ON HOLOMORPHIC CURVES INTO

ABELIAN VARIETIES

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Introduction.

Let A be an Abelian variety of dimension n and D an ample effective reduced divisor in A. let $f : \mathbb{C} \longrightarrow A$ be a holomorphic mapping from the one-dimensional complex numerical space $\mathbb{C}$ into A, which we call a holomorphic curve into A. Assume that $f(\mathbb{C}) \not\subset \mathrm{Supp}\, D$. We denote by $T_f(r,D)$ the characteristic function of f relative to a Kähler metric form contained in the Chern class of $[D]$ and $N(r,f^*D)$ the counting function for D counting multiplicities. The purpose of this paper is to prove the following inequality (8) of Second-Main-Theorem-type:

<u>Theorem 1</u>.[1)] Suppose f is algebraically non-degenerate. Then for any positive number ε we have

$$(8) \qquad T_f(r,D) \leq (1+\varepsilon)N(r,f^*D) + O(\log r + \log T_f(r,D)) \ .$$

The following result is a direct consequence of

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1) Many inequalities in the Nevanlinna theory including (8) is valid for r outisde a Borel set of finite Lebesgue measure. Since this does not affect our arguments, we do not mention this explicitly.

Theorem 1 and the solution to Bloch's conjecture.

Theorem 2. There exists no non-constant (entire) holomorphic curve into $A - D$.

A similar but stronger inequality of Second-Main-Theorem-type is conjectured in [N3] . Theorem 2 was conjectured by Lang and Griffiths (cf. Problem F. in [Gr]) and posed by Kobayashi (cf. Problem D.9. in [Ko]). Special cases of Theorem 2 have been considered by Ax [A] , Green [G] , Ochiai [O] and Noguchi [N2] . Namely, Ax proved Theorem 2 when f is a one-parameter subgroup, while Green proved Theorem 2 when D contains no non-trivial Abelian subvariety by showing that $A - D$ is complete hyperbolic and is hyperbolically embedded in A in the sense of Kobayashi (cf. [Kō]) . Ochiai and Noguchi proved Theorem 2 when D satisfies some cohomological condition, as bi-products of their attacks to Bloch's conjecture. On the other hand, our method for the proof of Theorem 2 is based on the Second Main Theorem established by Noguchi (cf. [N1]) . In fact, we reduce the problem to the simplest case of Noguchi's Second Main Theorem by a simple observation in elementary algebraic geometry. We should recall here that Noguchi's Second Main Theorem in [N1] is for meromorphic mappings of a finite analytic covering space over $\mathbb{C}^m$ into a projective variety of the same dimension.

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1. Preliminaries

Here we introduce the usual notations in the Nevanlinna theory and state Noguchi's Second Main Theorem not in its full generality but in a form sufficient for our purposes.

A holomorphic mapping $\pi : X \longrightarrow \mathbb{C}$ is a finite analytic covering over $\mathbb{C}$ if X is an irreducible Riemann surface and π is a surjective proper holomorphic mapping. If the fiber of π over a generic point consists of k points, we call $\pi : X \longrightarrow \mathbb{C}$ an analytic k-covering.

Let z be a natural coordinate in the numerical space $\mathbb{C}$ and set

$$\mathbb{C}(r) = \{z \in \mathbb{C} \ ; \ |z| < r\} ,$$

$$X(r) = \pi^{-1}(\mathbb{C}(r)) ,$$

$$\eta = (\sqrt{-1}/4)(\bar{\partial}-\partial)\log|z|^2 = d^c\log|z|^2 = d\theta/2\pi ,$$

where $z = re^{i\theta}$.

Let D be an effective divisor on an analytic k-covering X over $\mathbb{C}$ (resp. $\mathbb{C}$). We assume $\pi^{-1}(0) \cap D = \emptyset$ (resp. $0 \notin D$) for simplicity. The counting function for D on X (resp. $\mathbb{C}$) is

$$(1) \quad N(r,D) = (1/k)\Sigma_{a\in X(r)}\nu_{a,D}\log|r/a| \ (\text{resp. } \Sigma_{a\in\mathbb{C}(r)}\nu_{a,D}\log|r/a|) ,$$

where $\nu_{a,D} := \text{ord}_a(D)$, i.e., $\nu_{a,D}$ is zero if $a \notin \text{Supp } D$ and is equal to the coefficient of a if $a \in \text{Supp } D$.

