On Fractional Brownian Motion and Wavelets

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January 2, 2010

Abstract

Given a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, we associate with this a special family of represenions of Cuntz algebras related to frequency domains and wavelets. Vice versa, we consider a pair of Haar wavelets satisfying some compatibility conditions, and we construct the covariance functions of fBm with a fixed Hurst index. The Cuntz algebra representations enter the picture as filters of the associated wavelets. Extensions to $q-$ dependent covariance functions leading to a corresponding fBm process will also be discussed.

AMS-classification: 42C40, 46C05, 60G22, 47L90, Mathematical methods.
Key words: fractional Brownian motion, wavelets, Cuntz algebras, filters, Hilbert spaces.

1 Introduction

In this paper we study the role of fractional Brownian motion in wavelet decomposition theory. A novelty of our approach is the use of representations of a family of purely infinite C*-algebras, called the Cuntz algebras. A fractional Brownian motion (precise definitions are given below) is a stochastic process $X(t)$ (t being time) for which the usual measurement of the increments over arbitrary time intervals is assumed to satisfy certain axioms: roughly, it is a Gaussian process, and if a particular time interval is of length Δt, then the corresponding increment of the process $X(t)$ behaves for small values of Δt like $(\Delta t)^H$ where H is a fixed number (called the Hurst parameter) in the open interval $(0, 1)$. The case of $H = \frac{1}{2}$ is the more familiar Brownian motion. While the increments over disjoint intervals are independent in the standard Brownian case, this is not so for H different from $\frac{1}{2}$. The modification of H relative to the standard case of Brownian motion entails changes in the analysis of the associated stochastic integrals. In section 2, we present a way to deal with this making use of the theory of representations of the Cuntz algebras.

The connection between wavelets and fractional Brownian motion is two-fold.

Starting with the fractional Brownian motion (fBm) $X(t)$ for a fixed H, we show that it is possible to diagonalize $X(t)$ with the use of a chosen wavelet basis. A fractional Brownian motion with parameter H has the property that for all $c \in \mathbf{R}_+$, the two processes Xct and $c^H X(t)$ have the same distribution (similarity of scales). Conversely, the use of wavelet analysis allows us to gain new insight into the study of fractional Brownian motion (e.g. formula (38) in Lemma 4 (section 3)). And it amounts to a correspondence between points in a certain infinite-dimensional unitary group on the one side, and a family of representations of the Cuntz algebras on the other. The conclusion is that elements in the group of all $U(N)$-valued functions on T parameterize the family (up to unitary equivalence) of representations of O_N that we need in our construction. This wavelet analysis (section 4) is new and we feel of independent interest.

In recent work by Bratteli and Jorgensen and the co-authors, the discovery of a special family F of irreducible representations of the Cuntz algebras was made, with F modeling all the subband wavelet filters. In this case, the family F takes the form of an infinite-dimensional unitary group, or equivalently a $U(N)$-loop group; see section 3 below. While we make use of this part of non-commutative harmonic analysis, for perspective note the following difficulty arising in the study of representations of non-type I C*-algebras, also called Glimm C*-algebras, a class that includes the Cuntz algebras O_N. For O_N, Borel cross sections for the families of (equivalence classes of) irreducible representations are not available, and a classification of the representations of O_N seems to be entirely out of reach. It is even known for the Cuntz algebras that the set of equivalence classes of all irreducible representations does not admit a Borel labeling. Hence, there is no direct integral decompositions, and no reasonable harmonic analysis. Despite these difficulties, Bratteli and Jorgensen discovered a special family F of irreducible representations of the Cuntz algebras, with F modeling all the subband wavelet filters. In this case, the family F takes the form of an infinite-dimensional unitary group, or equivalently a U(N)-loop group.

In section 3 we show how the introduction of processes with increments which behave for small values of t as $(\Delta t)^H$ for H different from $\frac{1}{2}$ (fractional Brownian motion) dictates a subtle modification of the analysis and the representation theory for the O_N representations considered previously. We deal with this by building up our representations with the use of Fock space tools, in particular raising and lowering operators.

As well known, by work of Lévy and Ciesielski see, eg, ([Part]), the Haar functions seen as wavelets bases allow to construct Brownian motion by using them as a complete orthonormal basis. We are interested in looking correspondingly at the fractional Brownian motion processes via wavelets bases.
One of the early papers exploting the relations between operator theory and the theory of stochastic processes is ([Ne64]). Later papers on this theme include ([Cr07],[HSU07],[Un09], [Al]) and references therein, to mention only a few.

1.1 Motivation and Applications

We consider interconnections between three subjects which are not customarily thought to have much to do with one another: (1) the theory of stochastic processes, (2) the theory of wavelets, and (3) sub-band filters (in the sense of signal processing). While connections between (2) and (3) have gained recent prominence, see for example ([BrJo02]), applications of these ideas to stochastic integration is of more recent vintage. Nonetheless, there is always an element of noise in the processing of signals with systems of filters. But this has not yet been modeled with stochastic processes, and it has not previously been clear which processes do the best job. Recall however that the notion of low-pass and high-pass filters derives in part from probability theory. Here high and low refers to frequency bands, but there may well be more than two bands (not just high and low, but a finite range of bands). The idea behind this is that signals can be decomposed according to their frequencies, with each term in the decomposition corresponding to a range of a chosen frequency interval, for example high and low. Sub-band filtering amounts to an assignment of filter functions which accomplish this: each of the filters will then block signals in one band, and passes the others. This is known to allow for transmission of the signal over a medium, for example wireless. It was discovered recently (see ([BrJo02])), perhaps surprisingly, that the functions which give good filters in this context serve a different purpose as well: they offer the parameters which account for families of wavelet bases, for example families of bases functions in the Hilbert space $L^2(R)$. Indeed the simplest quadrature-mirror filter is known to produce the Haar wavelet basis in $L^2(R)$. It is further shown in ([BrJo02]) that both subjects (2) and (3) are governed by families of representations of one of the Cuntz algebras O_N, with the number N in the subscript equal to the number of sub-bands in the particular model. So for the Haar case, $N = 2$.

A main purpose in this paper is pointing out that fractional Brownian motion may be understood with the data in (2) and (3), and as a result that fBm may be understood with the use of a family of representations of O_N; albeit a quite different family of representations from those used in ([BrJo02]).

Let us start with a few definitions.

2 Definitions and Background

We begin this section with a few facts on the Wiener process that will be needed later on. The Wiener process represents an integral of a white noise process, and the latter enters, e.g., in the modeling of noise features in engineerings, errors in filtering theory and random forces in physics. Brownian motion, i.e., diffusion of minute particles suspended in fluid, and diffusion in general are described in terms of the Wiener process. It is also the basis for the study of stochastic integrals and rigorous path integration in quantum mechanics (see, eg., [Al]).

A (real-valued) stochastic process is an indexed system $X(t)$, $t \geq 0$ of random variables, i.e. (real-valued) measurable functions on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$

$$t \to X(t, \omega) \quad \omega \in \Omega, \tag{1}$$

is by definition a path of the process $(X(t), t \geq 0)$ We will make use of complex Hilbert spaces $\mathcal{H}$. When $\mathcal{H}$ is given, its inner product product is denoted $\langle ., . \rangle$. If more choices of Hilbert spaces are involved, we will use subscripts to make distinctions. A real-valued mean zero Gaussian process $W = (W(t), t \geq 0)$ with continuous paths and covariance function

$$E[W(t)W(s)] = t \wedge s \tag{2}$$

with $t, s \geq 0$ and where $t \wedge s$ stands for the minimum of t and s is called a standard Wiener process or standard Brownian motion. Let us recall a few facts about Brownian motion and the Haar functions as wavelets. The first proof of the existence of such process was given by Wiener (1923), based on Daniell's method (Daniell 1918) of constructing measures on infinite-dimensional spaces. In Paley and Wiener (1937) the Wiener process is constructed using Fourier series expansions and assuming the existence of a sequence of independent, identically distributed Gaussian random variables. The following construction is due to P. Lévy (1948), Z. Ciesielski (1961), Itô-Nisio (1968). In their construction the essential role is played by the Haar system connected with a dyadic partition of the interval $[0, 1]$. Namely, set $h_0 = 1$ and, for $2^n \leq k < 2^{n+1}$, $n \in \mathbf{N}$ set

$$\begin{aligned} h_k(t) &= \begin{cases} 2^{\frac{n}{2}} & \text{if } \frac{k-2^n}{2^n} \leq t < \frac{k-2^n}{2^n} + \frac{1}{2^{n+1}} \\ -2^{\frac{n}{2}} & \text{if } \frac{k-2^n}{2^n} + \frac{1}{2^{n+1}} \leq t < \frac{1}{2^n} \end{cases} \\ h_k(1) &= 0. \end{aligned}$$

h_k are the Haar functions and build an orthonormal basis in $L^2[0,1]$. One has the following:

Theorem 1 *Let $(X_k, k = 0, 1, \ldots)$ be a sequence of independent random variables with distribution $\mathcal{N}(0,1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then $\mathbf{P}$-a.s.,the series*

$$\sum_{k=0}^{\infty} X_k(\omega) \int_0^t h_k(s)ds, \quad t \in [0,1]$$

converges uniformly on $[0,1]$. A modification of this process is a standard Wiener process on $[0,1]$. (see,e.g., [Part] for the concept of modification and details). Note that $\int_0^t h_k(s)ds = \left\langle h_k, \chi_{[0,t]} \right\rangle$, where $\langle , \rangle$ is the scalar product in $L^2[0,1]$.

More generally, pick an orthonormal basis (ONB) ψ_k in $L^2(0,1)$. The unitary map that sends $\{\psi_k\}$ to $\{X_k\}$ sends the indicator function $\chi_{[0,t]}(.)$ of the interval $0 \leq s \leq t$, $t \leq 1$ considered as an element of $L^2[0,1]$ a random variable $X(t,\omega)$ in $L^2(\Omega, \mathcal{F}, P)$. Setting

$$X^n(t,\omega) = \sum_{k=0,\ldots,n} a_k(t) X_k(\omega) \tag{3}$$

where $a_k(t) =< \chi_{[0,t\,)}, \psi_k >$ (with $< ., . >$ being the scalar product given by (9)) are shown that $X^n(t,\omega)$ converges in $L^2(P)$ to random variables $X(t,\omega)$ which are centered and have the covariance function $E[X(s)X(t)] = s \wedge t$ of Brownian motion on $[0,1]$.

