Max-Planck-Institut für Mathematik
Bonn

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Michael E. Hoffman

Max-Planck-Institut für Mathematik
Preprint Series 2020 (36)

Date of submission: July 1, 2020

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Michael E. Hoffman

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

U. S. Naval Academy
Annapolis, MD 21402
USA

MPIM 20-36

Yamamoto's Formalism for Iterated Integrals and Multiple Zeta-Star Values

Michael E. Hoffman
U. S. Naval Academy
Annapolis, MD 21402 USA
and
Max-Planck-Institut für Mathematik
Vivatsgasse 7, 53111 Bonn Germany

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Abstract

S. Yamamoto developed an elegant way to represent iterated integrals involving the forms dt/t and $dt/(1-t)$ using 2-labeled posets. We show how Yamamoto's description leads to a "standard expansion" of any multiple zeta-star value $\zeta^\star(n_1,\dots,n_k)$. If $n_k \neq 1$, this expansion shows that the difference between $\zeta^\star(n_1,\dots,n_k)$ and the multiple zeta value $\zeta(n_k,\dots,n_1)$ is a sum of products of multiple zeta-star values and multiple zeta values. The standard expansion also leads to proofs of various identities for multiple zeta-star values, some known and some new.

1 Introduction

In [12], S. Yamamoto introduced an elegant formalism for writing iterated integrals. Let (X,δ) be a 2-labeled poset, i.e., a finite poset X together with a function $\delta : X \to \{0,1\}$. Call (X,δ) admissible if $\delta(x) = 1$ for all minimal $x \in X$ and $\delta(x) = 0$ for all maximal elements of X. Given an admissible

2-labeled poset (X, δ), the associated integral is

$$I(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(t_x), \tag{1}$$

where $\Delta(X) = \{(t_x)_{x \in X} \in [0,1]^X | \ t_x < t_y \text{ if } x < y \text{ in } X\}$, $\omega_0(t) = \frac{dt}{t}$, and $\omega_1(t) = \frac{dt}{1-t}$. Then admissibility ensures the integral is convergent, and the usual iterated integrals representing multiple zeta values are those associated with a chain (i.e., totally ordered set). For example,

$$I(\{x_1 > x_2 > \bar{x}_3 > \bar{x}_4\}) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} \int_0^{t_3} \frac{dt_4}{1-t_4} = \zeta(3,1),$$

where a bar over an element x indicates that $\delta(x) = 1$. What makes this formalism useful is the following result of [12].

Theorem 1. *Suppose X is an admissible 2-labeled poset. Then*

1. *If Y is another such poset, then $I(X \amalg Y) = I(X)I(Y)$.*

2. *For incomparable elements $a, b \in X$, let $X_{a<b}$ be X with the additional relation $a < b$. Then $I(X) = I(X_{a<b}) + I(X_{b<a})$.*

3. *If $X^\vee$ is X with the reversed order and the new labeling function $\delta^\vee$ given by $\delta^\vee(x) = 1 - \delta(x)$, then $I(X^\vee) = I(X)$.*

Yamamoto pointed out that several variant types of multiple zeta values are expressible as $I(X)$ for appropriate X. Most important for this paper, the multiple zeta-star value (MZSV)

$$\zeta^\star(k_1, \dots, k_r) = \sum_{i_1 \ge i_2 \ge \cdots \ge i_r \ge 1} \frac{1}{i_1^{k_1} i_2^{k_2} \cdots i_r^{k_r}}$$

is $I(X)$ for

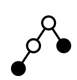
$$X = \{\bar{x}_1 < x_2 < \cdots < x_{k_1} > \bar{x}_{k_1+1} < x_{k_1+2} < \cdots < x_{k_1+k_2} > \cdots > \bar{x}_{k_1+\cdots+k_{r-1}+1} < x_{k_1+\cdots+k_{r-1}+2} < \cdots < x_{k_1+\cdots+k_r}\}.$$

In §2 we will show how this description of multiple zeta-star values, together with the use of Theorem 1, allows us to obtain for any admissible string $\mathbf{s}$ what we call the "standard expansion" of $\zeta^\star(\mathbf{s})$ as a sum of terms $\zeta^\star(\mathbf{r})\zeta(\mathbf{s})$

plus one (if the last element of $\mathbf{s}$ exceeds 1) or more (otherwise) terms $\zeta(\mathbf{t})$. In §3 we show how the standard expansion gives simple formulas for all MZSVs of weight n and depth $n-2$. We also show how to obtain identities for specific MZSVs: in §4 for $\zeta(\{2,\{1\}_n\}_m,1)$ and $\zeta(\{\{2\}_m,1\}_n)$ (where $\{\mathbf{s}\}_n$ means n repetitions of the string $\mathbf{s}$), in §5 for $\zeta^\star(\{2\}_n,3)$ and $\zeta^\star(3,\{2\}_n)$, and in §6 for $\zeta^\star(\{3,1\}_n,3)$ and $\zeta^\star(\{3,1\}_n,2)$.

2 The standard expansion of MZSVs

We shall identify a 2-labeled poset with its Hasse diagram, with the labels indicated by using solid points for elements having label 1 and open points for elements having label 0. For example, the 2-labeled poset $\{\bar{x}_1 < x_2 < x_3 > \bar{x}_4\}$ is represented by



From now on we abbreviate "admissible 2-labeled poset" to "poset."

We note a particular use of parts 1 and 2 of Theorem 1. Suppose P is a poset and $u,v \in P$ with $v > u$ so that removal of the relation $v > u$ disconnects P into components U, V with $u \in U$ and $v \in V$. Then parts 1 and 2 imply that

$$I(P) = I(U)I(V) - (U \amalg V)_{u>v},$$

where as before the subscript indicates an added relation. We call such a move "cut and subtract".

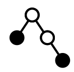
We already noted that if P is a chain, then $I(P)$ is the multiple zeta value $\zeta(w)$, where w is the word formed by reading P from top to bottom with x for each point labeled 0 and y for each point labeled 1, using the notation of [3]. Now we consider a 2-labeled poset of the form

$$P = \{\bar{x}_0 < x_1 > u_1 > u_2 > \cdots > u_k\},$$

where the u_i, $1 \le i < k$, can be either barred or unbarred. (We call such a poset a *hanger*.) By using part 2 of Yamamoto's theorem we can see that

$$I(P) = \zeta(\mathcal{D}(w)),$$

where w is the word formed from reading from x_1 to u_k, replacing each by x or y as before, and $\mathcal{D}(w) = y \sqcup\!\sqcup\, w - yw$ (We note that $\mathcal{D}$ is the derivation of $\mathfrak{H}^0$ sending x to xy and y to y^2). For example,

$I\,($  $) = \zeta(\mathcal{D}(x^2y)) = \zeta(xyxy + 2x^2y^2) = \zeta(2,2) + 2\zeta(3,1).$

More generally, we call a 2-labeled poset without branch points a *snake*; note that admissibility requires a snake to have label 1 at each minimum and label 0 at each maximum. The 2-labeled poset corresponding to a MZSV is a particular kind of snake (which we call a *well-groomed snake*) in which all the labels in ascending parts are 0's and all the labels in descending parts are 1's.

Given a word $w \in \mathfrak{H}^0$, there is a snake P so that $\zeta^\star(w) = I(P)$. There are two cases.

1. The word w ends in xy. This corresponds to the case where the right "tail" of P is pointing up.

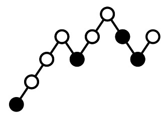
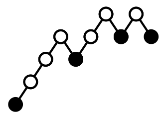
2. The word w has the form $w'y$, for $w' \in \mathfrak{H}^0$. This corresponds to the case where the right "tail" of P is pointing down.

Now starting at the right, make a cut in P at the position farthest left so that the right fragment is a chain. Then do a cut and subtract operation, so that

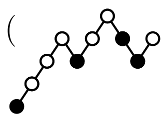
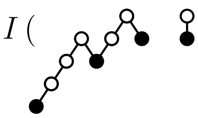
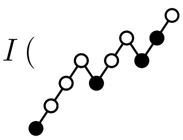
$$I(P) = I(P_1)I(P_2) - I(P_3)$$

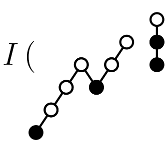
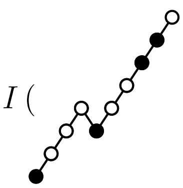
for P_1, P_2, P_3 snakes so that P_1 is well-groomed and P_2 is a chain. (Note that this means $I(P_1)$ is a MZSV and $I(P_2)$ is a MZV). Now repeat the process for P_3; although P_3 need not be well-groomed, it can be cut at a point further to the left than P was so that the right fragment is a chain and the left fragment is well-groomed. Eventually the left-over part from cut and subtract will either be be a chain (corresponding to the first case) or a hanger (corresponding to the second case).

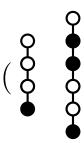
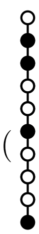
We illustrate the process for the examples $w_1 = x^3yx^2y^2xy = (4,3,1,2)$ and $w_2 = x^3yx^2yxy^2 = (4,3,2,1)$, corresponding to cases 1 and 2 above. The respective posets are

 and  .

