

# Pseudo toric lagrangian fibrations of toric and non toric Fano varieties

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## Introduction

In this paper we discuss lagrangian geometry of certain symplectic manifolds. More precise, we study lagrangian fibrations of certain symplectic manifolds. Thus mostly we are doing with symplectic toric manifolds which are the phase spaces of the corresponding completely integrable systems (called effective in [2]) and certain submanifolds of these toric manifolds. But the toric manifolds are not central in the story, providing just some appropriate framework where certain examples can be presented and certain results can be derived. The situation we are studying below is the following: let a symplectic manifold  $(X, \omega)$  of real dimension  $2n$  admits two intrinsically different pieces of data:

(**r**) (*real part*) a set of smooth functions  $(f_1, \dots, f_k)$  in involution,

(**c**) (*complex part*) a set of symplectic divisor pencils defined by fibrations  $\psi_i : M - B_i \rightarrow \mathbb{C}\mathbb{P}^1, i = k + 1, \dots, n$  "in involution" with compact generically smooth symplectic fibers, where  $B_i$  are the base set for the corresponding pencils,

such that the real and the complex parts are compatible in the following sense: for each  $f_i$  the Hamiltonian vector field  $X_{f_i}$  preserves the elements of each holomorphic pencil. For the pencils "in involution" means that the tangent spaces to the elements of pencils are symplectically orthogonal at each intersection point. It's not hard to see that for complex pencils it means that the fibers intersect each other transversally almost everywhere.

We will call such data *pseudo toric structure* on a symplectic manifold, see Definition 3.1 below.

In the first part of this work we consider the case of toric Fano varieties, and the first key point is that each toric Fano variety is automatically pseudo toric. At the same time one can show that it admits many different pseudo toric structures, but for our purposes we take the case when  $k = n - 1$  so we deal with a symplectic map  $\psi$  which means a fibration with symplectic fibers of  $M$  without a (real) codimension 4 subset (the base set of the pencil) over  $\mathbb{C}\mathbb{P}^1$  and a set of smooth Morse functions  $(f_1, \dots, f_{n-1})$  which are functionally independent almost everywhere and are in involution so  $\{f_i, f_j\} = 0$ . Again, the compatibility condition means that for each function  $f_i$  the Hamiltonian vector field  $X_{f_i}$  is parallel to every fiber

$$D_p = \psi^{-1}(p), p \in \mathbb{C}\mathbb{P}^1,$$

which is a generically smooth symplectic divisor in  $(M, \omega)$ .

One could understand this picture using an observation made in [9] : each smooth fiber  $D_p$  together with restricted functions  $(f_1|_{D_p}, \dots, f_{n-1}|_{D_p})$  gives a completely integrable system. Indeed, the commutation relation holds for the restrictions and thus one has a set of integrals for  $(D_p, \omega|_{D_p})$ . But here we have parameter  $p \in \mathbb{C}\mathbb{P}^1$  and totally it gives a complex family of integrable systems. On the other hand, the situation is already non classical, and looks like a theory of spin particles. Anyway, a symplectic manifold  $(X, \omega)$  endowed with the data  $(f_1, \dots, f_{n-1}, \psi)$ , which satisfy the compatibility condition, is not a completely integrable system, but can be reduced to a "completely integrable system with singularities", and different systems can be derived in this way. As it was shown in [4], the  $n$ th special "integral" can be added to the set  $(f_1, \dots, f_{n-1})$  if a smooth real function  $h \in C^\infty(\mathbb{C}\mathbb{P}^1, \mathbb{R})$  is chosen, and hence a lagrangian fibration of our given  $(X, \omega)$  is defined by the same additional datum. Thus we get a map from the space  $C^\infty(\mathbb{C}\mathbb{P}^1, \mathbb{R})$  to the set of possible lagrangian fibrations of  $(X, \omega)$  including singular, of course.

In all our examples toric geometry is basic, since even non toric Fano varieties studied below are given by complete intersections in the basic toric variety, the projective space. Recall, a toric symplectic manifold  $(X, \omega)$  is given by a set of moment map functions  $(f_1, \dots, f_n)$  with values in a convex polytop  $P_n \subset \mathbb{R}^n$ , and as it has been proven by T. Delzant, [7], there exists a complex toric manifold with an ample line bundle such that it is symplectically isomorphic to  $(X, \omega)$ . It means that a Kahler structure can be fixed,

and the functions  $f_i$  are quantizable in the sense of Rawnsley – Berezin quantization, see [10], so their Hamiltonian vector fields preserve the Kahler structure. According to geometric formulation of Quantum Mechanics, [1], these real smooth functions can be regarded as quantum observables (called Berezin symbols). Their infinitesimal Hamiltonian actions on holomorphic objects over  $X$  induces infinitesimal symmetries of these objects. Thus if one takes the projectivized spaces of holomorphic sections of the anticanonical bundle  $K_X^{-1}$  in any degree then each function  $f_i$  generates a Hamiltonian vector field  $\Theta_{f_i}^k$  on  $\mathbb{P}(H^0(X, K_X^{-k}))$ , and a divisor  $D \in |-k.K|$  is invariant with respect to  $X_{f_i}$  if and only if  $\Theta_{f_i}^k$  vanishes at the corresponding point  $p_D$ . It follows that there exists a smooth function  $F_{i,k}$  on the projective space, and an invariant pencil from the complete linear system  $|-k.K|$  exists if and only if  $F_{i,k}$  admits a critical projective line. It can happen if the dimension of the projective space is greater than  $\dim X$ . And since we are working with the anticanonical class and its degrees one can see that a toric Fano variety admits pseudo toric structure which can be exploited in the studying of displacability properties of standard fibers of the toric fibrations, which are lagrangian tori.

To do this one can apply the following scheme: fix a set of moment maps  $(f_1, \dots, f_n)$ , given by the standard toric structure, cancel one of them, say  $f_n$ , and then find a pencil of divisors which are invariant under the Hamiltonian action of  $f_1, \dots, f_{n-1}$ . The Fano condition shows that it can be done, thus the given data  $(f_1, \dots, f_n)$  are replaced by certain data  $(f_1, \dots, f_{n-1}, \psi)$  of the mixed type. At the same time these new data contain the given one: it can be shown that there exists a smooth function  $h_0 \in C^\infty(\mathbb{C}\mathbb{P}^1, \mathbb{R})$  such that the induced lagrangian fibration of  $X$  is the same as given by  $(f_1, \dots, f_n)$ . This remark shows that any standard lagrangian torus from the toric fibration can be reduced step by step to a chain of smooth loops on projective lines, and the Hamiltonian invariants of this torus can be estimated in terms of symplectic areas of the loops. Any smooth loop divides the projective line into two parts, and one calls a loop meridional if these parts have the same symplectic area. Then it is well known fact that a smooth loop is undisplacable if and only if it is meridional. Using this, one can see that a lagrangian fiber of a toric fibration of a toric Fano variety is undisplacable only if all the loops given by the reduction are meridional. It remains to compute the Hamiltonian invariants of the fiber in terms of the loops to establish the following fact

**Theorem 0.1** *Let  $S$  be a smooth lagrangian torus given as a fiber of a toric*

*lagrangian fibration of a toric Fano variety  $X$ . Then  $S$  is undisplacable only if  $S$  is Bohr – Sommerfeld with respect to the anticanonical class.*

Below we prove this theorem for the Clifford tori in the projective spaces (Proposition 2.1 and Proposition 2.2). Thus a small part of a conjecture, presented in [12] is proven. In full generality the theorem can be proven using the same method, and we will complete it in a subsequent paper. The reason to postpone the discussion is the following: we will concentrate on the case of general pseudo toric Fano variety and try to prove the same statement for it, then Theorem 0.1 would follow from this general statement as a corollary.

The second part of this paper is devoted to the case of non toric Fano varieties which admit pseudo toric structure. First we prove (Theorem 3.1) that a pseudo toric Fano variety can be fibered on isotropical submanifold such that a generic fiber is a smooth lagrangian torus. Thus pseudo toric structure looks like toric in the sense that it gives lagrangian fibrations. Then we prove (Theorem 3.2) that any smooth irreducible quadric admits pseudo toric structure. Recall that any quadric is a Fano variety but it is toric only in dimensions 1 and 2. At the end we discuss how pseudo toric structures can help in constructions of special lagrangian fibrations of Fano variety, introduced recently by D. Auroux, see [3].

The text below is organized as follows. Section 1 contains the story how the construction was found. It is not necessary for the resting part but it explains why we understand moment maps as Berezin symbols and lagrangian tori as quantum states. Section 2 starts with the construction and the computation for the "toy example" — the case of the projective plane, and we deduce there that if a Clifford torus in  $\mathbb{C}\mathbb{P}^2$  is not Bohr – Sommerfeld with respect to the anticanonical class then it is displaceable. After that in the same strategy is used for the case of the projective space of any dimension. The case of non toric but pseudo toric Fano varieties is studied in Section 3, where we first study certain examples and than give the definition of pseudo toric structure and show that any non singular quadric is pseudo toric. As an application we present a lagrangian fibration which suppose to be special in the sense of D. Auroux .

