The automorphism group of a generic Jacobian Kummer surface

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Abstruct

We shall prove that the automorphism group of the Kummer surface associated with a generic curve of genus two is generated by classical involutions and Keum's new automorphisms. The proof depends on a description of a fundamental domain of the reflection group of even unimodular lattice of signature (1,25) and a geometry of Leech lattice.

Introduction

Let C be a smooth curve of genus two over the complex number field. The Jacobian variety $J(C)$ of C has a natural involution ι and the quotient variety $J(C)/<\iota>$ has sixteen ordinary nodes. This surface can be embedded as a quartic surface F in $\mathbf{P}^3$. The minimal resolution X of $J(C)/<\iota>$ is called the *Kummer surface* associated with C which is a $K3$ *surface*, that is, a simply connected complex surface with trivial canonical line bundle. The X contains sixteen mutually disjoint smooth rational curves $\{N_\alpha\}$ obtained by the resolution of singularities. Also X contains another sixteen mutually disjoint smooth rational curves $\{T_\alpha\}$ which are the images of the theta divisor and its translations by 2-torsion points. The configuration of these thirty-two curves is called *Kummer* $(16)_6$ *configuration*. We call X *generic* if the Néron-Severi group of $J(C)$ is generated by the class of C.

At the last century it was known that a generic Kummer surface X has many involutions, that is, *sixteen translations* induced by those of $J(C)$ by 2-torsion points, *sixteen projections* of F from nodes, *sixteen corellations* by means of the tangent plane collinear to a trope, and a *switch* defined by the dual map of F (Klein [9]). In 1900, Hutchinson [6] found another sixty involutions associated with Göpel tetrads. Since Hutchinson, no other automorphisms have been provided, and it was conjectured that the automorphism group of X is generated by the above involutions.

However, recently, Keum [8] has given new 192 automorphisms of infinite order which are not included in the group generated by classical involutions. The purpose of this paper is to prove:

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Theorem. *The automorphism group of a generic Kummer surface is generated by classical involutions and Keum's automorphisms.*

The proof depends on a description of the reflection group of even unimodular lattice of signature (1,25) due to Conway [3] and is inspired by a paper of Borcherds [1] in which he calculated a fundamental domain of the reflection group of several lattices of sigunature $(1, r)$ and its automorphism group.

In the following we shall give a sketch of our proof. Let $S = S_X$ be the Picard lattice of X which has signature (1, 16) and determinant 64. Then S can be primitively embedded into an even unimodular lattice L of signature (1,25) such that the orthogonal complement T of S contains a root sublattice isomorphic to $A_3 \oplus A_1^{\oplus 6}$ (Lemma 4.1). A fundamental domain D of the reflection group of L is described explicitly by Conway [3] in terms of Leech lattice (see Proposition 2.5). After Borcherds, we consider the restriction $D' = D \cap P(S)$ where $P(S)$ is a connected component of the set $\{x \in S \otimes \mathbf{R} : \langle x, x\rangle > 0\}$. Then we can determine the hyperplanes which bound D' properly by using a geometry of Leech lattice due to Conway [2]. There are 316 $(= 32 + 32 + 60 + 192)$ such hyperplanes perpendicular to (-2)-, (-4)-, (-4)- or (-12)-vectors in S respectively (Lemmas 4.6 and 4.8). These hyperplanes naturally correspond to thirty-two smooth rational curves on X consisting Kummer $(16)_6$ configuration, sixteen projections and sixteen corellations, sixty Cremona transformations and 192 Keum's automorphisms respectively (§7). Also the projection of the Weyl vector of D corresponds to the ample divisor $\Sigma(N_\alpha + T_\alpha)/4$ on X (Lemma 7.1). Here we remark that each Keum's automorphism κ is not a reflection (it has infinite order), but it works like a reflection. More precisely it sends the half-space defined by the corresponding (-12)-vector to the opposite side of the half-space defined by the (-12)-vector which defines the inverse of κ (Lemma 6.5). We can also see that $Aut(D')$ is an extension of $(\mathbf{Z}/2\mathbf{Z})^5$ by $Sp(4, \mathbf{F}_2)$ where $(\mathbf{Z}/2\mathbf{Z})^5$ is generated by sixteen translations and a switch, and $Sp(4, \mathbf{F}_2) \simeq S_6$ is naturally isomorphic to the group of permutations of six Weierstrass points of the curve C (Lemma 7.2). Moreover D' is like a fundamental domain of the group generated by reflections corresponding to (-2)-vectors in S, projections, corellations, Cremona transformations and Keum's automorphisms (Lemma 7.3). We remark that the reflection group of S_X (corresponding to (-2)-, (-4)-vectors) is not of finite index in $O(S_X)$.

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§1. Lattices

A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\ ,\ \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. Also we denote by $L^{\oplus m}$ the orthogonal direct sum of m-copies of L. An isomorphism of lattices preserving the bilinear forms is called an *isometry*. For a lattice L, we denote by $O(L)$ the group of self-isometries of L. A sublattice S of L is called *primitive* if L/S is torsion free.

A lattice L is *even* if $\langle x, x\rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the discriminant $d(L)$ of its bilinear form is non zero, and *unimodular* if $d(L) = \pm 1$. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalues ± 1 for the quadratic form on $L \otimes \mathbf{R}$.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = Hom(L, \mathbf{Z})$. The factor group L^*/L, which is denoted by A_L, is an abelian group of order $|d(L)|$. We denote by $l(L)$ the number of minimal generators of A_L. We extend the bilinear form on L to the one on L^*, taking value in $\mathbf{Q}$, and define

$$q_L : A_L \to \mathbf{Q}/2\mathbf{Z},\ \ q_L(x + L) \ = \ \langle x, x\rangle + \ 2\mathbf{Z} \ \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L.

Let A be a finite abelian group. A *non-degenerate finite quadratic form* is a map $q : A \longrightarrow \mathbf{Q}/2\mathbf{Z}$ satisfying the following conditions:

(1) $q(na) = n^2 q(a)$ for all $n \in \mathbf{Z}$ and $a \in A$.

(2) $q(a + a') - q(a) - q(a') \equiv 2b(a, a') \bmod 2\mathbf{Z}$,

where $b : A \times A \longrightarrow \mathbf{Q}/2\mathbf{Z}$ is a non-degenerate symmetric bilinear form. We denote by u_2 (resp. v_2) a finite quadratic form on $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ defined by $q(x) = q(y) = 0, b(x, y) = 1/2$ (resp. $q(x) = q(y) = 1, b(x, y) = 1/2$). We also denote by $q_a(2^n)$ a finite quadratic form on $(\mathbf{Z}/2^n\mathbf{Z})$ defined by $q_a(2^n)(x) = a/2^n$. It is easy to see that $u_2^{\oplus 2} \simeq v_2^{\oplus 2}$.

We denote by U the hyperbolic lattice defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is an even unimodular lattice of signature (1,1), and by A_m, D_n or E_l an even negative definite lattice associated with the Cartan matrix of type A_m, D_n or E_l ($m \geq 1$, $n \geq 4$, l = 6,7,8). A *root* of an even lattice L is a (-2)-element in L. A *root lattice* of a negative definite lattice L is the sublattice of L generated by all roots in L, which is isometric to a direct sum of A_m, D_n, E_l. For a lattice L and an integer m, $L(m)$ is the lattice whose bilinear from is the one on L multiplied by m. For example $q_{A_1} \simeq q_{-1}(2), q_{A_3} \simeq q_{-3}(4), q_{D_4} \simeq v_2$ and $q_{U(2)} \simeq u_2$.

Next we recall a structure of the orthogonal group $O(L)$ of an even lattice L of signature $(1, r-1)$. Let $P(L)$ be a connected component of the set $\{x \in L \otimes \mathbf{R} : \langle x, x\rangle > 0\}$. Let Δ be the set of all roots in L. For each δ, an isometry s_δ defined by

$$s_\delta : x \to x \ + \langle x, \delta\rangle\delta$$

is called a *reflection* associated with δ. Let $W(L)^{(2)}$ be the group generated by all reflections associated with $\delta \in \Delta$. Let H_δ be the hyperplane orthogonal to $\delta \in \Delta$. Then $\{ \ H_\delta : \delta \in \Delta \ \}$ divide $P(L)$ into regions. The connected components of $P(L) \setminus (\bigcup H_\delta)$ are called the *chambers* each of which is a fundamental domain with respect to the action of $W(L)^{(2)}$ on $P(L)$. Fix a chamber $D(L)$. Then we have a partition $\Delta = \Delta^+ \cup \Delta^-$ such that Δ^+ consists of roots δ with $\langle \delta, \rho\rangle > 0$ where ρ is an interior point of $D(L)$. Let $G(L)$ be the group of symmetries of $D(L)$. Then $O(L)$ is a split extension of $\{\pm 1\} \cdot W(L)^{(2)}$ by $G(L)$. For more details we refer the reader to Vinberg [15].

