
\mathrm{Lie}_X(g)(Y,Z) &= g((D_X-\mathrm{Lie}_X)Y,Z)+g(Y,(D_X-\mathrm{Lie}_X)Z)\ , && (4.27)\\
\mathrm{Lie}_X(\kappa)(Y) &= (\mathrm{Lie}_X - D_X)(\kappa(Y))\ , && (4.28)\\
\mathrm{Lie}_{\overline{X}}(\kappa)(Y) &= \kappa((D_X-\mathrm{Lie}_X)Y)\ . && (4.29)
\end{aligned}$$

These statements for $X=e$ and the calculation

$$\begin{aligned}
(D_e-\mathrm{Lie}_e)(Y) &= D_e(Y\circ e)-[e,Y]\circ e\\
&= D_e(Y\circ e)-D_e(Y\circ)(e)+D_Y(e\circ)(e)\\
&= Y\circ D_e e && (4.30)
\end{aligned}$$

show the equivalences in (4.13)–(4.16). For (4.17) use (4.27) and $D_e(\mathcal{Q})=0$.
(c) The following calculation shows (4.19). It uses $\mathrm{Lie}_E(\circ)=\circ$ and $D'(C)=0$.

$$\begin{aligned}
& D'_X(\mathcal{U})-[C_X,\widetilde{\mathcal{Q}}]+C_X && (4.31)\\
=\ & D'_X(E\circ)+[X\circ,D'_E-\mathrm{Lie}_E]-X\circ\\
=\ & D'_X(E\circ)-D'_E(X\circ)+\mathrm{Lie}_E(X\circ)-X\circ\\
=\ & [X,E]\circ+\mathrm{Lie}_E(X)\circ+\mathrm{Lie}_E(\circ)(X,\cdot)-X\circ\\
=\ & 0\ .
\end{aligned}$$

One can rewrite $\widetilde{\mathcal{Q}}$ as $\widetilde{\mathcal{Q}} = D_{E\pm\overline{E}}-\mathrm{Lie}_{E\pm\overline{E}}-\frac{2-d}{2}\,\mathrm{id}$. Then

$$D_{E\pm\overline{E}}(\widetilde{\mathcal{Q}}) = [\mathrm{Lie}_{E\pm\overline{E}},D_{E\pm\overline{E}}] = \mathrm{Lie}_{E\pm\overline{E}}(\widetilde{\mathcal{Q}})\ . \qquad (4.32)$$

This shows (4.20). The two equivalences (4.21) and (4.22) follow from (4.25)–(4.27) for $X=E$.

Now suppose that $\mathrm{Lie}_{E-\overline{E}}(h)=0$ and $\mathrm{Lie}_E(g)=(2-d)g$ hold. Then $g=h(\cdot,\kappa\cdot)$ shows

$$\mathrm{Lie}_{E-\overline{E}}(\kappa)=(d-2)\kappa\ , \qquad (4.33)$$

and $\widetilde{\mathcal{Q}}+\kappa\widetilde{\mathcal{Q}}\kappa=0$ follows from the right hand sides of (4.21) and (4.22). In order to see that one obtains a CV-structure with $\widetilde{\mathcal{Q}}$ instead of $\mathcal{Q}$ it rests to make the following calculation for $X\in\mathcal{T}_M$. It uses curvature properties of the $(DC\widetilde{C})$-structure, $D(\kappa)=0$, (4.8), (4.33) and $[\mathrm{Lie}_Y,\mathrm{Lie}_Z]=\mathrm{Lie}_{[Y,Z]}$ for $Y,Z\in\mathcal{T}_M^{1,0}$.

$$\begin{aligned}
& D'_X(\widetilde{\mathcal{Q}})+[C_X,\kappa\mathcal{U}\kappa] && (4.34)\\
=\ & [D'_X,D'_E-\mathrm{Lie}_E]-[C_X,\widetilde{C}_{\overline{E}}]\\
=\ & D'_{[X,E]}-[D'_X,\mathrm{Lie}_E]+[D'_X,D''_{\overline{E}}]\\
=\ & D'_{[X,E]}-[D'_X,\mathrm{Lie}_{E-\overline{E}}]\\
=\ & D'_{[X,E]}-[\kappa D''_{\overline{X}}\kappa,\kappa\,\mathrm{Lie}_{\overline{E}-E}\,\kappa]\\
=\ & D'_{[X,E]}-\kappa[\mathrm{Lie}_{\overline{X}},\mathrm{Lie}_{\overline{E}-E}]\kappa\\
=\ & D'_{[X,E]}-\kappa\,\mathrm{Lie}_{[\overline{X},\overline{E}-E]}\,\kappa\\
=\ & D'_{[X,E]}-\kappa D''_{\overline{[X,E]}}\kappa\\
=\ & 0\ .
\end{aligned}$$

The properties of $\mathcal{Q}$ in (4.23) and (4.24) were already discussed in (2.68). The same arguments together with (4.20) give the properties of $\widetilde{\mathcal{Q}}$ in (4.23) and (4.24). □

Of course one wishes that the equivalent statements in (4.13)–(4.16) and (4.21), (4.22) are all satisfied and that $\mathcal{Q} = \widetilde{\mathcal{Q}}$. Furthermore one may wish that the holomorphic data $(M, \circ, e, E, g)$ yield a Frobenius manifold. If one starts with a CV-structure on a vector bundle $K \to M$ such that $\mathcal{O}(K)$ is a free $\mathcal{T}_M$-module of rank 1, one has to choose a generator $\zeta \in \mathcal{O}(K)$ very carefully. In section 5.4 it is discussed how to do this. Here we make a definition, motivated by theorem 4.5 and by [CV1][Du2].

Definition 4.6. A CDV-structure on a manifold M is a CV-structure $(TM, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ together with a Frobenius manifold structure $(M, \circ, e, E, g)$ such that $C_X Y = -X \circ Y$, $E = \mathcal{U}(e)$, $\mathrm{Lie}_E(g) = (2-d)g$, and

$$\mathcal{Q} = D_E - \mathrm{Lie}_E - \frac{2-d}{2}\,\mathrm{id} \tag{4.35}$$

for some $d \in \mathbb{R}$ and such that the equivalent statements in (4.13)–(4.16) hold.

Remarks 4.7. (i) The definition of a Frobenius manifold is recalled in definition 5.10. For a CDV-structure one needs besides the CV-structure with (4.35), $C_X C_Y = -C_{X \circ Y}$, $C_X e = -X$, $E = \mathcal{U}(e)$, $D_e e = 0$, only that the metric g is flat and that its Levi–Civita connection ∇^g satisfies the potentiality $\nabla^g(C) = 0$. Then $\nabla^g e = 0$ and $\mathrm{Lie}_E(g) = (2-d)g$ will follow automatically, as well as $\mathrm{Lie}_{E-\overline{E}}(h) = 0$.

(ii) If one starts with a Frobenius manifold one needs for a CDV-structure only one new ingredient, a real structure κ (or a hermitian pairing h). All the other structures, $D, \widetilde{C}, h$ (or κ), $\mathcal{Q}$ can be derived from it. But it has to satisfy a lot of properties. They are given in definition 1.2. In order to see the equivalence of definition 1.2 and definition 4.6 one needs (4.21) and (4.22).

5. Frobenius manifolds and tt^* geometry

In chapter 2 interesting geometry on a vector bundle $K \to M$ arose from a vector bundle $H \to \mathbb{P}^1 \times M$ with a flat connection on $\mathbb{C}^* \times M$. This connection induced additional structure on $H|_{\{0\}\times M} = K$ and on $H|_{\{\infty\}\times M}$; and K and $H|_{\{\infty\}\times M}$ could be identified under the assumption that H is a family of trivial bundles on $\mathbb{P}^1$. The same ideas will be used here. But instead of $(T\widetilde{T})$-structures here we will consider '(TL)-structures'. The antiholomorphic twin at $\{\infty\} \times M$ of a pole of Poincaré rank 1 will be replaced by a (holomorphic) logarithmic pole. Therefore the structure here is simpler in two aspects: it is holomorphic, and the logarithmic pole induces a simpler structure on K than (an antiholomorphic twin of) a pole of Poincaré rank 1. Still the whole discussion is very similar and many pieces coincide.

In section 5.1 logarithmic poles and the induced structures are treated. In section 5.2 the ideas above are carried out. In section 5.3 the resulting structures are lifted to the tangent bundle and give Frobenius manifolds. In section 5.4 CDV-structures are obtained by combining the discussion here with chapter 4.

5.1. (L)-structures and logarithmic poles.

In chapter 2 meromorphic connections were considered on $\mathbb{P}^1 \times M$ with poles along $\{0\} \times M$ and $\{\infty\} \times M$. It will be useful to separate inessential choices of coordinates from essential choices.

Definition 5.1. Let M_1 be a complex manifold, $M_2 \subset M_1$ a submanifold of codimension 1 and $f : M_1 \to \mathbb{C}$ a holomorphic submersion with $M_2 = f^{-1}(0)$. Let $H \to M_1$ be a holomorphic vector bundle.

(a) A holomorphic family of flat connections ∇ on $H|_{f^{-1}(t)}$, $t \in \mathbb{C}^*$, is said to have a *pole of Poincaré rank* $r \in \mathbb{Z}_{\geq 0}$ if

$$\nabla : \mathcal{O}(H) \to \frac{1}{f^r}\Omega^1_{M_1/\mathbb{C}} \otimes \mathcal{O}(H)\ . \tag{5.1}$$

If $r = 0$ then the pair (H, ∇) is called an (L)-structure ('L' for Logarithmic); if $r = 1$ then it is called a (T)-structure (cf. definition 2.3).

(b) A flat connection ∇ on $H|_{M_1 - M_2}$ has a *pole of Poincaré rank* r along M_2 if

$$\nabla : \mathcal{O}(H) \to \frac{1}{f^r}\Omega^1_{M_1}(\log M_2) \otimes \mathcal{O}(H)\ . \tag{5.2}$$

If $r = 0$ then the pair (H, ∇) is called an (LE)-structure. A pole of Poincaré rank 0 is called a logarithmic pole.

(c) Let $M_1 = \mathbb{C} \times M$, $M_2 = \{0\} \times M$, and $f = z$. Fix $w \in \mathbb{Z}$. The triple (H, ∇, P) is an $(LEP(w))$-structure if (H, ∇) is an (LE)-structure and P is a $\mathbb{C}$-bilinear pairing

$$P : H_{(z,t)} \times H_{(-z,t)} \to \mathbb{C} \ \text{ for any } (z, t) \in \mathbb{C}^* \times M \tag{5.3}$$

with all properties in definition 2.12 (d) except (2.35). That means, P is nondegenerate, $(-1)^w$-symmetric, ∇-flat, and on an open subset $U_1 \times U_2 \subset$

$\mathbb{C} \times M$ it extends to a nondegenerate pairing

$$P : \mathcal{O}(H)(U_1 \times U_2) \times \mathcal{O}(H)((-U_1) \times U_2) \to z^w \mathcal{O}_{\mathbb{C}\times M}(U_1 \times U_2) \quad (5.4)$$
$$(a, b) \mapsto ((z, t) \mapsto P(a(z, t), b(-z, t))) \ .$$

(d) Let $M_1 = \mathbb{C}\times M,\ M_2 = \{0\} \times M$, and $f = z$. The triple (H, ∇, P) is an $(LP(w))$-structure if (H, ∇) is an (L)-structure and P is a pairing as in (c).

Remarks 5.2. (i) The sheaf $\Omega^k_{M_1}(\log M_2)$ of logarithmic k-forms is $\Omega^k_{M_1} + \Omega^{k-1}_{M_1} \wedge \frac{\mathrm{d}f}{f}$. Part (b) is independent of the choice of f, but in part (a) the function f is essential.

(ii) We will also consider $LEP(w)$-structures with $M_1 = (\mathbb{P}^1 - \{0\}) \times M$, $M_2 = \{\infty\} \times M$, $f = \frac{1}{z}$. They are obtained from $(LEP(w))$-structures on $\mathbb{C} \times M \supset \{0\} \times M$ by the map $\mathbb{P}^1 \times M \to \mathbb{P}^1 \times M$, $(z, t) \mapsto (\frac{1}{z}, t)$. The structures in lemma 5.3 are shifted with it from $\{0\} \times M$ to $\{\infty\} \times M$.

(iii) In the same way $(LP(w))$-structures with $M_1 = (\mathbb{P}^1 - \{0\}) \times M$, $M_2 = \{\infty\} \times M$, and $f = \frac{1}{z}$ are defined.

Lemma 5.3. *Let M_1, M_2, f and $H \to M_1$ be as in definition 5.1.*

(a) Let (H, ∇) be an (L)-structure. Then ∇ restricts to a flat connection ∇^{res} on $H|_{M_2}$, the residual connection.

(b) Let (H, ∇) be an (LE)-structure. It contains an (L)-structure; let ∇^{res} be its residual connection on $H|_{M_2}$. There is a ∇^{res}-flat endomorphism $\mathcal{V}^{res}$ on $H|_{M_2}$, the residue endomorphism. Locally it is defined by

$$\mathcal{V}^{res}([a]) := \nabla_{f\cdot\zeta} a \mod f \cdot \mathcal{O}(H) \ , \quad (5.5)$$

where $a \in \mathcal{O}(H)$, $[a] \in \mathcal{O}(H|_{M_2})$, and $\zeta \in \mathcal{T}_{M_1}$ is a vector field with $\zeta(f) = 1$.

(c) Let $(H \to \mathbb{C} \times M, \nabla, P)$ be an $(LEP(w)$-structure. On $K := H|_{\{0\}\times M}$ one has ∇^{res} and $\mathcal{V}^{res}$ as in (b). There is a symmetric nondegenerate holomorphic pairing g on K defined by

$$g([a], [b]) := (z^{-w} P(a, b)) \mod z\mathcal{O}(H) \quad (5.6)$$

for $[a], [b] \in \mathcal{O}(K)$, and lifts $a, b \in \mathcal{O}(H)$. It is ∇^{res}-flat and satisfies

$$g(\mathcal{V}^{res}[a], [b]) + g([a], \mathcal{V}^{res}[b]) = wg([a], [b]) \ . \quad (5.7)$$

Proof. (a) Trivial.

(b) Suppose for simplicity $M_1 = \mathbb{C} \times M, M_2 = \{0\} \times M$ and $f = z$. Then one can choose $\zeta = \partial_z$ and lift vector fields $X \in \mathcal{T}_M$ canonically to M_1 such that $[X, \partial_z] = 0$. For $a \in \mathcal{O}(H)$, $[a] \in \mathcal{O}(H|_{\{0\}\times M})$, $X \in \mathcal{T}_M$

$$(\nabla^{res}_X \mathcal{V}^{res})([a]) = [\nabla_X, \nabla_{z\partial_z}](a) \mod z\mathcal{O}(H) = 0 \ . \quad (5.8)$$

Therefore $\mathcal{V}^{res}$ is ∇^{res}-flat.

(c) The pairing g is symmetric, nondegenerate and holomorphic as in lemma 2.14 (a). For $a, b \in \mathcal{O}(H)$ one finds with $\nabla(P) = 0$

$$\begin{aligned} & g(\mathcal{V}^{res}[a], [b]) + g([a], \mathcal{V}^{res}[b]) - wg([a], [b]) \quad (5.9)\\ = \ & z^{-w} P(\nabla_{z\partial_z} a, b) + z^{-w} P(a, \nabla_{z\partial_z} b) - w \cdot z^{-w} P(a, b) \mod z\mathcal{O}_{\mathbb{C}\times M} \\ = \ & z\partial_z(z^{-w} P(a, b)) \mod z\mathcal{O}_{\mathbb{C}\times M} = 0 \ . \end{aligned}$$

A similar calculation shows $\nabla^{res}(g) = 0$. □

Remarks 5.4. (i) The residue endomorphism of a connection with logarithmic pole is independent of a function f as in definition 5.1 (b). But the residual connection depends on it.

(ii) Given a flat bundle $\widetilde{H} \to \mathbb{C}^* \times M$, the extensions to vector bundles $H \to \mathbb{C} \times M$ whose connections have logarithmic poles along $\{0\} \times M$ are rather simple objects and can be classified and encoded by certain filtrations. See section 7.5.

5.2. $(TLEP)$**-structures and Frobenius type structures.**

Definition 5.5. Let $H \to \mathbb{P}^1 \times M$ be a holomorphic vector bundle such that $H|_{\mathbb{P}^1 \times \{t\}}$ is a trivial bundle for any $t \in M$.

(a) A $(trTL)$-structure is a pair (H, ∇) whose restriction to $\mathbb{C} \times M$ is a (T)-structure (definition 2.3 (i)) and whose restriction to $(\mathbb{P}^1 - \{0\}) \times M$ is an (L)-structure.

(b) A $(trTLP(w))$-structure is a triple (H, ∇, P) whose restriction to $\mathbb{C} \times M$ is a $(TP(w))$-structure (remark 2.13 (i)) and whose restriction to $(\mathbb{P}^1 - \{0\}) \times M$ is an $(LP(-w))$-structure (remark 5.2 (iii)).

(c) A $(trTLEP(w))$-structure is a triple (H, ∇, P) whose restriction to $\mathbb{C} \times M$ is a $(TEP(w))$-structure (remark 2.13 (i)) and whose restriction to $(\mathbb{P}^1 - \{0\}) \times M$ is an $(LEP(-w))$-structure (remark 5.2 (ii)).

Theorem 5.7 and corollary 5.8 will say that these structures induce on $K := H|_{\{0\} \times M}$ the structures in definition 5.6 and that they are equivalent to them.

Definition 5.6. (a) A $(\nabla^r C)$-structure is a holomorphic vector bundle $K \to M$ together with a flat holomorphic connection ∇^r and a Higgs field $C : \mathcal{O}(K) \to \Omega^1_M \otimes \mathcal{O}(K)$ such that $\nabla^r(C) = 0$; that means (cf. (2.6))

$$\nabla^r_X(C_Y) - \nabla^r_Y(C_X) - C_{[X,Y]} = 0 \quad \text{for } X, Y \in \mathcal{T}_M \tag{5.10}$$

(the ‘r’ in ∇^r stands for ‘residual’ or ‘restriction’).

(b) A $(\nabla^r C g)$-structure is a $(\nabla^r C)$ structure together with a symmetric nondegenerate ∇^r-flat pairing g on K with

$$g(C_X a, b) = g(a, C_X b) \tag{5.11}$$

for $X \in \mathcal{T}_M, a, b \in \mathcal{O}(K)$.

(c) A Frobenius type structure on a holomorphic vector bundle $K \to M$ consists of a flat holomorphic connection ∇^r on K, a Higgs field C on K with $\nabla^r(C) = 0$, an $\mathcal{O}_M$-linear endomorphism $\mathcal{U}$ of K with $[C, \mathcal{U}] = 0$, an $\mathcal{O}_M$-linear ∇^r-flat endomorphism $\mathcal{V}$ of K with

$$\nabla^r(\mathcal{U}) - [C, \mathcal{V}] + C = 0 \ , \tag{5.12}$$

and a symmetric nondegenerate ∇^r-flat pairing g on K with

$$\begin{aligned} g(C_X a, b) &= g(a, C_X b) \ , &(5.13)\\ g(\mathcal{U} a, b) &= g(a, \mathcal{U} b) \ , &(5.14)\\ g(\mathcal{V} a, b) &= -g(a, \mathcal{V} b) &(5.15) \end{aligned}$$

for $X \in \mathcal{T}_M, a, b \in \mathcal{O}(K)$.

Theorem 5.7. *Fix $w \in \mathbb{Z}$. There is a one-to-one correspondence between $(trTLEP(w))$-structures and Frobenius type structures on holomorphic vector bundles. It is given by the steps in (a) and (b). They are inverse to one another.*

(a) Let $(K \to M, \nabla^r, C, g, \mathcal{U}, \mathcal{V})$ be a Frobenius type structure on K. Let $\pi : \mathbb{P}^1 \times M \to M$ be the projection. Define $H := \pi^* K$, and let $\psi_z : H_{(z,t)} \to K_t$ for $z \in \mathbb{P}^1$ be the canonical projection. Extend $\nabla^r, C, g, \mathcal{U}, \mathcal{V}$ canonically to H. Define

$$\nabla := \nabla^r + \frac{1}{z} C + (\frac{1}{z}\mathcal{U} - \mathcal{V} + \frac{w}{2}\,\mathrm{id})\frac{\mathrm{d}z}{z} \ . \tag{5.16}$$

Define a pairing

$$\begin{aligned} P : H_{(z,t)} \times H_{(-z,t)} &\to \mathbb{C} \quad \text{for } (z,t) \in \mathbb{C}^* \times M \\ (a,b) &\mapsto z^w g(\psi_z a, \psi_{-z} b) \ . \end{aligned} \tag{5.17}$$

Then (H, ∇, P) is a $(trTLEP(w))$-structure.

(b) Let (H, ∇, P) be a $(trTLEP(w))$-structure. Define $K := H|_{\{0\}\times M}$. Let g and $\mathcal{U}$ on K come from the $(TEP(w))$-structure on $H|_{\mathbb{C}\times M}$ as in lemma 2.14. Let ∇^{res} and $\mathcal{V}^{res}$ be the residual connection and the residue endomorphism on $H|_{\{\infty\}\times M}$.

Because $H|_{\mathbb{P}^1\times\{t\}}$ is a trivial bundle for any $t \in M$, there is a canonical projection $\psi : H \to K$, and the bundles K and $H|_{\{\infty\}\times M}$ are canonically isomorphic. Structure on $H|_{\{\infty\}\times M}$ can be shifted to K. Let ∇^r on K be the shift of ∇^{res} and let $\mathcal{V}$ on K be the shift of $\mathcal{V}^{res} + \frac{w}{2}\,\mathrm{id}$. Then $(K \to M, \nabla^r, C, g, \mathcal{U}, \mathcal{V})$ is a Frobenius type structure and (5.16) holds.

Proof. Most of the proof is similar to the proof of theorem 2.19, but simpler.

(a) $\nabla^2 = 0$ holds because it splits into $(\nabla^r)^2 = 0$, $\nabla^r(C) = 0$, $C^2 = 0$, $[C, \mathcal{U}] = 0$, $\nabla^r(\mathcal{V}) = 0$, and (5.12). For the summand C in (5.12) observe that ∇^r in (5.16) contains a covariant derivative $\nabla^r_{\partial_z}$.

The pairing P is $(-1)^w$-symmetric because g is symmetric. The flatness $\nabla(P) = 0$ follows with a calculation similar to (2.87) from $\nabla^r(g) = 0$ and (5.13)–(5.15). By definition P satisfies the limit behaviour along $\{\infty\} \times M$ which is required for an $(LEP(-w))$-structure on $(\mathbb{P}^1 - \{0\}) \times M$ (formula (5.4)). The same holds for the $(TEP(w))$-structure on $\mathbb{C} \times M$.

(b) Lift $\nabla^r, C, \mathcal{U}$, and $\mathcal{V}$ canonically to H, using ψ. Consider

$$\Delta := \nabla - (\nabla^r + \frac{1}{z} C + (\frac{1}{z}\mathcal{U} - \mathcal{V} + \frac{w}{2}\,\mathrm{id})\frac{\mathrm{d}z}{z}) \ . \tag{5.18}$$

The $\mathcal{O}_{\mathbb{P}^1\times M}$-linear map Δ_X for $X \in \mathcal{T}_M$ maps sections in $\pi_*\mathcal{O}(H)$ to sections which have no pole along $\{0\} \times M$ because of the definition of C and which vanish along $\{\infty\} \times M$ because of the definition of ∇^r. Therefore $\Delta_X = 0$. The same holds for $\Delta_{z\partial_z}$, using the definitions of $\mathcal{U}$ and $\mathcal{V}$. Here observe $\frac{\mathrm{d}z}{z} = -\frac{\mathrm{d}(1/z)}{1/z}$. This shows $\Delta = 0$ and (5.16).