More generally, for a centered Gaussian process the covariance $k(s,t) = E[X(s)X(t)]$, $0 \leq s,t \leq 1$ determines the process uniquely (up to a natural equivalence for processes) and moreover, determines a complex Hilbert space $\mathcal{H}$ of functions obtained by completing the set of finite $\mathbf{C}$-linear combinations of $k(s,.)$ of the form

$$f(.) = \sum_{i=1,\ldots,n} a_i k(s_i,.), \quad a_i \in \mathbf{C} \tag{4}$$

with respect to the bilinear form

$$< f,g >= \sum_{i=1,\ldots,n} \sum_{j=1,\ldots,n} \bar{a}_i b_j k(s_i,s_j). \tag{5}$$

We recall the definition of reproducing kernel Hilbert space.

Definition 2 *A Hilbert space of functions $\mathcal{H}$ on a set S is said to be a reproducing kernel subspace if for every $s \in S$ the mapping $f \in \mathcal{H}; f \to f(s) \in \mathbf{C}$ is continuous.*

Note that by Riesz's lemma this means that for every s there is a unique element $i(s) \in \mathcal{H}$ such that

$$f(s) = \langle i(s)|f \rangle \tag{6}$$

(with the $< | >$ the scalar product in $\mathcal{H}$), and by Schwarz's inequality we get

$$|f(s)| \leq \|i(s)\| \|f\| \tag{7}$$

($\|.\|$ being the norm in $\mathcal{H}$).
As a variant we shall consider the case where instead of (7), we have only

$$|f(s_1) - f(s_2)| \leq const \, \|f\| \text{ for all } s_1, s_2 \in S. \tag{8}$$

As an example of (8), consider differentiable functions modulo constants defined on a subinterval $S \subset \mathcal{R}$ with inner product

$$< f_1, f_2 >= \int_S \bar{f}'_1(s) f'_2(s) ds. \tag{9}$$

Since by Schwarz's inequality

$$|f(s_1) - f(s_2)|^2 \leq |s_1 - s_2| \int |f'(t)|^2 \, dt$$

it follows that (8) is satisfied for the Hilbert space $\mathcal{H}$ with inner product (9).
The Hilbert space it is said to have $k(s,.)$ as a reproducing kernel $k\,(s,.)$, that is $k\,(s,.) \in \mathcal{H}$ and $< k\,(s,.)\,, g\,(.) >= g(s)$, for every $s \in [0,1]$.

In the case where $k(s,t) = s \wedge t$ the Gaussian process $B(t)$, $t \in [0,1]$ with mean 0 and covariance

$$E\,[B(s)B(t)] = k\,(s,t) \tag{10}$$

is a realization of standard Brownian motion (or Wiener process) on $[0,1]$, which we already described before. In this case the collection of finite linear combinations

$$F(t) = \sum_{i=1,\dots,n} f(s_i) s_i \wedge t, \tag{11}$$

with f as in (4) is precisely the collection of all piecewise linear functions F in $[0,1]$ with $F(0) = 0$.

It follows from the Lévy Ciesielski Ito Nisio construction (see Theorem 1 above and [Part]) that $t \to B(t)$ is continuous, so that the probability space can be taken to be $\Omega = C\,([0,1])$, $\mathcal{F} = \mathcal{B}\,([0,1])$. In this case the Ito-integral $\int f(t) dB(t)$ is well defined for any non-anticipating function such that

$$\int_0^T \|f(t)\|^2_{L^2(\Omega,\mathcal{P})} \, dt < \infty \tag{12}$$

and the Ito isometry holds:

$$\left\| \int_0^T f(t)dB(t) \right\|^2_{L^2(\Omega,\mathcal{P})} = \int_0^T \|f(t)\|^2_{L^2(\Omega,\mathcal{P})} \, dt \tag{13}$$

In the case where $k(s,t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H} \right)$, $H \in (0,1)$ the Gaussian process with mean 0 and covariance

$$E[B_H(t)B_H(s)] = k(s,t) \tag{14}$$

is a realization of fBm with Hurst index H (defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$). The extension of general centered Gaussian process given by a kernel $k(s,t)$ $B(t)$ respectively of $B_H(t)$ to $t \in [0,T]$, $T > 0$ arbitrary is immediate (see, e.g. ([Part]) for $B(t)$). Let $\mathcal{H}_T$ be the linear space closure in $L^2(\mathcal{P})$ of the (complex) linear space of a collection of random variables $B_H(t) : t \in [0,T]$ for some fixed time $T \geq 0$.

For a construction of fBm as functionals of the standard white noise see ([HuLi06],[CiKa]) and references therein.

Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing smooth function on $\mathbf{R}$ and $\mathcal{S}'(\mathbf{R})$ be the space of tempered distributions.

Denote by $\langle .,. \rangle$ the dual pairing on $\mathcal{S}'(\mathbf{R}) \times \mathcal{S}(\mathbf{R})$.

Fix a Hurst parameter H and define

$$\psi(s,t) = H(2H-1)\,|s-t|^{2H-2}, \quad s,t \in \mathbf{R}$$

where $c_H^2 = \frac{H(2H-1)}{B(H-\frac{1}{2},2-2H)}$, $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function, $K_\pm(t) = c_H t_\pm^{H-\frac{3}{2}}$, and $t_+ = t \wedge 0$, $t_- = -(t \wedge 0)$.. Moreover $\Gamma(x)$ denotes Euler's gamma function.

For $f, g \in \mathcal{S}(\mathbf{R})$ an inner product is defined in terms of above functions ψ by :

$$\langle f,g \rangle_\psi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s)g(t)\psi(s,t)dsdt.$$

By Ito's regularization theorem (see,e.g. [PeZa]), there exists a unique $\mathcal{S}'(\mathbf{R})$-valued random variable $T : \mathcal{S}'(\mathbf{R}) \to \mathcal{S}'(\mathbf{R})$ such that $\langle T\omega, \xi \rangle = \langle \omega, (K_- * f)(t) \rangle$ where $(K_- * f)(t) = c_H \int_u^\infty (s-u)^{H-\frac{3}{2}} f(s)ds$. Set $\Gamma_\psi f(u) = (K_- * f)(t)$ then we have:

Theorem 3 *Let $\mu_\psi = \mu \circ T^{-1}$ be the image of the measure of μ induced by the map T. Then, for any $\xi \in \mathcal{S}(\mathbf{R})$, the distribution of $\langle ., \xi \rangle$ under μ_ψ is the same as $\langle ., \Gamma_\psi \rangle$ under μ. Thus*

$$B_H(t) \equiv \left\langle \omega, \Gamma_\psi 1_{[0,t]} \right\rangle \tag{15}$$

$t \geq 0$, is (a realization of) the standard fractional Browian motion with Hurst constant H.

We now present our approach to stochastic integration with respect to fractional Brownian motion. In the case where X_t fBm $B_H(t)$, we have:

$$E[X(t)X(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{16}$$

If $H = \frac{1}{2}$ we have the covariance $\frac{1}{2}(|t| + |s| - |t-s|) = s \wedge t$ of Brownian motion and with $J_i = [s_i, t_i)$, $i = 1, 2$, we see that

$$E[(X(t_1) - X(s_1))\,(X(t_2) - X(s_2))] = \left|J_1 \bigcap J_2\right|, \tag{17}$$

i.e., we have independent increments, a property which is important for stochastic integration. But when $H \in (0,1) \backslash \left\{\frac{1}{2}\right\}$, things are more complicated; see details below and, eg., ([AL08].

2.1 Spectral representation

There exists several representations of the fBm that allow one to understand the structure of the linear space $\mathcal{H}_T$ defined above. One such representation is the spectral representation

$$E\left[B_H(t)B_H(s)\right] = \int_{\mathbf{R}} \frac{(\exp(i\lambda t) - 1)(\exp(-i\lambda s) - 1)}{\lambda^2} \mu(d\lambda) = \langle e_t, e_s \rangle_\mu . \tag{18}$$

where $\mu(d\lambda) = (2\pi)^{-1} \sin(\pi H)\,\Gamma(1+2H)\,|\lambda|^{1-2H}\, d\lambda$ is the spectral measure of the fBm, $e_t(\lambda) = \frac{\exp(i\lambda t)-1}{i\lambda}$ and $\langle , \rangle_\mu$ is the scalar product in $L^2(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mu) \equiv L^2(\mu)$.

Let $\mathcal{L}_T$ be the closure in $L^2(\mu)$ of the complex linear span of the collection of functions $e_t : t \in [0,T]$. This representation gives rise to an isometry between $\mathcal{H}_T$ and $\mathcal{L}_T$ by sending X_t to e_t. Let us denote by 1_t the indicator function of the interval $[0,t]$. Its Fourier transform is :

$$\begin{aligned} \hat{1}_\lambda(t) &= \int_{\mathbf{R}} 1_t \exp(i\lambda x)\, dx = \int_0^t \exp(i\lambda x)\, dx \\ &= \frac{(\exp(i\lambda t) - 1)}{i\lambda} = e_\lambda(t). \end{aligned}$$

We consider the class of functions

$$T_t = \left\{ f \in L^2[0,T]\,;\, \widehat{f} \in L^2(\mu) \right\}$$

with the inner product $\langle f, g \rangle_{T_t} = \left\langle \widehat{f}, \widehat{g} \right\rangle_\mu$. Then the spectral representation of B_H can be written as $E\left[B_H(s)B_H(t)\right] = \langle 1_s, 1_t \rangle_{T_t}$. In particular the mapping sending 1_t into $B_H(t)$ extends to a linear map $I : T_t \to H_t$ such that $I(1_t) =$