§2. Leech lattice

Let Ω be the projective line $PL(23)$ over the field $\mathbf{F}_{23}$: $\Omega = \{\infty, 0, 1, ..., 23\}$. We consider the set $P(\Omega)$ of all subsets of Ω with symmetric difference as a 24-dimensional vector space over $\mathbf{F}_2$. Let $\mathcal{C}$ be the binary Goray code which is a 12-dimensional subspace of $P(\Omega)$ defined by the 24 sets N_i where $N = \Omega \setminus \{x^2 : x \in \mathbf{F}_{23}\}$, $N_\infty = \Omega$ and $N_i = \{n - i : n \in N\}$ for $i \neq \infty$ (see Conway [2]). We call a set in $\mathcal{C}$ a *$\mathcal{C}$-set*. A $\mathcal{C}$-set consists of 0, 8, 12 or 16 elements. A 8-elements $\mathcal{C}$-set (resp. 12-elements $\mathcal{C}$-set) is called an *octad* (resp. *dodecad*), and a system of 6 tetrads is called a *sextet* if it satisfies that the union of any two tetrads is an octad. We denote by $\mathcal{C}(8)$ (resp. $\mathcal{C}(12)$) the set of all octads (resp. dodecads).

Proposition 2.1 (Conway [2], §3, Tables 1, 2). (1) *The number of octads through m-points is* $759, 253, 77, 21, 5$ *or* 1 *for* $m = 0, 1, 2, 3, 4$ *or* 5 *respectively.*

(2) *The number of dodecads through m-points is* $2576, 1288, 616, 280, 120, 48$ *or* 16 *for* $m = 0, 1, 2, 3, 4, 5$ *or* 6 *respectively.*

(3) *For any set of* 4 *elements, there exists a unique sextet.*

The stabilizer subgroup of the binary Goray code in the permutation group of Ω is called the *Mathieu group* (denoted by M_{24}) which is a sporadic simple group of order $2^{10}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11{\cdot}23$.

Next we recall the Leech lattice. Let $\mathbf{R}^{24}$ be spanned by the orthogonal basis ν_i $(i \in \Omega)$, and define ν_S to be $\Sigma\nu_i (i \in \Omega)$ for $S \subset \Omega$. Let Λ be the lattice spanned by the vectors $2\nu_K$ for $K \in \mathcal{C}(8)$ and $\nu_\Omega - 4\nu_\infty$.

Proposition 2.2 (Conway [2], §4, Theorem 2). *The vector* $(\xi_\infty, \xi_0, ..., \xi_{23})$ *is in* Λ *if and only if*

(i) *the coordinates* ξ_i *are all congruent modulo* 2*, to some* m;

(ii) *the set of* i *for which* ξ_i *takes any given value modulo* 4 *is a* $\mathcal{C}$*-set*;

(iii) *the coordinate-sum is congruent to* $4m$ *modulo* 8.

For $x, y \in \Lambda$ *the scalar product* $x{\cdot}y$ *is a multiple of* 8 *and* $x{\cdot}x$ *is a multiple of* 16.

For $x, y \in \Lambda$, define $\langle x, y\rangle = -x{\cdot}y/8$. Then $(\Lambda, \langle , \rangle)$ is an even negative-definite unimodular lattice which is called the *Leech lattice*. We denote by Λ_n the set of all vectors x in Λ with $\langle x, x\rangle = -n$. We remark that Λ_2 is empty.

Proposition 2.3 (Conway-Sloane [4], p.133, Table 4.13). *The complete lists of* Λ_4, Λ_6 *are as follows* :

$$\Lambda_4 = \{(\pm 2^8, 0^{16}), (\pm 3, \pm 1^{23}), (\pm 4^2, 0^{22})\},$$

$$\Lambda_6 = \{(\pm 2^{12}, 0^{12}), (\pm 3^3, \pm 1^{21}), (\pm 4, \pm 2^8, 0^{15}), (\pm 5, \pm 1^{23})\}$$

where the signs are taken to satisfy the conditions in Proposition 2.2.

Finally we recall a structure of the orthogonal group of even unimodular lattice of signature (1, 25). In the following in this paper we write L for an even unimodular lattice of signature (1, 25), which is unique up to isomorphisms and isomorphic to $\Lambda \oplus U$. Fix a decomposition $L = \Lambda \oplus U$. We write (λ, m, n) for a vector in L where λ in Λ, m, n are integers, the norm is given by $2mn + \langle \lambda, \lambda \rangle$. If $w = (0, 0, 1)$, then a root r in L with $\langle r, w \rangle = 1$ is called a *Leech root* and w is called a *Weyl vector.* The set of all Leech roots bijectively corresponds to the set Λ as follows (Conway-Sloane [4], Chap. 26, Theorem 3):

$$L \ni r = (\lambda, 1, -1 - \langle \lambda, \lambda \rangle / 2) \longleftrightarrow \lambda \in \Lambda.$$

Remark 2.4. For Leech roots $r, r' \in L$ and the corresponding vectors λ, λ' in Λ,

$$\langle r, r' \rangle = \begin{cases} 0, & \text{if } \lambda - \lambda' \in \Lambda_4; \\ 1, & \text{if } \lambda - \lambda' \in \Lambda_6. \end{cases}$$

Put

$$D = \{x \in P(L) \mid \langle x, r \rangle > 0 \text{ for any Leech root } r\}.$$

Proposition 2.5 (Conway [3]). *D is a fundamental domain of the reflection group $W(L)^{(2)}$ and $O(L)$ is a split extension of $W(L)^{(2)}$ by $Aut(D)$ so that $Aut(D)$ is isomorphic to the group $\cdot\infty$ of affine automorphisms of Λ, which is a split extension of $\mathbf{Z}^{24}$ by $\cdot 0 = O(\Lambda)$*

§3. Kummer surfaces

Let C be a compact Riemann surface of genus two. There is a double covering $\pi : C \longrightarrow \mathbf{P}^1$ ramified at six points $p_0, ..., p_5$. We denote by ι the covering transformation of π. Let $J(C)$ be the Jacobian variety of C. The map

$$\varphi : C \longrightarrow J(C)$$

defined by $\varphi(p) = [p - p_0]$ gives an embedding of C into $J(C)$, and the points

$$\mu_i = [p_i - p_0], i = 0, ..., 5, \ \mu_{ij} = [p_i + p_j - 2p_0], 1 \leq i < j \leq 5$$

are 2-torsion points of $J(C)$. We identify the group of 2-torsion points of $J(C)$ with the set of their indices, that is, $J(C)_2 = \{i, jk : 0 \leq i \leq 5, 1 \leq j < k \leq 5\}$. Let

$$\Theta = \Theta_0$$

be the theta divisor and put

$$\Theta_i = \Theta + \mu_i, \Theta_{ij} = \Theta + \mu_{ij}.$$

Then each of these divisors contains six 2-torsion points and, conversely, each 2-torsion point is contained in six of these divisors. The $J(C)$ has a natural involution $z \longrightarrow -z$ with $\varphi(\iota(p)) = -\varphi(p)$.

