Birationally rigid Fano double hypersurfaces

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Abstract

We prove that a general Fano double hypersurface V of index 1 ($\sigma: V \to Q_m \subset \mathbf{P}^{M+1}$ is a double cover, branched over a smooth divisor $W = W^*_{2l} \subset \mathbf{P}^{M+1}$, $m + l = M + 1 \geq 5$) is birationally superrigid; in particular, it permits no non-trivial structures of a fibration into uniruled varieties, its groups of birational and biregular automorphisms coincide and it is non-rational.

Introduction

The aim of this paper is to prove birational rigidity of a new family of Fano varieties of dimension 4 and higher. Recall [P2,P4] that a smooth Fano variety V is said to be *birationally superrigid*, if for any birational map $\chi: V - - \to V'$ onto a uniruled variety of the same dimension and any linear system of divisors Σ' on V' without fixed components the monotonicity condition

$$c(D, V) \leq c(D', V'),$$

is satisfied, where $D' \in \Sigma'$, $D \in \Sigma = (\chi^{-1})_* \Sigma'$, i.e. Σ is the proper inverse transform of Σ' with respect to χ, a linear system of divisors on V without fixed components. Here $c(Y, X)$ for a variety X, smooth in codimension 1, and a Weil divisor Y on it denotes the *threshold of canonical adjunction*

$$c(Y, X) = \sup\{\frac{b}{a} | b \in \mathbf{Z}_+, a \in \mathbf{Z}_+ \setminus \{0\}, |aY + bK_X| \neq \emptyset\}.$$

Any birationally superrigid variety V with $\mathrm{Pic}\, V \cong \mathbf{Z}$ has a number of standard properties. In particular,

(i) it has no non-trivial structures of a Fano fibration, i.e. there exists no rational map $\psi: V - - \to S$, $\dim S \geq 1$ and ψ is dominant, such that a general fiber $\psi^{-1}(s)$ is a uniruled variety (in particular, birationally equivalent to a Fano variety),

(ii) the groups of birational and biregular automorphisms of V coincide: $\mathrm{Bir}\, V = \mathrm{Aut}\, V$,

(iii) the variety V is non-rational.

The assertions (i)-(iii) follow from the monotonicity condition in a trivial way, see [P2-P4]. The majority of the known examples of rigid and superrigid varieties come from dimension three [IM,IP,P3,CPR,...], however, there are also higher-dimensional series [P1,P4].

In the present paper we prove birational superrigidity of the Fano double hypersurfaces of the index one. Fix integers $M \geq 4$, $m \geq 3$ and $l \geq 1$, satisfying the equality $m+l = M+1$. The symbol $\mathbf{P}$ stands for the projective space $\mathbf{P}^{M+1}$ over the field of complex numbers $\mathbf{C}$. Let $Q = Q_m \subset \mathbf{P}$ and $W^* = W^*_{2l} \subset \mathbf{P}$ be hypersurfaces of degrees m and $2l$, respectively, and $\sigma: V \to Q$ the double cover, branched over $W = W^* \cap Q$ (for a detailed description of the construction of V, see below Sec. 1). The variety V is a Fano variety of dimension M with the Picard group $\mathbf{Z}K_V$. Our principal result is given by the following

Theorem 1. *A general variety V is birationally superrigid.*

Here the generality means that the hypersurfaces Q and W^* are general in the sense of algebraic geometry: their polynomial equations should lie in a Zariski open subset of the space $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m)) \times H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(2l))$. As in [P4], our condition of being general is constructive: we require that the base sets of certain linear systems on $\mathbf{P}$ should be of the maximal possible codimension (see Sec. 1.3). Naturally, it is necessary to verify that these generality conditions define a *non-empty* Zariski open set. In this preprint we omit the verification, which will appear in the final publication of this paper. The relevant arguments are not too easy and take some tiresome work. However, there is no doubt that the claim of Theorem 1 is true for *any smooth* V, and even for any double cover with moderate singularities. Unfortunately, our techniques are not strong enough to cope with these cases. The conditions of general position are essential for the proof: they form a basis for the technique of hypertangent systems, which was introduced in [P4]. It is used in the present paper, too. For the double hypersurfaces we have to modify this technique (see Sec. 3).

Theorem 1 is proved by means of the method of maximal singularities combined with the above-mentioned technique of hypertangent systems. The method itself was started by the paper of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartics [IM]. Its importance can not be overestimated. In that article an effectively working scheme of arguments, which makes it possible to give an exhaustive description of birational correspondences of a three-fold with no global differential forms, was first developed. Gradually, the domain of applications of the method grew larger: from the quartics to other Fano three-folds [I], then to conic bundles [S], del Pezzo fibrations [P3], higher-dimensional Fanos [P1,P4], ... Many times it seemed that the method's potential had reached its limit. But every time a strengthening of its techniques increased its resources drastically. Today, with all the progress which has been made in the field, the method is far from being perfect. In the "boundary cases" the techniques of the method sometimes refuses to work. Undoubtedly, it is necessary to improve further both the local part of the method (to describe the "minimal" maximal singularities, or the extremal contractions to a point of a given type, as in [Ka] and [C]), and its global part (cycles with prescribed singularities on a given variety).

