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Abstract

We define 2-crossed module bundle 2-gerbes related to general Lie 2-crossed modules and discuss their properties. If $(L \to M \to N)$ is a Lie 2-crossed module and $Y \to X$ is a surjective submersion then an $(L \to M \to N)$-bundle 2-gerbe over X is defined in terms of a so called $(L \to M \to N)$-bundle gerbe over the fibre product $Y^{[2]} = Y \times_X Y$, which is an $(L \to M)$-bundle gerbe over $Y^{[2]}$ equipped with a trivialization under the change of its structure crossed module from $L \to M$ to $1 \to N$, and which is subject to further conditions on higher fibre products $Y^{[3]}$, $Y^{[4]}$ and $Y^{[5]}$. String structures can be described and classified using 2-crossed module bundle 2-gerbes.

Introduction

The modest purpose of this paper is to introduce nonabelian bundle 2-gerbes related to 2-crossed modules [20], simultaneously generalizing abelian bundle 2-gerbes [49], [50], [19] and crossed-module bundle gerbes [1], [30]. The idea is to describe objects in differential geometry, which would, in the terminology of [10], correspond to the Čech cohomology classes in $H^1(X, L \to M \to N)$, i.e., the first Čech cohomology classes on a manifold X with values in a Lie 2-crossed module $L \to M \to N$. What we want is a theory, which in the case of the 2-crossed module $U(1) \to 1 \to 1$ reproduces the theory of abelian bundle 2-gerbes and in the case of a 2-crossed module $1 \to M \to N$ reproduces the theory of crossed module bundle gerbes related to the crossed module $M \to N$ ($(M \to N)$-bundle gerbes). The latter requirement can slightly be generalized as follows. Let us assume given a crossed module $L \xrightarrow{\partial} M$. If we put $A := \ker \partial$ and $Q := \operatorname{coker}\partial$ then we have a 4-term exact sequence of Lie groups $0 \to A \to L \xrightarrow{\partial} M \to Q \to 1$ with abelian A. Let us assume that $A = U(1)$ is in the centre of L and that the restriction to $U(1)$ of the action of M on L is trivial. Then we want that an $(U(1) \to L \to M)$-bundle 2-gerbe is the same thing as an $(L \to M)$ bundle gerbe twisted with an abelian bundle 2-gerbe [2].

The paper is organized as follows. In section 2, we briefly recall the relevant notions of a Lie crossed module and Lie 2-crossed module. In section 3, relevant results on crossed module bundles and on crossed module bundle gerbes are collected. Let us mention that crossed module bundles are special kinds of bitorsors [27], [26], [7], [10] and that crossed module bundle gerbes can be seen as a special case of gerbes with constant bands (this follows, e.g., from discussion in section 4.2. of [10] commenting on the abelian bundle gerbes of [39], the cocycle bitorsors of [52], [53], and the bouquets of [24]). In section 4, 2-crossed module bundle gerbes are introduced as crossed module bundle bundles with an additional structure. 2-crossed module bundle gerbes are to 2-crossed module bundle 2-gerbes the same as crossed module bundles are to crossed module bundle gerbes. Finally, in section 5, 2-crossed module bundle 2-gerbes are introduced and their properties discussed, including their local description in terms of 3-cocycles similar to those of [23] and [9], [10]. The example of a lifting bundle 2-gerbe is described in some detail. Also, we discuss the relevance of 2-crossed module bundle 2-gerbes to string structures and their classification (see proposition 4.12 and remark 4.14). For the relevance of gerbes and abelian 2-gerbes to the string group and string structures see, e.g., [5], [17], [16], [30], [41], [48] and [54]. For discussions of abelian 2-gerbes in relation to quantum field theory and string theory see, e.g., [37], [18], [19], [2].

Let us mention that in [9] and [10] much more general 2-gerbes were introduced in the language of 2-stacks. These are generalizations of gerbes (defined as locally nonempty and locally connected stacks in groupoids [26], [38], [10], [7]) and seem to be related rather to crossed squares than to 2-crossed modules. We hope to return to a discussion concerning a possible relation of our bundle 2-gerbes and the 2-gerbes of [9] and [10] in the future. Also, we hope to discuss the relevant notion of a 2-bouquet elsewhere. Our task here is to describe nonabelian bundle 2-gerbes using a language very close to that of the classical reference books [34], [29]. This will allow us introduce connection, curvature, curving etc. in the forthcoming paper [32] using the language of differential geometry. For some further related work see, e.g., [45], [46], [47], [44], [25].

In this paper we work in the category of differentiable manifolds. In particular, all groups (with exception of the string group) are assumed to be Lie groups and all maps are assumed to be smooth maps. It would be possible to work with (for instance, paracompact Hausdorff) topological spaces, topological groups and continuous maps too. For this we would have to use a proper replacement of the notion of the surjective submersion $\pi : Y \to X$ in the definitions of crossed module bundle gerbes, 2-crossed module bundle gerbes and 2-crossed module bundle 2-gerbes. For instance, instead of surjective submersions we could consider surjective maps $\pi : Y \to X$ with the property that for each point $y \in Y$ there is a neighborhood O of $\pi(y)$ with a section $\sigma : O \to Y$, such that $s(\pi(y)) = y$. Such map may be called a surjective topological submersion.

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1. Crossed modules, 2-crossed modules

Let us recall the notion of a crossed module of Lie groups (see, e.g., [11],[15],[43]).

1.1. **Definition.** Let L and M be two Lie groups. We say that L is a crossed M-module if there is a Lie group morphism $\partial_1 : L \to M$ and a smooth action of M on L $(m, l) \mapsto {}^m l$ such that

$$^{\partial_1(l)}l' = ll'l^{-1} \text{ (Peiffer condition)}$$

for $l, l' \in L$, and

$$\partial_1({}^m l) = m\partial_1(l)m^{-1}$$

for $l \in L, m \in M$ hold true.

We will use the notation $L \overset{\partial_1}{\to} M$ or $L \to M$ for the crossed module.

Let us also recall that a crossed module is a special case of a pre-crossed module, in which the Peiffer condition doesn't necessarily hold. There is an obvious notion of a morphism of crossed modules.

1.2. **Definition.** A morphism between crossed modules $L \overset{\partial_1}{\to} M$ and $L' \overset{\partial_1'}{\to} M'$ is a pair of Lie group morphisms $\lambda : L \to L'$ and $\kappa : M \to M'$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\partial_1} & M \\ \lambda\downarrow & & \downarrow\kappa \\ L' & \xrightarrow{\partial_1'} & M' \end{array}$$

commutes, and for any $l \in L$ and $m \in M$ we have the following identity

$$\lambda({}^m l) =^{\kappa(m)} \lambda(l).$$

1.3. **Remark.** A crossed module of Lie groups defines naturally a strict Lie 2-group C (see, e.g., [6]). The Lie group of objects is $C_0 = \{*\}$, the Lie group of 1-arrows is $C_1 = M$ and the Lie group of 2-arrows is $C_2 = M \ltimes L$. The "vertical" multiplication is given on C_2 by

$$(m, l_1)(\partial_1(l_1)m, l_2) = (m, l_1 l_2)$$

and the "horizontal" multiplication is given by

$$(m_1, l_1)(m_2, l_2) = (m_1 m_2, l_1 {}^{m_1} l_2)$$

See, e.g., [12] and [14] for more details on the relation between crossed modules and strict Lie 2-groups.

1.4. **Definition.** The definition of a 2-crossed module of groups is due to Conduché [20]; see also, e.g., [21], [42], [13], [43], [44]). A Lie 2-crossed module is a complex of Lie groups

$$L \xrightarrow{\partial_1} M \xrightarrow{\partial_2} N \tag{1}$$

together with smooth left actions by automorphisms of N on L and M (and on N by conjugation), and the Peiffer lifting, which is an equivariant map $\{\,,\,\} : M \times M \to L$, i.e., ${}^n\{m_1, m_2\} = \{{}^n m_1, {}^n m_2\}$ such that:

i) (1) is a complex of N-modules, i.e., ∂_1 and ∂_2 are N-equivariant and $\partial_1\partial_2(l) = 1$ for $l \in L$,
ii) $m_1 m_2 m_1^{-1} = \partial_1\{m_1, m_2\}\, {}^{\partial_2(m_1)}m_2 =: \langle m_1, m_2\rangle$, for $m_1, m_2 \in M$,
iii) $[l_1, l_2] := l_1 l_2 l_1^{-1} l_2^{-1} = \{\partial_1 l_1, \partial_1 l_2\}$, for $l_1, l_2 \in L$,
iv) $\{m_1 m_2, m_3\} = \{m_1, m_2 m_3 m_2^{-1}\}\, {}^{\partial_2(m_1)}\{m_2, m_3\}$, for $m_1, m_2, m_3 \in M$,
v) $\{m_1, m_2 m_3\} = {}^{m_1 m_2 m_1^{-1}}\{m_1, m_3\}\{m_1, m_2\}$, for $m_1, m_2, m_3 \in M$,
vi) $\{\partial_1(l), m\}\{m, \partial_1(l)\} = l\, {}^{\partial_2(m)}(l^{-1})$, for $m \in M, l \in L$, and

wherein the notation ${}^n m$ and ${}^n l$ for left actions of the element $n \in N$ on elements $m \in M$ and $l \in L$ has been used. Also, let us note that ${}^m l := l\{\partial_1(l)^{-1}, m\}$ defines a left action of M on L by automorphisms. This is a consequence of the other axioms and is proved in [20], [13], where it is also shown that, equipped with this action, $L \xrightarrow{\partial_1} M$ defines a crossed module.

1.5. **Example.** Any crossed module $(L \xrightarrow{\delta} M)$ determines a 2-crossed module $A := \ker(\partial) \to L \to M$ with an abelian A.

1.6. **Remark.** In addition to the crossed module $L \xrightarrow{\partial_1} M$, there is another crossed module that can be associated with the 2-crossed module $L \xrightarrow{\partial_1} M \xrightarrow{\partial_2} N$. By definition, we see that $M \xrightarrow{\partial_2} N$ is a (special) pre-crossed module in which the Peiffer condition is satisfied only up to the Peiffer lifting. Hence, $M/\partial_1(L) \xrightarrow{\partial_2'} N$, with the induced Lie group homomorphism ∂_2' and with the induced action of N on $M/\partial_1(L)$, is a crossed module.

There is an obvious notion of a morphism of 2-crossed modules.

1.7. **Definition.** A morphism between 2-crossed modules $L \xrightarrow{\partial_1} M \xrightarrow{\partial_2} N$ and $L' \xrightarrow{\partial_1'} M' \xrightarrow{\partial_2'} N'$ is a triple of Lie group morphisms $L \to L'$, $M \to M'$ and $N \to N'$ making up, together with the maps ∂_1, ∂_1', ∂_2 and ∂_2' a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\partial_1} & M & \xrightarrow{\partial_2} & N \\ \lambda\downarrow & & \mu\downarrow & & \downarrow\nu \\ L' & \xrightarrow{\partial_1'} & M' & \xrightarrow{\partial_2'} & N' \end{array} \tag{2}$$

and being compatible with the actions of N on M and L and of N' on M' and L', respectively and with the respective Peiffer liftings.