Let $X = Km(J(C))$ be the minimal resolution of the quotient surface $J(C)/ < \iota >$, which is called the *Kummer surface* associated with C. The X has mutually disjoint sixteen smooth rational curves $\{N_\alpha\}$ obtained by resolution of singularities corresponding to 2-torsion points μ_α. Also X contains mutually disjoint sixteen smooth rational curves $\{T_\beta\}$ which are the images of $\Theta_\beta, \beta = i, ij$. Each member of $\{N_\alpha\}$ meets exactly six members of $\{T_\beta\}$ and vice versa. More precisely,

$$\langle N_\alpha, T_\beta \rangle = 1 \text{ if and only if } \alpha - \beta \in \{0, 1, 2, 3, 4, 5\}$$

This configuration is called *Kummer* $(16)_6$ *configuration.*

The linear system $\mid 2\Theta \mid$ gives a 2 to 1 morphism from $J(C)$ to a quartic surface F in $\mathbf{P}^3$ with sixteen ordinary nodes. The minimal resolution of F is isomorphic to $X = Km(J(C))$ and $\{N_\alpha\}$ are exceptional divisors. There are sixteen planes in $\mathbf{P}^3$ each of which touches F along a conic. These conics (called *tropes*) are the images of T_β. Let H be a divisor on X which is the pull back of the hyperplane section of F. Then the linear system

$$\mid 2H - (1/2)\Sigma N_\alpha \mid = \mid \Sigma(N_\alpha + T_\alpha)/4 \mid$$

gives an embedding of X onto a complete intersection of three quadrics in $\mathbf{P}^5$. For more details, we refer the reader to Hudson [7], Griffiths-Harris [5].

It is well known that $X = Km(J(C))$ is a $K3$ surface, i.e. it is simply connected and its canonical line bundle is trivial. There is a canonical structure of lattice on $H^2(X, \mathbf{Z})$ with the cup product which is even unimodular lattice of signature (3, 19), and hence isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$. Let S_X be the *Picard lattice* of X:

$$S_X = \{x \in H^2(X, \mathbf{Z}) : \langle x, \omega_X \rangle = 0\}$$

where ω_X is a nowhere vanishing holomorphic 2-form on X. Let T_X be the orthogonal complement of S_X, which is called the *transcendental lattice* of X. In our case, X is algebraic and contains linearly independent sixteen smooth rational curves. Hence $rank(S_X) \geq 17$. We call X *generic* if the Néron-Severi group of $J(C)$ is generated by the class of C. In this case $T_X \simeq U(2)^{\oplus 2} \oplus \langle -4 \rangle$ and S_X is an even lattice of signature (1, 16) with $q_{S_X} \simeq u_2^{\oplus 2} \oplus q_1(4)$.

Lemma 3.1 (Keum [8], Lemma 3.1). *Assume that X is generic. Then*

(i) *$\{H, N_\alpha\}$ is a $\mathbf{Q}$-base of S_X, and $\{N_\alpha, T_\beta\}$ generate S_X over $\mathbf{Z}$.*

(ii) *A_{S_X} is generated by* $(N_2 + N_{12} + N_3 + N_{13})/2, (N_1 + N_{12} + N_4 + N_{24})/2, (N_2 + N_{12} + N_4 + N_{14})/2, (N_1 + N_{12} + N_3 + N_{23})/2, H/4 + (N_0 + N_1 + N_2 + N_{12})/2$.

Lemma 3.2 (Keum [8], (1.8)). *T_α can be written as a linear combination of $\{H, N_\alpha\}$ as follows :*

$T_0 = (H - N_0 - N_1 - N_2 - N_3 - N_4 - N_5)/2$

$$T_1 = (H - N_0 - N_1 - N_{12} - N_{13} - N_{14} - N_{15})/2$$
$$T_2 = (H - N_0 - N_2 - N_{12} - N_{23} - N_{24} - N_{25})/2$$
$$T_{12} = (H - N_1 - N_2 - N_{12} - N_{45} - N_{35} - N_{34})/2$$
$$T_3 = (H - N_0 - N_3 - N_{13} - N_{23} - N_{35} - N_{34})/2$$
$$T_{13} = (H - N_1 - N_3 - N_{13} - N_{45} - N_{24} - N_{25})/2$$
$$T_{23} = (H - N_2 - N_3 - N_{23} - N_{45} - N_{14} - N_{15})/2$$
$$T_{45} = (H - N_{12} - N_{13} - N_{23} - N_{45} - N_4 - N_5)/2$$
$$T_4 = (H - N_0 - N_{45} - N_4 - N_{14} - N_{24} - N_{34})/2$$
$$T_{14} = (H - N_1 - N_{23} - N_4 - N_{14} - N_{35} - N_{25})/2$$
$$T_{24} = (H - N_2 - N_{13} - N_4 - N_{24} - N_{35} - N_{15})/2$$
$$T_{35} = (H - N_{12} - N_3 - N_{14} - N_{24} - N_{35} - N_5)/2$$
$$T_{34} = (H - N_{12} - N_3 - N_4 - N_{34} - N_{25} - N_{15})/2$$
$$T_{25} = (H - N_2 - N_{13} - N_{14} - N_{34} - N_{25} - N_5)/2$$
$$T_{15} = (H - N_1 - N_{23} - N_{24} - N_{34} - N_{15} - N_5)/2$$
$$T_5 = (H - N_0 - N_{45} - N_{35} - N_{25} - N_{15} - N_5)/2.$$

Definition 3.3. A *standard graph* Γ is a graph which is isomorphic to the dual graph of thirty-two nodal curves $\{N_\alpha, T_\beta\}$. We denote by $B(\Gamma)$ (resp. $R(\Gamma)$) the set of vertices (resp. edges) of Γ.

Let S be a lattice which has a generator system $\{\xi_a\}_{a \in B(\Gamma)}$ satisfying

$$\langle \xi_a, \xi_b \rangle = \begin{cases} -2, & \text{if } a = b; \\ 1, & \text{if } \{a, b\} \in R(\Gamma); \\ 0, & \text{if } a \neq b, \{a, b\} \notin R(\Gamma). \end{cases}$$

Obviously S is isometric to S_X for a generic Kummer surface X. We remark here that the automorphism group $Aut(\Gamma)$ of Γ is an extension of $(\mathbf{Z}/2\mathbf{Z})^5$ by S_6 (Nikulin [10], Corollary 2.7). By Lemma 3.1, $Aut(\Gamma) \subset O(S)$. It is known (Nikulin [10], Theorem 4.5) that

$$\xi = \Sigma_{a \in B(\Gamma)} \xi_a / 4 \in S.$$

Proposition 3.4 (Nikulin [10]). *Let X be a $K3$ surface whose Picard lattice S_X contains S as a sublattice. Assume that*

$$\langle \xi, r \rangle \neq 0 \textit{ for any } r \in S_X \textit{ with } \langle r, r \rangle = -2.$$

Then X is the Kummer surface associated with a curve of genus two. Moreover there exists an isometry ϕ in $W(S_X)^{(2)}$ such that $\{\phi(\xi_a)\}$ is the set of the classes of thirty-two nodal curves $\{N_\alpha, T_\beta\}$ on X.

Let $P(S_X)$ be the connected component of the cone $\{x \in S_X \otimes \mathbf{R} : \langle x, x \rangle > 0\}$ which contains an ample divisor. Put

$$D(S_X) = \{x \in P(S_X) : \langle x, \delta \rangle > 0 \text{ for any classes } \delta \text{ of smooth rational curves on } X\}$$

which is called the *Kähler cone* of X and a fundamental domain with respect to the natural action of $W(S_X)^{(2)}$ on $P(S_X)$. It follows from Torelli theorem for $K3$ surfaces due to Piatetsukii-Shapiro, Shafarevich [13] that $Aut(X)$ is isomorphic to $O(S_X)/W(S_X)^{(2)}$ up to finite groups. For any $g \in Aut(X)$, denote by $m(g)$ the order of $g^* \mid T_X$. Then it follows from Nikulin [11], Theorem 3.1 that $\varphi(m(g))$ divides the rank of T_X, where φ is the Euler function. If X is a generic Kummer surface, then $rank(T_X) = 5$. Hence $g^* \mid T_X = \pm 1$ as $\varphi(m)$ is even if $m > 2$. Hence we have an injection

$$Aut(X) \longrightarrow \Gamma(S_X)/W(S_X)^{(2)}$$

where $\Gamma(S_X) = \{g \in O(S_X) : g^* \mid A_{S_X} = \pm 1\}$. Moreover there exists an automorphism g of X with $g^* \mid A_{S_X} = -1$ (e.g. a switch σ in §5). It now follows from Nikulin [12], Proposition 1.6.1 and Torelli theorem that

Proposition 3.5 (Nikulin [10], Corollary 6.4, Keum [8], Theorem 4.1).

$$Aut(X) \simeq \Gamma(S_X)/W(S_X)^{(2)}.$$

We remark that $O(q_{S_X})$ is isomorphic to an extension of $\mathbf{Z}/2\mathbf{Z}$ by $Sp(4, \mathbf{F}_2)$ where $Sp(4, \mathbf{F}_2)$ is isomorphic to the orthogonal group of the form obtained from q_{S_X} by restricting it on the 2-elementary subgroup $(\mathbf{Z}/2\mathbf{Z})^5$ of A_{S_X}.