Now the flatness $\nabla^2 = 0$ splits into $(\nabla^r)^2 = 0$, $\nabla^r(C) = 0$, $C^2 = 0$, $[C, \mathcal{U}] = 0$, $\nabla^r(\mathcal{V}) = 0$, and (5.12). The pairing g is symmetric and nondegenerate and satisfies (5.13) and (5.14) by lemma 2.14. Consider sections $a, b \in \mathcal{O}(K)$ and their canonical lifts $\widetilde{a}, \widetilde{b} \in \pi_* \mathcal{O}(H)$ to H. Then by definition of a $(trTLEP(w))$-structure

$$P(\widetilde{a}, \widetilde{b}) \in z^w \cdot \pi_* \mathcal{O}_{\mathbb{P}^1 \times M} = z^w \cdot \mathcal{O}_M \ . \tag{5.19}$$

Therefore $P(\widetilde{a}, \widetilde{b}) = z^w g(a,b)$. The lift of g to H restricts on $H|_{\{\infty\}\times M}$ to the pairing in lemma 5.3 (c) of the $(LEP(-w))$-structure at $\{\infty\} \times M$. By lemma 5.3 (c) g is ∇^r-flat and satisfies (5.15). □

Corollary 5.8. *There is a one-to-one correspondence between $(trTL)$-structures and $(\nabla^r C)$-structures. For a fixed $w \in \mathbb{Z}$ there is a one-to-one correspondence between $(trTLP(w))$-structures and $(\nabla^r Cg)$-structures. In both cases one can read off from theorem 5.7 how to pass from one structure to the other.*

Proof. A part of the proof of theorem 5.7 gives the proof. □

Remark 5.9. Usually a $(TEP(w))$-structure comes from geometry, e.g. from oscillating integrals for singularities, and one wants to extend it to a $(trTLEP(w))$-structure. The choice of some extension to $\{\infty\} \times M$ with a logarithmic pole is cheap. But it is nontrivial to find extensions such that the restrictions $H|_{\mathbb{P}^1\times\{t\}}$ of the total bundle are trivial. This is called a Birkhoff problem (with parameters). A distinguished way to solve it takes the point of view of filtrations, which are associated to logarithmic extensions along $\{\infty\} \times M$ and to restrictions $(H|_{\mathbb{C}\times\{t\}}, \nabla)$ for special parameters $t \in M$. See section 7.5.

5.3. **Frobenius manifolds.** A Frobenius manifold M is a Frobenius type structure on the tangent bundle with some additional properties. By theorem 5.7, for any $w \in \mathbb{Z}$ it gives rise to a meromorphic connection and a pairing on the lifted tangent bundle $\pi^* TM$, where $\pi : \mathbb{P}^1 \times M \to M$, which together form a $(trTLEP(w))$-structure.

The connections are sometimes called first structures connections [Man]. They are most important. They are a gate to richer structures. They also give a frame to construct Frobenius manifolds: One has to construct $(trTLEP(w))$-structures on abstract bundles and shift them with some diligence to the lifted tangent bundle.

Here we recall the definition of Frobenius manifolds and their first structure connections and describe this frame for constructing Frobenius manifolds. Of course, this is essentially well known. The presentation here is influenced by papers of Barannikov, Dubrovin, Manin, Sabbah, and K. Saito.

Definition 5.10. (a) A *Frobenius manifold* $(M, \circ, e, E, g)$ is a complex manifold M of dimension ≥ 1 with a multiplication $\circ$ on the holomorphic tangent bundle TM , a unit field e, an *Euler field* E and a symmetric nondegenerate $\mathcal{O}_M$-bilinear pairing g on TM with the following properties.
The metric g is multiplication invariant, that means,

$$g(X \circ Y, Z) = g(Y, X \circ Z) \quad \text{for } X, Y, Z \in \mathcal{T}_M \ ; \tag{5.20}$$

the Levi–Civita connection ∇^g of the metric g is flat; together with the Higgs field $C : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ with $C_X Y := -X \circ Y$ it satisfies the potentiality condition $\nabla^g(C) = 0$; the unit field e is ∇^g-flat; and the Euler field satisfies $\mathrm{Lie}_E(\circ) = \circ$ and $\mathrm{Lie}_E(g) = (2-d)\cdot g$ for some $d \in \mathbb{C}$.

(b) A *Frobenius manifold without Euler field* consists of $(M, \circ, e, g)$ as in (a) with all properties without those involving E.

(c) A *Frobenius manifold without metric and Euler field* is a manifold $(M, \circ, e)$ with multiplication $\circ$, unit field e, and Higgs field C as in (a) together with a torsion free flat connection ∇^g with $\nabla^g(C) = 0$ and $\nabla^g(e) = 0$.

Lemma 5.11. *Let $(M, \circ, e, E, g)$ be a Frobenius manifold with $d \in \mathbb{C}$, Higgs field C, and Levi-Civita connection ∇^g as in definition 5.10. Define $\mathcal{U} := E\circ$ and*

$$\mathcal{V} := \nabla^g_\bullet E - \frac{2-d}{2}\,\mathrm{id} = \nabla^g_E - \mathrm{Lie}_E - \frac{2-d}{2}\,\mathrm{id}\ . \tag{5.21}$$

(a) Then $(TM, \nabla^g, C, g, \mathcal{U}, \mathcal{V})$ is a Frobenius type structure on TM.

(b) Fix $w \in \mathbb{Z}$. Let (π^*TM, ∇, P) be the $(trTLEP(w))$-structure which corresponds to this Frobenius type structure by theorem 5.7 (a). Then the canonical isomorphism $TM \to \pi^*TM|_{\{\infty\}\times M}$ maps ∇^g to the residual connection of ∇ at $\{\infty\}\times M$ and $\mathcal{V} - \frac{w}{2}\,\mathrm{id}$ to the residue connection of ∇ at $\{\infty\}\times M$. The unit field satisfies

$$\nabla^g e = 0 \quad \textit{and} \quad \mathcal{V}e = \frac{d}{2}e\ . \tag{5.22}$$

Proof. $\mathrm{Lie}_E(\circ)(e,e) = e$ shows $[e,E] = e$ and

$$\mathcal{V}(e) = \nabla^g_e E - \frac{2-d}{2}e = [e,E] - \frac{2-d}{2}e = \frac{d}{2}e\ . \tag{5.23}$$

The calculation (4.31) with D' replaced by ∇^g and $\widetilde{\mathcal{Q}}$ replaced by $\mathcal{V}$ shows (5.12). Because of $\nabla^g g = 0$ equation (5.15) is equivalent to $\mathrm{Lie}_E(g) = (2-d)\cdot g$. In order to see the flatness $\nabla^g(\mathcal{V}) = 0$ of $\mathcal{V}$ one has to show the symmetry and the skew symmetry

$$(\nabla^g_X\mathcal{V})(Y) = (\nabla^g_Y\mathcal{V})(X) \quad \text{for } X, Y \in \mathcal{T}_M\ , \tag{5.24}$$

$$g((\nabla^g_X\mathcal{V})(Y), Z) = -g((\nabla^g_X\mathcal{V})(Z), Y) \quad \text{for } X, Y, Z \in \mathcal{T}_M\ , \tag{5.25}$$

and use both three times in order to see $g((\nabla^g_X\mathcal{V})(Y), Z) = -g((\nabla^g_X\mathcal{V})(Y), Z) = 0$. The other statements are clear. □

These $(trTLEP(w))$-structures on π^*TM are called *first structure connections* of the Frobenius manifold.

The following gives an inverse to lemma 5.11 and a frame for constructing Frobenius manifolds.

Theorem 5.12. *Let $(H \to \mathbb{P}^1 \times M, \nabla, P)$ be a $(trTLEP(w))$-structure and let $(K = H|_{\{0\}\times M}, \nabla^r, C^K, g^K, \mathcal{U}^K, \mathcal{V}^K)$ be the corresponding Frobenius type structure on K.*

Let $\zeta \in \mathcal{O}(K)$ be a global section such that $v := -C^K_\bullet\zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism and such that $\nabla^r\zeta = 0$ and $\mathcal{V}^K\zeta = \frac{d}{2}\zeta$ for some $d \in \mathbb{C}$. Shift the

Frobenius type structure on K with the isomorphism v^{-1} to a Frobenius type structure $(TM, \nabla^g, C, g, \mathcal{U}, \mathcal{V})$ on TM.

Then this is the Frobenius type structure of a Frobenius manifold. Especially, ∇^g is torsion free, $e = v^{-1}(\zeta)$ is the unit field, $\mathrm{Lie}_E(g) = (2-d)\cdot g$, *and $\mathcal{V}$ satisfies* (5.21).

Proof. Define multiplication $\circ$ and unit field e as in lemma 4.1 and define $E := \mathcal{U}(e)$. By lemma 4.3 for $D' = \nabla^{(1,0)}$ and $\mathcal{Q} = \mathcal{V}$ the tuple $(M, \circ, e, E)$ is an F-manifold with Euler field. The unit field e is ∇^g-flat because of $\nabla^r\zeta = 0$.

With the isomorphism $\mathcal{O}(K) = \pi_*\mathcal{O}(H)$ elements of $\mathcal{O}(K)$ will also be considered as sections in H. By (5.16) for $X, Y \in \mathcal{T}_M$

$$\nabla_X v(Y) = \nabla^r_X v(Y) + \frac{1}{z} C^K_X v(Y) = v(\nabla^g_X Y) - \frac{1}{z} v(X \circ Y)\ , \tag{5.26}$$

$$\nabla_{z\partial_z} v(Y) = \frac{1}{z} v(E \circ Y) - v((\mathcal{V} - \frac{w}{2}\,\mathrm{id})Y)\ . \tag{5.27}$$

The connection ∇^g is torsion free because of the calculation

$$\begin{aligned} & v(\nabla^g_{[X,Y]} Z) - \frac{1}{z} v([X,Y]\circ Z) \\ = \ & \nabla_{[X,Y]} v(Z) = [\nabla_X, \nabla_Y] v(Z) \\ = \ & v([\nabla^g_X, \nabla^g_Y] Z) - \frac{1}{z} v((\nabla^g_X(Y\circ) - \nabla^g_Y(X\circ))(Z))\ . \end{aligned} \tag{5.28}$$

Therefore it is the Levi–Civita connection of g. The following calculation shows (5.21). It uses $\nabla^g e = 0$ and $\mathcal{V}e = \frac{d}{2} e$, which follows from $\mathcal{V}^K\zeta = \frac{d}{2}\zeta$.

$$\begin{aligned} & v(\nabla^g_X E) = \nabla_X v(E) + \frac{1}{z} v(X \circ E) \\ = \ & \nabla_X z \left(\nabla_{z\partial_z} v(e) + v(\frac{d-w}{2} e)\right) + \left(\nabla_{z\partial_z} v(X) + v((\mathcal{V} - \frac{w}{2}\,\mathrm{id})X)\right) \\ = \ & -\nabla_{z\partial_z} v(X) + v(X) - v(\frac{d-w}{2} X) + \nabla_{z\partial_z} v(X) + v((\mathcal{V} - \frac{w}{2}\,\mathrm{id})X) \\ = \ & v((\mathcal{V} + \frac{2-d}{2}\,\mathrm{id})X)\ . \end{aligned} \tag{5.29}$$

Now $\mathrm{Lie}_E(g) = (2-d)\cdot g$ follows from (5.21) and (5.15). □

The following two weaker cases are immediate consequences of the proof of theorem 5.12.

Corollary 5.13. *Consider the two cases:*

(a) a $(trTL)$-structure and the corresponding $(\nabla^r C)$-structure on a vector bundle $K \to M$;

(b) a $(trTLP(w))$-structure and the corresponding $(\nabla^r Cg)$-structure on a vector bundle $K \to M$.

Suppose in both cases that $\zeta \in \mathcal{O}(K)$ is a global section such that $-C^K_\bullet \zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism and $\nabla^r\zeta$=0.

In case (a) one obtains a Frobenius manifold without metric and Euler field. In case (b) one obtains a Frobenius manifold without Euler field.

Remarks 5.14. (i) In the construction of Frobenius manifolds in singularity theory [SK2][SK3][SM2], a partial Fourier transformation maps the Gauß-Manin connection to a $(TERP(w))$-structure. A primitive form of K. Saito yields an extension to a $(trTLEP(w))$-structure and is mapped itself to the lift to the total bundle of a section ζ as in theorem 5.14. Therefore one may call such sections also primitive forms. In a first structure connection the unit field plays the role of a primitive form.

(ii) In the case of M-tame functions on affine manifolds one can by work of Sabbah establish $(trTLEP(w))$-structures and sections $\zeta \in \mathcal{O}(K)$ with all properties in theorem 5.14 except for the property $\mathcal{V}^K\zeta = \frac{d}{2}\zeta$. This *homogeneity property* is only clear for special cases (Newton nondegenerate polynomials). In the general case one obtains in the moment only Frobenius manifolds without Euler field.

5.4. tt^* **geometry on a Frobenius manifold.** Now we will show how the wishes after theorem 4.5 can be satisfied. The construction in theorem 5.15 equips the Frobenius manifold with a real structure κ such that it becomes a CDV-structure, if the $(trTLEP(w))$-structure carries a suitable real structure. More precisely, one has the following.

Theorem 5.15. *Let $(H \to \mathbb{P}^1 \times M, \nabla, H_{\mathbb{R}}, P)$ be a tuple such that (H, ∇, P) is a $(trTLEP(w))$-structure (definition 5.5) and $(H|_{\mathbb{C}\times M}, \nabla, H_{\mathbb{R}}, P)$ is a $(trTERP(w)$-structure (definition 2.15).*
By theorem 2.19 the bundle $K := H|_{\{0\}\times M}$ carries a CV-structure; by theorem 5.8 it carries a Frobenius type structure $(K, \nabla^K, g^k, \mathcal{U}^K, \mathcal{V}^K)$.

Let $\zeta \in \mathcal{O}(K)$ be a global section such that $v := -C^K_\bullet\zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism and such that $\nabla^K\zeta = 0$ and $\mathcal{V}^K\zeta = \frac{d}{2}\zeta$ for some $d \in \mathbb{R}$.

Then the CV-structure and the Frobenius type structure on K are shifted with v^{-1} to a CDV-structure on the holomorphic tangent bundle TM.

Proof. The tensors $C^K, g^K, \mathcal{U}^K$ are also part of the CV-structure on K. Let $(TM, D, C, \widetilde{C}, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ be the induced CV-structure on TM and let $(TM, \circ, e, E, g, \nabla^g, C, \mathcal{U}, \mathcal{V})$ be the induced Frobenius type structure on TM and Frobenius manifold structure on M.

Let $\hat{H} \to \mathbb{P}^1 \times M$ be the bundle constructed in lemma 2.14 (e). The bundles H and $\hat{H}$ are different extensions of $H|_{\mathbb{C}\times M}$. Both are fiberwise trivial with respect to $\pi : \mathbb{P}^1 \times M \to M$. Let $\psi : \mathcal{O}(K) \to \pi_*\mathcal{O}(H)$ and $\hat{\psi} : \mathcal{O}(K) \to \pi_*\mathcal{O}(\hat{H})$ be the canonical isomorphisms. Then for $a \in \mathcal{O}(K)$

$$\psi(a) \equiv \hat{\psi}(a) \mod zC^{h\infty}(H|_{\mathbb{C}\times M}) \tag{5.30}$$

and for $Y \in \mathcal{T}_M$

$$\begin{aligned} \nabla\hat{\psi}v(Y) &= \hat{\psi}v(DY) + \frac{1}{z}\hat{\psi}v(CY) + z\hat{\psi}v(\widetilde{C}Y) \\ &+ \left(\frac{1}{z}\hat{\psi}v(\mathcal{U}Y) - \hat{\psi}v((\mathcal{Q} - \frac{w}{2}\,\mathrm{id})Y) - z\hat{\psi}v(\kappa\mathcal{U}\kappa Y)\right)\frac{\mathrm{d}z}{z}\ , \end{aligned} \tag{5.31}$$

and

$$\begin{aligned}\nabla\psi v(Y) &= \psi v(\nabla^g Y) + \frac{1}{z}\psi v(CY) \\ &+ \left(\frac{1}{z}\psi v(\mathcal{U}Y) - \psi v((\mathcal{V} - \frac{w}{2}\,\mathrm{id})Y)\right)\frac{\mathrm{d}z}{z}\ .\end{aligned} \tag{5.32}$$

Especially

$$\nabla_{z\partial_z+E}\hat\psi v(Y) = \hat\psi v\left(D_E Y - (\mathcal{Q} - \frac{w}{2}\,\mathrm{id})Y\right) - z\hat\psi v(\kappa\mathcal{U}\kappa Y)\ , \tag{5.33}$$

$$\nabla_{z\partial_z+E}\psi v(Y) = \psi v\left([E,Y] + \frac{w+2-d}{2}Y\right)\ . \tag{5.34}$$

$\nabla_{z\partial_z+E}$ maps $zC^{h\infty}(H|_{\mathbb{C}\times M})$ to itself. Therefore (5.30),(5.33), and (5.34) show

$$\mathcal{Q} = D_E - \mathrm{Lie}_E - \frac{2-d}{2}\,\mathrm{id}\ . \tag{5.35}$$

Write $\hat\psi v(e) = \psi v(e) + z\cdot a$ for some $a \in C^{h\infty}(H|_{\mathbb{C}\times M})$. Then

$$\begin{aligned}\hat\psi v(D_e e) &= \nabla_e\hat\psi v(e) + \frac{1}{z}\hat\psi v(e) \\ &= \nabla_e\psi v(e) + \frac{1}{z}\psi v(e) + z\nabla_e a + a \\ &= \psi v(\nabla^g_e e) + z\nabla_e a + a = z\nabla_e a + a \\ &\in zC^{h\infty}(H|_{\mathbb{C}\times M})\end{aligned} \tag{5.36}$$

because of $C_e = -\,\mathrm{id}$ and $\nabla^g_e e = 0$. This shows $D_e e = 0$ and (4.13)-(4.16). Finally, $\mathrm{Lie}_{E-\overline{E}}(h) = 0$ follows now from (5.35), (4.21), and (2.61). □

6. Remarks on the semisimple case

In [CV1][CV2] the tt^*-equations turn up in the study of moduli spaces of *massive* deformations of topological field theories. The corresponding CV⊕-structures are semisimple at generic points of the moduli spaces.

A CV-structure on a bundle $K \to M$ is called *semisimple* at $t \in M$, if the endomorphism $\mathcal{U}|_t$ is semisimple with eigenvalues with multiplicity one.

In [CV1][CV2] the CV-structures are not semisimple at the (most interesting) points of conformal field theories; at those points $\mathcal{U}|_t = 0$. A very intriguing moral in [CV1][CV2] is to study the CV-structures at semisimple points and get results for the points with $\mathcal{U}|_t = 0$ and their conformal field theories.

Here we will just make some remarks about the semisimple points.

Consider a $(TERP(w))$-structure $(H \to \mathbb{C} \times M, \nabla, H_{\mathbb{R}}, P)$, that is, (definition 2.12) a holomorphic vector bundle H with a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$, which has a pole of Poincaré rank 1 along $\{0\} \times M$, a real ∇-flat subbundle $H_{\mathbb{R}} \subset H|_{\mathbb{C}^* \times M}$, and a pairing P as in definition 2.12. In lemma 2.14 several objects are associated to it, the bundle $K := H|_{\{0\}\times M}$ over M with Higgs field C, holomorphic metric g and endomorphism $\mathcal{U}$, and an extension $\hat{H} \to \mathbb{P}^1 \times M$ of the bundle H with a fiberwise $\mathbb{C}$-antilinear automorphism τ, which maps $\hat{H}_{z,t}$ to $\hat{H}_{(1/\overline{z},t)}$.

The $(TERP(w))$-structure is called semisimple at $t \in M$ if $\mathcal{U}|_t$ is semisimple with eigenvalues of multiplicity one.

At semisimple points a connection with a pole of Poincaré rank 1 can be described in terms of Stokes data, at least up to tensoring it with a regular singular connection of rank 1. We refer to [Mal2][Mal3][Sab4] for definitions and discussions of this and just state some consequences.

Suppose the $(TERP(w))$-structure is semisimple everywhere. Its sections take the form $z^{w/2}$ times sections of a connection which is described by Stokes data. The Stokes data are given a priori by two complex upper triangular $\operatorname{rk} H \times \operatorname{rk} H$ matrices S and $\widetilde{S}$ with id on the diagonals. The monodromy of the $(TERP(w))$-structure is then given by $(-1)^w S \cdot \widetilde{S}^{-1}$. But the pairing P implies $\widetilde{S} = S^{tr}$, and the real subbundle implies that S is a real matrix. The Stokes matrix S together with the number w and the eigenvalues of $\mathcal{U}|_t$ for all $t \in M$ determines the $(TERP(w))$-structure completely [Mal3][Sab4].

In fact, for any matrix S as above one obtains a universal semisimple $(TERP(w))$-structure on

$$M_{univ} := \text{ universal covering of } \{(u_1, ..., u_m) \in \mathbb{C}^m \mid u_i \neq u_j \,\forall i \neq j\} \,. \quad (6.1)$$

Here the eigenvalues of $\mathcal{U}|_t$ are $u_1(t), ..., u_m(t)$ for any $t \in M_{univ}$. Consider such a universal $(TERP(w))$-structure $(H, \nabla, \mathbb{H}_{\mathbb{R}}, P)$. The functions $u_1, ..., u_m$ on $M = M_{univ}$ are local coordinates, the canonical coordinates. M is a semisimple F-manifold $(M, \circ, e)$ (section 4.1) with $e_i := \frac{\partial}{\partial u_i}$, $e = \sum_i e_i$, $e_i \circ e_j = \delta_{ij} e_i$. The Higgs bundle $(K = H|_{\{0\}\times M}, C)$ is a free $\mathcal{T}_M$-module of rank 1. There are unique global sections $v_i \in \mathcal{O}(K)$, $i = 1, ..., m$, with

$$\mathcal{O}_M \cdot v_i = \operatorname{Im}(C_{e_i} : \mathcal{O}(K) \to \mathcal{O}(K)) \quad (6.2)$$

and $g(e_i, e_i) = 1$. They satisfy

$$\bigoplus \mathcal{O}_M \cdot v_i = \mathcal{O}(K) \ , \tag{6.3}$$

$$g(v_i, v_i) = \delta_{ij} \ . \tag{6.4}$$

Now, being interested in CV-structures, one has to ask for which points $t \in M$ the bundle $\hat{H}|_{\mathbb{P}^1 \times \{t\}}$ is trivial

Theorem 6.1. *Let $(H \to \mathbb{C} \times M_{univ}, \nabla, H_{\mathbb{R}}, P)$ be a universal semisimple $(TERP(w))$-structure as above.*

(a) [Du2, Proposition 2.2] *There exists a bound $b > 0$ such that the restriction to $\{t \in M_{univ} \mid |u_i - u_j| > b \ \forall \ i \neq j\}$ is a $(trTERP(w))$-structure. The corresponding CV-structure there is a CV$\oplus$-structure, that means, the hermitian metric h is positive definite.*

(b) [CV1][CFIV][CV2, (4.5), (4.6)] *Let $\mathcal{Q}_{mat}$ and $\kappa_{mat} = h_{mat}$ be the matrices of the maps $\mathcal{Q}$ and κ and the metric h with respect to the sections v_i. Then along a path with $|u_i - u_j| \to \infty$ for $i \neq j$ one has $\mathcal{Q}_{mat} \to 0$ and $\kappa_{mat} = h_{mat} \to \mathrm{id}$.*

Remarks 6.2. (i) In [Du2, ch. 2] and [CV2, 4.2] the coefficients of the lifts of the v_i to $\hat{H}$ with respect to special ∇-flat bases of $H|_{\{\mathrm{Re} z > 0\}}$ and $H|_{\{\mathrm{Re} z < 0\}}$ are written down and solved in terms of integral equations.