$B_H(t)$ for $f, g \in T_t$ and $E[I(f), I(g)] = \left\langle \widehat{f}, \widehat{g} \right\rangle_\mu$.
We define the following kernel m_t, for $t \geq 0$

$$m_t(u) = \frac{1}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(H+\frac{3}{2}-H\right)} u^{\frac{1}{2}-H}(t-u)^{\frac{1}{2}-H} 1_t(u) \tag{19}$$

where $u \geq 0$ and Γ denotes Euler's gamma function. Then for every $t \in [0, T]$, $m_t \in T_t$, using the Poisson integral formula for the Bessel functions we get:

$$\widehat{m}_t(\lambda) = \begin{cases} \frac{\sqrt{\pi}}{2H\Gamma(H+\frac{1}{2})} \frac{t}{\lambda}^{1-H} \exp(i\lambda\frac{t}{2}) J_{1-H}\left(\frac{\lambda t}{2}\right) & if \lambda \neq 0 \\ \frac{\sqrt{\pi}}{2H\Gamma(H+\frac{1}{2})\Gamma(2-H)2^{2-2H}} & \lambda = 0, \end{cases} \tag{20}$$

where J_{1-H} is the Bessel function of the first kind of order $1-H$. To evaluate $\widehat{m}_t$ at $\lambda = 0$ we use the formula $z^{-\nu}J_\nu(z) \to \frac{1}{2^\nu\Gamma(\nu+1)}$, as $z \to 0$ (see, e.g. [Wa].)
For $t \in [0, T]$ we consider the random variable given by the formula:

$$M_t = \int m_t(u)\, dM_H(u) = \int_0^t m_t(u)\, dM_H(u) \qquad in\ H_T, \tag{21}$$

the integral being defined, eg., in ([DhFe]). We have that

$$E[M_s M_t] = \langle \widehat{m_s}, \widehat{m_t} \rangle = \widehat{m}_{s\wedge t}(0). \tag{22}$$

Hence the process M defined by (21) is a continuous Gaussian martingale with bracket $\langle M \rangle = \widehat{m}(0)$. The variance function V_t is given by $V_t = E[M_t^2] = d_H^2 t^{2-2H}$, where

$$d_H^2 = \frac{\Gamma\left(\frac{3}{2}-H\right)}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(3-2H)} \tag{23}$$

is a constant for every $t \geq 0$. M is called the fundamental martingale (associated with fBm).

We will be using the following parameters which are part of the wavelet-based synthesis of the fBm. Let $T = 2^M$ be the time duration $[0, T]$ of the synthesis of the fBm. Let 2^{-J} be the scale at which a final wavelet-based approximation of fBm is taken. This means that a wavelet-based approximation of fBm is taken at the following time points: $0, 2^{-J}, 2.2^{-J}, \ldots, 2^{M-J}+1$. We refer to 2^{-J} as a final approximation scale.

3 Cuntz algebras and Brownian motion

In the past few years (starting with the cited papers ([Br-Jo1],[BrJo02]) by Bratteli and Jorgensen), perhaps surprisingly, some areas of operator algebras have found applications in engineerings, more precisely in the parts of signal/image processing concerned with sub-band filtering. The reason for this is that the building of sub-band filters entails the use of specific systems of non-commuting operators in a Hilbert space of L^2-functions. These functions represent the frequency profile of the signals. In a particular model, the operator of up-sampling then becomes an isometry, and down-sampling becomes the corresponding adjoint co-isometric operator. In frequency space, the filters turn into multiplication operators. When the entire system is put together, what emerges is a representation of a C*-algebra. Perhaps surprisingly, in this case it turns out to be one of the C*-algebras O_N in a sequence of purely infinite C*-algebras, or the related Cuntz-Krieger algebras, indexed by an $N \times N$ transition matrix A.

In these applications, the number N is the number of frequency bands employed. Hence the study of representations of O_N , respectively of O_A, produces solutions to problems in signal/image processing. The use of filters this way has further been applied to the statistical analysis of signals. While the use of stochastic processes has been somewhat restricted in earlier papers, we turn here to the case of fractional Brownian motion (fBm). Indeed, it has been found that fBm provides a good model for the kind of noise observed in real-life transmission of signals over a medium.
As we shall see, the theory of representations of the Cuntz algebras is also well suited for applications to stochastic integration.
Let us give some preliminaries on wavelets and representations of the Cuntz algebra. We recall that one denotes by O_N the C^*-algebra generated by N, $N \in \mathbf{N}$, isometries $S_0, \ldots, S_{N-1}$ satisfying

$$S_i^* S_j = \delta_{ij} \mathbf{1} \tag{24}$$

and

$$\sum_{i=0}^{N-1} S_i S_i^* = \mathbf{1}. \tag{25}$$

where $i, j = 0, \ldots, N-1$.
The starting point for the multiresolution analysis from wavelet theory is a system U, $\{T_j\}_{j \in \mathbf{Z}}$, of unitary operators with the property that the underlying complex Hilbert space $\mathcal{H}$ with norm $\|.\|$ contains a vector $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, satisfying

$$U\varphi = \sum_{j \in \mathbf{Z}} a_j T_j \varphi \tag{26}$$

for some sequence $\{a_j\}$ of complex scalars. In addition, the operator system $\{U, T_j\}$ must satisfy a non-trivial commutation relation. In the case of wavelets, it is

$$UT_jU^{-1} = T_{N_j}, \qquad j \in \mathbf{Z}, \tag{27}$$

where $N \in \mathbf{N}_0$ is the *scaling number*, or equivalently the number of *subbands* in the corresponding multiresolution. These relations play a role in signal processing and wavelet analysis. When this system U,$(T_j)_{j \in \mathbf{Z}}$ is present, there is a way to recover the spectral structure of the problem at hand from representations of an associated C^*-algebra. In the case of orthogonal wavelets, we may take this C^*-algebra to be the Cuntz algebra. In that case, the operators T_j may be represented on $L^2(\mathbf{R})$ as translations,

$$(T_j\xi)(x) = \xi(x-j), \qquad \xi \in L^2(\mathbf{R}), x \in \mathbf{R},$$

and U may be taken as the scaling $(U\xi)(x) = N^{-1/2}\xi(x/N)$, $N \in \mathbf{N}$. This system clearly satisfies (27). In the wavelet case, a multiresolution is built from a $\varphi \in L^2(\mathbf{R})$ satisfying (26). The numbers $\{a_j\}_{j\in\mathbf{Z}}$ occurring in (26) must then satisfy the orthogonality relations

$$\sum_{k\in\mathbf{Z}} a_k = 1, \qquad \sum_{k\in\mathbf{Z}} \bar{a}_k a_{k+2m} = \delta_{0,m}, m \in \mathbf{Z} \tag{28}$$

In this case, the analysis is based on the Fourier transform: define m_0 as a map from S^1 to $\mathbf{C}$ by

$$m_0\left(e^{it}\right) = \sum_{k\in\mathbf{Z}} a_k e^{ikt}, t \in \mathbf{R}.$$

Then (in the wavelet case, following ([Dau]) a φ satisfying (26) in $\mathcal{H} = L^2(\mathbf{R})$ is given by the $L^2(\mathbf{R})$ product formula

$$\hat{\varphi}(t) = \prod_{j=1}^{\infty} m_0\left(t/N^j\right), \tag{29}$$

up to a constant multiple ($\hat{\varphi}$ denoting the Fourier trasform of φ). The Cuntz algebra O_N enters the picture as follows: formula (29) is not practical for computations, and the analysis of orthogonality relations is done better by reference to the Cuntz relations, see (24)–(25). Setting, for $\xi_j \in \mathbf{C}$, $j \in \mathbf{Z}$

$$W(\{\xi_j\}) := \sum_{j\in\mathbf{Z}} \xi_j \varphi(x-j), \tag{30}$$

and using (28), we get an isometry W of ℓ^2 into a subspace of $L^2(\mathbf{R})$, the resolution subspace. Setting

$$(S_0 f)(z) := \sqrt{N} m_0(z) f\left(z^N\right), \qquad f \in L^2(\mathbf{T}), \text{Borel measurable}, (\text{z} \in \mathbf{C}, |\mathbf{z}| = \mathbf{1}), \tag{31}$$

$\mathbf{T}$ being the 1-torus and using the isomorphism $L^2(\mathbf{T}) \cong \ell^2$ given by the Fourier series, we establish the following crucial intertwining identity:

$$WS_0 = UW, \tag{32}$$

so that U is a unitary extension of the isometry S_0. It was shown in Refs.([BEJ]) and ([BrJo02]) that functions $m_1, \ldots, m_{N-1} \in L^\infty(\mathbf{T})$ may then be chosen such that the corresponding matrix

$$\left(m_j\left(e^{i(t+k2\pi/N)}\right)\right)_{j,k=0}^{N-1} \tag{33}$$

is in $\mathrm{U}_N(\mathbf{C})$ for (Lebesgue) a.a. $t \in \mathbf{R}$. Then it follows that the operators

$$S_j f(z) := \sqrt{N} m_j(z) f\left(z^N\right), \qquad f \in L^2(\mathbf{T}), \tag{34}$$

will yield a representation of the Cuntz relations; see (24)–(25). Conversely, if (34) is given to satisfy the Cuntz relations, then the matrix in (33) takes values in $\mathrm{U}_N(\mathbf{C})$.

For the benefit of the reader we include the correspondence between the unitary matrix functions:

$$\mathcal{U} : \mathbf{T} \to \mathrm{U}_{\mathbf{N}}(\mathbf{C}), \tag{35}$$

(a loop) and the solutions (m_j), $j = 0, \ldots, N-1$, used in (33) and (34).

Lemma 4 *If $\mathcal{U}$ is a unitary matrix function, then the system (m_j) given by*

$$\begin{pmatrix} m_0(z) \\ m_1(z) \\ \ldots m_{N-1}(z) \end{pmatrix} = \mathcal{U}(z^N) \begin{pmatrix} 1 \\ z \\ z^2 \\ \ldots z^{N-1} \end{pmatrix} \tag{36}$$

satisfies the property (34), and vice versa. Column vector notation is used on the two sided of equality (36), and matrix multiplication is meant on the right hand side.

Proof. See ([BrJo02]). Indeed given (m_j) subject to (34), there is a direct formula for the loop group $\mathcal{U}(.)$ which solves (36). $\square$

The scaling could be between different resolutions in a sequence of closed subspaces of the underlying Hilbert space or it could refer to a system of frequency bands. Two structures are then present: a scaling from one band to the next and operations within each band. These two operations can be realized in a certain tensor factorization of the relevant Hilbert space. The representations we will consider are realized on Hilbert spaces $\mathcal{H} = L^2(X, \nu)$, where X is a measure space which will be specified later and ν is a probability measure on X.