For the first example we have

$\zeta^{\star}(4,3,1,2) = I\,($  $) = I\,($  $) - I\,($  $)$

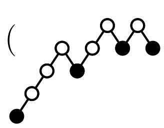
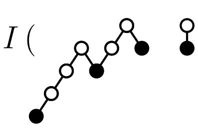
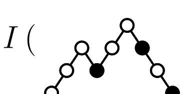
$= \zeta^{\star}(4,3,1)\zeta(2) - I\,($  $) + I\,($  $)$

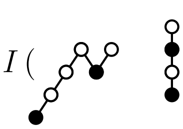
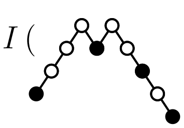
$= \zeta^{\star}(4,3,1)\zeta(2) - \zeta^{\star}(4,3)\zeta(2,1) + I\,($  $) - I\,($  $)$

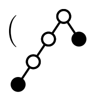
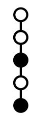
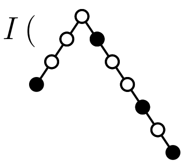
$$= \zeta^{\star}(4,3,1)\zeta(2) - \zeta^{\star}(4,3)\zeta(2,1) + \zeta^{\star}(4)\zeta(2,1,3) - \zeta(2,1,3,4).$$

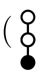
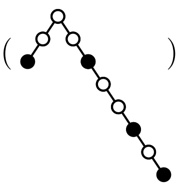
Note that (1) each cut is to the left of a solid dot, so that number of cuts required is $\ell(w_1)-1$; and (2) the last term is $(-1)^{\ell(w_1)-1}\zeta(y^{-1}R(w_1)y)$, where R reverses words.

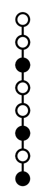
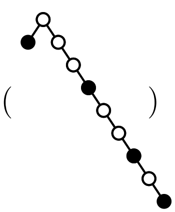
For the second example we have

$\zeta^{\star}(4,3,2,1) = I\,($  $) = I\,($  $) - I\,($  $)$

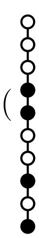
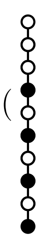
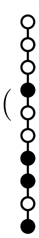
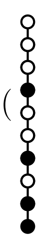
$= \zeta^{\star}(4,3,1)\zeta(2) - I\,($  $) + I\,($  $)$

$= \zeta^\star(4,3,1)\zeta(2) - \zeta^\star(4,2)\zeta(2,2) + I\,($   $) - I\,($  $)$

$= \zeta^\star(4,3,1)\zeta(2) - \zeta^\star(4,2)\zeta(2,2) + \zeta^\star(4,1)\zeta(3,2) - I\,($   $) + I\,($  $)$

$= \zeta^\star(4,3,1)\zeta(2) - \zeta^\star(4,2)\zeta(2,2) + \zeta^\star(4,1)\zeta(3,2) - \zeta^\star(3)\zeta(2,3,2) + I\,($   $) - I\,($  $)$

$= \zeta^\star(4,3,1)\zeta(2) - \zeta^\star(4,2)\zeta(2,2) + \zeta^\star(4,1)\zeta(3,2) - \zeta^\star(3)\zeta(2,3,2) + \zeta^\star(2)\zeta(3,3,2)$

$-I\,($  $) - I\,($  $) - 2I\,($  $) - I\,($  $) - 2I\,($  $) - 2I\,($  $)$

$$= \zeta^\star(4,3,1)\zeta(2) - \zeta^\star(4,2)\zeta(2,2) + \zeta^\star(4,1)\zeta(3,2) - \zeta^\star(3)\zeta(2,3,2) + \zeta^\star(2)\zeta(3,3,2)$$
$$-\zeta(2,3,3,2) - \zeta(3,2,3,2) - 2\zeta(4,1,3,2) - \zeta(4,2,2,2) - 2\zeta(4,3,1,2) - 2\zeta(4,3,2,1).$$

Note that (1) each cut is to the left of an open dot (except the leftmost one), so the number of cuts required is $|w_2| - \ell(w_2) - 1$; and (2) the last term is I

applied to a hanger, which is $(-1)^{|w_2|-\ell(w_2)-1}\zeta(\mathcal{D}(w_2 y^{-1}))$ by the discussion above.

We therefore have the following result.

Theorem 2. *Let $w \in \mathfrak{H}^0$.*

1. *If w ends in xy, then*

$$\zeta^\star(w) + (-1)^{\ell(w)}\zeta(y^{-1}R(w)y),$$

where $R(w)$ is the reverse of w, can be written as a sum of $\ell(w)-1$ terms of form $\zeta^\star(u)\zeta(v)$.

2. *If $w = w'y$ with w' ending in y, then*

$$\zeta^\star(w) + (-1)^{|w|-\ell(w)}\zeta(\mathcal{D}(w')), \tag{2}$$

where $\mathcal{D}$ is the derivation defined above, can be written as a sum of $|w|-\ell(w)-1$ terms of form $\zeta^\star(u)\zeta(v)$.

In the examples above, $w_1 = x^3yx^2y^2xy$ has $y^{-1}R(w_1)y = xy^2x^2x^3y$ (just the reverse in the sequence notation), and $w_2 = x^3yx^2yxy^2$ has

$$\mathcal{D}(w_2') = xyx^2yx^2yxy + x^2yxyx^2yxy + 2x^3y^2x^2yxy + x^3yxyxyxy + 2x^3yx^2y^2xy + 2x^3yx^2yx^2y^2,$$

corresponding to the expansions above.

Remark. The expression (2) above can be written in the alternative form

$$\zeta^\star(w) - (-1)^{|w|-\ell(w)}\zeta_{ш}(ywy^{-1}),$$

where $\zeta_{ш}$ is the shuffle regularization [8].

3 MZSVs of large depth

The usual expansion of MZSVs in terms of multiple zeta values, e.g.,

$$\zeta^\star(4,3,2) = \zeta(4,3,2) + \zeta(7,2) + \zeta(4,5) + \zeta(9) \tag{3}$$

grows longer with depth. Using the standard expansion of MZSVs we can get results that are the opposite, i.e., they are simplest when the depth is large. We shall also need the sum theorem for multiple zeta values, which implies that the sum of all MZSVs of weight n and depth d is $\binom{n-1}{d-1}\zeta(n)$. Now for $n \geq 2$ there is only one MZSV of weight n and depth $n-1$, namely $\zeta^\star(2,\{1\}_{n-1})$, so $\zeta^\star(2,\{1\}_{n-1}) = \binom{n-1}{n-2}\zeta(n) = (n-1)\zeta(n)$. Corresponding to weight $n \geq 3$ and depth $n-2$ there are $n-2$ MZSVs, i.e.,

$$\zeta^\star(3,\{1\}_{n-3}),\ \zeta^\star(2,2,\{1\}_{n-4}),\dots,\ \zeta^\star(2,\{1\}_{n-5},2,1),\ \zeta^\star(2,\{1\}_{n-4},2).$$

We can use the standard expansion to find three-term formulas for all of these. First, the standard expansion of $\zeta^\star(3,\{1\}_{n-3}) = \zeta^\star(x^2y^{n-2})$ is

$$\begin{aligned}\zeta^\star(3,\{1\}_{n-3}) &= \zeta^\star(2)\zeta(n-2) - \zeta(\mathcal{D}(x^2y^{n-3}))\\ &= \zeta(n) + \zeta(2,n-2) + \zeta(n-2,2) - \zeta(xyxy^{n-3} + (n-2)\zeta(x^2y^{n-2}))\\ &= \zeta(n) + \zeta(2,n-2) + \zeta(n-2,2) - \zeta(x^{n-3}yxy) - (n-2)\zeta(x^{n-2}y^2)\\ &= \zeta(n) + \zeta(2,n-2) - (n-2)\zeta(n-1,1).\end{aligned}$$

Similarly, for $\zeta^\star(2,2,\{1\}_{n-4}) = \zeta^\star(xyxy^{n-3})$ we have

$$\begin{aligned}\zeta^\star(2,2,\{1\}_{n-4}) &= \zeta^\star(2,1)\zeta(n-3) - \zeta(\mathcal{D}(xyxy^{n-4}))\\ &= 2\zeta(3)\zeta(n-3) - \zeta(2xy^2xy^{n-4} + (n-3)xyxy^{n-3}))\\ &= 2\zeta(n) + 2\zeta(3,n-3) + 2\zeta(n-3,3) - 2\zeta(x^{n-4}yx^2y) - (n-3)\zeta(x^{n-3}yxy)\\ &= 2\zeta(n) + 2\zeta(3,n-3) - (n-3)\zeta(n-2,2),\end{aligned}$$

and the pattern continues down to

$$\zeta^\star(2,\{1\}_{n-5},2,1) = (n-3)\zeta(n) + (n-3)\zeta(n-2,2) - 2\zeta(3,n-3).$$

If we add the $n-3$ equations obtained so far, noting cancellations, we have

$$\zeta^\star(3,\{1\}_{n-3}) + \zeta(2,2,\{1\}_{n-4}) + \cdots + \zeta(2,\{1\}_{n-5},2,1) = \binom{n-2}{2}\zeta(n) + \zeta(2,n-2) - (n-2)\zeta(n-1,1).$$

Now the sum of all $n-2$ MZSVs of weight n, depth $n-2$ is $\binom{n-1}{n-3}\zeta(n) = \binom{n-1}{2}\zeta(n)$, so it must be the case that

$$\zeta^\star(2,\{1\}_{n-4},2) = (n-2)\zeta(n) + (n-2)\zeta(n-1,2) - \zeta(2,n-2).$$

This proves the following result.