(ii) The estimates for $\mathcal{Q}_{mat}$ and $\kappa_{mat} = h_{mat}$ in [CV2, (4.5), (4.6)] are more precise. From the leading vanishing terms one can recover the Stokes matrix S.

(iii) The subset of M_{univ} where one does not have a CV-structure is $R := \{t \in M_{univ} \mid \hat{H}|_{\mathbb{P}^1 \times \{t\}}$ is not trivial$\}$. It is empty or a real analytic subvariety of M_{univ}, because $\hat{H}$ is a real analytic family of bundles on $\mathbb{P}^1$ (lemma 2.18 (a)). In section 8.3 examples will be given where R is not empty.

The monodromy of a $(TERP(w))$-structure with Stokes matrix S is $(-1)^w S (S^{-1})^{tr}$ in a suitable basis. The Stokes matrices and $(TERP(w))$-structures which turn up in [CV1][CV2] and in singularity theory have special properties: the matrix S has integer entries, and the monodromy is quasiunipotent.

In [CV2] a classification of such Stokes matrices was started. For $\mathrm{rk}\, H = 2$ and 3 it was carried out and corresponding field theories were found and discussed.

The manifold M_{univ} has the Euler field $E = \sum u_i e_i$. The renormalization group flow in [CV1][CV2] corresponds on M_{univ} to the flow of E or of $E + \overline{E}$. Recall that $D_{E-\overline{E}} \mathcal{Q} = 0$ (formula (2.68)). Of course, the moduli spaces of massive deformations of conformal field theories contain non semisimple points. The points t_0 with $\mathcal{U}|_{t_0} = 0$ are the fixed points of E and correspond to conformal field theories. The eigenvalues of $\mathcal{Q}|_{t_0}$ at such points are the *charges*, certain rational numbers α_j such that $(-1)^w e^{-2\pi i \alpha_j}$ are the eigenvalues of the monodromy. In the singularity case they are up to a shift the spectral numbers (= the exponents -1).

In [CV2] an intriguing conjecture is made about these numbers. It is a beautiful, although incomplete, example of how to use the geometry at semisimple points for points with $\mathcal{U}|_{t_0} = 0$. Consider an E-orbit M_{orb} of semisimple points whose closure contains a point t_0 with $\mathcal{U}|_{t_0} = 0$. Suppose that there exist a C^∞ one parameter family $S(r), r \in [0,1]$, of real Stokes matrices with $S(1) = S$ and $S(0) = \mathrm{id}$, such that all monodromies $(-1)^w S(r)(S(r)^{-1})^{tr}$ have all eigenvalues on the unit circle. Suppose further that for each $r \in [0,1]$ the $(TERP(w))$-structure on M_{orb} extends to a $(trTERP(w))$-structure on $\overline{M_{orb}}$. Then for each r the eigenvalues of $\mathcal{Q}(r)|_{t_0}$ are logarithms of the eigenvalues of $S(r)(S(r)^{-1})^{tr}$. For $r = 0$ they are all 0. Now passing from $r = 0$ to $r = 1$ and observing how the eigenvalues of $S(r)(S(r)^{-1})^{tr}$ move around on the unit circle, one can determine the charges α_i. Of course, these arguments are quite incomplete. Especially neither existence nor essential uniqueness of such a one parameter family of Stokes matrices is known. Still the argument is very intriguing.

7. Regular singular $(TERP)$-structures and Hodge filtrations

In chapter 3 a correspondence between CV$\oplus$-structures with $\mathcal{U} = 0$ and variations of polarized pure Hodge structures was presented. In the case of hypersurface singularities $(TERP(w))$-structures and CV-structures turn up such that at many points $t \in M$ the endomorphism $\mathcal{U}|_t$ is nilpotent, but not zero, the $(TERP(w))$-structure at $\mathbb{C} \times \{t\}$ is regular singular, but not logarithmic, and such that to these points polarized mixed Hodge structures are associated.

The purpose of this chapter is to study such $(TERP(w))$-structures and the arising filtrations. Sections 7.2–7.4 provide a general frame and a systematic discussion. Section 7.5 describes an important step in the construction of Frobenius manifolds in singularity theory. Section 7.1 presents a characterization of polarized mixed Hodge structures by nilpotent orbits which will be crucial in section 7.6. There it will be used together with the results of sections 7.2–7.4 to study CV-structures along an Euler field. The main result is theorem 7.20. Together with the conjectural inverse statement (remark 7.21) it shows that polarized mixed Hodge structures turn up most naturally in CV-structures. It will be used in the case of singularities (theorem 8.4 (b)).

7.1. **Polarized mixed Hodge structures and nilpotent orbits.** Throughout the whole section, w will be an integer, H a complex vector space of finite dimension, $H_\mathbb{R}$ a real subspace with $H = H_\mathbb{R} \oplus iH_\mathbb{R}$, and S a nondegenerate $(-1)^w$-symmetric pairing on H with real values on $H_\mathbb{R}$.

The definition below of a polarized mixed Hodge structure is motivated by Schmid's limit mixed Hodge structure [Sch, Theorem 16.6]. It requires a nilpotent endomorphism N of $H_\mathbb{R}$ and of H which is an infinitesimal isometry of S. From it one obtains a weight filtration $W_\bullet$ in the following way.

Lemma 7.1. [Sch, Lemma 6.4] *Let $(H, H_\mathbb{R}, S, N, w)$ be as above.*

(a) There exists a unique exhaustive increasing filtration $W_\bullet$ on $H_\mathbb{R}$ such that $N(W_l) \subset W_{l-2}$ and such that $N^l : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l}$ is an isomorphism.

(b) The filtration satisfies $S(W_l, W_{l'}) = 0$ for $l + l' < w$.

(c) A nondegenerate $(-1)^{w+l}$-symmetric bilinear form S_l is well defined on Gr^W_{w+l} for $l \geq 0$ by $S_l(a, b) := S(\widetilde{a}, N^l\widetilde{b})$ for $a, b \in \mathrm{Gr}^W_{w+l}$ with representatives $\widetilde{a}, \widetilde{b} \in W_{w+l}$.

(d) The primitive subspace $P_{w+l} \subset \mathrm{Gr}^W_{w+l}$ is defined by

$$P_{w+l} := \ker(N^{l+1} : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l-2}) \tag{7.1}$$

for $l \geq 0$ and by $P_{w+l} := 0$ for $l < 0$. Then

$$\mathrm{Gr}^W_{w+l} = \bigoplus_{i \geq 0} N^i P_{w+l+2i}\ , \tag{7.2}$$

and this decomposition is orthogonal with respect to S_l if $l \geq 0$.

Definition 7.2. [CaK1][He3] A *polarized mixed Hodge structure of weight w* (abbreviation PMHS) consists of data $H, H_\mathbb{R}, S, N$ and $W_\bullet$ as above and an exhaustive decreasing *Hodge filtration* $F^\bullet$ on H with the following properties.

(a) $F^\bullet \mathrm{Gr}^W_k$ gives a pure Hodge structure of weight k, that means, $\mathrm{Gr}^W_k = F^p \mathrm{Gr}^W_k \oplus \overline{F^{k+1-p} \mathrm{Gr}^W_k}$.

(b) $N(F^p) \subset F^{p-1}$, i.e. N is a $(-1,-1)$-morphism of mixed Hodge structures.

(c) $S(F^p, F^{w+1-p}) = 0$.

(d) The pure Hodge structure $F^\bullet P_{w+l}$ of weight $w+l$ on P_{w+l} is polarized by S_l, that means,

$$S_l(F^p P_{w+l}, F^{w+l+1-p} P_{w+l}) = 0 \ , \tag{7.3}$$

$$i^{p-(w+l-p)} S_l(a, \overline{a}) > 0 \ \text{ for } \ a \in F^p P_{w+l} \cap \overline{F^{w+l-p} P_{w+l}} - \{0\} \ . \tag{7.4}$$

A PMHS comes equipped with a generalized Hodge decomposition, with Deligne's $I^{p,q}$.

Lemma 7.3. [De, (1.2.8)][CaK1][He3, 2.3] *For a PMHS as above define*

$$I^{p,q} \ := \ (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \sum_{j>0} \overline{F^{q-j}} \cap W_{p+q-j-1}) \ , \tag{7.5}$$

$$I_0^{p,q} \ := \ \ker(N^{p+q-w+1} : I^{p,q} \to I^{w-q-1, w-p-1}) \ . \tag{7.6}$$

Then

$$F^p = \bigoplus_{i,q:\ i \geq p} I^{i,q} \ , \quad W_l = \bigoplus_{p+q \leq l} I^{p,q} \ , \tag{7.7}$$

$$N(I^{p,q}) \subset I^{p-1,q-1} \ , \quad I^{p,q} = \bigoplus_{j \geq 0} N^j I_0^{p+j,q+j} \ , \tag{7.8}$$

$$S(I^{p,q}, I^{r,s}) = 0 \ \textit{for} \ \ (r,s) \neq (w-p, w-q) \ , \tag{7.9}$$

$$S(N^i I_0^{p,q}, N^j I_0^{r,s}) = 0 \ \textit{for} \ \ (r,s,i+j) \neq (q,p,p+q-w) \ , \tag{7.10}$$

$$I^{q,p} \cong \overline{I^{p,q}} \quad \mod W_{p+q-2} \ , \tag{7.11}$$

$$I_0^{q,p} \cong \overline{I_0^{p,q}} \quad \mod W_{p+q-2} \ . \tag{7.12}$$

A polarized Hodge structure is a polarized mixed Hodge structure with $N = 0$. Let us fix one reference polarized Hodge structure $(H, H_\mathbb{R}, S, F_0^\bullet)$ of weight w. The space

$$\check{D} := \{\text{filtrations } F^\bullet \subset H \mid \dim F^p = \dim F_0^p, \ S(F^p, F^{w+1-p}) = 0\} \tag{7.13}$$

is a complex homogeneous space and a projective manifold, and the subspace

$$D := \{F^\bullet \in \check{D} \mid F^\bullet \text{ is part of a polarized Hodge structure}\} \tag{7.14}$$

is an open submanifold and a real homogeneous space [Sch]. It is a classifying space for polarized Hodge structures with fixed Hodge numbers.

Definition 7.4. A pair $(F^\bullet, N)$ is said *to give rise to a nilpotent orbit* if $F^\bullet \in \check{D}$, the endomorphism N of $H_\mathbb{R}$ is nilpotent and an infinitesimal isometry with $N(F^p) \subset F^{p-1}$, and there exists a bound $b \in \mathbb{R}$ such that

$$e^{zN} F^\bullet \in D \text{ for } \ \mathrm{Im} z > b \ . \tag{7.15}$$

Then the set $\{e^{zN} F^\bullet \mid z \in \mathbb{C}\}$ is called *a nilpotent orbit.*

Nilpotent orbits play a fundamental role in Schmid's work [Sch]. They give rise to and in fact characterize PMHS's. This will be crucial for the applications to CV-structures in section 7.6.

Theorem 7.5. [Sch][CaK1][CaKS][CaK2] *Let $(H, H_{\mathbb{R}}, S)$ be as above.*

(a) The tuple $(H, H_{\mathbb{R}}, S, F^\bullet, N)$ is a PMHS of weight w
$\iff$ the pair $(F^\bullet, N)$ gives rise to a nilpotent orbit.
(b) If $(H, H_{\mathbb{R}}, S, F^\bullet, N)$ is a PMHS with $I^{q,p} = \overline{I^{p,q}}$ then $e^{zN}F^\bullet \in D$ for $\mathrm{Im} z > 0$.

Proof. '$\Longleftarrow$' is [Sch, Theorem 16.6.]. It is a consequence of the SL_2-orbit theorem. '$\Longrightarrow$' is [CaKS, Corollary 3.13]. Short proofs of both directions are given in [CaK2, Theorem 3.13]. The special case (b) is proved in [CaK1, Proposition 2.18] and in [CaKS, Lemma 3.12].

Theorem 7.20 will generalize '$\Longrightarrow$'. As a preparation for its proof, we will now sketch a rather elementary proof of '$\Longrightarrow$', which uses (b).

The first and longest part of the proof is devoted to showing that $e^{zN}F^\bullet$ gives a pure Hodge structure for $z \in \mathbb{C}$ with $|\mathrm{Im} z|$ sufficiently large, forgetting the polarization. Equivalently, we want to show

$$e^{(z-\overline{z})N}F^p \oplus \overline{F^{w+1-p}} = H \text{ for } |z - \overline{z}| \text{ large.} \tag{7.16}$$

We choose a Jordan block basis of H with respect to N, which respects the $I_0^{p,q}$ and the real structure as good as possible. Namely, we choose a basis $\{v_a\}_{a\in A}$ of $\bigoplus_{p,q} I_0^{p,q}$ such that

$$v_a \in I_0^{p(a),q(a)} \ , \tag{7.17}$$

$$\overline{v_a} \equiv v_{k(a)} \mod W_{p(a)+q(a)-2} \tag{7.18}$$

$$N^{n(a)+1}v_a = 0 \ , \ N^{n(a)}v_a \neq 0 \ . \tag{7.19}$$

Then

$$p(a) + q(a) - w = n(a) \ , \ k(k(a)) = a \ , \ n(k(a)) = n(a) \ , \tag{7.20}$$

$$p(k(a)) = q(a), \ q(k(a)) = p(a) \ . \tag{7.21}$$

The set

$$\bigcup_{a\in A} \{N^j v_a \mid j = 0, 1, .., n(a)\} \tag{7.22}$$

is a basis of H, the set

$$\bigcup_{a\in A} \{e^{(z-\overline{z})N} N^j v_a \mid j = 0, 1, .., p(a) - p\} \tag{7.23}$$

is a basis of $e^{(z-\overline{z})N}F^p$, and the set

$$\bigcup_{a\in A} \{N^j \overline{v_{k(a)}} \mid j = 0, 1, ..., p(k(a)) - (w + 1 - p)\} \tag{7.24}$$

is a basis of $\overline{F^{w+1-p}}$.

Consider the coefficient matrix B with which the bases in (7.23) and (7.24) are written in terms of that in (7.22). We claim

$$\det B = c_0 \cdot (z - \overline{z})^{c_1} + \text{ smaller powers of } (z - \overline{z}) \tag{7.25}$$

where $c_0 \neq 0$ depends only on the numbers $p, p(a), q(a), n(a)$ $(a \in A)$ and

$$c_1 = \sum_{a:\ p(a)\geq p} c_1(a) \tag{7.26}$$

with $c_1(a) := (p(a) - p + 1)(n(a) - p(a) + p)$ if $0 \leq p(a) - p \leq n(a)$ and $c_1(a) := 0$ else.

First consider the case of a PMHS with $I^{q,p} = \overline{I^{p,q}}$. Then $\overline{v_a} = v_{k(a)}$. The matrix B consists of blocks in the diagonal, one for each $a \in A$, of size $(n(a)+1) \times (n(a)+1)$. One easily sees that each diagonal block contributes a monomial $c_0(a)(z - \overline{z})^{c_1(a)}$ to $\det B$. Because of [CaK1, Proposition 2.18] (or by elementary calculation) $c_0 = \prod_{a\in A} c_0(a) \neq 0$.

More precisely, for a diagonal block of a fixed $a \in A$, the subsquare of coefficients of the part of a in (7.23) with respect to the largest powers of N in $\{N^j v_a \mid j = 0, 1, .., n(a) + 1\}$ contributes $c_0(a)(z - \overline{z})^{c_1(a)}$, the dual subsquare in the diagonal block is the identity matrix and contributes 1.

Going over to the general case with $I^{q,p} \neq \overline{I^{p,q}}$, only the coefficients in B which express (7.24) in terms of (7.22) change. The leading monomial $\prod_{a\in A} c_0(a)(z - \overline{z})^{c_1(a)}$ is contributed by the same subsquares of the diagonal blocks. But now a priori more pieces than only the dual subsquares are involved in the coefficients of this monomial.

Nevertheless, by an induction on the size of the Jordan blocks one sees that still the only contribution comes from the dual subsquares and is 1. The details are left to the reader.

Now the claim (7.25) is proved. It shows (7.16). It rests to see $e^{zN}F^\bullet \in D$ for large $\mathrm{Im} z$. One can choose a continuous one parameter family of PMHS's with Hodge filtrations $F^\bullet(r)$, $r \in [0, 1]$, such that $F^\bullet(1) = F^\bullet$ and $F^\bullet(0)$ satisfies $I^{q,p}(0) = \overline{I^{p,q}(0)}$. For $z \in \mathbb{C}$ with $\mathrm{Im} z$ sufficiently large $\det B \neq 0$ for all $r \in [0, 1]$, and $e^{zN}F^\bullet(r)$ is a Hodge filtration and has a nondegenerate hermitian form $h(r)$ as in (3.8)+(3.9). By [CaK1, Proposition 2.18] $e^{zN}F^\bullet(0) \in D$ and $h(0)$ is positive definite. Therefore all $h(r)$ are positive definite and all $e^{zN}F^\bullet(r) \in D$. □

7.2. **Structures around connections on $\mathbb{C}^*$.** The following definitions are modelled after the singularity case. In section 8.1 the objects will be identified with topological data of a singularity. Section 7.3 will discuss objects which correspond to nontopological data of a singularity.

Throughout this section we will fix the following data $(H, \nabla, H_{\mathbb{R}}, P, w)$ and develop some useful structures around them:

(a) A flat complex vector bundle $H \to \mathbb{C}^*$ of rank $\mu \in \mathbb{N}$ with connection ∇ and monodromy $M_{mon} = M_s \cdot M_u$ with semisimple part M_s, unipotent part M_u and nilpotent part $N := \log M_u$ on each fiber H_z, $z \in \mathbb{C}^*$, such that the eigenvalues of M_{mon} have absolute value 1.

(b) A real ∇-flat subbundle $H_{\mathbb{R}} \to \mathbb{C}^*$ with $H_z = H_{\mathbb{R},z} \oplus iH_{\mathbb{R},z}$ for $z \in \mathbb{C}^*$.

(c) An integer w and a ∇-flat nondegenerate bilinear pairing

$$P : H_z \times H_{-z} \to \mathbb{C} \text{ for } \ z \in \mathbb{C}^* \tag{7.27}$$

which takes values in $i^w\mathbb{R}$ on $H_{\mathbb{R}}$.

First we derive from this structure a tuple $(H^\infty, H^\infty_{\mathbb{R}}, M_s, N, S)$ of 'topological' data which together with certain filtrations may give PMHS's.

(d) H^∞ is the space of manyvalued global ∇-flat sections in H. More precisely, define

$$\mathbf{e} : \mathbb{C} \to \mathbb{C}^*,\ \zeta \mapsto e^{2\pi i \zeta} = z\ , \tag{7.28}$$

and let $pr_H : \mathbf{e}^* H \to H$ be the canonical projection. Then

$$H^\infty = \{A : \mathbb{C} \to H \mid A = pr_H \circ \widetilde{A},\ \widetilde{A} \text{ is a flat global section of } \mathbf{e}^* H\} \tag{7.29}$$

(e) The space H^∞ is equipped with the real subspace $H^\infty_{\mathbb{R}}$ of manyvalued ∇-flat sections in $H_{\mathbb{R}}$ and with a monodromy which is also denoted M_{mon}, with semisimple part M_s and nilpotent part N. In fact, all monodromy invariant structure on a fiber H_z can be shifted to H^∞. Define

$$H^\infty_\lambda := \ker(M_s - \lambda\,\mathrm{id} : H^\infty \to H^\infty) \text{ for } \lambda \in S^1\ , \tag{7.30}$$

$$H^\infty_{\neq 1} := \bigoplus_{\lambda \neq 1} H^\infty_\lambda\ . \tag{7.31}$$

The decomposition $H^\infty = H^\infty_1 \oplus H^\infty_{\neq 1}$ is already defined for $H_{\mathbb{R}}$.

(f) A *polarizing form* S on H^∞ is defined in the following way. It may look artificial, but it is precisely the right way (cf. remark 7.8; for the singularity case see section 8.1). Let $\gamma_\pi : H_z \to H_{-z}$ be the ∇-flat shift in mathematical positive direction; so $\gamma_\pi \circ \gamma_\pi = M_{mon}$. Define

$$L : H_z \times H_z \to \mathbb{C},\ (a,b) \mapsto P(a, \gamma_\pi(b))\ . \tag{7.32}$$

Then

$$\begin{aligned} L(a,b) &= P(a,\gamma_\pi(b)) = P(\gamma_\pi(a), \gamma_\pi \circ \gamma_\pi(b)) = (-1)^w P(M_{mon} b, \gamma_\pi(a)) \\ &= (-1)^w L(M_{mon} b, a)\ , \end{aligned} \tag{7.33}$$

$$L(a,b) = L(M_{mon} a, M_{mon} b)\ . \tag{7.34}$$

Then L is monodromy invariant and can be shifted to H^∞. Define $S : H^\infty \times H^\infty \to \mathbb{C}$ by

$$S(a,b) := (-1)(2\pi i)^w L(a, \frac{1}{M_{mon} - \mathrm{id}} b) \text{ for } a \in H^\infty, b \in H^\infty_{\neq 1}\ , \tag{7.35}$$

$$S(a,b) := (-1)(2\pi i)^w L(a, \frac{-N}{M_{mon} - \mathrm{id}} b) \text{ for } a \in H^\infty, b \in H^\infty_1\ , \tag{7.36}$$

where $\frac{-N}{M_{mon} - \mathrm{id}} := -(\sum_{k \geq 1} \frac{1}{k!} N^{k-1})^{-1}$ on H^∞_1 is an automorphism of H^∞_1.

Lemma 7.6. *The pairing S is monodromy invariant and nondegenerate. The restriction to $H^\infty_{\neq 1}$ is $(-1)^{w-1}$-symmetric, the restriction to H^∞_1 is $(-1)^w$-symmetric. It takes real values on $H^\infty_{\mathbb{R}}$.*

Proof. Only the symmetry properties are not completely trivial. For this proof denote $c := (-1)(2\pi i)^w$. For $a, b \in H^\infty_{\neq 1}$ and $\widetilde{a} = (M_{mon} - \mathrm{id})a$, $\widetilde{b} =$

$(M_{mon} - \mathrm{id})b$

$$\begin{aligned} S(\widetilde{b}, \widetilde{a}) &= c \cdot L((M_{mon} - \mathrm{id})b, a) \\ &= c \cdot (-1)^w L(a, b) - c \cdot (-1)^w L(M_{mon} a, b) \\ &= c \cdot (-1)^{w-1} L((M_{mon} - \mathrm{id})a, b) \\ &= (-1)^{w-1} S(\widetilde{a}, \widetilde{b}) \ . \end{aligned} \tag{7.37}$$

(7.34) implies $L(Na, b) = L(a, -Nb)$. This is used in the next calculation. For $a, b \in H_1^\infty$

$$\begin{aligned} S(b, a) &= c \cdot L(b, \frac{-N}{e^N - \mathrm{id}} a) = c \cdot L(\frac{N}{e^{-N} - \mathrm{id}} b, a) \\ &= c \cdot L(M_{mon} \frac{-N}{e^N - \mathrm{id}} b, a) = c \cdot (-1)^w L(a, \frac{-N}{e^N - \mathrm{id}} b) \\ &= (-1)^w S(a, b) \ . \end{aligned} \tag{7.38}$$

□

In section 7.3 filtrations in H^∞ will be related to additional structure on $H \to \mathbb{C}^*$. For this we need the following sections $es(A, \alpha)$ and spaces C^α and V^α.