The representations can be defined also in terms of maps

$$\sigma_i \colon X \longrightarrow X \quad \text{such that} \quad X = \bigcup_{i=0}^{N-1} \sigma_i(X) \quad \text{and} \quad (\sigma_i(X) \cap \sigma_j(X)) = \emptyset \quad \text{for all } i \neq j.$$

A system (σ_i) as above is called an iterated function system, and X is called invariant under the system σ_i. An iterated function system (IFS) σ_i is said to be continuous in some metric d on X if there is a constant $c < 1$ such that

$$d(\sigma_i(x), \sigma_i(y)) \leq c d(x, y)$$

holds for all $i \in \{0, \ldots, N-1\}$ and $x, y \in X$. In that case, it is known that there is a unique probability Borel measure μ on X which is invariant in the following sense:

$$\mu = \frac{1}{N} \sum_{i=0,\dots,N-1} \mu \circ \sigma_i^{-1},$$

or equivalently

$$\int f d\mu = \frac{1}{N} \sum_{i=0,\dots,N-1} \int f \circ \sigma_i d\mu$$

holds for all Borel functions f which are $\mu-$integrable.
In this case, we take $X := supp(\mu)$. Let $H \in (0,1)$ and for some fixed time horizon $T \geq 0$ let us consider the Bessel functions J_{1-H} of order $\nu = 1 - H$, J_{1-H} has $N(T)$ zeros in the interval $[0,T]$. Let us order them and denote them by j_i, $i = 1,\dots,N(T)$, such that $0 \leq j_1 \leq j_2 \leq \dots \leq j_{N(T)} \leq T$.
Our goal is to express the fundamental martingale associated to a fractional Brownian motion process by operators that in a particular case generate the Cuntz algebra. By ([Wa])

$$\int_0^\infty J_{1-H}\left(\frac{\lambda j_h}{2}\right) J_{1-H}\left(\frac{\lambda j_k}{2}\right) d\lambda = \delta_{h,k}, h,k \in \{1,\dots,N(T)\}.$$

In the following we shall see how the covariance of the fundamental martingale of a fractional Brownian process can be associated to a family of operators in a Hilbert space. $\mathcal{B}(\mathbf{C})$ be the space of bounded operators in tye field $\mathbf{C}$ of complex numbers. Consider a family of operators $S_t \in \mathcal{B}(\mathbf{C})$ defined by

$$S_t e_\lambda = J_{1-H}\left(\frac{\lambda t}{2}\right) e_{\lambda t}, t \in (0,\infty). \tag{37}$$

where $e_\lambda = \exp(i\lambda)$, $\lambda \in \mathbf{R}$.

Then by taking the inner product in $L^2(\mathbf{C})$ we get:

$$\langle S_t^\star S_s e_\lambda, e'_\lambda\rangle = \langle S_s e_\lambda, S_t e'_\lambda\rangle = \left\langle J_{1-H}\left(\frac{\lambda t}{2}\right) e_{\lambda t}, J_{1-H}\left(\frac{\lambda s}{2}\right) e'_{\lambda s}\right\rangle. \tag{38}$$

For $H \neq \frac{1}{2}$, $s,t > 0$ we get

$$\int_0^\infty J_{1-H}\left(\frac{t\lambda}{2}\right) J_{1-H}\left(\frac{s\lambda}{2}\right) \frac{1}{\lambda} \cos(a\lambda)\, d\lambda = \left(\frac{s}{t}\right)^{1-H} \frac{1}{2-2H}. \tag{39}$$

Choosing $a = t - s$ it is shown in ([Dz-Za]) that this gives the covariance function of the fundamental martingale associated to a fractional Brownian motion. Taking real parts in (38) resp (39) as in ([Dz-Za]) we get:

$$Re\left(\langle S_s, S_t\rangle\right) = \frac{1}{2-2H}\left(\frac{s}{t}\right)^{2H}. \tag{40}$$

This is the covariance $E\left[M_t M_s\right]$ of the fundamental (centered) martingale M_t of the fBm process in terms of the operators S_i and its adjoint. Hence

$$Re\left(\langle S_s, S_t\rangle_\lambda\right) = E\left[M_t M_s\right]. \tag{41}$$

In the particular case where $s = j_k$ and $t = j_h$ are zeroes of the Bessel function in $[0, T]$ the operators S_k satisfy the Cuntz algebra relations, eg,

$$\langle S_k, S_h\rangle = \left\langle J_{1-H}\left(\frac{\lambda j_k}{2}\right), J_{1-H}\left(\frac{\lambda j_h}{2}\right)\right\rangle = \delta_{k,h}, \tag{42}$$

by using ([Wa]) the orthogonality of the Bessel functions. This is the realization of the covariance of the fundamental martingale for an operator representation of fBm, as we shall discuss in the next section.

4 Fractional Brownian motion and Haar wavelets

As we recalled in Section 2, Haar functions seen as wavelets basis can be used to construct the standard Brownian motion using them as a complete orthonormal basis. In this section we look similarly at the fBm processes via wavelets bases.

Theorem 5 *Consider a pair of representations S_i and $\tilde{S}_j$ of the Cuntz algebra as in Section 3 associated to the ground wavelets ψ_i and φ_j satisfying (25). Then there exists a centered fBm process $B_H(t)$ such that*

$$E\left[B_H(t)\, B_H(s)\right] = \tag{43}$$

$$\Gamma(H)\left(< \chi_{(0,t]}, \psi_k > + < \chi_{(0,s]}, \varphi_k > - \left|< \chi_{(0,t]}\chi_{(0,s]}, \psi_k\varphi_h >\right|\right) = \tag{44}$$

$$\frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right). \tag{45}$$

Proof. We construct a process whose covariance function is the one of a (centered) fBm process. This is built by using a pair of wavelet functions and relating them to representations of the Cuntz algebras. Let $\{\Psi_k\}$, $\{\Phi_k\}$, be two families of the Haar wavelets (as in Section 2.1). Assume that the coefficients $\{a_k(t)\}$, $\{b_k(t)\}$ relative to the filters of Haar wavelets $\{\Psi_k\}$, $\{\Phi_k\}$ respectively are given by

$$\langle \chi_{[0,t]}, \Psi_k\rangle = \int \chi_{[0,t]}(x)\Psi_k(x)(t-x)^{2H-1}dx = \delta_{t,k}\frac{t^{2H}}{2\Gamma(H)} \tag{46}$$

and

$$\langle \chi_{[0,t]}, \Phi_h\rangle = \int \chi_{[0,t]}(x)\Phi_h(x)(t-x)^{2H-1}dx = \delta_{t,h}\frac{t^{2H}}{2\Gamma(H)} \tag{47}$$

with $t \in [0, 1]$ and $k, h \in \mathbf{Z}_+$.

To calculate

$$\langle \chi_{[0,t]}\chi_{[0,s]}, \Psi_k \Phi_h \rangle$$

we distinguish two cases, for $t > s$ and $t < s$ respectively. In the former one, we have

$$\langle \chi_{[0,t]}\chi_{[0,s]}, \Psi_k \Phi_h \rangle = \langle \chi_{[0,t-s]}, \Psi_k \Phi_h \rangle = \delta_{t,t-s} \frac{(t-s)^{2H}}{2\Gamma(H)} \tag{48}$$

while in the latter case we have

$$\langle \chi_{[0,t]}\chi_{[0,s]}, \Psi_k \Phi_h \rangle = \langle \chi_{[0,s-t]}, \Psi_k \Phi_h \rangle = \delta_{t,s-t} \frac{(s-t)^{2H}}{2\Gamma(H)} \tag{49}$$

Combining (46) and (47) with (48) and (49) we get

$$\frac{1}{2}\left(\frac{|t|^H}{\Gamma(H)} + \frac{|s|^H}{\Gamma(H)} - \frac{|t-s|^{2H}}{\Gamma(H)} = \right) = \frac{1}{2\Gamma(H)}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) = E[B_H(t)B_H(s)]$$

the covariance of a process $B_H(t)$ which is a realization of the fractional Brownian motion in the sense that it is a Gaussian mean zero process with the covariance function $E\left(B_H(s)B_H(t)\right) = k(s,t)$ of the fBm (14) in Section 2. □

Remark.
Let

$$X^n(t,\omega) = \sum_{k=0}^{n} a_k(t) X_k(\omega) = \sum_{k=0}^{n} \left\langle \chi_{[0,t]}, \Psi_k \right\rangle X_k(\omega) \tag{50}$$

with $a_k(t) \equiv \left\langle \chi_{[0,t]}, \Psi_k \right\rangle$, X_k the k-th coordinate function as in Section 2.1.
Set correspondingly

$$Y^n(t,\omega) = \sum_{k=0}^{n} b_k(t) Y_k(\omega) = \sum_{k=0}^{n} \langle \chi_{[0,t]}, \Phi_k \rangle Y_k(\omega) \tag{51}$$

with $b_k(t) \equiv \left\langle \chi_{[0,t]}, \Psi_k \right\rangle$, Y_k being also the k-th coordinate function. Both $\{X^n(t,\omega)\}$, $\{Y^n(t,\omega)\}$ are approximations of two independent standard Brownian motions X_t, Y_t such that their covariance functions are given by

$$E[X_t X_s] = s \wedge t \quad \text{and} \quad E[Y_t Y_s] = s \wedge t,$$

We have

$$\langle \chi_{[0,t]}, \Psi_k \rangle = \int \chi_{[0,t]}(x) \Psi_k(x) (t-x)^{2H-1} dx = \delta_{t,k} \frac{t^{2H}}{2\Gamma(H)} \tag{52}$$

and

$$\langle \chi_{[0,t]}, \Phi_h \rangle = \int \chi_{[0,t]}(x) \Phi_h(x) (t-x)^{2H-1} dx = \delta_{t,h} \frac{t^{2H}}{2\Gamma(H)} \tag{53}$$

with $t \in [0,1]$ and $k, h \in \mathbf{Z}_+$.