1.8. **Remark.** A 2-crossed module of Lie groups defines naturally a Gray (Lie) 3-groupoid with a single object. For the construction and for more details on the relation between 2-crossed modules and Gray 3-groupoids see [33], [13], [42], [25]. There are two "vertical" multiplications and one "horizontal" multiplication on triples (3-cells) $(n, m, l) \in N \times M \times L$. The vertical multiplications are determined by the crossed module $L \to M$. The two vertical multiplications are given by

$$(n, m, l_1)(n, \partial_1(l_1)m, l_2) = (n, m, l_1 l_2)$$

and

$$(n, m_1, l_1)(\partial_2(m_1)n, m_2, l_2) = (n, m_1 m_2, l_1\, {}^{m_1}l_2)$$

and the horizontal multiplication is given by

$$(n_1, m_1, l_1)(n_2, m_2, l_2) = (n_1 n_2, m_1\, {}^{n_1}m_2, l_1\, {}^{m_1}({}^{n_1}l_2))$$

2. Crossed module bundle gerbes

Let X be a (smooth) manifold. Crossed module bundle gerbes have been introduced, for instance, in [30], [1]. These can be seen as generalizations of abelian bundle gerbes [39], [40]. If $(L \stackrel{\partial_1}{\to} M)$ is a crossed module of Lie groups, X a manifold and $P \to X$ a left principal L-bundle, we can change the structure group of P from L to M, in order to obtain a left principal M-bundle $P' = M \times_{\partial_1} P$ defined as follows. Points $p' \in P'$ correspond to equivalence classes $[m, p] \in M \times_{\partial_1} P$ with the equivalence relation on $M \times P$ given by $(m, p) \sim (m\partial_1(l), l^{-1}p)$. Obviously, the principal left M-action is given by $M \times P' \to P'$, $m' \times [m, p] \mapsto [m'm, p]$.

2.1. **Definition.** Let $(L \stackrel{\partial_1}{\to} M)$ be a crossed module of Lie groups and X a manifold. Let $P \to X$ be a left principal L-bundle, such that the principal M-bundle $M \times_{\partial_1} P$ is trivial with a trivialization defined by a section (i.e. a left L-equivariant smooth map) $\boldsymbol{m} : P \to M$. We call the pair $(P, \boldsymbol{m})$ an $(L \to M)$-bundle.

2.2. **Remark.** If we think about the crossed module $L \to M$ as a groupoid with the Lie group of objects M and the Lie group of arrows $M \ltimes L$ then a crossed module bundle is the same thing as a principal groupoid bundle.

2.3. **Definition.** Two $(L \to M)$-bundles $(P, \boldsymbol{m})$ and $(P', \boldsymbol{m}')$ over X are isomorphic if they are isomorphic as left L-bundles by an isomorphism $\boldsymbol{\ell} : P \to P'$ such that $\boldsymbol{m}'\boldsymbol{\ell} = \boldsymbol{m}$. An $(L \to M)$-bundle is trivial if it is isomorphic to the trivial $(L \to M)$-bundle $(X \times L, \partial_1 \mathrm{pr}_L)$.

2.4. **Example.** Notice that a general $(L \to M)$-bundle is not necessarily locally trivial, although it is locally trivial as a left principal L-bundle. For instance, for a function $m : X \to M$ such that $\mathrm{Im}(m)$ is not a subset of $\mathrm{Im}(\partial_1)$ the $(L \to M)$-bundle $(X \times L, \partial_1 \mathrm{pr}_L.m\mathrm{pr}_X)$ is locally non-trivial. We will refer to such an $(L \to M)$-bundle as an $(L \to M)$-bundle defined by the M-valued function m. Two such $(L \to M)$-bundles are isomorphic iff their respective sections m and m' are related by an L-valued function l on X by $m' = \partial_1(l)m$. Obviously, compositions of isomorphisms corresponds to multiplication of the respective L-valued function defining them.

2.5. **Example.** A $(1 \to G)$-bundle is the same thing as a G-valued function.

2.6. **Example.** A pair $(T, \mathbf{l})$, where T is a trivial left principal L-bundle and $\mathbf{l} : T \to L$ its trivializing section, defines an $(L \to M)$-bundle with the section $\boldsymbol{m} = \partial_1 \mathbf{l} : T \to M$. $(T, \mathbf{l})$ is a trivial $(L \to M)$-bundle.

2.7. **Example.** Let L be a normal subgroup of M. The adjoint action of M restricted to L defines a crossed module structure on $L \to M$ with $\ker \partial_1 = 1$. Let L be also a closed subgroup of M and assume M to be finite dimensional. We put $G := L/M$, so that we have an exact sequence of Lie groups $1 \to L \to M \stackrel{\bar{\pi}}{\to} G \to 1$. It follows that $M \to G$ is a left principal L-bundle over G [34] (hence, admitting smooth local sections).[1] Moreover, $(M \to G, \boldsymbol{m})$ with $\boldsymbol{m} = \mathrm{id}_M$ is an $(L \to M)$-bundle.

2.8. **Pullback.** Obviously, a pullback of an $(L \to M)$-bundle is again an $(L \to M)$-bundle. Pullbacks preserve isomorphisms of crossed module bundles, in particular a pullback of a trivial $(L \to M)$-bundle is again a trivial $(L \to M)$-bundle.

2.9. **Change of the structure crossed module.** If $(L \to M) \to (L' \to M')$ is a morphism of crossed modules, there is an obvious way to construct, starting from an $(L \to M)$-bundle $(P, \boldsymbol{m})$, an $(L' \to M')$-bundle $(L' \times_\lambda P, \kappa\boldsymbol{m})$ where $\lambda : L \to L'$ and $\kappa : M \to M'$ define the morphism of the two crossed modules. Obviously, the change of the structure crossed module preserves isomorphisms of crossed module bundles.

[1] More generally, to assure the existence of smooth local sections of $\bar{\pi}$ in a short exact sequence of topological groups $1 \to L \to M \stackrel{\bar{\pi}}{\to} G \to 1$, we would have to ask the projection $\bar{\pi}$ to be a Hurewicz fibration.

2.10. **1-cocycles.** Consider an $(L \to M)$-bundle $(P, \boldsymbol{m})$ and a trivializing covering $\coprod O_i = X$ of the left principal L-bundle P. Let $\sigma_i : P|O_i \to L$ be the trivializing sections of L and $l_{ij} = \sigma_i^{-1}\sigma_j : O_i \cap O_j \to L$ be the corresponding transition functions. We put $m_i = \partial_1(\sigma_i)^{-1}\boldsymbol{m}$, which obviously gives an L-valued function on O_i. We have $\partial_1(l_{ij}) = m_i m_j^{-1}$. Hence the $(L \to M)$-bundle $(P, \boldsymbol{m})$ can be described by a 1-cocycle given by transition functions (m_i, l_{ij}), $m_i : O_i \to M$, $l_{ij} : O_{ij} = O_i \cap O_j \to L$ satisfying on nonempty O_{ij}

$$\partial_1(l_{ij}) = m_i m_j^{-1}$$

and on nonempty $O_{ijk} = O_i \cap O_j \cap O_k$

$$l_{ij} l_{jk} = l_{ik}$$

Transition functions (m_i, l_{ij}) and (m'_i, l'_{ij}) corresponding to two isomorphic $(L \to M)$-bundles are related by

$$m'_i = \partial_1(l_i) m_i$$

and

$$l'_{ij} = l_i l_{ij} l_i^{-1}$$

We say that two 1-cocycles (m_i, l_{ij}) and (m'_i, l'_{ij}), related as above, are equivalent. We will denote by $H^0(X, L \to M)$ the set of corresponding equivalence classes. A trivial $(L \to M)$-bundle is described by transition functions $(\partial_1(l_i), l_i l_j^{-1})$.

On the other hand, given transition functions (m_i, l_{ij}) we can reconstruct an $(L \to M)$-bundle. We define a left principal L-bundle P with the total space formed by equivalence classes of triples $[x, l, i]$ with $x \in O_i$, $l \in L$ under the equivalence relation $(x, l, i) \sim (x', l', j)$ iff $x = x'$ and $l' = l l_{ij}$. The principal left L-action is given by $l'[x, l, i] = [x, l'l, i]$. Now we put $\boldsymbol{m}([x, l, i]) = \partial_1(l) m_i(x)$. $(P, \boldsymbol{m})$ is an $(L \to M)$-bundle.

With the two above constructions it is not difficult to prove that the isomorphism classes of $(L \to M)$-bundles are 1-1 with elements of $H^0(X, L \to M)$.

2.11. **Lifting crossed module bundle.** Let L and M be as above in (2.7). Consider a G-valued function $g : X \to G$. The pullback $g^*(M, \mathrm{id})$ of the $(L \to M)$-bundle $\pi : M \to G$ is an $(L \to M)$-bundle on X (the lifting crossed module bundle). It is the obstruction to a lifting of the G-valued function G to some M-valued function. To go in the opposite direction, we note that we have an obvious morphism of crossed modules $(L \to M) \to (1 \to G)$. Under the change of the structure crossed module of an $(L \to M)$-bundle $(P, \boldsymbol{m})$ to $(1 \to G)$, the section $\boldsymbol{m}$ becomes an L-invariant G- valued function $\pi\boldsymbol{m}$ on P. Hence, it is identified with an G-valued function g on X. Two isomorphic $(L \to M)$-bundles give the same function. The two constructions are inverse to each other up to an isomorphism of $(L \to M)$-bundles.

It is now easy to give a local description of lifting crossed module bundles. Let $\{O_i\}_i$ be an open covering on X. Let P be an $(L \to M)$-bundle described by transition function (l_{ij}, m_i). Since $\pi\partial_1 = 1$, we have $\pi(m_i) = \pi(m_j)$. Hence, the collection of local functions $\{\pi(m_i)\}_i$ defines a G-valued function on X. To go in the opposite direction, let g be a G-valued function on X. Let $\{O_i\}_i$ be a trivializing covering of the pullback principal bundle $g^*(M)$. The function g can be now described by a collection of local functions $g_i : O_i \to G$ such that $g_i = g_j$ on O_{ij}. Hence, we have local functions $m_i : O_i \to M$ the local sections of $g^*(M)$ such that $\pi(m_i) = g_i$, which are related on double intersections O_{ij} by $m_i = \partial_1(l_{ij}) m_j$ with L-valued functions $l_{ij} : O_{ij} \to L$, the transition functions of the principal L-bundle $g^*(M)$, fulfilling the 1-cocycle condition $l_{ij} l_{jk} = l_{ik}$ on O_{ijk}.

Concerning crossed module bundles, we have the following lemma and proposition (see [1]).

Lemma 2.1. *The $(L \to M)$-bundle $(P, \boldsymbol{m})$ is also a right principal L-bundle with the right action of L given by $p.l = {}^{\boldsymbol{m}(p)}(l).p$ for $p \in P, l \in L$. The left and right actions commute. The section $\boldsymbol{m}$ is L-biequivariant.*

Proposition 2.1. *Let $\mathcal{P} = (P, \boldsymbol{m})$ and $\tilde{\mathcal{P}} = (\tilde{P}, \tilde{\boldsymbol{m}})$ are two $(L \to M)$-bundles over X. Let us define an equivalence relation on the Whitney sum $P \oplus \tilde{P} = P \times_X \tilde{P}$ by $(pl, \tilde{p}) \sim (p, l\tilde{p})$, for $(p, \tilde{p}) \in P \oplus \tilde{P}$ and $l \in L$. Then $\mathcal{P}\tilde{\mathcal{P}} := (P\tilde{P} := (P \oplus \tilde{P})/\sim, \boldsymbol{m}\tilde{\boldsymbol{m}})$ with $\boldsymbol{m}\tilde{\boldsymbol{m}}([p, \tilde{p}]) := \boldsymbol{m}(p)\tilde{\boldsymbol{m}}(\tilde{p})$ is an $(L \to M)$-bundle.*

2.12. **Remark.** Obviously, if $\mathcal{P} \cong \mathcal{Q}$ and $\tilde{\mathcal{P}} \cong \tilde{\mathcal{Q}}$ then also $\mathcal{P}\tilde{\mathcal{P}} \cong \mathcal{Q}\tilde{\mathcal{Q}}$. The set of isomorphism classes of $(L \to M)$-bundles equipped with the above defined product is a group. The unit is given by the class of the trivial bundle $(X \times L, \partial_1 \mathrm{pr_L})$. The inverse is given by the class of $(L \to M)$-bundle $(P^{-1}, \boldsymbol{m}^{-1})$ with P^{-1} having the same total space as P, the left L-action on P^{-1} being the inverse of the right L-action on P and the trivializing section $\boldsymbol{m}^{-1}$ being the composition of the inverse in M with the trivializing section $\boldsymbol{m}$. Let us note that in the case of an exact sequence $1 \to L \to M \to N \to 1$ as above (2.7) this group structure is compatible with the group structure of $G = M/L$-valued functions with point-wise multiplication.