I started my work on the Fano double covers in 1997 during my stay at Max-Planck-Institut für Mathematik in Bonn (Deutschland). The strategy of arguments, which was found there [P4], for the first time made it possible to overcome the "degree barrier", and Theorem 1 of the present paper is also a realization of this strategy. I would like to use this opportunity and thank the Directors and the staff of the Institute for hospitality.

1 Fano double hypersurfaces

1.1 Construction

Fix integers $m\geq 3$, $l\geq 1$, $M\geq 4$ such that $m+l=M+1$, and a pair of hypersurfaces $Q=Q_m\subset \mathbf{P}=\mathbf{P}^{M+1}$ and $W^*=W^*_{2l}\subset \mathbf{P}$ of degrees m and $2l$, respectively, where the hypersurface Q and the complete intersection $W=W_{m\cdot 2l}=W^*\cap Q$ are smooth. The object of our study is the double cover $\sigma\colon V\to Q$, branched over the smooth divisor W. The Lefschetz theorem implies that the Picard group of the smooth variety V is $\operatorname{Pic} V=\mathbf{Z}L$, where $L=\sigma^*H$ and $H\in \operatorname{Pic} Q$ is the class of a hyperplane section of Q, generating $\operatorname{Pic} Q$. The canonical class of the variety V is $K_V=-L$, so that V is a Fano variety of the index 1 and dimension ≥ 4. The anticanonical linear system $|-K_V|=|L|$ for $m\leq M-1$ determines exactly the double cover σ. For $m=M$, $l=1$ the variety V can be realized by means of the anticanonical system $|-K_V|$ as a complete intersection in $\mathbf{P}^{M+2}$ of a cone over the hypersurface Q and a quadric, which does not pass through its vertex. The "degree" of the variety V is $(-K_V)^M=2m$.

Explicitly, let

$$A=\mathbf{Spec}\,\mathcal{O}_Q(l)[t]$$

be an $\mathbf{A}^1$-fibration over the hypersurface Q, and take a section $s\in H^0(Q,\mathcal{O}_Q(2l))$, corresponding to the divisor $W\subset Q$, i.e. $W=\{s=0\}$. Now V is realized in the $(M+1)$-dimensional variety A as a hypersurface with the equation $t^2=s$.

For an arbitrary point $p\in Q$ and a system of affine coordinates on $\mathbf{A}^{M+1}\subset \mathbf{P}$ with the origin at p, $(z_1,\dots,z_{M+1})$, denote the affine equation of Q by $f=q_1+\ldots+q_m$, and the equation of W^* by $g=w_0+\ldots+w_{2l}$. If $p\notin W$, then for conveniency we shall assume that $w_0=1$. In this case the σ-preimage of any local system of parameters on Q at the point p makes a local system of parameters at any of the two points in $\sigma^{-1}(p)\subset V$.

1.2 Weighted complete intersections

The variety V can be realized as a complete intersection, i.e. the variety of zeros of the system of equations

$$\left\{ \begin{array}{l} \tilde{f} = 0, \\ u^2 = \tilde{g}, \end{array} \right.$$

where $\tilde{f}$, $\tilde{g}$ are the homogeneous polynomials of degrees m and $2l$ on $\mathbf{P}$, corresponding to f and g, in the weighted projective space [D,Fl]

$$\mathbf{P}(\underbrace{1, 1, \dots, 1}_{M+2}, l) = \mathbf{C}^{M+3}/\mathbf{C}^*,$$

where $(1, \dots, 1, l)$ denotes the weights of $\mathbf{C}^*$-action on $\mathbf{C}^{M+3}$. If $(x_0 : x_1 : \dots : x_{M+1})$ are homogeneous coordinates on $\mathbf{P}$, then $(x_0 : x_1 : \dots : x_{M+1} : u)$ are homogeneous coordinates on the weighted space $\mathbf{P}(1, \dots, l)$. Here u is of the weight l, i.e. represents a section of the sheaf $\mathcal{O}_{\mathbf{P}}(l)$, inducing a section of $\mathcal{O}_V(l)$. For $l = 1$ we get the above-mentioned presentation of V as a complete intersection of a cone and a quadric in $\mathbf{P}^{M+2}$.

The only singular point of the weighted space $\mathbf{P}(\cdot)$ has the coordinates $(0 : \dots : 0 : 1)$ and so does not lie on V. In one of the standard affine charts, say, $x_0 \neq 0$, we get a system of affine coordinates $(z_1, \dots, z_{M+1}, y)$, setting $z_i = x_i/x_0$, $y = u/x_0^l$. This chart is isomorphic to $\mathbf{A}^{M+2}$ and with respect to the coordinates (z_*, y) the variety V is given by a system of equations

$$\left\{ \begin{array}{l} f = q_1 + \dots + q_m = 0, \\ y^2 = w_0 + \dots + w_{2l}. \end{array} \right.$$

Here y is of the weight 0, i.e. it is an ordinary function on V.