§4. A relation between the Picard lattice and Leech roots

In the following we fix a decomposition $L = \Lambda \oplus U$ and take $w = (0, 0, 1)$ as a Weyl vector. Consider the following vectors in Λ:

$$X = 4\nu_\infty + \nu_\Omega, Y = 4\nu_0 + \nu_\Omega, Z = 0, P = 4\nu_\infty + 4\nu_0, Q_i = 2\nu_{K_i} \ (i = 1,..., 5)$$

where $K_i \in \mathcal{C}(8), K_i \ni 0, \infty$ and $\mid K_i \cap K_j \mid = 4$ for $i \neq j$, i.e. $K_1, ..., K_5$ are the five octads containing common four points. Obviously

$\langle X, X \rangle = \langle Y, Y \rangle = -6, \langle P, P \rangle = \langle Q_i, Q_i \rangle = \langle X, Y \rangle = -4$ and $\langle P, Q_i \rangle = \langle Q_i, Q_j \rangle = -2$ for $i \neq j$.

The corresponding Leech roots

$$x = (X, 1, 2), y = (Y, 1, 2), z = (0, 1, -1), x_0 = (P, 1, 1) \text{ and } x_i = (Q_i, 1, 1)$$

genetare a root lattice R isomorphic to $A_3 \oplus A_1^{\oplus 6}$. Put

$$t = (x + 2z + 3y)/4, t_i = x_i/2 \ (i = 0,...,5).$$

Then $\{t, t_0, ..., t_5\}$ is a generator of A_R. Note that $q_R \simeq q_{-3}(4) \oplus (q_{-1}(2))^{\oplus 6}$. Let

$$\theta = 2t + t_0 + \cdots + t_5$$

which is isotropic with respect to q_R. Hence we have an over lattice T of R corresponding to θ : T is generated by R and θ. Note that $q_T \simeq q \mid \theta^{\perp}/\theta \simeq < t_0+t_1, t_0+t_2, t_3+t_4, t_3+t_5, t+t_0+t_1+t_2 > \bmod T \simeq v_2^{\oplus 2} \oplus q_{-1}(4) \simeq u_2^{\oplus 2} \oplus q_{-1}(4) \simeq q_{T_X}$.

Lemma 4.1. *T is a primitive sublattice of L whose root lattice is R.*

Proof. First we shall see that $T = T^* \cap L \bmod T$. Let

$$\theta^{'} = at + \Sigma a_i t_i \ (0 \le a < 4, 0 \le a_i < 2) \in T^* \cap L.$$

Since $\theta' \in T^*, \langle \theta, \theta' \rangle = -3a/2 - \Sigma a_i/2 \in \mathbf{Z}$. By definition,

$$\theta^{'} = ((a+2a_0)\nu_\infty + (3a+2a_0)\nu_0 + a\nu_\Omega + \Sigma a_j \nu_{K_j}, (3a+\Sigma a_i)/2, (3a+\Sigma a_i)/2) \in \Lambda \oplus U.$$

It is easy to see that if a is odd, then $\theta^{'}$ does not satisfy the condition (i) in Proposition 2.2. Hence $a = 0$ or 2. Again by Proposition 2.2, (i), and the fact that $\Sigma a_i \equiv 0 \bmod 2$, we can see that all a_i are even or odd. If $a_i = 0$ for all i, then $\theta^{'} \in R$ or $\theta^{'} \equiv (2\nu_\infty + 6\nu_0, 0, 0)$ mod L. In the last case, $\theta^{'}$ does not satisfy the condition (ii) in Proposition 2.2. If $a_i = 1$ for all i, then

$$\theta^{'} = ((a+2)\nu_\infty + (3a+2)\nu_0 + a\nu_\Omega + \Sigma \nu_{K_j}, (3a+6)/2, (3a+6)/2).$$

If $a = 0$, then θ' does not satisfy the condition (ii) in Proposition 2.2. If $a = 2$, then $\theta' = \theta \in T$.

Next we shall see that the root sublattice of T is R. For $r \in T$, write $r = ax + by + cz + \Sigma a_i x_i$ $(a, b, c, a_i \in \mathbf{Q})$. If $r \notin R$, then no a_i vanish, and hence $r^2 < -2$. This implies the assersion. *q.e.d*

Let S be the orthogonal complement of T in L. Since $q_T \simeq q_{T_X}$, $S \simeq S_X$ (Nikulin [12], Theorem 1.14.2). It is easy to see that the symmetry group $\mathbf{Z}/2\mathbf{Z} \times S_6$ of the Dynkin diagram R can be extended to isometries of T because it preserves θ. Moreover they generate $O(q_T) \simeq O(q_S)$. Hence the natural map $O(T) \longrightarrow O(q_T)$ is surjective. It follows from Nikulin [12], Proposition 1.6.1 that any $g \in O(S)$ can be extended to an isometry in $O(L)$.

Now put

$$D' = D \cap P(S).$$

Then it is known that D' is non-empty, has only a finite number of faces and of finite volume. Moreover D' contains the projection w' of the Weyl vector $w = (0,0,1) \in \Lambda \oplus U$ as an interior point (Borcherds [1], Lemmas 4.1, 4.2, 4.3). First we determine the Leech roots orthogonal to R.

Lemma 4.2. *There are exactly thirty-two Leech roots which are orthogonal to R.*

Proof. By Proposition 2.3 and Remark 2.4, we can see that such Leech root r corresponds to a vector

$$2\nu_K \text{ in } \Lambda \text{ where } K \in \mathcal{C}(8), K \ni 0, \infty, \mid K \cap K_i \mid = 4, i = 1, ..., 5.$$

The number of these octads is counted as follows: Take a point A in $K_1 \cap K_2$ with $A \neq 0, \infty$ and a point B in $K_1 \setminus K_2$. Then the set $\{0, \infty, A, B\}$ determines a unique sextet, in other words, there exists exactly five octads containing 0, ∞, A, B. Note that any two distinct octads meet 0, 2 or 4 points because $\mathcal{C}$-set consists of 0, 8, 12, 16 or 24 elements. It follows easily from this fact that the above five octads satisfy the condition $\mid K \cap K_i \mid = 4$ for all i. We should exclude the octad K_1 from the set of these five octads. Hence the number of desired octads is $\binom{2}{1} \times \binom{4}{1} \times 4 = 32$. *q.e.d.*

Remark 4.3, (i). In his paper [14], Todd gave the 759 octads of the Steiner system $S(5,8,24)$ (p.219, Table I). The octads in the proof of Lemma 4.2 correspond to the following $\{E_\alpha, L_\beta\}$ in his table. Here we assume

$K_1 = \{\infty,0,1,2,3,5,14,17\}, K_2 = \{\infty,0,1,2,4,13,16,22\}, K_3 = \{\infty,0,1,2,6,7,19,21\},$