2.13. **Example.** If $\mathcal{P} = (P = X \times L, \partial_1 \mathrm{pr}_L.m\mathrm{pr}_X)$ and $\mathcal{P}' = (P' = X \times L, \partial_1 \mathrm{pr}_L.m'\mathrm{pr}_X)$ are $(L \to M)$-bundles defined by two respective M-valued functions m and m' on X (2.4) then the product $\mathcal{P}\mathcal{P}'$ is explicitly described again as an $(L \to M)$-bundle defined by the function mm' by identifying $[(x, l), (x, l')] \in PP'$ with $(x, l\, {}^m l') \in X \times L$.

2.14. **Product on 1-cocycles.** Transition functions $(\bar{m}_i, \bar{l}_{ij})$ of the product of two $(L \to M)$-bundles described by transition functions (m_i, l_{ij}) and $(\tilde{m}_i, \tilde{l}_{ij})$ are given by

$$\bar{m}_i = m_i \tilde{m}_i$$

and

$$\bar{l}_{ij} = l_{ij}\, {}^{m_i}\tilde{l}_{ij}$$

Transition functions of the inverse crossed module bundle are $({}^{m_j^{-1}} l_{ij}^{-1} = {}^{m_i^{-1}} l_{ij}^{-1}, m_i^{-1})$.

2.15. **1-cocycles as functors.** The crossed module $(L \to M)$ defines naturally a topological category (groupoid) $\mathcal{C}$ with the set of objects $C_0 = L$ and the set of arrows $C_1 = M \times L$. Let us consider the topological category $\mathcal{O}$ (groupoid) defined by the good covering $\{O_i\}$ of X with objects $x_i := (x, i | \, x \in O_i)$ and exactly one arrow from x_i to y_j iff $x = y$. Then a 1-cocycle is the same thing as a continuous functor from $\mathcal{O}$ to $\mathcal{C}$. Further, if $2\mathcal{B}$ is a strict topological 2-category, then the category of 2-arrows with the vertical composition is naturally a topological category $\mathcal{B}$. The horizontal composition in $2\mathcal{B}$ defines a continuous functor from the cartesian product $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$. Thus, in case $\mathcal{B} = \mathcal{C}$ it defines naturally a multiplication on functors $\mathcal{O} \to \mathcal{C}$ (i.e., on transition functions), which is the same as the one defined above (2.1).

2.16. **Crossed module bundle gerbes.** Let Y be a manifold. Consider a surjective submersion $\pi \;:\; Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e., $\pi\sigma_i = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \ldots \times_X Y$, the n-fold fibre product of Y, i.e., $Y^{[n]} := \{(y_1, \ldots y_n) \in Y^n \mid \pi(y_1) = \pi(y_2) = \ldots \pi(y_n)\}$. Given an $(L \to M)$-bundle $\mathcal{P} = (P, \boldsymbol{m})$ over $Y^{[2]}$ we denote by $\mathcal{P}_{12} = p_{12}^*(\mathcal{P})$ the crossed module bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{P}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$. Consider a quadruple $(\mathcal{P}, Y, X, \boldsymbol{\ell})$, where $\mathcal{P} = (P, \boldsymbol{m})$ is a crossed module bundle, $Y \to X$ a surjective submersion and $\boldsymbol{\ell}$ an isomorphism of crossed module bundles $\boldsymbol{\ell} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$. We now consider bundles $\mathcal{P}_{12}$, $\mathcal{P}_{23}$, $\mathcal{P}_{13}$, $\mathcal{P}_{24}$, $\mathcal{P}_{34}$, $\mathcal{P}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} \;:\; Y^{[4]} \to Y^{[2]}$ etc. and also the crossed module isomorphisms $\boldsymbol{\ell}_{123}$, $\boldsymbol{\ell}_{124}$, $\boldsymbol{\ell}_{123}$, $\boldsymbol{\ell}_{234}$ induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc.

2.17. **Definition.** The quadruple $(\mathcal{P}, Y, X, \boldsymbol{\ell})$, where $Y \to X$ is a surjective submersion, $\mathcal{P}$ is a crossed module bundle over $Y^{[2]}$, and $\boldsymbol{\ell} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ an isomorphism of crossed module bundles over $Y^{[3]}$, is called a crossed module bundle gerbe if $\boldsymbol{\ell}$ satisfies the cocycle condition (associativity) on $Y^{[4]}$

$$\begin{array}{ccc} \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} & \xrightarrow{\boldsymbol{\ell}_{234}} & \mathcal{P}_{12}\mathcal{P}_{24} \\ \boldsymbol{\ell}_{123}\big\downarrow & & \big\downarrow \boldsymbol{\ell}_{124} \\ \mathcal{P}_{13}\mathcal{P}_{34} & \xrightarrow{\boldsymbol{\ell}_{134}} & \mathcal{P}_{14} \end{array} \tag{3}$$

2.18. **Abelian bundle gerbes.** Abelian bundle gerbes as introduced in [39], [40] are $(U(1) \to 1)$-bundle gerbes. More generally, if $A \to 1$ is a crossed module then A is necessarily an abelian group and an abelian bundle gerbe can be identified as an $(A \to 1)$-bundle gerbe.

2.19. **Example.** A $(1 \to G)$-bundle gerbe is the same thing as a G-valued function g on $Y^{[2]}$ (2.5) satisfying on $Y^{[3]}$ the cocycle relation $g_{12}g_{23} = g_{23}$ and hence, a principal G-bundle on X (more precisely a descent datum of a principal G-bundle).

2.20. **Pullback.** If $f : X \to X'$ is a map then we can pullback $Y \to X$ to $f^*(Y) \to X'$ with a map $\tilde{f} : f^*(Y) \to Y$ covering f. There are induced maps $\tilde{f}^{[n]} : f^*(Y)^{[n]} \to Y^{[n]}$. Then the pullback $f^*(\mathcal{P}, Y, X, \boldsymbol{\ell}) := (\tilde{f}^{[2]*}\mathcal{P}, f^*(Y), f(X), \tilde{f}^{[3]*}\boldsymbol{\ell})$ is again an $(L \to M)$-bundle gerbe.

2.21. **Definition.** Two crossed module bundle gerbes $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ and $(\mathcal{P}', Y', X, \boldsymbol{\ell}')$ are stably isomorphic if there exists a crossed module bundle $\mathcal{Q} \to \bar{Y} = Y \times_X Y'$ such that over $\bar{Y}^{[2]}$ the crossed module bundles $q^*\mathcal{P}$ and $\mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ are isomorphic. The corresponding isomorphism $\tilde{\boldsymbol{\ell}} : q^*\mathcal{P} \to \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ should satisfy on $\bar{Y}^{[3]}$ (with an obvious abuse of notation) the condition

$$\tilde{\boldsymbol{\ell}}_{13}\boldsymbol{\ell} = \boldsymbol{\ell}'\tilde{\boldsymbol{\ell}}_{23}\tilde{\boldsymbol{\ell}}_{12} \tag{4}$$

Here q and q' are projections onto the first and second factor of $\bar{Y} = Y \times_X Y'$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the pullbacks of $\mathcal{Q} \to \bar{Y}$ to $\bar{Y}^{[2]}$ under the respective projections from $\bar{Y}^{[2]}$ to $\bar{Y}$ etc.

A crossed module bundle gerbe $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ is called trivial if it is stably isomorphic to the trivial crossed module bundle gerbe $((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1)$. Pullbacks preserve stable isomorphisms, in particular a pullback of a trivial crossed module bundle gerbe is again a trivial crossed module bundle gerbe. If $Y = X$ then the crossed module bundle gerbe is trivial.

2.22. **Definition.** Let $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ and $(\mathcal{P}', Y', X, \boldsymbol{\ell}')$ be two crossed module bundle gerbes and $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ two stable isomorphisms between them. We call $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ isomorphic if there is an isomorphism $\underline{\boldsymbol{\ell}} : \mathcal{Q} \to \mathcal{R}$ of crossed module bundles on $\bar{Y} = Y \times_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{Q}}} & \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1} \\ \mathrm{id}\big\downarrow & & \big\downarrow \underline{\boldsymbol{\ell}}_1 \underline{\boldsymbol{\ell}}_2{}^{-1} \\ q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{R}}} & \mathcal{R}_1 q'^*\mathcal{P}'\mathcal{R}_2^{-1} \end{array} \tag{5}$$

is commutative.

2.23. **Remark.** Let $\pi' : Y' \to X$ be an another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the crossed module bundle gerbes $\mathcal{G}_f = (f^*\mathcal{P}, Y', X, f^{[3]*}\boldsymbol{\ell})$ and $\mathcal{G} = (\mathcal{P}, Y, X, \boldsymbol{\ell})$ are stably isomorphic. This can be easily seen by noticing first that $\mathcal{G}$ is stably isomorphic to itself and then using the obvious fact that pullbacks of crossed module bundles commute with their products [1]. It follows that locally each crossed module bundle gerbe $\mathcal{G}$ is trivial. For this, take a point $x \in X$ and its neighborhood $O \subset X$ such that there exists a local section $\sigma : O \to Y$ of π. Over O we have the bundle gerbe $\mathcal{G}_O$, the restriction of $\mathcal{G}$ to O. Now we can put $Y' := O$ and $\pi' := \mathrm{id}_O$ and we have $\pi\sigma = \pi'$. It follows that $\mathcal{G}_\sigma$ is stably isomorphic to $\mathcal{G}_O$. However $\mathcal{G}_\sigma$ is trivial, because of $Y' = O$.

2.24. **Change of the structure crossed module.** If $(L \to M) \to (L' \to M')$ is a morphism of crossed modules, there is an obvious way to construct starting from an $(L \to M)$-bundle gerbe an $(L' \to M')$-bundle gerbe by changing the structure crossed module of the corresponding $(L \to M)$-bundle over $Y^{[2]}$. Obviously, the change of the structure crossed module preserves stable isomorphisms of crossed module bundle gerbes.

2.25. **2-cocycles.** Locally, crossed module bundle gerbes can be described in terms of 2-cocycles as follows. First, let us note that the trivializing cover $\{O_i\}$ of the map $\pi : Y \to X$ defines a new surjective submersion $\pi' : Y' = \coprod O_i \to X$. The local sections of $Y \to X$ define a map $f : Y' \to Y$, which is compatible with the maps π and π', i.e., such that $\pi f = \pi'$. We know that the crossed module bundle gerbes $\mathcal{G}_f$ and $\mathcal{G}$ are stably isomorphic. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one. Then the crossed module bundle gerbe can be described by a 2-cocycle (m_{ij}, l_{ijk}) where the maps $m_{ij} : O_{ij} \to M$ and $l_{ijk} : O_{ijk} \to L$ fulfill the following conditions

$$m_{ij} m_{jk} = \partial_1(l_{ijk}) m_{ik} \quad \text{on} \quad O_{ijk}$$

and

$$l_{ijk} l_{ikl} = {}^{m_{ij}}l_{jkl} l_{ijl} \quad \text{on} \quad O_{ijkl}$$

Two crossed module bundle gerbes are stably isomorphic if their respective 2-cocycles (m_{ij}, l_{ijk}) and (m'_{ij}, l'_{ijk}) are related (equivalent) by

$$m'_{ij} = m_i \partial_1(l_{ij}) m_{ij} m_j^{-1} \tag{6}$$

and

$$l'_{ijk} = {}^{m_i}l_{ij} \, {}^{m_i m_{ij}}l_{jk} \, {}^{m_i}l_{ijk} \, {}^{m_i}l_{ik}^{-1} \tag{7}$$

with $m_i : O_i \to M$ and $l_{ij} : O_{ij} \to L$

We will denote by $H^1(X, L \to M)$ the set of the corresponding equivalence classes of 2-cocycles.