The random series

$$X_t(\omega) = \sum_{n=1}^{\infty} X^n(t,\omega) \tag{54}$$

and

$$Y_t(\omega) = \sum_{n=1}^{\infty} Y^n(t,\omega) \tag{55}$$

represent two Gaussian processes, and $X_t(\omega)$ and $Y_t(\omega)$ should be understood as sums of the series in the $L^2(\mathcal{P})$ sense. From (54), (55) and (50), (51) we get

$$\begin{aligned} X_t(\omega)Y_s(\omega) &= \frac{1}{2\Gamma(H)} \sum_{h,k=1}^{\infty} \langle \chi_{[0,t-s]}, \Psi_k \Phi_h \rangle X_k(t,\omega) Y_h(s,\omega) \\ &= \frac{1}{2\Gamma(H)} \sum_{h=1}^{\infty} \delta_{h,k} \left(t-s\right)^{2H} X_k(t,\omega) Y_h(s,\omega) \end{aligned}$$

and

$$\begin{aligned} X_t(\omega)Y_s(\omega) = \sum_{h,k=1}^{\infty} \langle \chi_{[0,s-t]}, \Psi_k \Phi_h \rangle X_k(t,\omega) Y_h(s,\omega) = \\ \sum_{h=1}^{\infty} \delta_{h,k} \frac{1}{2\Gamma(H)} \left(s-t\right)^{2H} X_k(t,\omega) Y_h(s,\omega). \end{aligned}$$

(with convergence in the $L^2(\mathcal{P})-$sense) which implies

$$X_t(\omega)Y_s(\omega) = |s-t|^{2H} \sum_{h=1}^{\infty} X_h(s,\omega) Y_h(t,\omega)$$

Now we introduce a construction of a twisted full Fock space suitable to study a pair of Cuntz algebra representations in connections with fBm and wavelets.

Definition 6 . *The full Fock space over $\mathbf{C}^N$, where N is a fixed positive integer with $N \geq 2$, is the orthogonal direct sum of Hilbert spaces given by*

$$\mathcal{K} = \left(\sum_{k=-\infty}^{-1} \oplus \left(\mathbf{C}^N\right)^{\otimes -k} \right) \oplus \mathbf{C}$$

Let $\{\xi_1, ..., \xi_N\}$ be a fixed orthonormal basis for $\mathbf{C}^N$ and let $\Omega_{0=\{1,0,...,0\}}$ be the vacuum vector. Then $\mathcal{K}$ is an infinite-dimensional complex Hilbert space with orthonormal basis given by $\{\xi_{i_1}, ..., \xi_{i_k} : 1 \leq i_1, ..., i_k \leq N,\ k \geq 1\} \cup \{\Omega_0\}$. We take the tensor product of the full Fock space $\mathcal{K}$ with a given complex Hilbert space $\mathcal{H}$, then we define a "new" inner product $\langle\cdot,\cdot\rangle_\Phi$ by using a completely positive map Φ from the complex matrices into $B(\mathcal{H})$ (i.e. a positive matrix with entries in $B(\mathcal{H})$). The completely positive map Φ can be identified with the positive matrices $P = [p_{i,j}] \in M_N(\mathcal{B}(\mathcal{H}))$ where the correspondence is given by

$$P = \Phi^{(N)}([e_{i,j}]) = [\Phi(e_{i,j})]$$

where P is the Choi matrix [GK09], [Ch80] associated with Φ, $P = [\Phi(e_{i,j})]$. Every completely positive matrix can be naturally extended to the matrix algebras M_{N^k} by $\widetilde{\Phi}(a_1 \otimes \ldots \otimes a_k) = \Phi(a_1)\ldots\ \Phi(a_k)$. We take the matrix P associated to the map Φ as the row matrix $P = \begin{bmatrix} S & \widetilde{S} \end{bmatrix}$. From [Jo-Kr] a new Fock space is defined as follows. We start with the $N-$variables pre-Fock space over $\mathcal{K}$ to be the vector space of finite sums

$$T_N(\mathcal{H}) = \left\{ \sum_{|w|\leq k} w \otimes h_w : w \in \mathbf{F}_N^+,\ k \geq 1,\ h_w \in \mathcal{H} \right\},$$

where $\mathbf{F}_N^+$ is the unital free-semigroup on N, $N \in \mathbf{Z}$, non-commuting letters $\{1, 2, ..., N\}$ with unit e. We can think of the full Fock space $\mathcal{K}$ as $l^2\left(\mathbf{P}_N^+\right)$ where an orthonormal basis is given by the vectors $\left\{\xi_w : w \in \mathbf{P}_N^+\right\}$ corresponding to the words w, $|w|$ is the word of length zero or empty word, $\xi_w = \xi_{i_1} \otimes \ldots \otimes \xi_{i_n}$, $w = i_1...i_k \in \mathbf{F}_N^+$.. Then a vector $(i_1, ..., i_k) \otimes h$ with $w = i_1...i_k \in \mathbf{F}_N^+$ corresponds to the vector $\xi_{i_1} \otimes \ldots \otimes \xi_{i_n} \otimes h$ in $\left(\mathbf{C}^N\right)^{\otimes k} \otimes \mathcal{H}$. Hence the action of the creation operators can be written by the short statement:

$$L_i(w) = iw \quad for \quad w \in \mathbf{F}_N^+$$

Let Φ be the completely positive map $\Phi : M_N \longrightarrow B(\mathcal{H})$. Let us define a form

$$\langle\cdot,\cdot\rangle_\Phi : T_N(\mathcal{H}) \times T_N(\mathcal{H}) \longrightarrow \mathbf{C}$$

as follows. For $w, w' \in \mathbf{F}_N^+$, $h, h' \in \mathcal{H}$

i) $\langle e \otimes h, e' \otimes h' \rangle_\Phi = \langle h|h \rangle$;

ii) If $|w| \neq |w'|$ then $\langle w \bigotimes h, w' \bigotimes h' \rangle_\Phi = 0$;

iii) if $w = i_1...i_k$, $w' = i'_1...i'_k$, then

$$\langle w \otimes h, w' \otimes h' \rangle_\Phi$$
$$= \langle h|\Phi\left(e_{i_1\sigma\left(i'_1\right)} \otimes \ldots \otimes e_{i_k\sigma\left(i'_k\right)}\right) h' \rangle,$$

with $i(\sigma) = \#\left\{(i,j) \in \{1, ..., N\}^2 : i < j, \sigma(i) > \sigma(j)\right\}$

Since Φ is completely positive we have that $\langle \cdot, \cdot \rangle_\Phi$ is positive semi-definite. We extend then $\langle \cdot, \cdot \rangle_\Phi$ to $T_N(\mathcal{H}) \times T_N(\mathcal{H})$ as a map linear in the first variable and a map conjugate linear in the second one. From Theorem 4.5 of [Jo-Kr] the form $\langle \cdot, \cdot \rangle_\Phi$ is positive semi-definite on $T_N(\mathcal{H})$.

Definition 7 . *Let $N_\Phi = \{x \in T_N(\mathcal{H}) : \langle x | x \rangle_\Phi = 0\}$ be the kernel of the form $\langle \cdot, \cdot \rangle_\Phi$. The Fock space of Φ over $\mathcal{H}$ is the Hilbert space completion*

$$F_N(\mathcal{H}, \Phi) = \overline{T_N(\mathcal{H}) / N_\Phi}^{\langle \cdot, \cdot \rangle_\Phi}.$$

The left creation operators $T = (T_1, ..., T_N)$ on $\mathcal{F}_N(\mathcal{H}, \Phi)$ are (unbounded) linear transformations densely defined by

$$T_i \left(w \bigotimes h + N_\Phi \right) = (iw) \bigotimes h + N_\Phi.$$

These operators are well-defined and $T_i(N_\Phi) \subset N_\Phi$, $1 \leq i \leq N$.

Moreover:

$$T_N(\mathcal{H}) = \left\{ \sum_{|w| \leq k} w \bigotimes h_w : w \in \mathbf{F}_N^+,\ k \geq 1,\ h_w \in \mathcal{H} \right\}$$

is as before.

Following ([Ar]) we give some preliminaries on concrete product systems. This will be used to construct a covariance given by operator representations. Let E be a standard Borel space and let

$$p : E \to (0, +\infty), \tag{56}$$

- a measurable function from E onto $(0, +\infty)$ such that each fiber $E_t = E(t) = p^{-1}(t)$, $t > 0$, is a separable infinite dimensional Hilbert space, a t-th copy of a fixed Hilbert space,
- the inner product is measurable (considered as a complex-valued function defined on the Borel subset of $E \times E$ given by $\{(x, y) \in E \times E : p(x) = p(y)\}$).
- A condition of local triviality: there is a Hilbert space H_0 such that E is isomorphic ($\cong$) to the trivial family, i.e.

$$E \cong (0, +\infty) \times H_0. \tag{57}$$

We also require that there is on E a jointly measurable binary associative operation:

$$(x, y) \in E \times E \mapsto xy \in E \tag{58}$$

satisfying the conditions

1. $p(xy) = p(x) + p(y)$, and
2. for every $s, t > 0$, $E(s+t)$ is spanned by $E(s)E(t)$ and we have $< xy, x'y' >=< x, x' >< y, y' >$ for all $x, x' \in E(s)$, $y, y' \in E(t)$ (with the scalar product $\langle ., . \rangle$ the scalar product in the space of respective arguments).

Condition 2 asserts that there is a unique unitary operator (eg. the multiplication defined on the fibers determines the operator W) $W_{s,t} : E(s) \otimes E(t) \to E(s+t)$ which satisfies the condition $W_{s,t}(x \otimes y) = xy$, $x \in E(s)$, $y \in E(y)$.

Definition 8 *A structure* $p : E \to (0, +\infty)$ *satisfying (56),(57),(58) is called a product system (e.g. continuous product system).*

We use the notation $\{E(t) = E_t : t > 0\}$ *for a product system* $p : E \to (0, +\infty)$ *having fiber spaces* $E(t) = E_t = p^{-1}(t)$.

A representation of a product system E is a measurable operator-valued function $\psi : E \to \mathcal{B}(\mathcal{H})$ (with $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$) such that:

1. $\psi(v)^\star \psi(u) =< u, v > 1_\mathcal{H}$ with $1_\mathcal{H}$ the identity operator on $\mathcal{H}$, $u, v \in E_t$, $t > 0$ and
2. $\psi(u)\psi(v) = \psi(uv)$, $u, v \in E$.

We use the notation ψ_t for the restriction of ψ to the fiber E_t.
Following ([Ar]) a *concrete product system* is a Borel subset $\mathcal{E}$ of the cartesian product $(0, +\infty) \times \mathcal{B}(\mathcal{H})$ with the following properties, (where $p : \mathcal{E} \to (0, +\infty)$ is the projection $p(t, T) = t$, required to be surjective):

1. for each $t > 0$, the set of operators $\mathcal{E}_t = p^{-1}(t)$ is a norm closed linear subspace of $\mathcal{B}(\mathcal{H})$ such that $B^\star A$ is a scalar for every $A, B \in \mathcal{E}_t$, and
2. for every $s, t > 0$, $\mathcal{E}_{s+t}$ is the norm-closed linear span of $\{AB : A \in \mathcal{E}_s, B \in \mathcal{E}_t\}$.

