MULTIPLICITIES OF DUPIN HYPERSURFACES

Stephan Stolz

University of Notre Dame and Max–Planck–Institut für Mathematik, Bonn

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Abstract. All known examples of isoparametric hypersurfaces of spheres are of the following two types:

(1) The homogeneous examples, which are principal orbits of isotropy representations of symmetric spaces of rank 2.
(2) The Clifford examples of Ferus–Karcher–Münzner, which are constructed from modules over Clifford algebras.

Our main result, which extends work of Münzner, Thorbergsson and Grove–Halperin, is that any isoparametric hypersurface has the same number of principal curvatures and the same multiplicities as some hypersurface of type (1) or (2). More generally, this holds for any Dupin hypersurface. The proof is homotopy theoretic; the main tools are the Hopf invariant and the EHP–sequence.

1. Introduction

From a geometric point of view the simplest hypersurfaces of spheres are the *isoparametric hypersurfaces*. We recall that a hypersurface $M^n \subset S^{n+1}$ is called isoparametric if its principal curvatures (the eigenvalues of the shape operator) are constant. A more general class of hypersurfaces are the *Dupin hypersurfaces* for which the number of distinct eigenvalues of the shape operator at $x \in M$ is required to be independent of x and the i-th eigenvalue $\lambda_i(x)$ is constant on the leaves of the foliation defined by the eigenspaces corresponding to $\lambda_i(x)$.

Elie Cartan classified those isoparametric hypersurfaces whose number g of distinct principal curvatures is equal to 1, 2, or 3 [Ca] (all of them are homogeneous). Moreover, it was shown that all Dupin hypersurfaces with $g = 1, 2, 3$ are Lie equivalent to isoparametric surfaces (this is classical for $g = 1$, due to Cecil and Ryan [CR] for $g = 2$, and to Miyaoka [Miy] for $g = 3$).

For $g > 3$ a classification of isoparametric hypersurfaces or Dupin hypersurfaces is not known. However we have a lot of information about the number g and the possible multiplicities m_i (m_i is the dimension of the eigenspace corresponding to the eigenvalue λ_i for $i = 1, 2, \dots, g$; here we assume that the eigenvalues are ordered according to size: $\lambda_1 < \lambda_2 < \cdots < \lambda_g$). Münzner has shown for isoparametric

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Hypersurfaces that $g = 1, 2, 3, 4$, or 6 and that $m_{i+2} = m_i$, $i \in \mathbb{Z}/g$ [Mü1, Satz 1(b)]; in particular, all multiplicities m_i are determined by m_1 and m_2; moreover, $m_1 = m_2$ for g odd. These statements continue to hold for Dupin hypersurfaces by work of Thorbergsson [Th, Remark on p. 497] and Grove-Halperin [GH, §2]. For $g = 6$, Grove and Halperin proved that the only possible multiplicities are $m_1 = m_2 = 1$ or $m_1 = m_2 = 2$ [GH, Thm. C(iv)]; they also show: if $g = 4$ and $m_1 = m_2$, then $m_1 = 1$ or 2; if $g = 4$, and $2 \leq m_1 < m_2$, then $m = m_1 + m_2$ is odd. Stronger restrictions for isoparametric surfaces have been obtained by Abresch [Ab], and very recently, by Fang [Fa].

We note that the shape operator A of the embedding $M \subset S^{n+1}$ involves the choice of a normal direction; making the opposite choice replaces A by $-A$ and hence switches m_1 and m_2. Hence we may assume $m_1 \leq m_2$ without loss of generality. The main result of this paper is the following theorem.

1.1 Theorem. *Let $M^n \subset S^{n+1}$ be a Dupin hypersurface with $g = 4$ distinct principal curvatures and multiplicities $m_1 \leq m_2$. Then $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^{\phi(m_1-1)}$. Here $\phi(l)$ is the number of integers s with $1 \leq s \leq l$ and $s \equiv 0, 1, 2, 4 \mod 8$.*

The above restrictions on the multiplicities are *sharp*. To show this we recall that the known examples of isoparametric hypersurfaces are of the following two types.

(1) Hsiang and Lawson noticed that every homogeneous isoparametric hypersurface occurs as a principal orbit of the isotropy representation of some symmetric space of rank 2 [HL]. These were studied in detail by Takagi and Takahashi [TT]. They show in particular that for $g = 6$ the multiplicities $(1, 1)$ and $(2, 2)$ occur, and for $g = 4$ among others the multiplicities $(2, 2)$ and $(4, 5)$.
(2) Extending work of Ozeki-Takeuchi [OT], Ferus-Karcher-Münzner [FKM] associate to each module V over the Clifford algebra C_l an isoparametric hypersurface with $g = 4$, $m_1 = l+1$, and $m_2 = \dim V - m_1 - 1$ (cf. [KFM, 3.3 and Satz 4.1]). We note that the dimension of the irreducible modules over the Clifford algebra C_l is $2^{\phi(l)}$ (there are two irreducible modules for $l \equiv 0 \mod 4$, and one otherwise). It follows that positive integers (m_1, m_2) with $m_1 \leq m_2$ occur as multiplicities of isoparametric hypersurfaces of Clifford type if and only if $m_1 + m_2 + 1$ is a multiple of $2^{\phi(m_1-1)}$.

Summarizing the above discussion we conclude:

1.2 Corollary. *For any Dupin hypersurface $M^n \subset S^{n+1}$ there is a homogeneous isoparametric hypersurface or an isoparametric hypersurface of Clifford type with the same multiplicities.*

This result possibly suggests that every isoparametric hypersurface in a sphere is equivalent to a homogeneous isoparametric hypersurface or an isoparametric hypersurface of Clifford type. However we should mention that there are Dupin hypersurfaces not Lie equivalent to any isoparametric hypersurface; examples were provided by Pinkall and Thorbergsson [PT] and Miyaoka and Ozawa [MO].

In the next section we will outline the proof of Theorem 1.1; a description of the contents of the subsequent sections can be found at the end of section 2.

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§2. Outline of the proof of Theorem 1.1

Münzner showed that an isoparametric hypersurface $M^n \subset S^{n+1}$ leads to a decomposition of the sphere

$$S^{n+1} = D\gamma_1 \cup_M D\gamma_2, \tag{2.1}$$

as the union of two linear disc bundles $D\gamma_i \to M_i$ over smooth manifolds M_i with common boundary $\partial D\gamma_1 = \partial D\gamma_2 = M$ [Mü1, Satz 4(d)], where the fibers of the fiber bundle $p_i\colon M = \partial D\gamma_i \to M_i$ are spheres of dimension m_i. Geometrically the manifolds M_i (considered as submanifolds of $S^{n+1} \subset M$ via the identification (2.1)) are focal manifolds of M, and γ_i is the normal bundle of M_i. The above statements continue to be true for Dupin hypersurfaces by work of Thorbergsson [Th] and Grove-Halperin [GH, §2].

Let M be a Dupin hypersurface of dimension n with g distinct principal curvatures and multiplicities $m_1, \dots, m_g$. Then $n = m_1 + \cdots + m_g$, and since as mentioned in the introduction $m_{i+2} = m_i$ for $i \in \mathbb{Z}/g$, we have

$$n = \frac{gm}{2} \quad \text{for} \quad m = m_1 + m_2. \tag{2.2}$$

The results of Münzner, Thorbergsson and Grove-Halperin mentioned in the introduction concerning the number of principal curvatures of isoparametric (resp. Dupin) hypersurfaces and their multiplicities are consequences of this decomposition result, and the following purely topological result.

2.3 Theorem. *Assume that S^{n+1} decomposes in the form (2.1), and let g be given by equation (2.2). Then*

1. *(Münzner* [Mü1, Satz A]*) $g = 1, 2, 3, 4,$ or 6.*
2. *(Grove-Halperin* [GH, Thm. C(iv)(d)]*) If $g = 6$, then $m_1 = m_2 = 1$ or $m_1 = m_2 = 2$.*
3. *(Grove-Halperin* [GH, Thm. C(iv)(c)]*) If $g = 4$ and $m_1 = m_2$, then $m_1 = 1$ or 2; if $g = 4$, and $2 \le m_1 < m_2$, then $m = m_1 + m_2$ is odd.*

Similarly, our main result Theorem 1.1 is a consequence of the decomposition result and the following theorem.

2.4 Theorem. *Assume that S^{n+1} decomposes in the form (2.1), and let g be given by equation (2.2). Assume that $g = 4$ and $m_1 \le m_2$. Then $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^{\phi(m_1 - 1)}$.*

We would like to draw attention to the similarities between this theorem and Adams' result concerning vector fields on spheres. In terms of the numerical function $\phi(l)$, Adams' result can be phrased as follows [Ad, Thm. 1.1] (cf. [Hu, Prop. 11.2]).

2.5 Theorem (Adams). *If there are l linearly independent vector fields on the sphere S^m then $m+1$ is a multiple of $2^{\phi(l)}$.*

We recall that the classical Hurwitz-Radon-Eckmann construction produces l linearly independent vector fields on the sphere S^m starting with a module V over the Clifford algebra C_l with $\dim V = m+1$ (cf. [Hu, Chap. 11, Thm. 2.2, Thm. 2.4 and 2.5]). Adams' result shows that the maximal possible number of linearly independent vector fields is achieved by this construction. Similarly, Theorem 2.4 shows that all possible multiplicities of Dupin hypersurfaces except $(m_1, m_2) = (2,2), (4,5)$ are realized by the Clifford examples of Ferus-Karcher-Münzner. In fact, the Hurwitz-Radon-Eckmann construction and the Ferus-Karcher-Münzner construction are closely related: if β is the subbundle of the tangent bundle TS^m of vectors orthogonal to the l vector fields produced by the Hurwitz-Radon-Eckmann construction, then its sphere bundle $S(\beta)$ is diffeomorphic to a focal manifold of the corresponding isoparametric hypersurface of Clifford type (cf. [FKM, Satz 4.2(ii)]).