A trivial crossed module bundle gerbe is described by transition functions

$$m_{ij} = m_i \partial_1(l_{ij}) m_j^{-1}$$

and

$$l_{ijk} = {}^{m_i}l_{ij} \, {}^{m_i}l_{jk} \, {}^{m_i}l_{ik}^{-1}$$

Two collections of stable isomorphism data (m_i, l_{ij}) and (m'_i, l'_{ij}) are isomorphic if

$$m'_i = \partial_1(l_i) m_i$$
$$l'_{ij} = l_i l_{ij} \, {}^{m_{ij}}l_j^{-1}$$

Now we briefly describe how an $(L \to M)$-bundle gerbe can be reconstructed from transition functions (m_{ij}, l_{ijk}). Put $Y = \coprod O_i$. On each nonempty O_{ij} consider the $(L \to M)$-bundle $\mathcal{P}_{ij}$ defined by the function $m_{ij} : O_{ij} \to M$ as in (2.4). Hence, on $Y^{[2]}$ we have the $(L \to M)$-bundle given by $\mathcal{P} = \coprod_{ij} \mathcal{P}_{ij}$. Now we recall the explicit descriptions of the multiplication (2.13) and isomorphisms (2.4) of two $(L \to M)$-bundles defined by their respective M-valued functions. Using the 2-cocycle relations, it is now straightforward to show that the collection of functions

l_{ijk} defines an isomorphism of $\mathcal{P}_{12}\mathcal{P}_{23}$ and $\mathcal{P}_{13}$ on $Y^{[3]}$ satisfying the associativity condition on $Y^{[4]}$ (compare, e.g., to Theorem 3.1 in [38]).

Further, two crossed module bundle gerbes corresponding to two equivalent 2-cocycles are stably isomorphic. To show it let us denote, similarly as above, by $\mathcal{P}$ and $\mathcal{P}'$ the two $(L \to M)$-bundles over $Y^{[2]}$ defined by the two respective collections of local functions m_{ij} and m'_{ij}. Note that according to (2.4), the local M-valued functions m_i in (6) define an $(L \to M)$-bundle $\mathcal{Q}$ over $Y = \coprod O_i$ and that the local L-valued functions ${}^{m_i}l_{ij}$ define on $Y^{[2]}$ an isomorphism $\tilde{\boldsymbol{\ell}}$ of the $(L \to M)$ bundles $\mathcal{P}'$ and $\mathcal{Q}_1\mathcal{P}\mathcal{Q}_2^{-1}$. Finally, the relation (7) tells us that the isomorphism $\tilde{\boldsymbol{\ell}}$ fulfils the requested compatibility condition (4) (cf. the last statement in Example 2.4 concerning the composition of isomorphisms).

Hence, the above discussion of 2-cocycles proves the following proposition.

Proposition 2.2. *Stable isomorphism classes of $(L \to M)$-bundle gerbes are in a bijective correspondence with elements of $H^1(X, L \to M)$.*

2.26. **Remark.** Actually, when considering isomorphisms of stable isomorphisms, we have the respective 2-categories of $(L \to M)$-bundle gerbes and transition functions. The correspondence between $(L \to M)$-bundle gerbes and the transition functions can be formulated in the framework of 2-categories similarly to [9], but we will not do this here. Further, if we consider the topological category $\mathcal{O}$ defined by the good covering $\{O_i\}$ of X. Then a 2-valued cocycle can be seen as a continuous normal pseudo-functor from $\mathcal{O}$ to the bicategory defined by the crossed module $L \to M$.

2.27. **Lifting crossed module bundle gerbe.** Let $L \to M$ be a crossed module associated with a normal subgroup L of M (cf. Example 2.7). We have a Lie group extension

$$1 \to L \xrightarrow{\partial_1} M \xrightarrow{\bar{\pi}} G \to 1$$

and also the $(L \to M)$-bundle $M \xrightarrow{\bar{\pi}} G$.

The following statement has appeared in the literature before. In [5] (cf. Lemma 2) a version of it is attributed to Larry Breen.

Proposition 2.3. *Let $L \to M \xrightarrow{\bar{\pi}} G$ be an Lie group extension. Let us also assume that the conditions for M being a principal L-bundle over G are satisfied.*[2] *Then the isomorphism classes of G-principal bundles are in bijective correspondence with stable isomorphism classes of $(L \to M)$-bundle gerbes.*

Proof. Let $E \to X$ be a (locally trivial) left principal G-bundle over X. As a principal G-bundle E defines a (division) map $g : E^{[2]} \to G$ which gives for two elements in a fibre of E the group element relating them. The pullback $\mathcal{P} = g^*(M, \mathrm{id}_M)$ of the $(L \to M)$-bundle $M \to G$ gives an $(L \to M)$-bundle on $E^{[2]}$; $\mathcal{P}$ is by definition the lifting $(L \to M)$-bundle corresponding to the division map g (2.11). It follows that the crossed module bundles $\mathcal{P}_{12}\mathcal{P}_{23}$ and $\mathcal{P}_{13}$ are isomorphic on $E^{[3]}$. This follows from the above mentioned fact that, in case of Lie groups L, M and G as above, isomorphism classes of $(L \to M)$-bundles are one to one to G-valued functions and that this correspondence respects the respective multiplications (2.11). Such an isomorphism $\boldsymbol{\ell}$ fulfills the requested associativity condition because of $\ker(\partial_1) = 1$. Hence, we have a crossed module bundle gerbe $\mathcal{G}$, which can be seen as an obstruction to a lifting of the principal G bundle E to some principal M-bundle. Also, it is easily seen that lifting two isomorphic G-bundle leads to stably isomorphic $(L \to M)$ bundle gerbes. On the other hand, if we have a crossed module $L \to M$ with a trivial kernel of ∂_1 and hence fitting the exact sequence with $G = \mathrm{coker}\partial_1$ we can change the structure crossed module from $L \to M$ to $1 \to G$ in a crossed module bundle gerbe in order to get a principal G-bundle on X. These two

[2]cf. Example 2.4

constructions are inverse to each other on sets of stable isomorphism classes of $(L \to M)$-bundle gerbes (with $(L \to M)$ as above) and isomorphism classes of principal G-bundles.

2.28. **Remark.** It is also easy to give a local description of lifting crossed module bundle gerbes. Let again $\{O_i\}_i$ be a good covering of X. Let us consider an $(L \to M)$-bundle gerbe described by transition functions (m_{ij}, l_{ijk}). Then $\bar{\pi}(m_{ij})\bar{\pi}(m_{jk}) = \bar{\pi}(m_{ik})$. Hence, we have a principal G-bundle with transition functions $g_{ij} = \bar{\pi}(m_{ij})$. To go in the opposite direction, let g_{ij} be the transition functions of a principal G-bundle. Since the double intersections O_{ij} are contractible we can choose lifts m_{ij} of the transition functions g_{ij}. On O_{ijk} these will be related by $m_{ij}m_{jk} = \partial_1(l_{ijk})m_{ik}$ with L-valued functions l_{ijk} which, because of $\ker \partial_1 = 1$, necessarily satisfy the required compatibility condition on O_{ijkl} (2.16).

2.29. **Remark.** Note that given three principal G-bundles E, E'' and E''' and isomorphisms $E \xrightarrow{f} E'$, $E' \xrightarrow{f'} E''$ and $E'' \xrightarrow{f''} E'''$ such that $f'f = f''$ the corresponding lifting crossed module bundle gerbes $\mathcal{G}, \mathcal{G}''$ and $\mathcal{G}'''$ will be stably isomorphic, but the respective stable isomorphisms $\boldsymbol{f}\boldsymbol{f}'$ and $\boldsymbol{f}''$ will be only isomorphic in general.

3. 2-crossed module bundle gerbes

Let $(L \to M \to N)$ be a Lie 2-crossed module and $\mathcal{G}$ be an $(L \to M)$-bundle gerbe over X. From the definition of the 2-crossed module we see immediately that the maps $L \to 1$ and $\partial_2 : M \to N$ define a morphism of crossed modules $\mu : (L \xrightarrow{\partial_1} M) \to (1 \to N)$. Thus, we have the following trivial lemma (2.5):

Lemma 3.1. *$\mu(\mathcal{G})$ is a principal N-bundle on X. If $\mathcal{G}$ and $\mathcal{G}'$ are stably isomorphic, then $\mu(\mathcal{G})$ and $\mu(\mathcal{G}')$ are isomorphic.*

3.1. **Definition.** Let $\mathcal{G}$ be an $L \to M$-bundle gerbe such that the principal bundle $\mu(\mathcal{G})$ over X is trivial with a section $\boldsymbol{n} : \mu(\mathcal{G}) \to N$. We call the pair $(\mathcal{G}, \boldsymbol{n})$ a 2-crossed module bundle gerbe.

3.2. **Remark.** The following interpretation of the trivializing section $\boldsymbol{n}$ will be useful later. For the $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ the trivializing section $\boldsymbol{n}$ of the left principal N-bundle $\mu(\mathcal{G})$ is the same thing as an N-valued function $\mathbf{n}$ on Y such that $\partial_2(\boldsymbol{m}) = \mathbf{n}_1\mathbf{n}_2^{-1}$.

3.3. **Remark.** If we think about the 2-crossed module $L \to M \to N$ as a 2-groupoid with objects in L, 1-arrows in $L \times M$ and 2-arrows in $L \times N \times M$ then $L \to M \to N$-bundle gerbes should give an example of the bigroupoid 2-torsors introduced in [3].

3.4. **Pullback.** If $f : X \to X'$ then we put $f^*(\mathcal{G}, \boldsymbol{n}) = (f^*(\mathcal{G}), f^*\boldsymbol{n})$; this will again be a 2-crossed module bundle gerbe.

3.5. **Definition.** We call two $(L \to M \to N)$-bundle gerbes $(\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}', \boldsymbol{n}')$ over the same manifold X stably isomorphic if there exists a stable isomorphism $\boldsymbol{q} := (\mathcal{Q}, \tilde{\boldsymbol{\ell}}) : \mathcal{G} \to \mathcal{G}'$ of $(L \to M)$-bundle gerbes such that $\boldsymbol{n}'\mu(\boldsymbol{q}) = \boldsymbol{n}$ holds true for the induced isomorphism of trivial bundles $\mu(\boldsymbol{q}) : \mu(\mathcal{G}) \to \mu(\mathcal{G}')$. An $(L \to M \to N)$-bundle gerbe is trivial if it is stably isomorphic to the trivial $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N)$.

Pullbacks preserve stable isomorphisms, in particular a pullback of a trivial 2-crossed module bundle gerbe is again a trivial 2-crossed module bundle gerbe.

3.6. **Example.** Note that a general $(L \to M \to N)$-bundle gerbe is not necessarily locally trivial, although it is locally trivial as an $(L \to M)$-bundle gerbe. For a function $n : X \to N$ such that $\mathrm{Im}(n)$ is not a subset of $\mathrm{Im}(\partial_2)$ the $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N.n\mathrm{pr}_X)$ is locally non-trivial. We will refer to such a 2-crossed module as the 2-crossed module bundle gerbe defined by the N-valued function n on X. Two such 2-crossed module bundle gerbes are stably isomorphic iff their respective functions n and n' are related by an M-valued function m by $n' = \partial_2(m)n$. We will refer to such a stable isomorphism as being defined by the function m. Further, two such stable isomorphisms defined by respective functions m and m' are isomorphic iff they are related by an L-valued function l on X by $m' = \partial_1(l)m$.