Theorem 3. *Let $n \geq 3$ be a positive integer. Then for $0 \leq j \leq n-3$,*

$$\zeta^\star(xy^jxy^{n-2-j}) = (j+1)\zeta(n)+(j+1)\zeta(j+2, n-j-2)-(n-j-2)\zeta(n-1-j, j+1).$$

Remark. The same strategy works for weight n, depth $n-3$, etc., but the results don't fit such a simple pattern. The following, however, is true: MZSVs of weight n and depth $n-d$ can be written in terms of multiple zeta values of weight n and depth at most d. As a demonstration of the utility of this technique, we give in the Appendix formulas for all MZSVs of weight 9 as rational polynomials in $\zeta(2), \ldots, \zeta(9)$, based on existing tables of multiple zeta values (as computed by Bigotte *et. al.* [2]). While these could have been computed from equations of the form (3), this would be extremely tedious for the larger depths. In fact we used the standard expansion for depths exceeding 4.

4 Formulas for $\zeta^\star(\{2, \{1\}_n\}_m, 1)$ and $\zeta^\star(\{\{2\}_m, 1\}_n)$

In this section we give a new proof of a result first proved by Zlobin [15] using generating functions; it also follows from the cyclic sum theorem for multiple star-zeta values of Ohno and Wakabayashi [10]. We then give a "pictorial" proof of a result of Zhao [13] on the way to establishing a formula for $\zeta^\star(\{\{2\}_m, 1\}_n)$.

Theorem 4. *For nonnegative integers n and positive integers m,*

$$\zeta^\star(\{2, \{1\}_n\}_m, 1) = (n+2)\zeta(m(n+2)+1).$$

Proof. We induct on m, with $m = 1$ being $\zeta^\star(2, \{1\}_{n+1}) = (n+2)\zeta(n+3)$: this follows from the sum theorem for multiple zeta values. Now suppose the result holds for $m < M$, $M > 1$, and consider the standard expansion of $\zeta^\star(\{2, \{1\}_n\}_M, 1) = \zeta^\star((xy^{n+1})^My)$. This can be written

$$\sum_{j=1}^{M-1}(-1)^{j-1}\zeta^\star(\{2, \{1\}_n\}_{M-j})\zeta(\{n+2\}_j) + (-1)^{M-1}\zeta(\mathcal{D}((xy^{n+1})^M). \quad (4)$$

Now

$$\mathcal{D}((xy^{n+1})^M) = (n+2)x^{n+2}y(xy^{n+1})^{M-1} + (n+2)x^{n+1}yx^{n+2}y(xy^{n+1})^{M-2} + \cdots + (n+2)(xy^{n+1})^{M-1}xy^{n+2}$$

whose dual is

$$(n+2)[(x^{n+1}y)^{M-1}xy^{n+2}+(x^{n+1}y)^{M-2}x^{n+2}yx^{n+1}y+\cdots+x^{n+2}y(x^{n+1}y)^{M-1}],$$

so by duality of multiple zeta values the last term of expression (4) can be written

$$(-1)^{M-1}(n+2)[\zeta(n+3,\{n+2\}_{M-1})+\zeta(n+2,n+3,\{n+2\}_{M-2})+\cdots+\zeta(\{n+2\}_{M-1},n+3)].$$

The preceding term of (4) is $(-1)^{M-2}\zeta^\star(2,\{1\}_{n+1})\zeta(\{n+2\}_{M-1})$, which by the induction hypothesis and stuffle product is

$$\begin{aligned}&(-1)^M(n+2)\zeta(n+3)\zeta(\{n+2\}_{M-1}) = \\ (-1)^M(n+2)[&\zeta(n+3,\{n+2\}_{M-1}+\zeta(n+2,n+3,\{n+2\}_{M-2})+\cdots+\zeta(\{n+2\}_{M-1},n+3) \\ &+\zeta(2n+5,\{n+2\}_{M-2})+\cdots+\zeta(\{n+2\}_{M-2},2n+5)],\end{aligned}$$

and combining the two leaves

$$(-1)^{M-2}(n+2)[\zeta(2n+5,\{n+2\}_{M-2})+\cdots+\zeta(\{n+2\}_{M-2},2n+5)].$$

Now combine this with the next term in (4), namely

$$(-1)^{M-3}\zeta^\star(\{\{2,\{1\}_n\}_2,1)\zeta(\{n+2\}_{M-2}) = (-1)^{M-3}(n+2)\zeta(2n+5)\zeta(\{n+2\}_{M-2}),$$

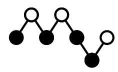
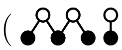
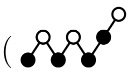
and continue in this way to obtain the conclusion. □

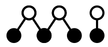
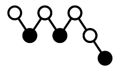
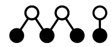
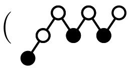
We now give a pictorial proof of [13, Theorem 10.9.4] which, unlike the proof in [13], does not rely on the two-one formula (for which see below). We use two facts: (1) part 3 of Theorem 1, which says that $I(P)$ doesn't change when P is flipped top-to-bottom; and (2) $I(P)=I(\bar{P})$, where $\bar{P}$ is the left-to-right reversal of P (since P and $\bar{P}$ represent the same poset).

Theorem 5. *For positive integers m,n,*

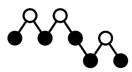
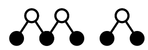
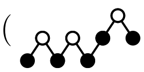
1. $\zeta^\star(\{2\}_m,1,\{2\}_n)=\zeta^\star(\{2\}_m,1)\zeta^\star(\{2\}_n)-\zeta^\star(\{2\}_{n-1},3,\{2\}_m)$
2. $\zeta^\star(\{2\}_m,1,\{2\}_n,1)=\zeta^\star(\{2\}_m,1)\zeta^\star(\{2\}_n,1)-\zeta^\star(\{2\}_n,1,\{2\}_m,1)$
3. $\zeta^\star(\{2\}_m,3,\{2\}_n,1)=\zeta^\star(\{2\}_{m+1})\zeta^\star(\{2\}_{n+1})-\zeta^\star(\{2\}_n,3,\{2\}_m,1)$.

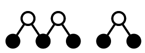
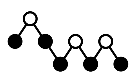
Proof. To keep the pictures simple we assume $m = 2$, $n = 1$. In each case we make a single cut and subtract move. For the first part,

$\zeta^\star(2,2,1,2) = I\,($  $) = I\,($  $) - I\,($  $) =$

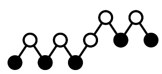
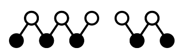
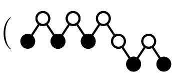
$I\,($  $) - I\,($  $) = I\,($  $) - I\,($  $) = \zeta^\star(2,2,1)\zeta^\star(2) - \zeta^\star(3,2,2).$

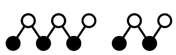
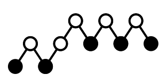
For the second part,

$\zeta^\star(2,2,1,2,1) = I\,($  $) = I\,($  $) - I\,($  $) =$

$I\,($  $) - I\,($  $) = \zeta^\star(2,2,1)\zeta^\star(2,1) - \zeta^\star(2,1,2,2,1).$

For the third part,

$\zeta^\star(2,2,3,2,1) = I\,($  $) = I\,($  $) - I\,($  $) =$

$I\,($  $) - I\,($  $) = \zeta^\star(2,2,2)\zeta^\star(2,2) - \zeta^\star(2,3,2,2,1).$

□

Taking $n = 0$ in Theorem 4,

$$\zeta^\star(\{2\}_m, 1) = 2\zeta(2m+1). \tag{5}$$

Using this with part 2 of the preceding result, we have

$$\zeta^\star(\{2\}_m, 1, \{2\}_n, 1) + \zeta^\star(\{2\}_n, 1, \{2\}_m, 1) = 4\zeta(2m+1)\zeta(2n+1)$$

which in the case $n = m$ becomes

$$\zeta^\star(\{2\}_m, 1, \{2\}_m, 1) = 2\zeta(2m+1)^2. \tag{6}$$

Using the two-one formula of Zhao [13, Prop. 10.8.3], we can extend Eqs. (5) and (6) as follows.

Theorem 6. *For positive integers m, n,*

$$\zeta^\star(\{\{2\}_m,1\}_n) = \sum_{i_1+3i_3+5i_5+\cdots=n} \frac{2^{i_1+i_3+i_5+\cdots}\zeta(2m+1)^{i_1}\zeta(3(2m+1))^{i_3}\zeta(5(2m+1))^{i_5}\cdots}{1^{i_1}i_1!3^{i_3}i_3!5^{i_5}i_5!\cdots}.$$

Proof. By the two-one formula,

$$\zeta^\star(\{\{2\}_m,1\}_n) = 2^n\zeta^{\frac{1}{2}}(\{2m+1\}_n),$$

where we have expressed the right-hand side using Yamamoto's interpolated multiple zeta values [11]. Since by [4, Eq. (8.7)]

$$\zeta^{\frac{1}{2}}(\{a\}_n) = 2^{-n} \sum_{i_1+3i_3+5i_5+\cdots=n} \frac{2^{i_1+i_3+i_5+\cdots}\zeta(a)^{i_1}\zeta(3a)^{i_3}\zeta(5a)^{i_5}\cdots}{1^{i_1}i_1!3^{i_3}i_3!5^{i_5}i_5!\cdots},$$

the result follows. □

Remark. The case $m = 1$ is proved as [6, Eq. (39)] by another method.