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## 1 Berezin symbols on the projective space

Quantum Mechanics can be formulated in pure geometric terms, [1], more precisely — in terms of algebraic geometry. As it has been discussed several times, f.e. in [10], the phase space of a quantum mechanical system is presented by an algebraic variety endowed with a Kahler metric of the Hodge type, and for this one quantum observables are given by real smooth functions whose Hamiltonian vector fields preserve both the complex and the symplectic structures. Thus they present infinitesimal symmetries of the Kahler structure. They are called symbols in [1]. The basic example is the projective space with the standard Fubini – Study metric; in this case each symbol is given by a self adjoint operator on the Hilbert space whose projectivization is our given projective space. To reconstruct a self adjoint operator on a Hilbert space from the data  $(\mathbb{C}\mathbb{P}^n, \omega, g_{FS})$  and a symbol  $f$  one can apply a method of geometric quantization, namely — the Rawnsley – Berezin method. Since  $f$  preserves by its Hamiltonian action the Kahler structure one can extend it to the section space of the line bundle  $\mathcal{O}(1)$  choosing a hermitian connection with the curvature form proportional to the symplectic form. Then the corresponding Souriau – Konstant operator  $A_f$  preserves the subspace of holomorphic sections of  $\mathcal{O}(1)$  and this is the space and the self adjoint operator (see, f.e. [10]). Therefore for a given projective space  $\mathbb{C}\mathbb{P}^n$  the space of symbols is completely described; it's not hard to see that it is a Poisson subalgebra in  $C^\infty(\mathbb{C}\mathbb{P}^n, \mathbb{R})$  with respect to the Poisson brackets, defined by the symplectic form  $\omega$ . Every (nondegenerated) symbol  $f$  can be included in a commutative subalgebra of the Poisson algebra spanned by  $(f = f_1, \dots, f_n)$  — a set of symbols. Note that this set comes from a set of commuting self adjoint operators; the maximal number of these operators is  $n + 1$  but since the identical operator goes to the constant function under the projectivization the rank of the set of commuting algebraically independent almost everywhere symbols is  $n$ . For the projective spaces these symbols are the moment maps in the language of toric geometry.

But it is natural to extend this geometric approach to a wider class of algebraic varieties. Let  $X$  be such a variety and  $\omega$  is the Kahler form of a metric of the Hodge type on  $X$ . Then it is not hard to see that a symbol exists on  $X$  if and only if the following holds: take the corresponding ample line bundle  $L \rightarrow X$ , consider the embedding  $\phi : X \rightarrow \mathbb{P}(H^0(X, L^k)^*)$ , then there exists a symbol  $f$  on  $\mathbb{P}(H^0(X, L^k)^*)$  such that the Hamiltonian vector field  $X_f$  is tangent to the image  $\phi(X)$  at each point of it. This observation simplifies

the problem if one would like to find all algebraic varieties with symbols, so all possible quantum phase space of the generalized quantum mechanics. It follows that these algebraic varieties are submanifolds of  $\mathbb{C}\mathbb{P}^n$  parallel to the Hamiltonian vector fields of certain symbols. In this formulation the problem was studied by S. Belyov in his Master diploma [5]. For example, in the case of  $\mathbb{C}\mathbb{P}^2$  he found that a smooth curve  $C \subset \mathbb{C}\mathbb{P}^2$  admits symbols if and only if it is rational. And at the same time an interesting effect appears — if a non degenerated conic is preserved by the Hamiltonian action of certain symbol  $f$  then there exists a pencil of conics, including the given one, which are invariant under the Hamiltonian action of the same symbol. And it is a common principle for irreducible divisors in the projective space as it was shown in [5]. Indeed, let  $D$  be an irreducible divisor in  $\mathbb{C}\mathbb{P}^n$  and  $f$  be a symbol, which preserves  $D$ . Since each symbol corresponds to a self adjoint operator on  $\mathbb{C}^{n+1}$  there exists a homogenous coordinate system  $[z_0 : \dots : z_n]$  such that  $f$  has the "diagonal" form

$$f = \frac{\sum_{i=0}^n \lambda_i |z_i|^2}{\sum_{i=0}^n |z_i|^2}$$

where  $\lambda_i$  are the critical values of  $f$ . Since  $D$  is irreducible, in these coordinates it is defined by a polynomial  $P(z)$  which consists of at least two summands, and since  $D$  is invariant the sum of weights  $\lambda_i$  for each summands is constant. This means that if we take any linear combination of these summands and consider the zero set of the corresponding polynomial it must be again an invariant divisor. It follows that if an invariant irreducible divisor exists then an invariant pencil exists as well. Note that in these arguments we've exploited the toric structure on  $\mathbb{C}\mathbb{P}^n$ , and it is crucial since a symbol *a priori* doesn't generate  $U(1)$  action on a given algebraic variety, thus doesn't give closed orbits etc. while a moment map does by the definition. But it hints that a generalization of lagrangian torus fibration construction to the case of non toric Fano varieties should follow almost the same scheme: if an  $X$  admits a set of symbols  $f_1, \dots, f_{n-1}$  and doesn't admit a complete set being non toric but there is an invariant pencil then it would give a set of examples. One could try to exploit this scheme for del Pezzo surface  $\mathbb{C}\mathbb{P}_4^2$  which is not toric, [6].

Now come back to the case of the projective plane. In [4] we present as the first example the following one: the pencil on  $\mathbb{C}\mathbb{P}^2$  is given by the equation

$$\alpha z_0^2 + \beta z_1 \cdot z_2 = 0 \tag{1}$$

and the corresponding symbol is given in the coordinates  $Z_0, Z_1, Z_2$  of the associated  $\mathbb{C}^3$  as the diagonal operator with eigenvalues  $(1, 0, -2)$ . In the homogenous coordinates  $[z_0 : z_1 : z_2]$  the function reads as

$$f = \frac{|z_0|^2 - 2|z_2|^2}{|z|^2}, \quad (2)$$

where

$$|z|^2 = \sum_{i=0}^2 |z_i|^2$$

as usual.

The base set of the pencil  $\mathbb{CP}_{\alpha,\beta}^1$  is contained by the singular set of  $f$ , and the singular fibers of the pencil contains all the critical points of  $f$ ; we will see below that it is a general principle for invariant symplectic pencils. The pencil  $\mathbb{CP}_{\alpha,\beta}^1$  contains two singular conics:  $D_{1:0}$  — double line,  $D_{0:1}$  — two lines, thus we have two distinguished points on  $\mathbb{CP}_{\alpha,\beta}^1$  with coordinates  $[1 : 0]$  and  $[0 : 1]$ . Consider first a smooth function  $h$  on  $\mathbb{CP}_{\alpha,\beta}^1$  defined by the following equation

$$h = \frac{|\alpha|^2 - |\beta|^2}{|\alpha|^2 + |\beta|^2}, \quad (3)$$

and which is a symbol on  $\mathbb{CP}_{\alpha,\beta}^1$ . It has exactly two critical points which correspond to degenerated elements of the pencil. It was shown in [4] that this function  $h$  gives us the standard Clifford fibration on  $\mathbb{CP}^2$ . Indeed, a smooth level set

$$\gamma_c = \{h = c\} \subset \mathbb{CP}_{\alpha,\beta}^1, \quad -1 < c < 1,$$

is defined by the conditions  $|\alpha| = \text{const}$ ,  $|\beta| = \text{const}$ , and it follows from the pencil equation that the corresponding sets in  $\mathbb{CP}^2$  are defined by the Clifford conditions  $|z_i| = \text{const}$ . Thus for this  $h$  we just restore the toric lagrangian fibration.

Now consider another function  $h$  defined by the equation

$$h = \frac{|\alpha|^2}{|\alpha|^2 + |a\beta - \alpha|^2}, \quad (4)$$

where  $a$  is a positive real parameter. This function has two critical points, which correspond to the following element of the pencil: again double line  $2l = \{z_0^2 = 0\}$  and a non degenerated conic  $Q = \{z_0^2 = -az_1.z_2\}$ . Thus the resulting fibration of  $\mathbb{CP}^2$  has degeneration on a reducible cubic line  $l \cup D$  and

contains one singular torus. The point is that this fibration is exactly the non toric fibration constructed by D. Auroux in [3]. And the construction above can be understood as a generalization of the Auroux method. This construction gives us a set of non toric lagrangian fibrations of  $\mathbb{C}\mathbb{P}^2$ . It can be easily generalized to the case of any toric Fano variety, and it is the main result of [4].

But our initial aim for the present paper is to study certain Hamiltonian invariants of fibers of toric fibrations and their relation to the displacability property. And at the next section we take the Clifford fibration of  $\mathbb{C}\mathbb{P}^2$  as a toy example and prove that for a Clifford torus the following holds: it is undisplacable only if it is Bohr – Sommerfeld with respect to the anticanonical class.

## 2 Clifford tori in $\mathbb{C}\mathbb{P}^2$ and $\mathbb{C}\mathbb{P}^n$

Consider  $\mathbb{C}\mathbb{P}^2$  with the standard Fubini – Study metric and the corresponding symplectic form  $\omega$ . Thus  $\mathbb{C}\mathbb{P}^2$  is equipped with the hermitian triple and the same is for its holomorphic tangent bundle. It follows that its determinant  $-K$  is equipped with the corresponding hermitian structure, and one can consider the space of hermitian connections  $\mathcal{A}_h(-K)$ . Since the canonical class is proportional to the class of  $[\omega]$ , an orbit is distinguished in  $\mathcal{A}_h(-K)$  defined by the condition that the curvature form is proportional to the symplectic form. Take any connection  $a$  from this orbit and consider the restriction of the pair  $(-K, a)$  to a lagrangian torus  $S \subset \mathbb{C}\mathbb{P}^2$ . Then one says that the torus is Bohr – Sommerfeld with respect to the anticanonical class if the restriction admits covariantly constant sections. This definition appears for any monotone symplectic manifold, for example for any Fano variety, see [11].