We will prove Theorem 2.4 by reducing it to a homotopy theoretic result of Adams which is the technical core of his theorem on vector fields. To state this result we need some notation. Let H_l be the real Hopf line bundle over the real projective space $\mathbb{RP}^l$ of dimension l, and let $T(kH_l)$ be the Thom space of the Whitney sum of k copies of H_l (we recall that the Thom space of a vector bundle α is obtained from the disc bundle of α by collapsing the sphere bundle to a point). We remark that the standard CW-structure on $\mathbb{RP}^l$ (with one cell in every dimension s for $0 \le s \le l$) gives $T(kH_l)$ the structure of a CW-complex with one cell in every dimension s for $k \le s \le k+l$.

2.6 Theorem (Adams). *Assume that the top cell of $T((m-l)H_l)$ splits off stably, i.e., there is a stable map $s\colon S^m \to T((m-l)H_l)$ such that the composition*

$$S^m \xrightarrow{s} T((m-l)H_l) \xrightarrow{p} S^m$$

has mapping degree one, where p is the projection map which collapses the $(m-1)$-skeleton of $T((m-l)H_l)$ to a point. Then $m+1$ is a multiple of $2^{\phi(l)}$.

We recall that a *stable map* from a pointed space X to a pointed space Y is a map $f\colon \Sigma^k X \to \Sigma^k Y$ between suitable supensions of X and Y. To facilitate comparison with the literature we want to mention that results like Theorem 2.6 above are frequently phrased in terms of statements about the stunted projective space $\mathbb{RP}^m_{m-l} = \mathbb{RP}^m/\mathbb{RP}^{m-l-1}$ (obtained from $\mathbb{RP}^m$ by collapsing the subspace $\mathbb{RP}^{m-l-1} \subset \mathbb{RP}^m$ to a point) which is homeomorphic to $T((m-l)H_l)$ (cf. [Hu, Chap. 15, Thm. 1.8]).

Roughly speaking, Theorem 2.6 is the Spanier-Whitehead dual of Theorem 1.2 in [Ad]. For the convenience of the reader we will make this relationship precise at the end of this section.

It is a classical result of James that the existence of l linearly independent vector fields on S^m implies that the top cell of $T((m-l)H_l)$ splits off stably (cf. [Hu, Chap. 15, Thm. 9.4]). It shows that Adams' technical result Theorem 2.6 implies his result on vector fields on spheres (Theorem 2.5).

One way of proving James' result proceeds in the following two steps.

(1) If S^m admits l linearly independent vector fields, then the Thom space $T(\tau S^m)$ of the tangent bundle of S^m desuspends l times (i.e., $T(\tau S^m)$ is homotopy equivalent to $\Sigma^l X$ for some space X).

(2) If $T(\tau S^m)$ desuspends l times then the top cell of $T((m-l)H_l)$ splits off stably.

We remark that step 1 is rather obvious (cf. [Hu, Chap. 15, Cor. 1.6]). The following propositions are analogous to steps (1) resp. (2) above.

2.7 Proposition. *Assume that S^{n+1} decomposes in the form (2.1) with $2 \le m_1 < m_2$, $n = 2m$, $m = m_1 + m_2$. If $(m_1, m_2) \neq (2,2), (4,5)$, then $(M_1 \wedge M_2)^{(2m)}$ (the $2m$-skeleton of $M_1 \wedge M_2$) desuspends l times for some l with $\phi(l) \ge \phi(m_1 - 1)$.*

2.8 Proposition. *Assume that S^{n+1} decomposes in the form (2.1) with $2 \le m_1 < m_2$, $n = 2m$, $m = m_1 + m_2$. If $(M_1 \wedge M_2)^{(2m)}$ desuspends at least l times, $0 < l \le m_1 - 1$, then the top cell of $T((m-l)H_l)$ splits off stably.*

Together with Adams' technical result (2.6) these propositions imply Theorem 2.4. We note that we may assume for the proof of Theorem 2.4 that $m_1 \ge 2$, since otherwise the conclusion that $m+1$ is a multiple of $2^{\phi(m_1-1)}$ is trivially satisfied. We can also assume $m_1 < m_2$, since the case $m_1 = m_2$ was settled by Grove-Halperin (cf. Thm. 2.3(3)). We observe that the condition $n = 2m$, $m = m_1 + m_2$ in Propositions 2.7 and 2.8 is equivalent to $g = 4$ by equation (2.2).

For the proof of Proposition 2.7 we will need the following result which is a corollary of Proposition 2.8 and Adams' result (2.6).

2.9 Corollary. *If $(M_1 \wedge M_2)^{(2m)}$ desuspends at least l times, $0 < l \le m_1 - 1$, then $m+1$ is a multiple of $2^{\phi(l)}$.*

Comments on the Proof of Theorem 2.6. We recall that a closed manifold M is Spanier-Whitehead dual to the Thom space $T(-\tau)$, where τ is the tangent bundle of M, and we write $-\tau$ for any bundle which is an inverse to τ in the sense that its Whitney sum with τ is trivial (e.g., we could take $-\tau$ to be the normal bundle of some embedding of M into Euclidean space). More generally, the Thom space $T(\alpha)$ of some vector bundle over M is Spanier-Whitehead dual to $T(-(\alpha \oplus \tau))$ (cf. [Hu, Chap. 15, Thm. 3.7]). In particular, if M is the real projective space $\mathbb{RP}^l$ of dimension l, then τ is stably (i.e., after adding trivial bundles) isomorphic to $(l+1)H$, and hence $T((m-l)H_l)$ is Spanier-Whitehead dual to $T(-(m+1)H_l)$.

The assumption that the top cell of $T((m-l)H_l)$ splits off stably implies that in the Spanier-Whitehead dual space $T(-(m+1)H_l)$ the bottom cell splits off stably (the space $T(-(m+1)H_l)$ is called s-coreducible in the terminology of [Ad], [Hu]). This implies that the vector bundle $-(m+1)H_l$ represents the trivial element in the group $J(\mathbb{RP}^l)$ consisting of stable J-equivalence classes of vector bundles over $\mathbb{RP}^l$ (cf. [Hu, Chap. 15, Thm. 7.7]). We recall that two vector bundles α, β are *stably J-equivalent* if and only if the sphere bundles $S(\alpha \oplus \epsilon^r)$ and $S(\beta \oplus \epsilon^s)$ are fiber homotopy equivalent for some r, s, where ϵ^r denotes the trivial vector bundle of dimension r.

Adams showed that $J(\mathbb{RP}^l)$ is the cyclic group of order $2^{\phi(l)}$ generated by H_l. In particular, if $-(m+1)H_l$ represents the trivial element of $J(\mathbb{RP}^l)$, then $m+1$ is a multiple of $2^{\phi(l)}$. $\square$

The paper is organized as follows. Sections 3 and 4 are devoted to preliminaries; we recall Münzner's results concerning the cohomology of the manifolds M, M_1 and M_2 in section 3, while in section 4 we discuss the EHP-sequence. This is our main homotopy theoretic tool. We prove Proposition 2.8 in section 5. Section 6 contains the proof of Proposition 2.7 under the additional assumption $(m_1, m_2) \neq (8,9), (8,11)$. In section 7 we show that the cases $(m_1, m_2) = (8,9), (8,11)$ can't occur.

3. The cohomology of M, M_1, and M_2

The goal of this section is to recall some results of Münzner concerning the cohomology of the pieces M, M_1, and M_2 of a decomposition (2.1) of the sphere S^{n+1}. Following his conventions we write $H^*(\quad)$ for the cohomology with coefficients in R, where $R = \mathbb{Z}$ if M_1 and M_2 are both orientable, and $R = \mathbb{Z}/2$ otherwise.

3.1 Lemma. *If m_1 and m_2 are both larger than one, then M, M_1 and M_2 are simply connected; in particular, they are orientable.*

Proof. Apply van Kampen's Theorem to the decomposition (2.1). □

Applying the Mayer-Vietoris sequence to the decomposition (2.1) we conclude that

$$H^q(M_1) \oplus H^q(M_2) \xrightarrow{p_1^* \oplus p_2^*} H^q(M) \tag{3.2}$$

is an isomorphism for $0 < q < n$. This shows in particular, that the homomorphisms p_1^* and p_2^* are injective; we will use these maps to identify $H^*(M_i)$ with a subring of $H^*(M)$. The cohomology of the manifolds M_i is (additively) given as follows.

3.3 Proposition ([Mü2, Satz 5]). *Assume that S^{n+1} decomposes in the form (2.1), and let $m = m_1 + m_2$. Then the cohomology groups of M_i, $i = 1, 2$, are given by*

$$H^q(M_i) = \begin{cases} R & q \equiv 0, m_{i+1} \mod m, \quad 0 \leq q < n \\ 0 & \text{otherwise} \end{cases}$$

This implies in particular that $\dim H^*(M_i) = \frac{1}{2} \dim H^*(M) = g$, where g is defined by equation (2.2), i.e., $n = gm/2$.

Next we discuss some of the multiplicative structure of $H^*(M)$. Let $1 \in H^0(M)$ the unit of the cohomology ring $H^*(M)$. For $g \geq 2$, let $a_i \in H^{m_{i+1}}(M_i) \cong R$ be a generator. The Leray–Hirsch Theorem applied to the fiber bundle $M \to M_i$ with fiber S^{m_i} leads to the following results.