Let B be a smooth complex projective variety, $L \longrightarrow V$ a holomoprhic line bundle with a Hermitian metric $\| \ \|$ whose curvature form is Ω , and $f : X \longrightarrow V$ (resp. $f : \mathbb{C} \longrightarrow V$) a holomorphic curve. The <u>characteristic function</u> of f with respect to the line bundle L is

$$(2) \quad T_f(r,L) := (1/k)\int_1^r \frac{dt}{t} \int_{X(t)} f^*\Omega \quad (\text{resp. } \int_1^r \frac{dt}{t} \int_{\mathbb{C}(t)} f^*\Omega) .$$

For $D \in |L|$ which does not contain the whole image of f , we define the <u>proximity function</u> of f with respect to the effective divisor D by

$$(3) \quad m_f(r,D) := (1/k) \int_{\partial X(r)} \log(1/\| \sigma\circ f \|)\pi^*\eta$$

$$(\text{resp. } \int_{\partial\mathbb{C}(r)} \log(1/\| \sigma\circ f \|)\eta) ,$$

where σ is a holomorphic section of L such that $(\sigma) = D$ and $\| \sigma \| \leq 1$. Since $[D] = L$, we often write $T_f(r,L) = T_f(r,D) = T_f(r,L,\Omega) = T_f(r,D,\Omega)$.

Now let us assume for simplicity that $f(0) \notin D$. Let $\| \ \|_t$ be a family of Hermitian metrics for L such that the curvature forms Ω_t converge to D in the sense of currents, i.e., $\text{Supp } \Omega_t$ converge to D in the limit $t \longrightarrow \infty$. Letting $t \longrightarrow \infty$ in

$$(4) \quad T_f(r,D,\Omega) = (T_f(r,D,\Omega) - T_f(r,D,\Omega_t)) + T_f(r,D,\Omega_t) ,$$

and noticing that $\Omega_t = dd^c \log(1/\|\sigma\|_t^2)$ and that $\|\sigma\|_t$ goes to a positive constant outside of D in the limit, we obtain the First Main Theorem

$$(5) \quad T_f(r,L) = m_f(r,D) + N(r,f^*D) - m_f(1,D) \geq N(r,f^*D) + O(1) .$$

Here we have used the Jensen formula to the first term of the right hand side of (4).

On the other hand, a theorem of Second-Main-Theorem-type gives us quantitive information on how often a holomorphic curve intersects a divisor, i.e., an inequality estimating N by T <u>from below</u>. Now Noguchi's Second Main Theorem is stated as follows (cf. [N1]) .

<u>The Second Main Theorem</u>. Let $\pi : X \longrightarrow \mathbb{C}$ be an analytic k-covering and $f : X \longrightarrow V$ a holomorphic mapping to a compact Riemann surface V . Assume that there exists a point $z \in \mathbb{C}$ such that $d\pi \neq 0$ at every point of $\pi^{-1}(z)$ and $f(x) \neq f(y)$ for any distinct points x,y of $\pi^{-1}(z)$. Then for any reduced effective divisor $\Sigma_{i=1}^q p_i$ such that $q + 2(g(V) - 1) > 0$, we have

$$(6) \quad \{q - 2(k - 1)\}T_f(r,L) + T_f(r,K_V)$$

$$\leq \Sigma_{i=1}^q N(r,f^*p_i) - N(r,R_f) + O(\log r + \log T_f(r,L)) ,$$

where $g(V)$ is the genus of V, $L \longrightarrow V$ is a holomorphic line bundle of degree 1, and R_f is the divisor determined by df .

Since $K_V = 2(g(V) - 1)L$ in $H^2(V, \mathbb{R})$, we have the following inequality:

$$(7) \quad \{q + 2(g(V) - k)\}T_f(r,L) \leq \Sigma_{i=1}^{q} N(r, f^*p_i) + O(\log r + \log T_f(r,L)) ,$$

under the same assumption as Noguchi's Second Main Theorem.