We can define an inner product $< ., . >$ on each fiber space $\mathcal{E}_t$ by:

$$B^\star A =< A, B > 1 \text{ when } A, B \in \mathcal{E}_t.$$

Thus each $\mathcal{E}_t$ is a Hilbert space. Note that since the inner product $\langle ., . \rangle$ is measurable then $p : \mathcal{E} \to (0, +\infty)$ is measurable. From Prop. 1.9 ([Ar]) any abstract product system E is isomorphic to a concrete product system $\mathcal{E}$ via a representation π of the abstract product system.

Thus $\{(t, A) : t > 0, A \in \pi(E_t)\} \subseteq (0, +\infty) \times \mathcal{B}(\mathcal{H})$ is a concrete product system which we still denote by E for convenience. With the concrete product system E there is an associated Hilbert space $L^2(E)$ consisting of all measurable sections $f : t \in (0, \infty) \to f(t) \in E_t$ satisfying

$$\|f\|^2 = \int \|f(t)\|^2 \, dt < \infty$$

The inner product on $L^2(E)$ is defined by

$$\langle f,g\rangle = \int_0^\infty \langle f(t),g(t)\rangle\, dt$$

so $L^2(E)$ is analogous to a full Fock space (as described above, in Section 4) with no one-particle subspace. We consider the Hilbert space $\tilde{\mathcal{H}} = L^2(E)\otimes\mathcal{H}$ with the inner product given by:

$$\langle f_t\otimes h, f_s\otimes h'\rangle_{\tilde{\mathcal{H}}} := \langle f_t,f_s\rangle_{L^2(E)}\,\langle h,h'\rangle_{\mathcal{H}}$$

Let $\mathcal{F}_N(\tilde{\mathcal{H}},\Phi)$, $N\in\mathbf{N_0}$ be the Fock space constructed in ([Jo-Kr]) over the Hilbert space $\tilde{\mathcal{H}}$ as in Definition 6. Let $x = f_t\otimes h\in\tilde{\mathcal{H}}$ where $f_t\in L^2(E)$ and $h\in\mathcal{H}$.
Let $R,V\in\mathcal{F}_N(\tilde{\mathcal{H}})$, $\tilde{h},\tilde{h}'\in\tilde{\mathcal{H}}$ then

$$< R(e_{i_1}\otimes\ldots\otimes e_{i_k},\tilde{h}), V(e_{j_1}\otimes\ldots\otimes e_{j_k},\tilde{h}') >_{\Phi,E_t\otimes\tilde{\mathcal{H}}} = < \tilde{h},\Phi(e_{i_1,j_1}\otimes\ldots\otimes e_{i_k,j_k})\tilde{h}' >_{\tilde{\mathcal{H}}} .$$

Let $\mathbf{T}$ be the one-torus (or equivalently the circle). Now fix N. (In the application, N is the number of subbands.) We use the term "loop group" for the infinite-dimensional group $L(N)$ of all functions from $\mathbf{T}$ into the group $U(N) = U(N,\mathbf{C})$ of all unitary N by N complex matrices. Thus an element in $L(N)$ is a loop in the group $U(N)$.

Let us consider two pairs of scaling functions plus wavelets Φ, Ψ and $\tilde{\Phi}$, $\tilde{\Psi}$ as the Haar family used in Theorem 4, defined by

$$\begin{aligned}
\hat{\Phi}(\xi) &= m_0(\xi/N)\,\hat{\Phi}(\xi/N), \quad \hat{\Psi}(\xi) = m_1(\xi/N)\,\hat{\Psi}(\xi/N),\\
\hat{\tilde{\Phi}}(\xi) &= \tilde{m}_0(\xi/N)\,\hat{\tilde{\Phi}}(\xi/N), \quad \hat{\tilde{\Psi}}(\xi) = \tilde{m}_1(\xi/N)\,\hat{\tilde{\Phi}}(\xi/N).
\end{aligned}$$

where m_0 and m_1 are the wavelets filters, see ([Jo-Kr]). Given the filters $m_i(t)$ and $\tilde{m}_j(t)$ $i,j = 0,\ldots,N-1$ (in the sense of,e.g., ([Jo-Kr]) we have the following two matrix functions A and $\tilde{A}$, $z\in\mathbf{T}, w\in\mathbf{T}$ ($\mathbf{T}$ being the one-torus).

$$A_{k,l}(t,z) = \frac{1}{N}\sum_{w^N=z} w^{-l} m_k(t,w)$$

and

$$\tilde{A}_{k,l}(t,z) = \frac{1}{N}\sum_{w^N=z} w^{-l}\tilde{m}_k(t,w)$$

respectively, where $t\in[0,T]$. and $k,l\in\{0,\ldots,N-1\}$ (the sum indicated by $w^N = z$ should be understood as sum over w such that $w^N = z$).

Then,

$$A_{l,k}(z,t) = \sum_{w^N=z} w^{-l} m_k(w,t) = \sum_{w^N=z}\sum_i a_{k,i}(t) w^{k-l} \tag{59}$$

$$= \sum_{w^N=z}\sum_i \left\langle \chi_{[0,t]},\Psi_{i,k}\right\rangle w^{k-l} \tag{60}$$

and

$$\widetilde{A}_{l,k}(z,t) = \sum_{w'^N = z} w'^{-l} \tilde{m}_k(w',t) = \sum_{(w')^N = z} \sum_i b_{k,i}(t)(w')^{k-l} \tag{61}$$

$$= \sum_{w'^N = z} \sum_j \left\langle \chi_{[0,t]}, \Phi_{j,k} \right\rangle w'^{k-l} \tag{62}$$

where $z \in \mathbf{T}$. It is an easy computation to see that the loop matrices for the choice of the filters coefficients as in (45)-(46) and (51)-(52) satisfy the following relations:

$$\sum_{k=0}^{N-1} [\overline{A}_{k,i}(t,z)\tilde{A}_{k,j}(s,z) + \tilde{A}_{k,j}(t,z)\overline{A}_{k,i}(s,z)] = \delta_{i,j}|s-t|^{2H}, \tag{63}$$

for $s,t \in [0,T]$, $i,j = 0,\ldots N-1$ and

$$\frac{1}{N} \sum_{w^N = z} \overline{m_i(t,w)} m_j(s,w) = \delta_{i,j} \left| s \wedge t \right|^{2H}, i,j = 0,\ldots,N-1 \tag{64}$$

$$\frac{1}{N} \sum_{w^N = z} \overline{\tilde{m}_i(t,w)} \tilde{m}_j(s,w) = \delta_{i,j} \left| s \vee t \right|^{2H}. \tag{65}$$

The Fock space previously defined yields creation operators which reduce to the Cuntz algebra isometries in the Hilbert space $\mathcal{H}$.

Consider now the map Φ to be given by the matrix $P = \mathcal{S}^\star \mathcal{S}$ in $\mathcal{M}_{2N}\mathcal{B}(\mathcal{H})$. It is determined by the row matrix $\mathcal{S} = [S \quad \tilde{S}]$ associated with S, $\tilde{S}$, where S, $\tilde{S}$ are two representations of O_N arising from wavelets as described in section 3. Let

$$T = \left(T_1, \ldots, T_N, \tilde{T}_1, \ldots, \tilde{T}_N\right) \tag{66}$$

be the creation operator acting on $\mathcal{F}_{2N}(\tilde{\mathcal{H}}, \mathcal{P})$

For every $t \in [0,T]$ the operators $T_i(t)$ and $\tilde{T}_j(t)$ are creation operators in $\mathcal{F}_N(\tilde{\mathcal{H}}, \Phi)$. and are given by:

$$T_i(t)\left(\xi_{i_1} \otimes \ldots \otimes \xi_{i_k} \otimes e_a \otimes h\right) = \xi_i \xi_{i_1} \otimes \ldots \otimes \xi_{i_k} \otimes e_a \otimes h,$$

and similarly

$$\tilde{T}_j(t)\left(\xi_{i_1} \otimes \ldots \otimes \xi_{i_k} \otimes e_a \otimes h\right) = \xi_j \xi_{i_1} \otimes \ldots \otimes \xi_{i_k} \otimes e_a \otimes h$$

where $e_a \in L^2(E)$, $h \in \tilde{\mathcal{H}}$ and $\xi_{i_1} \otimes \ldots \xi_{i_k} \in \mathbf{C}^{\mathbf{N}^{\otimes \mathbf{k}}}$ is a vector corresponding to $i_1 \ldots i_k \in \mathbf{F}_{\mathbf{N}}^+$

We have:

$$T = (T_1, \ldots, T_N) \tag{67}$$

on $\mathcal{F}_N\left(\tilde{\mathcal{H}}, \Phi\right)$ is defined by

$$T_i(\omega \otimes \tilde{h} + N_\Phi = (i\omega) \otimes \tilde{h} + N_\Phi$$

from Prop. 5.2 ([Jo-Kr]) it follows that the T_i are well defined. We can describe the action of the operators $T_i^\star$ on the spanning vectors by looking at

$$\left\langle T_i^\star(e \otimes \tilde{h}), \omega \otimes \tilde{h}'\right\rangle = 0 \tag{68}$$

where e is the unit in the unital free semigroup $\mathbf{F}_\mathbf{N}^+$on N letters $\{1, \ldots, N\}$ and for every words $\omega \in \mathbf{F}_\mathbf{N}^+$. Thus $T_i^\star \mathcal{H} = 0$.

Further

$$\left\langle T_i^\star(\omega \otimes \tilde{h}), \omega' \otimes \tilde{h}'\right\rangle = 0 \tag{69}$$

unless $|\omega| = |\omega'| + 1$, where $|\omega|$ stands for the length of the word ω.