Fix $A \in H_\lambda^\infty$ and $\alpha \in \mathbb{R}$ with $e^{-2\pi i \alpha} = \lambda$. Define a global holomorphic section $es(A, \alpha)$ of $H \to \mathbb{C}^*$ by

$$es(A, \alpha)(z) := e^{2\pi i \alpha \zeta} \exp(-\zeta N) A(\zeta) \text{ for } \zeta \text{ with } e^{2\pi i \zeta} = z \tag{7.39}$$

$$= \text{“} z^\alpha \exp(\log z \cdot \frac{-N}{2\pi i}) A\text{”} \ . \tag{7.40}$$

One sees that the right hand side of (7.39) is independent of the choice of ζ with $e^{2\pi i \zeta} = z$. The section $es(A, \alpha)$ is called *an elementary section of order* α. Let C^α be the space of all elementary sections of order α. The map

$$\psi_\alpha : H_\lambda^\infty \to C^\alpha \ , \ A \mapsto es(A, \alpha) \tag{7.41}$$

is an isomorphism of vector spaces. By definition one has

$$\begin{aligned} z \cdot es(A, \alpha) &= es(A, \alpha + 1) \ , \ z \circ \psi_\alpha = \psi_{\alpha+1} \ , \\ \nabla_{\partial_z} es(A, \alpha) &= \alpha \cdot es(A, \alpha - 1) + es(\frac{-N}{2\pi i} A, \alpha - 1) \ , \\ (\nabla_{z\partial_z} - \alpha) es(A, \alpha) &= es(\frac{-N}{2\pi i} A, \alpha) = \frac{-N}{2\pi i} es(A, \alpha) \ , \end{aligned} \tag{7.42}$$

$$\begin{aligned} z \ &: \ C^\alpha \to C^{\alpha+1} \quad \text{bijective,} \\ \nabla_{\partial_z} \ &: \ C^\alpha \to C^{\alpha-1} \quad \text{bijective iff } \alpha \neq 0 \ , \\ z\nabla_{\partial_z} - \alpha = \frac{-N}{2\pi i} \ &: \ C^\alpha \to C^\alpha \quad \text{nilpotent.} \end{aligned} \tag{7.43}$$

Let $i : \mathbb{C}^* \to \mathbb{C}$ be the inclusion. The space

$$V^{>-\infty} := \sum_\alpha \mathbb{C}\{t\} C^\alpha \subset (i_* \mathcal{O}(H))_0 \tag{7.44}$$

is the space of all germs at 0 of sections in $\mathcal{O}(H)$ of moderate growth. It is a $\mathbb{C}\{z\}[z^{-1}]$-vector space with dimension equal to $\operatorname{rk} H$. The Kashiwara–Malgrange filtration on $V^{>-\infty}$ is a decreasing filtration, indexed by the set $\{\alpha \in \mathbb{R} \mid e^{-2\pi i\alpha}$ is an eigenvalue of $M_{mon}\}$, with subspaces

$$V^\alpha := \sum_{\beta\geq\alpha} \mathbb{C}\{z\}C^\beta = \bigoplus_{\alpha\leq\beta<\alpha+1} \mathbb{C}\{z\}C^\beta \tag{7.45}$$

and additionally

$$V^{>\alpha} = \sum_{\beta>\alpha} \mathbb{C}\{z\}C^\beta = \bigoplus_{\alpha<\beta\leq\alpha+1} \mathbb{C}\{z\}C^\beta\ . \tag{7.46}$$

Then $\mathrm{Gr}_V^\alpha := V^\alpha/V^{>\alpha} \cong C^\alpha$.

Finally, we will need a Fourier–Laplace transformation on elementary sections. The following automorphisms $G^{(\alpha)}$ of $H^\infty_{e^{-2\pi i\alpha}}$ for $\alpha > 0$ are central in the next proposition,

$$\begin{aligned} G^{(\alpha)} &:= \sum_{k\geq 0} \frac{1}{k!}\Gamma^{(k)}(\alpha)\left(\frac{-N}{2\pi i}\right)^k : H^\infty_{e^{-2\pi i\alpha}} \to H^\infty_{e^{-2\pi i\alpha}} & (7.47)\\ &= \text{“}\Gamma(\alpha\cdot\mathrm{id}+\frac{-N}{2\pi i})\text{”}\ . & (7.48)\end{aligned}$$

Here $\Gamma^{(k)}$ is the k-th derivative of the gamma function.

Proposition 7.7. *(a) Let τ and z both be coordinates on $\mathbb{C}$. Fiz $\alpha > 0$ and $A \in H^\infty_{e^{-2\pi i\alpha}}$. The Fourier–Laplace transformation FL with*

$$FL(es(A,\alpha-1)(\tau))(z) := \int_0^{\infty\cdot z} e^{-\tau/z}\cdot es(A,\alpha-1)(\tau)\mathrm{d}\tau \tag{7.49}$$

is well defined and gives the elementary section

$$FL(es(A,\alpha-1)(\tau))(z) = es(G^{(\alpha)}A,\alpha)(z)\ . \tag{7.50}$$

(b) For α,β with $0<\alpha,\beta<1$ and for $A\in H^\infty_{e^{-2\pi i\alpha}}, B\in H^\infty_{e^{-2\pi i\beta}}$

$$P\left(es(G^{(\alpha)}A,\alpha)(z),\ es(G^{(\beta)}B,\beta)(-z)\right) = z\frac{1}{(2\pi i)^{w-1}}\cdot S(A,B)\ . \tag{7.51}$$

For $\alpha=\beta=1$ and for $A,B\in H^\infty_1$

$$P\left(es(G^{(1)}A,1)(z),\ es(G^{(1)}B,1)(-z)\right) = z^2\frac{-1}{(2\pi i)^w}\cdot S(A,B)\ . \tag{7.52}$$

Proof. We omit the details. They are straightforward computations. Part (a) uses the identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^k(\Gamma(\alpha+1)z^{\alpha+1}) = \int_0^\infty e^{-\tau/z}\tau^\alpha(\log\tau)^k\mathrm{d}\tau\ , \tag{7.53}$$

part (b) uses

$$\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}-x) = \frac{\pi}{\cos\pi x} \tag{7.54}$$

and (which is a consequence of (7.54))

$$\Gamma(1+x)\Gamma(1-x) = \frac{\pi x}{\sin \pi x}\ . \tag{7.55}$$

□

Remark 7.8. The formulas (7.51) and (7.52) show that the definition above of the polarizing form S is a good one and not as artificial as it might have looked.

7.3. **Filtrations from regular singular** $(TERP)$**-structures.** We stick to the situation considered in section 7.2, with all the objects $H, \mu, \nabla, H_{\mathbb{R}}, P, w, H^\infty, H^\infty_{\mathbb{R}}, M_s, N, S, es(A,\alpha), C^\alpha, V^\alpha, V^{>-\infty}, G^{(\alpha)}$. The purpose of this section is to describe how an extension of the bundle $H \to \mathbb{C}^*$ over $\{0\}$ may give rise to a filtration $F^\bullet$ on H^∞ which together with $(H^\infty, H^\infty_{\mathbb{R}}, M_s, -N, S)$ might give a sum of two PMHS's of weight w and $w-1$ with an automorphism M_s.

In the following, $\widetilde{H}$ will always mean a vector bundle on $\mathbb{C}$ which extends H, that means, $\widetilde{H}|_{\mathbb{C}^*} = H$ and which satisfies the following two assumptions.

- (α) $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)$ is a $(TERP(w))$-structure.
- (β) The pair $(\widetilde{H}, \nabla)$ has a regular singularity at 0, that means, $\mathcal{O}(\widetilde{H})_0 \subset V^{>-\infty}$, in other words, the holomorphic sections of $\widetilde{H}$ have moderate growth at 0.

A $(TERP(w))$-structure on $M = \{pt\}$ with (β) is called a *regular singular* $(TERP(w))$-structure. Assumption (β) implies that the endomorphism $\mathcal{U} : \widetilde{H}_0 \to \widetilde{H}_0$ is nilpotent (see e.g. [Sab4, II.4.1]).

Remark 7.9. In the case of a $(TERP(w))$-structure $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)$ without assumption (β), Sabbah [Sab2] proposed a way to define a filtration on H^∞ from $\widetilde{H}$. In the case of tame functions on affine manifolds this filtration yields a mixed Hodge structure [Sab1][Sab2]. The way is similar, but also dual to the way described below. We refer to [Sab2] for it.

The following way to define a filtration from a pair $(\widetilde{H}, \nabla)$ with regular singularity is essentially already used in [Sch] and [Va1]. Consider the isomorphisms $\psi_\alpha : H^\infty_\lambda \to C^\alpha$ from (7.41) and $C^\alpha \cong \mathrm{Gr}^\alpha_V$. Define two M_s-invariant decreasing filtrations $F^\bullet$ and $\widetilde{F}^\bullet$ on H^∞ by

$$F^p H^\infty_\lambda \ := \ \psi_\alpha^{-1} z^{-w+1+p}\, \mathrm{Gr}_V^{\alpha+w-1-p}\, \mathcal{O}(H)_0\ , \tag{7.56}$$
$$\widetilde{F}^p H^\infty_\lambda \ := \ (G^{(\alpha)})^{-1} F^p H^\infty_\lambda \tag{7.57}$$

for $\alpha \in (0,1]$ with $e^{-2\pi i \alpha} = \lambda$.

Lemma 7.10. *The filtrations satisfy*

$$\dim F^p H_\lambda^\infty = \dim \widetilde{F}^p H_\lambda^\infty \ , \tag{7.58}$$

$$\dim \widetilde{F}^p H_\lambda^\infty + \dim \widetilde{F}^{w-p} H_{\overline{\lambda}}^\infty = \dim H_\lambda^\infty \quad \textit{for} \quad \lambda \neq 1 \ , \tag{7.59}$$

$$\dim \widetilde{F}^p H_1^\infty + \dim \widetilde{F}^{w+1-p} H_1^\infty = \dim H_1^\infty \ , \tag{7.60}$$

$$N(F^p) \subset F^{p-1} \ , \ N(\widetilde{F}^p) \subset \widetilde{F}^{p-1} \ , \tag{7.61}$$

$$S(\widetilde{F}^p H_\lambda^\infty, \, \widetilde{F}^{w-p} H_{\overline{\lambda}}^\infty) = 0 \quad \textit{for} \quad \lambda \neq 1 \ , \tag{7.62}$$

$$S(\widetilde{F}^p H_1^\infty, \, \widetilde{F}^{w+1-p} H_1^\infty) = 0 \ . \tag{7.63}$$

Proof. The definition of $F^\bullet$ can be rewritten as follows. Consider any $\beta \in \mathbb{R}$.

$$A \in F^{w+[-\beta]} H^\infty_{e^{-2\pi i \beta}} \iff es(A, \beta) \in \operatorname{Gr}_V^\beta \mathcal{O}(\widetilde{H})_0 \ . \tag{7.64}$$

Now $N(F^p) \subset F^{p-1}$ follows from the pole of Poincaré rank 1,

$$\begin{aligned} es(\frac{-N}{2\pi i} A, \beta + 1) &= (\nabla_{z\partial_z} - (\beta+1)) es(A, \beta+1) \\ &\in (\nabla_{z\partial_z} - (\beta+1)) z \operatorname{Gr}_V^\beta \mathcal{O}(\widetilde{H})_0 \\ &= \operatorname{Gr}_V^{\beta+1} \Big((\nabla_{z\partial_z} - (\beta+1)) z \mathcal{O}(\widetilde{H})_0 \Big) \\ &\subset \operatorname{Gr}_V^{\beta+1} \mathcal{O}(\widetilde{H})_0 \ . \end{aligned} \tag{7.65}$$

The maps $G^{(\alpha)}$ are automorphisms with $NG^{(\alpha)} = G^{(\alpha)} N$. This shows (7.61) and (7.58). The pairing P satisfies by definition

$$P : C^\alpha \times C^\beta \to 0 \ \text{ if } \alpha + \beta \notin \mathbb{Z} \ , \tag{7.66}$$

$$P : C^\alpha \times C^\beta \to z^{\alpha+\beta} \mathbb{C} \ \text{ nondegenerate if } \alpha + \beta \in \mathbb{Z} \ . \tag{7.67}$$

Because of the $(TERP(w))$-structure one has for $b \in V^{>-\infty}$

$$P(\mathcal{O}(\widetilde{H})_0, b) \subset z^w \mathcal{O}_{\mathbb{C},0} \iff b \in \mathcal{O}(\widetilde{H})_0 \ . \tag{7.68}$$

This implies for $\beta \in \mathbb{R}$ and $b \in C^\beta$

$$P(\operatorname{Gr}_V^{w-1-\beta} \mathcal{O}(\widetilde{H})_0, b) = 0 \iff b \in \operatorname{Gr}_V^\beta \mathcal{O}(\widetilde{H})_0 \ . \tag{7.69}$$

With (7.64) and (7.51) this shows for $\lambda \neq 1$ and $B \in H_{\overline{\lambda}}^\infty$

$$S(\widetilde{F}^p H_\lambda^\infty, B) = 0 \iff B \in \widetilde{F}^{w-p} H_{\overline{\lambda}}^\infty \ . \tag{7.70}$$

This gives (7.62) and (7.59), because S is nondegenerate. (7.63) and (7.60) follow analogously with (7.52). □

The lemma shows that the filtrations $F^\bullet$ and $\widetilde{F}^\bullet$ have some properties of Hodge filtrations of (mixed) Hodge structures of weight w on H_1^∞ and weight $w-1$ on $H_{\neq 1}^\infty$. In fact, in the singularity case $\widetilde{F}^\bullet$ is the Hodge filtration of such PMHS's ([He4, Theorem 10.29], [He3, Theorem 3.5], and section 8.1 step 4 (γ)).

In general the given germ $\mathcal{O}(\widetilde{H})_0$ is richer than the filtered parts $\operatorname{Gr}_V^\alpha \mathcal{O}(\widetilde{H})_0$, which give the filtrations. The proof of the lemma shows that the pole of Poincaré rank 1 and the relation (7.68) are natural generalizations of the properties (7.59)–(7.63). So one can consider regular singular $(TERP(w))$-structures $(\widetilde{H} \to \mathbb{C}, \nabla, H_{\mathbb{R}}, P)$ such that $\widetilde{F}^\bullet$ is a Hodge filtration of PMHS's as a natural generalization of PMHS's.

This point of view was taken in [He3]; but there the Fourier–Laplace duals of $(TERP(w))$-structures with $\mathcal{O}(\widetilde{H})_0 \subset V^{>0}$ were considered and called Brieskorn lattices.

A regular singular $(TERP(w))$-structure has important discrete invariants, its exponents. Their definition is modelled after the singularity case ([Va1], cf. also [He3][He4]). The *exponents* are real numbers $\alpha_1, ..., \alpha_\mu$ such that

$$\sharp(i \mid \alpha = \alpha_i) = \dim \mathrm{Gr}_V^\alpha \mathcal{O}(\widetilde{H})_0 - \dim \mathrm{Gr}_V^{\alpha-1} \mathcal{O}(\widetilde{H})_0 \ . \tag{7.71}$$

They are ordered by $\alpha_1 \leq ... \leq \alpha_\mu$. From (7.69) one derives the symmetry

$$\alpha_i + \alpha_{\mu+1-i} = w \ . \tag{7.72}$$

One also sees easily

$$V^{\alpha_1} \supset \mathcal{O}(\widetilde{H})_0 \supset V^{>\alpha_\mu - 1} \ , \tag{7.73}$$

$$\mathcal{O}(\widetilde{H})_0 = \left(\mathcal{O}(\widetilde{H})_0 \cap \sum_{\alpha_1 \leq \beta \leq \alpha_\mu - 1} C^\beta \right) \oplus V^{>\alpha_\mu - 1} \ . \tag{7.74}$$

7.4. $(TERP)$**-structures generated by elementary sections.** We stick to the situation considered in sections 7.2 and 7.3. Especially we fix $(H, \nabla, H_{\mathbb{R}}, P)$ and (H^∞, M_s, N, S). If $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)$ is a $(TERP(w))$-structure with regular singularity at 0 then in general one cannot recover from the filtration $F^\bullet$ the $\mathbb{C}\{z\}$-lattice $\mathcal{O}(\widetilde{H})_0$ and the $(TERP(w))$-structure. But one can recover the quotients $\mathrm{Gr}_V^\alpha \mathcal{O}(\widetilde{H})_0$ and define with them a slightly simpler $(TERP(w))$-structure. One obtains a correspondence (corollary 7.11) and can describe precisely when such a $(TERP(w))$-structure is a $(trTERP(w))$-structure (lemma 7.12).

Let $F^\bullet$ and $\widetilde{F}^\bullet$ be decreasing M_s-invariant filtrations on H^∞ with (7.57) and with the properties in lemma (7.10). Let $\widetilde{H} \to \mathbb{C}$ be the extension of $H \to \mathbb{C}^*$ such that the sheaf $\mathcal{O}(\widetilde{H})$ is generated by the elementary sections in

$$\{es(A, \beta) \mid \beta \in \mathbb{R},\ A \in F^{w+[-\beta]} \cap H^\infty_{e^{-2\pi i \beta}}\} \ . \tag{7.75}$$

Then the proof of lemma 7.10 and especially formula (7.64) show that $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)$ is a $(TERP(w))$-structure with a regular singularity at 0 and that $F^\bullet$ and $\widetilde{F}^\bullet$ are the corresponding filtrations, i.e. (7.56) and (7.57) hold. If $\alpha_1, ..., \alpha_\mu$ are the exponents of this $(TERP(w))$-structure then

$$\mathcal{O}(\widetilde{H})_0 = \bigoplus_{\alpha_1 \leq \beta \leq \alpha_\mu - 1} \mathrm{Gr}_V^\beta \mathcal{O}(\widetilde{H})_0 \oplus V^{>\alpha_\mu - 1} \ . \tag{7.76}$$

The proof of lemma 7.10 yields the following correspondence.

Corollary 7.11. *The formulas* (7.56) *and* (7.75) *give a one-to-one correspondence between the following data,*

(a) *decreasing M_s-invariant filtrations $F^\bullet$ and $\widetilde{F}^\bullet$ with* (7.57) *and the properties in lemma 7.10.*

(b) $(TERP(w))$*-structures* $(\widetilde{H} \to \mathbb{C}, \nabla, H_{\mathbb{R}}, P)$ *such that $\mathcal{O}(\widetilde{H})_0$ is generated by elementary sections.*

For such a $(TERP(w))$-structure the conditions to have a $(trTERP(w))$-structure and thus a CV-structure or even a CV$\oplus$-structure can be expressed in properties of the filtration $F^\bullet$. This generalizes in the case $M = \{pt\}$ the results of chapter 3 (because now $\mathcal{U}$ is nilpotent, but not necessarily zero) and contains the viewpoint of $(TERP(w))$-structures, which was not considered in chapter 3.

Lemma 7.12. *Let $(\widetilde{H} \to \mathbb{C}, \nabla, H_{\mathbb{R}}, P)$ be a $(TERP(w))$-structure such that $\mathcal{O}(\widetilde{H})$ is generated by elementary sections and let $F^\bullet$ be the corresponding filtration defined in* (7.56)*.*

(a) The $(TERP(w))$-structure is a $(trTERP(w))$-structure if and only if $F^\bullet H^\infty_{\neq 1}$ gives a Hodge structure of weight $w-1$ and $F^\bullet H^\infty_1$ gives a Hodge structure of weight w.

(b) In that case let $(K = \widetilde{H}_0, \kappa, h, g, \mathcal{U}, \mathcal{Q})$ be the corresponding CV-structure (on $M = \{pt\}$; therefore $D, C, \widetilde{C}$ vanish) and let $\hat{H} \to \mathbb{P}^1$ be the extension of $\widetilde{H}$ constructed in lemma 2.14 (e). The Hodge decomposition can be written as

$$H^\infty = \bigoplus_{\beta\in\mathbb{R}} H^\infty(\beta) \ , \tag{7.77}$$

$$H^\infty(\beta) := F^{w+[-\beta]} H^\infty_{e^{-2\pi i\beta}} \cap \overline{F^{[\beta]} H^\infty_{e^{2\pi i\beta}}} \ . \tag{7.78}$$

Then the space of global holomorphic sections of $\hat{H}$ is

$$\pi_* \mathcal{O}(\hat{H}) = \bigoplus_{\beta\in\mathbb{R}} \{es(A,\beta) \mid A \in H^\infty(\beta)\} \ . \tag{7.79}$$

This yields a canonical isomorphism $\Psi : H^\infty \to K$, which is the composition of

$$H^\infty = \bigoplus_{\beta\in\mathbb{R}} H^\infty(\beta) \to \pi_*\mathcal{O}(\hat{H}) \quad \text{with} \quad \psi_\beta : H^\infty(\beta) \to \pi_*\mathcal{O}(\hat{H}) \tag{7.80}$$

and the isomorphism $\pi_*\mathcal{O}(\hat{H}) \to K$.

The isomorphism Ψ maps the complex conjugation on H^∞ to κ. The hermitian form $h^\infty := h \circ (\Psi \times \Psi)$ on H^∞ is given as follows. Define a bilinear form S^∞ on H^∞ by

$$S^\infty(A,B) := S((G^{(\alpha)})^{-1}A, (G^{(\beta)})^{-1}B) \tag{7.81}$$

for $A \in H^\infty_{e^{-2\pi i\alpha}}, B \in H^\infty_{e^{-2\pi i\beta}}$, $\alpha, \beta \in (0,1]$. *The forms* S^∞ *and* h^∞ *are monodromy invariant, and* h^∞ *satisfies*

$$h^\infty : H^\infty(\alpha) \times H^\infty(\beta) \to 0 \quad \textit{for} \quad \alpha \neq \beta \ , \tag{7.82}$$

$$h^\infty : (H^\infty)^{p,q} \times (H^\infty)^{r,s} \to 0 \quad \textit{for} \quad (p,q) \neq (r,s) \ , \tag{7.83}$$

$$h^\infty(A,B) = \frac{(-1)^p}{(2\pi i)^{w-1}} S^\infty(A, \overline{B}) \quad \textit{for} \quad A, B \in (H^\infty_{\neq 1})^{p,w-1-p} \ , \tag{7.84}$$

$$h^\infty(A,B) = \frac{(-1)^p}{(2\pi i)^{w}} S^\infty(A, \overline{B}) \quad \textit{for} \quad A, B \in (H^\infty_{1})^{p,w-p} \ . \tag{7.85}$$

Proof. (a) The map τ from lemma 2.14 (d) maps elementary sections to elementary sections, in the following way,

$$\begin{aligned} \tau(es(A,\alpha)) &= \tau\left(z \mapsto z^\alpha \exp(\log z \frac{-N}{2\pi i}) A \right) \\ &= \left(\frac{1}{\overline{z}} \mapsto \overline{z^{-w} z^\alpha \exp(\log z \frac{-N}{2\pi i}) A} \right) \\ &= \left(z \mapsto z^{-\alpha+w} \exp(\log z \frac{-N}{2\pi i}) \overline{A} \right) \\ &= es(\overline{A}, -\alpha + w) \ . \end{aligned} \tag{7.86}$$

Let $\hat{H} \to \mathbb{P}^1$ be the extension of $\widetilde{H} \to \mathbb{C}$ constructed in lemma 2.14 (e). Because $\mathcal{O}(\widetilde{H})$ is generated by elementary sections, the same holds for $\mathcal{O}(\hat{H}|_{\mathbb{P}^1 - \{0\}})$. Therefore also the space $\pi_* \mathcal{O}(\hat{H})$ is generated by elementary sections. The formulas (7.64) and (7.86) show (7.79), with $H^\infty(\beta)$ defined by (7.78). Therefore $\hat{H}$ is a trivial bundle if and only if (7.77) holds.