1.3 Regularity conditions

By analogy with [P4], we say that a point $p \in Q$ is *regular* (as a point on this hypersurface, which makes some difference from the definition given below), if for some (and thus for any) system of linear coordinates $(z_1, \dots, z_{M+1})$ with the origin at p the sequence of homogeneous polynomials $q_1, \dots, q_m$ (where $f = q_1 + \dots + q_m$ is the equation of Q) is regular in $\mathcal{O}_{p,\mathbf{P}}$. The hypersurface Q is said to be regular, if any point on it is regular. A dimension count (see [P4]) shows that for $m \leq M$ a general hypersurface Q is regular. However, for our technique of proving Theorem 1 the condition of Q being regular is not sufficient. We need some extra condition of general position for the double cover.

Take $p \in Q \setminus W$. Choose a system of coordinates $(z_1, \dots, z_{M+1})$ with the origin at the point p. Let $g = w_0 + w_1 + \dots + w_{2l}$, $w_0 = 1$, be the equation of W^* with respect to this coordinate system. Consider a point $x \in \sigma^{-1}(p) \subset V$. To be definite, we assume that $y(x) = 1$.

Consider the power series $(1+a)^{1/2} = 1 + \sum \gamma_i a^i = 1 + \frac{1}{2}a + \ldots$, where a is just a formal variable. Set

$$\sqrt{g} = (1 + w_1 + \ldots + w_{2l})^{1/2} = 1 + \sum_{i=1}^{\infty} \gamma_i (w_1 + \ldots + w_{2l})^i = $$
$$= 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \ldots, w_{2l}),$$

where $\Phi_i(w_1(z_*), \ldots, w_{2l}(z_*))$ are homogeneous degree i in z_*. Obviously, $\Phi_i(w_*) = w_i + \ldots$ for $i \leq 2l$, where the dots stand for a polynomial in $w_1, \ldots, w_{i-1}$. For $k \geq 1$ set

$$[\sqrt{g}]_k = 1 + \sum_{i=1}^{k} \Phi_i(w_*), \quad g^{(k)} = g - [\sqrt{g}]_k^2.$$

It is easy to see that the first non-zero homogeneous component in $g^{(k)}$ is of degree $k+1$ in z_*. This component for $k+1 \leq 2l$ is of the form $w_{k+1} + \ldots$, where the dots stand for a polynomial in $w_1, \ldots, w_k$. For instance, $g^{(1)} = w_2 - \frac{1}{4}w_1^2 +$ (components of higher order in z_*).

Definition 1. For a point $l \geq 2$ a point $x \in V$, $p = \sigma(x) \notin W$, is said to be a *regular point* of the double cover V, if the set of polynomials (*)

$$q_1, \ldots, q_m, g^{(l)}, \ldots, g^{(2l-2)}$$

makes a regular sequence in $\mathcal{O}_{p,\mathbf{P}}$. For $l = 1$ a point $x \in V$, $p = \sigma(x) \notin W$, is regular on V, if the point p is regular on Q and, moreover, on the hyperplane $q_1 = 0$ the quadric cones $q_2 = 0$ and $w_2 - \frac{1}{4}w_1^2 = 0$ have no common irreducible components.

Let us draw the reader's attention to the *non*-invariance of the regularity condition: with respect to another coordinate system the sequence (*) may well become no more regular in its segment $g^{(l)}, \ldots, g^{(2l-2)}$. Consequently, we speak about the regularity with respect to a *certain* (and thus a general) coordinate system. However, this non-invariance brings no additional problems.

Geometrically the regularity condition for $l \geq 2$ means that in a neighborhood of the point $p = \sigma(x)$ the system of equations

$$q_1 = \ldots = q_m = g^{(l)} = \ldots = g^{(2l-2)} = 0$$

defines a curve in $\mathbf{P}$.

Definition 2. A point $x \in V$, $\sigma(x) = p \in W$, is said to be a *regular point* of the double cover V, if p is regular on the hypersurface Q and the quadratic form q_2 does not vanish identically on the plane $q_1 = w_1 = 0$.

The plane $\{q_1 = w_1 = 0\}$ is the tangent space to the surface W at the point p.

Definition 3. The double cover $\sigma\colon V \to Q$ is said to be *regular*, if any point $x \in V$ is regular, either in the sense of Definition 1 or in the sense of Definition 2.

Now we can make the claim of Theorem 1 more precise.

Theorem 1*. *A regular double cover V is birationally superrigid.*

Theorem 1* implies Theorem 1 by the following

Proposition 1.1. *A general double hypersurface V is regular: there exists a non-empty Zariski open subset $U \subset H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(m)) \times H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(2l))$, such that the* *double cover* $\sigma: V \to Q = \{\tilde f = 0\}$, *branched over the surface* $W = Q \cap \{\tilde g = 0\}$, *is regular for any pair* $(\tilde f, \tilde g) \in U$.