3.7. **Example.** Consider an $(1 \to G \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$. As a $(1 \to G)$-bundle gerbe $\mathcal{G}$ gives a principal G-bundle P (more precisely a G-valued function g on $Y^{[2]}$ satisfying the 1-cocycle relation on $Y^{[3]}$). The trivializing section $\boldsymbol{n}$ then gives an N valued function $\mathbf{n}$ (3.2) on Y such that $\partial_2 g_{12} \mathbf{n}_2 = \mathbf{n}_1$ on $Y^{[2]}$ and hence, a trivialization of the left principal G-bundle P under the map $G \to N$. Hence, a $(1 \to G \to N)$-bundle gerbe is the same thing as a $(G \to N)$-bundle.

Obviously, isomorphic $(G \to N)$-bundles correspond to stably isomorphic $(1 \to G \to N)$-bundle gerbes.

3.8. **Remark.** Let $\pi' : Y' \to X$ be an another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the 2-crossed module bundle gerbes $(f^*\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}, \boldsymbol{n})$ are stably isomorphic. This can be shown in a completely analogous way to the case of a crossed module bundle gerbe (2.23).

3.9. **Change of the structure 2-crossed module.** If $(L \to M \to N) \to (L' \to M' \to N')$ is a morphism of 2-crossed modules, there is an obvious way to construct starting from an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$ an $(L' \to M' \to N')$-bundle gerbe $(\mathcal{G}', \boldsymbol{n}')$ by changing the structure crossed module of $\mathcal{G}$ from $L \to M$ to $L' \to M'$ and putting $\boldsymbol{n}' = \nu \boldsymbol{n}$ where ν is the morphism $\nu : N \to N'$ entering the definition of the morphism of two 2-crossed modules. Obviously, change of the structure 2-crossed module preserves stable isomorphisms of 2-crossed module bundle gerbes.

3.10. **Definition.** Let $((\mathcal{P}, Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $((\mathcal{P}', Y', X, \boldsymbol{\ell}'), \boldsymbol{n}')$ be two 2-crossed module bundle gerbes and $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ two stable isomorphisms between them; see (2.21). We call $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ isomorphic if there is an isomorphism $\underline{\boldsymbol{\ell}} : \mathcal{Q} \to \mathcal{R}$ of crossed module bundles on $\bar{Y} = Y \times_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{Q}}} & \mathcal{Q}_1 q'^* \mathcal{P}' \mathcal{Q}_2^{-1} \\ \mathrm{id}\downarrow & & \downarrow \underline{\boldsymbol{\ell}}_1 \underline{\boldsymbol{\ell}}_2^{-1} \\ q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{R}}} & \mathcal{R}_1 q'^* \mathcal{P}' \mathcal{R}_2^{-1} \end{array} \tag{8}$$

is commutative. Obviously, pullbacks preserve isomorphisms of stable isomorphisms.

3.11. **2-cocycles.** Let $\pi : Y \to X$ be the surjective submersion, which was implicitly contained in the above definition of a 2-crossed module bundle gerbe. Since also for 2-crossed module bundle gerbes it holds true that 2-crossed module bundle gerbes $(f^*\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}, \boldsymbol{n})$ are stably isomorphic if the respective maps π and π' are related by a compatible map, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one, in which case the $(L \to M \to N)$-bundle gerbe is characterized by transition functions (n_i, m_{ij}, l_{ijk}), $n_i : O_i \to N$, $m_{ij} : O_{ij} \to M$, $l_{ijk} : O_{ijk} \to L$ fulfilling 2-cocycle relations

$$n_i = \partial_2(m_{ij})n_j$$

$$m_{ij}m_{jk} = \partial_1(l_{ijk})m_{ik}$$
$$l_{ijk}l_{ikl} = {}^{m_{ij}}l_{jkl}l_{ijl}$$

on O_{ij}, O_{ijk} and O_{ijkl}, respectively.

In terms of 2-cocycles the stable isomorphism $(l_{ijk}, m_{ij}, n_i) \sim (l'_{ijk}, m'_{ij}, n'_i)$ is expressed by relations

$$n'_i = \partial_2(m_i)n_i \tag{9}$$
$$m'_{ij} = m_i\partial_1(l_{ij})m_{ij}m_j^{-1} \tag{10}$$
$${}^{m_i^{-1}}l'_{ijk} = l_{ij}\,{}^{m_{ij}}l_{jk}l_{ijk}l_{ik}^{-1} \tag{11}$$

Two $(L \to M \to N)$ valued 2-cocycles related as above will be called equivalent. The corresponding set of equivalence classes will be denoted by $H^0(X, L \to M \to N)$.

A trivial 2-crossed module bundle gerbe is described by transition functions

$$n_i = \partial_2(m_i)$$
$$m_{ij} = m_i\partial_1(l_{ij})m_j^{-1}$$

and

$${}^{m_i^{-1}}l_{ijk} = l_{ij}l_{jk}l_{ik}^{-1}$$

Locally, two collections of stable isomorphism data (m_i, l_{ij}) and (m'_i, l'_{ij}) are isomorphic if

$$m'_i = \partial_1(l_i)m_i$$
$$l'_{ij} = l_i l_{ij}\,{}^{m_{ij}}l_j^{-1}$$

An $(L \to M \to N)$-bundle gerbe can be reconstructed from transition functions (n_i, m_{ij}, l_{ijk}) in complete analogy with the case of an $(L \to M)$-bundle gerbe. Starting from (10) and (11) we can reconstruct an $(L \to M)$-bundle gerbe $\mathcal{G}$ as in (2.25). Further, the collection of N-valued local functions n_i appearing in (9) defines a trivial principal N-bundle $\mathcal{N}$ on X with transition functions $n_i n_j^{-1}$. The relation (9) then guaranties that the principal N-bundle $\mu(\mathcal{G})$ is isomorphic to $\mathcal{N}$. Hence, $\mathcal{G}$ is an $(L \to M \to N)$ bundle.

Further, two 2-crossed module bundle gerbes corresponding to two equivalent 2-cocycles are stably isomorphic. Starting from two equivalent 2-cocycles (9-11) we will construct as above the two respective $(L \to M \to N)$-bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$. It follows from (2.25) that $\mathcal{G}$ and $\mathcal{G}'$ will be stably isomorphic as $(L \to M)$-bundle gerbes and because of the relation (9) they will be also stably isomorphic as $(L \to M \to N)$-bundle gerbes.

Hence, the above discussion of 2-cocycles proves the following proposition.

Proposition 3.1. *Stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are in a bijective correspondence with the set $H^0(X, L \to M \to N)$.*

Similarly to the case of crossed module bundles (2.25), we can consider a 2-category of $(L \to M \to N)$-bundle gerbes, with 1-arrows being stable isomorphisms and 2-arrows being isomorphism of stable automorphisms and similarly a 2-category of 2-cocycles, but we will not use these.

3.12. **Lifting 2-crossed module bundle gerbe.** Consider a Lie 2-crossed module $L \to M \to N$ such that $\ker(\partial_1) = 1$ and $\ker(\partial_2) = \mathrm{Im}(\partial_1)$. Put $G := M/L$ and $Q := N/G$. Assume that the conditions are satisfied for having extensions of Lie groups

$$1 \to L \stackrel{\partial_1}{\to} M \stackrel{\partial_2}{\to} N \stackrel{\pi_2}{\to} Q \to 1 \tag{12}$$
$$1 \to L \stackrel{\partial_1}{\to} M \stackrel{\pi_1}{\to} G \to 1 \tag{13}$$

and

$$1 \to G \stackrel{\partial'_2}{\to} N \stackrel{\pi_2}{\to} Q \to 1 \tag{14}$$

such that $M \xrightarrow{\pi_1} G$ is an $(L \to M)$-bundle and $N \xrightarrow{\pi_1} Q$ is an $(G \to N)$-bundle (cf. Example 2.7). Also, we have an exact sequence of pre-crossed modules

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \xrightarrow{\partial_1} & M & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \partial_2\downarrow & & \partial_2'\downarrow & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

where G is a normal subgroup of N and also a morphism of 2-crossed modules

$$\begin{array}{ccccc} L & \xrightarrow{\partial_1} & M & \xrightarrow{\partial_2} & N \\ \downarrow & & \pi_1\downarrow & & \mathrm{id}\downarrow \\ 1 & \longrightarrow & G & \xrightarrow{\partial_2'} & N \end{array}$$

Considering the above extension of Lie groups, we have proved the following proposition.

Proposition 3.2. *Consider an exact sequence (12) of Lie groups such that the the exact sequences (13) and (14) define an $(L \to M)$-bundle and an $(G \to N)$-bundle, respectively.*[3] *Then the stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are in bijective correspondence with the isomorphism classes of $(G \to N)$-bundles.*

Proof. Let us first note that given a $(G \to N)$-bundle $\mathcal{P} = (P, \tilde{\boldsymbol{n}})$ on X, the left principal G-bundle P can be lifted to an $(L \to M)$-bundle gerbe $\mathcal{G}$ (2.27), which will be actually an $(L \to M \to N)$-bundle gerbe. This is because of the identity $\partial_2'\pi_1 = \partial_2$ the trivialization $\tilde{\boldsymbol{n}}$ of $\mathcal{P}$ under ∂_2' naturally defines a trivialization of the principal N-bundle $\mu(\mathcal{G})$ by putting $\mathbf{n} := \tilde{\boldsymbol{n}}$ (cf. (3.2). Due to the identification $\mathbf{n} := \tilde{\boldsymbol{n}}$, two isomorphic $(G \to N)$-bundles will lead to two stably isomorphic $(L \to M \to N)$-bundle gerbes. On the other hand, starting with an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$ with the 2-crossed module as above, we can change its structure 2-crossed module to $1 \to G \to N$ in order to obtain a principal $(G \to N)$-bundle $\mathcal{P}$. The N-valued function $\mathbf{n}$ on Y defined by the trivialization $\boldsymbol{n}$ of $\mu(\mathcal{G})$ will give a trivialization of $\partial_2'(\mathcal{P})$, cf. (3.7). From this it is again easy to see that stably isomorphic $(L \to M \to N)$-bundle gerbes will lead to isomorphic $(G \to N)$-bundles. Lifting an principal $(G \to N)$-bundle to an $(L \to M \to N)$-bundle gerbe followed by the change of structure 2-crossed modules $(L \to M \to N) \to (G \to N)$ will give back the original $(G \to N)$-bundle.

The local description of the above correspondence is similar to the case of crossed module bundle gerbes (2.28).

Because of (2.7) we also have the following corollary.

Corollary 3.1. *Under the hypothesis of proposition 3.2, stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in bijective correspondence with Q-valued functions.*

Concerning the corresponding cocycles we have the following isomorphisms of sets.

Corollary 3.2. *Under the hypothesis of proposition 3.2, we have*

$$H^0(X, L \to M \to N) \cong H^0(X, G \to N) \cong H^0(X, Q)$$

3.13. **Remark.** Note that given three $(G \to N)$-bundles $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$ and isomorphisms $\mathcal{P} \xrightarrow{f} \mathcal{P}'$, $\mathcal{P}' \xrightarrow{f'} \mathcal{P}''$ and $\mathcal{P} \xrightarrow{f''} \mathcal{P}''$ such that $f'f = f''$ the corresponding lifting 2-crossed module bundle gerbes will be stably isomorphic, but the respective stable isomorphisms $\boldsymbol{f}\boldsymbol{f}'$ and $\boldsymbol{f}''$ will be only isomorphic in general.