$K_4 = \{\infty,0,1,2,8,11,12,18\}, K_5 = \{\infty,0,1,2,9,10,15,20\}.$

Then

$E_{45} = \{\infty,0,1,3,4,11,19,20\}, E_5 = \{\infty,0,1,3,6,8,10,13\},$

$E_{24} = \{\infty,0,1,3,7,9,16,18\}, E_2 = \{\infty,0,1,3,12,15,21,22\},$

$E_3 = \{\infty,0,1,4,5,7,8,15\}, E_{35} = \{\infty,0,1,4,6,9,12,17\},$

$E_4 = \{\infty,0,1,4,10,14,18,21\}, E_1 = \{\infty,0,1,5,6,18,20,22\},$

$E_{34} = \{\infty,0,1,5,9,11,13,21\}, E_{14} = \{\infty,0,1,5,10,12,16,19\},$

$E_{13} = \{\infty,0,1,6,11,14,15,16\}, E_{15} = \{\infty,0,1,7,10,11,17,22\},$

$E_0 = \{\infty,0,1,7,12,13,14,20\}, E_{25} = \{\infty,0,1,8,9,14,19,22\},$

$E_{23} = \{\infty,0,1,8,16,17,20,21\}, E_{12} = \{\infty,0,1,13,15,17,18,19\},$

$L_1 = \{\infty,0,2,3,4,8,9,21\}, L_{34} = \{\infty,0,2,3,6,12,16,20\},$

$L_{14} = \{\infty,0,2,3,7,11,13,15\}, L_3 = \{\infty,0,2,3,10,18,19,22\},$

$L_2 = \{\infty,0,2,4,5,6,10,11\}, L_{25} = \{\infty,0,2,4,7,17,18,20\},$

$L_{15} = \{\infty,0,2,4,12,14,15,19\}, L_{45} = \{\infty,0,2,5,7,9,12,22\},$

$L_{24} = \{\infty,0,2,5,8,13,19,20\}, L_5 = \{\infty,0,2,5,15,16,18,21\},$

$L_4 = \{\infty,0,2,6,8,15,17,22\}, L_{23} = \{\infty,0,2,6,9,13,14,18\},$

$L_{12} = \{\infty,0,2,7,8,10,14,16\}, L_0 = \{\infty,0,2,9,11,16,17,19\},$

$L_{13} = \{\infty,0,2,10,12,13,17,21\}, L_{35} = \{\infty,0,2,11,14,20,21,22\}$

(ii) We use the same notation as in Remark 4.3, (i). For simplicity we denote by the same symbols $\{E_\alpha, L_\beta\}$ for the corresponding Leech roots. Then $\{E_\alpha\}$ and $\{L_\beta\}$ are both mutually orthogonal sixteen roots and they form a standard graph Γ (see Definition 3.3). Moreover $\{E_\alpha\}$ and $\{L_\beta\}$ are switched by the isometry σ induced from the permutation

$$(0,\infty)(1,2)(3,5)(4,22)(6,21)(7,19)(8,18)(9,20)(10,15)(11,12)(13,16)(14,17)$$

on Ω (Todd [14], §3, (xvii)) : $\sigma(E_\alpha) = L_\alpha$ for any α. Consider the following involutions in M_{24} which commute with σ, fixes $\infty, 0, 1, 2$ and preserves five octads $K_1, ..., K_5$:

$$\tau_1 = (3,17)(4,16)(5,14)(6,7)(8,11)(12,18)(13,22)(19,21),$$

$$\tau_2 = (3,14)(4,16)(5,17)(6,19)(7,21)(9,10)(13,22)(15,20),$$

$$\tau_3 = (3,17)(4,13)(5,14)(8,12)(9,10)(11,18)(15,20)(16,22),$$

$$\tau_4 = (4,13)(6,19)(7,21)(8,11)(9,15)(10,20)(12,18)(16,22),$$

$$\tau_5 = (3,14)(5,17)(6,7)(8,12)(9,15)(10,20)(11,18)(19,21).$$

It is easy to see that $\{\tau_1, ..., \tau_5, \sigma\}$ generate a 2-elementary abelian group of order 32 with $\tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4 \circ \tau_5 = 1$. The set of indices appeared in $\{E_\alpha\}$ has a group structure such that the translation by α is equal to τ_α, $\alpha = 1, ..., 5$, that is, it coincides with $J(C)_2 = \{i, jk : 0 \leq i \leq 5, 1 \leq j < k \leq 5\}$.

Lemma 4.4. *Let w' be the projection of the Weyl vector $w = (0,0,1) \in \Lambda \oplus U$. Then*

$$w' = (\Sigma E_\alpha + \Sigma L_\beta)/4 \in S.$$

Proof. First note that $w = w' + x^* + y^* + z^* + \Sigma x_i^*$ and $w^2 = 0$ where $x^* = -(3x + y + 2z)/4$, $y^* = -(x + 3y + 2z)/4$, $z^* = -(x + y + 2z)/2$ and $x_i^* = -x_i/2$. Hence $(w')^2 = 8$. On the other hand, $\langle w', E_\alpha \rangle = \langle w', L_\beta \rangle = 1$. The right hand side of the equation in Lemma 4.4 also has this property. Since $\{E_\alpha, L_\alpha\}$ generate S (Lemma 3.1), this implies the assersion. *q.e.d*

We denote by $Aut(D')$ the group of isometries of S which preserves D'. As mentioned before, each isometry $\varphi \in Aut(D')$ can be extended to an isometry of L which preserves D. Hence $Aut(D')$ is isomorphic to the subgroup G of $Aut(D)$ which preserves R.

Lemma 4.5. *$Aut(D')$ is an extension of 2-elementary abelian group of order 32 by S_6 in which a 2-elementary abelian subgroup of order 16 acts on R trivially and the quotient by it is the symmetry group of the Dynkin diagram of R.*

Proof. Obviously the restriction of G on R is a subgroup of the symmetry group $\mathbf{Z}/2\mathbf{Z} \times S_6$ of the Dynkin diagram of R. Let G_0 be the subgroup of G which acts trivially on R. Then G_0 is a subgroup of M_{22} because G_0 fixes X, Y, Z and P (see Conway-Sloane [4], Chap.10, §3.5). Moreover G_0 preserves five octads K_i. By the table of maximal subgroups of M_{22} (Conway [2], Table 3), G_0 is isomorphic to a 2-elementary abelian group of order 16. Thus G is a subgroup of a group $2^5 \cdot S_6$. On the other hand, $O(S)$ contains a subgroup $(\mathbf{Z}/2\mathbf{Z})^5 \cdot S_6$ induced from the automorphism group $Aut(\Gamma)$ of the standard graph Γ whose vertices are $\{E_\alpha, L_\beta\}$ (see §3). By Lemma 4.4, they preserves w' and hence D'. Thus $Aut(D') \simeq G \simeq (\mathbf{Z}/2\mathbf{Z})^5 \cdot S_6$. *q.e.d.*

Next we shall classify the hyperplanes bounding D'. By Lemma 4.1, the root sublattice of T is R. Since T is primitive in L, there are no sublattice of rank 9 in L generated by some roots which contains R properly. Hence it suffices to consider Leech roots r such that r and R generate a root lattice of rank 10 because otherwise the hyperplane H_r in $P(L)$ does not meet $P(S)$.

Lemma 4.6. *Let r be a Leech root. Assume r and R generate a root lattice R' of rank 10. Then the following cases occur :*

(0) $R' = A_3 \oplus A_1^{\oplus 7}$, (1) $R' = D_4 \oplus A_1^{\oplus 6}$, (2) $R' = A_3 \oplus A_3 \oplus A_1^{\oplus 4}$, (3) $R' = A_5 \oplus A_1^{\oplus 5}$.

Moreover the number of Leech roots of type (0), (1), (2) *or* (3) *is* 32, 32, 60 *or* 192 *respectively.*

Proof. It is obvious that R' is one of the following nine cases:

$(0) A_3 \oplus A_1^{\oplus 7}, (1) D_4 \oplus A_1^{\oplus 6}, (2) A_3 \oplus A_3 \oplus A_1^{\oplus 4}, (3) A_5 \oplus A_1^{\oplus 5}, (4) A_4 \oplus A_1^{\oplus 6}, (5) A_3 \oplus A_2 \oplus A_1^{\oplus 5}$, $(6) D_5 \oplus A_1^{\oplus 5}, (7) D_6 \oplus A_1^{\oplus 4}, (8) A_3 \oplus D_4 \oplus A_1^{\oplus 3}$.

We have seen the Case (0) in Lemma 4.2.

Case (1): By Proposition 2.3 and Remark 2.4, in this case, the root r corresponds to a vector

$$(\xi_0, \xi_\infty, \xi_{j_1}, ..., \xi_{j_6}, \xi_{j_7}, ..., \xi_{j_{22}}) = (3, 3, 3, -1, ..., -1, 1, ..., 1)$$

in Λ where $K = \{0, \infty, j_1, ..., j_6\}$ is an octad satisfying $\mid K \cap K_i \mid = 4$ and $K_i \ni j_1$ for any i.

By the same reason as in Case (0), the number of desired Leech roots is equal to 32.