Recall that a lagrangian submanifold  $S$  is *displacable* if there exists a Hamiltonian isotopy  $\phi_t$  such that

$$S \cap \phi_t(S) = \emptyset$$

for certain  $t$ . Otherwise it is called *undisplacable*.

A conjecture, presented [12], says that a regular fiber of a toric fibration of a toric Fano variety is undisplacable if and only if it is Bohr – Sommerfeld with respect to the anticanonical bundle. In particular, it states that a Clifford torus in  $\mathbb{C}\mathbb{P}^2$  is undisplacable if and only if it is Bohr – Sommerfeld



with respect to  $-K$ . In this section we prove a part of the last sentence using the construction with invariant pencils from above.

**Proposition 2.1** *Every undisplacable Clifford torus in  $\mathbb{C}\mathbb{P}^2$  is Bohr – Sommerfeld with respect to the anticanonical class.*

The proof is as follows. Come back to the picture from section 1. A regular Clifford torus  $S \subset \mathbb{C}\mathbb{P}^2$  is realized by the conditions  $f = c_1, h = c_2$  such that  $c_1$  is not critical for  $f$  so  $c_1 \neq -2, 1$  and  $c_2$  is not critical for  $h$  so  $c_2 \neq -1, 1$  for the functions  $f$  (2) and  $h$  (3). Note that the Bohr – Sommerfeld with respect to the anticanonical bundle fiber corresponds to the values  $c_1 = 0, c_2 = 0$ . Suppose that a fiber is not Bohr – Sommerfeld with respect to the anticanonical class and denote it as  $S_{c_1, c_2}$ . Then we claim that if  $c_2$  is not equal to 0, so the level loop  $\gamma_{c_2} = \{h = c_2\} \subset \mathbb{C}\mathbb{P}_{\alpha, \beta}^1$  is not meridional, then there exists a Hamiltonian transformation  $\phi_t$  of  $\mathbb{C}\mathbb{P}^2$  which moves  $S_{c_1, c_2}$  to certain  $\phi_t(S_{c_1, c_2})$  such that

$$S_{c_1, c_2} \cap \phi_t(S_{c_1, c_2}) = \emptyset.$$

To prove this implication we construct explicitly a real smooth function  $F$  on  $\mathbb{C}\mathbb{P}^2$  whose Hamiltonian vector field generates desired transformation.

The base set of the pencil (4) consists of two critical points of  $f$  namely  $[0 : 1 : 0]$  and  $[0 : 0 : 1]$  and the pencil contains a singular quadric with singularity at  $[1 : 0 : 0]$ . Choose a small real number  $\varepsilon$  such that

$$\varepsilon \ll \min(|c_1 - 1|, |c_1|, |c_1 - 2|).$$

Then the balls with radius  $\varepsilon$  centred in  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$  and  $[0 : 0 : 1]$  don't intersect our torus  $S_{c_1, c_2}$ . Denote these balls as  $B_1, B_2$  and  $B_3$ . Consider the holomorphic map

$$\psi : \mathbb{C}\mathbb{P}^2 - B \rightarrow \mathbb{C}\mathbb{P}_{\alpha, \beta}^1, \tag{5}$$

defined by our pencil (1), where  $B$  is the base set,

$$B = [0 : 1 : 0] \cup [0 : 0 : 1].$$

Then any smooth real function  $h_c : \mathbb{C}\mathbb{P}_{\alpha, \beta}^1 \rightarrow \mathbb{R}$  can be lifted to  $\mathbb{C}\mathbb{P}^2 - (B_1 \cup B_2)$  and then can be continued to a smooth function  $F_{h_c}$  on whole  $\mathbb{C}\mathbb{P}^2$ . Indeed, let's take smaller balls  $B'_1, B'_2, B'_3$  with the same centers but of radius  $\frac{1}{2}\varepsilon$  and define  $F = F_{h_c}$  on five different pieces of  $\mathbb{C}\mathbb{P}^2$  as follows

- $F = \psi^* h_c$  on  $\mathbb{C}\mathbb{P}^2 - (B_1 \cup B_2 \cup B_3)$ ;
- $F = 0$  on  $B'_i, i = 1, 2, 3$ ;
- and on  $B_i - B'_i$  our function linearly descends from the big boundary  $\partial B_i$  to the small boundary  $B'_i$  such that  $F$  is smooth on  $\mathbb{C}\mathbb{P}^2$ .

The Hamiltonian vector field induced by this function  $F$  is symplectically orthogonal to "the central non singular parts" of the fibers, if one understands that this central part for a fiber  $D_p$  is given by

$$D_p^c = D_p - (D_p \cap (B_1 \cup B_2 \cup B_3)).$$

Note that we must exclude  $B_1$  since the Hamiltonian vector field of  $\psi^* h_c$  has singularity at point  $[1 : 0 : 0]$ .

Thus the Hamiltonian action  $\phi_t(F)$  moves "central non singular parts" of smooth fibers to "central non singular parts" of smooth fibers and twists the neighborhoods  $B_1$  and  $B_2$  of the base set. Thus the circle action from below, generated by a Morse function  $h_c$  with only two critical points on  $\mathbb{C}\mathbb{P}^1_{\alpha,\beta}$ , is lifted to a "quasi circle" action which twists the neighborhood of the base set and the singular point, leaving them unmoved, and interchanges "central non singular parts"  $D_p^c$  of the smooth fibers.

Now we define the function  $h_c$  on  $\mathbb{C}\mathbb{P}^1_{\alpha,\beta}$  which will generate the transformation we need. This function has the form

$$h_c = \frac{2 \operatorname{Re} \alpha \bar{\beta}}{|\alpha|^2 + |\beta|^2}; \tag{6}$$

it has two critical points on  $\mathbb{C}\mathbb{P}^1_{\alpha,\beta}$  with coordinates  $[1 : 1]$  and  $[1 : -1]$ , and the corresponding Hamiltonian motion induced by  $X_{h_c}$  on  $\mathbb{C}\mathbb{P}^1_{\alpha,\beta}$  is rotation with fixed poles  $[1 : 1]$  and  $[1 : -1]$ . Under this rotation for the time  $t = \pi$  the loop  $\gamma_{c_2} = \{h = c_2\}$  moves to the "mirror" loop  $\gamma_{-c_2} = \{h = -c_2\}$  which doesn't intersect  $\gamma_{c_2}$  if it is not meridional.

The argument can be extended to the function  $F$ ; the Hamiltonian motion  $\phi_t$  generated by  $X_F$  on  $\mathbb{C}\mathbb{P}^2$  transports fibers to fibers except the parts which are close either to the base set or to the singular point. This means that our lagrangian torus  $S_{c_1, c_2}$  is transported by  $\phi_\pi$  to the torus  $S_{c_1, -c_2}$  and the last one doesn't intersect the given one unless  $c_2$  is equal to zero. Note, that  $c_2 = 0$  is equivalent to  $|\alpha| = |\beta|$ .

On the other hand, we can consider any permutation in  $(z_0, z_1, z_2)$  and substitute in the picture another symbol and another pencil, say,

$$\alpha' z_1^2 + \beta' z_0 \cdot z_2 = 0$$

and repeat the story for these data. Then the fiber would be displaceable if it corresponds to the level loop of the corresponding function with  $|\alpha'| \neq |\beta'|$ . Every Clifford torus is defined uniquely by two level loops in  $\mathbb{C}\mathbb{P}_{\alpha,\beta}^1$  and  $\mathbb{C}\mathbb{P}_{\alpha',\beta'}^1$ , thus the arguments above ensure that if a Clifford torus is undisplaceable then it can be described by the system

$$z_0^2 = e^{i\phi_1} z_1 z_2 \quad z_1^2 = e^{i\phi_2} z_0 z_2,$$

where  $\phi_i$  are real parameters. And as it was shown in [12] it is exactly the Bohr – Sommerfeld with respect to the anticanonical class fiber of the Clifford fibration.

On the other hand, we prove in the same papers that this Bohr – Sommerfeld fiber is monotone, thus one has

**Corollary 2.1** *If a Clifford torus is undisplaceable it must be monotone.*

One expects that the converse is also true.

Consider now the case of  $\mathbb{C}\mathbb{P}^n$  where  $n > 2$ . In this case we can reproduce the same scheme to prove that

**Proposition 2.2** *Let  $S$  be a Clifford torus in  $\mathbb{C}\mathbb{P}^n$ . Then it is undisplaceable only if it is Bohr – Sommerfeld with respect to the anticanonical class.*

Take  $\mathbb{C}\mathbb{P}^n$  and consider a set of integrals  $(f_1, \dots, f_n)$  which define the action variables for the corresponding Clifford fibration (according to the famous Arnold – Liouville theorem such a set exists). Under the action map

$$(f_1, \dots, f_n) : \mathbb{C}\mathbb{P}^n \rightarrow P_n \subset \mathbb{R}^n$$

the image  $P_n$  is given by the conditions

$$x_1 + \dots + x_n = 1, \quad x_i \geq 0,$$

and for the corresponding universal basis of  $H_1(T^n, \mathbb{Z})$  the values of  $x_1, \dots, x_n$  are the periods of the corresponding loops with respect to the symplectic form. A Clifford torus is Bohr – Sommerfeld with respect to the anticanonical class if and only if its periods are represented by integer numbers multiplied by  $\frac{1}{n+1}$ . It shows that there is unique such a torus of dimension  $n$  (but there are many isotropical Clifford tori which are Bohr – Sommerfeld with respect to the anticanonical class). Note that  $f_i$  are degenerated as symbols in our

terminology since each  $f_i$  has non isolated critical points, which form the corresponding divisors.