The proof is similar (although more complicated) to the proof of Proposition 1, Sec. 1 in [P4]. The details will be given elsewhere.

During the proof of Theorem 1^* (Sec. 2,3), every time when a point $x \in V$ is considered and the regularity condition is used, we assume without special comments that a suitable system of coordinates $(z_1, \dots, z_M)$ is fixed, and all the equations $q_i, g^{(j)}$ are taken with respect to this coordinate system.

2 The method of maximal singularities

This and the next sections contain a proof of Theorem 1^*. The arguments are based upon the method of maximal singularities. The scheme of the proof and the techniques of the method have been published many times in full details [IM,IP,P2,P4,...]. For this reason we omit here the Nöther-Fano inequality, the local analysis of a maximal singularity and the $4n^2$-inequality. All of this can be found in any of the above-mentioned papers. We reduce application of the method to verifying some technical condition on the degrees and multiplicities of the cycles of codimension 2.

2.1 A condition of birational rigidity

Let X be a smooth projective Fano variety with $\operatorname{Pic} X = \mathbf{Z}K_X$. For a cycle $Y \subset X$ of pure dimension k the integer $\deg Y = (-1)^k (Y \cdot K_X^k)$ is said to be its *degree*.

The next proposition can be looked at as a summary of the method of maximal singularities for those Fano varieties which have no non-trivial birational automorphisms.

Proposition 2.1. *Assume that the variety X satisfy the following two conditions:*

(i) for any irreducible cycle $Y \subset X$ of codimension 2 and any linear system of divisors $\Sigma \subset |-nK_X|$, which is free from fixed components, the following estimate is true

$$\operatorname{mult}_Y \Sigma \leq n,$$

(ii) for any irreducible cycle Y of codimension 2 and any point $x \in X$ the estimate

$$\operatorname{mult}_x Y \leq \frac{4}{\deg X} \deg Y$$

holds.

Then X is a birationally superrigid Fano variety.

In accordance with the definition of degree, given above, here $\deg X$ is equal to $(-K_X)^{\dim X} > 0$.

Proof. See [P2-P4]. If X is not superrigid, then on X exists a linear system $\Sigma \subset |-nK_X|$, free in codimension one, which has a maximal singularity $\nu \in \mathcal{N}(X)$ ([P4, Sec. 2]). If the centre of the discrete valuation ν, $Z(V, \nu) \subset X$, is an irreducible cycle $Y \subset X$ of codimension 2, then the inequality $\operatorname{mult}_Y \Sigma > n$ is true, which contradicts condition (i). If $\operatorname{codim}_Y Z(V,\nu) \geq 3$, then for any point $x \in Z(V,\nu)$ we get $\operatorname{mult}_x Z > 4n^2$, where $Z = (D_1 \bullet D_2)$, $D_i \in \Sigma$, is the algebraic cycle of the scheme-theoretic intersection of two general divisors from Σ ([P4, Appendix B]). Obviously, $\deg Z = n^2 \deg X$, so that there exists an irreducible component Y of the cycle Z, which satisfies the inequality

$$\frac{\operatorname{mult}_x}{\deg} Y > \frac{4}{\deg X}.$$

However, this fact contradicts the condition (ii). Q.E.D. for the proposition.

Note that if $Y \subset X$ is an irreducible cycle of codimension 2, satisfying the inequality $\operatorname{mult}_Y \Sigma > n$, where $\Sigma \subset |-nK_X|$ is a linear system, free in codimension 1, then in any case $\deg Y \leq \deg X - 1$, because $\deg Z = n^2 \deg X$, $Z = (D_1 \bullet D_2)$, $D_i \in \Sigma$, and Y comes into Z with a multiplicity strictly greater than n^2.

We prove Theorem 1^*, verifying the conditions (i) and (ii) of Proposition 2.1. The condition (i) should be verified for the irreducible cycles of codimension 2 and degree $\leq \deg V - 1 = 2m - 1$. The condition (ii) in our case takes the form of the estimate

$$\operatorname{mult}_x Y \leq \frac{2}{m} \deg Y.$$

2.2 Maximal cycles of codimension two

Assume that a system Σ, which is free in codimension one, has a maximal cycle $Y \subset V$ of codimension 2. Then $\deg Y \leq 2m-1$, and for the cycle $B = \sigma(Y) \subset Q$ there are two possible cases: either $\deg B = \deg Y$, i.e. the morphism $\sigma\colon Y \to B$ is birational, or $2\deg B = \deg Y$, i.e. $Y = \sigma^{-1}(B)$. If $\dim V \geq 5$, then by the Lefschetz theorem, applied to $B \subset Q$, we see that $B = Q \cap P$, $P \subset \mathbf{P}$ is a subspace of codimension 2. Moreover, $\deg Y = m$, i.e. the double cover $\sigma^{-1}(B) \to B$ is reducible. Consequently, the intersection $B \cap W$ is singular at every point. But $B \cap W = P \cap W^* \cap Q$ and by Lemma 2 from Sec. 3.4 in [IP] the cycle $B \cap W$ can have at most a curve of singular points. We have come to a contradiction, which proves (i) for the double hypersurfaces of dimension 5 and higher.