5 Identities for $\zeta^\star(2,\ldots,2,3)$ and $\zeta^\star(3,2,\ldots,2)$

Lemma 1.

$$\sum_{j=1}^{n}(-1)^{j-1}\zeta^\star(\{2\}_{n-j})\zeta(\{2\}_j) = \zeta^\star(\{2\}_n).$$

Proof. Apply the standard expansion to $\zeta^\star(\{2\}_n)$. □

Theorem 7. *For $n \ge 0$,*

$$\zeta^\star(\{2\}_n,3) = 2(2n+1-(n+1)2^{-2n-1})\zeta(2n+3) - 4\sum_{r=1}^{n}(1-2^{-2n+2r-1})\zeta(2r+1)\zeta(2n+2-2r).$$

Proof. From the standard expansion we have

$$\zeta^\star(\{2\}_n,3) = \sum_{j=0}^{n}(-1)^j\zeta^\star(\{2\}_{n-j})\zeta(3,\{2\}_j),$$

where in the last term $\zeta^\star(\emptyset)$ means 1. Now from [13],

$$\zeta(3,\{2\}_j) = -4\sum_{r=1}^{j}(-1)^r(1-2^{-2r})r\zeta(2r+1)\zeta(\{2\}_{j+1-r}) + 2(-1)^j[2j+1-(j+1)2^{-2j-1}]\zeta(2j+3).$$

Hence

$$\zeta^\star(\{2\}_n,3)=\zeta^\star(\{2\}_n)\zeta(3)-4\sum_{j=1}^{n}\sum_{r=1}^{j}(-1)^{j+r}(1-2^{-2r})r\zeta(2r+1)\zeta^\star(\{2\}_{n-j})\zeta(\{2\}_{j+1-r})$$
$$+2\sum_{k=1}^{n}(-1)^k[2k+1-(k+1)2^{-2k-1}]\zeta(2k+3)\zeta^\star(\{2\}_{n-k}).$$

In the double sum change the order of summation and then use the lemma to get

$$-4\sum_{r=1}^{n}(1-2^{-2r})r\zeta(2r+1)\zeta^\star(\{2\}_{n+1-r}).$$

Thus we have

$$\zeta^\star(\{2\}_n,3)=\zeta^\star(\{2\}_n)\zeta(3)-4\sum_{r=1}^{n}(1-2^{-2r})r\zeta(2r+1)\zeta^\star(\{2\}_{n+1-r})$$
$$+2\sum_{k=1}^{n}(-1)^k[2k+1-(k+1)2^{-2k-1}]\zeta(2k+3)\zeta^\star(\{2\}_{n-k})$$
$$=\zeta^\star(\{2\}_n)\zeta(3)-3\zeta(3)\zeta^\star(\{2\}_n)-4\sum_{r=2}^{n}(1-2^{-2r})r\zeta(2r+1)\zeta^\star(\{2\}_{n+1-r})$$
$$-4\sum_{r=2}^{n}(-1)^r\left[r-\frac{1}{2}-r2^{-2r}\right]\zeta(2r+1)\zeta^\star(\{2\}_{n-r+1})+2(-1)^n[2n+1-(n+1)2^{-2n-1}]\zeta(2n+3)$$
$$=-2\zeta(3)\zeta^\star(\{2\}_n)-4\sum_{r=2}^{n}\frac{1}{2}\zeta^\star(\{2\}_{n-r+1})\zeta(2r+1)+2(-1)^n[2n+1-(n+1)2^{-2n-1}]\zeta(2n+3)$$
$$=-4\sum_{r=1}^{n}(1-2^{-2n+2r-1})\zeta(2r+1)\zeta(2n-2r+2)+2(-1)^n[2n+1-(n+1)2^{-2n-1}]\zeta(2n+3),$$

where in the last step we used the identity $\zeta^\star(\{2\}_j)=(2-2^{2-2j})\zeta(2j)$. □

Remark. The preceding result can be deduced from results of [4] and [7] as follows. From [4, Theorem 13] follows

$$\sum_{j=1}^{n+1}\zeta^\star(\{2\}_{j-1},3,\{2\}_{n+1-j})=\sum_{j=1}^{n+1}\zeta(2j+1)\zeta^\star(\{2\}_{n+1-j}),$$

which when compared to [7, Theorem 2], i.e.,

$$\sum_{j=1}^{n+1} \zeta^\star(\{2\}_{j-1}, 3, \{2\}_{n+1-j}) = 2(n+1)(1-2^{-2n-2})\zeta(2n+3) - \frac{1}{2}\zeta^\star(\{2\}_n, 3),$$

gives the result.

Theorem 8. *For $n \geq 0$,*

$$\zeta^\star(3, \{2\}_n) = \sum_{k=1}^{n} 4a_k(n-k+1)(1-2^{2k-2-2n})\zeta(2n+3-k)\zeta(\{2\}_k)$$
$$- (n+1)(4n-2+2^{-2n})\zeta(2n+3),$$

where $\{a_n\}$ is the sequence of rational numbers defined by $a_1 = 1$ and

$$a_{n+1} = (-1)^n \left[1 - \frac{a_1}{3}\binom{2n+3}{2} + \cdots + \frac{(-1)^n a_n}{2n+1}\binom{2n+3}{2n}\right]$$

for $n \geq 1$.

Proof. We use induction on n, the case $n = 0$ being immediate. From the standard expansion,

$$\zeta^\star(3, \{2\}_n) = \sum_{j=1}^{n} (-1)^{j-1}\zeta(3, \{2\}_{n-j})\zeta(\{2\}_j) + (-1)^n\zeta(\{2\}_n, 3).$$

From [13] we have

$$\zeta(\{2\}_n, 3) = \sum_{r=1}^{n} (-1)^r 2r(2r-1)\zeta(2r+1)\zeta(\{2\}_{n+1-r})$$
$$- (-1)^n(n+1)(4n-2+2^{-2n})\zeta(2n+3).$$

Hence by the induction hypothesis, $\zeta^\star(3, \{2\}_n)$ is

$$\sum_{j=1}^{n}\sum_{i=1}^{n-j} (-1)^{j-1}4a_i(n-j-i+1)(1-2^{-2(n-j-i+1)})\zeta(\{2\}_i)\zeta(\{2\}_j)\zeta(2n+3-2i-2j)$$
$$+ \sum_{j=1}^{n} (-1)^j(n-j+1)(4(n-j)-2+2^{-2n+2j})\zeta(2n-2j+3)\zeta(\{2\}_j)$$
$$+\sum_{r=1}^{n} (-1)^{r+n}2r(2r-1)\zeta(2r+1)\zeta(\{2\}_{n+1-r}) - (n+1)(4n-2+2^{-2n})\zeta(2n+3).$$

The coefficient of $\zeta(2n+3)$ is evidently correct. Now collect the terms contributing to the coefficient of $\zeta(2n-2k+3)\zeta(\{2\}_k)$, $1 \le k \le n$, noting that

$$\zeta(\{2\}_i)\zeta(\{2\}_j) = \frac{(2i+2j+1)!}{(2i+1)!(2j+1)!}\zeta(\{2\}_{i+j}) = \binom{2i+2j+1}{2i}\frac{\zeta(\{2\}_{i+j})}{2i+1}.$$

The conclusion then follows from the definition of the sequence $\{a_n\}$. $\square$

The sequence $\{a_n\}$ begins

$$1,\ \frac{7}{3},\ \frac{31}{3},\ \frac{381}{5},\ \frac{2555}{3},\ \frac{1414477}{105},\ 286685, \ldots$$

Thus, e.g,

$$\zeta^\star(3,2,2,2,2) = \frac{1143}{5}\zeta(3)\zeta(2,2,2,2) + \frac{155}{2}\zeta(5)\zeta(2,2,2) + \frac{441}{16}\zeta(7)\zeta(2,2) + \frac{255}{16}\zeta(9)\zeta(2) - \frac{17925}{256}\zeta(11).$$

Remark. We note that by the results of [13] and multiple zeta value duality,

1. the multiple zeta value of any argument string consisting of n 2's and a single 3 can be written in the form

$$c_0\zeta(2n+3) + \sum_{j=1}^{n} c_j\zeta(2j)\zeta(2n+3-2j)$$

with $c_j \in \mathbb{Q}$, $0 \le j \le n$;

2. the multiple zeta value of any (admissible) argument string consisting of $n+1$ 2's and a single 1 can be written in the form

$$c_0\zeta(2n+3) + \sum_{j=1}^{n} c_j\zeta(2j)\zeta(2n+3-2j)$$

with $c_j \in \mathbb{Q}$, $0 \le j \le n$.

By using the standard expansion, both statements remain true if "multiple zeta value" is replaced by "multiple zeta-star value".