[3] cf. Example 2.4

3.14. **Remark.** Similarly to lifting crossed module bundles (cf. (2.11)), also lifting 2-crossed module bundle gerbes can be interpreted as pullbacks. Starting from exact sequences (13) and (14) we have a lifting $(L \to M)$-bundle gerbe over Q. due to equality $\partial_2' = \partial_2\pi_1$ this will actually be and $(L \to M \to N)$-bundle gerbe $\mathfrak{Q}$ over Q. A representative of class of a lifting 2-crossed module bundle gerbe over X corresponding to a Q-valued function $q : X \to Q$ can be obtained as the pullback $q^*(\mathfrak{Q})$.

3.15. **Remark.** For an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ we recall from (3.2) that the trivializing section $\boldsymbol{n}$ of the left principal N-bundle $\mu(\mathcal{G})$ defines an N-valued function $\mathbf{n}$ on Y such that $\partial_2(\boldsymbol{m}) = \mathbf{n}_1\mathbf{n}_2^{-1}$. Let us recall that on the left principal L-bundle P there is a compatible principal right L-action. Using the N-valued function $\mathbf{n}$ we can introduce a further principal right L-action on P, which will again commute with the principal left L-action. We will use the notation $(p, l) \mapsto p._{\mathbf{n}}l$ for $p \in P$, $l \in L$ for this principal right action of L and put $p._{\mathbf{n}}l := p^{\,\mathbf{n}_2(y_1,y_2)}l$, where p lies in the fibre over $(y_1, y_2) \in Y^{[2]}$ and $\mathbf{n}_2$ is the pullback to $Y^{[2]}$ of $\mathbf{n}$ under the projection to the second factor of $Y^{[2]}$. It is easy to check that this formula indeed defines a principal right L-action commuting with the principal left L-action on P.

Let now $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = ((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$ be two 2-crossed module bundle gerbes. Let us again consider on $Y^{[2]}$ the Whitney sum $P \oplus \tilde{P}$ and introduce an equivalence relation on $P \oplus \tilde{P}$ by

$$(p._{\mathbf{n}}l, \tilde{p}) \sim_{\mathbf{n}} (p, l\tilde{p})$$

and define $\bar{P} = P._{\mathbf{n}}\tilde{P} = P \oplus \tilde{P}/\sim_{\mathbf{n}}$. We will denote an element of $P._{\mathbf{n}}\tilde{P}$ defined by the equivalence class of $(p, \tilde{p}) \in P \oplus \tilde{P}$ as $[p, \tilde{p}]_{\mathbf{n}}$ in order to distinguish it from equivalence class $[p, \tilde{p}] \in P\tilde{P}$ defined previously in (2.1). Also, put

$$\bar{\boldsymbol{m}} = \boldsymbol{m}^{\,\mathbf{n}_2}\tilde{\boldsymbol{m}}$$

It is easy to see that $\bar{\mathcal{P}} := (\bar{P}, \bar{\boldsymbol{m}})$ is an $(L \to M)$-bundle on $Y^{[2]}$. Let us note that also $\partial_2(\bar{\boldsymbol{m}}) = \bar{\mathbf{n}}_1\bar{\mathbf{n}}_2$ on $Y^{[2]}$ with

$$\bar{\boldsymbol{n}} = \boldsymbol{n}\tilde{\boldsymbol{n}}$$

Now on $Y^{[3]}$ we do have the pullbacks $\mathcal{P}_{12}$, $\tilde{\mathcal{P}}_{12}$, $\bar{\mathcal{P}}_{12}$, etc. An element of $\bar{P}_{12}\bar{P}_{23}$ is then given by $((y_1, y_2, y_3), [[p, \tilde{p}]_{\mathbf{n}}, [p', \tilde{p}']_{\mathbf{n}}])$ with $(y_1, y_2, y_3) \in Y^{[3]}$, $p \in P$ and $\tilde{p} \in \tilde{P}$ in the respective fibres of P and $\tilde{P}$ over $(y_1, y_2) \in Y^{[2]}$, and $p' \in P$ and $\tilde{p}' \in \tilde{P}$ are in the respective fibres of P and $\tilde{P}$ over $(y_2, y_3) \in Y^{[2]}$. Finally, we define $\bar{\boldsymbol{\ell}} : \bar{P}_{12}\bar{P}_{23} \to \bar{P}_{13}$ as

$$\bar{\boldsymbol{\ell}}((y_1, y_2, y_3), [[p, \tilde{p}]_{\mathbf{n}}, [p', \tilde{p}']_{\mathbf{n}}]) := ((y_1, y_2, y_3), [\boldsymbol{\ell}([p, p'], \tilde{\boldsymbol{\ell}}[\tilde{p}, \tilde{p}']]_{\mathbf{n}})$$

Now it is a rather lengthy but straightforward check to establish the following proposition.

Proposition 3.3. $(\bar{\mathcal{G}}, \bar{\boldsymbol{n}}) := ((\bar{P}, \bar{\mathbf{m}}), Y, X, \bar{\boldsymbol{\ell}}), \bar{\boldsymbol{n}})$ *defines an* $(L \to M \to N)$*-bundle gerbe, the product of* $(L \to M \to N)$*-bundle gerbes* $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ *and* $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = ((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$.

3.16. **Example.** If $(\mathcal{G}, \boldsymbol{n}) = (((P = Y^{[2]} \times L, \partial_1\mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N.n\mathrm{pr}_X)$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = (((\tilde{P}Y^{[2]} \times L, \partial_1\mathrm{pr}_L), Y, X,), \mathrm{pr}_N.\tilde{n}\mathrm{pr}_X)$ are two $(L \to M \to N)$-bundle gerbes defined by two respective N-valued functions n and $\tilde{n}$ on X (3.6) then their product is explicitly described again as an $(L \to M \to N)$-bundle gerbe defined by the function $n\tilde{n}$ by identifying $[(y_1, y_2, l), (y_1, y_2, \tilde{l})] \in PP'$ with $(y_1, y_2, l^{\,n(x)}\tilde{l}) \in Y^{[2]} \times L)$. Here $(y_1, y_2) \in Y^{[2]}$ live in the fibre over $x \in X$.

3.17. **Remark.** The above product defines a group structure on stable isomorphism classes of $(L \to M \to N)$-bundle gerbes. The unit is given by the class of the trivial $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1\mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N)$. We will give an explicit (local) formula for the inverse later. Let us note that the relation between the stable isomorphism classes of lifting $(L \to M \to N)$-bundle gerbes described above (3.12) and Q-valued functions (and stable isomorphism classes of $(G \to N)$-bundle gerbes) is compatible with the respective multiplications.

3.18. **Product on 2-cocycles.** The product formulas for the corresponding transition functions (2-cocycles) of the product $\bar{\mathcal{G}} = \mathcal{G}\tilde{\mathcal{G}}$ of two 2-crossed module bundles are given by

$$\bar{n}_i = n_i \tilde{n}_i$$
$$\bar{m}_{ij} = m_{ij}\, {}^{n_j}\tilde{m}_{ij}$$
$$\bar{l}_{ijk} = l_{ijk}\, {}^{m_{ik}}\{m_{jk}{}^{-1}, {}^{n_j}\tilde{m}_{ij}\}\, {}^{n_i}\tilde{l}_{ijk}$$

The inverse $(n_i, m_{ij}, l_{ijk})^{-1}$ is given by

$$(n_i^{-1}, {}^{n_j^{-1}}m_{ij}^{-1}, {}^{n_k^{-1}}\{m_{jk}^{-1}, m_{ij}^{-1}\}^{-1}\, {}^{n_i^{-1}}l_{ijk}^{-1})$$

3.19. **Remark.** Let us forget, for the moment, about the "horizontal" composition in the Gray 3-groupoid corresponding to the 2-crossed module $L \to M \to N$. The two "vertical" compositions define a strict 2-groupoid (a strict topological 2-category), which we will denote as $2\mathcal{C}$. Let us again consider the topological $\mathcal{O}$ category defined by the good covering $\{O_i\}$. A 2-cocycle is the same thing as a continuous, normal pseudo-functor from $\mathcal{O}$ to $2\mathcal{C}$. Now we can use the fact that the horizontal composition in a topological Gray 3-category defines a continuous cubical functor $\mathfrak{F} : 2\mathcal{C} \times 2\mathcal{C} \to 2\mathcal{C}$ from the cartesian product $2\mathcal{C} \times 2\mathcal{C}$ to $2\mathcal{C}$ [22]. We may use the following property of cubical functors, which follows almost immediately from definition. If $\mathcal{F}$ and $\mathcal{G}$ are two continuous normal pseudo-functors from $\mathcal{O}$ to $2\mathcal{C}$ then $\mathfrak{F}(\mathcal{F}, \mathcal{G})$ is a pseudo-functor from $\mathcal{O}$ to $2\mathcal{C}$. Hence, we obtain a product on 2-cocycles, which is the same as the one given above (3.3).

4. 2-crossed module bundle 2-gerbes

Consider again a surjective submersion $\pi : Y \to X$. Let, as before, $p_{ij} : Y^{[3]} \to Y^{[2]}$ denote the projection to the i-th and j-th component, and similarly for projections of higher fibred powers $Y^{[n]}$ of Y. Let $L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N$ be a 2-crossed module.

4.1. **Definition.** A 2-crossed module bundle 2-gerbe is defined by a quintuple $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$, where $\mathfrak{G} = (\mathcal{G}, \boldsymbol{n})$ is a 2-crossed module bundle gerbe over $Y^{[2]}$,

$$\boldsymbol{m} : \mathfrak{G}_{12}\mathfrak{G}_{23} \to \mathfrak{G}_{13}$$

is a stable isomorphism on $Y^{[3]}$ of the product $\mathfrak{G}_{12}\mathfrak{G}_{23}$ of the pullback 2-crossed module bundle gerbes $\mathfrak{G}_{12} = p_{12}^*\mathfrak{G}$ and $\mathfrak{G}_{23} = p_{23}^*\mathfrak{G}$ and the pullback 2-crossed module bundle gerbe $\mathfrak{G}_{13} = p_{13}^*\mathfrak{G}$, and

$$\boldsymbol{\ell} : \boldsymbol{m}_{124}\boldsymbol{m}_{234} \to \boldsymbol{m}_{134}\boldsymbol{m}_{123}$$

is an isomorphism of the composition of pullbacks of stable isomorphisms $p_{124}^*\boldsymbol{m}$ and $p_{234}^*\boldsymbol{m}$ and the composition of pullbacks of stable isomorphisms $p_{123}^*\boldsymbol{m}$ and $p_{134}^*\boldsymbol{m}$ on $Y^{[4]}$. On $Y^{[5]}$, the isomorphism $\boldsymbol{\ell}$ should satisfy the obvious coherence relation

$$\boldsymbol{\ell}_{1345}\boldsymbol{\ell}_{1235} = \boldsymbol{\ell}_{1234}\boldsymbol{\ell}_{1245}\boldsymbol{\ell}_{2345}.$$

4.2. **Abelian bundle 2-gerbes.** Abelian bundle 2-gerbes as introduced in [49], [50], [19] are $(U(1) \to 1 \to 1)$-bundle 2-gerbes. If $A \to 1 \to 1$ is a 2-crossed module then A is necessarily an abelian group and an abelian bundle 2-gerbe can be identified as an $(A \to 1 \to 1)$-bundle 2-gerbe.

4.3. **Example.** Consider an $(1 \to G \to N)$-bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$. The $(1 \to G \to N)$-bundle gerbe on $Y^{[2]}$ gives a $(G \to N)$-bundle $\mathcal{P}$ on $Y^{[2]}$. The stable isomorphism $\boldsymbol{m} : \mathfrak{G}_{12}\mathfrak{G}_{23} \to \mathfrak{G}_{13}$ gives on $Y^{[3]}$ an isomorphism $\boldsymbol{g} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ satisfying on $Y^{[4]}$ the associativity condition $\boldsymbol{g}_{124}\boldsymbol{g}_{234} = \boldsymbol{g}_{134}\boldsymbol{g}_{123}$ since the first Lie group of the 2-crossed module $(1 \to G \to N)$ is trivial. Hence, a $(1 \to G \to N)$-bundle 2-gerbe is the same thing as a $(G \to N)$-bundle gerbe. Obviously, stably isomorphic $(1 \to G \to N)$-bundle 2-gerbes correspond to stably isomorphic $(G \to N)$-bundle gerbes.