3.4 Proposition ([Mü2, Satz 6]). *For $g \geq 2$ the cohomology ring $H^*(M)$ is a free module over $H^*(M_i)$ with basis $\{1, a_{i+1}\}$, $i = 1, 2$.*

3.5 Corollary. *For $g \geq 3$ there is a choice of generators $b_i \in H^m(M_i) \cong R$ such that $a_1 a_2 = b_1 + b_2$.*

Proof. The elements b_1, b_2 form a basis of $H^m(M)$ due to the isomorphism (3.2). Hence $a_1 a_2 = \lambda_1 b_1 + \lambda_2 b_2$ with $\lambda_i \in R$. Proposition 3.3 implies that also the elements b_1, $a_1 a_2$ form a basis and hence $\lambda_2 = \pm 1$. By the same argument $\lambda_1 = \pm 1$, which proves the lemma. □

4. Hopf invariant and EHP-sequence

In this section we recall some well-known constructions from homotopy theory. In particular, we describe the (generalized) EHP-sequence, the main tool in the proof of Propositions 2.7 and 2.8. We include a discussion of how the EHP-sequence follows from the configuration space model for $\Omega^k\Sigma^k X$, since we will need the geometric formulation of the EHP-sequence (Corollary 4.6, used in the proof of Lemma 7.2) and compatibilty of the Hopf invariant with suspensions (Lemma 4.7, used in the proof of Proposition 2.8). Also, this discussion might help to make the EHP-sequence less mysterious for non-experts.

4.1 Suspension homomorphism. Let X, Y be pointed spaces (i.e., X, Y are topological spaces with distinguished points $x_0 \in X$, $y_0 \in Y$). For technical reasons we will work in the category of compactly generated spaces with non-degenerate base points (cf. [Wh, Ch. I, §§4–6]). We recall that the *smash product* $X \wedge Y$ is the pointed space obtained from the Cartesian product $X \times Y$ by collapsing the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ to a point (this is the distinguished point in $X \wedge Y$). The smash product $S^k \wedge X$ is often denoted $\Sigma^k X$ and is called the *k-fold (reduced) suspension* of X. If $f: X \to Y$ is a base point preserving map, then we get a (base point preserving) map $1\wedge f: S^k \wedge X \to S^k \wedge Y$, which is called the *$k$-fold suspension of f* and denoted $\Sigma^k f$. This construction gives in particular a homomorphism

$$\Sigma^k: \pi_q(X) \to \pi_{q+k}(\Sigma^k X). \tag{4.2}$$

We recall that Freundenthal's Suspension Theorem says that if the space X is $(l-1)$-connected (i.e., if $\pi_i(X) = 0$ for $i < l$), then the *suspension homomorphism* (4.2) is surjective for $q < 2l$, and an isomorphism for $q < 2l-1$ (the range $q < 2l-1$ is called the *stable range*). The EHP-sequence described below is a vast generalization of Freundenthal's Theorem. It describes in the larger range $q < 3l-2$ (called the *metastable range*) the kernel and cokernel of the k-fold suspension map by embedding it into a long exact sequence.

The direct limit of the groups $\pi_{q+k}(\Sigma^k X)$ for $k \to \infty$ is called the q-th *stable homotopy group of* X and is denoted by $\pi_q^s(X)$. The homomorphism (4.2) induces a homomorphism $\Sigma^\infty: \pi_q(X) \to \pi_q^s(X)$; by Freudenthal's Theorem, it is an isomorphism for $q < 2l-1$. Following the usual custom, we write π_q^s for $\pi_q^s(S^0)$.

For a pointed space X, let $\Omega^k X$ be the space of base point preserving maps $S^k \to X$; we furnish it with a modification of the compact-open topology (cf. [Wh, p. 20]) and take the constant map as the base point in $\Omega^k X$. If Y is a pointed space, then there is a natural homeomorphism between (base point preserving) maps $Y \to \Omega^k X$ and maps $\Sigma^k Y \to X$ [Wh, Ch. III, (2.9)]. In particular, $\pi_{q+k}(\Sigma^k X)$ can be identified with $\pi_q(\Omega^k\Sigma^k X)$; using this inclusion, the suspension homomorphism (4.2) can be identified with the map induced on π_q by the inclusion map

$$i: X \to \Omega^k\Sigma^k X \tag{4.3}$$

(for $x \in X$ the map $i(x): S^k \to S^k \wedge X$ sends v to $v \wedge x$).

The following theorem is a corollary of results of May, Milgram and Segal concerning combinatorial models for $\Omega^k\Sigma^k X$.

4.4 Theorem. *Let X be an $(l-1)$-connected space, $l \geq 1$. Then the inclusion (4.3) can be factored in the form*

$$X \subset F_2C_k(X) \xrightarrow{\alpha_k} \Omega^k\Sigma^k X,$$

where α_k is a $(3l-1)$-equivalence, and $(F_2C_k(X), X)$ is an NDR-pair (cf. [Wh, Ch. I, §5]*). Moreover, the quotient $F_2C_k(X)/X$ is homotopy equivalent to*

$$D_{k,2}(X) \stackrel{\text{def}}{=} \left(S^{k-1}_+ \wedge X \wedge X\right)/\mathbb{Z}/2.$$

Here S^{k-1}_+ is the sphere S^{k-1} equipped with a disjoint base point; $\mathbb{Z}/2$ acts on S^{k-1}_+ by the antipodal map and on $X \wedge X$ by permuting factors.

Before showing how this result follows from the approximation results for $\Omega^k\Sigma^k X$ (cf. 4.13 below), we derive some corollaries. Theorem 4.4 shows that in the metastable range $q < 3l-2$ we can replace $\Omega^k\Sigma^k X$ by its approximation $F_2C_k(X)$, and the inclusion $i\colon X \to \Omega^k\Sigma^k X$ by the inclusion $X \to F_2C_k(X)$. Abusing notation, we will hence also write i for the latter. We will denote by p the composition of the projection map $F_2C_k(X) \to F_2C_k(X)/X$ with the homotopy equivalence $F_2C_k(X)/X \sim D_{k,2}(X)$.

We note that for an $(l-1)$-connected space X the space $D_{k,2}(X)$ is $(2l-1)$-connected. (If X is a pointed CW-complex with only cells of dimension $\geq l$, then the usual CW structure on $X \wedge X$ has cells of dimension $\geq 2l$. We note that we can think of $D_{k,2}(X)$ as the quotient of the total space of the fiber bundle

$$\left(S^{k-1} \times X \wedge X\right)/\mathbb{Z}/2 \to S^{k-1}/\mathbb{Z}/2 = \mathbb{RP}^{k-1}$$

with fiber $X \wedge X$ modulo the canonical section given by the base points of the fibers. It follows that $D_{k,2}(X)$ has a CW structure with cells of dimension $\geq 2l$.)

We recall the following result, which is a special case of the Blakers-Massey Theorem (cf.[Wh, Chap. VII, Thm. 7.12]). Let (Y, A) be a $(k-1)$-connected NDR-pair, and suppose that A is $(l-1)$-connected, $k \geq 2$, $l \geq 2$. Then the projection map $(Y, A) \to (Y/A, *)$ induces an isomorphism on π_q for $q < k+l-1$ and an epimorphism for $q = k+l-1$. Applying this inductively to $(Y, A) = (F_2C_k(X), X)$, the fact that $F_2C_k(X)/X \sim D_{k,2}(X)$ is $(2l-1)$-connected implies that $(F_2C_k(X), X)$ is $(2l-1)$-connected. Applying the result again shows that the projection map

$$(F_2C_k(X), X) \to (F_2C_k(X)/X, *)$$

is a $(3l-1)$-equivalence.

Now consider the long exact homotopy sequence of the pair $(F_2C_k(X), X)$; identifying for $q < 3l-1$ the groups $\pi_q(F_2C_k(X))$ with $\pi_q(\Omega^k\Sigma^k X) = \pi_{q+k}(\Sigma^k X)$, and $\pi_q(F_2C_k(X), X))$ with $\pi_q(D_{k,2}(X))$ leads to the following exact sequence.

4.5 Corollary (Generalized EHP-sequence). *If X is an $(l-1)$-connected space, then for $q < 3l-1$ there is an exact sequence*

$$\pi_q(X) \xrightarrow{\Sigma^k} \pi_{q+k}(\Sigma^k X) \xrightarrow{H^k} \pi_q(D_{k,2}(X)) \xrightarrow{\partial} \pi_{q-1}(X) \xrightarrow{\Sigma^k} \pi_{q+k-1}(\Sigma^k X) \dots .$$

For $k = 1$, this is the classical EHP-sequence due to James (cf. [Wh, Chap. XII, Thm. 2.2]). The homomorphism H^k is called the (generalized) *Hopf invariant.* The letters EHP stand for "Einhängung" (the German word for suspension), "Hopf invariant" and "product", respectively (if X is a sphere and $k = 1$, the boundary map ∂ can be interpreted as a Whitehead product).

We note that the inclusion map $i: X \to F_2C_k(X)$ lifts to the homotopy fiber of the map $p: F_2C_k(X) \to D_{k,2}(X)$, and that this lift is a $(3l-2)$-equivalence (this follows by comparing the long exact homotopy sequence of the pair $(F_2C_k(X), X)$ with the long exact sequence of the homotopy fibration). Hence we conclude:

4.6 Corollary. *If X is an $(l-1)$-connected space, then*

$$X \xrightarrow{i} F_2C_k(X) \xrightarrow{p} D_{k,2}(X)$$

is a homotopy fibration in the metastable range (i.e., in dimensions $q < 3l-2$); the corresponding exact sequence of homotopy groups can be identified with the EHP-sequence.

For our proof of Proposition 2.8 it is crucial that the Hopf invariant is compatible with suspensions in the following sense.