Fix a set of homogenous coordinates  $[z_0 : \dots : z_n]$  such that each  $f_i$  degenerates at the hyperplane  $\{z_i = 0\}$ . Remove a symbol from the set  $(f_1, \dots, f_n)$ , say,  $f_n$ . The cases when  $n$  is even and odd are different since one takes different pencils and symbols for the cases.

If  $n$  is odd, then one takes a homogenous coordinate system  $[z_0 : \dots : z_n]$  and consider the pencil

$$\{D_u\} = \{u_0 z_0 \dots z_{(n-1)/2} + u_1 z_{(n+1)/2} \dots z_n = 0\} \quad (7o)$$

which defines a holomorphic map

$$\psi : \mathbb{C}\mathbb{P}^n - B \rightarrow \mathbb{C}\mathbb{P}_u^1,$$

where  $B$  is the base set formed by  $(n/2)^2$  hyperplanes of dimension  $n - 2$ . The set of symbols (moment maps) which preserve the pencil  $\{D_u\}$  is defined by the condition

$$\lambda_0 + \dots + \lambda_{(n-1)/2} = \lambda_{(n+1)/2} + \dots + \lambda_n \quad (8o)$$

for the critical values  $\lambda_i$ .

If  $n$  is even we take the pencil

$$\{D_u\} = \{u_0 z_0^2 z_1 \dots z_{n/2-1} + u_1 z_{n/2} \dots z_n\}, \quad (7e)$$

and the symbols must satisfy

$$2\lambda_0 + \lambda_1 + \dots + \lambda_{n/2-1} = \lambda_{n/2} + \dots + \lambda_n. \quad (8e)$$

Both the cases can be treated further simultaneously.

It's clear that our  $f_1, \dots, f_{n-1}$  can be changed by certain linear transformation to the set of non degenerated  $f'_1, \dots, f'_{n-1}$  such that each  $f'_i$  satisfies the condition (8o) or (8e) above and therefore each  $f'_i$  preserves each  $D_u$ . The degeneration simplex  $\Delta_{n-1} \subset \mathbb{C}\mathbb{P}^n$  of the set  $\{f'_1, \dots, f'_{n-1}\}$  consists of  $\frac{(n+1)n}{2}$  hyperplanes of dimension  $n - 2$ . The fibers of  $\psi$  are generically smooth, and there are exactly two singular fibers which go to the points  $[0 : 1]$  and  $[1 : 0]$  in  $\mathbb{C}\mathbb{P}_u^1$ . Consider the singular sets of fibers  $D_{[1:0]}$  and  $D_{[0:1]}$  and denote these ones as  $Sing_0$  and  $Sing_1$ . Then it's not hard to see that

$$\Delta_{n-1} = B \cup Sing_0 \cup Sing_1.$$

Take a function  $h$  on  $\mathbb{C}\mathbb{P}_u^1$  of the form (3). We claim that the fibration of  $\mathbb{C}\mathbb{P}^n - B$  defined by  $(f'_1, \dots, f'_{n-1}, \psi^*h)$  is the same Clifford fibration. Indeed, let's take a Clifford torus  $T_{(c_1, \dots, c_n)}$  which is defined by the conditions

$$|z_i|^2 = c_i,$$

so the  $c_i$ 's are regular values of the action moment maps  $f_1, \dots, f_n$  (therefore they must be positive). Since  $f'_j$ 's are given by the linear transformation of  $f_i$ 's it means that the same transformation maps the set  $(c_1, \dots, c_{n-1})$  to the corresponding values of  $f'_j$ 's which we denote as  $(c'_1, \dots, c'_{n-1})$ . The corresponding value of  $h$  is given by

$$c = \frac{c_{(n+1)/2} \dots c_n}{c_1 \dots c_{(n-1)/2}} - 1$$

for the odd case or by

$$c = \frac{c_{n/2} \dots c_n}{c_1 \dots c_{n/2-1}}$$

for the even one. Then it's obvious that

$$S_{(c'_1, \dots, c'_{n-1}, c)} = \{f'_j = c'_j, \psi^*h = c\} = T_{(c_1, \dots, c_n)}.$$

Thus without the loss of generality we can label the Clifford tori by numbers  $(c'_j, c)$  instead of  $(c_i)$ .

Our next step is to construct a smooth function  $F$  on whole  $\mathbb{C}\mathbb{P}^n$  whose Hamiltonian vector field will move a Clifford torus  $T_{(c'_1, \dots, c'_{n-1}, c)}$  to the Clifford torus  $T_{(c'_1, \dots, c'_{n-1}, -c)}$ . It would imply that a Clifford torus is undisplacable only if it is projected by  $\psi$  to the meridional circle in  $\mathbb{C}\mathbb{P}_u^1$ .

Such an  $F$  is constructed in the same way as in the proof of Proposition 2.1. Consider small neighborhood  $\mathcal{O}_1(\Delta_{n-1})$  of the degeneration simplex  $\Delta_{n-1}$  of radius  $\varepsilon$  such that  $T_{(c'_1, \dots, c'_{n-1}, c)}$  doesn't intersect  $\mathcal{O}_1(\Delta_{n-1})$  and a smaller neighborhood  $\mathcal{O}_2(\Delta_{n-1})$  with the same center of radius  $\frac{1}{2}\varepsilon$ . Take the function  $h_c$  of the form (6) on  $\mathbb{C}\mathbb{P}_u^1$  and construct a global function  $F$  on  $\mathbb{C}\mathbb{P}^n$  by the same rules:

- $F$  vanishes inside of  $\mathcal{O}_2(\Delta_{n-1})$ ;
- $F$  equals to  $\psi^*h_c$  outside of  $\mathcal{O}_1(\Delta_{n-1})$ ;
- it linearly descends from the boundary of  $\mathcal{O}_1(\Delta_{n-1})$  to the boundary of  $\mathcal{O}_2(\Delta_{n-1})$ .

The property of this function  $F$  is the same as of the function from the proof of Proposition 2.1 above: it Hamiltonian action moves the "central

parts" of the fibers to the "central parts" of the fibers, and therefore it generates Hamiltonian isotopy of  $T_{(c'_1, \dots, c'_{n-1}, c)}$  to  $T_{(c'_1, \dots, c'_{n-1}, -c)}$  which means what we claimed above.

Thus a Clifford torus is undisplacable only if  $c = 0$  and consequently either

$$c_1 \dots c_{(n-1)/2} = c(n+1)/2 \dots c_n$$

if  $n$  is odd or

$$c_1 \dots c_{n/2-1} = c_{n/2+1} \dots c_n$$

for the even case. But we can remove another  $f_i$  and repeat all the construction for this case. At the end one gets that all  $c_i$  must be the same which means that an undisplacable Clifford torus must be Bohr – Sommerfeld with respect to the anticanonical class. This complete the proof.

We can summarize the discussion on the Clifford tori in  $\mathbb{C}\mathbb{P}^n$  by the following

**Corollary 2.2** *Every undisplacable Clifford torus in  $\mathbb{C}\mathbb{P}^n$  is monotone.*

**Remark (on the Chekanov and the Clifford tori in  $\mathbb{C}\mathbb{P}^2$ ).** In [3] one presents two types of lagrangian tori in  $\mathbb{C}\mathbb{P}^2$ , namely the Clifford type and the Chekanov type, and these two types are separated by a singular torus in the Auroux fibration. After [4] we can say that tori of the Chekanov type appear if the function  $h$  on  $\mathbb{C}\mathbb{P}^1_{\alpha, \beta}$  has two critical points  $p_{\max}, p_{\min} \in \mathbb{C}\mathbb{P}^1_{\alpha, \beta}$  such that only one of them corresponds to a singular element from the pencil, namely to the double line  $\{z_0^2 = 0\}$ . In this case one has not two, but three distinguished points in  $\mathbb{C}\mathbb{P}^1_{\alpha, \beta}$ ,

$$p_{\max}, p_{\min}, p_{\text{sing}},$$

where the last one corresponds to the singular fiber  $\{z_1 z_2 = 0\}$  as it happens for  $h$  given by (4). As one claims in [4] any real Morse function on  $\mathbb{C}\mathbb{P}^1_{\alpha, \beta}$  gives a lagrangian fibration, and the fibers of the fibration are projected by the map (5) to the level lines of this function. The set of level lines of our  $h$  are separated by point  $p_{\text{sing}}$ , and the tori from the different chambers are called of the Clifford and of the Chekanov types. The previous construction with the invariant pencil on  $\mathbb{C}\mathbb{P}^2$  shows that in this situation the Clifford and the Chekanov tori can be moved each to the other by a *symplectomorphism* but not by a Hamiltonian isotopy. Indeed, for any *rotation* of the projective line

$\mathbb{C}\mathbb{P}_{\alpha,\beta}^1$  one has a corresponding twisting of the fibers of  $\psi_L$  which gives a symplectomorphism of  $\mathbb{C}\mathbb{P}^2$ . But the singular point  $[1 : 0 : 0]$  is a singular point of the lifted Hamiltonian vector field and then this twist symplectomorphism is not a Hamiltonian isotopy. Resuming, we see that the Clifford tori and the Chekanov tori with the same Hamiltonian invariants are symplectomorphic but not Hamiltonian isotopic.

### 3 Non toric manifolds

In this section we show that the construction with invariant pencils can be performed in certain cases for non toric Fano varieties. The result given in this way however is rather similar to the toric case. At the same time below we present the situation when one has several invariant pencils combined with real integrals.