6 Identities for $\zeta^\star(3,1,\ldots,3,1,3)$ and $\zeta^\star(3,1,\ldots,3,1,2)$

From the standard expansion we have

$$\begin{aligned}
\zeta^\star(3,1,3) &= \zeta^\star(3,1)\zeta(3) - \zeta^\star(3)\zeta(3,1) + \zeta(3,1,3)\\
\zeta^\star(3,1,3,1,3) &= \zeta^\star(3,1,3,1)\zeta(3) - \zeta^\star(3,1,3)\zeta(3,1) + \zeta^\star(3,1)\zeta(3,1,3) - \zeta^\star(3)\zeta(3,1,3,1)\\
&\quad + \zeta(3,1,3,1,3)\\
\zeta^\star(3,1,3,1,3,1,3) &= \zeta^\star(3,1,3,1,3,1)\zeta(3) - \zeta^\star(3,1,3,1,3)\zeta(3,1) + \zeta^\star(3,1,3,1)\zeta(3,1,3)\\
&\quad - \zeta^\star(3,1,3)\zeta(3,1,3,1) + \zeta^\star(3,1)\zeta(3,1,3,1,3) - \zeta^\star(3)\zeta(3,1,3,1,3,1)\\
&\quad + \zeta(3,1,3,1,3,1,3)
\end{aligned}$$

and so on. These relations imply

$$\left[\sum_{n\ge 0}\zeta^\star(\{3,1\}_n)t^{4n} + \sum_{n\ge 0}\zeta^\star(\{3,1\}_n,3)t^{4n+3}\right]\left[\sum_{n\ge 0}\zeta(\{3,1\}_n)t^{4n} - \sum_{n\ge 0}\zeta(\{3,1\}_n,3)t^{4n+3}\right]$$

is an even function. Theorem 1 of [1] gives

$$\sum_{n\ge 0}\zeta(\{3,1\}_n,3)t^{4n+3} = t^3\sum_{n\ge 0}\zeta(\{3,1\}_n)t^{4n}\sum_{m\ge 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m,$$

so it follows that

$$\left[\sum_{n\ge 0}\zeta^\star(\{3,1\}_n)t^{4n} + \sum_{n\ge 0}\zeta^\star(\{3,1\}_n,3)t^{4n+3}\right]\left[1 - t^3\sum_{m\ge 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m\right]$$

is even. Multiplying this out, we have that

$$\begin{aligned}
&\sum_{n\ge 0}\zeta^\star(\{3,1\}_n)t^{4n} - t^3\sum_{n\ge 0}\zeta^\star(\{3,1\}_n)t^{4n}\sum_{m\ge 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m\\
&+ t^3\sum_{n\ge 0}\zeta^\star(\{3,1\}_n,3)t^{4n} - t^6\sum_{n\ge 0}\zeta^\star(\{3,1\}_n,3)t^{4n}\sum_{m\ge 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m
\end{aligned}$$

is even, and hence that

$$\sum_{n\ge 0}\zeta^\star(\{3,1\}_n,3)t^{4n} = \sum_{n\ge 0}\zeta^\star(\{3,1\}_n)t^{4n}\sum_{m\ge 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m.$$

We have proved the following result.

Theorem 9. *For $n \geq 0$,*

$$\zeta^\star(\{3,1\}_n, 3) = \sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k \zeta(4k+3)\zeta^\star(\{3,1\}_{n-k}).$$

In connection with the right-hand side of Theorem 9, we note that the generating function

$$\sum_{n\geq 0} \zeta^\star(\{3,1\}_n) t^{4n} = \frac{\cosh(\pi t) - \cos(\pi t)}{\sinh(\pi t)\sin(\pi t)}$$

is obtained in [6, §6.1], and an explicit formula for $\zeta^\star(\{3,1\}_n)$ in terms of Bernoulli numbers was given earlier by Muneta [9]. Thus we have, e.g.,

$$\zeta^\star(3,1,3,1,3,1,3) = -\frac{1}{64}\zeta(15) + \frac{5}{64}\zeta(11)\zeta(4) - \frac{265}{768}\zeta(7)\zeta(8) + \frac{981955}{707584}\zeta(3)\zeta(12).$$

A similar formula can be obtained for MZSVs of the form $\zeta(3,1,3,\ldots,3,1,2)$. Again using the standard expansion, we have

$$\begin{aligned}
\zeta^\star(3,1,2) &= \zeta^\star(3,1)\zeta(2) - \zeta^\star(3)\zeta(3) + \zeta(2,1,3)\\
\zeta^\star(3,1,3,1,2) &= \zeta^\star(3,1,3,1)\zeta(2) - \zeta^\star(3,1,3)\zeta(3) + \zeta^\star(3,1)\zeta(2,1,3) - \zeta^\star(3)\zeta(3,1,3)\\
&\quad + \zeta(2,1,3,1,3)\\
\zeta^\star(3,1,3,1,3,1,2) &= \zeta^\star(3,1,3,1,3,1)\zeta(2) - \zeta^\star(3,1,3,1,3)\zeta(3) + \zeta^\star(3,1,3,1)\zeta(2,1,3)\\
&\quad - \zeta^\star(3,1,3)\zeta(3,1,3) + \zeta^\star(3,1)\zeta(2,1,3,1,3) - \zeta^\star(3)\zeta(3,1,3,1,3)\\
&\quad + \zeta(2,1,3,1,3,1,3)
\end{aligned}$$

and so on. This means that the coefficients of all exponents of the form t^{4n+2} in the product

$$\left[\sum_{n\geq 0}\zeta^\star(\{3,1\}_n)t^{4n} + \sum_{n\geq 0}\zeta^\star(\{3,1\}_n,2)t^{2n+2} + \sum_{n\geq 0}\zeta^\star(\{3,1\}_n,3)t^{2n+3}\right]\times$$
$$\left[1 - \sum_{n\geq 0}\zeta(2,\{1,3\}_n)t^{4n+2} + \sum_{n\geq 0}\zeta(\{3,1\}_n,3)t^{4n+3}\right] \quad (7)$$

vanish. If we write

$$\mathcal{A}(t) = \sum_{n\geq 0}\zeta(\{3,1\}_n)t^{4n},$$

then Theorem 2 of [1] says that

$$\sum_{n\geq 0}\zeta(2,\{1,3\}_n)t^{4n+2} =$$

$$t^2\mathcal{A}(t)\sum_{m\geq 0}\left[(4m+1)\zeta(4m+2)-4\sum_{j=1}^{m}\zeta(4j-1)\zeta(4m-4j+3)\right]\left(-\frac{t^4}{4}\right)^m$$

Setting

$$\mathcal{B}(t)=\sum_{n\geq 0}\zeta^\star(\{3,1\}_n)t^{4n}$$

and

$$\mathcal{F}(t)=\sum_{m\geq 0}\left[(4m+1)\zeta(4m+2)-4\sum_{j=1}^{m}\zeta(4j-1)\zeta(4m-4j+3)\right]\left(-\frac{t^4}{4}\right)^m,$$

we can write the product (7) as

$$\left[\mathcal{B}(t)+t^3\mathcal{B}(t)\sum_{m\geq 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m+\sum_{n\geq 0}\zeta^\star(\{3,1\}_n,2)t^{4n+2}\right]\times$$
$$\left[1+t^3\zeta\mathcal{A}(t)\sum_{m\geq 0}\zeta(4m+3)\left(-\frac{t^4}{4}\right)^m-t^2\mathcal{A}(t)\mathcal{F}(t)\right],$$

and by an argument similar to that above we get

$$\begin{aligned}\sum_{n\geq 0}\zeta^\star(\{3,1\}_n,2)t^{4n} &= \mathcal{A}(t)\mathcal{B}(t)\mathcal{F}(t)-t^4\mathcal{A}(t)\mathcal{B}(t)\left(\sum_{n\geq 0}\zeta(4n+3)\left(-\frac{t^4}{4}\right)^n\right)^2\\ &= \mathcal{A}(t)\mathcal{B}(t)\left[\mathcal{F}(t)-t^4\left(\sum_{n\geq 0}\zeta(4n+3)\left(-\frac{t^4}{4}\right)^n\right)^2\right].\end{aligned}$$

But

$$t^4\left(\sum_{n\geq 0}\zeta(4n+3)\left(\frac{-t^4}{4}\right)^n\right)^2=-\sum_{m\geq 1}\left(-\frac{t^4}{4}\right)^m\sum_{i+j=m-1}4\zeta(4i+3)\zeta(4j+3),$$

so

$$\mathcal{F}(t) - t^4 \left(\sum_{m \geq 0} \zeta(4n+3) \left(-\frac{t^4}{4} \right)^m \right)^2 = \sum_{m \geq 0} (4m+1)\zeta(4m+2) \left(-\frac{t^4}{4} \right)^m$$

and

$$\sum_{n \geq 0} \zeta^\star(\{3,1\}_n, 2) t^{4n} = \mathcal{A}(t)\mathcal{B}(t) \sum_{m \geq 0} (4m+1)\zeta(4m+2) \left(-\frac{t^4}{4} \right)^m . \qquad (8)$$

Thus we have the following result (cf. [14, Theorem 10.9.5(i)]).