4.7 Lemma. *If X is an $(l-1)$-connected space, then in the metastable range $q < 3l-2$ the following diagram commutes*

$$\begin{array}{ccccc}
\pi_{q+k}(\Sigma^k X) & \xrightarrow{H^k} & \pi_q(D_{k,2}(X)) & \xrightarrow{\Sigma^r} & \pi_{q+r}(\Sigma^r D_{k,2}(X)) \\
\Sigma^r \big\downarrow & & & & \big\downarrow \Delta_* \\
\pi_{q+k+r}(\Sigma^{k+r} X) & & \xrightarrow{\quad H^k \quad} & & \pi_{q+r}(D_{k,2}(\Sigma^r X))
\end{array}$$

Here the lower horizontal map is the Hopf invariant for the space $\Sigma^r X$, and the map $\Delta: \Sigma^r D_{k,2}(X) \to D_{k,2}(\Sigma^r X)$ sends $v \wedge [e \wedge x \wedge y]$ to $[e \wedge (v \wedge x) \wedge (v \wedge y)]$ for $v \in S^r$, $e \in S^{k-1}_+$, $x, y \in X$.

We defer the proof of this lemma to the end of the section.

For calculations with the EHP-sequence, it is of course important to identify the space $D_{k,2}(X)$. We will now discuss the well-known description of $D_{k,2}(S^n)$ in terms of other spaces. It is easy to see that $D_{k,2}(S^n) = \big(S^{k-1}_+ \wedge S^n \wedge S^n\big) \big/ \mathbb{Z}/2$ is homeomorphic to the Thom space of the vector bundle

$$\big(S^{k-1} \times (\mathbb{R}^n \oplus \mathbb{R}^n)\big) \big/ \mathbb{Z}/2 \tag{4.8}$$

over $S^{k-1} / \mathbb{Z}/2 = \mathbb{RP}^{k-1}$. Here $\mathbb{Z}/2$ acts on $\mathbb{R}^n \oplus \mathbb{R}^n$ by permuting the summands. As a representation of $\mathbb{Z}/2$, this is isomorphic to the direct sum of n copies of the non-trivial one dimensional representation $\mathbb{R}^-$ and n copies of the trivial representation. Hence the vector bundle (4.8) is isomorphic to $nH_{k-1} \oplus n\underline{\mathbb{R}}$, where $H_{k-1} = \big(S^{k-1} \times \mathbb{R}^-\big) \big/ \mathbb{Z}/2$ is the Hopf line bundle over $\mathbb{RP}^{k-1}$, and $\underline{\mathbb{R}}$ is the one dimensional trivial real line bundle. It follows that the Thom space of the vector bundle (4.8) is homeomorphic to $T(nH_{k-1} \oplus n\underline{\mathbb{R}})$, which in turn is homeomorphic to $\Sigma^n T(nH_{k-1})$ (cf. [Hu, Chap. 13, Cor. 1.6]). Moreover, the Thom space $T(nH_{k-1})$ is homeomorphic to the stunted projective space $\mathbb{RP}^{n+k-1}_n = \mathbb{RP}^{n+k-1} \big/ \mathbb{RP}^{n-1}$ (cf. [Hu, Chap. 13, Example 1.7]). For later reference we summarize our discussion in the following lemma.

4.9 Lemma. *There are homeomorphisms*

$$D_{k,2}(S^n) \approx \Sigma^n T(nH_{k-1}) \approx \Sigma^n \mathbb{RP}_n^{n+k-1}.$$

Remark 4.10. Results (4.5)–(4.9) are statements for a fixed k. It is not hard to see that they are compatible for different k in the sense that there are commutative diagrams relating the results for k and $k+1$. Passing to the direct limit as k approaches infinity, we get a statement for "$k=\infty$". E.g., Corollary 4.5 leads to the exact sequence

$$\pi_q(X) \xrightarrow{\Sigma^\infty} \pi_q^s(X) \xrightarrow{H} \pi_q(D_2(X)) \xrightarrow{\partial} \pi_{q-1}(X) \xrightarrow{\Sigma^\infty} \pi_{q-1}^s(X) \dots,$$

where $D_2(X) = \left(S_+^\infty \wedge X \wedge X\right) / \mathbb{Z}/2$ is the direct limit of the spaces $D_{k,2}(X)$. Corollary 4.6 leads to a homotopy fibration

$$X \xrightarrow{i} F_2C(X) \xrightarrow{p} D_2(X),$$

where $F_2C(X)$ is the direct limit of the spaces $F_2C_k(X)$. Lemma 4.9 leads to homeomorphisms

$$D_2(S^n) \approx \Sigma^n T(nH) \approx \Sigma^n \mathbb{RP}_n,$$

where H is the Hopf line bundle over $\mathbb{RP}^\infty$, and $\mathbb{RP}_n$ is the stunted projective space $\mathbb{RP}_n = \mathbb{RP}^\infty / \mathbb{RP}^{n-1}$.

Theorem 4.4 is a consequence of the existence of 'combinatorial models' for $\Omega^k\Sigma^k X$, which we want to describe now.

4.11 Configuration space models for $\Omega^k\Sigma^k X$. May, Milgram and Segal have developed combinatorial "models" for $\Omega^k\Sigma^k X$ (cf. [Ma], [Mil], [Se]); here we want to describe the configuration space model of May and Segal. Let $\widetilde{C}_{k,s} \subset \mathbb{R}^k \times \cdots \times \mathbb{R}^k$ be the subspace of consisting of those tuples $(v_1, \dots, v_s)$, $v_i \in \mathbb{R}^k$, for which the distance between any pair of points is ≥ 2. Given a pointed space X, define

$$C_k(X) = \left(\coprod_{s \geq 1} \widetilde{C}_{k,s} \times X \times \cdots \times X \right) \Big/ \sim .$$

Here the equivalence relation $\sim$ is generated by declaring $(v_1, \dots, v_s, x_1, \dots, x_s)$ equivalent to $(v_{\sigma(1)}, \dots, v_{\sigma(s)}, x_{\sigma(1)}, \dots, x_{\sigma(s)})$ for any permutation σ; moreover, if $x_i \in X$ is the base point of X, then $(v_1, \dots, v_s, x_1, \dots, x_s)$ is equivalent to $(v_1, \dots, \widehat{v}_i, \dots, v_s, x_1, \dots, \widehat{x}_i, \dots, x_s)$ (obtained by deleting v_i and x_i). A point $[v_1, \dots, v_s, x_1, \dots, x_s] \in C_k(X)$ can be thought of as a "finite subset of $\mathbb{R}^k$ labelled by X" (think of $x_i \in X$ as a label for the point $v_i \in \mathbb{R}^k$).

It is more customary to allow in the definition of $\widetilde{C}_{k,s}$ all tuples $(v_1, \dots, v_s)$ for which the v_i's are pairwise distinct. These two definitions of $C_n(X)$ give homotopy equivalent spaces (cf. [Se, §3]). The advantage of the above definition is that it is easy to define a map

$$\alpha_k : C_k(X) \to \Omega^k \Sigma^k X \tag{4.12}$$

as follows (cf. [Se, §3]). A point $[v_1, \ldots, v_s, x_1, \ldots, x_s] \in C_k(X)$ determines a map $\phi\colon \mathbb{R}^\infty \cup \infty \to \Sigma^k X = D^k / \partial D^k \wedge X$ defined by

$$\phi(v) = \begin{cases} T_{v_i}^{-1}(v) \wedge x_i \in D^k / \partial D^k \wedge X & \text{if } v \in B_1(v_i) \\ * & \text{otherwise} \end{cases} .$$

Here $*$ is the base point of $D^k / \partial D^k \wedge X$, and $T_{v_i}\colon D^k \to B_1(v_i)$ is translation by v_i ($B_1(v_i)$ denotes the ball of radius 1 around v_i). We note that ϕ is well-defined, since the interiors of the balls $B_1(v_i)$ are disjoint due to the assumption that the distance between any pair of points v_i, v_j is ≥ 2. Identifying $\mathbb{R}^k \cup \infty$ with S^k the above construction defines the map (4.12). According to results of May α_k is a weak homotopy equivalence for path connected X (i.e., it induces an isomorphism on homotopy groups, cf. [Se, Thm. 3]).

4.13 Proof of Theorem 4.4. Let $F_s C_k(X)$ be the subspace of $C_k(X)$ represented by tuples $(v_1, \ldots, v_r, x_1, \ldots, x_r)$ with $r \leq s$. Consider the cofibration sequence

$$F_1 C_k(X) \xrightarrow{i} F_2 C_k(X) \xrightarrow{p} F_2 C_k(X) / F_1 C_k(X).$$

We remark:

(1) $F_1 C_k(X)$ is homotopy equivalent to X.

(2) $F_s C_k(X) / F_{s-1} C_k(X)$ is homeomorphic to $\Big((\widetilde{C}_{k,s})_+ \wedge X \wedge \cdots \wedge X \Big) \Big/ \Sigma_s$, where the symmetric group Σ_s acts on $(\widetilde{C}_{k,s})_+$ and $X \wedge \cdots \wedge X$ by permutations. We note that this quotient is $(sl-1)$-connected (cf. discussion of connectivity of $D_{k,2}(X)$ following Theorem 4.4), and hence the composition $F_2 C_k(X) \subset C_k(X) \xrightarrow{\alpha_k} \Omega^k \Sigma^k X$ is a $(3l-1)$-equivalence.

(3) $\widetilde{C}_{k,2}$ is $\mathbb{Z}/2$-equivariantly homotopy equivalent to S^{k-1} with the antipodal action, and hence (2) implies that $F_2 C_k(X) / X \sim F_2 C_k(X) / F_1 C_k(X)$ is homotopy equivalent to $D_{k,2} X$.

