SYMMETRIC PRODUCTS OF CYCLES

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If G is a group and X is a finite G-set, then the k-th symmetric product of X (denoted by S_kX) is defined as follows. The symmetric group S_k on k letters operates on the k-fold Cartesian product X^k by permutation of coordinates. This action commutes with the diagonal action of G on X^k, so this means that the orbit space $S_kX = X^k/S_k$ is a finite G-set. In this note we determine the structure of S_kX for $G = (Z,+)$, the additive group of the integers. The topological motivation for studying this special case comes from Dold[2].

We first notice that it is enough to determine S_kX for G-sets of the form G/H. This is because we have:

LEMMA. If $X \sqcup Y$ denotes the disjoint sum of X and Y, then $S_k(X \sqcup Y)$ is G-equivalent to

$$\coprod_{j=0}^{k} S_jX \times S_{k-j}Y$$

with the convention that S_0X is a single point.

We now focus on the special case of $G=(Z,+)$. The finite orbits of Z are the sets of the form $\textcircled{n} = Z/(n)$ with the action $k.(x + (n))= k+x + (n)$. In other words, $\textcircled{n}$ is the standard n-cycle, and every finite Z-set decomposes into a disjoint sum of cycles. According to the lemma above it is enough to determine $S_k\textcircled{n}$ in order to determine S_kX for any finite Z-set X. We first of all determine the number of elements in $S_k\textcircled{n}$.

PROPOSITION. The symmetric product $S_k\, n$ has precisely $\binom{n+k-1}{k}$ elements.

<u>Proof</u>. Let $s_k(n)$ be the number of elements in $S_k\textcircled{n}$. We notice that it satisfies the initial condition $s_1(n)=n$. The recursive condition for $s_k(n)$ is easy to derive: let $0 \in \textcircled{n}$ be the coset of zero, then the subset of $S_k\textcircled{n}$ of elements having 0 as one or more coordinates is a copy of $S_{k-1}\textcircled{n}$, and the complementary set is a copy of $S_k\textcircled{n-1}$. Of course, the two subsets are not Z-subsets, but after all we are just

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counting the number of elements in $S_k\textcircled{n}$, so this does not matter.. This means that $s_k(n)$ satisfies the recursive condition $s_k(n) = s_{k-1}(n) + s_k(n-1)$. But this means that $s_k(n) = \binom{n+k-1}{k}$, as claimed.

We now determine the structure of $S_k\textcircled{n}$ as a Z-set. The main step is to determine the structure of the fixed point set

$$\text{Fix } T^r = \left\{ x \in S_k\textcircled{n} \mid T^r x = x \right\},$$

where $T = 1. : S_k\textcircled{n} \longrightarrow S_k\textcircled{n}$. Since $\textcircled{n}$ is a $Z/(n)$ -set, so is $S_k\textcircled{n}$, and so we are really interested in Fix T^r for r dividing n . Now if $n = rs$, we let $Z/(s)$ act on $\textcircled{rs}$ by setting $1.x= x+r \bmod (n)$ and we know that $\textcircled{rs}$ splits into r cycles $\textcircled{s}$. Indeed, if we let

$$A = \left\{0, r, \ldots, (s-1)r\right\},$$

then the r cycles are given by $t^i A = \left\{i, r+i, \ldots, (s-1)r+i\right\}$ for $i = 0,1, \ldots, r-1$, where $t = 1. : \textcircled{n} \longrightarrow \textcircled{n}$ is the original actic of Z on $\textcircled{n}$, namely $t(x) = x + 1 \bmod (n)$. Now let's take an element x in $S_k\textcircled{rs}$ and find out what it means that $x \in \text{Fix } T^r$. Let us write $a \in x$ if $a \in \textcircled{rs}$ is a coordinate of a point in $\textcircled{rs}^k$ representing the orbit x in $S_k\textcircled{rs} = \textcircled{rs}^k/S_k$. A necessary and sufficient condition tha $x \in \text{Fix } T^r \subset S_k\textcircled{rs}$ is that $a \in x$ always implies $t^r a \in x$. If we write $x = [a_1, \ldots, a_k]$ = the orbit of $(a_1, \ldots, a_k)$ under S_k, this means that the a_i fall into complete orbits of $Z/(s)$ under the action $1.x = t^r(x) = x+r \bmod (n)$. This implies two things: first, if k is not a multiple of s, Fix T^r is empty; second, if $k=ms$, then we can write eac x in Fix T^r uniquely in the form $x = [t^{x}1.A, \ldots, t^{x}m.A]$, where $A = \{0, r, \ldots, (s-1)r\}$ is the basic s-cycle considered above, and $0 \leq x_1 \leq \cdots \leq x_m < r$. This proves the basic result that we need:

THEOREM. If k is not a multiple of s, then Fix $T^r = \emptyset$ in S_k rs . If $k = ms$, then the map

$$F : S_m\textcircled{r} \longrightarrow \text{Fix } T^r \text{ in } S_{ms}\textcircled{rs}$$

given by $F[x_1, \ldots, x_m] = [t^{x}1.A, \ldots, t^{x}m.A]$ is an equivalence of $Z/(r)$ -sets, where $A = \{0, r, \ldots, (s-1)r\}$ is the basic s-cycle in $\textcircled{rs}$.

This theorem allows an efficient recursive algorithm for calculatin the $Z/(n)$ -structure of the set $S_k\textcircled{n}$:

COROLLARY. The multiplicity of ⓡ in S_k (rs) is zero if s does not divide k. If k = ms, then the multiplicity of ⓡ in S_{ms} (rs) is the same as the multiplicity of ⓡ in S_mⓡ .

The proof of the corollary is immediate, for the cycle ⓡ appears in Fix $T^r \subset S_{ms}$ (rs) .

We let $f_k(n)$ be the multiplicity of the cycle ⓝ in S_kⓝ . We can restate the preceding corollary as follows:

COROLLARY. The integers $f_k(n)$ are determined by the initial condition $f_1(n) = 1$ with the recursive relations

$$\binom{n+k-1}{k} = \sum_{\substack{n=rs\\k=ms}} f_m(r).r \quad .$$

We plot the values of $f_k(n)$ for $1 \leq k,n \leq 10$

TABLE OF $f_k(n)$

k ↓ \ n →	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5
3	1	2	3	5	7	9	12	15	18	22
4	1	2	5	8	14	20	30	40	55	70
5	1	3	7	14	25	42	66	99	143	200
6	1	3	9	20	42	75	132	212	333	497
7	1	4	12	30	66	132	245	429	715	1144
8	1	4	15	40	99	212	429	800	1430	2424
9	1	5	18	55	143	333	715	1430	2700	4862
10	1	5	22	70	200	497	1144	2424	4862	9225

The table suggests:

RECIPROCITY CONJECTURE. For each k and n we have $f_k(n) = f_n(k)$, that is the number of cycles ⓝ in S_kⓝ is the same as the number of cycles ⓚ in S_nⓚ .

The motivation for this work is the recent paper by A.Dold [2] which presents a new model for the universal λ-ring on one generator. Indeed, let PER^+ be the set of isomorphism classes of permutations of finite type on a countable set (that is, for each natural number n there are only a finite number of n-cycles). Addition in PER^+ is induced by disjoint sum, product by Cartesian product and diagonal action. Denote PER the completion of PER^+ to a ring, and $L : PER \to \Lambda = 1 + tZ[[t]]$ the Lefshetz power series homomorphism. Theorem 2.16 of [2] gives us that L is an isomorphism of λ-rings, where the λ-ring structure of the target is classical (see [1] or [3]) , and the λ- operations in PER are defined in terms of symmetric power operations in PER^+, that is

$$\lambda^n x = \sum_{i=1}^{n} (-1)^{i+1} (\lambda^{n-i} x)(S_i x)$$

with the initial conditions $S_0 x=1$, $\lambda^0 x=1$.

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