2. Proof of Theorems

Let A be an Abelian variety of dimension n, D an ample effective reduced divisor in A, and $f : \mathbb{C} \longrightarrow A$ a holomorphic curve which is algebraically non-degenerate. We always choose a Hermitian metric $\| \ \|$ on D and a holomorphic section $\sigma \in H^0(A,[D])$ such that $(\sigma) = D$ $\|\sigma\| \leq 1$.

<u>Theorem 1</u>. For any positive number ε , we have

$$(8) \quad T_f(r,D) \leq (1 + \varepsilon)N(r,f^*D) + O(\log r + \log T_f(r,D)) .$$

<u>Proof</u>. We first assume that D is an irreducible ample hypersurface in the Abelian variety A . Let $p \in A$ be the identity element of the group A . Choose N smooth curves $S_1, \ldots, S_N$ in A through p such that the tangent vectors to S_i's at p span $\mathbb{C}^n$, where a curve means a one-dimensional compact closed subvariety. Here, N is chosen to be sufficiently large so that the following arguments make sense (especially the inequality (9)). If D has at worst normal crossings, then $N = n$ is enough. We may further assume that none of S_i's are contained in parallel translations of $-D$, where $-D$ is the image of D under the involution $i : A \longrightarrow A$, $i(z) = -z$. Let $X_i = X_i(f,D)$ be an analytic finite covering over $\mathbb{C}$ defined by

$$X_i(f,D) = \{(z,q);\ z \in \mathbb{C}\ ,\ q \in D \text{ such that } f(z) - q \in S_i\}$$

for $i = 1, \ldots, N$. Let k_i be the converging number for $\pi_i : X_i(f,D) \longrightarrow \mathbb{C}$, $\pi_i(z,q) = z$. Then we have $k_i = (-D)\cdot S_i$. Since D is ample, k_i is positive and $X_i(f,D) \neq \emptyset$. We define n holomorphic mappings

$$f_i : X_i(f,D) \longrightarrow S_i$$

by $f_i(z,q) = f(z) - q$ for $i = 1, \ldots, N$. Suppose $f(z)$ is very close to D for some $z \in \mathbb{C}$, i.e., $\|\sigma\circ f\|(z)$ is very small. Let $(z,q_{i\nu(i)})(\nu(i) = 1, \ldots, k_i)$ be the points in $X_i(f,D)$ over z . Then for some $(z,q_{i\nu(i)})$, $f_i(z,q_{i\nu(i)}) = f(z) - q_{i\nu(i)}$ must be very small, i.e., $f_i(z,q_{i\nu(i)})$ is very close to the identity element p in A with respect to, for example, the Euclidean metric on A . Let σ_i be a section of $[p]$ on S_i and $\|\ \|_i$ a Hermitian metric for $[p]$ such that $\|\sigma_i\|_i \leq 1$. Replacing $\|\ \|_i$'s by some constant multiples if necessary, we have the following string of inequalities for arbitrary f with $f(\mathbb{C}) \not\subset \operatorname{Supp} D$:

$$(9)\quad m_f(r,D) = \int_{\partial\mathbb{C}(r)} \log(1/\|\sigma\circ f\|)\eta \qquad \text{(from (3))}$$

$$\leq \Sigma_{i=1}^{N} \int_{\partial X_i(r)} \log(1/\|\sigma_i\circ f\|_i)\pi_i^*\eta$$

(from the construction)

$$\leq \Sigma_{i=1}^{N} T_{f_i}(r,p) \qquad \text{(from (5))}$$

$$\leq \Sigma_{i=1}^{N} \frac{N(r,f_i{}^*p)}{1+2(g(S_i)-k_i)} + O(\log r + \log T_{f_i}(r,p))$$

$$\text{(from (7))}$$

$$\leq \Sigma_{i=1}^{N} \frac{N(r,f^*D)}{1+2(g(S_i)-k_i)} + O(\log r + \log T_f(r,D)) \ .$$