As it was shown in [4] a non degenerated 2 - dimensional quadric admits invariant pencils and can be fibered by lagrangian tori in different ways. But it is toric, and if we increase the dimension and consider a 3 - dimensional non degenerated quadric  $Q$  in  $\mathbb{C}\mathbb{P}^4$  it is already non toric. Let us show that nevertheless  $Q$  can be sliced by lagrangian tori (the symplectic structure in what follows is induced by the restriction from the projective space).

**Example 1.** Take a quadric  $Q \subset \mathbb{C}\mathbb{P}^4$  defined by the equation

$$z_0^2 + z_1z_2 + z_3z_4 = 0$$

in a homogeneous coordinate system  $[z_0 : \dots : z_4]$ . Then one has a subspace in the moment map space (or  $C_q^\infty(\mathbb{C}\mathbb{P}^4, \mathbb{R})$ ) consists of functions whose Hamiltonian vector fields preserve  $Q$ . If a symbol  $f$  has critical values  $(\lambda_0, \dots, \lambda_4)$  then it preserves the quadric if and only if

$$2\lambda_0 = \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4.$$

It's clear that there are only two functionally independent symbols which preserve  $Q$  thus  $Q$  is non toric. To proceed with our construction one needs to find a pencil of divisors on  $Q$  which are invariant with respect to these two symbols. A pencil can be derived as follows: note that the last condition on  $\lambda_i$ 's is satisfied by a family of quadric, not by only the given one. Then if we fix a pair of moment maps  $f_1, f_2$  say of the form

$$f_1 \mapsto (0, 1, -1, 2, -2), \quad f_2 \mapsto (0, 2, -2, 1, -1)$$

which preserve  $Q$  then the same function must preserve any quadric

$$Q_w = \{w_0 z_0^2 + w_1 z_1 z_2 + w_2 z_3 z_4 = 0\}.$$

Thus the family of invariant quadrics is  $\mathbb{C}\mathbb{P}^2$  with coordinates  $[w_0 : w_1 : w_2]$ . Choose in this  $\mathbb{C}\mathbb{P}^2$  a line which doesn't pass through the point  $[1 : 1 : 1]$  and which consists of generically non degenerated quadrics (so its equation is not of the form  $w_i = 0$ ). Denote this line as  $\mathbb{C}\mathbb{P}_u^1$  and the quadrics presented by its points as  $Q_u$ . Then the intersections  $D_u = Q \cap Q_u \subset Q$  are generically smooth divisors which form an invariant pencil with respect to  $f_1|_Q, f_2|_Q$  without base components. The proof of this fact is a simple exercise in basic algebraic geometry. The base set  $B$  of the pencil  $D_u$  is given by the equations

$$z_0 = z_1 z_2 = z_3 z_4 = 0,$$

and is presented by 4 projective lines contained by  $Q$ . The pencil contains three singular elements, and it means that the map

$$\psi : Q - B \rightarrow \mathbb{C}\mathbb{P}_u^1$$

marks three points on  $\mathbb{C}\mathbb{P}_u^1$  which correspond to these singular fibers. If it were two singular fibers the source space should be toric. But now for any Morse function  $h : \mathbb{C}\mathbb{P}_u^1 \rightarrow \mathbb{R}$  the induced fibration must have singular fibers. Let's treat the example more carefully and consider the following pencil of quadrics  $Q_w$ :

$$\mathbb{C}\mathbb{P}_u^1 = \{w_0 - w_1 + w_2 = 0\}.$$

Then we have three quadrics which give singular divisors  $D_u$  being intersected with  $Q$  :

- (1)  $Q_1 = \{z_1 z_2 + z_3 z_4 = 0\}$ ;
- (2)  $Q_2 = \{z_0^2 + z_1 z_2 = 0\}$ ;
- (3)  $Q_3 = \{z_0^2 - z_3 z_4 = 0\}$ .

The corresponding divisors are

- (1)  $D_1$  — double 2 -dimensional quadric (with smooth support);
- (2)  $D_2$  — two cones over the same conic  $q_1 = \{z_0^2 + z_1 z_2 = 0, z_3 = z_4 = 0\}$

which is the singular set of this fiber;

- (2)  $D_3$  — two cones over the same conic  $q_2 = \{z_0^2 - z_3 z_4 = 0, z_1 = 0, z_2 = 0\}$  which is the singular set of this fiber.

The degeneration simplex of  $f_1|_Q, f_2|_Q$  consists of 4 lines which form the base set and these two conics  $q_i$ .



Consider now a Morse function  $h$  on  $\mathbb{C}\mathbb{P}_u^1$  given by the equation

$$h = \frac{|w_1 - w_2|^2 - |w_1|^2}{|w_0 - w_1 + w_2|^2 + |w_1|^2 + |w_1 - w_2|^2}$$

on the line  $\mathbb{C}\mathbb{P}_u^1$ . This function has two critical points

— the maximal  $p_1 = [0 : 1 : 1]$  and the minimal  $p_2 = [1 : 0 : -1]$ , corresponding to  $D_1$  and  $D_3$ . Thus the choice of  $h$  defines a lagrangian fibration of  $Q - (D_1 \cup D_3)$ . Since we have a singular fiber  $D_2$  which is projected to a point on the level set

$$\gamma_0 = \{h = 0\},$$

there is a one - dimensional subfamily of singular tori modeled by a three - dimensional torus with shrunked two - subtorus to a loop. The singular loops lie on the conic  $q_1$  and slice it outside of two points which are the intersection with the base set.

**Digression: the monodromy.** Here we would like to mention an important advantage of our construction: it is possible to calculate the monodromy of the lagrangian fibration around the singular tori. The monodromy is the obstruction to the existence of the global action — angle coordinates so it measures how far is our system to be completely integrable in usual sense.

Choose an open part of  $Q - B$  consisting of the preimage under the map

$$\psi : Q - B \rightarrow \mathbb{C}\mathbb{P}_u^1$$

of an annulus

$$\mathbb{C}\mathbb{P}_u^1 \supset A_\delta = \{-\delta < h < \delta\}$$

where  $0 < \delta < 1$  is a real parameter. Then the lagrangian fibration on  $Q$  given by the triple  $(f_1|_Q, f_2|_Q, \psi_L^* h)$  of commuting functions can be projected by the "action map"

$$F_{act} = (f_1|_Q, f_2|_Q, \psi^* h) : \psi^{-1}(A_\delta) \rightarrow \mathbb{R}^3 = \mathbb{R} \langle x, y, z \rangle,$$

and there one has a one dimensional set  $I$  in

$$\text{Im } F_{act} \subset \mathbb{R}^3 \langle x, y, z \rangle,$$

which is given by the segment

$$I = \{2x = y, \quad z = 0, \quad -1 < x < 1\}.$$

Thus the three dimensional space of lagrangian fibers  $\text{Im } F_{act}$  contains a one dimensional subfamily of singular lagrangian tori. Consider a loop  $\gamma \subset \text{Im } F_{act}$  which passes around the singular string. The natural question arises — what is the monodromy along this loop which is an element of  $\text{End } H_1(T^3, \mathbb{Z})$ . The answer is given in the following

**Proposition 3.1** *The monodromy operator  $\text{Mon}_\gamma \in \text{End } H^1(T^3, \mathbb{Z})$  can be presented in terms of "vanishing cycle" of the singular fiber  $D_2^0 = D_2 - B$ .*

For a smooth generic fiber in  $Q - B$  which is a smooth two dimensional quadric without 4 lines so it is homeomorphic to  $C^* \times C^*$  the homology group is the direct sum  $\mathbb{Z}[h_1] \oplus \mathbb{Z}[h_2]$ ; and for the singular fiber  $D_2$  without the intersection with the same lines the group is reduced to  $\mathbb{Z}[h]$  as for one dimensional quadric without two points. So our "vanishing cycle" is a generator in  $H_1(D_u^0, \mathbb{Z})$  which vanishes when one passes to the singular fiber. In our situation the vanishing cycle defines the monodromy operator  $\text{Mon}_{van}$  from  $\text{End } H_1(D_u^0, \mathbb{Z})$ , and since we have the inclusion

$$H_1(D_u^0, \mathbb{Z}) \subset H_1(T^3, \mathbb{Z})$$

where  $T^3$  is a smooth fiber of  $\text{Im } F_{act}$ , the monodromy  $\text{Mon}_\gamma$  is just the direct sum

$$\text{Mon}_\gamma = \text{Id}_1 \oplus \text{Mon}_{van} .$$

Note that this monodromic relationship always takes place if we consider a lagrangian fibration given by invariant holomorphic pencils.

Now we extend the last example and consider the case when a number of invariant pencils must be taken to define a lagrangian fibration.