This implies Theorem 4.4. $\square$

Proof of Lemma 4.6. The definition of the map $\alpha_k\colon C_k(X) \to \Omega^k \Sigma^k X$ (cf. 4.12) shows that the following diagram commutes.

$$\begin{array}{ccc} \Sigma^r C_k(X) & \xrightarrow{\Sigma^r \alpha_k} & \Sigma^r \Omega^k \Sigma^k X \\ {\scriptstyle \Delta}\big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ C_k(\Sigma^r X) & \xrightarrow{\alpha_k} & \Omega^k \Sigma^k (\Sigma^r X) \end{array} .$$

Here the vertical map Δ on the left sends $v \wedge [v_1, \ldots v_s, x_1, \ldots, x_s] \in \Sigma^r C_k(X)$ to $[v_1, \ldots v_s, (v \wedge x_1), \ldots, (v \wedge x_s)] \in C_k(\Sigma^r X)$. The vertical map Δ on the right sends $v \wedge f \in \Sigma^r \Omega^k \Sigma^k X$ to the element of $\Omega^k \Sigma^k (\Sigma^r X)$ given by the map $S^k \to \Sigma^k(\Sigma^r X) = S^r \wedge \Sigma^k X$ which maps $w \in S^k$ to $v \wedge f(w)$.

The map Δ on the left respects the filtration of the configuration spaces; in particular, it restricts to a map $\Delta\colon \Sigma^r F_2 C_k(X) \to F_2 C_k(\Sigma^r X)$. It is compatible with the map Δ defined in Lemma 4.7; i.e., the following diagram is commutative, where as in Corollary 4.6, p is the map defined after Theorem 4.4:

$$\begin{array}{ccc}
\Sigma^r F_2C_k(X) & \xrightarrow{\Sigma^r p} & \Sigma^r D_{k,2}(X) \\
\Delta\big\downarrow & & \big\downarrow\Delta \\
F_2C_k(\Sigma^r X) & \xrightarrow{p} & D_{k,2}(\Sigma^r X)
\end{array}$$

Combining the two diagrams above and replacing the maps Δ by their adjoints Δ', we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\Omega^k\Sigma^k X & \xleftarrow{\alpha_k} & F_2C_k(X) & \xrightarrow{p} & D_{k,2}(X) \\
\Delta'\big\downarrow & & \Delta'\big\downarrow & & \big\downarrow\Delta' \\
\Omega^{k+r}\Sigma^{k+r}X & \xleftarrow{\Omega^r\alpha_k} & \Omega^r F_2C_k(\Sigma^r X) & \xrightarrow{\Omega^r p} & \Omega^r D_{k,2}(\Sigma^r X)
\end{array}$$

Now we consider the functor π_q (the q-th homotopy group) applied to the diagram above for $q < 3l-2$. We note that

(1) The maps α_k and $\Omega^r\alpha_k$ induce isomorphisms.
(2) The homomorphisms $p_*\circ(\alpha_k)_*^{-1}$ (resp. $\Omega^r p_*\circ(\Omega^r\alpha_k)_*^{-1}$) coincide with the Hopf invariant H^k for the space X (resp. $\Sigma^r X$).
(3) The homomorphism induced by the left Δ' can be identified with the suspension homomorphism $\Sigma^r\colon \pi_{q+k}(\Sigma^k X) \to \pi_{q+k+r}(\Sigma^{k+r}X)$.
(4) The homomorphism induced by the right Δ' can be identified with the composition $\pi_q(D_{k,2}(X)) \xrightarrow{\Sigma^r} \pi_{q+r}(\Sigma^r D_{k,2}(X)) \xrightarrow{\Delta_*} \pi_{q+r}(D_{k,2}(\Sigma^r X)$.

This proves Lemma 4.6. $\square$

5. Proof of Proposition 2.8

The main tool for the proof of Proposition 2.8 is the Hopf invariant described in the last section.

5.1 Definition of the Thom map. Let M^n be a *framed manifold*, i.e., a manifold equipped with an embedding of M into some sphere S^{n+k} and a trivialization of the normal bundle of this embedding. This allows us to identify a tubular neighbourhood T of M with $D^k\times M$ and the quotient space $T/\partial T$ with $\big(D^k\times M\big)/\big(\partial D^k\times M\big) = D^k/\partial D^k\wedge M_+ = S^k\wedge M_+$. The *Thom map* is the map

$$c\colon S^{n+k}\to T/\partial T = S^k\wedge M_+,$$

which is the identity restricted to $T\subset S^{n+k}$ and sends points outside of T to the base point. We will regard c as a stable map $c\colon S^n\to M_+$.

Throughout the rest of this section we will assume given a decomposition of S^{n+1} of the form (2.1) with $2\le m_1 < m_2$, $n = 2m$, $m = m_1+m_2$.

5.2 Proposition. *Let c be the Thom map of the hypersurface $M^{2m} \subset S^{2m+1}$. Then the Hopf invariant of the composition*

$$S^{2m} \xrightarrow{c} M^{2m} \xrightarrow{p_1 \wedge p_2} (M_1 \wedge M_2)^{(2m)} \subset M_1 \wedge M_2$$

is non-trivial.

Before proving Proposition 5.2 we want to use this result to deduce Proposition 2.8.

Proof of Proposition 2.8. By assumption there is a space X such that $\Sigma^l X$ is homotopy equivalent to $(M_1 \wedge M_2)^{(2m)}$. Consider the following diagram which is commutative by Lemma 4.7 (applied for $k = \infty$; cf. remark 4.10).

$$\begin{array}{ccccc}
\pi^s_{2m-l}(X) & \xrightarrow{H} & \pi_{2m-l}(D_2(X)) & \xrightarrow{\Sigma^l} & \pi_{2m}(\Sigma^l D_2(X)) \\
\Sigma^l \big\downarrow & & & & \big\downarrow \Delta_* \\
\pi^s_{2m}(\Sigma^{k+r} X) & & \xrightarrow{\quad\quad H \quad\quad} & & \pi_{2m}(D_2(\Sigma^l X))
\end{array}$$

Let $f \in \pi^s_{2m-l}(X)$ be the element whose l-fold suspension $\Sigma^l f \in \pi^s_{2m}(\Sigma^l X) = \pi^s_{2m}((M_1 \wedge M_2)^{(2m)})$ is the composition $(p_1 \wedge p_2) \circ c$ of Proposition 5.2.

Claim 1. There is a map $i: S^{m-l} \to X$ ('inclusion of the bottom cell') such that $\Sigma^l D_2(i)_*: \pi_{2m}(\Sigma^l D_2(S^{m-l})) \to \pi_{2m}(\Sigma^l D_2(X))$ is surjective.

We recall from Lemma 4.9 and Remark 4.10 that $D_2(S^{m-l}) = D_{\infty,2}(S^{m-l})$ is homeomorphic to the suspended Thom space $\Sigma^{m-l} T((m-l)H)$ (where H is the Hopf line bundle over $\mathbb{RP}^\infty$). In particular, we can identify $\pi_{2m}(\Sigma^l D_2(S^{m-l}))$ with $\pi_{2m}(\Sigma^m T((m-l)H)) \cong \pi^s_m(T((m-l)H))$.

Let

$$s': S^m \to T((m-l)H)$$

be a stable map with $\Delta_*(\Sigma^l D_2(i)_*(s')) = H(f)$. The Hopf invariant of f is non-trivial by Proposition 5.2; since the bottom cell of $D_2(\Sigma^l X)$ is in dimension $2m$, this implies that $H(f)$ induces a non-trivial map in integral homology. We conclude that hence also s' induces a non-trivial map in $H_m(\quad; \mathbb{Z})$.

We observe that that $T((m-l)H_l)$ ($H_l =$ Hopf bundle over $\mathbb{RP}^l$) is the m-skeleton of $T((m-l)H)$, and hence up to homotopy s' factors through a stable map $s: S^m \to T((m-l)H_l)$.

Claim 2. For a suitable choice of $s: S^m \to T((m-l)H_l)$, the composition with the projection map $p: T((m-l)H_l) \to S^m$ onto the top cell has mapping degree one.

To finish the proof of Proposition 2.8, it now remains to prove claims 1 and 2. For the proof of claim 1 we recall from section 3 that M_i is simply connected and that $H^q(M_i; \mathbb{Z}) = \mathbb{Z}$ for $q = 0, m_{i+1}, m, m + m_{i+1}$ and trivial for all other q. Then obstruction theory shows that M_i is homotopy equivalent to a CW complex with one cell of dimension m_{i+1} and other cells of dimension $\geq m$; it follows that we have the following homotopy equivalences

$$\begin{aligned}
(M_1 \wedge M_2)^{(2m)} &\sim S^m \cup \text{cells of dimension} \geq m + m_1 \\
X &\sim S^{m-l} \cup \text{cells of dimension} \geq m + m_1 - l \\
X \wedge X &\sim S^{m-l} \wedge S^{m-l} \cup \text{cells of dimension} \geq 2m + m_1 - 2l \\
D_2(X) &\sim D_2(S^{m-l}) \cup \text{cells of dimension} \geq 2m + m_1 - 2l
\end{aligned}$$

Here the last line follows from the previous one by the remarks concerning a CW structure for $D_2(X)$ following Theorem 4.4. The last line implies that the map

$$\pi_{2m}(\Sigma^l D_2(S^{m-l})) \xrightarrow{\Sigma^l D_2(i)_*} \pi_{2m}(\Sigma^l D_2(X))$$

induced by the inclusion $i: S^{m-l} \to X$ of the bottom cell of X is surjective provided $2m < 2m + m_1 - l$, which is guaranteed by the assumption $l \leq m_1 - 1$. This proves claim 1.