Case (2): If r meets x_0 and x_i, then r corresponds to a vector

$$\nu_\Omega - 4\nu_k (k \neq 0, \infty), k \in K_i, k \notin K_j \text{ for } j \neq i,$$

and if r meets x_i and x_j, then r corresponds to a Leech vector

$$2\nu_K, K \in \mathcal{C}(8), K \ni 0, \infty, \mid K \cap K_i \mid = \mid K \cap K_j \mid = 2, \mid K \cap K_l \mid = 4 \text{ for } l \neq i, j.$$

Obviously the number of the above roots r meeting x_0 and x_i is equal to 4. It follows from Lemma 4.5 that the number of desired Leech roots is $\binom{6}{2} \times 4 = 60$.

Case (3): If r meets y and x_0, then r corresponds to a vector

$$2\nu_K, \infty \in K, 0 \notin K, \mid K \cap K_i \mid = 4 \text{ for all } i,$$

and if r meets y and x_i, then r corresponds to

$$(\xi_\infty, \xi_{j_1}, ..., \xi_{j_7}, \xi_{j_8}, ..., \xi_{j_{22}}, \xi_0) = (3, -1, ..., -1, 1, ..., 1)$$

where $K = \{\infty, j_1, ..., j_7\}$ is an octad with $\mid K \cap K_i \mid = 4$ and $\mid K \cap K_j \mid = 2$ for $j \neq i$. Now we count the number of Leech roots. We may assume that r meets y and x_0. Take a point $A \in K_1 \setminus K_2$ and consider the five octads containing the four points ∞ , A, $K_1 \cap K_2 \setminus \{0, \infty\}$. Each of them satisfies the above condition, however, in which we should exclude the octad K_1. The number of such octads is equal to $\binom{4}{1} \times 4 = 16$. Hence the total number of the desired Leech roots is $\binom{2}{1} \times \binom{6}{1} \times 16 = 192$.

In other cases (4)-(8), by Proposition 2.3 and case-by-case argument, we can see that there are no Leech roots r satisfying $< r, R > \simeq R'$. We leave the proof to the reader. Thus we have now finished the proof. *q.e.d.*

Remark 4.7. We use the same notation as in Remark 4.3. Then the octads appeared in the proof of Lemma 4.6 correspond to the followings in Todd's list [14]:

Case (1) : The same octads as in Remark 4.3.

Case (2) : 40 octads from $\{\infty, 0, 3, 4, 5, 12, 13, 18\}$ to $\{\infty, 0, 12, 14, 16, 17, 18, 22\}$ in p. 219.

Case (3) : Sixteen octads from $\{\infty, 1, 2, 3, 4, 6, 15, 18\}$ to $\{\infty, 1, 2, 14, 16, 18, 19, 20\}$ in p.219, 96 octads from $\{\infty, 3, 4, 5, 6, 8, 17, 20\}$ to $\{0, 1, 2, 12, 17, 19, 20, 22\}$ in p.220 and 80 octads from $\{0, 3, 4, 5, 7, 9, 11, 17\}$ to $\{0, 11, 12, 13, 14, 15, 18, 21\}$ in p. 221.

Lemma 4.8. *Let r be a Leech root as in Lemma* 4.6. *Let $r = r' + r''$ where $r' \in S^*$ and $r'' \in T^*$. Then*

Case (0) : $r = r'$; *Cases* (1), (2) : $\langle r', r' \rangle = -1$; *Case* (3) : $\langle r', r' \rangle = -3/4$.

Proof. We shall see the assersion in each case:

Case (0): obvious.

Case (1): Let $\{x, y, z\}$ be a base of A_3 as above. Then $r = z^* + r'$ where $z^* = -(x + 2z + y)/2$ is the dual of z. Since $\langle r, r \rangle = -2$, $\langle r', r' \rangle = -1$.

Case (2): We may assume that r meets x_1 and x_2. Then $r = -x_1/2 - x_2/2 + r'$. Hence $\langle r', r' \rangle = -1$.

Case (3): We may assume that r meets y and x_1. Then $r = y^* - x_1/2 + r'$ where $y^* = -(x + 2z + 3y)/4$ is the dual of y. Hence $\langle r', r' \rangle = -3/4$. *q.e.d*

§5. Classical involutions

In this section we recall classical involutions of X. Let F be a quartic surface in $\mathbf{P}^3$ with sixteen nodes $\{n_\alpha\}$ which is the image of $J(C)$ by the map given by $\mid 2\Theta \mid$.

(1) *Sixteen translations* (Klein [9]):

A 2-torsion point $\mu \in J(C)_2$ acts on $J(C)$ as translations. These involutions descend to involutions t_μ on X which are induced from projective automorphisms of $\mathbf{P}^3$. There are no other projective automorphisms if $rank(S_X) = 17$ (Keum [8]).

(2) *Sixteen switches* (Klein [9]):

The dual surface F^* is also a quartic surface with sixteen ordinary nodes (corresponding to sixteen tropes) and is projectively isomorphic to F (e.g. Griffiths-Harris [5], Chap.6). Hence we have an involution σ which sends $\{N_\alpha\}$ to $\{T_\alpha\}$. We call this involution a *switch*. Let σ be a switch which sends N_α to T_α for any α. Then sixteen switches are of the form

$\sigma \circ t_\mu, \mu \in J(C)_2$. We use the same symbol σ for a switch and for the isometry stated in Remark 4.3, (ii) because they coincide under the identification of $\{N_\alpha, T_\beta\}$ and $\{E_\alpha, L_\beta\}$ (see §6).

(3) *Sixteen projections* (Klein [9]):

The projection from a node n_α on F gives an involution p_α of X.

(4) *Sixteen correlations* (Klein [9]):

A projection of F^* induces an involution of X : $\sigma \circ p_\alpha \circ \sigma$.

(5) *Sixty Cremona transformations referred to Göpel tetrads* (Hutchinson [6]):

A *Göpel tetrad* is a set of four nodes $\{n_\alpha, n_\beta, n_\gamma, n_\delta\}$ such that no tropes contain three of four nodes. It is known that there are sixty Göpel tetrads of F. For each Göpel tetrad, F can be defined by the equation:

$A(x^2t^2 + y^2z^2) + B(y^2t^2 + z^2x^2) + C(z^2t^2 + x^2y^2) + Dxyzt + F(yt + zx)(zt + xy)+$

$G(zt + xy)(xt + yz) + H(xt + yz)(yt + zx) = 0.$

The Cremona transformation

$$(x, y, z, t) \longrightarrow (yzt, ztx, txy, xyz)$$

preserves F and so we have sixty involutions $c_{\alpha,\beta,\gamma,\delta}$ of X.

By definitions and Lemma 3.2, we have:

Proposition 5.1 (Keum [8], §4, Table). *Assume that X is generic. Then the actions of classical involutions on S_X are as follows* :

(1) $t_\alpha(N_\beta) = N_{\alpha+\beta}, t_\alpha(H) = H$.

(2) $\sigma(N_\alpha) = T_\alpha, \sigma(H) = 3H - \Sigma N_\alpha$.

(3) p_α *acts on the Picard lattice as reflection with respect to* (-4)*-root* $H - 2N_\alpha$.

(4) $c_{\alpha,\beta,\gamma,\delta} = r_{\alpha,\beta,\gamma,\delta} \circ t_{\alpha,\beta,\gamma,\delta}$ *where* $r_{\alpha,\beta,\gamma,\delta}$ *is a reflection with respect to* (-4)*-root* $H - N_\alpha - N_\beta - N_\gamma - N_\delta$ *and* $t = t_{\alpha,\beta,\gamma,\delta}$ *fixes* $H, N_\alpha, N_\beta, N_\gamma, N_\delta$ *and permutes the remaining* 12 *nodal curves as follows* : t *interchanges the two tropes containing any two of the four nodes* $n_\alpha, n_\beta, n_\gamma, n_\delta$. *This determines uniquely the action of* t *on sixteen nodal curves* N_α.

Remark 5.2(Keum [8], §6, Warning). Neither $r_{\alpha,\beta,\gamma,\delta}$ nor $t_{\alpha,\beta,\gamma,\delta}$ can be realized as an automorphism of X.