(b) It rests to show $\Psi \circ \overline{(\cdot)} = \kappa \circ \Psi$ and the formulas for h^∞. The compatibility $\Psi \circ \overline{(\cdot)} = \kappa \circ \Psi$ follows from the definition of κ in theorem 2.19 (b) and from (7.86) and (7.77)–(7.79). Consider $A \in H^\infty(\alpha), B \in H^\infty(\beta)$; then

$$\begin{aligned} h^\infty(A,B) &= h(\Psi(A), \Psi(B)) = g(\Psi(A), \kappa \Psi(B)) \\ &= z^{-w} \cdot P(es(A,\alpha)(z),\, es(\overline{B}, -\beta + w)(-z)) \ , \end{aligned} \tag{7.87}$$

and this vanishes for $\alpha \neq \beta$ because of (7.66)+(7.67). Suppose $\alpha = \beta$; then (7.51)+(7.52) and the z-sesquilinearity of P show for $\alpha \notin \mathbb{Z}$

$$h^\infty(A,B) = \frac{(-1)^{w+[-\alpha]}}{(2\pi i)^{w-1}} S\left((G^{(\alpha - [\alpha])})^{-1} A,\, (G^{(1+[\alpha]-\alpha)})^{-1} \overline{B} \right) \ , \tag{7.88}$$

and for $\alpha \in \mathbb{Z}$

$$h^\infty(A,B) = \frac{(-1)^{w+[-\alpha]}}{(2\pi i)^{w}} S\left((G^{(1)})^{-1} A,\, (G^{(1)})^{-1} \overline{B} \right) \ . \tag{7.89}$$

This shows (7.82)–(7.85). The forms S^∞ and h^∞ are monodromy invariant because S is monodromy invariant. □

Remarks 7.13. (i) The form S^∞ shares many properties with S; it is monodromy invariant, nondegenerate, $(-1)^{w-1}$-symmetric on $H^\infty_{\neq 1}$ and $(-1)^w$-symmetric on H^∞_1. But for $N \neq 0$ it does not take real values on $H_{\mathbb{R}}$, because then $G^{(\alpha)}$ does not respect $H_{\mathbb{R}}$.

(ii) The CV-structure in lemma 7.12 is a CV$\oplus$-structure if and only if h^∞ is positive definite. The formulas (7.84) and (7.85) have the same form as (3.9), but with S^∞ instead of S. Therefore for $N \neq 0$ the condition to have a CV$\oplus$-structure does not mean that $F^\bullet$ gives polarized Hodge structures on $H^\infty_{\neq 1}$ and H^∞_1. For $N \neq 0$ it does also not mean that $\widetilde{F}^\bullet$ gives polarized Hodge structures on $H^\infty_{\neq 1}$ and H^∞_1, because $G^{(\alpha)} \neq const. \cdot \overline{G^{(1-\alpha)}}$ for $0 < \alpha < 1$ if $N \neq 0$. But the condition that h^∞ is positive definite is close enough to a polarizing condition. This will be used in the proof of theorem 7.20.

Finally, consider the case $N = 0$. Then $G^{(\alpha)} = \Gamma(\alpha) \cdot \mathrm{id}$, and it respects the real structure. The pairings S^∞ and S coincide on $H^\infty_\lambda \times H^\infty_{\overline{\lambda}}$ up to the factor $\frac{\pi}{\cos \pi(\alpha - \frac{1}{2})}$ where $\alpha \in (0,1)$ and $e^{-2\pi i \alpha} = \lambda \neq 1$. They coincide on $H^\infty_1 \times H^\infty_1$. Two filtrations $F^\bullet$ and $\widetilde{F}^\bullet$ on H^∞ with (7.57) are equal. The results from sections 3.2 for $M = \{pt\}$ are recovered in the following corollary, now including the point of view of $(TERP(w))$-structures.

Corollary 7.14. *Let $(H \to \mathbb{C}^*, \nabla, H_\mathbb{R}, P)$ and (H^∞, M_s, N, S) be as at the beginning of section 7.2. Suppose that $N = 0$.*

Let $(\widetilde{H} \to \mathbb{C}, \nabla, H_\mathbb{R}, P)$ be a $(TERP(w))$-structure with $\widetilde{H}|_{\mathbb{C}^*} = H$.

(a) The following conditions are equivalent.

(α) *The sheaf $\mathcal{O}(\widetilde{H})$ is generated by elementary sections.*

(β) *The pair $(\widetilde{H}, \nabla)$ has a logarithmic pole at 0.*

(γ) *The endomorphism $\mathcal{U} : \widetilde{H}_0 \to \widetilde{H}_0$ is $\mathcal{U} = 0$.*

(b) Corollary 7.11 gives a one-to-one correspondence between $(TERP(w))$-structures with (α)–(γ) *and decreasing monodromy invariant filtrations $\widetilde{F}^\bullet$ on H^∞ as in lemma 7.10.*

(c) The correspondence restricts to a one-to-one correspondence between $(trTERP(w))$-structures with (α)–(γ) *and filtrations $\widetilde{F}^\bullet$ on H^∞ which give Hodge structures of weight $w-1$ on $H^\infty_{\neq 1}$ and of weight w on H^∞_1 and which satisfy* (7.62) *and* (7.63).

(d) The CV-structures of $(trTERP(w))$-structures with (α)–(γ) *are CV$\oplus$-structures if and only if the corresponding Hodge structures on $H^\infty_{\neq 1}$ and H^∞_1 are polarized.*

(e) In the situation in (d) theorem 3.1 and lemma 3.4 give polarized Hodge structures on $\widetilde{H}_0 = K$ of weight $w-1$ and w. The isomorphism $\Psi : H^\infty \to K$ maps the filtration $\widetilde{F}^\bullet$ to the filtration in lemma 3.4, the pairing S^∞ to the pairing in lemma 3.4 and the automorphism M_s to $(-1)^w e^{2\pi i \mathcal{Q}}$.

Proof. Everything follows easily from the discussion above. □

Remarks 7.15. In the case of a variation of polarized Hodge structures on a bundle on M as in section 3.2 the corresponding $(trTERP(w))$-structure has a pole of Poincaré rank 1 along $\{0\} \times M$, but because of $\mathcal{U} = 0$ each pair $(\widetilde{H}|_{\mathbb{C} \times \{t\}}, \nabla)$ for $t \in M$ has a logarithmic pole and determines a Hodge filtration $\widetilde{F}^\bullet(t)$ on H^∞ as above. The pole of Poincaré rank 1 for derivations ∇_X with $X \in \mathcal{T}_M$ just rephrases the Griffiths transversality.

7.5. **Filtrations and logarithmic poles.** In this section the situation considered in sections 7.2, 7.3 and 7.4 is extended along a manifold M. Now $H \to \mathbb{C}^* \times M$ is a flat complex vector bundle with connection ∇ and monodromy with eigenvalues in S^1 and with ∇-flat real subspace $H_\mathbb{R}$ and a ∇-flat pairing P as that in section 7.2 (c) for some integer w. The vector space H^∞ and the objects $H^\infty_\mathbb{R}, M_s, N, S, G^{(\alpha)}$ are defined as in section 7.2. Also elementary sections $es(A,\alpha)$ on $\mathbb{C}^* \times M$ are defined as in section 7.2. Such a section satisfies $\nabla_X es(A,\alpha) = 0$ for $X \in \mathcal{T}_M$ by definition. Here $X \in \mathcal{T}_M$ is identified with the canonical lift to $\mathbb{C}^* \times M$.

Extensions of H to vector bundles $\widetilde{H} \to \mathbb{C} \times M$ with logarithmic pole along $\{0\} \times M$ are simple objects. The sheaves $\mathcal{O}(\widetilde{H})$ are generated by elementary sections, and therefore they can be encoded in filtrations on H^∞. More precisely, the following holds.

Proposition 7.16. *There is a one-to-one correspondence between $(LEP(w))$-structures $(\widetilde{H} \to \mathbb{C} \times M, \nabla, P)$ (definition 5.1 (c)) with $\widetilde{H}|_{\mathbb{C}^* \times M} = H$ and decreasing monodromy invariant filtrations $F^\bullet = \widetilde{F}^\bullet$ with the properties in lemma 7.10.*

For any $t \in M$ one recovers $F^\bullet$ from $(H|_{\mathbb{C}\times\{t\}}, \nabla)$ with (7.56). The sheaf $\mathcal{O}(\widetilde{H})$ is generated by the elementary sections in (7.75).

Proof. One can also formulate a correspondence without the pairings P and S. This is well known. Precise statements and proofs can for example be found in [Sab4, III.1.4][He4, 8.2]. The extension to a correspondence with P and S follows from the proof of lemma 7.10. □

An important step in the construction of a Frobenius manifold is to extend a given $(TEP(w))$-structure, or better a $(TERP(w))$-structure (definition 2.12), to a $(trTLEP(w))$-structure (definition 5.5). The construction in singularity theory by K. Saito and M. Saito takes the point of view of filtrations.

Theorem 7.17. *Let $(\widetilde{H} \to \mathbb{C} \times M, \nabla, H_\mathbb{R}, P)$ be a $(TERP(w))$-structure with $\widetilde{H}|_{\mathbb{C}^* \times M} = H$ such that for some $t_0 \in M$ the restricted $(TERP(w))$-structure $(\widetilde{H}, \nabla, H_\mathbb{R}, P)|_{\mathbb{C}\times\{t_0\}}$ is regular singular and the filtration $\widetilde{F}^\bullet$ defined in (7.57) for this $(TERP(w))$-structure gives PMHS's on $H^\infty_{\neq 1}$ and H^∞_1 of weights $w-1$ and w.*

Then any opposite filtration (definition 7.18 below) gives via proposition 7.16 (applying it to $\{\infty\} \times M$ instead of $\{0\} \times M$) rise to an extension of $\widetilde{H}$ to a bundle $\hat{H} \to \mathbb{P}^1 \times M$ such that $(\hat{H}, \nabla, P)$ is a $(trTLEP(w))$-structure in a neighborhood of $t_0 \in M$.

The endomorphism $\mathcal{V} : K \to K$ of the corresponding Frobenius type structure (theorem 5.7) is semisimple and has eigenvalues $-\alpha_1 + \frac{w}{2}, ..., -\alpha_\mu + \frac{w}{2}$, where $\alpha_1, ..., \alpha_\mu$ are the exponents of the restricted $(TERP(w))$-structure on $\mathbb{C}\times\{t_0\}$. One has for $\alpha \in \mathbb{R}$

$$\sum_{\beta \geq \alpha} \ker(\mathcal{V} - (-\beta + \frac{w}{2}\,\mathrm{id}) : K_{t_0} \to K_{t_0}) \tag{7.90}$$

$$= \{[a] \in K_{t_0} \mid a \in \mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{t_0\}})_0 \cap V^\alpha\} \ .$$

Proof. One has to adopt the proof of [He4, Theorem 7.16] and include the pairings P and S, using arguments in the proof of lemma 7.10. □

Opposite filtrations exist because of Deligne's $I^{p,q}$ (lemma 7.3); the space of all opposite filtrations is isomorphic to $\mathbb{C}^{N_{opp}}$ as algebraic manifold for some $N_{opp} \in \mathbb{Z}_{\geq 0}$ [He4, Lemma 10.21]. The following definition is adapted to the situation here.

Definition 7.18. Let $(H^\infty, H^\infty_\mathbb{R}, M_s, N, S)$ be as above and let $\widetilde{F}^\bullet$ be an M_s-invariant filtration on H^∞ such that the restrictions to $H^\infty_{\neq 1}$ and H^∞_1 give PMHS's of weight $w-1$ and w. A filtration $U^\bullet$ on H^∞ is called *opposite to* $\widetilde{F}^\bullet$ if it is decreasing and monodromy invariant with

$$N(U^p) \subset U^{p+1} \ , \tag{7.91}$$

$$S(U^p H^\infty_{\neq 1}, U^{-w-p} H^\infty_{\neq 1}) = 0 \ , \tag{7.92}$$

$$S(U^p H^\infty_1, U^{-w+1-p} H^\infty_1) = 0 \ , \tag{7.93}$$

$$H^\infty_{\neq 1} = \bigoplus_p \widetilde{F}^p \cap U^{-1-p} \cap H^\infty_{\neq 1} \ , \tag{7.94}$$

$$H^\infty_1 = \bigoplus_p \widetilde{F}^p \cap U^{-p} \cap H^\infty_1 \ . \tag{7.95}$$

7.6. E**-orbits of regular singular** $(TERP)$**-structures.** We consider $(H \to \mathbb{C}^* \times M, \nabla, H_\mathbb{R}, P)$ and $(H^\infty, H^\infty_\mathbb{R}, M_s, N, S)$ and elementary sections $es(A, \alpha)$ on $\mathbb{C}^* \times M$ as in section 7.5.

Suppose for a moment $M = \{pt\}$. Because of lemma 7.10 the filtration $\widetilde{F}^\bullet$ in section 7.3 of a regular singular $(TERP(w))$-structure $(\widetilde{H} \to \mathbb{C}, \nabla, H_\mathbb{R}, P)$ has a good chance to be the Hodge filtration of PMHS's on $H^\infty_{\neq 1}$ and H^∞_1.

Theorem 7.5 says that PMHS's correspond to nilpotent orbits of Hodge like filtrations. In theorem 7.20 a part of this correspondence is generalized to regular singular $(TERP(w))$-structures. Nilpotent orbits generalize to Euler field orbits. Theorem 7.20 is the main result of chapter 7. Together with the conjectural inverse statement (remark 7.21) it shows the close relationship between PMHS's and CV-structures.

Suppose now that $M \subset \mathbb{C}$ is a ball around $0 \in \mathbb{C}$ with coordinate ρ on $\mathbb{C}$. A *$(TERP(w))$-structure on M with Euler field* is a $(TERP(w))$-structure $(\widetilde{H} \to \mathbb{C} \times M, \nabla, H_\mathbb{R}, P)$ with $\widetilde{H}|_{\mathbb{C}^* \times M} = H$ such that $E := \frac{\partial}{\partial \rho}$ satisfies $\mathcal{U} = -C_E$. The vector field E is called *Euler field.*

Lemma 7.19. *(a) A $(TERP(w))$-structure on M with Euler field $E = \frac{\partial}{\partial \rho}$ is determined by its restriction to $0 \in M$ in the following way. Let*

$$\gamma_\rho : H_{(e^{-\rho}z, 0)} \to H_{(z,\rho)} \quad \textit{for } (z, \rho) \in \mathbb{C}^* \times M \tag{7.96}$$

be the isomorphism by parallel transport along the orbit of $z\partial_z + E$. Let $(z \mapsto \sigma(z, 0) \in H_{(z,0)})$ be a section in $\mathcal{O}(\widetilde{H}|_{\mathbb{C} \times \{0\}})$ in a neighborhood of 0. Then

$$(z, \rho) \mapsto \sigma(z, \rho) := \gamma_\rho \sigma(e^{-\rho} z, 0) \in H_{(z,\rho)} \tag{7.97}$$

is a section in $\mathcal{O}(\widetilde{H})$ in a neighborhood of $\{0\} \times M$. Such sections generate $\mathcal{O}(\widetilde{H})$.

(b) The $(TERP(w))$-structure in (a) extends canonically to $\mathbb{C} \supset M$.

(c) If the restriction of the $(TERP(w))$-structure in (a) to one parameter $\rho \in M$ is regular singular then the restriction to any parameter is regular singular. If the restriction to one parameter is generated by elementary sections then the restriction to any parameter is generated by elementary sections.

Proof. Let $\widetilde{\sigma_1}, ..., \widetilde{\sigma_\mu}$ be an $\mathcal{O}_{\mathbb{C}\times M,0}$-basis of sections of the germ $\mathcal{O}(\widetilde{H})_0$ and let $\sigma_1, ..., \sigma_\mu$ be the sections on $\widetilde{H}|_{\mathbb{C}^*\times M}$ in a neighborhood of $0 \in \mathbb{C} \times M$ with (7.97) and with $\sigma_i(z,0) = \widetilde{\sigma_i}(z,0)$ for $z \in \mathbb{C}^*$ close to 0. Write $\widetilde{\sigma} := (\widetilde{\sigma_1}, ..., \widetilde{\sigma_\mu})$ and $\sigma := (\sigma_1, ..., \sigma_\mu)$. Then

$$\sigma = \widetilde{\sigma} \cdot A \quad \text{for some } A \in \mathrm{Gl}(\mu, \mathcal{O}_{\mathbb{C}^*\times M}) \tag{7.98}$$

with $A|_{\rho=0} = \mathrm{id}$. The orbits of $z\partial_z + E$ are the sets $\{(z,\rho) \mid e^{-\rho}z = \text{const.}\}$. By definition $\nabla_{z\partial_z+E}\sigma = 0$, and because of $\mathcal{U} = -C_E$

$$\nabla_{z\partial_z+E}\widetilde{\sigma} = \widetilde{\sigma} \cdot B \quad \text{for some } B \in M(\mu\times\mu, \mathcal{O}_{\mathbb{C}\times M,0}) \ . \tag{7.99}$$

Therefore

$$(z\partial_z + E)A = -B \cdot A \ . \tag{7.100}$$

This shows $A \in \mathrm{Gl}(\mu, \mathcal{O}_{\mathbb{C}\times M,0})$ and part (a). The parts (b) and (c) are now trivial. □

Theorem 7.20. *Let $(\widetilde{H} \to \mathbb{C}\times\mathbb{C}, \nabla, H_\mathbb{R}, P)$ be a $(TERP(w))$-structure on $M = \mathbb{C}$ (with coordinate ρ) with Euler field $E = \frac{\partial}{\partial\rho}$. Suppose that the restrictions to fixed parameters $\rho \in M$ are regular singular $(TERP(w))$-structures.*

(a) Let $F^\bullet(\rho)$ and $\widetilde{F}^\bullet(\rho)$ be the corresponding filtrations from (7.56) and (7.57). Then

$$F^\bullet(\rho) = \exp(\frac{\rho}{2\pi i}N)F^\bullet(0) \ , \tag{7.101}$$
$$\widetilde{F}^\bullet(\rho) = \exp(\frac{\rho}{2\pi i}N)\widetilde{F}^\bullet(0) \ . \tag{7.102}$$

(b) Suppose that $(H^\infty_{\neq 1}, S, -N, \widetilde{F}^\bullet(0))$ and $(H^\infty_1, S, -N, \widetilde{F}^\bullet(0))$ are PMHS's of weights $w-1$ and w. Then a bound $b \in \mathbb{R}$ exists such that the restriction of the $(TERP(w))$-structure to $\{\rho \mid \mathrm{Re}\rho > b\}$ is a $(trTERP(w))$-structure with positive definite metric h.

(c) Consider the situation in (b). Let $\mathcal{Q}$ be the endomorphism of the corresponding $CV\oplus$-structure on $\{\rho \mid \mathrm{Re}\rho > b\}$. Let $\alpha_1, ..., \alpha_\mu$ be the exponents of the $(TERP(w))$-structure at $\rho = 0$. Then the eigenvalues of $\mathcal{Q}$ tend to $-\alpha_1 + \frac{w}{2}, ..., -\alpha_\mu + \frac{w}{2}$ for $\mathrm{Re}\rho \to \infty$.

Remark 7.21. Part (b) generalizes the direction '$\Longrightarrow$' in the correspondence between PMHS's and nilpotent orbits in theorem 7.5 (a). Probably also the inverse statement to part (b) is true. It would generalize the direction '$\Longleftarrow$' in theorem 7.5 (a), which is the most important consequence of Schmid's SL_2-orbit theorem. We do not use it and do not prove it here.

Proof of theorem 7.20. (a) An elementary section $es(A,\alpha)$ on $H \to \mathbb{C}^* \times M$ with $A \in H^\infty_{e^{-2\pi i\alpha}}$ transforms as follows under the shift γ_ρ from (7.96),

$$\gamma_\rho es(A,\alpha)(e^{-\rho}z,0) = e^{-\rho\alpha}\exp\left(\frac{\rho}{2\pi i}N\right) es(A,\alpha)(z,\rho)\ . \tag{7.103}$$

Therefore the filtrations $F^\bullet(\rho)$ and $\widetilde{F}^\bullet(\rho)$ from (7.56) and (7.57) satisfy (7.101) and (7.102).

(b) and (c) The proof is divided into five parts. In the parts (I)–(III) parts (b) and (c) are proved for the case when the $(TERP(w))$-structures for fixed $\rho \in M$ are generated by elementary sections, in (IV) and (V) they are proved in the general case.

(I) By theorem 7.5 (a) the tuples $(H^\infty_{\neq 1}, S, \widetilde{F}^\bullet(\rho))$ and $(H^\infty_1, S, \widetilde{F}^\bullet(\rho))$ are pure polarized Hodge structures of weight $w-1$ and w, respectively, if $\mathrm{Re}\rho$ is sufficiently large. But in order to apply lemma 7.12 (b) we need statements about $F^\bullet(\rho)$. We have to connect $\widetilde{F}^\bullet(\rho)$ and $F^\bullet(\rho)$ by a one parameter family of filtrations.

With a new parameter $r \in [0,1]$ the automorphism $G^{(\alpha)} : H^\infty_{e^{-2\pi i\alpha}} \to H^\infty_{e^{-2\pi i\alpha}}$ from (7.47) is generalized to

$$G^{(\alpha,r)} := \sum_{k\geq 0} \frac{1}{k!}\Gamma^{(k)}(\alpha)\left(r\frac{-N}{2\pi i}\right)^k : H^\infty_{e^{-2\pi i\alpha}} \to H^\infty_{e^{-2\pi i\alpha}}\ ; \tag{7.104}$$

then $G^{(\alpha,1)} = G^{(\alpha)}$ and $G^{(\alpha,0)} = \Gamma(\alpha)\cdot\mathrm{id}$. The filtrations $F^\bullet(\rho)$ are generalized to

$$F^\bullet_{\rho,r} H^\infty_{e^{-2\pi i\alpha}} := G^{(\alpha,r)}\widetilde{F}^\bullet(\rho) H^\infty_{e^{-2\pi i\alpha}} \tag{7.105}$$

for $r \in [0,1]$ and $\alpha \in (0,1]$; then $F^\bullet_{\rho,0} = \widetilde{F}^\bullet(\rho)$ and $F^\bullet_{\rho,1} = F^\bullet(\rho)$.

All these filtrations define mixed Hodge structures on $H^\infty_{\neq 1}$ and H^∞_1 with the weight filtration from $-N$, because they coincide on the quotients of the weight filtration. The proof of theorem 7.5 shows that a bound $b_1 \in \mathbb{R}$ exists such that for $\mathrm{Re}\rho > b_1$ they define pure Hodge structures of weight $w-1$ on $H^\infty_{\neq 1}$ and of weight w on H^∞_1. (Here it is important that the first part of the proof of theorem 7.5 does not use any pairings.)