For the proof of claim 2 we note that by Lemma 4.9 and Remark 4.10 we can identify the inclusion $T((m-l)H_l) \to T((m-l)H)$ with the inclusion $\mathbb{RP}^m_{m-l} \to \mathbb{RP}_{m-l}$ of stunted projective spaces. Our assumption $l > 0$ implies

$$H_m(\mathbb{RP}_{m-l}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2 & m \text{ odd} \\ 0 & m \text{ even} \end{cases}.$$

The fact that $s': S^m \to \mathbb{RP}_{m-l}$ induces a non-trivial map in integral homology then implies that m is odd and that for any lift s the composition $p \circ s$ has odd degree. Let $g: S^m \to \mathbb{RP}^m_{m-l}$ be the attaching map for the top cell of $\mathbb{RP}^{m+1}_{m-l}$. We note that for m odd the composition $S^m \xrightarrow{g} \mathbb{RP}^m_{m-l} \xrightarrow{p} S^m$ has mapping degree 2. The long exact sequence of stable homotopy groups associated to the cofibration

$$S^m \xrightarrow{g} \mathbb{RP}^m_{m-l} \xrightarrow{i} \mathbb{RP}^{m+1}_{m-l}$$

then shows that $s' \in \pi^s_m(\mathbb{RP}_{m-l}) \cong \pi^s_m(\mathbb{RP}^{m+1}_{m-l})$ can be factored through some s whose composition with p has mapping degree one. $\square$

To prove Proposition 5.2 we will show the stronger statement that the Hopf invariant of the composition

$$g: S^{2m} \xrightarrow{c} M \xrightarrow{p_1 \wedge p_2} (M_1 \wedge M_2)^{(2m)} \xrightarrow{h} K_m \tag{5.3}$$

is non-trivial. Here K_m is the Eilenberg MacLane space $K_m = K(\mathbb{Z}/2, m)$ (which up to homotopy equivalence is characterized by $\pi_m(K_m) = \mathbb{Z}/2$, and the vanishing of all other homotopy groups), and h is the map corresponding to the generator of $H^m((M_1 \wedge M_2)^{(2m)}; \mathbb{Z}/2) \cong \mathbb{Z}/2$. We recall that for any space X the map

$$[X, K_m] \to H^m(X, \mathbb{Z}/2) \qquad x \mapsto x^*(\iota)$$

is an isomorphism. Here $[M, K_m]$ is the group of homotopy classes of maps to K_m, and $\iota \in H^m(K_m; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator. We use this isomorphism to identify cohomology classes with (homotopy classes of) maps to K_m.

For the proof that the Hopf invariant of the map (5.3) is non-trivial, it is useful to consider the following more general situation.

5.4 Definition. Let M be a framed manifold of dimension $2m$, and let $c\colon S^{2m} \to M$ be its stable Thom map. Let

$$\varphi\colon H^m(M;\mathbb{Z}/2) \to \mathbb{Z}/2$$

be the map which sends a map $x\colon M \to K_m$ to the Hopf invariant of the stable map

$$S^{2m} \xrightarrow{c} M \xrightarrow{x} K_m.$$

We note that $H(x \circ c)$ is an element of $\pi_{2m}(D_2(K_m)) = \mathbb{Z}/2$ (cf. [Mil, table 10.3]).

The above construction of φ is a special case of a much more general construction by E. Brown [Br, Cor. 1.11] (we deal here with the special case where the Thom spectrum MG is the sphere spectrum, and consider manifolds instead of Poincaré complexes). Specializing Lemma 1.14 and Corollary 1.11 of [Br] to our situation, we obtain the following properties for φ.

5.5 Proposition. *Let M be a framed manifold of dimension $2m$. Then φ has the following properties.*

(1) *If M is the boundary of a framed manifold W and x is in the image of the restriciton map $H^m(W;\mathbb{Z}/2) \to H^m(M;\mathbb{Z}/2)$ then $\varphi(x) = 0$.*
(2) *$\varphi(x+y) = \varphi(x) + \varphi(y) + \lambda(x,y)$ for $x, y \in H^m(M;\mathbb{Z}/2)$. Here $\lambda(x,y) = \langle x \cup y, [M]\rangle \in \mathbb{Z}/2$ is the intersection form ($[M] \in H_{2m}(M;\mathbb{Z}/2)$ is the mod 2 fundamental class of M).*

Proof of Proposition 5.2. By the discussion before Definition 5.4 it suffices to show $\varphi(x) \neq 0$, where $x \in H^m(M;\mathbb{Z}/2)$ is the cohomology class corresponding to the composition

$$M \xrightarrow{p_1 \wedge p_2} (M_1 \wedge M_2)^{(2m)} \xrightarrow{h} K_m.$$

Let $\bar{a}_1 \in H^{m_2}(M_1;\mathbb{Z}/2)$, $\bar{a}_2 \in H^{m_1}(M_2;\mathbb{Z}/2)$, and $\bar{b}_i \in H^m(M_i;\mathbb{Z}/2)$ be the mod 2 reduction of the integral cohomology classes a_i, b_i of section 3. We note that x is the product $\bar{a}_1\bar{a}_2$ (as in section 3 we identify here $H^*(M_i)$ with its image under p_i^* in $H^*(M)$), which is equal to $\bar{b}_1 + \bar{b}_2$ by Corollary 3.4.

The class $\bar{b}_i$ is a restriction of a cohomology class in $H^m(D\gamma_i;\mathbb{Z}/2)$. Hence $\lambda(\bar{b}_i, \bar{b}_i) = 0$ and $\varphi(\bar{b}_i) = 0$ by Proposition 5.5(1). Since the intersection form λ is unimodular we get $\lambda(\bar{b}_1, \bar{b}_2) = 1$ and hence we conclude using part (2) of Proposition 5.5

$$\varphi(\bar{a}_1\bar{a}_2) = \varphi(\bar{b}_1 + \bar{b}_2) = \varphi(\bar{b}_1) + \varphi(\bar{b}_2) + \lambda(b_1, b_2) = 1.$$

□

6. Desuspensions of $(M_1 \wedge M_2)^{(2m)}$

We recall that Proposition 2.7 claims that $(M_1 \wedge M_2)^{(2m)}$ desuspends l times for some l with $\phi(l) \geq \phi(m_1 - 1)$ except for $(m_1, m_2) = (2,2), (4,5)$. In this section we prove Proposition 2.7 assuming $(m_1, m_2) \neq (8,9), (8,11)$; in the next section we will show that these cases don't occur. Throughout the section we assume given a decomposition of S^{n+1} of the form (2.1) with $2 \leq m_1 < m_2$, $n = 2m$, $m = m_1 + m_2$.

We recall from section 3 that the assumption $2 \leq m_1$ implies that the manifolds M_1 and M_2 are simply connected (Lemma 3.1) and that $H^q(M_i;\mathbb{Z}) = \mathbb{Z}$ for $q = 0, m_{i+1}, m, m+m_{i+1}$ and $H^q(M_i;\mathbb{Z}) = 0$ for all other q (Lemma 3.3). It follows from obstruction theory that – up to homotopy equivalence – M_i allows a CW decomposition of the form

$$M_i = (S^{m_{i+1}} \cup_{f_i} e^m) \cup_{g_i} e^{m+m_{i+1}} \tag{6.1}$$

with attaching maps

$$f_i \in \pi_{m-1}(S^{m_{i+1}}) \qquad \text{and} \qquad g_i \in \pi_{m+m_{i+1}-1}(S^{m_{i+1}} \cup_{f_i} e^m).$$

For the proof of Proposition 2.7 we need a number of lemmas.

6.2 Lemma. *Let $M_i^{(m)} = S^{m_{i+1}} \cup_{f_i} e^m$ be the m-skeleton of M_i. If $M_1^{(m)} \wedge M_2^{(m)}$ desuspends l times with $l \leq m_1$, then so does $(M_1 \wedge M_2)^{(2m)}$.*

Proof. The CW-structures (6.1) on M_1 resp. M_2 induce a CW-structure on $M_1 \wedge M_2$ with three cells of dimension $2m$; namely the m_2-cell in M_1 times the top cell in M_2, the top cell in M_1 times the m_1-cell in M_2, and the m-cell in M_1 times the m-cell in M_2. We note that the first two of these cells are contained in the subcomplexes $S^{m_2} \wedge M_2 \subset M_1 \wedge M_2$ and $M_1 \wedge S^{m_1} \subset M_1 \wedge M_2$, respectively, which obviously desuspend m_1 times. Hence the attaching maps of these cells desuspend m_1 times.

The third $2m$-cell of $M_1 \wedge M_2$ is contained in the subcomplex $M_1^{(m)} \wedge M_2^{(m)} \subset M_1 \wedge M_2$. Hence its attaching desuspends l times due to our assumption that $M_1^{(m)} \wedge M_2^{(m)}$ desuspends l times. □

6.3 Lemma. *$\Sigma M_i^{(m)}$ is homotopy equivalent to the Thom space $T(\bar\gamma_{i+1})$, where $\bar\gamma_{i+1}$ is the restriction of the vector bundle γ_{i+1} to $S^{m_i} \subset M_{i+1}$.*

We remark that the image of the well-studied J-homomorphism consists precisely of the attaching maps of Thom spaces of vector bundles over spheres. In particular, Lemma 6.3 implies that Σf_1 is in the image of the J-homomorphism.