For the 4-dimensional case, we have to consider quite a few possible cases. The final result is the same: no maximal surface can occur. The detailed arguments will be given elsewhere.

3 Cycles of codimension two

We prove that a regular hypersurface satisfies the condition (ii) of Proposition 2.1.

3.1 Hypertangent systems

Fix an irreducible cycle $Y \subset V$ of codimension 2 and a point $x \in Y$. We must show that the following inequality

$$\frac{\operatorname{mult}_x}{\deg} Y \leq \frac{2}{m} \tag{1}$$

is true.

We first consider the case $\sigma(x) \notin W$ and assume that $l \geq 2$.

Let us arrange the set of polynomials $q_1, \ldots, q_m, g^{(l)}, \ldots, g^{(2l-2)}$ as a diagram of one (and only one) of the following three types: for $m < l$

$$\begin{array}{cccccc} q_1, & \ldots, & q_m, & & & \\ & & & g^{(l)}, & \ldots, & g^{(2l-2)} \end{array}$$

(we call this situation the $\vee$-case), for $l \leq m < 2l-2$

$$\begin{array}{ccccccccc} q_1, & \ldots, & q_l, & q_{l+1}, & \ldots, & q_m, & & & \\ & & g^{(l)}, & g^{(l+1)}, & \ldots, & g^{(m)}, & g^{(m+1)}, & \ldots, & g^{(2l-2)} \end{array}$$

(we call this situation the $\cap$-case), and for $2l-2 \leq m$

$$\begin{array}{cccccccc} q_1, & \ldots, & q_l, & q_{l+1}, & \ldots, & q_{2l-2}, & \ldots, & q_m. \\ & & g^{(l)}, & g^{(l+1)}, & \ldots, & g^{(2l-2)}, & & \end{array}$$

(we call this situation the $\subset$-case).

In each of these cases we construct a family of *hypertangent linear systems* in the following way. Let s_k denote an arbitrary homogeneous polynomial of degree k in the variables $z_1, \ldots, z_{M+1}$. For $k \leq m$ the symbol f_k stands for the truncated equation of Q, $f_k = q_1 + \ldots + q_k$. To begin with, let us construct certain linear spaces Π_k of polynomials in z_*, y. For conveniency of notations and definitions, which are somewhat different in the $\vee$-, $\cap$- and $\subset$-cases, we introduce two sets of indices:

$$\mathcal{M} = \{k | 1 \leq k \leq m-1\} \quad \text{and} \quad \mathcal{L} = \{k | l \leq k \leq 2l-2\}.$$

The spaces Π_k are indexed by the integers $k \in \mathcal{M} \cup \mathcal{L}$. Set

$$\Phi_k = \left\{ \sum_{i=1}^{\min\{k, m-1\}} f_i s_{k-i} \right\} \qquad \text{for} \quad k \geq 1,$$

$$\Gamma_k = \left\{ \sum_{i=l}^{\min\{k, 2l-2\}} (y - [\sqrt{g}]_i) s_{k-i} \right\} \quad \text{for} \quad k \geq l.$$

Now for $k \in \mathcal{M} \setminus \mathcal{L}$ set $\Pi_k = \Phi_k$, for any other $k \in \mathcal{M} \cup \mathcal{L}$ set $\Pi_k = \Phi_k + \Gamma_k$.

Let $\varphi \colon \widetilde{V} \to V$ be the blowing up of the point $x \in V$, $E = \varphi^{-1}(x) \cong \mathbf{P}^{M-1}$ the exceptional divisor.

Definition 1. The linear system Λ_k of divisors on V, the proper inverse image $\widetilde{\Lambda}_k$ of which on $\widetilde{V}$ is $|k(\varphi^* L) - (k+1)E|$, is said to be the *k-th hypertangent linear system* at the point x.

Proposition 3.1. *The following inclusion holds:*

$$\Pi_k|_V \subset \Lambda_k.$$

Proof. As in [P4], for $k \leq m-1$ the polynomial f_k is of degree k and $\mathrm{mult}_x\, \sigma^*(f_k|_Q) \geq k+1$. For $k \in \mathcal{L}$ we get: $(y - [\sqrt{g}]_k)$ represents a section of the sheaf $\sigma^*\mathcal{O}_Q(k)$ (since u is of degree $l \leq k$), and the formal series $y-\sqrt{g}$ vanishes identically, whereas the components of degree $\leq k$ in $[\sqrt{g}]_k$ coincide with those in $\sqrt{g}$. Therefore, $\mathrm{mult}_x(y-[\sqrt{g}]_k) \geq k+1$, which is what we need.