With the parameter $r \in [0,1]$ the pairing S^∞ on H^∞ which was defined in lemma 7.12 is generalized to pairings $S^{\infty,r}$ by formula (7.81) with $G^{(\alpha)}$ and $G^{(\beta)}$ replaced by $G^{(\alpha,r)}$ and $G^{(\beta,r)}$.

For $r \in [0,1]$ and $\rho \in \mathbb{C}$ with $\mathrm{Re}\rho > b_1$ the pairing h^∞ in lemma 7.12 is generalized to pairings $h^{\infty,r,\rho}$ by the formulas (7.82)–(7.85) with $S^{\infty,r}$ instead of S^∞ and with the Hodge decomposition of the filtration $F^\bullet_{\rho,r}$. Then $h^{\infty,0,\rho}$ is a positive definite hermitian form for $\widetilde{F}^\bullet(\rho)$.

One sees easily that all the forms $h^{\infty,r,\rho}$ for ρ with $\mathrm{Re}\rho > b_1$ are hermitian and nondegenerate. Because $h^{\infty,0,\rho}$ is positive definite, they all are positive definite.

As in section 7.4, the filtration $F^\bullet(\rho)$ gives rise to a $(TERP(w))$-structure which is generated by elementary sections. Now lemma 7.12 shows that for ρ with $\mathrm{Re}\rho > b_1$ this is a $(trTERP(w))$-structure with positive definite metric.

This shows (b) if the $(TERP(w))$-structures $(\widetilde{H},\nabla,H_{\mathbb{R}},P)|_\rho$ for $\rho\in\mathbb{C}$ are all (one is sufficient by lemma 7.19 (c)) generated by elementary sections.

(II) Suppose that the $(TERP(w))$-structure at $\rho=0$ is generated by elementary sections and that Deligne's $I^{p,q}(0)$ for the mixed Hodge structure $(H^\infty,-N,F^\bullet(0))$ satisfy $I^{q,p}(0)=\overline{I^{p,q}(0)}$.

We will see that then the bound b in (b) can be chosen as $b=0$ and that the endomorphisms $\mathcal{Q},\mathcal{U}$ and $\kappa\mathcal{U}\kappa$ of the CV$\oplus$-structure can be calculated explicitely. As in the proof of theorem 7.5 we choose a basis $\{v_a\}_{a\in A}$ of $\bigoplus_{p,q}I_0^{p,q}(0)$ with (7.17), $\overline{v_a}=v_{k(a)}$, (7.19). We can even choose $v_a\in I_0^{p(a),q(a)}(0)\cap H^\infty_{\lambda(a)}$. Then $\lambda(k(a))=\overline{\lambda(a)}$. Define

$$b_{j,k,n}:=(-1)^j\binom{k}{j}\frac{1}{n(n-1)...(n-j+1)} \tag{7.106}$$

for $n\in\mathbb{Z}_{\geq 0}$, $0\leq k\leq n$, $0\leq j\leq k$. Define for $a\in A$ and $p\in\mathbb{Z}$ with $0\leq p(a)-p\leq n(a)$

$$\varphi_{a,p}(\rho):=\exp\left(\frac{\rho}{2\pi i}N\right)\sum_{j=0}^{p(a)-p}b_{j,p(a)-p,n(a)}\cdot\left(\frac{\mathrm{Re}\rho}{\pi i}N\right)^j v_a\ . \tag{7.107}$$

For $a\in A$ and $p\in\mathbb{Z}$ without $0\leq p(a)-p\leq n(a)$ define $\varphi_{a,p}(\rho):=0$.

Claim 1: *For* $\mathrm{Re}\rho>0$ *the pairs* $(H^\infty_{\neq 1},F^\bullet(\rho))$ *and* $(H^\infty_1,F^\bullet(\rho))$ *are pure Hodge structures of weight* $w-1$ *and* w, *and*

$$F^p(\rho)\cap\overline{F^{w-1-p}(\rho)}\cap H^\infty_{\neq 1}=\bigoplus_{a:\lambda(a)\neq 1,\,0\leq p(a)-p\leq n(a)}\mathbb{C}\cdot\varphi_{a,p}(\rho)\ , \tag{7.108}$$

$$F^p(\rho)\cap\overline{F^{w-p}(\rho)}\cap H^\infty_{1}=\bigoplus_{a:\lambda(a)=1,\,0\leq p(a)-p\leq n(a)}\mathbb{C}\cdot\varphi_{a,p}(\rho)\ , \tag{7.109}$$

This is an easy consequence of (7.23), (7.24) and the following elementary claim. We leave the proof of this claim to the reader.

Claim 2: *For* $n\in\mathbb{Z}_{\geq 0}$, $0\leq k\leq n$,

$$\begin{aligned}&e^x(1+b_1x+b_2x^2+...+b_kx^k)\in\mathbb{C}\oplus\mathbb{C}x\oplus...\oplus\mathbb{C}x^{n-k}\quad\mathrm{mod}\ x^{n+1}\\ \iff\ & b_j=b_{j,k,n}\ \ \textit{for}\ \ 1\leq j\leq k\ .\end{aligned} \tag{7.110}$$

Also the proof of the following claim is left to the reader.

Claim 3:

$$\begin{aligned}\left(\frac{\mathrm{Re}\rho}{\pi i}N\right)\varphi_{a,p}(\rho) &= (p(a)-p-n(a))\varphi_{a,p-1}(\rho)\\ &\quad +(n(a)-2p(a)+2p)\varphi_{a,p}(\rho)\\ &\quad +(p(a)-p)\varphi_{a,p+1}(\rho)\ .\end{aligned} \tag{7.111}$$

For $a\in A$ and $p\in\mathbb{Z}$ define $\alpha(a,p)\in\mathbb{R}$ as the number such that

$$\lambda(a)=e^{-2\pi i\alpha(a,p)}\ \ \text{and}\ \ w+[-\alpha(a,p)]=p\ . \tag{7.112}$$

The numbers $\alpha(a,p)$ for $a\in A$ and $p\in\mathbb{Z}$ with $0\leq p(a)-p\leq n(a)$ form the exponents of the $(TERP(w))$-structure at $\rho=0$.

Because of claim 1 and lemma 7.12, the $(TERP(w))$-structure $(\widetilde{H},\nabla,H_{\mathbb{R}},P)$ restricts on $\{\rho \mid \mathrm{Re}\rho>0\}$ to a $(trTERP(w))$-structure. Let $\hat{H}\to\mathbb{P}^1\times\{\rho\mid\mathrm{Re}\rho>0\}$ be the extension of $\widetilde{H}|_{\mathbb{C}\times\{\rho\ \mid\ \mathrm{Re}\rho>0\}}$ constructed in lemma 2.14 (e). Because of (7.79)

$$\pi_*\mathcal{O}(\hat{H}) = \bigoplus_{a\in A,p:\,0\leq p(a)-p\leq n(a)} \mathcal{O}_M\cdot es(\varphi_{a,p},\alpha(a,p))\ . \tag{7.113}$$

The endomorphisms $\mathcal{Q},\mathcal{U}$ and $\kappa\mathcal{U}\kappa$ on $K:=\hat{H}|_{\{0\}\times\{\rho\ \mid\ \mathrm{Re}\rho>0\}}$ lift canonically to $\hat{H}$ and $\pi_*\mathcal{O}(\hat{H})$; for $\sigma\in\pi_*\mathcal{O}(\hat{H})$

$$\nabla_{z\partial_z}\sigma = \frac{1}{z}\mathcal{U}\sigma-(\mathcal{Q}-\frac{w}{2}\,\mathrm{id})\sigma - z\kappa\mathcal{U}\kappa\sigma\ . \tag{7.114}$$

For $\sigma = es(\varphi_{a,p},\alpha(a,p))$ we obtain with claim 3

$$\begin{aligned}
&\nabla_{z\partial_z} es(\varphi_{a,p},\alpha(a,p)) = es((\alpha(a,p)\,\mathrm{id}+\frac{-N}{2\pi i})\varphi_{a,p},\alpha(a,p))\\
= \ & \frac{p(a)-p-n(a)}{-2\mathrm{Re}\rho}\cdot\frac{1}{z}\cdot es(\varphi_{a,p-1},\alpha(a,p-1))\\
+ \ & \left(\alpha(a,p)+\frac{n(a)-2p(a)+2p}{-2\mathrm{Re}\rho}\right)\cdot es(\varphi_{a,p},\alpha(a,p))\\
+ \ & \frac{p(a)-p}{-2\mathrm{Re}\rho}\cdot z\cdot es(\varphi_{a,p+1},\alpha(a,p+1))\ .
\end{aligned} \tag{7.115}$$

For $\mathrm{Re}\rho\to\infty$, the matrices of $\mathcal{U}$ and $\kappa\mathcal{U}\kappa$ with respect to the basis of $\pi_*\mathcal{O}(\hat{H})$ in (7.113) tend to 0, the matrix of $\mathcal{Q}$ tends to the diagonal matrix with eigenvalues $-\alpha(a,p)+\frac{w}{2}$. This shows part (c) of the theorem under the assumptions above.

(III) Suppose that the $(TERP(w))$-structure at $\rho=0$ is generated by elementary sections. Because of (I) a bound $b\in\mathbb{R}$ exists such that for ρ with $\mathrm{Re}\rho>b$ the filtrations $F^\bullet(\rho)$ and $\widetilde{F}^\bullet(\rho)$ give pure Hodge structures on $H^\infty_{\neq 1}$ and H^∞_1 and the $(TERP(w))$-structure at ρ is a $(trTERP(w))$-structure. It rests to estimate $\mathcal{Q}$ for $\mathrm{Re}\rho\to\infty$.

As in the proof of theorem 7.5, we choose a basis $\{v_a\}_{a\in A}$ of $\bigoplus_{p,q}I_0^{p,q}(0)$ with (7.17)–(7.19) and with $v_a\in I_0^{p(a),q(a)}\cap H^\infty_{\lambda(a)}$. We define $b_{j,k,n}$, $\varphi_{a,p}(\rho)$ and $\alpha(a,p)$ by (7.106), (7.107) and (7.112). We define as in (7.78) for ρ with $\mathrm{Re}\rho>b$

$$H^\infty(\beta,\rho) := F^{w+[-\beta]}(\rho)H^\infty_{e^{-2\pi i\beta}}\cap\overline{F^{[\beta]}(\rho)H^\infty_{e^{2\pi i\beta}}}\ . \tag{7.116}$$

Then $H^\infty=\bigoplus_\beta H^\infty(\beta,\rho)$, and the numbers β with multiplicities $\dim H^\infty(\beta,\rho)$ are the exponents $\alpha_1,...,\alpha_\mu$.

Claim 4: *A bound $b_2\geq b$ exists such that for ρ with* $\mathrm{Re}\rho>b_2$ *and any $a\in A,p\in\mathbb{Z}$ with $0\leq p(a)-p\leq n(a)$ the set $H^\infty(\alpha(a,p),\rho)$ contains a unique element $\psi_{a,p}(\rho)$ with*

$$\psi_{a,p}(\rho)=\varphi_{a,p}(\rho)+\sum_{b,q:\,\alpha(b,q)\neq\alpha(a,p)} c_{a,p,b,q}(\rho)\varphi_{b,q}(\rho)\ . \tag{7.117}$$

The coefficients $c_{a,p,b,q}(\rho)$ are of order $O((\mathrm{Re}\rho)^{-\varepsilon})$ for some $\varepsilon > 0$. The elements $\psi_{a,p}(\rho)$ for fixed ρ form a basis of H^∞.

It can be proved be refining the arguments in the proof of theorem 7.5. One uses that the coefficients of the matrix B in the proof of theorem 7.5 which give the leading contribution to its determinant are the same in the special case $I^{q,p} = \overline{I^{p,q}}$ and in the general case $I^{q,p} \neq \overline{I^{p,q}}$. The details are left to the reader.

Because of lemma 7.10 $N(F^p(\rho)) \subset F^{p-1}(\rho)$ and $N(H^\infty(\beta,\rho)) \subset \bigoplus_{\gamma-\beta\in\{-1,0,1\}} H^\infty(\gamma,\rho)$. Claim 3 and claim 4 show

$$\begin{aligned}\left(\frac{\mathrm{Re}\rho}{\pi i}N\right)\psi_{a,p}(\rho) &= (p(a)-p-n(a))\psi_{a,p-1}(\rho) \\ &\quad +(n(a)-2p(a)+2p)\psi_{a,p}(\rho) \\ &\quad +(p(a)-p)\psi_{a,p+1}(\rho) \\ &\quad + \sum_{b,q:\,\alpha(b,q)-\alpha(a,p)\in\{-1,0,1\}} \widetilde{c}_{a,p,b,q}(\rho)\psi_{b,q}(\rho)\ ,\end{aligned} \tag{7.118}$$

and the coefficients $\widetilde{c}_{a,p,b,q}(\rho)$ are of order $O((\mathrm{Re}\rho)^{-\varepsilon})$.

One can now repeat the arguments in (II) after claim 3 with $\psi_{a,p}(\rho)$ instead of $\varphi_{a,p}(\rho)$. One obtains formula (7.115) with an additional sum on the right hand side with coefficients of order $O((\mathrm{Re}\rho)^{-1-\varepsilon})$. This shows part (c) if the $(TERP(w))$-structure at $\rho = 0$ is generated by elementary sections.

(IV) Now part (b) will be proved in the general case. We choose a basis $\{v_a\}_{a\in A}$ of $\bigoplus_{p,q} I_0^{p,q}(0)$ as in step (III) and define $\varphi_{a,p}(\rho)$, $\alpha(a,p)$, $H^\infty(\beta,\rho)$, $\psi_{a,p}(\rho)$ and the bound $b_2 \geq b$ as in step (III).

By step (II) and corollary 7.11, for ρ with $\mathrm{Re}\rho > b_2$ the filtration $F^\bullet(\rho)$ corresponds to a $(trTERP(w))$-structure which is generated by elementary sections. Its sheaf of fiberwise global sections on $\mathbb{P}^1 \times \{\rho \mid \mathrm{Re}\rho > b_2\}$ is

$$\bigoplus_{a\in A,p:\,0\leq p(a)-p\leq n(a)} \mathcal{O}_M \cdot es(\psi_{a,p},\alpha(a,p))\ . \tag{7.119}$$

We want to show that for ρ with large $\mathrm{Re}\rho$ the $(TERP(w))$-structure $(\widetilde{H},\nabla,H_{\mathbb{R}},P)|_\rho$ is so close to it that it is also a $(trTERP(w))$-structure.

Claim 5: *For ρ with $\mathrm{Re}\rho > b_2$, $a\in A$, $p\in\mathbb{Z}$ with $0\leq p(a)-p\leq n(a)$, the germ $\mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{\rho\}})_0$ contains a unique element $\chi_{a,p}(\rho)$ with*

$$\begin{aligned}\chi_{a,p}(\rho) &= es(\psi_{a,p}(\rho),\alpha(a,p)) \\ &\quad + \sum_{(b,q,k)\in B_1(a,p)} c^{(k)}_{a,p,b,q}(\rho)\cdot z^{-k} es(\psi_{b,q}(\rho),\alpha(b,q))\ ,\end{aligned} \tag{7.120}$$

where

$$B_1(a,p) = \{(b,q,k) \mid b\in A, q\in\mathbb{Z}, k\in\mathbb{Z}_{\geq 1}, \alpha(b,q)-k > \alpha(a,p)\}. \tag{7.121}$$

The sets $B_1(a,p)$ are finite. The sections $\chi_{a,p}(\rho)$ form a $\mathbb{C}\{z\}$-basis of the germ $\mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{\rho\}})_0$. The coefficients $c^{(k)}_{a,p,b,q}(\rho)$ are of an exponential order $O(e^{-\varepsilon_2\cdot\mathrm{Re}\rho})$ for some $\varepsilon_2 > 0$.

Proof of claim 5. An elementary section is the leading elementary section of a section in $\mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{\rho\}})_0$ if and only if it is a linear combination of sections $z^k es(\psi_{b,q}(\rho), \alpha(b,q))$ with $k \geq 0$. This follows from the definition of the filtration $F^\bullet(\rho)$. Because of (7.74) the germ $\mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{\rho\}})_0$ is generated by sections which are linear combinations of finitely many elementary sections.

Now one can inductively find a section as in (7.120). The uniqueness is also clear; just consider the leading elementary section of the difference of two such sections. The set $B_1(a,p)$ is obviously finite.

Fix $\rho_0 \in \mathbb{C}$ with $\mathrm{Re}\rho_0 > b_2$. Because of lemma 7.19 and formula (7.103), one $\mathbb{C}\{z\}$-basis of $\mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{\rho\}})_0$ for ρ with $\mathrm{Re}\rho > b_2$ is given by the sections

$$\begin{aligned} & es(\exp\left(\frac{\rho-\rho_0}{2\pi i}N\right)\psi_{a,p}(\rho_0), \alpha(a,p)) \\ + & \sum_{(b,q,k)\in B_1(a,p)} e^{-(\rho-\rho_0)(\alpha(b,q)-k-\alpha(a,p))} \\ & \cdot c^{(k)}_{a,p,b,q}(\rho_0)\cdot z^{-k} es(\exp\left(\frac{\rho-\rho_0}{2\pi i}N\right)\psi_{b,q}(\rho_0), \alpha(b,q)) \ . \end{aligned} \tag{7.122}$$

If one expresses the sections $\exp\left(\frac{\rho-\rho_0}{2\pi i}N\right)\psi_{a,p}(\rho_0)$ in terms of the sections $\psi_{a,p}(\rho)$ or vice versa, the coefficients which turn up have at most polynomial order in $\mathrm{Re}\rho$. This follows from claim 4 and (7.107). If one constructs the $\mathbb{C}\{z\}$-basis $\chi_{a,p}(\rho)$ starting from the basis in (7.122), the exponential terms $e^{-(\rho-\rho_0)(\alpha(b,q)-k-\alpha(a,p))}$ in (7.122) also dominate the coefficients of the $\chi_{a,p}(\rho)$. This shows claim 5. □

Let $\hat{H} \to \mathbb{P}^1 \times \{\rho \mid \mathrm{Re}\rho > b_2\}$ be the extension of the $(TERP(w))$-structure $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)$ on $\mathbb{C}\times\{\rho \mid \mathrm{Re}\rho > b_2\}$ which was constructed in lemma 2.14 (e). For ρ with $\mathrm{Re}\rho \to \infty$ the coefficients $c^{(k)}_{a,p,b,q}(\rho)$ tend exponentially fast to 0. The bundle $\hat{H}|_{\mathbb{P}^1\times\{\rho\}}$ tends to the trivial bundle on $\mathbb{P}^1\times\{\rho\}$ with global sections in (7.119). Because triviality is an open condition, a bound $b_3 \geq b_2$ exists such that $\hat{H}|_{\mathbb{P}^1\times\{\rho \mid \mathrm{Re}\rho > b_3\}}$ is a trivial bundle. This shows part (b) in the general case. It rests to prove part (c).

(V) One can construct fiberwise global sections on $\hat{H}|_{\mathbb{P}^1\times\{\rho \mid \mathrm{Re}\rho > b_3\}}$ for some $b_3 \geq b_2$ starting with the sections $z^k\chi_{a,p}$ for $k \geq 0$ and their images under the map τ. By an infinite iteration one easily sees the following.

Claim 6: *For ρ with $\mathrm{Re}\rho > b_3$ for some $b_3 \geq b_2$ the space $\pi_*\mathcal{O}(\hat{H}|_{\mathbb{P}^1\times\{\rho\}})$ is generated by unique sections*

$$\begin{aligned} & es(\psi_{a,p}(\rho), \alpha(a,p)) \\ + & \sum_{(b,q,k)\in B_2(a,p)} \widetilde{c}^{(k)}_{a,p,b,q}(\rho)\cdot z^{-k} es(\psi_{b,q}(\rho), \alpha(b,q)) \ , \end{aligned} \tag{7.123}$$

where

$$B_2(a,p) = \{(b,q,k) \mid b \in A, q \in \mathbb{Z}, k \in \mathbb{Z}-\{0\}, \alpha_1 \leq \alpha(b,q)-k \leq \alpha_\mu\} \tag{7.124}$$

The sets $B_2(a,p)$ are finite. The coefficients $\widetilde{c}^{(k)}_{a,p,b,q}(\rho)$ are of exponential order $O(e^{-\varepsilon_2\cdot \mathrm{Re}\rho})$.

As in step (II) and step (III) one applies $\nabla_{z\partial_z}$ to these sections in order to determine $\mathcal{Q},\mathcal{U}$ and $\kappa\mathcal{U}\kappa$. The leading coefficients are the same as in formula (7.115). There are additional terms with coefficients of order $O((\mathrm{Re}\rho)^{-1-\varepsilon})$ from step (III) and additional terms with coefficients of order $O(e^{-\varepsilon_2\cdot\mathrm{Re}\rho})$ from claim 6. This shows part (c) in the general case and finishes the proof of theorem 7.20. □

Lemma 7.22. *Let $(\widetilde{H}\to\mathbb{C}\times\mathbb{C},\nabla,H_{\mathbb{R}},P)$ be a $(TERP(w))$-structure on $M=\mathbb{C}$ (with coordinate ρ) with Euler field $E=\frac{\partial}{\partial\rho}$. Let $\hat{H}\to\mathbb{P}^1\times\mathbb{C}$ be the extension of $\widetilde{H}$ constructed in lemma 2.14 (e).*

Two bundles $\hat{H}|_{\mathbb{P}^1\times\{\rho_1\}}$ and $\hat{H}|_{\mathbb{P}^1\times\{\rho_2\}}$ are canonically isomorphic if $\mathrm{Re}\rho_1=\mathrm{Re}\rho_2$. The set $\{\rho\mid \hat{H}|_{\mathbb{P}^1\times\{\rho\}}$ is not trivial$\}$ is either $\mathbb{C}$ or empty or a discrete union of real lines $\{\rho\mid\mathrm{Re}\rho=const.\}$.

Proof. Consider $\rho_1=0$ and $\rho_2=ir$ with $r\in\mathbb{R}$.

Recall the maps $\gamma:\mathbb{P}^1\to\mathbb{P}^1$, $z\mapsto\frac{1}{\overline{z}}$, γ_∇, τ_{real} and τ from lemma 2.14. The flow of $\mathrm{Im}(z\partial_z)$ on $\mathbb{C}^*$ commutes with γ. The ∇-parallel transport in $H\to\mathbb{C}^*\times M$ along the orbits of $\mathrm{Im}(z\partial_z)+\mathrm{Im}(E)$ commutes with γ_∇ and τ_{real}. It nearly commutes with τ; more precisely, for $\rho=ir$ with $r\in\mathbb{R}$,

$$\tau\circ\gamma_{ir}=e^{irw}\cdot\gamma_{ir}\circ\tau\ . \tag{7.125}$$

Because of this and (7.97), the maps γ_ρ in (7.96) extend for $\rho=ir$ with $r\in\mathbb{R}$ to isomorphisms

$$\hat{H}_{(e^{-ir}z,0)}\to\hat{H}_{(z,ir)}\ \text{ for } (z,ir)\in\mathbb{P}^1\times i\mathbb{R}\ . \tag{7.126}$$

This shows the first part of the lemma. The second part follows from the first part and from the fact that the set $\{\rho\mid\hat{H}|_{\mathbb{P}^1\times\{\rho\}}$ is not trivial$\}$ is real analytic. □

Remarks 7.23. (i) Let $(\widetilde{H}\to\mathbb{C}\times M,\nabla,H_{\mathbb{R}},P)$ be a $(TERP(w))$-structure on $M=\mathbb{C}$ (with coordinate ρ) with Euler field $E=\frac{\partial}{\partial\rho}$. It induces a $(TERP(w))$-structure on the quotient $M_{quot}:=M/2\pi i\mathbb{Z}$;

$$\mathbb{C}=M\to M_{quot}\cong\mathbb{C}^*,\ \rho\mapsto e^{-\rho}=:\zeta\ . \tag{7.127}$$

The deck transformation $\rho\mapsto\rho+2\pi i$ on M is lifted with the maps $\gamma_{2\pi i+\rho}\circ\gamma_\rho^{-1}$ to an automorphism of the $(TERP(w))$-structure on M. This gives the $(TERP(w))$-structure on M_{quot}. The coordinate ζ on $M_{quot}\cong\mathbb{C}^*$ is chosen such that E is mapped to $-\zeta\partial_\zeta$. The flow of $\mathrm{Re}(-\zeta\partial_\zeta)$ tends to 0. Also with this choice of the coordinate on M_{quot}, the monodromy on $\{z\}\times\mathbb{C}^*$ is equal to the usual monodromy on $\mathbb{C}^*\times\{\zeta\}$.