Proof. The decomposition (2.1) implies that the quotient space $S^{n+1} / D(\gamma_i)$ is homeomorphic to the Thom space $T(\gamma_{i+1})$. The associated Puppe sequence

$$D(\gamma_i) \to S^{n+1} \to T(\gamma_{i+1}) \to \Sigma D(\gamma_i) \to$$

shows that $\Sigma D(\gamma_i)$ and hence ΣM_i is homotopy equivalent to $T(\gamma_{i+1})$ in dimensions $\leq n$. This implies the lemma. □

6.4 Proof of Proposition 2.7 for $m_1 \not\equiv 0,1,2,4 \mod 8$. We note that $\widetilde{KO}(S^{m_1})$ is trivial for $m_1 \not\equiv 0,1,2,4 \mod 8$ by Bott periodicity. This implies that the vector bundle $\bar\gamma_2$ (the restriction of γ_2 to $S^{m_1} \subset M_2$) is trivial, since its fiber dimension is $m_2 + 1 > m_1$. Hence its Thom space $T(\bar\gamma_2)$ is $\Sigma^{m_2+1} S^{m_1}_+$. It follows from Lemma 6.3 that $\Sigma M_1^{(m)}$ desuspends (m_2+1) times. This implies that $M_1^{(m)}$ desuspends m_2 times, since the condition $m_1 < m_2$ guarantees that the attaching map $f_1 \in \pi_{m-1}(S^{m_2})$ lives in a stable group (i.e., $\Sigma^\infty \colon \pi_{m-1}(S^{m_2}) \to \pi^s_{m-1}(S^{m_2})$ is an isomorphism). □

In the cases $m_1 \equiv 0,1,2,4 \mod 8$ we need the following lemma.

6.5 Lemma. *Let γ be a vector bundle over S^{m_1}, $m_1 \geq 9$, of dimension m_1+k+1. Then the Thom space $T(\gamma)$ desuspends $k+4$ times.*

Proof. Barratt and Mahowald prove that the map $\pi_q(O(n)) \to \pi_q(O)$ is surjective for $n \geq 13$, $q < 2(n-1)$ [BM, Thm. 2]. Via the clutching construction which interprets elements of $\pi_q(O(n))$ as (isomorphism classes of) n-dimensional vector bundles over S^{q+1}, this implies that every vector bundle over S^{q+1} of dimension $> q+1$ is obtained from an n-dimensional bundle by adding a trivial vector bundle of the appropriate dimension. This implies the lemma for $m_1 \geq 16$.

For $m_1 = 9, 10, 12$ this follows from explicit calculations of Toda. We note that the Thom space $T(\gamma)$ is a CW complex with one cell in dimension m_1+k+1 and one cell in dimension $2m_1+k+1$, and hence its attaching map is an element of $\pi_{2m_1+k}(S^{m_1+k+1})$. By Toda's results the suspension homomorphism

$$\Sigma^{k+4}\colon \pi_{2m_1-4}(S^{m_1-3}) \to \pi_{2m_1+k}(S^{m_1+k+1})$$

is surjective for $m_1 = 9, 10, 12$ (cf. Theorems 7.1 resp. 7.2 resp. 7.4 of [To]), and hence $T(\gamma)$ desuspends $k+2$ times. This proves the lemma (for $m_1 \not\equiv 0, 1, 2, 4 \mod 8$ the bundle γ is trivial thanks to Bott periodicity). $\square$

6.6 Proof of Proposition 2.8 for $m_1 \equiv 0, 1, 2, 4 \mod 8$, $(m_1, m_2) \neq (8, 9), (8, 11)$.

First we assume $m_1 \geq 9$. In that case Lemma 6.5 implies that the Thom space $T(\bar{\gamma}_2)$ desuspends $m_2 - m_1 + 4$ times and hence $M_1^{(m)} = \Sigma^{m_2-m_1+3} Y$ for some space Y by Lemma 6.3. In particular, $(M_1 \wedge M_2)^{(2m)}$ desuspends at least 4 times by Lemma 6.2, and hence $m+1$ is a multiple of $2^{\phi(4)} = 8$ by Corollary 2.9. It follows that $m_2 \equiv -m_1 - 1 \equiv 7, 6, 5, 3 \mod 8$.

Now we write $M_1^{(m)} \wedge M_2^{(m)}$ in the form

$$M_1^{(m)} \wedge M_2^{(m)} = Y \wedge \Sigma^{m_2-m_1+3} M_2^{(m)} = Y \wedge T(\bar{\gamma}_1 \oplus \underline{\mathbb{R}}^{m_2-m_1+2}),$$

where $\underline{\mathbb{R}}^{m_2-m_1+2}$ denotes the trivial vector bundle over S^{m_2} of dimension $m_2 - m_1 + 2$. We note that $\bar{\gamma}_1 \oplus \underline{\mathbb{R}}^{m_2-m_1+2}$ is a vector bundle over S^{m_2} of dimension m_2+3. Since $m_2 \not\equiv 0, 1, 2, 4 \mod 8$, Bott periodicity implies that this vector bundle is trivial and hence its Thom space desuspends m_2+3 times.

Now we study the cases $m_1 = 2, 4, 8$. We note that by Freudenthal's Suspension Theorem the Suspension homomorphism $\Sigma^\infty\colon \pi_{2m_1-1}(S^{m_1}) \to \pi^s_{m_1}$ is surjective. In particular, the element $f_1 \in \pi_{m-1}(S^{m_2})$, hence $M_1^{(m)}$ and thus $(M_1 \wedge M_2)^{(2m)}$ desuspend at least $l = m_2 - m_1$ times. We note that

(1) For $m_1 = 2$ we have $\phi(m_2 - m_1) \geq \phi(m_1 - 1) = 1$ provided $m_2 \geq 3$.
(2) For $m_1 = 4$ we have $\phi(m_2 - m_1) \geq \phi(m_1 - 1) = 2$ provided $m_2 \geq 6$.
(3) For $m_1 = 8$ we have $\phi(m_2 - m_1) \geq \phi(m_1 - 1) = 3$ provided $m_2 \geq 12$.

This covers all cases with $m_1 < m_2$ except $(m_1, m_2) = (4, 5)$ (which is excluded by assumption) and $(m_1, m_2) = (8, 9), (8, 10), (8, 11)$. It's easy to see that $(8, 10)$ doesn't occur: by the above argument $(M_1 \wedge M_2)^{(2m)}$ desuspends $l = m_1 - m_2 = 2$ times and hence Corollary 2.9 leads to the contradiction that $m+1 = 19$ is a multiple of $2^{\phi(2)} = 4$. $\square$

7. Non-existence of multiplicities $(m_1, m_2) = (8,9), (8,11)$

In this section we show that there is no decomposition (2.1) of the sphere S^{n+1} with $n = 2m$, $m = m_1 + m_2$, and $(m_1, m_2) = (8,9)$ or $(m_1, m_2) = (8,11)$. We keep the notation from the previous section; in particular, f_i and g_i are the attaching maps of the CW-decomposition (6.1) of M_i.

To rule out the case $(8,9)$, we need the following two lemmas.

7.1 Lemma. *The composition*

$$S^{m+m_1-1} \xrightarrow{\Sigma^{m_1} f_1} S^m \xrightarrow{\Sigma f_2} S^{m_1+1}$$

is homotopic to the constant map.

Proof. Consider the Puppe sequence associated to the map f_2:

$$S^{m-1} \xrightarrow{f_2} S^{m_1} \to S^{m_1} \cup_{f_2} e^m \xrightarrow{p} \Sigma S^{m-1} \xrightarrow{\Sigma f_2} \Sigma S^{m_1} \to .$$

Any two consecutive maps in this sequence have trivial composition; in particular, the composition $\Sigma f_2 \circ p \circ g_2$ is trivial.

We note that the map $p \circ g_2 \in \pi_{m+m_1-1}(S^m)$ describes how the top cell of M_2 is attached to the m-cell. Since M_2 is Spanier-Whitehead dual to M_1, this map is stably equivalent to f_1. This implies $p \circ g_2 = \Sigma^{m_1} f_1$, since the homotopy group $\pi_{m+m_1-1}(S^m)$ is stable (i.e., the suspension homomorphism from this group to $\pi^s_{m_1-1}$ is an isomorphism). This proves the lemma. $\square$

7.2 Lemma.

(1) *The index two subgroup of $\pi^s_7 \cong \mathbb{Z}/24$ is in the image of $\Sigma^\infty\colon \pi_{14}(S^7) \to \pi^s_7$.*

(2) *Let $m_1 = 8$, $m_2 = 9$. If $f_1 \in \pi_{16}(S^9) \cong \pi^s_7$ is an odd multiple of the generator, then $\Sigma M_2^{(m)}$ desuspends at least twice.*

Proof. The first statement follows from the EHP-sequence

$$\to \pi_{14}(S^7) \xrightarrow{\Sigma^\infty} \pi^s_{14}(S^7) \xrightarrow{H} \pi_{14}(D_2(S^7)) \to,$$

since $\pi_{14}(D_2(S^7)) = \pi_{14}(\Sigma^7 \mathbb{RP}_7) = H_{14}(\Sigma^7 \mathbb{RP}_7; \mathbb{Z}) = \mathbb{Z}/2$.