Theorem 10. *For $n \geq 0$,*

$$\zeta^\star(\{3,1\}_n, 2) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \left(-\frac{1}{4} \right)^k (4k+1)\zeta(4k+2)\zeta(\{3,1\}_i)\zeta^\star(\{3,1\}_j).$$

In particular, $\zeta^\star(\{3,1\}_n, 2)$ is a rational multiple of π^{4n+2}.

We note that

$$\mathcal{A}(t)\mathcal{B}(t) = \frac{(\cosh(\pi t) - \cos(\pi t))^2}{\pi^2 t^2 \sinh(\pi t) \sin(\pi t)} = 1 + \frac{\pi^4 t^4}{60} + \frac{\pi^8 t^8}{5400} + \frac{1423\pi^{12} t^{12}}{742996800} + \cdots ,$$

so it may be more efficient to read off the coefficients of the power series in Eq. (8), e.g.,

$$\zeta^\star(3,1,3,1,2) = \frac{\pi^8 \zeta(2)}{5400} - \frac{5\zeta(6)}{4} \cdot \frac{\pi^4}{60} + \frac{9\zeta(10)}{16} = \frac{37\pi^{10}}{2494800} = \frac{111}{80}\zeta(10).$$

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Multiple zeta-star values of weight 9

Depth 2

$$\zeta^\star(8,1)=5\zeta(9)-\zeta(2)\zeta(7)-\zeta(3)\zeta(6)-\zeta(4)\zeta(5)$$
$$\zeta^\star(7,2)=-\frac{35}{2}\zeta(9)+7\zeta(2)\zeta(7)+2\zeta(3)\zeta(6)+4\zeta(4)\zeta(5)$$
$$\zeta^\star(6,3)=\frac{85}{2}\zeta(9)-21\zeta(2)\zeta(7)-6\zeta(4)\zeta(5)$$
$$\zeta^\star(5,4)=-\frac{125}{2}\zeta(9)+35\zeta(2)\zeta(7)+5\zeta(4)\zeta(5)$$
$$\zeta^\star(4,5)=\frac{127}{2}\zeta(9)-35\zeta(2)\zeta(7)-4\zeta(4)\zeta(5)$$
$$\zeta^\star(3,6)=-\frac{83}{2}\zeta(9)+21\zeta(2)\zeta(7)+\zeta(3)\zeta(6)+6\zeta(4)\zeta(5)$$
$$\zeta^\star(2,7)=\frac{37}{2}\zeta(9)-6\zeta(2)\zeta(7)-2\zeta(3)\zeta(6)-4\zeta(4)\zeta(5)$$

Depth 3

$$\zeta^\star(7,1,1)=-\frac{25}{6}\zeta(9)+3\zeta(2)\zeta(7)-\frac{3}{4}\zeta(3)\zeta(6)+\frac{3}{4}\zeta(4)\zeta(5)+\frac{1}{6}\zeta(3)^3$$
$$\zeta^\star(6,2,1)=\frac{1159}{72}\zeta(9)-11\zeta(2)\zeta(7)+\frac{7}{2}\zeta(3)\zeta(6)-\frac{1}{2}\zeta(4)\zeta(5)-\frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(6,1,2)=\frac{551}{36}\zeta(9)-7\zeta(2)\zeta(7)+\frac{1}{3}\zeta(3)\zeta(6)-\frac{9}{4}\zeta(4)\zeta(5)-\frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(5,3,1)=-\frac{559}{24}\zeta(9)+17\zeta(2)\zeta(7)-\frac{7}{4}\zeta(3)\zeta(6)-\frac{7}{4}\zeta(4)\zeta(5)+\frac{1}{6}\zeta(3)^3$$
$$\zeta^\star(5,2,2)=-\frac{3319}{72}\zeta(9)+21\zeta(2)\zeta(7)-\frac{2}{3}\zeta(3)\zeta(6)+\frac{43}{4}\zeta(4)\zeta(5)+\frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(5,1,3)=-\frac{131}{12}\zeta(9)+7\zeta(2)\zeta(7)+\frac{5}{4}\zeta(3)\zeta(6)-\frac{1}{2}\zeta(4)\zeta(5)-\frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(4,4,1)=\frac{559}{18}\zeta(9)-18\zeta(2)\zeta(7)+\frac{1}{3}\zeta(3)\zeta(6)-\frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(4,3,2)=\frac{1567}{36}\zeta(9)-14\zeta(2)\zeta(7)-\frac{2}{3}\zeta(3)\zeta(6)-\frac{35}{2}\zeta(4)\zeta(5)+\frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(4,2,3)=46\zeta(9)-28\zeta(2)\zeta(7)+\frac{5}{3}\zeta(3)\zeta(6)-\frac{1}{3}\zeta(3)^3$$

$$\zeta^\star(4,1,4) = \frac{115}{18}\zeta(9) - \frac{41}{12}\zeta(3)\zeta(6) - 2\zeta(4)\zeta(5) + \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(3,5,1) = -\frac{559}{24}\zeta(9) + 10\zeta(2)\zeta(7) + \frac{5}{4}\zeta(3)\zeta(6) + \frac{25}{4}\zeta(4)\zeta(5) - \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(3,4,2) = -\frac{1567}{36}\zeta(9) + 14\zeta(2)\zeta(7) + \frac{2}{3}\zeta(3)\zeta(6) + \frac{35}{2}\zeta(4)\zeta(5) + \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(3,3,3) = \frac{1}{3}\zeta(9) + \frac{1}{2}\zeta(3)\zeta(6) + \frac{1}{6}\zeta(3)^3$$
$$\zeta^\star(3,2,4) = -59\zeta(9) + 28\zeta(2)\zeta(7) + \frac{33}{4}\zeta(3)\zeta(6) + \frac{11}{2}\zeta(4)\zeta(5) - \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(3,1,5) = \frac{121}{12}\zeta(9) - 7\zeta(2)\zeta(7) + \frac{1}{2}\zeta(3)\zeta(6) + \frac{7}{4}\zeta(4)\zeta(5) + \frac{1}{6}\zeta(3)^3$$
$$\zeta^\star(2,6,1) = \frac{1159}{72}\zeta(9) - \frac{79}{12}\zeta(3)\zeta(6) - \frac{27}{4}\zeta(4)\zeta(5) + \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,5,2) = \frac{439}{36}\zeta(9) - 10\zeta(2)\zeta(7) + \frac{41}{6}\zeta(3)\zeta(6) - \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,4,3) = \frac{593}{36}\zeta(9) + 4\zeta(2)\zeta(7) - \zeta(3)\zeta(6) - \frac{35}{2}\zeta(4)\zeta(5) - \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(2,3,4) = -\frac{53}{36}\zeta(9) - 3\zeta(2)\zeta(7) - \frac{37}{6}\zeta(3)\zeta(6) + 13\zeta(4)\zeta(5) + \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,2,5) = \frac{2513}{72}\zeta(9) - 10\zeta(2)\zeta(7) - \frac{37}{6}\zeta(3)\zeta(6) - 9\zeta(4)\zeta(5) + \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,6) = -\frac{313}{36}\zeta(9) + 4\zeta(2)\zeta(7) + \frac{25}{12}\zeta(3)\zeta(6) + \frac{9}{4}\zeta(4)\zeta(5) - \frac{1}{3}\zeta(3)^3$$

Depth 4

$$\zeta^\star(6,1,1,1) = \frac{881}{72}\zeta(9) - 5\zeta(2)\zeta(7) - \frac{5}{48}\zeta(3)\zeta(6) - \frac{5}{2}\zeta(4)\zeta(5)$$
$$\zeta^\star(5,2,1,1) = -\frac{5375}{144}\zeta(9) + \frac{221}{16}\zeta(2)\zeta(7) + \frac{5}{24}\zeta(3)\zeta(6) + \frac{109}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(5,1,2,1) = \frac{19}{48}\zeta(9) - \frac{5}{8}\zeta(2)\zeta(7) + \frac{395}{48}\zeta(3)\zeta(6) - \frac{41}{8}\zeta(4)\zeta(5) - \frac{3}{2}\zeta(3)^3$$
$$\zeta^\star(5,1,1,2) = -\frac{3227}{144}\zeta(9) + \frac{189}{16}\zeta(2)\zeta(7) - \frac{83}{24}\zeta(3)\zeta(6) + \frac{13}{2}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(4,3,1,1) = \frac{6167}{144}\zeta(9) - \frac{61}{8}\zeta(2)\zeta(7) - \frac{35}{16}\zeta(3)\zeta(6) - \frac{195}{8}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$