In the case $\omega = i_1 \ldots i_k$ and $\omega' = i_1' \ldots i_k'$, we have

$$\left\langle T_i^\star(\omega \otimes \tilde{h}), \omega' \otimes \tilde{h}'\right\rangle =$$
$$\left\langle \omega \otimes \tilde{h}, (i\omega') \otimes \tilde{h}'\right\rangle = \left\langle \tilde{h}, p_{i_1, i_1'} \ldots p_{i_k i_k'} \omega' \otimes \tilde{h}'\right\rangle.$$

Thus,

$$\left\langle T_i^\star T_j(\omega \otimes \tilde{h}), \omega' \otimes \tilde{h}'\right\rangle =$$
$$\left\langle (j\omega) \otimes \tilde{h}, (i\omega) \otimes \tilde{h}'\right\rangle = \left\langle \tilde{h}, p_{i,j} p_{i_1, i_1'} \ldots p_{i_k i_k'} \tilde{h}'\right\rangle =$$
$$\left\langle \omega \otimes p_{i,j} \tilde{h}, \omega' \otimes \tilde{h}'\right\rangle$$

since the vectors $\omega \otimes \tilde{h}$ span (not necessarily orthogonally) the Fock space $\mathcal{F}_N(\tilde{\mathcal{H}}, \Phi)$, it suffices to look at the inner product

$$\left\langle T_i^\star T_j(\omega \otimes \tilde{h}), \omega' \otimes \tilde{h}'\right\rangle$$

for every $\omega, \omega' \in \mathbf{F}_N^+$ and $\tilde{h}, \tilde{h}' \in \mathcal{H}$.
If ω, ω' are words of different lengths the above inner product is $= 0$, on the other hand if $\omega = i_1 \ldots i_k$ and $\omega' = i_1{}' \ldots i_k{}'$ then we get the relation

$$\left\langle T_i^\star T_j(\omega \otimes \tilde{h}), \omega' \otimes \tilde{h}'\right\rangle = \left\langle \omega \otimes p_{i,j} \tilde{h}, \omega' \otimes \tilde{h}'\right\rangle \tag{70}$$

Theorem 9 *Let $S = (S_0, S_1, \ldots.., S_{N-1})$ and $\widetilde{S} = (\widetilde{S}_0, \widetilde{S}_1, \ldots.., \widetilde{S}_{N-1})$ be a pair of wavelet representations on $\mathcal{H} = L^2(\mathbf{C})$ with invertible loop matrices A and $\widetilde{A}$ respectively satisfying (56),(57),(58). Let $\mathcal{S} = \left[\begin{array}{cc} S & \tilde{S} \end{array}\right]$ be the row matrix associated to S and $\tilde{S}$ and let $P = \mathcal{S}^\star \mathcal{S}$.*

Let $Q = (T_0(t), T_2(t), \ldots, T_{N-1}(t), \tilde{T}_0(t), \tilde{T}_2(t), \ldots \tilde{T}_{N-1}(t))$ be the creation operator on $\mathcal{F}_{2N}(\tilde{\mathcal{H}}, P)$, with the $T_i(t)$ and $\tilde{T}_j(t)$ as defined above, then:

$$T_i^\star(t)T_j(s)|_{\tilde{\mathcal{H}}} = \left(S_{i-1}^\star(t)S_{j-1}(s)\right) = (AA^\star)_{i,j}(t,s) = \delta_{i,j}\,|s \wedge t|^{2H} \tag{71}$$

and

$$\tilde{T}_i^\star(t)\tilde{T}_j(s)|_{\tilde{\mathcal{H}}} = \left(\tilde{S}_{i-1}^\star(t)\tilde{S}_{j-1}(s)\right) = \left(\tilde{A}\tilde{A}^\star\right)_{i,j}(t,s) = \delta_{i,j}\,|s \vee t|^{2H} \tag{72}$$

on a dense domain.
Hence

$$T_i(t)\tilde{T}_j^\star(s)|_{\tilde{\mathcal{H}}} + \tilde{T}_i^\star(t)T_j(s)|_{\tilde{\mathcal{H}}} = \delta_{i,j}|s-t|^{2H} \tag{73}$$

on a dense domain.

Proof. Recall that from (56) and (57) we have

$$A_{l,k}(z,t) = \sum_{w^N=z}\sum_i \left\langle \chi_{[0,t]}, \Psi_{i,k}\right\rangle w^{k-l}$$

and

$$\tilde{A}_{l,k}(z,t) = \sum_{w'^N=z}\sum_j \left\langle \chi_{[0,t]}, \Phi_{j,k}\right\rangle w'^{k-l}$$

where $z \in \mathbf{T}$, $t \geq 0$. Let us take the representation of $S_i(t)$ and $\tilde{S}_j(t)$ associated to the filters $m_i(t)$ and $\tilde{m}_j(t)$.
Then the relations (71) and (72) follow, since, on a dense domain:

$$S_i^\star(t)S_j(t) = \sum_{h,k} \bar{m}_i(z^h,t)m_j(z^k,t) = \delta_{i,j}\,|t|^{2H} \tag{74}$$

and similarly

$$\tilde{S}_i^\star(s)\tilde{S}_j(s) = \sum_{h,k} \bar{\tilde{m}}_i(z^h,s)\tilde{m}_j(z^k,s) = \delta_{i,j}\,|s|^{2H} \tag{75}$$

To prove (73), let us consider, on a dense domain:

$$\tilde{S}_i^\star(t)S_j(s) - S_j(t)\tilde{S}_i^\star(s) \tag{76}$$

$$= \sum_{w^N=z,(w')^N=z} w^{-i}(\bar{w}')^{-j}\left(\sum_{l,k}\bar{\tilde{m}}_i(z'^l,t)m_j(z^k,s) - \sum_{h,k} m_i(z^l,t)\bar{\tilde{m}}_j(z'^k,s)\right) \tag{77}$$

$$= \sum_{w^N=z,(w')^N=z} \bar{w}^{-i}(w')^{-j}\sum_{h,k}\left\langle \chi_{[0,t]}, \Psi_{h,j}\right\rangle\left\langle \chi_{[0,s]}, \Phi_{k,i}\right\rangle (z\prime)^{h-j}z^{k-i} -$$
$$\sum_{w^N=z,(w')^N=z}\sum_{h,k} \bar{w}^{k-i}(w')^{h-j}\left\langle \chi_{[0,t]}, \Phi_{k,i}\right\rangle\left\langle \chi_{[0,s]}, \Psi_{h,j}\right\rangle z'^j\bar{z}^i z'^j\bar{z}^i$$

where we used (68) respectively (69) and the relations expressing m_i respectively $\tilde{m}_j$ that are equivalent to the one (58) written in terms of the loop matrices. Thus, using (70)-(73) we have, on a dense domain:

$$\tilde{S}_i^\star(s)S_j(t) - S_i(s)\tilde{S}_j^\star(t) = \delta_{i,j}\left[(s\wedge t)^{2H} - (t\wedge s)^{2H}\right] = \delta_{i,j}\,|s-t|^{2H}\,,$$

which is the desired result. □

Corollary 10 *Given Q as in Theorem 9, then the covariance of the fBm process $B_H(t)$ can be written as :*

$$E\left[B_H(t)B_H(s)\right] = T_i^\star(t)T_j(s) + \tilde{T}_i^\star(t)\tilde{T}_j(s) - \left(\tilde{T}_i^\star(t)T_j(s) - T_j(t)\tilde{T}_i^\star(s)\right) \quad (78)$$

Proof. Let $T_i(t)$ and $\tilde{T}_j(t)$ be the operators on the Fock space as in Theorem 8 on $\mathcal{F}_{2N}(\tilde{\mathcal{H}}, P)$
Thus from Theorem 9 we have

$$T_i^\star(t)T_j(s) = \delta_{i,j}\,|s\wedge t|^{2H} \quad (79)$$

and

$$\tilde{T}_i^\star(t)\tilde{T}_j(s) = \delta_{i,j}\,|s\vee t|^{2H}\,. \quad (80)$$

and

$$\tilde{T}_i^\star(t)T_j(s) - T_j^\star(t)\tilde{T}_i^\star(s) = \delta_{i,j}\,|s-t|^{2H} \quad (81)$$

where $i \neq j$. Then this is the covariance of a fBm, i.e.:

$$E\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right) =$$
$$T_i^\star(t)T_j(s) + \tilde{T}_i^\star(t)\tilde{T}_j(s) - \left(\tilde{T}_i^\star(t)T_j(s) - T_j(t)\tilde{T}_i^\star(s)\right)$$

Then the statement follows. □

A construction of a conditional expectation for the full Fock space can be found in ([Ba-Vo]).

5 q-Fractional Brownian motion

In this Section we generalize the construction done for the fBm process to the one of a process having a covariance dependent on a parameter $0 < q < 1$

A construction of wavelets satisfying alternatives to the vanishing moments conditions which gives orthonormal basis functions with scale dependent properties is found in ([PaScSc]). This allows us to construct a q dependent covariance. The underlying loop group conditions are then related to wavelets which are scale q-dependent. Let $s_j(x) = q^{\omega_j x}$, where $x \in \mathbf{R}$ and ω_j is a sequence of complex parameters for $j \in I$, I is a suitable index set. This means that the orthonormality of the integer translates of the scaling function ψ, eg.

$\int_{\mathbf{R}} \psi(x-l)\psi(x)dx = \delta_{0,l}$, $l = 0, \ldots, N-1$, $N \in \mathbf{N}$; requires the filter coefficients h_i to satisfy.

$$\sum_{k\in\mathbf{Z}} h_k h_{k-2l} = \delta_{0,l}, \quad l = 0, \ldots, N-1$$

Moreover the wavelet φ is assumed to have N vanishing moments, i.e.,

$$\int_{\mathbf{R}} x^l \varphi(x)dx = 0$$

which requires the filters coefficients to satisfy

$$\sum_{k\in bfZ} h_k h_{k-2l} = \delta_{0,l}, l = 0, \ldots, N-1.$$

We consider wavelets that satisfy a q- dependent condition on the vanishing moments. We keep the condition that the wavelet and the scaling function are compactly supported and that the scaling function is orthogonal to its integer translates on every scale. In other words, we consider family of functions $\{s_j : j \in I\}$ on $\mathbf{R}$ such that for all $j \in I$ and all integers k

$$\int_{\mathbf{R}} s_j(x)\varphi_{m,k}(x)dx = 0 \tag{82}$$

where $\varphi_{m,k}$ is a family of compacted wavelets. The following condition on the filter sequence is needed to ensure that (70) is satisfied

$$\sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} s_j(2^{m-1}k) = 0,$$

for all $j \in I$. In our case we take $\omega_j = 1$ and the set $I = \{0\}$. Let the loop matrix be defined by using the filters associated to the wavelets in which the scale is q-dependent. This means we take (cf. the proof of Theorem 9):

$$A_{l,k}(z,t) = \sum_{w^N=z} \sum_i \left\langle \chi_{[0,t]} q^x, \Psi_{i,k}(x) \right\rangle w^{k-l} \tag{83}$$

and

$$\tilde{A}_{l,k}(z,t) = \sum_{w'^N=z} \sum_j \left\langle \chi_{[0,t]} q^x, \Phi_{j,k}(x) \right\rangle {w'}^{k-l} \tag{84}$$

where $z \in \mathbf{T}$ and $\Phi_{j,k}, \Psi_{i,k}$ are compactly supported wavelets in $[0,1]$.