(ii) Suppose that for any (one is sufficient) ρ and ζ the $(TERP(w))$-structures in (i) are regular singular. Then $\mathcal{U}=-C_E$ is nilpotent. Then the $(TERP(w))$-structure on M_{quot} is *tame* in the sense of [Si2, ch. 2]. The estimates for $\mathcal{U}$ and $\kappa\mathcal{U}\kappa$ in the proof of theorem 7.20 can also be obtained from [Si2, Theorem 1], but not the estimate for $\mathcal{Q}$.

8. The case of hypersurface singularities

In section 8.1 a version of the construction of Frobenius manifolds for hypersurface singularities is presented. A central step is the construction of a $(TERP)$-structure, using essentially oscillating integrals. In section 8.2 this is extended to a construction of tt^* geometry and of CDV-structures. With chapter 6 and theorem 7.20 a part of a conjecture about the behaviour of the tt^* geometry along the real vector field $E+\overline{E}$ is proved. Some examples in section 8.3 illustrate this behaviour.

8.1. **Construction of Frobenius manifolds with oscillating integrals.** In the beginning of the 80ies K. Saito [SK2][SK3] studied the Gauß–Manin connection of hypersurface singularities and developed the notion of primitive forms. His work was completed by M. Saito [SM2] and resulted in a construction of Frobenius manifolds. A detailed and simplified account is given in [He4].

In [Sab3][Sab4] a modified version is outlined which uses a Fourier–Laplace dual of the Gauß–Manin connection, a $(TERP)$-structure in our notations. Two center pieces of this construction are recalled in the theorems 5.12 and 7.17. But Sabbah considered the case of global functions on affine manifolds. In this section we will sketch his version for the classical case of function germs. The following data will be constructed in four steps.

Step 1: A function $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$, a semiuniversal unfolding F, the base space M as an F-manifold (definition 4.2), the μ-constant stratum $S_\mu\subset M$; the integer w of the earlier chapters will be $w=n+1$.

Step 2: The topological part $(H\to\mathbb{C}^*\times M,\nabla,H_\mathbb{R},P)$ of a $(TERP(n+1))$-structure and the data (H^∞,M_s,N,S) (section 7.2).

Step 3: A $(TERP(n+1))$-structure $(\widetilde{H}\to\mathbb{C}\times M,\nabla,H_\mathbb{R},P)$ (definition 2.12) with $\widetilde{H}|_{\mathbb{C}^*\times M}=H$, the bundle $K:=\widetilde{H}|_{\{0\}\times M}$ with Higgs field C^K, endomorphism $\mathcal{U}^K$ and holomorphic metric g^K (lemma 2.14).

Step 4: A Hodge filtration $\widetilde{F}^\bullet$ for PMHS's on $H^\infty_{\neq 1}$ and H^∞_1 as in section 7.3, an extension of the $(TERP(n+1))$-structure to a $(trTLEP(n+1))$-structure $(\widetilde{H}^{(U^\bullet)}\to\mathbb{P}^1\times M,\nabla,P)$ with theorem 7.17, a Frobenius manifold structure on M with theorem 5.12.

Step 1: Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a holomorphic function germ with an isolated singularity at 0. Its Milnor number μ is the dimension of the Jacobi algebra $\mathcal{O}_{\mathbb{C}^{n+1},0}/\left(\frac{\partial f}{\partial x_0},...,\frac{\partial f}{\partial x_n}\right)$.

Let $F:(\mathbb{C}^{n+1}\times\mathbb{C}^\mu,0)\to(\mathbb{C},0)$ be a semiuniversal unfolding. That means that $F|_{(\mathbb{C}^{n+1}\times\{0\},0)}=f$ and that the derivations $\frac{\partial F}{\partial t_i}|_{(\mathbb{C}^{n+1}\times\{0\},0)}$, $i=1,...,\mu$, represent a basis of the Jacobi algebra of f; here $(x,t)=(x_0,...,x_n,t_1,...,t_\mu)\in\mathbb{C}^{n+1}\times\mathbb{C}^\mu$.

One can choose a representative $F:\mathcal{X}\to\Delta$ with $\Delta=B^1_\eta=\{\tau\mid|\tau|<\eta\}\subset\mathbb{C}$, $M=B^\mu_\theta\subset\mathbb{C}^\mu$ and $\mathcal{X}=F^{-1}(\Delta)\cap(B^{n+1}_\varepsilon\times M)\subset\mathbb{C}^{n+1}\times\mathbb{C}^\mu$ for suitable small $\varepsilon,\eta,\theta>0$. This F is a good representative in the sense of [Lo, ch. 2D], and the map

$$\varphi : \mathcal{X} \to \Delta \times M,\ (x,t) \mapsto (F(x,t),t)\ , \tag{8.1}$$

is a C^∞-fibration of Milnor fibers outside of a discriminant $\check{\mathcal{D}} \subset \Delta \times M$. The critical space $C \subset \mathcal{X}$ of φ is defined by the ideal $\left(\frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_n}\right)$. The discriminant is $\check{\mathcal{D}} = \varphi(C)$. The projection $pr_{C,M} : C \to M$ is finite and flat of degree μ. The Kodaira–Spencer map

$$\mathbf{a}_1 : \mathcal{T}_M \to (pr_{C,M})_* \mathcal{O}_C,\ \frac{\partial}{\partial t_i} \mapsto \frac{\partial F}{\partial t_i}|_C \tag{8.2}$$

is an isomorphism of free $\mathcal{O}_M$-modules of rank μ, because the unfolding F is semiuniversal. Here $M = B^\mu_\theta$ has to be chosen sufficiently small. The multiplication on the right hand side of (8.2) induces a multiplication $\circ$ on $\mathcal{T}_M$. With $e := \mathbf{a}_1^{-1}(1|_C)$ and $E := \mathbf{a}_1^{-1}(F|_C)$, the tuple $(M, \circ, e, E)$ is an F-manifold with Euler field E [He4, theorem 5.3]. This will also follow from the $(TERP(n+1))$-structure in step 3 and from lemma 4.3. One should consider F as a family of functions

$$F_t : \mathcal{X}_t \to \Delta,\ \text{where } \mathcal{X}_t := \mathcal{X} \cap \mathbb{C}^{n+1} \times \{t\}\ , \tag{8.3}$$

parameterized by $t \in M$, with $F_0 = f$. The Kodaira–Spencer map gives for each tangent space T_tM an isomorphism

$$(T_tM, \circ, E|_t) \cong (\bigoplus_{x \in Sing(F_t)} \text{Jacobi algebra of } (F_t, x), \text{mult.}, [F_t])\ . \tag{8.4}$$

The F-manifold is generically semisimple, because for generic t the function F_t has μ A_1-singularities.

The μ-constant stratum $S_\mu \subset M$ is the subvariety

$$S_\mu = \{t \in M \mid Sing(F_t) = \{x\} \text{ for some point } x \in F_t^{-1}(0)\}\ . \tag{8.5}$$

For singularities with $n \geq 2$ it is in general not smooth. Define

$$\mathcal{U} := E\circ : \mathcal{T}_M \to \mathcal{T}_M\ . \tag{8.6}$$

Then

$$S_\mu = \{t \in M \mid \mathcal{U} : T_tM \to T_tM \text{ is nilpotent}\}\ . \tag{8.7}$$

The inclusion '$\subset$' is obvious because of (8.4). The inclusion '$\supset$' follows from the result of Gabrielov, Lazzeri and Lê that the singularities of F_t are not all in the same fiber if F_t has several singularities (cf. [He4, Theorem 12.1] and references there).

Step 2: The bundle $H \to \mathbb{C}^* \times M$ will be the dual bundle to a bundle of homology classes of Lefschetz thimbles. Fix for a moment $(z,t) \in \mathbb{C}^* \times M$. A Lefschetz thimble in $\mathcal{X}_t$ with boundary in $F_t^{-1}\left(\eta \cdot \frac{z}{|z|}\right) \subset \mathcal{X}_t$ is an $(n+1)$-cycle $\Gamma = \bigcup_{\tau \in \gamma([0,1])} \delta(\tau)$, where $\gamma : [0,1] \to \overline{\Delta}$ is a path with $\gamma(0) \in F_t(Sing(F_t))$, $\gamma((0,1)) \cap F_t(Sing(F_t)) = \emptyset$, $\gamma(1) = \eta \cdot \frac{z}{|z|}$ and $\delta(\tau)$ for $\tau \in \gamma([0,1])$ is a continuous family of n-cycles in $F_t^{-1}(\tau)$ which vanish for $\tau \to \gamma(0)$. For more details see [Ph3][Ph4].

If t is generic then F_t has μ A_1-singularities with different critical values. In that case standard arguments (e.g. [AGV, ch. 2], [Lo, (5.11)]) show that one obtains a space homotopy equivalent to $\mathcal{X}_t$ if one glues μ Lefschetz thimbles to the fiber $F_t^{-1}(\eta\cdot\frac{z}{|z|})$. Therefore for generic t

$$H^{Lef}_{(z,t),\mathbb{Z}} := H_{n+1}(\mathcal{X}_t, F_t^{-1}(\eta\cdot\frac{z}{|z|}),\mathbb{Z}) \cong \mathbb{Z}^\mu \ . \tag{8.8}$$

Because $\mathcal{X}_t$ is contractible, the homology sequence of the pair $(\mathcal{X}_t, F_t^{-1}(\eta\cdot\frac{z}{|z|}))$ shows that the map

$$H^{Lef}_{(z,t),\mathbb{Z}} \to H_n(F_t^{-1}(\eta\cdot\frac{z}{|z|}),\mathbb{Z}),\ [\Gamma]\to[\delta(\gamma(1))] \tag{8.9}$$

is an isomorphism.

Because the pairs $(\mathcal{X}_t, F_t^{-1}(\eta\cdot\frac{z}{|z|}))$ are diffeomorphic for all $(z,t)\in\mathbb{C}^*\times M$, (8.8) and (8.9) hold for all $(z,t)\in\mathbb{C}^*\times M$, and $\bigcup_{(z,t)\in\mathbb{C}^*\times M} H^{Lef}_{(z,t),\mathbb{Z}}$ is a $\mathbb{Z}$-bundle of rank μ. Therefore

$$H := \bigcup_{(z,t)\in\mathbb{C}^*\times M} \mathrm{Hom}(H^{Lef}_{(z,t),\mathbb{C}},\mathbb{C}) \tag{8.10}$$

is a flat vector bundle with a connection ∇ and a real flat subbundle $H_\mathbb{R}$. The monodromy M_{mon} is the same as the monodromy of the Milnor fibration of f, because of the isomorphism in (8.9). It is quasiunipotent. Semisimple and unipotent part are called M_s and M_u, the nilpotent part is $N:=\log M_u$.

The intersection form for Lefschetz thimbles with boundaries in $F_t^{-1}(\eta\cdot\frac{z}{|z|})$ and in $F_t^{-1}(\eta\cdot\frac{-z}{|z|})$ gives a well defined nondegenerate pairing [Ph4]

$$<\cdot,\cdot>: H^{Lef}_{(z,t),\mathbb{Z}}\times H^{Lef}_{(-z,t),\mathbb{Z}} \to \mathbb{Z} \ . \tag{8.11}$$

It induces an isomorphism $H^{Lef}_{(z,t),\mathbb{Z}} \to \mathrm{Hom}(H^{Lef}_{(-z,t),\mathbb{Z}},\mathbb{Z})$. With this isomorphism for z and for $-z$ one obtains a pairing $<\cdot,\cdot>^*$ on the dual spaces. Define the pairing

$$\begin{aligned} P \ &: \ H_{(z,t)}\times H_{(-z,t)} \to \mathbb{C}\ , \\ &(a,b)\mapsto (-1)^{\frac{n(n+1)}{2}}\frac{1}{(2\pi i)^{n+1}} <a,b>^* \ . \end{aligned} \tag{8.12}$$

It is ∇-flat, nondegenerate, $(-1)^{n+1}$-symmetric and takes values in $i^{n+1}\mathbb{R}$ on $H_\mathbb{R}$.

Now the data $(H^\infty, H^\infty_\mathbb{R}, M_s, N, S)$ are defined as in section 7.2. The space H^∞ is the space of manyvalued global flat sections of the bundle H, see (7.29). The space $H^\infty_\mathbb{R}\subset H^\infty$ is the real subspace of sections in $H_\mathbb{R}$. The induced monodromy operator on H^∞ is also called M_{mon} with semisimple part M_s and nilpotent part N. The eigenspaces of M_s are the H^∞_λ; and $H^\infty_{\neq 1} := \bigoplus_{\lambda\neq 1} H^\infty_\lambda$.

A pairing S on H^∞ is defined as in section 7.2 (f). It satisfies the properties in lemma 7.6 for $w=n+1$. In [He4, ch. 10] another space H^∞ with a pairing S was defined. The spaces H^∞ are canonically isomorphic by the isomorphism in (8.9). Using topological arguments from [Ph4] or [AGV] one can show that the pairings S coincide.

Step 3: One has to start with the Gauß–Manin connection of the map $\varphi : \mathcal{X} \to \Delta \times M$ in (8.1). Detailed discussions from different points of view are given in [Mal1], [Gr], [Ph1], [SK2][SK3], [Lo], [Od], [SM2], [AGV], [He4]. The sketchy version here is close to [He4].

Work of Brieskorn on the Gauß–Manin connection for f motivated the definition of the following sheaf by Greuel and K. Saito. The sheaf

$$\mathcal{H}^{(GM)} := \varphi_* \Omega^{n+1}_{\mathcal{X}/M} / \mathrm{d}F \wedge \mathrm{d}\varphi_* \Omega^{n-1}_{\mathcal{X}/M} \tag{8.13}$$

is a coherent and even a free $\mathcal{O}_{\Delta \times M}$-module of rank μ [Gr]. Let $H^{(GM)}$ be the corresponding vector bundle. The restriction of $H^{(GM)}$ to $\Delta \times M - \check{\mathcal{D}}$ is canonically isomorphic to the cohomology bundle

$$\bigcup_{(\tau,t) \in \Delta \times M - \check{\mathcal{D}}} H^n(F_t^{-1}(\tau), \mathbb{C}) \ . \tag{8.14}$$

The flat Gauß–Manin connection $\nabla^{(GM)}$ on this cohomology bundle has a logarithmic pole along the discriminant $\check{\mathcal{D}}$ with respect to the bundle $H^{(GM)}$ [SK2][SK3].

In order to carry out a Fourier–Laplace transformation in the sense of [Sab4, V.2.c] one has to make things algebraic in the τ-direction, where now τ is the coordinate on $\Delta \subset \mathbb{C}$ used in (8.14). Because $\check{\mathcal{D}}$ does not meet $(\partial\Delta) \times M$, one can extend the bundle $H^{(GM)} \to \Delta \times M$ to a bundle $\widetilde{H}^{(GM)} \to \mathbb{C} \times M$ with flat connection outside of $\check{\mathcal{D}}$.

The sheaf $\mathcal{O}(\widetilde{H}^{(GM)})$ is a free $\mathcal{O}_{\mathbb{C}\times M}$-module of rank μ and extends the sheaf $\mathcal{H}^{(GM)}$. It contains the sheaf $\mathcal{O}^{mod}(\widetilde{H}^{(GM)})$ of sections of $\widetilde{H}^{(GM)}$ which have moderate growth along $\{\infty\} \times M$. Let $\pi_1 : \mathbb{C} \times M \to M$ be the projection. Then the sheaf $\mathcal{H}^{alg,GM} := (\pi_1)_* \mathcal{O}^{mod}(\widetilde{H}^{(GM)})$ is a free $\mathcal{O}_M[\tau]$-module of rank μ. It has a logarithmic pole along $\check{\mathcal{D}}$ and a regular singularity along $\{\infty\} \times M$.

Let $\pi_2 : \Delta \times M \to M$ be the projection. Any element of $(\pi_2)_* \mathcal{H}^{(GM)}$ has a unique preimage in $(\pi_2)_* \mathcal{H}^{(GM)}$ under $\nabla^{(GM)}_{\partial_\tau}$ ([SK2][SK3][Od] [He4, Theorem 10.7]). Therefore $(\nabla^{(GM)}_{\partial_\tau})^{-1}$ acts on $(\pi_2)_* \mathcal{H}^{(GM)}$ and also on $\mathcal{H}^{alg,GM}$.

With the following Fourier–Laplace transformation one obtains an $\mathcal{O}_M[z]$-module $\mathcal{H}^{alg,osc}$ with a meromorphic connection $\nabla^{(osc)}$ [Sab4, V.2]: As sheaves $\mathcal{H}^{alg,osc} = \mathcal{H}^{alg,GM}$. Functions in $\mathcal{O}_M$ and vector fields $X \in \mathcal{T}_M$ are lifted canonically to $\mathbb{C} \times M$. Their actions on $\mathcal{H}^{alg,osc}$ and $\mathcal{H}^{alg,GM}$ coincide,

$$\nabla^{(GM)}_X = \nabla^{(osc)}_X \quad \text{for } X \in \mathcal{T}_M \ . \tag{8.15}$$

The actions of z and of $\nabla^{(osc)}_{\partial_z}$ on $\mathcal{H}^{alg,osc}$ are given by

$$z \ = \ (\nabla^{(GM)}_{\partial_\tau})^{-1} \ , \tag{8.16}$$

$$z^2 \nabla^{(osc)}_{\partial_z} \ = \ \tau \ . \tag{8.17}$$

Then $\mathcal{H}^{alg,osc}$ is a free $\mathcal{O}_M[z]$-module of rank μ and $\nabla^{(osc)}$ is a flat connection on it with regular singularity along $\{\infty\} \times M$ and pole of Poincaré rank 1 along $\{0\} \times M$, see [Sab4, V.2.7] for the case without parameters $t_1, ..., t_\mu$. The order 1 pole for derivations $\nabla^{(osc)}_{\partial/\partial t_i}$ follows from formula (8.23) below. The sheaf $\mathcal{H}^{alg,osc}$ is the sheaf $(\pi_1)_* \mathcal{O}^{(mod)}(\widetilde{H})$ of fiberwise global sections with moderate growth along $\{\infty\}\times M$ of a vector bundle $\widetilde{H}\to\mathbb{C}\times M$ with flat connection $\nabla^{(osc)}$ on $\widetilde{H}|_{\mathbb{C}^*\times M}$. We claim that there is a canonical isomorphism

$$(\widetilde{H}|_{\mathbb{C}^*\times M},\nabla^{(osc)})\cong (H,\nabla) \tag{8.18}$$

with (H,∇) from step 2. Formula (8.19) will make this isomorphism explicit. Consider a section $\omega_{GM}\in\mathcal{H}^{alg,GM}$, the corresponding section $\omega_{osc}\in\mathcal{H}^{alg,osc}$, a point $(z,t)\in\mathbb{C}^*\times M$ and a Lefschetz thimble $\Gamma=\bigcup_{\tau\in\gamma([0,1])}\delta(\tau)$ with $\delta(\tau)$ and $\gamma:[0,1]\to\overline{\Delta}$ as in step 2. Then $[\Gamma]\in H_{n+1}(\mathcal{X}_t,F_t^{-1}(\eta\cdot\frac{z}{|z|}),\mathbb{Z})$ and we want to define a number $\omega_{osc}([\Gamma])$.

Extend γ to a path $\gamma:[0,\infty)\to\mathbb{C}$ with $\gamma([1,\infty))$ the half line from $\gamma(1)$ to $\infty\cdot z$. Extend the homology bundle which is dual to the cohomology bundle in (8.14) to a flat bundle on $\mathbb{C}\times M-\check{\mathcal{D}}$. Extend the family $[\delta(\tau)]$ of homology classes of cycles to a flat section in this bundle over $\gamma([0,\infty))$. Define

$$\omega_{osc}([\Gamma]):=\int_{\gamma([0,\infty))}e^{-\tau/z}\omega_{GM}([\delta(\tau)])\mathrm{d}\tau\ . \tag{8.19}$$

This imitates *oscillating integrals*. One has to show that it is well defined and that ω_{osc} gives in this way a holomorphic section in H which has moderate growth along $\{\infty\}\times M$. Then it is easy to see that this makes the Fourier–Laplace transformation above explicit. We will not carry out the details here. Closely related considerations (but without the extension to $\{\infty\}\times M$) are presented in [Ph3][Ph4].

Also the pairing P is treated there. In [Ph4, 2ième partie 4] it is shown that the pairing P on $\mathcal{H}^{alg,osc}$ which is defined in (8.12) is the image under the Fourier–Laplace transformation of K. Saito's higher residue pairings ([SK1], cf. also [He4, 10.4]). This implies that $(\widetilde{H},\nabla,H_{\mathbb{R}},P)$ is a $(TERP(n+1))$-structure.

Let us describe the bundle $K:=\widetilde{H}|_{\{0\}\times M}$, its Higgs field C^K, the endomorphism $\mathcal{U}^K$ and the pairing g^K (lemma 2.14). The sheaf $\mathcal{O}(K)$ is

$$\begin{aligned}\mathcal{O}(K) &\cong \mathcal{H}^{alg,osc}/z\mathcal{H}^{alg,osc}\cong\mathcal{H}^{alg,GM}/(\nabla^{(GM)}_{\partial_\tau})^{-1}\mathcal{H}^{alg,GM}\\ &\cong (\pi_2)_*\Omega^{n+1}_{\mathcal{X}/\Delta\times M}=:\Omega_F\ ,\end{aligned} \tag{8.20}$$

where $\pi_2:\Delta\times M\to M$ is the projection [SK2][SK3][He4]. Let us denote this isomorphism by $\mathbf{a}_2:\mathcal{O}(K)\to\Omega_F$. The sheaf Ω_F is an $\mathcal{O}_M$-module of rank μ and a free $(pr_{C,M})_*\mathcal{O}_C$-module of rank 1. With $\mathbf{a}_1$ as in (8.2), the Higgs bundle structure on K is given by

$$\mathbf{a}_2(C^K_{\partial/\partial t_i}b)=-\mathbf{a}_1(\frac{\partial}{\partial t_i})\cdot\mathbf{a}_2(b) \tag{8.21}$$

for $b\in\mathcal{O}(K)$. Therefore $\mathcal{O}(K)$ is a free $\mathcal{T}_M$-module of rank 1 and

$$C^K_XC^K_Y=-C^K_{X\circ Y}\text{ and }C^K_e=-\,\mathrm{id}\ . \tag{8.22}$$

Formula (8.21) follows from the formula for the Gauß–Manin connection (e.g. [He4, Theorem 10.5]),

$$\nabla^{(GM)}_{\partial/\partial t_i}[\omega]=[\mathrm{Lie}_{\partial/\partial t_i}\,\omega]-\nabla^{(GM)}_{\partial_\tau}[\frac{\partial F}{\partial t_i}\omega] \tag{8.23}$$

for $\omega \in \Omega^{n+1}_{\mathcal{X}}$ with $[\omega] \in \mathcal{H}^{(GM)}$.