4.4. **Pullback.** If $f : X \to X'$ is a map then we can pullback $Y \to X$ to $f^*(Y) \to X'$ with a map $\tilde{f} : f^*(Y) \to Y$ covering f. There are induced maps $\tilde{f}^{[n]} : f^*(Y)^{[n]} \to Y^{[n]}$. The pullback $f^*(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell}) := (\tilde{f}^{[2]*}\mathfrak{G}, f^*(Y), f(X), \tilde{f}^{[3]*}\boldsymbol{m}, \tilde{f}^{[4]*}\boldsymbol{\ell})$ is again an $(L \to M \to N)$-bundle 2-gerbe.

4.5. **Definition.** Two 2-crossed module bundle 2-gerbes $((\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ and $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}')$ are stably isomorphic if there exists a 2-crossed module bundle gerbe $\mathfrak{Q} \to \bar{Y} = Y \times_X Y'$ such that over $\bar{Y}^{[2]}$ the 2-crossed module bundle gerbes $q^*\mathfrak{G}$ and $\mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$ are stably isomorphic. Let $\tilde{\boldsymbol{m}}$ be the stable isomorphism $\tilde{\boldsymbol{m}} : q^*\mathfrak{G} \to \mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$. Then we ask on $Y^{[3]}$ (with an obvious abuse of notation) for the existence of an isomorphism $\tilde{\boldsymbol{\ell}}$ of stable isomorphisms

$$\tilde{\boldsymbol{\ell}} : \boldsymbol{m}'\tilde{\boldsymbol{m}}_{23}\tilde{\boldsymbol{m}}_{12} \to \tilde{\boldsymbol{m}}_{13}\boldsymbol{m}$$

fulfilling on $Y^{[4]}$

$$\boldsymbol{\ell}_{1234}\tilde{\boldsymbol{\ell}}_{124}\tilde{\boldsymbol{\ell}}_{234} = \tilde{\boldsymbol{\ell}}_{134}\tilde{\boldsymbol{\ell}}_{123}\boldsymbol{\ell}'_{1234}$$

Here q and q' are projections onto first and second factor of $\bar{Y} = Y \times_X Y'$ and $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are the pullbacks of $\mathfrak{Q}$ to $\bar{Y}^{[2]}$ under respective projections p_1, p_2 form $\bar{Y}^{[2]}$ to $\bar{Y}$, etc.

A 2-crossed module bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ is called trivial if it is stably isomorphic to the trivial 2-crossed module bundle 2-gerbe $(\mathcal{T}, Y, X, 1, 1)$, where $\mathcal{T}$ is the trivial 2-crossed module bundle gerbe $(((Z^{[2]} \times L, \partial_1 \mathrm{pr}_L), Z, Y^{[2]}, 1), \mathrm{pr}_N))$. Pullbacks preserve stable isomorphisms, a pullback of a trivial 2- crossed module bundle 2-gerbe is again a trivial 2-crossed module bundle 2-gerbe.

If $Y = X$ then the 2-crossed module bundle 2-gerbe is trivial.

4.6. **Definition.** Let $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ and $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}')$ be two 2-crossed module bundle 2-gerbes and $(\mathfrak{Q}, \tilde{\boldsymbol{m}}_{\mathfrak{Q}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{Q}})$ and $(\mathfrak{R}, \tilde{\boldsymbol{m}}_{\mathfrak{R}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{R}})$ two stable isomorphisms between them. We call these two stable isomorphisms stably isomorphic if there is a stable isomorphism $\underline{\boldsymbol{m}} : \mathfrak{Q} \to \mathfrak{R}$ of 2-crossed module bundles on $\bar{Y} = Y \times_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{Q}}} & \mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle \underline{\boldsymbol{m}}_1 \underline{\boldsymbol{m}}_2{}^{-1}} \\ q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{R}}} & \mathfrak{R}_1 q'^*\mathfrak{G}'\mathfrak{R}_2^{-1} \end{array}$$

commutes up to an isomorphism of stable isomorphisms

$$\underline{\boldsymbol{\ell}} : \tilde{\boldsymbol{m}}_{\mathfrak{Q}}\underline{\boldsymbol{m}}_1\underline{\boldsymbol{m}}_2{}^{-1} \to \tilde{\boldsymbol{m}}_{\mathfrak{R}}$$

on $\bar{Y}^{[2]}$, fulfilling on $\bar{Y}^{[3]}$

$$\tilde{\boldsymbol{\ell}}_{\mathfrak{Q}}\underline{\boldsymbol{\ell}}_{13} = \underline{\boldsymbol{\ell}}_{12}\underline{\boldsymbol{\ell}}_{23}\tilde{\boldsymbol{\ell}}_{\mathfrak{R}}$$

4.7. **Remark.** Let $\pi' : Y' \to X$ be another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the 2-crossed module bundle 2-gerbes $(f^*\mathfrak{G}, Y', X, f^{[3]*}\boldsymbol{m}, f^{[4]*}\boldsymbol{\ell})$ and $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ are stably isomorphic. It follows that locally each 2-crossed module bundle 2-gerbe is trivial. The arguments to show the above two statements are completely analogous to the case of a crossed module bundle gerbe (3.8).

4.8. **Change of the structure 2-crossed module.** If $(L \to M \to N) \to (L' \to M' \to N')$ is a morphism of crossed modules, there is an obvious way to construct, starting from an $(L \to M \to N)$-bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$, an $(L' \to M' \to N')$-bundle 2-gerbe $(\mathfrak{G}', Y, X, \boldsymbol{m}', \boldsymbol{\ell}')$ by changing the structure 2-crossed module of $\mathfrak{G}$ from $L \to M \to N$ to $L' \to M' \to N'$.

4.9. **3-cocycles.** Let $\pi : Y \to X$ be the surjective submersion, which was implicitly contained in the above definition of a 2-crossed module bundle 2-gerbe. Let us recall (4.7) that also for 2-crossed module bundle 2-gerbes it holds true that 2-crossed module bundle 2-gerbes $(f^*\mathfrak{G}, Y', X, f^{[3]*}\boldsymbol{m}, f^{[4]*}\boldsymbol{\ell})$ and $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ are stably isomorphic if the respective maps π and π' are related by a compatible map f. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one, in which case the $(L \to M \to N)$-bundle gerbe can be described by transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$ $n_{ij} : O_{ij} \to N$, $m_{ijk} : O_{ijk} \to M$ and $l_{ijkl} : O_{ijkl} \to L$ satisfying

$$\begin{aligned} n_{ij}n_{jk} &= \partial_2(m_{ijk})n_{ik} \\ m_{ijk}m_{ikl} &= \partial_1(l_{ijkl})\, {}^{n_{ij}}m_{jkl}m_{ijl} \\ l_{ijkl}\ {}^{n_{ij}m_{jkl}}(l_{ijlm})\, {}^{n_{ij}}l_{jklm} &= {}^{m_{ijk}}l_{iklm}\{m_{ijk},\, {}^{n_{ik}}m_{klm}\}\ {}^{n_{ij}n_{jk}m_{klm}}(l_{ijkm}) \end{aligned} \tag{15}$$

We shall not give explicit formulas relating transition functions (3-cocycles) of two stably isomorphic 2-crossed module bundle 2-gerbes. We introduce the notation $H^1(X, L \to M \to N)$ for the equivalence classes of 3-cocycles. We just give the formulas for transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$ of a trivial 2-crossed module bundle 2-gerbe:

$$\begin{aligned} n_{ik} &= n_i^{-1}\partial_2(m_{ij})n_j \\ {}^{n_i}m_{ijl} &= \partial_1(l_{ijk}^{-1})\, m_{ij}m_{jk}m_{ik}^{-1} \\ {}^{n_i}l_{ijkl} &= {}^{n_i m_{ijk}}(l_{ikl}^{-1})l_{ijk}^{-1}\, {}^{m_{ij}}l_{jkl}\{m_{ij},\, {}^{n_j}m_{jkl}\}\ {}^{n_i n_{ij} m_{jkl}}(l_{ijl}) \end{aligned} \tag{16}$$

We introduce the notation $H^1(X, L \to M \to N)$ for the corresponding equivalence classes of 3-cocycles.

Now we briefly describe how an $(L \to M \to N)$-bundle 2-gerbe can be reconstructed from transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$. This is analogous to the case of an $(L \to M)$-bundle gerbe (2.25). Put $Y = \coprod O_i$. On each nonempty O_{ij} consider the $(L \to M \to N)$-bundle gerbe $\mathfrak{G}_{ij}$ defined by the function $n_{ij} : O_{ij} \to N$ as in (3.6). Hence, on $Y^{[2]}$ we have the $(L \to M \to N)$-bundle gerbe given by $\mathfrak{G} = \coprod_{ij} \mathfrak{G}_{ij}$. Now, we recall the explicit descriptions of the multiplication (3.16) and stable isomorphisms (3.6) of two $(L \to M \to M)$-bundle gerbes defined by their respective N-valued functions. Also, recall the description of isomorphisms of stable isomorphism in case of such $(L \to M \to M)$-bundle gerbes. Using the 3-cocycle relations, it is now straightforward to show that the collection of functions m_{ijk} defines a stable isomorphism of $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ on $Y^{[3]}$ satisfying on $Y^{[4]}$ the associativity condition up to the an isomorphism defined by the collection of functions l_{ijkl}, which fulfils the coherence relation on $Y^{[5]}$. It is now clear that, in a complete analogy to the case of a crossed module bundle gerbe (2.25), starting from two equivalent 3-cocycles, we obtain stably isomorphic 2-crossed module bundle 2-gerbes. This is however a tedious check and we shall omit it.

Hence, we can summarize the discussion in the following proposition.

Proposition 4.1. *Stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in a bijective correspondence with the set $H^1(X, L \to M \to N)$.*

It might be interesting to examine possible 3-categorical aspects of the above constructions.