§6. Keum's automorphisms

Recently Keum [8] has found 192 automorphisms of a generic Kummer surface X which are not contained in the group generated by the classical automorphisms. In this section we shall give the Definition and properties of Keum's automorphisms. Consider thirty-two roots $\{E_\alpha, L_\beta\}$ stated in Remark 4.3. Then $\{E_\alpha, L_\beta\}$ forms a standard graph Γ (see Definition 3.3). By Lemma 4.4 and Proposition 3.4, we can identify $\{E_\alpha, L_\beta\}$ to thirty-two nodal curves $\{N_\alpha, T_\beta\}$ on a generic Kummer surface X associated with a curve of genus two. Under this identification, S_X is a primitive sublattice of L. Note that $D' = D \cap P(S_X) \subset D(S_X)$ because D' contains an ample divisor $w' = \Sigma(N_\alpha + T_\alpha)/4$ (Lemma 4.4).

Lemma 6.1. *Let r be a Leech root as in Lemma 4.6, Case (3). Then there exists exactly two hexads $\Psi_r, \Phi_r \subset J(C)_2$ which satisfy the following :*

(1) $\langle r, N_\alpha\rangle = \langle r, T_\beta\rangle = \begin{cases} 1 & \textit{for } \alpha \in \Psi_r, \beta \in \Phi_r; \\ 0 & \textit{for } \alpha \notin \Psi_r, \beta \notin \Phi_r. \end{cases}$

(2) $\mid \Psi_r \cap \Phi_r \mid = 1.$

(3) *For each $\alpha \in \Phi_r$, there exists a unique $\beta_\alpha \in \Psi_r$ with $\langle N_\alpha, T_{\beta_\alpha}\rangle = 1$, and vice versa.*

(4) *For each $\alpha \in \Psi_r \setminus \Phi_r$, N_α meets exactly three members $T_\alpha, T_{\alpha_1}, T_{\alpha_2}$ in $\{T_\beta : \beta \in \Psi_r\}$ and $I(T_{\alpha_1}) \cap I(T_{\alpha_2}) = \{N_\alpha, N_{\alpha'}\}$ where $\alpha' \in J(C)_2 \setminus (\Psi_r \cup \Phi_r)$ and $I(T_\beta) = \{N_\alpha : \langle N_\alpha, T_\beta\rangle = 1\}$.*

(5) *For each $\alpha \in J(C)_2 \setminus (\Psi_r \cup \Phi_r)$, N_α meets exactly three members $T_{\Psi_r \cap \Phi_r}, T_{\beta_1}, T_{\beta_2}$ in $\{T_\beta : \beta \in \Phi_r\}$ and $I(T_{\beta_1}) \cap I(T_{\beta_2}) = \{N_\alpha, N_{\alpha''}\}$ where $\alpha'' \in \Phi_r \setminus \Psi_r$.*

(6) *Let σ be the isometry defined in Remark 4.5. Then*

$$\Psi_{\sigma(r)} = \Phi_r, \Phi_{\sigma(r)} = \Psi_r.$$

Proof. Consider the action of $Aut(D')$ on the set of Leech roots as in Case (3). First we shall show that the stabilizer of a such Leech root is isomorphic to S_5. Let r be a Leech root corresponding to a vector $2\nu_K, \infty \in K, 0 \notin K, \mid K \cap K_i \mid = 4$ for all i. The stabilizer of $2\nu_K$ is the subgroup of M_{22} which preserves K , fixes two points $0, \infty$ and has three orbits on K of sizes 1, 2, 5. It is known (Conway [2], §3, Theorem 6) that this group is isomorphic to S_5. Since $[Aut(D') : S_5] = 192$, $Aut(D')$ acts on this set transitively. Therefore it suffices to prove the assersion for one such Leech root.

Now we use the same notation as in Remarks 4.3, 4.7. Let $K = \{\infty, 1, 2, 3, 4, 6, 15, 18\}$ as in Remark 4.7, Case (3). Let r be the Leech root corresponding to $2\nu_K$. Then

$$\Psi_r = \{0, 14, 15, 23, 25, 34\}, \Phi_r = \{0, 12, 13, 24, 35, 45\}.$$

The assersions (1)-(2) are now obvious. The assersion (3) follows from Lemma 3.2. The assersion (4) follows from the facts that $\Psi_r \setminus \Phi_r = \{14, 15, 23, 25, 34\}$ and

$$I(T_{23}) \cap I(T_{25}) = \{N_2, N_{14}\}, I(T_{23}) \cap I(T_{34}) = \{N_3, N_{15}\}, I(T_{14}) \cap I(T_{15}) = \{N_1, N_{23}\},$$
$$I(T_{14}) \cap I(T_{34}) = \{N_4, N_{25}\}, I(T_{15}) \cap I(T_{25}) = \{N_5, N_{34}\}.$$

Also the assersion (5) follows from the facts that $J(C)_2 \setminus (\Psi_r \cup \Phi_r) = \{1, 2, 3, 4, 5\}$ and

$I(T_{12}) \cap I(T_{13}) = \{N_1, N_{45}\}, I(T_{12}) \cap I(T_{24}) = \{N_2, N_{35}\}, I(T_{13}) \cap I(T_{35}) = \{N_3, N_{24}\},$
$I(T_{24}) \cap I(T_{45}) = \{N_4, N_{13}\}, I(T_{35}) \cap I(T_{45}) = \{N_5, N_{12}\}.$

Finally the assersion (6) follows from the fact that $\sigma(N_\alpha) = T_\alpha$ for any α. *q.e.d*

Definition 6.2. Let r be a Leech root as in Lemma 4.6, Case (3), and Ψ_r, Φ_r as in Lemma 6.1. Put

$$K_r = \Sigma_{\alpha \in \Psi_r} N_\alpha.$$

We define an isometry ψ_r of $S_X \otimes \mathbf{Q}$ by the following equations:

$$\psi_r(H) = 7H - 4K_r,$$

$$\psi_r(N_\alpha) = \begin{cases} 2H - K_r - N_{\beta_\alpha} & \text{for } \alpha \in \Phi_r, \\ N_{\alpha'} & \text{for } \alpha \in \Psi_r \setminus \Phi_r, \\ N_{\alpha''} & \text{for } \alpha \in J(C)_2 \setminus (\Psi_r \cup \Phi_r), \end{cases}$$

where $\beta_\alpha, \alpha', \alpha''$ are as in Lemma 6.1.

Remark 6.3. (1) Let r' be the projection of r into S_X^*. By Lemma 4.8, $(r')^2 = -3/4$. In particular, $4r' = 3H - 2K_r$.

(2) By using the relations in Lemma 3.2, we can see that $\psi_r(T_\beta) = \sigma(\psi_{\sigma(r)}(N_\beta))$. In particular ψ_r preserves S_X.

(3) $\psi_{\sigma(r)} = \psi_r^{-1}$.

Lemma 6.4. *ψ_r is an isometry of S_X which acts on A_{S_X} as identity.*

Proof. By using Lemma 3.1, (ii) and Lemma 3.2, we can see that ψ_r acts on A_{S_X} as identity. *q.e.d.*

Lemma 6.5 (Keum). *Let D be a divisor on X. Then $\langle D, 3H - 2K_{\sigma(r)} \rangle \geq 0$ if and only if $\langle \psi_r(D), 3H - 2K_r \rangle \leq 0$.*

Proof. Write D as $D = aH - \Sigma b_\alpha N_\alpha$. Then $\langle D, 3H - 2K_{\sigma(r)} \rangle = 12a - 4\Sigma_{\alpha \in \Phi_r} b_\alpha$ and $\langle \psi_r(D), 3H - 2K_r \rangle = -12a + 4\Sigma_{\alpha \in \Phi_r} b_\alpha$. *q.e.d*

Proposition 6.6 (Keum [8]). *ψ_r preserves the Kähler cone $D(S_X)$ of X.*

Proof. See Keum [8], the proof of Proposition 6.16. *q.e.d.*

Definition 6.7. By Lemma 6.4 and Proposition 6.6, ψ_r can be extended to a Hodge isometry of $H^2(X, \mathbf{Z})$ which preserves the Kähler cone. It follows from the Torelli theorem for $K3$ surfaces (Piatetsukii-Shapiro, Shafarevich [13]) that ψ_r is induced from an automorphism $\kappa_r : \psi_r = \kappa_r^*$. We call κ_r *Keum's automorphism* associated with a Leech root r.