**Example 2.** Take a smooth non degenerated 5 - dimensional quadric  $Q \subset \mathbb{C}\mathbb{P}^6$  with the equation

$$Q = \{z_0^2 + z_1z_2 + z_3z_4 + z_5z_6 = 0\}$$

If a symbol preserves this one by its Hamiltonian action this implies that its critical values satisfy

$$2\lambda_0 = \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = \lambda_5 + \lambda_6,$$

and there exists exactly 3 functionally independent symbols which do preserve it. Denote them as  $f_1, f_2, f_3$ . Thus to approach the situation when our

construction can be applied we need to find two invariant pencils. And the point is it can be done. Indeed, the quadric  $Q$  is contained by a big family of quadrics which are invariant with respect to  $X_{f_i}$  for each  $i$ . Thus we have

$$\mathbb{C}\mathbb{P}_w^3 = \{Q_w | Q_w = \{w_0z_0^2 + w_1z_1z_2 + w_2z_3z_4 + w_3z_5z_6 = 0\} \subset \mathbb{C}\mathbb{P}^6\}.$$

Take in this  $\mathbb{C}\mathbb{P}_w^3$  two skew projective lines  $\mathbb{C}\mathbb{P}_u^1, \mathbb{C}\mathbb{P}_v^1$  such that none is passing through the point  $[1 : 1 : 1 : 1]$ . The quadrics parameterized by  $\mathbb{C}\mathbb{P}_u^1, \mathbb{C}\mathbb{P}_v^1$  are denoted as  $Q_u$  and  $Q_v$ . Each pencil defines the corresponding pencil of intersections

$$D_u = Q_u \cap Q, \quad D_v = Q_v \cap Q,$$

and we denote as  $B_1$  and  $B_2$  the base set of  $D_u$  and  $D_v$ , the mutual base set is denoted by  $B = B_1 \cap B_2$  and

$$B_0 = B_1 \cup B_2.$$

Outside of  $B_0$  we have two holomorphic maps  $\psi_i : Q - B_0 \rightarrow \mathbb{C}\mathbb{P}_i^1, i = 1, 2$ , and the compatibility condition shows that the fibers of  $\psi_1$  and  $\psi_2$  intersect each other transversally outside of the degeneration set of  $f_1, f_2, f_3$ . Indeed, at any intersection point the tangent space is the complex span of three real linearly independent vectors  $X_{f_1}, X_{f_2}, X_{f_3}$ , and since these three vectors form an isotropic subspace the intersection must be transversal. Each map  $\psi_i$  admits singular fibers and it is not hard to see that the singular sets must be contained by the degeneration set of  $f_1, f_2, f_3$ .

One has the following

**Proposition 3.2** *The choice of two Morse functions  $h_1$  and  $h_2$  on  $\mathbb{C}\mathbb{P}_1^1$  and  $\mathbb{C}\mathbb{P}_2^1$  defines a fibration on  $Q - B_0$  whose generic fiber is a lagrangian torus.*

Indeed, for the open part  $Q - B_0$  we have a collection of functions  $(f_1, f_2, f_3, \psi_1^*h_1, \psi_2^*h_2)$  and the point is that they commute. The commutation relation  $\{f_i, f_j\}_\omega = 0$  is obvious, the commutation relation for  $f_i$  and  $\psi_j^*h_j$  follows from the same argument as in Theorem 1 from [4], and finally the commutation relation for  $\psi_1^*h_1$  and  $\psi_2^*h_2$  follows from the fact that the fibers of  $p_1$  and  $p_2$  are symplectically orthogonal being complex and transversal. Thus the open part of  $Q$  carries a complete set of first integrals which doesn't mean in general that a general fiber must be compact. But here in our situation a generic fiber can be constructed just by hands. To do this first choose two loops  $\gamma_i \subset \mathbb{C}\mathbb{P}_i^1$  which are the level sets for certain values of

$h_1$  and  $h_2$  such that  $\gamma_i$  avoids the points which correspond to singular fibers. Since the number of singular fibers is finite for both  $\psi_i$  we do not leave the general case having such suggestions. Fix a pair of points  $p_1 \in \mathbb{C}\mathbb{P}_1^1, p_2 \in \mathbb{C}\mathbb{P}_1^1$  and consider the intersection

$$N^0(p_1, p_2) = \psi_1^*(p_1) \cap \psi_2^*(p_2) \subset Q - B_0$$

which is a part of a smooth complex submanifold  $N(p_1, p_2) \subset Q$  which is already *toric* with the integrals  $f_i|_{N(p_1, p_2)}$ . Choose certain regular values of  $f_1, f_2, f_3$  and consider the mutual level set

$$N(p_1, p_2) \supset T_{(p_1, p_2)} = \{f_i|_{N(p_1, p_2)} = c_i\}.$$

Note that for general values  $c_i$  this subset lies in  $N^0(p_1, p_2)$ .

Now let  $p_i$  moves along  $\gamma_i \subset \mathbb{C}\mathbb{P}_i^1$ . Collect

$$T = \cup_{p_1 \in \mathbb{C}\mathbb{P}_1^1, p_2 \in \mathbb{C}\mathbb{P}_2^1} T_{(p_1, p_2)}$$

to a smooth compact 5 - dimensional torus. And this torus is lagrangian since the tangent space  $T_s T$  at each point is spanned by the Hamiltonian vector fields  $X_{f_i}, X_{\psi_j^* h_j}$  of commuting functions. It completes the proof.

**Remark.** The hamiltonian vector field  $X_{\psi_j^* h_j}$  is ill defined at the singular points of the fibers. The point is that the symplectic connection defined in the fibers of  $\psi_i$  is not defined at that points. But at the same time the singular sets of singular fibers are contained by the degeneration sets of the integrals  $f_1, \dots, f_k$ , restricted to singular fibers. Thus the dimension of the singular sets is less that  $k - 1$ .

Note that now we can apply for this situation all we did do above to establish

— displacability of the fibers in connection with the Bohr – Sommerfeld condition;

— the monodromy computation in terms of vanishing generators.

To formalize the story we need the following

**Definition 3.1** Let  $X$  be a Fano variety of complex dimension  $n$  endowed with a compatible symplectic structure coming from the pluri anticanonical embedding. Let  $(f_1, \dots, f_k, \{D_{u_1}^1\}, \dots, \{D_{u_{n-k}}^{n-k}\})$  be a set of data combined from

(**r**) functionally independent real Morse functions  $f_i$  whose Hamiltonian vector fields preserve the Kahler structure, which commute to each other;

(c) pencils  $\{D_{u_j}^j\}$  without base component of generically smooth divisors in involution with the associated holomorphic maps  $\psi_j : X - B_j \rightarrow \mathbb{C}\mathbb{P}_{u_j}^1$  (here  $B_j$  is the corresponding base set), where "in involution" means that the elements of pencils are transversal outside of singular and base sets;

such that the compatibility condition holds: every  $f_i$  preserves every element of each pencil. Then we will call the data  $(f_1, \dots, f_k, \{D_{u_1}^1\}, \dots, \{D_{u_{n-k}}^{n-k}\})$  a pseudo toric structure on  $X$ .

A Fano variety  $X$  is called *pseudo toric* if it admits such a structure.

One can attach to a pseudo toric structure certain integer valued invariants. Here we introduce

**Definition 3.2** The number of invariant pencils  $\psi_j$  is called the rank of the pseudo toric structure.

Note that the intersection of generic elements of the invariant pencils

$$D_{u_1}^{k+1} \cap \dots \cap D_{u_n}^n$$

is a smooth toric variety, so the meaning of the definition is that we slice our non toric  $X$  by toric varieties and then we exploit some toric geometry to get lagrangian submanifolds inside of  $X$  or to get lagrangian fibrations on it.

Then we have the following reminiscence of Proposition 3.3 above

**Theorem 3.1** *Let  $X$  be a pseudo toric Fano variety with a fixed pseudo toric structure  $(f_1, \dots, f_k, \{D_{u_1}^1\}, \dots, \{D_{u_{n-k}}^{n-k}\})$  and the projective line  $\mathbb{C}\mathbb{P}_{u_j}^1$  parameterizes the corresponding pencil  $\{D_{u_j}^j\}$ . Then the choice of a set of Morse functions  $h_j \in C^\infty(\mathbb{C}\mathbb{P}_{u_j}^1, \mathbb{R})$  on each  $\mathbb{C}\mathbb{P}_{u_j}^1$  induces an isotropic fibration of  $X$  whose generic fiber is a smooth lagrangian torus.*

The proof repeats the arguments we used in Theorem 1 in [4] and Proposition 3.3 above. Let  $(h_1, \dots, h_{n-k})$  is a set of Morse functions on  $\mathbb{C}\mathbb{P}_{u_1}^1, \dots, \mathbb{C}\mathbb{P}_{u_{n-k}}^1$ . For the associated to the pencils  $\{D_{u_j}^j\}$  holomorphic maps  $\psi_j$  consider the liftings  $\psi_j^* h_j$  which are functions on  $X - B_0$  where

$$B_0 = B_1 \cup \dots \cup B_{n-k}$$

is the union of the base sets. Note that the functions  $\psi_j^* h_j$  are not smooth on  $X - B_0$  — the singular set of a singular fiber from  $\{D_{u_j}^j\}$  automatically is the singular set of the function. But we suppose that the fibers are generically

smooth (so the set of singular fibers is finite since the parameterizing space is the projective line). Denote the union of singular sets of all singular fibers as  $\text{Sing}$ , so  $\text{Sing} \subset X$ . Thus the complement  $X - (B_0 \cup \text{Sing})$  carries the set  $(f_1, \dots, f_k, \psi_1^* h_1, \dots, \psi_{n-k}^* h_{n-k})$  of smooth functions. Moreover, it is not hard to see that the functions commute. Indeed,  $f_j$  and  $f_j$  commute by the definition of pseudo toric structure. The Hamiltonian vector fields  $X_{\psi_j^* h_j}$  can be derived by the following observation: since the fibers of a map  $\psi_j$  are holomorphic and therefore symplectic there exists a natural symplectic connection  $\nabla_j$  which is defined outside of  $\text{Sing}$ . Then the Hamiltonian vector field  $X_{\psi_j^* h_j}$  on  $X - (B_0 \cup \text{Sing})$  coincides with the lifting of the Hamiltonian vector field  $X_{h_j}$  from  $\mathbb{C}\mathbb{P}_{u_j}^1$  defined by the symplectic connection  $\nabla_j$ . Consequently, the Hamiltonian vector field  $X_{\psi_j^* h_j}$  is symplectically orthogonal to the fibers of  $\psi_j$  while the Hamiltonian vector fields  $X_{f_i}$ 's are parallel to the fibers. It implies that

$$\omega(X_{f_i}, X_{\psi_j^* h_j}) = 0 = \{f_i, \psi_j^* h_j\}_\omega$$

on  $X - (B_0 \cup \text{Sing})$ . Furthermore, the fibers of  $\psi_i$  and  $\psi_j$  are holomorphic and transversal by the definition and therefore they are symplectically orthogonal which implies that

$$\omega(X_{\psi_i^* h_i}, X_{\psi_j^* h_j}) = 0 = \{\psi_i^* h_i, \psi_j^* h_j\}_\omega$$

on  $X - (B_0 \cup \text{Sing})$ .