Theorem 11 *Let $S = (S_0(t), S_1(t), \ldots, S_{N-1}(t))$ and $\tilde{S} = (\tilde{S}_0(t), \tilde{S}_1(t), \ldots, \tilde{S}_{N-1}(t))$ be a pair of representations of O_N on $\tilde{\mathcal{H}} = L^2(E) \otimes \mathcal{H}$ with A and $\widetilde{A}$ the invertible loop matrices arising from the q-dependent scale wavelet as in ([PaScSc]). Let $\{e_{i,j}\}_{i,j} \in \{0, \ldots, N-1\}$ be matrices units for the set of $2N \times 2N$ complex matrices with values in $\tilde{\mathcal{H}}$. Choose the map Φ to be given by the $2N \times 2N$-*

matrix $P = [\Phi(e_{i,j})] = \mathcal{S}^\star\mathcal{S}$*, where* $\mathcal{S} = \begin{bmatrix} S & \tilde{S} \end{bmatrix}$ *is the matrix associated to* S *and* $\tilde{S}$*.*

Then
$T = (T_1(t), T_2(t), \ldots, T_{N-1}(t), \tilde{T}_1(t), \tilde{T}_2(t), \ldots, \tilde{T}_{N-1}(t))$ *is the natural creation operator on* $\mathcal{F}_{2N}\left(\tilde{\mathcal{H}}, P\right)$ *:*

$$T_i(t)\left(\xi_{i_1}\ldots\xi_{i_k}\otimes\tilde{h}\right) = \xi_i\xi_{i_1}\ldots\xi_{i_k}\otimes\tilde{h}$$

and similarly

$$\tilde{T}_j(t)\left(\xi_{i_1}\ldots\xi_{i_k}\otimes\tilde{h}\right) = \xi_j\xi_{i_1}\ldots\xi_{i_k}\otimes\tilde{h}$$

such that on a dense domain:

$$T_i^\star(t)T_j(s)|_{\mathcal{H}} = \left(S_{i-1}^\star(t)S_{j-1}(s)\right) = \left(AA^\star\right)_{i,j}(t,s) = \delta_{i,j}\left[t\wedge s\right]_q^{2H} \tag{85}$$

and

$$\tilde{T}_i^\star(t)\tilde{T}_j(s)|_{\mathcal{H}} = \left(\tilde{S}_{i-1}^\star(t)\tilde{S}_{j-1}(s)\right) = \left(\tilde{A}\tilde{A}^\star\right)_{i,j}(t,s) = \delta_{i,j}\left[t\vee s\right]_q^{2H}, \tag{86}$$

where we used the notation $[t]_q = \frac{q^t-1}{q-1}$*. Moreover relations corresponding to (73) and (74) hold on a dense domain:*

$$T_i(t)\tilde{T}_j^\star(s) = \left(\tilde{A}^\star A\right)_{i,j}(s,t) = \delta_{i,j}\frac{q^{|s-t|}-1}{q-1} \tag{87}$$

and

$$\tilde{T}_j^\star(s)T_i(t) = \left(A\tilde{A}^\star\right)_{i,j}(s,t) = \delta_{i,j}\frac{q^{-|s-t|}-1}{q-1} \tag{88}$$

Thus the pair of operators T_i *and* $\tilde{T}_j$ *satisfy the following q-algebra relations:*

$$T_i(s)\tilde{T}_j^\star(t)|_{\tilde{\mathcal{H}}} + \tilde{T}_j^\star(s)T_i(t)|_{\tilde{\mathcal{H}}} = \delta_{i,j}[|s-t|]_{q^2}^{2H} \tag{89}$$

on a dense domain.

Proof. Let S and $\tilde{S}$ be as stated. Let A and $\tilde{A}$ be their invertible loop matrices respectively. Let $P = \mathcal{S}^\star\mathcal{S}$ be the $2N\times 2N$ matrix in $\mathcal{B}(\tilde{\mathcal{H}})$. Let $T = (T_1, T_2, \ldots, T_{N-1}, \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_{N-1})$ be the creation operators defined in $\mathcal{F}_{2N}\left(\tilde{\mathcal{H}}, P\right)$. Then in our setting for $\tilde{h}, \tilde{h}' \in \tilde{H}$ and by using the $\langle .,.\rangle_\Phi$ as defined in Section 4, we have

$$< T_i(t)\omega\otimes\tilde{h}, T_j(s)\omega'\otimes\tilde{h}' >_\Phi = \delta_{i,j}[t\wedge s]_q\left\langle\tilde{h},\tilde{h}'\right\rangle_{\tilde{\mathcal{H}}}$$

and

$$< \tilde{T}_i(t)\omega \otimes \tilde{h}, \tilde{T}_j(s)\omega' \otimes \tilde{h}' >_\Phi = \delta_{i,j}[t \vee s]_q \left\langle \tilde{h}, \tilde{h}' \right\rangle_{\tilde{\mathcal{H}}} \tag{90}$$

which implies that

$$T_i(t)^\star T_j(s)|_{\tilde{\mathcal{H}}} = \delta_{i,j}[s \wedge t]_q^{2H}$$

where $[t]_q = \frac{q^t-1}{q-1}$. Similarly for the operators $\tilde{T}_i$ we have:

$$\tilde{T}_i(s)^\star \tilde{T}_j(t)|_{\tilde{\mathcal{H}}} = \delta_{i,j}[s \vee t]_q^{2H}$$

To prove that T_i and the $\tilde{T}_j$ satisfy the q-relations (85)-(86) we observe that if $s > t$ then

$$\tilde{T}_j^\star(t)T_i(s) = \left(A\tilde{A}^\star\right)_{i,j}(t,s) = \delta_{i,j}\frac{q^{t-s}-1}{q-1} \tag{91}$$

and

$$T_i(t)\tilde{T}_j^\star(s) = \left(\tilde{A}^\star A\right)_{i,j}(t,s) = \delta_{i,j}\frac{q^{s-t}-1}{q-1} = \delta_{i,j}\frac{q^{-(t-s)}-1}{q-1} \tag{92}$$

Similarly if $t > s$ then

$$\widetilde{T}_j^\star(t)T_i(s) = \left(A\widetilde{A}^\star\right)_{i,j}(t,s) = \delta_{i,j}\frac{q^{s-t}-1}{q-1} \tag{93}$$

and

$$T_i(t)\widetilde{T}_j^\star(s) = \left(\widetilde{A}^\star A\right)_{i,j}(t,s) = \delta_{i,j}\frac{q^{t-s}}{q-q^{-1}} = \delta_{i,j}\frac{q^{-(s-t)}-1}{q-1} \tag{94}$$

From (87),(89) the theorem follows. □

The above result will allow to construct an extension of the classical fBm depending on a parameter q.

Theorem 12 *Let $T_i(t,q)$ and $\tilde{T}_j(t,q)$ be a pair of operators associated to the operators S_i, $\tilde{S}_j$ arising from wavelet representations under the conditions of Theorem 11. Then there is a fBm process $B_{H,q}(s)$ depending on a parameter q with mean zero and whose covariance is*

$$E\left[B_{H,q}(t)B_{H,q}(s)\right] = [t]_q^{2H} + [s]_q^{2H} - [|t-s|]_{q^2}^{2H}$$

Proof.

This follows from the following computation and by using Theorem 11:

$$\begin{aligned} E\left[B_{H,q}(t)B_{H,q}(s)\right] = \\ T_i^\star(t)T_j(s) + \tilde{T}_i^\star(t)\tilde{T}_j(s) + \\ -\tilde{T}_j^\star(t)T_i(s) + T_i(t)\tilde{T}_j^\star(s) \\ = \delta_{i,j}\left([t]_q^{2H} + [s]_q^{2H} - [|t-s|]_{q^2}^{2H}\right) \end{aligned}$$

The definite positivity follows from the definite positivity of the map Φ (see Theorem 4.5 [Jo-Kr]) in the constructed Fock space. $\square$

We observe that the covariance function of the above process coincides with the one for the classical fBm when $q = 1$. In this sense $B_{H,q}$ is a generalization of the fBm when $q \neq 1$

6 Conclusions

We studied the role of fractional Brownian motion in wavelet decomposition theory, making use of representations of a family of purely infinite C*-algebras, called the Cuntz algebras.
A fractional Brownian motion is a Gaussian process such that for a time interval of length Δt by definition the corresponding increment of the process $X(t)$ behaves like $(\Delta t)^H$ where H is a number in the open interval $(0, 1)$, called Hurst index. In the non-Brownian case, the increments over disjoint intervals are not independent.
We established a two-fold connection between wavelets and fractional Brownian motion:

(1)Starting with the fractional Brownian motion $B_H(t)$ for a fixed H, we showed that it is possible to diagonalize $B_H(t)$ with the use of a chosen wavelet basis.

(2)Conversely, the use of wavelet analysis allows us to gain new insight into the properties of fractional Brownian motion. We extended a recent discovery of a special family F of irreducible representations of the Cuntz algebras, with the set of equivalence classes F modeling all the subband wavelet filters. In this case, the family F takes the form of in infinite-dimensional unitary group, or equivalently a $U(N)$-loop group.

We showed moreover that the introduction of fractional Brownian processes yield new wavelet filters and new representations. We did this by building up our representations with the use of Fock space tools, in particular raising and lowering operators.

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7 Acknowledgments

P.J. was supported in part by a grant from the National Science Foundation (USA). A.M.P. would like to thank the Max-Planck Institut für Mathematik in Bonn for support and excellent working conditions. .

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