Remark 6.8. Our definition of ψ_r is slightly different from that of Keum. In [8], he defines an isometry f associated with a *good hexad* $\{\alpha, \beta, \gamma, \delta, \varepsilon, \lambda\} \subset J(C)_2$, that is, no four of six elements are contained in a trope or a Göpel tetrad. The above hexads Ψ_r, Φ_r are good hexads in this sense. For example, if $\Psi_r = \{0, 14, 15, 23, 25, 34\}$, then $\psi_r = f \circ t_{34}$ where t_{34} is the translation by 2-torsion point μ_{34}. He also showed that f^{-1} is induced by the linear system $\mid 7H - 4(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\varepsilon + N_\lambda) \mid$ by using Torelli theorem for $K3$ surfaces.

§7. The automorphism group of a generic Kummer surface

In this section we shall complete a proof of the main theorem. We identify thirty-two Leech roots $\{E_\alpha, L_\beta\}$ to thirty-two nodal curves $\{N_\alpha, T_\beta\}$ on X .

Lemma 7.1. *Let w' be the projection of the Weyl vector w in S_X. Then*

$$w' = (\Sigma N_\alpha + \Sigma T_\beta)/4 = 2H - \Sigma N_\alpha/2.$$

Proof. We have shown that the first equality and $(w')^2 = 8$ (Lemma 4.4). The second assersion follows from the fact $\langle w', N_\alpha \rangle = 1$. *q.e.d.*

Lemma 7.2. *$Aut(D')$ is generated by sixteen translations t_μ, a swiche σ and the permutations of six Weierstrass points of the curve C.*

Proof. Recall that $O(S_X)$ contains isometries induced from sixteen translations t_α, a switch σ, and the permutations of six Weierstrass points of the curve C. It follows from Proposition 7.1 that these isometries preserve w' and hence D'. Hence these isometies generate the group $Aut(D') \simeq (\mathbf{Z}/2\mathbf{Z})^5 \cdot S_6$ (see Lemma 4.5). *q.e.d*

Next we consider automorphisms corresponding to Leech roots in Lemma 4.6, Cases (1) and (2).

Case (1): Let r' be the projection of a Leech root r into S_X^*. Let λ in Λ corresponding to r (see Remark 2.4). Recall that r is defined by an octad. This octad also defines a (-2)-vector in S (see Remark 4.7). First we assume that this octad defines N_α. Then $2\nu_{N_\beta} - \lambda$ is a vector of the following form:

$$(3^5, -1^{19}) \text{ if } \alpha = \beta \text{ and } (3, \pm 1^{23}) \text{ if } \alpha \neq \beta.$$

Hence $\langle r, N_\beta \rangle = 2\delta_{\alpha,\beta}$. If we write

$$2r' = aH - \Sigma b_\alpha N_\alpha,$$

then the above implies

$$2r' = H - 2N_\alpha$$

because $(2r')^2 = -4$. Hence the action of the reflection $s_{r'}$ on S_X coincides with the action of the projection p_α (see Proposition 5.1).

Next we assume that the above octad defines T_α. Then $2\nu_{N_\beta} - \lambda$ is:

$$(\pm 3^3, \pm 1^{21}) \text{ if } \langle T_\alpha, N_\beta \rangle = 0,\ (-3, \pm 1^{23}) \text{ if } \langle T_\alpha, N_\beta \rangle = 1.$$

Hence

$$\langle N_\beta, r\rangle = \begin{cases} 1 & \text{if } \langle T_\alpha, N_\beta\rangle = 0; \\ 0 & \text{if } \langle T_\alpha, N_\beta\rangle = 1. \end{cases}$$

This implies

$$2r' = 3H - \Sigma N_\nu - 2T_\alpha = \sigma(H - 2N_\alpha).$$

Thus $s_{r'} = \sigma \circ p_\alpha^* \circ \sigma$.

Case (2): First we shall see that for each r there are exactly four nodal curves $N_\alpha, N_\beta, N_\gamma, N_\delta$ in $\{N_\nu\}$ which meet r. We use the same notation as in the proof of Lemma 4.6. It suffices to see the claim in the case that r corresponding to a vector $\nu_\Omega - 4\nu_k$ because there exists an isometry in $Aut(D')$ which sends $\nu_\Omega - 4\nu_k$ to $2\nu_K$. Then $\langle r, N_\alpha\rangle = 1$ if and only if the corresponding octad to N_α contains k. The number of such octads is obviously 4 (see the proof of Lemma 4.2). Next we shall see that $\{N_\alpha, ..., N_\delta\}$ is a Göpel tetrad. We may assume that $k = 3$. Then $\{N_\alpha, ..., N_\delta\} = \{N_2, N_5, N_{24}, N_{45}\}$ (see Remark 4.3, (i)) and

$$I(N_2) = \{T_0, T_2, T_{12}, T_{23}, T_{24}, T_{25}\},\ I(N_5) = \{T_0, T_5, T_{15}, T_{25}, T_{35}, T_{45}\},$$
$$I(N_{24}) = \{T_2, T_4, T_{13}, T_{15}, T_{24}, T_{35}\},\ I(N_{45}) = \{T_4, T_5, T_{12}, T_{13}, T_{23}, T_{45}\},$$

where $I(N_\alpha) = \{T_\beta : \langle N_\alpha, T_\beta\rangle = 1\}$. Now the assersion is obvious.

Since $(r')^2 = -1$ (Lemma 4.8), we have

$$2r' = H - (N_\alpha + N_\beta + N_\gamma + N_\delta).$$

Hence $s_{r'} = r_{\alpha,\beta,\gamma,\delta}$ (see Proposition 5.1).

Let r be a Leech root in Lemma 4.6, Case (1), (2) or (3). Denote by ι_r the isometry induced from the corresponding automorphism (i.e. classical involution or Keum's automorphism). Let N be the group generated by all ι_r. Obviously N acts on the Kähler cone $D(S_X)$.

Lemma 7.3. *Let ϕ be an isometry of S_X which preserves $D(S_X)$. Then there exists a $g \in N$ such that $g \circ \phi \in Aut(D')$.*

Proof. Let w' be the projection of the Weyl vector w : $w' = 2H - \Sigma N_\nu/2$. Put $v = \phi(w')$. Take an element $g \in N$ which attains the minimum $min\{\langle g(v), w'\rangle : g \in N\}$. Then for any r,

$$\langle g(v), w'\rangle \le \langle \iota_r(g(v)), w'\rangle = \langle g(v), \iota_r^{-1}(w')\rangle.$$

If ι_r corresponds to a classical involution, then by Proposition 5.1,

$$\iota_r^{-1}(w') = \iota_r(w') = w' + 2\langle w', r'\rangle r'.$$

Hence

$$\langle g(v), w'\rangle \le \langle g(v), w'\rangle + 2\langle w', r'\rangle\langle r', g(v)\rangle.$$

Since $\langle w', r'\rangle > 0$, we have $\langle r', g(v)\rangle > 0$.

If ι_r corresponds to a Keum's automorphism, then by Definition 6.2 and Remark 6.3, (1),

$$\iota_r^{-1}(w') = w' + 8\sigma(r').$$

Hence

$$\langle g(v), w' \rangle \leq \langle g(v), w' \rangle + 8\langle g(v), \sigma(r') \rangle.$$

Thus $\langle g(v), \sigma(r') \rangle > 0$, and hence $g(v) \in D'$. *q.e.d.*

Combining Proposition 3.5, Lemmas 7.2, 7.3, we have:

Theorem 7.4. *The automorphism group of a generic Kummer surface is a split extension of* 2*-elementary abelian group* 2^5 *of oreder* 32 *by a group* N *where* 2^5 *is generated by sixteen translations* t_μ *and a switch* σ*, and* N *is generated by sixteen projections* p_α*, sixteen corellations* $\sigma \circ p_\alpha \circ \sigma$*, sixty Cremona transformations* $c_{\alpha,\beta,\gamma,\delta}$ *and* 192 *Keum's automorphisms* κ_r*.*

Remark 7.5. The method is applicable to other $K3$ surfaces. For example, by taking E_6, D_6 as a root lattice R $(= T)$, we have the two singular $K3$ surfaces with

$$T_X \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

respectively. In these cases, the reflection group is of finite index in $O(S_X)$ and the automorphism groups of these $K3$ surfaces were calculated by Vinberg [16] and also by Borcherds [1] by our method.

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