For any positive number ε , we can find the above S_i's which satisfy the additional conditions:

$$0 < \frac{1}{1+2(g(S_i)-k_i)} < \varepsilon \ .$$

Now let D be as in Theorem 1 and $D = \Sigma_{j=1}^{d} D_j$ the decomposition of D into irreducible components. From Chap.VI of [We] , there exist an Abelian variety A_j of positive dimension n_j , an ample irreducible hypersurface D_j' in A_j and a surjective homomorphism $\rho_j : A \longrightarrow A_j$ such that $D_j = \rho_j^* D_j'$. Find N_j smooth curves $S_{j\mu(j)}$ $(\mu(j) = 1, \ldots, N_j)$ in A_j through the identity element p_j of A_j so that the tangent vectors to $S_{j\mu(j)}$'s at p_j span n_j-dimensional complex numerical space and

$$0 < \frac{1}{1+2(g(S_{j\mu(j)})-k_{j\mu(j)})} < \frac{\varepsilon}{d\ N_j} \ ,$$

where $k_{j\mu(j)} = (-D_j')\cdot S_{j\mu(j)} > 0$. Here, N_j is chosen to be sufficiently large so that we can use the inequality (9). It thus follows from (9) that

$$(10)\quad m_f(r,D) = \Sigma_{j=1}^{d} m_f(r,D_j) = \Sigma_{j=1}^{d} m_{\rho_j\circ f}(r,D_j')$$

$$\leq \Sigma_{j=1}^{d}\Sigma_{\mu(j)=1}^{N_j} \frac{N(r,(\rho_j\circ f)^*D_j')}{1+2(S_{j\mu(j)})-k_{j\mu(j)})} + O(\log r + \log T_{\rho_j\circ f}(r,D_j'))$$

$$\leq \varepsilon N(r,f^*D) + O(\log r + \log T_f(r,D)) \ ,$$

because $f : \mathbb{C} \longrightarrow A$ is algebraically non-degenerate. Combining the inequality (10) with the First Main Theorem (5), we obtain an inequality of Second-Main-Theorem-type:

$$(8)\quad T_f(r,D) \leq (1+\varepsilon)N(r,f^*D) + O(\log r + \log T_f(r,D)) \ .$$

Q.E.D.

Let $f : \mathbb{C} \longrightarrow \mathbb{C}^m/\Gamma$ be a non-constant holomorphic curve into a complex torus $\mathbb{C}^m/\Gamma$. Then, by [N2] , for any Kähler form Ω on the complex torus, there exist positive constants C and r_0 such that

$$(11)\quad T_f(r,\Omega) \geq C\, r^2$$

holds for $r \geq r_0$.

We introduce the Nevanlinna defect of f with respect to D :

$$\delta_f(D) := 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, f^*D)}{T_f(r, D)} ,$$

which has the following properties:

$0 \leq \delta_f(D) \leq 1$, and

$\delta_f(D) = 1$ if $f(\mathbb{C})$ does not meet D .

Combining (8) and (11), we have the following

Corollary. Let f be a holomorphic curve into an Abelian variety A and D an ample effective reduced divisor in A . Suppose f is algebraically non-degnerate. Then $\delta_f(D) = 0$.

Corollary means that any non-degenerate holomorphic curve into an Abelian variety meets ample divisors as often as possible.

Remark. The following result is actually proved by the proof of Theorem 1:

Theorem 1'. Let A and D be as in Theorem 1. Then there exists a proper algebraic subvariety D' determined only by D such that for any holomorphic curve $f : \mathbb{C} \longrightarrow A$ satisfying $f(\mathbb{C}) \not\subset \mathrm{Supp}\, D'$ the inequality (8) holds.

Theorem 2. Let A and D be as in Theorem 1. Then there exists no non-constant (entire) holomorphic curve into $A - D$.

Proof. Suppose there exists an algebraically non-degenerate holomorphic curve $f : \mathbb{C} \longrightarrow A-D$. We have $\delta_f(D) = 1$ from (8) and (11), but it contradicts Corollary. Therefore any non-constant holomorphic curve f omitting D must be algebraically degenerate. From the solution to Bloch's conjecture due to Ochiai, Green, Kawamata and Wong (cf. [O], [Ka] and [Wo]), it follows that the Zariski closure of $f(\mathbb{C})$ must be the parallel translation of a proper Abelian subvariety. On the other hand, Theorem 2 is clear if A is an elliptic curve. Hence Theorem 2 is proved by the induction on $\dim A$.

Q.E.D.

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