The endomorphism $\mathcal{U}^K$ is defined by the action of $z^2\nabla_{\partial_z}$ on $\mathcal{O}(K)$ (lemma 2.14). The definition of the Euler field, (8.17) and (8.21) show

$$\mathcal{U}^K = -C^K_E \ . \tag{8.24}$$

From Pham's result about the pairing P and the higher residue pairings and from their properties, it follows that the pairing g^K on K is mapped with $\mathbf{a}_2$ to Grothendieck's residue pairing on Ω_F (see e.g. [He4, 10.4]).

Step 4: The pair $(\mathcal{H}^{(GM)}, \nabla^{(GM)})$ has a logarithmic pole along the discriminant $\check{\mathcal{D}}$. Therefore the restriction to $t=0$ has a regular singularity at 0. The germ $\mathcal{O}(H^{(GM)}|_{\Delta\times\{0\}})_0$ at 0 is called Brieskorn lattice and has been studied a lot. Its properties are put together in [He4, 10.6]. They can be translated to properties of the restricted $(TERP(n+1))$-structure $(\widetilde{H}, \nabla, H_{\mathbb{R}}, P)|_{\mathbb{C}\times\{0\}}$. Some of them are the following:

(α) This $(TERP(n+1))$-structure is regular singular at 0; therefore all the notions in sections 7.2–7.4 can be used to discuss it.

(β) Its exponents $\alpha_1, ..., \alpha_\mu$ (defined in (7.71)) satisfy [Mal1]

$$0 < \alpha_1 \leq ... \leq \alpha_\mu < n+1 \ , \ \ \alpha_i + \alpha_{\mu+1-i} = n+1 \ . \tag{8.25}$$

(γ) Define $F^\bullet$ and $\widetilde{F}^\bullet$ for this $(TERP(n+1))$-structure with (7.56) and (7.57). Then $(H^\infty_{\neq 1}, M_s, -N, S, \widetilde{F}^\bullet)$ is a PMHS of weight n with automorphism M_s, and $(H^\infty_1, -N, S, \widetilde{F}^\bullet)$ is a PMHS of weight $n+1$ [He4, Theorem 10.29] [He3, Theorem 3.5] (the minus sign in $-N$ stems from the different definitions of the monodromy here and in [Sch]).

(δ) The smallest exponent has multiplicity one, i.e. $\alpha_1 < \alpha_2$, and for $\omega \in \mathcal{O}(\widetilde{H}|_{\mathbb{C}\times\{0\}})_0$ with $[\omega] \in K_0$

$$\begin{aligned} &\text{the map } C^K_\bullet[\omega] : T_0M \to K_0 \text{ is an isomorphism} \\ &\iff \omega \in V^{\alpha_1} - V^{\alpha_2} \ . \end{aligned} \tag{8.26}$$

[SM3, 3.11] (cf. [He4, Theorem 10.32]) (V^α was defined in (7.45)).

Because of (γ) one can apply theorem 7.17. The choice of any filtration $U^\bullet$ on H^∞ which is opposite to $\widetilde{F}^\bullet$ (definition 7.18) yields an extension $\widetilde{H}^{(U^\bullet)} \to \mathbb{P}^1 \times M$ of $\widetilde{H} \to \mathbb{C}\times M$ such that the tuple $(\widetilde{H}^{(U^\bullet)}, \nabla, H_{\mathbb{R}}, P)$ is a $(trTLEP(n+1))$-structure.

Let $(K, \nabla^r, C^K, g^K, \mathcal{U}^K, \mathcal{V}^K)$ be the corresponding Frobenius type structure on K. By theorem 7.17 $\mathcal{V}^K$ is semisimple with eigenvalues $-\alpha_1 + \frac{n+1}{2}, ..., -\alpha_\mu + \frac{n+1}{2}$. In order to obtain a Frobenius manifold one wants to apply theorem 5.12. One needs a ∇^r-flat section $\zeta \in \mathcal{O}(K)$ such that $C^K_\bullet\zeta : \mathcal{T}_M \to \mathcal{O}(K)$ is an isomorphism and ζ is an eigenvector of $\mathcal{V}^K$. Because of (7.90) and (δ) above, the space of all such sections is $\{\zeta \in \mathcal{O}(K) \mid \nabla^r\zeta = 0, \ \mathcal{V}^K\zeta = (-\alpha_1 + \frac{n+1}{2})\zeta\}$, and it is one dimensional. This recovers [He4, Theorem 11.1].

Theorem 8.1. *Each choice $(U^\bullet, \zeta)$ of an opposite filtration $U^\bullet$ and a generator ζ of a one dimensional space gives a Frobenius manifold structure $(M, \circ, e, E, g)$ on M. The flat endomorphism $\nabla^g_\bullet E$ of TM is semisimple with* *eigenvalues* $1+\alpha_1-\alpha_i$, $i=1,...,\mu$; *and* $\mathrm{Lie}_E(g)=(2-(\alpha_\mu-\alpha_1))g$. *The choice* $(U^\bullet, c\zeta)$ *for* $c\in\mathbb{C}^*$ *gives the Frobenius manifold* $(M,\circ,e,E,c^2g)$.

8.2. tt^* **geometry in the singularity case.** In step 3 in section 8.1 a $(TERP(n{+}1))$-structure $(\widetilde{H}\to\mathbb{C}\times M,\nabla,H_{\mathbb{R}},P)$ was constructed on the base space M of a semiuniversal unfolding F of a singularity $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$, using essentially oscillating integrals. The results in chapter 6 show that it is generically a $(trTERP(n+1))$-structure.

Theorem 8.2. *(a) The set* R *of points in* M *where the* $(TERP(n+1))$*-structure is not a* $(trTERP(n{+}1))$*-structure is a real analytic subvariety* $R\subset M$, $R\neq M$. *The set* R *is invariant under the flow of* $E-\overline{E}$. *The restriction to* $M-R$ *of the bundle* $K=\widetilde{H}|_{\{0\}\times M}$ *carries a canonical CV-structure.*

(b) If $(M,\circ,e,E,g)$ *is one of the Frobenius manifold structures in theorem 8.1, then on* $M-R$ *a real structure* κ *exists such that* $(M-R,\circ,e,E,g,\kappa)$ *gives a CDV-structure. The hermitian metric* h *and the endomorphism* $\mathcal{Q}$ *of this CDV-structure are invariant under the flow of* $E-\overline{E}$.

Proof. (a) The F-manifold $(M,\circ,e,E)$ is generically semisimple. The Higgs bundle $(K=\widetilde{H}|_{\{0\}\times M},C^K)$ is a free TM-module of rank 1. Therefore the $(TERP(n+1))$-structure is at generic points locally isomorphic to a universal semisimple $(TERP(n+1))$-structure in the sense of chapter 6. Theorem 6.1 (a) and remark 6.2 (iii) apply. The set R is invariant under the flow of $E-\overline{E}$ because of lemma 7.22.

(b) This follows from theorem 5.15, from the construction of the Frobenius manifold structure and from the definition of a CDV-structure. □

In [CV1][CV2] an important aspect of the study of the CV-structures there is the behaviour with respect to the renormalization group flow. Lemma 8.6 will show that it corresponds here to the real vector field $E+\overline{E}$.

Especially interesting are the behaviour of the hermitian metric h^K and the endomorphism $\mathcal{Q}^K$ of the CV-structure on $K|_{M-R}$. We formulate conjecture 8.3 and prove a part of it in theorem 8.4. Part (b) of theorem 8.4 is the main result of this chapter. It uses theorem 7.20, which showed the relevance of PMHS's for CV-structures.

The restriction F_t of the semiuniversal unfolding F to a point $t\in M$ has a finite set $Sing(F_t)$ of singularities. For each singularity $x\in Sing(F_t)$ one has a tuple of exponents $Exp(F_t,x)\subset\mathbb{Q}\cap(0,n{+}1)$. They are defined precisely as the exponents $\alpha_1,...,\alpha_\mu$ for f. They are symmetric around $\frac{n+1}{2}$. Let $Exp(F_t):=\bigcup_{x\in Sing(F_t)}Exp(F_t,x)$ be the union. Along an orbit of E this tuple is constant because there the singularities do not split and because the exponents are constant along μ-constant deformations [Va2].

Conjecture 8.3. *Consider any* $t\in M$. *The set* R *does not contain the* $E+\overline{E}$ *orbit of* t. *If one goes far enough along the flow of* $E+\overline{E}$ *then one will not meet anymore the set* R, *the hermitian metric will be positive definite, and the eigenvalues of* $\mathcal{Q}^K$ *will tend to* $Exp(F_t)-\frac{n+1}{2}$.

Theorem 8.4. *(a) Conjecture 8.3 is true for points $t \in M$ such that F_t has μ different critical values.*

(b) Conjecture 8.3 is true for points t in the μ-constant stratum S_μ.

The points in (a) are the points t such that $\mathcal{U}|_t$ has μ different eigenvalues, the points in (b) are the points t such that $\mathcal{U}|_t$ is nilpotent.

Proof of theorem 8.4. (a) The eigenvalues of $\mathcal{U}$ are at generic points t canonical coordinates $u_1, ..., u_\mu$ with $E = \sum_i u_i \frac{\partial}{\partial u_i}$. Along the flow of $E + \overline{E}$ their phases are constant and their absolute values increase. Theorem 6.1 (b) applies.

(b) It is sufficient to prove the conjecture for the point $0 \in S_\mu \subset M$. At $t = 0$ the $(TERP(n+1))$-structure from step 3 in section 8.1 is regular singular and induces PMHS's $(H^\infty_{\neq 1}, M_s, -N, S, \widetilde{F}^\bullet)$ and $(H^\infty_1, -N, S, \widetilde{F}^\bullet)$ of weight n and $n+1$ (section 8.1, step 4 (γ)).

The Euler field orbit of 0 is a point, i.e. $E|_0 = 0$, if and only if the singularity is quasihomogeneous. Then $N = 0$, and $\widetilde{F}^\bullet$ gives pure polarized Hodge structures. Corollary 7.14 (d) applies.

If $E|_0 \neq 0$ then lemma 7.19 and theorem 7.20 apply. □

Remark 8.5. The base space M of a semiuniversal unfolding was constructed in section 8.1 as a small ball in $\mathbb{C}^\mu$. Within M one cannot go very far along the flow of E.

But if $E|_t \neq 0$ for all $t \in M$ then one can extend M uniquely to a manifold M_{orb} with Euler field E such that each E-orbit in M_{orb} is isomorphic to $(\mathbb{C}, \frac{\partial}{\partial \rho})$ (with ρ as coordinate on $\mathbb{C}$) and such that each E-orbit in M_{orb} meets M. Lemma 7.19 works also with additional holomorphic parameters and shows that the $(TERP(n+1))$-structure on M extends to a $(TERP(n+1))$-structure on M_{orb}.

If $E|_t = 0$ for some t then

$$\{t \in M \mid E|_t = 0\} = \{t \in M \mid F_t \text{ is a quasihomogeneous singularity}\}\ .$$

In that case one can choose new coordinates τ_i on M such that the Euler field takes the form

$$E = \sum_{i=1}^{\mu} d_i \tau_i \partial_{\tau_i} \tag{8.27}$$

where $d_i \in \mathbb{Q}_{<1}$, and the numbers $1-d_i$ are the weighted degrees of a monomial basis of the Jacobi algebra of a quasihomogeneous singularity. Then the closure of $\tau(M) \subset \mathbb{C}^\mu$ under the flow of E is an open subset $M_{orb} \subset \mathbb{C}^\mu$ such that all E-orbits with Euler fields are either points or are isomorphic to $(\mathbb{C}^*, c \cdot \zeta \partial_\zeta)$ with ζ as coordinate on $\mathbb{C}$ and $c \in \mathbb{Q}^*$ (c may vary with the orbits). Again the $(TERP(n+1))$-structure on $\tau(M)$ extends to a $(TERP(n+1))$-structure on M_{orb}. Therefore, in any case one can extend M and the $(TERP(n+1))$-structure on M arbitrarily along E-orbits.

A semiuniversal unfolding F of a singularity is a family of functions F_t, $t \in M$. In [CV1][CV2] an additional parameter λ and one parameter unfoldings

$\lambda \cdot F_t$, $\lambda \in \mathbb{C}^*$ (or $\mathbb{R}^*$), are considered and related to the renormalization group flow. Lemma 8.6 states the relation with the Euler field E.

Lemma 8.6. *Choose in the base space M of a semiuniversal unfolding F a parameter t. Consider the one parameter unfolding $e^\rho \cdot F_t$, $\rho \in \mathbb{C}$, of F_t.*

A neighborhood $U \subset \mathbb{C}$ of $\rho = 0$ and a unique map $u : U \to M$ with $u(0) = t$ exist such that u^*F and the unfolding $e^\rho \cdot F_t$, $\rho \in U$, are isomorphic in the following sense: a holomorphic family of (multi-germ) isomorphisms φ_ρ of suitable neighborhoods of the critical points of $u^*F|_\rho$ and $e^\rho \cdot F_t$ exists such that $u^*F|_\rho = e^\rho \cdot F_t \circ \varphi_\rho$ in these neighborhoods and such that $\varphi_0 = \mathrm{id}$.

The image $u(U) \subset M$ is an E-orbit, and $\mathrm{d}u(\frac{\partial}{\partial\rho}) = E$ on $u(U)$.

Proof. The first part follows from the fact that the unfolding F is for any $t \in M$ a semiuniversal unfolding of F_t as a multi-germ around its singular points. The set $u(U) \subset M$ is an E-orbit with $\mathrm{d}u(\frac{\partial}{\partial\rho}) = E$ because of the calculation

$$\frac{\partial}{\partial\rho}(e^\rho \cdot F_t) = e^\rho \cdot F_t \tag{8.28}$$

and because of the Kodaira–Spencer map in (8.2). □

8.3. **Examples.** In [CV1][CV2] quasihomogeneous singularities are considered. A semiuniversal unfolding of a quasihomogeneous singularity $f \in \mathbb{C}[x_0, ..., x_n]$ can be chosen as

$$F = f + \sum_{i=1}^{\mu} t_i m_i \ , \tag{8.29}$$

where $m_1, ..., m_\mu$ are weighted homogeneous polynomials which represent a basis of the Jacobi algebra of f. The m_i have weighted degrees $\deg_w m_i \in \mathbb{Q}_{\geq 0}$, where $\deg_w f = 1$, the t_i inherit weighted degrees $d_i := \deg_w t_i := 1 - \deg_w m_i$. The Euler field is

$$E = \sum_{i=1}^{\mu} d_i t_i \partial_{t_i} \ . \tag{8.30}$$

In [CV1][CV2] a deformation F_t of f is called *relevant* if $t_i = 0$ for $d_i \leq 0$, *marginal* if $t_i = 0$ for $d_i \neq 0$, *irrelevant* if $t_i = 0$ for $d_i \geq 0$.

In [CV1][CV2] mainly deformations F_t with $t_i = 0$ for $d_i < 0$ are considered. One reason is that such deformations have as global functions on $\mathbb{C}^{n+1}$ still only the singularities which come from the singularity 0 of $f = F_0$, so that $\sum_{x\in\mathbb{C}^{n+1}} \mu(F_t, x) = \mu$. Deformations F_t with $t_i \neq 0$ for some $d_i < 0$ have in general additional singularities coming from infinity, so that $\sum_{x\in\mathbb{C}^{n+1}} \mu(F_t, x) > \mu$.

The deformations F_t with $t_i = 0$ for $d_i > 0$ are the μ-constant deformations. The results in chapter 7 and theorem 8.4 (b) apply precisely to these deformations and thus form a complement to [CV1][CV2]. Also, they apply to nonquasihomogeneous singularities, as well.

Theorem 8.4 (b) will be illustrated by the simplest examples of singularities with nontrivial μ-constant stratum. There are three classes of singularities with one dimensional μ-constant stratum:

(A) the simple elliptic singularities $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$;
(B) the hyperbolic singularities T_{pqr} for $p, q, r \in \mathbb{Z}_{\geq 2}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$;
(C) the 14 exceptional unimodal singularities.
There are also three classes of singularities with two dimensional μ-constant stratum. One consists of
(D) the 14 exceptional bimodal singularities.

The singularities in (A) are all quasihomogeneous. We consider the singularities in (B), (C) and (D) and describe their μ-constant strata S_μ, the subsets $R\cap S_\mu$ of points where the $(TERP(n+1))$-structure is not a $(trTERP(n+1))$-structure, the metric h and the endomorphism $\mathcal{Q}$ of the CV-structure on the complement $S_\mu - R$. In [He1][He2] (see also [Ku]) the Gauß–Manin connections for these singularities were studied carefully. We use the results.

In the following S_μ will be the parameter space of a standard μ-constant family $\widetilde{F}_t$, $t\in S_\mu$, which contains representatives of each right equivalence class of singularities of the given topological type. In (B) it is not simply connected. We work with surface singularities ($n=2$), but that is not important.

(B) The hyperbolic surface singularities $T_{p,q,r}$ for $p,q,r\in\mathbb{Z}_{\geq 2}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ form a one parameter family

$$\widetilde{F}_t = x_0^p + x_1^q + x_2^r + t\cdot x_0x_1x_2,\ t\in S_\mu = \mathbb{C}^* \ , \tag{8.31}$$

with $\mu = p+q+r-1$ and exponents $\alpha_1, ..., \alpha_\mu$ with $1=\alpha_1<\alpha_2\leq ...\leq \alpha_{\mu-1}<\alpha_\mu = 2$. The Euler field on S_μ is

$$E = (1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r})t\partial_t \ . \tag{8.32}$$

The nilpotent endomorphism N of H^∞ vanishes on $H^\infty_{\neq 1}$, but it has a 2×2-Jordan block on the two dimensional space H^∞_1. For each $t\in S_\mu$ the $(TERP(n+1))$-structure at t is generated by elementary sections. It splits into one $(TERP(n+1))$-structure for $H^\infty_{\neq 1}$ and one for H^∞_1.

The one for $H^\infty_{\neq 1}$ is constant along S_μ, it is a $(trTERP(n+1))$-structure, its metric h is positive definite, its endomorphism $\mathcal{Q}$ has eigenvalues $\alpha_2 - \frac{3}{2}, ..., \alpha_{\mu-1}-\frac{3}{2}$ everywhere.

Using the calculations in [He1][He2] and doing some more, one obtains the following. The $(TERP(n+1))$-structure for H^∞_1 is not a $(trTERP(n+1))$-structure on the set

$$S_\mu\cap R = \{t\in S_\mu \mid |t| = r_1\} \tag{8.33}$$

for some $r_1\in\mathbb{R}_{>0}$. The metric h of its CV-structure on $S_\mu - R$ is positive definite on $\{t\in S_\mu \mid |t|>r_1\}$ and negative definite on $\{t\in S_\mu \mid |t|<r_1\}$. Its endomorphism $\mathcal{Q}$ is on $S_\mu - R$ semisimple with eigenvalues

$$\pm\left(\frac{1}{2}+\left(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}\right)\frac{1}{2}\cdot\frac{1}{\log|t|-\log r_1}\right) \ . \tag{8.34}$$

(C) The 14 exceptional unimodal surface singularities form each a one parameter family

$$\widetilde{F}_t = f + t \cdot m \ , \quad t \in S_\mu = \mathbb{C} \ , \tag{8.35}$$

where f is quasihomogeneous and m is a monomial of weighted degree $\deg_w m > 1$. The exponents satisfy

$$\alpha_1 < 1,\ 1 + (1-\alpha_1) < \alpha_2,\ \alpha_{\mu+1-i} = 3 - \alpha_i \ . \tag{8.36}$$

The Euler field E on S_μ is

$$E = -2(1-\alpha_1) t \partial_t \ . \tag{8.37}$$

The $(TERP(n+1))$-structure splits into two $(TERP(n+1))$-structures, one for $\sum_{j=2}^{\mu-1} H^\infty_{e^{-2\pi i \alpha_j}}$, the other for the two dimensional space $\bigoplus_{j\in\{1,\mu\}} H^\infty_{e^{-2\pi i\alpha_j}}$. The first one has the same properties as the first one in (B).

With the calculations in [He1][He2] and some more one finds the following. The second one is not a $(trTERP(n+1))$-structure on

$$S_\mu \cap R = \{ t \in S_\mu \mid |t| = r_2 \} \tag{8.38}$$

for some $r_2 \in \mathbb{R}_{>0}$. The metric h of its CV-structure on $S_\mu - R$ is positive definite on $\{t \in S_\mu \mid |t| < r_2\}$ and negative definite on $\{t \in S_\mu \mid |t| > r_2\}$. Its endomorphism $\mathcal{Q}$ is on $S_\mu - R$ semisimple with eigenvalues

$$\pm \frac{\alpha_1 - \frac{3}{2} - |t|^2(\frac{1}{2} - \alpha_1)}{1 - |t|^2} \ . \tag{8.39}$$

(D) The 14 exceptional bimodal surface singularities form each a two parameter family

$$\widetilde{F}_t = f + t_1 \cdot m_1 + t_2 \cdot m_2 \ , \quad t \in S_\mu = C^2 \ , \tag{8.40}$$

where f is quasihomogeneous and m_1 and m_2 are monomials of weighted degrees > 1. The exponents satisfy

$$\alpha_1 < 1,\ 1 < \alpha_2 < 1 + (1-\alpha_1) < \alpha_3,\ \alpha_{\mu+1-i} = 3 - \alpha_i \ . \tag{8.41}$$

The Euler field E on S_μ is

$$E = -2(1-\alpha_1) t_1 \partial_{t_1} + (-2 + \alpha_1 + \alpha_2) t_2 \partial_{t_2} \ . \tag{8.42}$$

The $(TERP(n+1))$-structure splits into two $(TERP(n+1))$-structures, one for $\sum_{j=3}^{\mu-2} H^\infty_{e^{-2\pi i\alpha_j}}$, the other for the four dimensional space $\bigoplus_{j\in\{1,2,\mu-1,\mu\}} H^\infty_{e^{-2\pi i\alpha_j}}$. The first one has the same properties as the first ones in (B) and (C). Its endomorphism $\mathcal{Q}$ has eigenvalues $\alpha_j - \frac{3}{2}$, $j \in \{3, ..., \mu-2\}$.

With the calculations in [He1][He2] and some more one finds the following. The second one is not a $(trTERP(n+1))$-structure on

$$S_\mu \cap R = \{ (t_1,t_2) \in S_\mu \mid r_3 |t_1 + r_5 t_2^{r_6}|^2 - (1 - r_4|t_2|^2)^2 = 0 \} \tag{8.43}$$

for some $r_3, r_4 \in \mathbb{R}_{>0}$, $r_5 \in \mathbb{R}$, $r_6 = \deg_w t_2 / \deg_w t_1 \in \{4, 3, \frac{5}{2}\}$ and $r_5 = 0$ for $r_6 = \frac{5}{2}$. The set $S_\mu \cap R$ has singularities. It splits $S_\mu - R$ into three connectivity components. I did not determine the metric h and the endomorphism

$\mathcal{Q}$ precisely. It is clear that the metric h is positive definite on the component which contains $t = 0$, and that the eigenvalues of $\mathcal{Q}$ tend to $\alpha_j - \frac{3}{2}$ for $j \in \{1, 2, \mu - 1, \mu\}$ for $t \to 0$.

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