Note, that the Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_k}, X_{\psi_1^* h_1}, \dots, X_{\psi_{n-k}^* h_{n-k}}$  are linear independent almost everywhere; the last fields degenerate at the fibers over critical points of  $h_j$ . These critical fibers can be jointed in own big reducible divisor

$$D = D_{p_1}^1 \cup \dots \cup D_{p_{l_1}}^1 \cup \dots \cup D_{p_1}^{n-k} \cup \dots \cup D_{p_{l_{n-k}}}^{n-k}$$

where  $p_1, \dots, p_{l_j}$  are the critical points of  $h_j$  on  $\mathbb{C}\mathbb{P}_{u_j}^1$ . If we denote as  $\Delta_k$  the "degeneration simplex" of the set  $(f_1, \dots, f_k)$  so

$$\Delta_k = \{X_{f_1} \wedge \dots \wedge X_{f_k} = 0\},$$

then the complement  $X - (B_0 \cup D_0 \cup \Delta_k)$  carries a real polarization so a lagrangian distribution spanned by the Hamiltonian vector fields. However one needs certain additional arguments since not every lagrangian distribution is integrable.

First observation concerns the base sets of the pencils  $\{D_{u_j}^j\}$  whose elements are preserved by the functions  $f_1, \dots, f_k$ . The compatibility condition implies that each base set  $B_j$  is invariant under the Hamiltonian action of each  $f_i$ . Moreover, the same condition implies that the singular set  $\text{Sing}_0$  is invariant under each  $f_i$ . The real codimension of each  $B_j$  is 4. The real codimension of each  $\text{Sing}_j$  is less or equal to 4. Consider the "action map"

$$F_a = (f_1, \dots, f_k) : X \rightarrow \mathbb{R}^k$$

and denote the image  $F_a(X)$  as  $P_k$ . It is a connected bounded  $k$ -dimensional domain in  $\mathbb{R}^k$ ; the image of  $B_0$  is the union of  $F_a(B_j)$  and the invariance of  $B_j$  with respect to each  $X_{f_i}$  means that each  $F_a(B_j)$  has real codimension 2 in  $P_k$ . The same reason gives the real codimension of  $F_a(\text{Sing}_j)$  is less or equal to 2. Consequently for each generic inner point of  $P_k$  representing by a set of non critical values  $(c_1, \dots, c_k)$  the mutual level set

$$S_{(c_1, \dots, c_k)} = \{f_i = c_i\}$$

is a smooth compact coisotropical manifold which intersects neither  $B_0$  nor  $\text{Sing}_0$ . Indeed, the images of  $B_0$  and  $\text{Sing}_0$  have codimensions greater than one, and this means that a generic point lies neither in  $F_a(B_0)$  nor in  $F_a(\text{Sing}_0)$ . At the same time since the boundary  $\partial P_k$  is the image of  $\Delta_k$ , every  $X_{f_i}$  is non vanishing on  $S_{(c_1, \dots, c_k)}$ .

Further, since  $S_{(c_1, \dots, c_k)}$  doesn't touch  $B_0$  and  $\text{Sing}_0$  the functions  $(f_{k+1}, \dots, f_n)$  such that

$$f_{k+j} = \psi_j^* h_j|_{S_{(c_1, \dots, c_k)}}$$

are correctly defined and smooth; the commutation relations for these restricted functions still hold with respect to the Poisson brackets. Now choose a set of non critical values  $(c_{k+1}, \dots, c_n)$  of  $h_1, \dots, h_{n-k}$  which possess the following property: no level loop

$$\mathbb{C}\mathbb{P}_{u_j}^1 \supset \gamma_j = \{h_j = c_{k+j}\}$$

passes through a point corresponding to the singular fiber (and obviously this choice is generic). Consider the mutual level set

$$S_{(c_1, \dots, c_k)} \supset T_{(c_1, \dots, c_k, c_{k+1}, \dots, c_n)} = \{f_{k+i} = c_{k+i}, i = 1, \dots, n - k\}.$$

The set is smooth compact submanifold of  $X$  of real dimension  $n$ . The smoothness follows from the fact that non critical level sets of commuting algebraically independent Morse functions must intersect each other transversally. On the other hand there are  $n$  linearly independent non vanishing

Hamiltonian vector field, parallel to  $T_{(c_1, \dots, c_n)}$ , and this shows that it is a lagrangian torus.

What remains in  $X$  outside of smooth lagrangian tori? Either singular tori which corresponds to saddle critical points of  $h_j$  and to level loops which pass through the points which correspond to singular fibers or isotropical submanifolds which slice the divisors which lie over focal critical points of  $h_j$ . Anyway the fibration of whole  $X$  is rather complicated and singular, but it is defined. This complete the proof.

We will call a smooth lagrangian torus  $T \subset X$  *pseudo toric fiber* if it is given by the construction for a set of compatible data  $(f_1, \dots, f_k, \{D_{u_1}^{k+1}\}, \{D_{u_{n-k}}^n\})$  for certain Morse functions  $h_1, \dots, h_{n-k}$  on  $\mathbb{C}\mathbb{P}_{u_1}^1, \dots, \mathbb{C}\mathbb{P}_{u_{n-k}}^1$ . We will call the corresponding pseudo toric fibration *simple* if it is defined by Morse functions  $h_1, \dots, h_{n-k}$  such that each  $h_j$  has exactly two critical points. The simplicity implies that the singularities of lagrangian tori come only from the singularities of the pencils. The functions  $f_1, \dots, f_k$  in the definition can be taken of more general type namely with non isolated critical points but this case requires more delicate analysis.

It is natural to extend the statement of Theorem 0.1 to the following

**Conjecture.** *A pseudo toric fiber  $T \subset X$  in a pseudo toric Fano variety is undisplacable only if it is Bohr – Sommerfeld with respect to the anticanonical class.*

But let's come back to our examples. Examples 1 and 2 can be easily summarized by the following

**Theorem 3.2** *Any smooth quadric  $Q$  admits pseudo toric structure.*

The example with 5 - dimensional quadric in  $\mathbb{C}\mathbb{P}^6$  was detailed above and the proof of the theorem just follows the pattern. Dependent on the parity of  $n$  the equation of the quadric is either

$$z_0^2 + z_1z_2 + \dots + z_{n-1}z_n = 0$$

if  $n$  is even, or

$$z_0z_1 + \dots + z_{n-1}z_n = 0$$

if it is odd. In the first case the space of symbols which preserve  $Q$  is  $\frac{n}{2}$  - dimensional; in the second it is  $\frac{n+1}{2}$  - dimensional. Putting the coefficients

$$w_0z_0^2 + w_1z_1z_2 + \dots + w_{\frac{n}{2}}z_{n-1}z_n = 0$$



in the first case and

$$w_0 z_0 z_1 + \dots + w_{\frac{n-1}{2}} z_{n-1} z_n = 0$$

in the second we get a  $\mathbb{C}\mathbb{P}_w^{\frac{n}{2}}$  of quadrics which are invariant with respect to the hamiltonian action of the same symbols in the first case and a  $\mathbb{C}\mathbb{P}_w^{\frac{n-1}{2}}$  in the second. For the first case one can take in this  $\mathbb{C}\mathbb{P}_w^{\frac{n}{2}}$  a set of  $\frac{n}{2} - 1$  skew projective lines which do not pass through the point  $[1 : \dots : 1]$ . These lines will be our invariant pencils. For the second case one can take in this  $\mathbb{C}\mathbb{P}_w^{\frac{n-1}{2}}$  a set of  $\frac{n-3}{2}$  skew projective lines which do not pass the point with the same coordinates, and for this case these lines will be our invariant pencils. The transversality of the corresponding divisors follows from the same arguments as were placed for the dimension 5.

Now a natural question arises:

**Problem:** *Which Fano varieties admits pseudo toric structure?*

It's not hard to see that any toric Fano variety is automatically pseudo toric. The case of non degenerated quadric considered in Theorem 3.2 can be extended to complete intersections, projectivizations of certain vector bundles. At the same time another natural question arises:

**Problem:** *How many different pseudo toric structures a Fano variety can admit?*

**Example 3.** Consider 4 - dimensional non singular quadric  $Q \subset \mathbb{C}\mathbb{P}^5$  and choose homogenous coordinates  $[z_0 : \dots : z_5]$  such that  $Q$  is given by the equation

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4 z_5 = 0.$$

There is only one diagonal symbol  $f$  which preserve  $Q$  given by

$$f = \frac{|z_4|^2 - |z_5|^2}{|z|^2},$$

and it is degenerated — it has one dimensional critical subset  $q_{cr} \subset Q$ . However there are several invariant pencils defined by the intersections

$$D_{u_1} = Q \cap \{\alpha z_0 + \beta z_1 = 0\},$$

$$D_{u_2} = Q \cap \{\alpha z_1 + \beta z_2 = 0\}$$

and

$$D_{u_3} = Q \cap \{\alpha z_2 + \beta z_3\}.$$

It's not hard to see that the set  $(f, \psi_1, \psi_2, \psi_3)$  where  $\psi_i$  is given by the pencil  $\{D_{u_i}\}$  defines a pseudo toric structure on  $Q$ . And this structure obviously is different from the one given by Theorem 3.2: they have different ranks (and of course moreover they have different singular fibers, different monodromy etc.).