Proposition 3.2. *The following estimates hold:*

$$\mathrm{codim}_V \,\mathrm{Bs}\,\Lambda_k \geq \min\{k, m-1\}$$

for $k \in \mathcal{M} \setminus \mathcal{L}$,

$$\mathrm{codim}_V \,\mathrm{Bs}\,\Lambda_k \geq \min\{k, m-1\} + \min\{k, 2l-2\} + 1 - l$$

for all the other $k \in \mathcal{M} \cup \mathcal{L}$.

Proof. Let $\tau \in \mathrm{Aut}\, V$ be the Galois involution, permuting the sheets of the double cover $\sigma\colon V \to Q$. Obviously,

$$\tau^*(y-[\sqrt{g}]_k) = -y - [\sqrt{g}]_k,$$

so that the image of the divisor $y=[\sqrt{g}]_k$ on V with respect to σ is given on Q by the equation $g^{(k)}=0$. Set

$$\Gamma'_k = \left\{ \sum_{i=l}^{\min\{k,2l-2\}} g^{(i)} s_{k-i} \right\}$$

for $k \geq l$. Since σ is a finite morphism, it suffices to show that

$$\mathrm{codim}_Q \,\mathrm{Bs}(\Phi_k) \geq \min\{k, m-1\}$$

for $k \in \mathcal{M} \setminus \mathcal{L}$,

$$\mathrm{codim}_Q \,\mathrm{Bs}(\Phi_k + \Gamma'_k) \geq \min\{k, m-1\} + \min\{k, 2l-2\} + 1 - l$$

for all the other $k \in \mathcal{M} \cup \mathcal{L}$. In fact, in both cases the equalities hold. Let us prove these estimates.

The system of equations $f_1 = \ldots = f_k = 0$ is equivalent to the system $q_1 = \ldots = q_k = 0$. The latter defines a set of cones of codimension k in $\mathbf{P}$. If $k \leq m-1$, then neither of these cones can lie on Q, since otherwise all the polynomials $q_{k+1}, \ldots, q_m$ vanish on it, which is impossible by the assumption of regularity of the sequence $q_1, \ldots, q_m$. Thus the first estimate (for $k \in \mathcal{M} \setminus \mathcal{L}$) has been proved.

We emphasize, that the only non-trivial point in our proof is the fact that the codimension of the base set does not jump by one when we restrict the linear system from $\mathbf{P}$ to Q. Generally speaking, when we restrict the linear system onto a subvariety of codimension $a \geq 1$, the codimension of the base set may jump by $0 \leq i \leq a$. In our case this does not happen due to the presence of q_m in the regularity condition.

Assume that $k \geq l$. The base set of the system $\Phi_k + \Gamma'_k$ is the set of zeros of the following set of polynomials

$$\mathcal{P}_k = \{q_i | 1 \leq i \leq \min\{k, m-1\}\} \cup \{g^{(i)} | l \leq i \leq \min\{k, 2l-2\}\}.$$

If some irreducible component $T \subset \mathrm{Bs}(\Phi_k + \Gamma'_k)$ lies entirely in Q, then the "residual" polynomial

$$r = \sum_{i=\min\{k,m-1\}+1}^{m} q_i$$

vanishes on T.

If r has only one homogeneous component q_m, then we come to a contradiction: the system of functions $\mathcal{P}_k \cup \{q_m\}$ does not make a regular sequence. If $r = q_{k+1} + \ldots + q_m$ has more than two components, then the system of functions

$$\mathcal{P}_k \cup \{r\} \cup \{q_{k+1}, \ldots, q_{m-1}\}$$

does not make a regular sequence, whence it follows that

$$\mathcal{P}_k \cup \{q_{k+1}, \ldots, q_m\}$$

does not make a regular sequence, too. Again we come to a contradiction. This proves Proposition 3.2 completely.

3.2 Making curves from higher-dimensional cycles

For $i \in \mathcal{M}$ choose a divisor D_i in the linear system Λ_i, for $j \in \mathcal{L}$ choose a divisor $D_j^* \in \Lambda_j$, so that for $i \in \mathcal{L} \cap \mathcal{M}$ in the linear system Λ_i we choose *two* divisors. Formally speaking, we define a map

$$\beta \colon \mathcal{L} \bigvee \mathcal{M} \to \prod_{i \in \mathcal{L}} \Lambda_i \times \prod_{i \in \mathcal{M}} \Lambda_i$$

from the disjoint union of $\mathcal{L}$ and $\mathcal{M}$ to the product of the corresponding linear systems. Assume for the moment that $m \geq 4$.

Proposition 3.3. *For a general set $(D_i, i \in \mathcal{M}, D_j^*, j \in \mathcal{L})$ the set-theoretic intersection*

$$Y \bigcap \left(\bigcap_{i=3}^{m-1} D_i \right) \bigcap \left(\bigcap_{i=l}^{2l-2} D_i^* \right)$$

is one-dimensional in a neighborhood of the point x.