4.10. **Lifting 2-crossed module bundle 2-gerbe.** As before (cf. (3.12)), consider a Lie 2-crossed module $L \to M \to N$ such that $\ker(\partial_1) = 1$ and $\ker(\partial_2) = \mathrm{Im}(\partial_1)$. Put $G := M/L$ and $Q := N/G$. Assume that the conditions are satisfied for having extensions of Lie groups

$$1 \to L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N \overset{\pi_2}{\to} Q \to 1 \tag{17}$$

$$1 \to L \overset{\partial_1}{\to} M \overset{\pi_1}{\to} G \to 1 \tag{18}$$

and

$$1 \to G \overset{\partial_2'}{\to} N \overset{\pi_2}{\to} Q \to 1 \tag{19}$$

such that $M \stackrel{\pi_1}{\to} G$ is an $(L \to M)$-bundle and $N \stackrel{\pi_1}{\to} Q$ is an $(G \to N)$-bundle (cf. Example 2.7). Also, we have an exact sequence of pre-crossed modules

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \xrightarrow{\partial_1} & M & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_2'} & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

where G is a normal subgroup of N and also a morphism of 2-crossed modules

$$\begin{array}{ccccc} L & \xrightarrow{\partial_1} & M & \xrightarrow{\partial_2} & N \\ \downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \mathrm{id}} \\ 1 & \longrightarrow & G & \xrightarrow{\partial_2'} & N \end{array}$$

Proposition 4.2. *Consider an exact sequence (17) of Lie groups such that the the exact sequences (18) and (19) define an $(L \to M)$-bundle and an $(G \to N)$-bundle, respectively.*[4] *Then the stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in bijective correspondence with the isomorphism classes of Q-bundles and hence also with $(G \to N)$-bundle gerbes.*

Proof. Recall that in accordance with (3.14) from the 3-term exact sequence $1 \to L \stackrel{\partial_1}{\to} M \stackrel{\pi_1}{\to} G \to 1$ and the right principal $(G \to N)$-bundle $N \to Q$ (given by the 3-term exact sequence $1 \to G \stackrel{\partial_2'}{\to} N \stackrel{\pi_2}{\to} Q \to 1$) we can construct a lifting $(L \to M)$-bundle gerbe on Q. This lifting bundle gerbe will actually be an $(L \to M \to N)$-bundle gerbe $\mathfrak{G}$ (3.12). If now P is a left principal Q-bundle over X then we can use the corresponding division map $d : P^{[2]} \to G$ to pullback the 2-crossed module gerbe $\mathfrak{G}$ from G to $P^{[2]}$. It follows that the 2-crossed module bundle gerbes $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ are stably isomorphic on $P^{[3]}$. This follows from the above mentioned fact that, in case of Lie groups L, M N, and Q as above, stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are one to one to Q-valued functions (3.1) and that this correspondence respects the respective multiplications. Such an stable isomorphism in general fulfills on $P^{[4]}$ the associativity condition only up to an isomorphism, which however, due to $\ker(\partial_1) = 1$, will fulfil the requested coherence condition on $P^{[5]}$. Hence, we have obtained 2-crossed module bundle 2-gerbe, the so called lifting 2-crossed module bundle 2-gerbe. Starting from an isomorphic principal Q-bundle P' we obtain a stably isomorphic $(L \to M \to N)$-bundle 2-gerbes. This follows from the fact that on $\bar{P}^{[2]}$, where $\bar{P} := P \times_X P'$ the pullbacks of respective division functions d and d' are related by $d(\bar{p}_1, \bar{p}_2) = \bar{d}(\bar{p}_1) d'(\bar{p}_1, \bar{p}_2) \bar{d}(\bar{p}_2)^{-1}$ with some Q-valued function $\bar{d}$ on $\bar{P}$. To finish the argumentation, we refer again to the 1-1 relation between stable isomorphism classes of $(L \to M \to N)$-bundle gerbes to Q-valued functions (3.1) and the fact that this respects the respective multiplication.

Going in the opposite direction, let us consider an $(L \to M \to N)$-bundle 2-gerbe $(\mathfrak{G}, X, Y, \boldsymbol{m}, \boldsymbol{\ell})$ with the 2-crossed module $(L \to M \to N)$ as above. Changing the structure 2-crossed module to $1 \to G \to N$, we obtain a $G \to N$-bundle gerbe $(\mathcal{G}, n)$ on X. After changing its structure crossed module to $1 \to Q$ we obtain a left principal Q-bundle on X. Since all steps in the construction preserve the respective stable isomorphisms and isomorphisms, starting from stably isomorphic 2-crossed module bundle 2-gerbes we will obtain isomorphic Q-bundles.

It is a rather tedious task to check that staring from a principal Q-bundle, constructing the lifting 2-crossed module bundle 2-gerbe and going back will result in the same principal Q-bundle.

Corollary 4.1. *Under the hypothesis of the above proposition,*

$$H^1(X, L \to M \to N) \cong H^1(X, G \to N) \cong H^1(X, Q)$$

[4]cf. Example 2.4

4.11. **Remark.** We can also reinterpret the above described lifting 2-crossed module bundle 2-gerbe as follows. We start again with a principal Q-bundle P as above. Let us consider the corresponding lifting $(G \to N)$-bundle gerbe $\mathfrak{P}$. This in particular means that on $P^{[2]}$ we have a $(G \to N)$-bundle $\mathcal{P}$ which can be lifted to an $(L \to M \to N)$-bundle gerbe $\mathfrak{G}$ on $P^{[2]}$ (3.12). It follows that the 2-crossed module bundle gerbes $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ are stably isomorphic with a stable isomorphism $\boldsymbol{m}$. This follows, again, from the above mentioned fact that in case of Lie groups L, M, N and Q as above stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are one to one with Q-valued functions and that this correspondence respects the respective multiplications. Again, such a stable isomorphism $\boldsymbol{m}$ fulfills the associativity condition on $P^{[4]}$ only up to an isomorphism fulfilling the coherence relation on $P^{[5]}$ because of $\ker \partial_1 = 1$. This way we obtain an 2-crossed module bundle 2-gerbe stably isomorphic to the lifting 2-crossed module bundle 2-gerbe (4.10).

4.12. **Twisting crossed module bundle gerbes with abelian bundle 2-gerbes.** Twisted crossed module bundle gerbes as discussed here were introduced in [2]. A more general concept of twisting has been introduced recently in [47].

Let us consider a 2-crossed module $A \to L \xrightarrow{\delta} M$ associated to the crossed module $L \to M$ (1.5). Recall that in this case A is necessarily abelian. Putting $Q := \operatorname{coker}\delta$ we have an exact sequence

$$0 \to A \xrightarrow{\partial} L \xrightarrow{\delta} M \to Q \to 1$$

Assume, similarly to (4.10), that extensions of Lie groups

$$1 \to A \xrightarrow{\partial} L \xrightarrow{\pi_1} G \to 1 \tag{20}$$

and

$$1 \to G \xrightarrow{\delta'} M \xrightarrow{\pi_2} Q \to 1 \tag{21}$$

define an $(A \to L)$-bundle and $(G \to M)$-bundle, respectively (cf. (2.7)). However, now we have an exact sequence of crossed modules

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\partial} & L & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \delta\downarrow & & \delta'\downarrow & & \downarrow \\
1 & \longrightarrow & 1 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 1
\end{array}$$

As before, we have a morphism of 2-crossed modules

$$\begin{array}{ccccc}
A & \xrightarrow{\partial} & L & \xrightarrow{\delta} & M \\
\downarrow & & \pi_1\downarrow & & \mathrm{id}\downarrow \\
1 & \longrightarrow & G & \xrightarrow{\delta'} & M
\end{array}$$

Hence, starting from a principal Q-bundle P we can construct a lifting $(A \to L \xrightarrow{\delta} M)$-bundle 2-gerbe (recall that according to (4.2) there is a 1-1 correspondence between stable isomorphism classes of $(A \to L \xrightarrow{\delta} M)$-bundle 2-gerbes and isomorphism classes of principal Q-bundles). Let us further assume that what we have here is a central extension of L by A, and that M acts trivially on A. Let us assume that the lifting bundle 2-gerbe G) is described locally, with respect to a good covering, by a 3-cocycle $(m_{ij}, l_{ijk}, a_{ijkl})$

$$\begin{gathered}
m_{ij}m_{jk} = \delta(l_{ijk})m_{ik} \\
l_{ijk}l_{ikl} = \partial(a_{ijkl})^{m_{ij}}l_{jkl}l_{ijl} \\
a_{ijkl}a_{ijlm}a_{jklm} = a_{iklm}a_{ijkm}.
\end{gathered} \tag{22}$$

The collection of A-valued functions a_{ijkl} on the quadruple intersections represents a Čech class in $H^3(X,A)$ (which in the case $A = U(1)$ would correspond to a class in $H^4(X,\mathbb{Z})$). We may think of it as representing an abelian bundle 2-gerbe A. If we assume that A is trivial, we have

$$a_{ijkl} = \tilde{a}_{ijk}\tilde{a}_{ikl}\tilde{a}^{-1}_{jkl}\tilde{a}^{-1}_{ijl}.$$

Also, we see that we have a 2-cocycle $(l_{ijk}\partial(a_{ijk})^{-1}, m_{ij})$ representing a possibly non-trivial $(L \to M)$-bundle gerbe $\mathcal{G}$. Obviously, the $(A \to 1 \to 1)$-bundle 2-gerbes represented by non-trivial classes in $H^3(X,A)$ represent obstructions to lift a $(G \to M)$-bundle gerbe (and hence also a Q-bundle) to an $(L \to M)$-bundle gerbe. Further, if $\tilde{a}_{ijk}$ and $\tilde{a}'_{ijk}$ represent two trivializations of a_{ijkl} then $\tilde{a}_{ijk}(\tilde{a}'_{ijk})^{-1}$ represents a Čech class in $H^2(X,A)$. We may think of it as representing an abelian bundle gerbe, i.e, the $(A \to 1)$-bundle gerbe, $\mathcal{A}$. We can summarize the above discussion in the following proposition.

Proposition 4.3. *Let $A \to L \xrightarrow{\delta} M$ be a 2-crossed module originating from the crossed module $L \xrightarrow{\delta} M$*[5] *such that the extensions of Lie groups (20) and (21) define an $(A \to L)$-bundle and $(G \to M)$-bundle, respectively.*[6] *Let us also assume that (20) is a central extension of L by A and that M acts trivially on A*

i) A principal Q-bundle on X can be lifted to an $(L \to M)$-bundle gerbe if and only if the corresponding obstruction $(A \to 1 \to 1)$-bundle 2-gerbe A is trivial.

ii) If non-empty, the set of stable isomorphism classes of those $(L \to M)$-bundle gerbes, which are liftings of Q-principal bundles from the same isomorphism class, is freely and transitively acted on by the group of stable isomorphism classes of $(A \to 1)$-bundle gerbes.

Corollary 4.2. *Under the assumptions of Proposition 4.3, there is an exact sequence*

$$H^1(X, A \to 1 \to 1) \to H^1(X, L \to M) \to H^1(X,Q)$$

The above proposition and corollary remain true also in case when the principal Q-bundles and their isomorphisms classes are replaced by $(G \to M)$-bundle gerbes and their stable isomorphism classes (cf. 2.27).

4.13. **Remark.** Of course, the above lifting always exists when the 4-term exact sequence $1 \to A \to L \to M \to Q \to 1$ corresponds to a trivial class in $H^3(Q,A)$ [36],[11], the third Q-cohomology with values in A. The above lifting also trivially exists when X doesn't admit nontrivial $(A \to 1 \to 1)$-bundle 2-gerbe, i.e., when $[X, B^2A]$ is trivial.

4.14. **A remark on string structures.** Let Q be a simply-connected compact simple Lie group. Associated to Q there is a crossed module $L \to M$ of infinite dimensional Fréchet Lie groups with $L := \widehat{\Omega Q}$ and $M := P_0Q$, where $\widehat{\Omega Q}$ is the centrally extended group of based smooth loops in Q and P_0Q is the group of smooth paths in Q that start at the identity [4]. Hence in the notation of (4.12) we have $A = U(1)$, and $G = \Omega Q$. Let us note (see [51], [4], [28]) that, in the situation as above (4.12), the classifying space $BU(1) = K(\mathbb{Z},2)$ can be equipped with a proper group structure and a topological group $String(Q)$ can be defined fitting an exact sequence of groups $1 \to K(\mathbb{Z},2) \to String(Q) \to Q \to 1$. Also, it is known [30], [5] that the categories of $(L \to M)$-bundle gerbes and principal $String(Q)$-bundles are equivalent. A string structure is, by definition, a lift of a principal Q-bundle to a principal $String(Q)$-bundle and hence equivalently a lift of a $(G \to M)$-bundle gerbe to an $(L \to M)$-bundle gerbe. Thus, the above discussion applies to the existence of string structures and their classification as well.

[5] cf. Example 1.5

[6] cf. Example 2.4

4.15. **Remark.** A crossed square $(L \to A) \to (B \to N)$ [35] of Lie groups gives a 2-crossed module, namely $L \to A \rtimes B \to N$ (see, e.g., [42]). A definition of a crossed square bundle 2-gerbe could possibly be read of from [8], [10] [9]. It would be interesting to compare these bundle 2-gerbes with $L \to A \rtimes B \to N$-bundle 2-gerbes defined in this paper.

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