## Final remarks

Recall that all the story started at the Auroux example where the projective plane was fibered by *special* lagrangian tori, see [3]. The specialty condition for a lagrangian tori is famous in the Mirror Symmetry setup, see f.e. [8], but it arises for lagrangian tori in Calabi – Yau manifolds. D. Auroux proposes a way how to generalize the notion of special lagrangian tori for the case of Fano variety (or, more precisely, for *an open Calabi – Yau manifold*). If a Fano variety  $X$  is endowed with an element of the anticanonical system  $D \subset |-K_X|$ , then the complement  $X - D$  is endowed with a top holomorphic form  $\theta_D$ , defined up to scaling. This form can be extended to a meromorphic top form with pole at  $D$ . Then if the complement  $X - D$  is fibered by lagrangian tori then one can impose on the fibers the specialty condition with respect to  $\theta_D$ . The main conjecture, presented in [3], states that for a Fano variety  $X$  and an element  $D$  there exists a fibration on special lagrangian tori (with possible singular fibers, of course). Therefore our main interest in the pseudo toric setup is in the examples of pseudo toric lagrangian fibrations of Fano varieties without elements from the anticanonical systems.

If a Fano variety  $X$  admits a pseudo toric structure  $(f_1, \dots, f_k, \{D_{u_1}^1\}, \dots, \{D_{u_{n-k}}^{n-k}\})$  then one can ask is it possible to realize a special lagrangian fibration in the way presented by the choice of certain  $h_j$ 's. The first example is given in [3] by D. Auroux, and here we present another

**Example 5.** Consider a non degenerated 4 - dimensional quadric  $Q \subset \mathbb{C}\mathbb{P}^5$  and a homogenous coordinate system  $[z_0 : \dots : z_5]$  where  $Q$  is given by the equation

$$z_0z_1 + z_2z_3 + z_4z_5 = 0.$$

Take three non degenerated symbols (moment maps) which preserve the quadric  $f_1, f_2, f_3$  as in the proof of Theorem 3.2; the degeneration simplex  $\Delta_3 \subset Q$  is formed by 8 two dimensional planes which lie on  $Q$  plus 3 two dimensional quadrics. We distinguish these two parts of  $\Delta_3$  denoting as  $B$  the union of these planes and as  $Sing$  the union of these 3 quadrics. Removing

$B$  from  $Q$ , we can consider holomorphic map

$$\psi : Q - B \rightarrow \mathbb{C}\mathbb{P}_u^1,$$

where  $\mathbb{C}\mathbb{P}_u^1$  is a projective line in  $\mathbb{C}\mathbb{P}_w^2$  with homogenous coordinates  $[w_0 : \dots : w_2]$  defined by the equation

$$\mathbb{C}\mathbb{P}_u^1 = \{w_0 + w_1 + w_2 = 0\} \subset \mathbb{C}\mathbb{P}_w^2.$$

The map  $\psi$  is given by

$$w_0 = z_0 z_1, \quad w_1 = z_2 z_3, \quad w_2 = z_4 z_5$$

and is ill defined exactly on  $B$ . It's not hard to see that the fibers of  $\psi$  are invariant with respect to the Hamiltonian action of each  $f_i$  and that each fiber can be compactified to the zero set of a section of the line bundle  $2H$  over  $Q$  and our 8 projective planes are the base set of the corresponding pencil of invariant divisors  $\{D_u\}$  (therefore we denoted it as  $B$ ). This pencil has exactly three distinguished elements:

$$— \psi^{-1}([0 : 1 : -1]), \psi^{-1}([1 : 0 : -1]), \psi^{-1}([-1 : 1 : 0]),$$

each of them is presented by two cones over a non degenerated two dimensional quadric which is the singular set of the fiber, and it is the reason why we denoted this part of  $\Delta_3$  as Sing. The smooth fibers of  $\psi$  have  $H_1(D_u^0, \mathbb{Z})$  of rank three, and passing to a singular fiber one kills a generator of  $H_1(D_u^0, \mathbb{Z})$  which is the corresponding vanishing generator of the sigular fiber.

Now we claim that the choice of any symbol (moment map)  $h$  on  $\mathbb{C}\mathbb{P}_u^1$  induces a special lagrangian fibration on  $Q$ . To see this note first that a symbol  $h$  has exactly two critical points  $p_{\max}, p_{\min}$  on  $\mathbb{C}\mathbb{P}_u^1$  and this critical points give two elements  $D_{\max}, D_{\min}$  in our pencil  $\{D_u\}$ . Therefore according to Theorem 3.1 the choice of  $h$  defines a lagrangian fibration of  $Q - (D_{\max} \cup D_{\min})$ . The union

$$D = D_{\max} \cup D_{\min}$$

is a reducible divisor from the anticanonical system  $|-K_Q|$ . Indeed, by the adjunction formula

$$-K_Q = 4H,$$

and each  $D_*$  presents a section of  $2H$ .

To prove the statement we take the corresponding holomorphic form  $\theta_D$  and observe that it is invariant under the action of the moment maps  $f_1, f_2, f_3$

(since the divisor  $D$  is moved to itself under the action). At the same time we have a family of holomorphic automorphisms of  $Q - B$  defined by the rotations generated by our  $h$ , and the point is that  $\theta_D$  is invariant as well with respect to these automorphisms. Thus the restriction of  $\theta_D$  to a torus is a constant multiple of the volume form, but the norm of this constant is different for different level loops of  $h$  on  $\mathbb{C}\mathbb{P}_u^1$  since the Fubini - Study metric on  $\mathbb{C}\mathbb{P}_u^1$  and the Kahler metric on  $Q - B$  are related by the scaling on a multiple of the norm of the Hamiltonian vector field  $X_h$  on  $\mathbb{C}\mathbb{P}_u^1$ . However the phase is the same for all smooth lagrangian torus in the pseudo toric fibration, and this means that the fibration is special.

Note that the construction of Theorem 3.2 doesn't lead to the same observation for any quadric  $Q$  due to the cohomological reason, but it can be exploited in the studies of special lagrangian fibrations of  $n$  - dimensional quadric. At the same time recall that  $Q \subset \mathbb{C}\mathbb{P}^5$  is the Grassman variety  $Gr(2, 4)$ , and the same technique can be applied in the case of  $Gr(2, k)$ .

The introduction of pseudo toric structures makes it possible to extend a number of approaches to several problems and conjectures adopted for toric Fano varieties.

For Geometric Quantization programme applied to a given Fano variety  $X$  which admits pseudo toric structures one can take sufficiently generic pseudo toric fibrations and consider as usual the fibers which are Bohr - Sommerfeld with respect to certain appropriate polarization (f.e. with respect to the anticanonical bundle and its powers). Then a natural and very interesting question arises on the number of Bohr - Sommerfeld fibers which must be the same for different pseudo toric fibrations.

For Mirror Symmetry programme applied to a non toric Fano variety  $X$  which admits pseudo toric structures one can extend a standard approach taking the fibers of a pseudo toric fibrations which have non trivial Floer cohomology to itself and then construct certain  $A_\infty$  category based on these distinguished fibers.

For the integrable systems as well pseudo toric structures can help in solving real dynamical systems on homogenous spaces, grassmanian or certain spherical varieties. But for us the main interest comes with the following speculations. Having a pseudo toric structure  $(f_1, \dots, f_k, \{D_{u_1}^1\}, \dots, \{D_{u_{n-k}}^{n-k}\})$  one can try to find a characteristic classes  $c_1, \dots, c_{n-k}$  such that

$$c_i \in H^{2i}(X, \mathbb{Z}).$$

Indeed, for a pseudo toric structure one has a distinguished set in  $H^2(X, \mathbb{Z})$

which contains  $n - k$  classes Poincare dual to the homology classes of  $D_{u_j}$ . We will call these classes *the Chern roots* of the corresponding pseudo toric structure. Twisting these data by certain relations coming from the topology of the base sets intersections one can derive classes  $c_i$  which are analogies of the Chern classes of a vector bundle. These data can be combined together in the form of a vector

$$(r, c_1, \dots, c_r) \in H^{2*}(X, \mathbb{Z}).$$

Such a vector can be realized by pseudo toric lagrangian fibrations with the fixed topological data, and one can consider an equivalence relation on the space of all such fibrations, given by the Hamiltonian isotopies. Then certain *moduli spaces* appear which can be studied...

Why this idea comes? Sometimes ago one understood Mirror Symmetry as certain duality between vector bundles and lagrangian cycles. But as far as we know from Geometric Quantization vector bundle is not equal to lagrangian cycle since a quantum state is represented either by a *section* (up to scaling) of a vector bundle or by a lagrangian cycle. This means that a vector bundle can be compared with a lagrangian fibration. And as we guess above certain lagrangian fibrations — namely pseudo toric ones — admit some topological characteristics rather familiar for the theory of moduli spaces of vector bundles.

All the arguments are rather speculative, but we hope to continue this work.

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