$$\zeta^\star(4,2,2,1) = \frac{661}{36}\zeta(9) - \frac{157}{16}\zeta(2)\zeta(7) - \frac{143}{24}\zeta(3)\zeta(6) + \frac{5}{2}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(4,2,1,2) = \frac{2125}{36}\zeta(9) - \frac{483}{16}\zeta(2)\zeta(7) + \frac{15}{4}\zeta(3)\zeta(4) - \frac{77}{8}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(4,1,3,1) = \frac{1421}{144}\zeta(9) + \frac{1}{4}\zeta(2)\zeta(7) - \frac{145}{24}\zeta(3)\zeta(6) - \frac{25}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(4,1,2,2) = -\frac{839}{72}\zeta(9) + \frac{189}{16}\zeta(2)\zeta(7) - \frac{47}{16}\zeta(3)\zeta(6) - \frac{35}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(4,1,1,3) = \frac{397}{16}\zeta(9) - \frac{231}{16}\zeta(2)\zeta(7) + \frac{11}{12}\zeta(3)\zeta(6) - \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(3,4,1,1) = -\frac{2123}{48}\zeta(9) + \frac{109}{16}\zeta(2)\zeta(7) + \frac{16}{3}\zeta(3)\zeta(6) + \frac{105}{4}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(3,3,2,1) = \frac{97}{6}\zeta(9) - \frac{75}{8}\zeta(2)\zeta(7) - \frac{143}{16}\zeta(3)\zeta(6) + \frac{45}{8}\zeta(4)\zeta(5) + 3\zeta(3)^3$$
$$\zeta^\star(3,3,1,2) = -\frac{217}{8}\zeta(9) + \frac{189}{16}\zeta(2)\zeta(7) + \frac{63}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(3,2,3,1) = -\frac{1235}{24}\zeta(9) + \frac{291}{16}\zeta(2)\zeta(7) + \frac{433}{16}\zeta(3)\zeta(6) - 6\zeta(3)^3$$
$$\zeta^\star(3,2,2,2) = -\frac{641}{16}\zeta(9) + \frac{189}{16}\zeta(2)\zeta(7) + \frac{93}{16}\zeta(3)\zeta(6) + \frac{105}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(3,2,1,3) = -\frac{3389}{72}\zeta(9) + \frac{441}{16}\zeta(2)\zeta(7) + \frac{5}{6}\zeta(3)\zeta(6) + \zeta(3)^3$$
$$\zeta^\star(3,1,4,1) = \frac{1421}{144}\zeta(9) - \frac{61}{8}\zeta(2)\zeta(7) - \frac{209}{48}\zeta(3)\zeta(6) + \frac{55}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(3,1,3,2) = -\frac{31}{8}\zeta(9) + \frac{63}{16}\zeta(2)\zeta(7) + \frac{33}{8}\zeta(3)\zeta(6) - \frac{45}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(3,1,2,3) = \frac{2879}{72}\zeta(9) - \frac{441}{16}\zeta(2)\zeta(7) - \frac{5}{4}\zeta(3)\zeta(6) + \frac{55}{8}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(3,1,1,4) = -\frac{3721}{144}\zeta(9) + \frac{231}{16}\zeta(2)\zeta(7) + \frac{95}{24}\zeta(3)\zeta(6) - \frac{1}{2}\zeta(4)\zeta(5) - \frac{1}{2}\zeta(3)^3$$

$$\zeta^\star(2,5,1,1) = \frac{3019}{144}\zeta(9) - \frac{20}{3}\zeta(3)\zeta(6) - \frac{23}{2}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(2,4,2,1) = -\frac{187}{72}\zeta(9) + \frac{433}{24}\zeta(3)\zeta(6) - \frac{77}{8}\zeta(4)\zeta(5) - 4\zeta(3)^3$$
$$\zeta^\star(2,4,1,2) = \frac{59}{2}\zeta(9) - \frac{109}{16}\zeta(2)\zeta(7) - \frac{75}{16}\zeta(3)\zeta(6) - \frac{77}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,3,3,1) = \frac{97}{6}\zeta(9) - \frac{145}{8}\zeta(3)\zeta(6) + \frac{9}{4}\zeta(4)\zeta(5) + 3\zeta(3)^3$$
$$\zeta^\star(2,3,2,2) = -\frac{455}{16}\zeta(9) + \frac{75}{8}\zeta(2)\zeta(7) + \frac{105}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,3,1,3) = \frac{251}{24}\zeta(9) - \frac{67}{16}\zeta(2)\zeta(7) + \frac{35}{48}\zeta(3)\zeta(6) - \frac{3}{2}\zeta(3)^3$$
$$\zeta^\star(2,2,4,1) = \frac{661}{36}\zeta(9) - \frac{673}{48}\zeta(3)\zeta(6) - \frac{5}{2}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(2,2,3,2) = \frac{889}{16}\zeta(9) - \frac{291}{16}\zeta(2)\zeta(7) - 21\zeta(4)\zeta(5)$$
$$\zeta^\star(2,2,2,3) = \frac{223}{16}\zeta(9) - 2\zeta(2)\zeta(7) - \frac{31}{8}\zeta(3)\zeta(6) - \frac{7}{2}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,2,1,4) = \frac{1187}{36}\zeta(9) - \frac{179}{16}\zeta(2)\zeta(7) - \frac{117}{16}\zeta(3)\zeta(6) - \frac{37}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(2,1,5,1) = \frac{19}{48}\zeta(9) + \frac{121}{48}\zeta(3)\zeta(6) + \frac{5}{8}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,1,4,2) = -\frac{1009}{36}\zeta(9) + \frac{61}{8}\zeta(2)\zeta(7) + \frac{93}{16}\zeta(3)\zeta(6) + \frac{77}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,3,3) = \frac{1205}{72}\zeta(9) - \frac{67}{16}\zeta(2)\zeta(7) + \frac{61}{48}\zeta(3)\zeta(6) - \frac{63}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,2,4) = -\frac{1501}{36}\zeta(9) + \frac{131}{8}\zeta(2)\zeta(7) + \frac{21}{8}\zeta(3)\zeta(6) + \frac{113}{8}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,1,1,5) = \frac{2765}{144}\zeta(9) - \frac{109}{16}\zeta(2)\zeta(7) - \frac{21}{16}\zeta(3)\zeta(6) - \frac{7}{2}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$

Depth 5

$$\zeta^\star(5,1,1,1,1) = -\frac{1949}{144}\zeta(9) + 5\zeta(2)\zeta(7) + \frac{85}{48}\zeta(3)\zeta(6) + \frac{9}{2}\zeta(4)\zeta(5) - \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(4,2,1,1,1) = \frac{1345}{36}\zeta(9) - \frac{221}{16}\zeta(2)\zeta(7) - \frac{85}{24}\zeta(3)\zeta(6) - \frac{77}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(4,1,2,1,1) = \frac{149}{24}\zeta(9) + \frac{61}{8}\zeta(2)\zeta(7) - \frac{151}{48}\zeta(3)\zeta(6) - \frac{105}{8}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(4,1,1,2,1) = \frac{265}{36}\zeta(9) - \frac{109}{16}\zeta(2)\zeta(7) + \frac{35}{48}\zeta(3)\zeta(6) + \frac{15}{4}\zeta(4)\zeta(5)$$
$$\zeta^\star(4,1,1,1,2) = \frac{3227}{144}\zeta(9) - 7\zeta(2)\zeta(7) - \frac{259}{48}\zeta(3)\zeta(6) - \frac{7}{2}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(3,3,1,1,1) = -\frac{493}{36}\zeta(9) + \frac{5}{8}\zeta(2)\zeta(7) - \frac{143}{48}\zeta(3)\zeta(6) + \frac{113}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(3,2,2,1,1) = -\frac{8063}{144}\zeta(9) + \frac{157}{16}\zeta(2)\zeta(7) + \frac{503}{24}\zeta(3)\zeta(6) + 20\zeta(4)\zeta(5) - 4\zeta(3)^3$$
$$\zeta^\star(3,2,1,2,1) = -\frac{1637}{144}\zeta(9) + \frac{75}{8}\zeta(2)\zeta(7) - \frac{45}{8}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(3,2,1,1,2) = -\frac{2125}{36}\zeta(9) + 21\zeta(2)\zeta(7) + \frac{719}{48}\zeta(3)\zeta(6) + \frac{77}{8}\zeta(4)\zeta(5) - 2\zeta(3)^3$$
$$\zeta^\star(3,1,3,1,1) = \frac{1}{8}\zeta(9) - \frac{1}{4}\zeta(2)\zeta(7) + \frac{11}{8}\zeta(3)\zeta(6)$$
$$\zeta^\star(3,1,2,2,1) = \frac{1283}{48}\zeta(9) - \frac{291}{16}\zeta(2)\zeta(7) - \frac{143}{16}\zeta(3)\zeta(6) + \frac{75}{8}\zeta(4)\zeta(5) + 3\zeta(3)^3$$
$$\zeta^\star(3,1,2,1,2) = \frac{217}{8}\zeta(9) - 14\zeta(2)\zeta(7) - \frac{9}{4}\zeta(4)\zeta(5)$$
$$\zeta^\star(3,1,1,3,1) = -\frac{1295}{72}\zeta(9) + \frac{61}{8}\zeta(2)\zeta(7) + \frac{433}{48}\zeta(3)\zeta(6) - \frac{5}{8}\zeta(4)\zeta(5) - 2\zeta(3)^3$$
$$\zeta^\star(3,1,1,2,2) = -\frac{59}{2}\zeta(9) + 14\zeta(2)\zeta(7) + \frac{75}{16}\zeta(3)\zeta(6) + \frac{15}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(3,1,1,1,3) = -\frac{391}{144}\zeta(9) + \frac{23}{48}\zeta(3)\zeta(6) + \frac{5}{2}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(2,4,1,1,1) = \frac{1181}{72}\zeta(9) + \frac{7}{8}\zeta(3)\zeta(6) - \frac{25}{2}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,3,2,1,1) = \frac{331}{144}\zeta(9) - \frac{145}{12}\zeta(3)\zeta(6) + \frac{77}{8}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(2,3,1,2,1) = -\frac{403}{144}\zeta(9) + \frac{145}{8}\zeta(3)\zeta(6) - \frac{63}{8}\zeta(4)\zeta(5) - 5\zeta(3)^3$$
$$\zeta^\star(2,3,1,1,2) = \frac{839}{72}\zeta(9) - \frac{5}{8}\zeta(2)\zeta(7) - \frac{905}{48}\zeta(3)\zeta(6) + \frac{77}{8}\zeta(4)\zeta(5) + 2\zeta(3)^3$$