Proof. For any index $i \in \mathcal{L} \cup \mathcal{M}$, $i \geq \min\{3, l\}$, set

$$Y_i = Y \bigcap \left(\bigcap_{j=3}^{\min\{i,m-1\}} D_j \right) \bigcap \left(\bigcap_{j=l}^{\min\{i,2l-2\}} D_j^* \right).$$

Let us prove by ascending induction on $i \in \mathcal{L} \cup \mathcal{M}$ that

> (*) for a general set $(D_j, 1 \leq j \leq \min\{i, m-1\}, D_j^*, l \leq j \leq \min\{i, 2l-2\})$ the algebraic set Y_i in a neighborhood of the point x is of codimension
>
> $$\begin{array}{ll} \min\{i, m-1\} & \text{for} \quad i < l, \\ \min\{i, m-1\} + \min\{i, 2l-2\} + 1 - l & \text{for} \quad i \geq l. \end{array}$$

Taking $i = \max\{j \in \mathcal{M} \cup \mathcal{L}\}$, we get exactly the claim of our proposition.

Firstly, let us prove that (*) holds for the minimal value of the index $i = \min\{3, l\}$. There are three possible cases:

1) $i = l = 2$. Here $\operatorname{codim} \operatorname{Bs} \Lambda_2 \geq 3$ (the codimensions $\operatorname{Bs} \Lambda_i$ are taken with respect to V) and thus $Y \not\subset D_2$ for a general divisor $D_2 \in \Lambda_2$. Therefore, $\operatorname{codim} Y_2 = 3$, which is what we need.

2) $i = l = 3$. Here $\operatorname{codim} \operatorname{Bs} \Lambda_3 \geq 4$ and for a general pair of divisors $D_3, D_3^* \in \Lambda_3$ we have $Y \not\subset D_3$ and $Y \cap D_3 \not\subset D_3^*$. This implies that $\operatorname{codim} Y_3 = 4$.

3) $l \geq 4$. Here $\operatorname{codim} \operatorname{Bs} \Lambda_3 \geq 3$, $Y \not\subset D_3$ and $\operatorname{codim} Y_3 = 3$.

Before making the step of induction, we would like to make a remark. The set of integers $\mathcal{L} \cup \mathcal{M}$ can have a "hole": in the $\vee$-case $l \geq (m-1) + 2$.

Now, assume that (*) is proved for $i \leq k$, and let $b \geq k+1$ be the next index in $\mathcal{L} \cup \mathcal{M}$, i.e.

$$b = \min\{j \in \mathcal{L} \cup \mathcal{M}, j \geq k+1\}.$$

As we pointed out, it may well happen that $b \neq k+1$. There are two possible cases: A) $b \in \mathcal{L} \cap \mathcal{M}$, B) $b \in \mathcal{L} \setminus \mathcal{M}$ or $b \in \mathcal{M} \setminus \mathcal{L}$. Let us consider them separately.

A) Here $b \leq m-1$ and $l \leq b \leq 2l-2$, so that we have the estimate

$$\operatorname{codim} \operatorname{Bs} \Lambda_b \geq \operatorname{codim} Y_k + 2.$$

Consequently, the linear system $\Lambda_b|_{Y_k}$ is free from fixed components (in a neighborhood of the point x) and therefore for a general pair of divisors $D_b, D_b^* \in \Lambda_b$ we have $\operatorname{codim}(Y_k \cap D_b \cap D_b^*) = \operatorname{codim} Y_k + 2$, which is what we need.

B) Assume, for instance, that $b \in \mathcal{L} \setminus \mathcal{M}$ (the symmetric case $b \in \mathcal{M} \setminus \mathcal{L}$ is completely analogous). Here $b \geq m$, so that the estimate

$$\operatorname{codim} \operatorname{Bs} \Lambda_b \geq \operatorname{codim} Y_k + 1$$

holds. Thus a general divisor $D_b^* \in \Lambda_b$ does not contain Y_k and therefore $\operatorname{codim} Y_b = \operatorname{codim} Y_k + 1$.

Q.E.D. for Proposition 3.3.

Finally, let us prove the estimate (1).

For an irreducible cycle C of an arbitrary codimension $k \in \mathbf{Z}_+$ set

$$\beta(C) = \frac{\operatorname{mult}_x}{\deg} C.$$

Assume that $\beta(C) > 0$ and take a hypertangent system Λ_a, satifying the condition $\operatorname{codim} \operatorname{Bs} \Lambda_a \geq k+1$ near the point x.

Proposition 3.4. *The algebraic cycle $(C \bullet T)$ of the scheme-theoretic intersection of C and a general divisor $T \in \Lambda_a$ is an effective cycle of codimension $k+1$ on V. One of its components C_1 satisfies the inequality*

$$\beta(C_1) \geq \frac{a+1}{a} \beta(C).$$

Proof. It follows directly from the properties of the hypertangent systems.