$$\zeta^\star(2,2,3,1,1) = \frac{5707}{144}\zeta(9) - \frac{209}{16}\zeta(3)\zeta(6) - \frac{45}{2}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(2,2,2,2,1) = 2\zeta(9)$$
$$\zeta^\star(2,2,2,1,2) = \frac{641}{16}\zeta(9) - \frac{157}{16}\zeta(2)\zeta(7) - \frac{93}{16}\zeta(3)\zeta(6) - \frac{105}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,2,1,3,1) = \frac{1283}{48}\zeta(9) - \frac{145}{8}\zeta(3)\zeta(6) - \frac{55}{8}\zeta(4)\zeta(5) + 3\zeta(3)^3$$
$$\zeta^\star(2,2,1,2,2) = \frac{455}{16}\zeta(9) - \frac{75}{8}\zeta(2)\zeta(7) - \frac{77}{8}\zeta(4)\zeta(5)$$
$$\zeta^\star(2,2,1,1,3) = \frac{803}{36}\zeta(9) - \frac{115}{16}\zeta(2)\zeta(7) - \frac{23}{48}\zeta(3)\zeta(6) - \frac{21}{4}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,1,4,1,1) = -\frac{61}{6}\zeta(9) + \frac{59}{12}\zeta(3)\zeta(6) + \frac{15}{2}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,1,3,2,1) = -\frac{403}{144}\zeta(9) + \frac{143}{48}\zeta(3)\zeta(6) + \zeta(3)^3$$
$$\zeta^\star(2,1,3,1,2) = \frac{31}{8}\zeta(9) + \frac{1}{4}\zeta(2)\zeta(7) - \frac{11}{8}\zeta(3)\zeta(6)$$
$$\zeta^\star(2,1,2,3,1) = -\frac{1637}{144}\zeta(9) + \frac{145}{12}\zeta(3)\zeta(6) + \frac{45}{8}\zeta(4)\zeta(5) - 4\zeta(3)^3$$
$$\zeta^\star(2,1,2,2,2) = -\frac{889}{16}\zeta(9) + \frac{291}{16}\zeta(2)\zeta(7) + \frac{31}{8}\zeta(3)\zeta(6) + 21\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,2,1,3) = -\frac{91}{6}\zeta(9) + \frac{51}{8}\zeta(2)\zeta(7) - \frac{85}{48}\zeta(3)\zeta(6) + \frac{63}{8}\zeta(4)\zeta(5) + \frac{1}{2}\zeta(3)^3$$
$$\zeta^\star(2,1,1,4,1) = \frac{265}{36}\zeta(9) - \frac{263}{24}\zeta(3)\zeta(6) + \frac{21}{4}\zeta(4)\zeta(5) + 2\zeta(3)^3$$
$$\zeta^\star(2,1,1,3,2) = \frac{1009}{36}\zeta(9) - \frac{61}{8}\zeta(2)\zeta(7) + \frac{205}{16}\zeta(3)\zeta(6) - \frac{185}{8}\zeta(4)\zeta(5) - \zeta(3)^3$$
$$\zeta^\star(2,1,1,2,3) = \frac{989}{72}\zeta(9) - \frac{115}{16}\zeta(2)\zeta(7) - \frac{151}{24}\zeta(3)\zeta(6) + \frac{55}{8}\zeta(4)\zeta(5) + \zeta(3)^3$$
$$\zeta^\star(2,1,1,1,4) = \frac{265}{144}\zeta(9) + 2\zeta(2)\zeta(7) + \frac{151}{48}\zeta(3)\zeta(6) - \frac{7}{2}\zeta(4)\zeta(5) - \frac{1}{2}\zeta(3)^3$$

Depth 6

$$\zeta^\star(4,1,1,1,1,1) = \frac{1217}{72}\zeta(9) - 3\zeta(2)\zeta(7) - \frac{59}{12}\zeta(3)\zeta(6) - \frac{19}{4}\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(3,2,1,1,1,1) = -\frac{1681}{36}\zeta(9) + 11\zeta(2)\zeta(7) + \frac{89}{6}\zeta(3)\zeta(6) + \frac{23}{2}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(3,1,2,1,1,1) = \frac{1081}{36}\zeta(9) - 17\zeta(2)\zeta(7) + \frac{1}{12}\zeta(3)\zeta(6) - \zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(3,1,1,2,1,1) = -\frac{1297}{36}\zeta(9) + 18\zeta(2)\zeta(7) + 5\zeta(3)\zeta(6) + \frac{5}{2}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(3,1,1,1,2,1) = \frac{619}{72}\zeta(9) - 10\zeta(2)\zeta(7) + \frac{1}{6}\zeta(3)\zeta(6) + \frac{25}{4}\zeta(4)\zeta(5) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(3,1,1,1,1,2) = -\frac{551}{36}\zeta(9) + 7\zeta(2)\zeta(7) + \frac{29}{6}\zeta(3)\zeta(6) + \frac{1}{2}\zeta(5)\zeta(4) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,3,1,1,1,1) = \frac{835}{72}\zeta(9) - 10\zeta(3)\zeta(6) + \frac{13}{4}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,2,2,1,1,1) = \frac{56}{3}\zeta(9) - 10\zeta(3)\zeta(6) - 6\zeta(4)\zeta(5) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,2,1,2,1,1) = \frac{137}{3}\zeta(9) - 10\zeta(3)\zeta(6) - 30\zeta(4)\zeta(5) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,2,1,1,2,1) = -\frac{16}{3}\zeta(9) + 10\zeta(3)\zeta(6) - \frac{8}{3}\zeta(3)^3$$
$$\zeta^\star(2,2,1,1,1,2) = \frac{3319}{72}\zeta(9) - 11\zeta(2)\zeta(7) - \frac{89}{6}\zeta(3)\zeta(6) - \frac{35}{4}\zeta(4)\zeta(5) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,3,1,1,1) = \frac{1}{2}\zeta(9) - \frac{1}{12}\zeta(3)\zeta(6) + \frac{3}{2}\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,2,2,1,1) = -\frac{115}{3}\zeta(9) + 10\zeta(3)\zeta(6) + 30\zeta(4)\zeta(5) - \frac{8}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,2,1,2,1) = \frac{2}{3}\zeta(9) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,2,1,1,2) = -\frac{1567}{36}\zeta(9) + 17\zeta(2)\zeta(7) + \frac{35}{2}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,1,3,1,1) = \frac{43}{2}\zeta(9) - \frac{1}{12}\zeta(3)\zeta(6) - \frac{33}{2}\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,1,2,2,1) = -\frac{16}{3}\zeta(9) + 6\zeta(4)\zeta(5) + \frac{4}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,1,2,1,2) = \frac{1567}{36}\zeta(9) - 18\zeta(2)\zeta(7) - \frac{59}{12}\zeta(3)\zeta(6) - 4\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,1,1,3,1) = \frac{619}{72}\zeta(9) - \frac{11}{4}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$

$$\zeta^\star(2,1,1,1,2,2) = -\frac{439}{36}\zeta(9) + 10\zeta(2)\zeta(7) + \frac{59}{6}\zeta(3)\zeta(6) - \frac{19}{2}\zeta(4)\zeta(5) - \frac{2}{3}\zeta(3)^3$$
$$\zeta^\star(2,1,1,1,1,3) = \frac{461}{72}\zeta(9) - 4\zeta(2)\zeta(7) + \frac{1}{12}\zeta(3)\zeta(6) + \frac{17}{4}\zeta(4)\zeta(5) + \frac{1}{3}\zeta(3)^3$$

Depth 7

$$\zeta^\star(3,1,1,1,1,1,1) = -\frac{19}{2}\zeta(9) + \zeta(2)\zeta(7) + 5\zeta(3)\zeta(6) + 3\zeta(4)\zeta(5)$$
$$\zeta^\star(2,2,1,1,1,1,1) = 28\zeta(9) - 10\zeta(3)\zeta(6) - 12\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,2,1,1,1,1) = -17\zeta(9) + 18\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,1,2,1,1,1) = 4\zeta(9)$$
$$\zeta^\star(2,1,1,1,2,1,1) = 25\zeta(9) - 18\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,1,1,1,2,1) = -20\zeta(9) + 10\zeta(3)\zeta(6) + 12\zeta(4)\zeta(5)$$
$$\zeta^\star(2,1,1,1,1,1,2) = \frac{35}{2}\zeta(9) - \zeta(2)\zeta(7) - 5\zeta(3)\zeta(6) - 3\zeta(4)\zeta(5)$$

Depth 8

$$\zeta^\star(2,1,1,1,1,1,1,1) = 8\zeta(9)$$