Let us construct a sequence of irreducible cycles $C_i \subset V$, $i = 2, \dots, M-1$, of codimension $\mathrm{codim}_V C_i = i$, in the following way. Set $C_2 = Y$. Now for $j = 3, \dots, m-1$ we set inductively:$C_j =$ an irreducible component of the cycle $(C_{j-1} \bullet D_j)$, satisfying the inequality

$$\beta(C_j) \geq \frac{j+1}{j} \beta(C_{j-1}).$$

For $j = l, \dots, 2l-2$ we set inductively: C_{m-l+j} is an irreducible component of the cycle $(C_{m-l+j-1} \bullet D_j)$ such that the inequality

$$\beta(C_{m-l+j}) \geq \frac{j+1}{j} \beta(C_{m-l+j-1})$$

holds. Propositions 3.3 and 3.4 make sure that this construction is possible. The cycle $C_{M-1} \subset V$ is an irreducible curve. We get

$$\begin{array}{rcl} \frac{\mathrm{mult}_x}{\deg} C_{M-1} & \geq & \frac{\mathrm{mult}_x}{\deg} Y \cdot \frac{4}{3} \cdot \dots \cdot \frac{m}{m-1} \cdot \frac{l+1}{l} \cdot \dots \cdot \frac{2l-1}{2l-2} \\ & & \qquad \| \\ & & \frac{\mathrm{mult}_x}{\deg} Y \cdot \frac{m}{3} \cdot \frac{2l-1}{l}. \end{array}$$

The linear system $|L - E|$ on $\widetilde{V}$ is free, therefore the ratio $(\mathrm{mult}_x / \deg)(\cdot)$ can never exceed one. Consequently,

$$\frac{\mathrm{mult}_x}{\deg} Y \leq \frac{3l}{m(2l-1)} < \frac{2}{m},$$

which is what we need.

3.3 The exceptional cases

In the course of our arguments we put aside the case $m = 3$. In this case instead of Proposition 3.3 one should use its modification: the set-theoretic intersection

$$Y \bigcap \left(\bigcap_{i=l}^{2l-2} D_i^* \right)$$

is one-dimensional in a neighborhood of the point x. The rest of the arguments is completely analogous to the general case. The final estimate takes the form

$$\frac{\mathrm{mult}_x}{\deg} Y \leq \frac{l}{2l-1} < \frac{2}{3}.$$

Finally, let us consider the exceptional case $l = 1$, $m = M$. In this case the linear system $|L - 2E|$ is two-dimensional (V can be realized as a complete intersection in $\mathbf{P}^{M+2}$ and thus has two tangent hyperplanes at every point). The regularity condition implies that the cycle $(D_1 \bullet D_1^*)$, $D_1, D_1^* \in |L - 2E|$, is of multiplicity exactly 4 at the point x. Obviously, $(D_1 \bullet D_1^*)$ is irreducible. Thus if $\beta(Y) > 2/M$, then one of the divisors D_1, D_1^* does not contain Y. Let it be D_1. Then the cycle $(Y \bullet D_1)$ has a component Y^* such that $\beta(Y^*) > 4/M$. Applying the above-described procedure to the sequence of general divisors $D_4, \dots, D_{M-1}$ and the cycle Y^* of codimension 3, we get a curve $Y_{M-1} \subset V$, for which the ratio $\mathrm{mult}_x / \deg$ is strictly higher than one. A contradiction. For the case $\sigma(x) \notin W$ the proof is complete.

Now let us consider the case $\sigma(x) \in W$. Here our arguments repeat the proof, which has just been given for $\sigma(x) \notin W$, $l = 1$, almost word for word. Indeed, the linear system $|L - 2E|$ on $\widetilde{V}$ is two-dimensional again (here $\varphi: \widetilde{V} \to V$ is the blow up of the point x, $E = \varphi^{-1}(x)$), since the inverse image of any hyperplane section $T \subset Q$, which is tangent to W at $\sigma(x)$, is singular at x. Again the regularity condition (Definition 2 from Sec. 1.3) guarantees that the multiplicity of the cycle $(D_1 \bullet D_1^*)$, where $\widetilde{D}_1, \widetilde{D}_1^* \in |L - 2E|$, at the point x is equal to 4 exactly. It is easy to see that $(D_1 \bullet D_1^*) = \sigma^{-1}(T_x W \cap Q)$ is an irreducible cycle and thus if $(\mathrm{mult}_x / \deg) Y > 2/m$, then Y is not contained, for instance, in D_1. Now there is a component Y^* of the cycle $(Y \bullet D_1)$ of codimension 3 with $\mathrm{mult}_x / \deg > 4/m$.

From this point on we repeat the arguments for $l = 1$ word for word. Content-Length: 2449 X-Lines: 95 Status: RO

The property (ii) for the regular double hypersurface V is proved in full.

Q.E.D. for Theorem 1^*.

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