To prove the second part, we need to show that $\Sigma f_2 \in \pi_{17}(S^9)$ desuspends twice. Consider the following homotopy commutative diagram.

$$\begin{array}{ccccccc}
S^{22} & \xrightarrow{\Sigma^6 f_1} & S^{15} & \xrightarrow{f_2'} & F_2C_2(S^7) & \xrightarrow{p} & D_{2,2}(S^7) \approx \Sigma^7 \mathbb{RP}^8_7 \\
{\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha_2} & & \\
\Omega^2\Sigma^2 S^{22} & \xrightarrow{\Omega^2\Sigma^8 f_1} & \Omega^2\Sigma^2 S^{15} & \xrightarrow{\Omega^2\Sigma f_2} & \Omega^2\Sigma^2 S^7 & &
\end{array}$$

Here f_2' is a factorization up to homotopy of $\Omega^2\Sigma f_2 \circ i$ through the map α_2 which is a 20-equivalence by Theorem 4.4. Suppose that Σf_2 does not desuspend twice or, equivalently, that f_2' does not factor through the inclusion $i\colon S^7 \to F_2C_2(S^7)$. This implies by Corollary 4.6 that the composition $p \circ f_2'$ is non-trivial. We note that $\mathbb{RP}^8_7$ is the Moore space $S^7 \cup_2 e^8$ obtained by attaching to S^7 an 8-cell whose attaching map $S^7 \to S^7$ has degree 2. This leads to the exact sequence

$$\to \pi^s_q(S^7) \xrightarrow{2} \pi^s_q(S^7) \xrightarrow{j_*} \pi^s_q(\mathbb{RP}^8_7) \xrightarrow{\partial} \pi^s_{q-1}(S^7) \xrightarrow{2} \pi^s_{q-1}(S^7) \to, .$$

where $j: S^7 \to \mathbb{RP}^8_7$ is the inclusion map. Using $\pi^s_7(S^7) \cong \mathbb{Z}$ and $\pi^s_8(S^7) \cong \mathbb{Z}/2$ generated by η (the suspension of the Hopf map), the sequence shows that $\pi^s_8(\mathbb{RP}^8_7)$ is isomorphic to $\mathbb{Z}/2$ generated by $j_*\eta$.

Assuming that $\Sigma^\infty f_1 = k\sigma \in \pi^s_7$, where k is odd and σ is the generator, then the composition

$$p \circ f_2' \circ \Sigma^6 f_1 \in \pi_{22}(\Sigma^7 \mathbb{RP}^8_7) = \pi^s_{15}(\mathbb{RP}^8_7)$$

is equal to $kj_*(\eta\,\sigma)$. This is a non-trivial element since $\eta\sigma \in \pi^s_8$ is a non-trivial element not divisible by 2 (cf. [To, Thm. 7.1 and equation 7.4]).

On the other hand, the projection map $p: F_2C_2(S^7) \to D_{2,2}(S^7)$ factors *stably* in the form $F_2C_2(S^7) \xrightarrow{\alpha_2} \Omega^2\Sigma^2 S^7 \to D_{2,2}(S^7)$ by a result due to Snaith [Sn].

This factorization, and the fact that the composition $\Sigma f_2 \circ \Sigma^8 f_1$ is trivial (by Lemma 7.1) show that $p \circ f_2' \circ \Sigma^6 f_1$ is stably trivial, which is the desired contradiction. □

7.3 Non-existence of $(m_1, m_2) = (8, 9)$. If $f_1 \in \pi_{16}(S^9) \cong \mathbb{Z}/24$ is an even multiple of the generator, then part (1) of Lemma 7.2 implies that f_1 desuspends twice; hence so does $M_1^{(m)}$. If f_1 is an odd multiple of the generator, then f_1 desuspends once by Freudenthal's Suspension Theorem; hence $M_1^{(m)} = \Sigma X$, and so $M_1^{(m)} \wedge M_2^{(m)} = X \wedge \Sigma M_2^{(m)}$ desuspends twice by part (2) of Lemma 7.2.

In either case, $M_1^{(m)} \wedge M_2^{(m)}$ desuspends at least twice, and hence so does $(M_1 \wedge M_2)^{(2m)}$ by Lemma 6.2. Then Corollary 2.9 leads to the contradictory statement that $m + 1 = 18$ is a multiple of $2^{\phi(2)} = 4$. □

Now we turn to the case $(m_1, m_2) = (8, 11)$. The first part of Lemma 7.2 shows that if $f_1 \in \pi_{18}(S^{11}) \cong \pi^s_7 \cong \mathbb{Z}/24$ is an *even* multiple of the generator, then it is in the image of

$$\Sigma^4: \pi_{14}(S^7) \to \pi_{18}(S^{11}).$$

Hence $M_1^{(m)}$ desuspends 4 times and so does $(M_1 \wedge M_2)^{(2m)}$ by Lemma 6.2. This leads by Corollary 2.9 to the contradictory statement that $m+1 = 20$ is a multiple of $2^{\phi(4)} = 8$.

Hence from now on we will assume that f_1 is an *odd* multiple of the generator. We recall that the mod 2 Hopf invariant is the homomorphism $H: \pi^s_q \to \mathbb{Z}/2$ which sends a stable map $f: S^{k+q} \to S^k$ to the non-trivial element if and only if the Steenrod operation Sq^{q+1} acts non-trivially on the cohomology of the mapping cone of f. E.g., Sq^8 acts non-trivially on the cohomology of the Cayley plane, which shows that its attaching map has non-trivial mod 2 Hopf invariant. In particular, $H: \pi^s_7 \to \mathbb{Z}/2$ is non-trivial. This implies $H(f_1) \neq 0$; in other words,

$$Sq^8: H^{11}(X) \to H^{19}(X) \tag{7.4}$$

is non-trivial, where X is the mapping cone of f_1; i.e., $X = S^{11} \cup_{f_1} e^{19} = M_1^{(19)}$. We recall from (6.1) that $M_1 = X \cup_{g_1} e^{30}$.

7.5 Lemma. *The attaching map g_1 is stably trivial, i.e., it maps to zero under the homomorphism*

$$\Sigma^\infty\colon \pi_{29}(X) \to \pi_{29}^s(X).$$

Proof. We note that the reduced cohomology of M_1 is trivial except in dimensions 11, 19, and 30. This shows that the E_2-term

$$E_2^{p,q}(M_1) = \widetilde{H}^p(M_1; KO^q)$$

of the Atiyah-Hirzebruch spectral sequence converging to $\widetilde{KO}^{p+q}(M_1)$ is trivial for $p+q=0$, since KO^q is trivial for $-q \not\equiv 0,1,2,4 \mod 8$. Hence any vector bundle over M_1 of large enough dimension is trivial. In particular, the normal bundle of M_1 in a sufficiently high dimensional euclidean space is trivial, and it is well known that this implies that the attaching map of the top cell is stably trivial.

For the convenience of the reader we recall the argument: a closed n-manifold M is Spanier-Whitehead dual to the Thom space $T(\gamma)$, where γ is the normal bundle of some embedding of M into euclidean space. The projection map onto the top cell of M is dual to the inclusion of the bottom cell of $T(\gamma)$. If γ has dimension k, the bottom cell of $T(\gamma)$ has dimension k, and the inclusion of the bottom cell $i\colon S^k \to T(\gamma)$ is given geometrically by thinking of S^k as the Thom space of γ restricted to the base point. A trivialization of γ can be used to construct a map $s\colon T(\gamma) \to S^k$ which is a left inverse to i (i.e., the composition $s \circ i$ is the identity). Dually, there is a stable map $S^n \to M$ which is a right inverse to the projection map onto the top cell. This implies that the attaching map of the top cell is stably trivial. $\square$

We recall from Corollary 4.6 (applied for $k = \infty$; cf. Remark 4.10) that the cofibration sequence

$$X \xrightarrow{i} F_2C(X) \xrightarrow{p} D_2(X) \tag{7.6}$$

is a homotopy fibration in the metastable range $q < 3l-2 = 31$, since the bottom cell of X is in dimension $l = 11$. We also recall that $i_*\colon \pi_q(X) \to \pi_q(F_2C(X)) \cong \pi_q^s(X)$ is the stabilization map Σ^∞ for $q < 3l-1 = 32$ and that the homotopy exact sequence of the fibration (7.6) is the EHP sequence. Hence Lemma 7.5 implies:

7.7 Corollary. *There is an element $\tilde{g}_1 \in \pi_{30}(D_2(X))$ which maps to g_1 under the boundary map $\partial\colon \pi_{30}(D_2(X)) \to \pi_{29}(X)$ of the fibration (7.6).*

7.8 Lemma. *The map $\tilde{g}_1$ induces a non-trivial homomorphism on reduced cohomology with $\mathbb{Z}/2$-coefficients.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "i"'] \\
X \cup_{g_1} e^{30} \arrow[r, "f"] \arrow[d, "p"'] & F_2C(X) \arrow[d, "p"'] \\
S^{30} \arrow[r, "\tilde{g}_1"] & D_2(X)
\end{tikzcd}$$

Here the map f is constructed on e^{30} as a lift of $D^{30} \xrightarrow{pr} D^{30}/S^{29} = S^{30} \xrightarrow{g_1} D_2(X)$; the restriction of this lift to $S^{29} \subset D^{30}$ is $g_1 = \partial \tilde{g}_1$ (by definition of the boundary map), and hence the lift and the map $i: X \to F_2C(X)$ fit together to define the map f.

Let $x_i \in H^i(M_1) \cong H^i(X)$, $i = 11, 19$, be non-trivial elements (all cohomology groups here have $\mathbb{Z}/2$ as coefficients, and since we work in the category of pointed spaces, we write $H^*(X)$ for the reduced cohomology of a pointed space X). The exact sequence

$$H^i(D_2(X)) \xrightarrow{p^*} H^*(F_2C(X)) \xrightarrow{i^*} H^*(X) \xrightarrow{\partial} H^{i_1}(D_2(X)) \to .$$

of the cofibration (7.6) implies that there are elements $y_i \in H^i(F_2C(X))$ with $i^*(y_i) = x_i$. Again the exact sequence shows that there is an element $y_{30} \in H^{30}(D_2(X))$ with $p^*(y_{30}) = y_{11}y_{19}$. Poincaré duality for $M_1 = X \cup_{g_1} e^{30}$ shows that the cup product $x_{11}x_{19} \in H^{30}(M_1)$ is non-trivial. This implies by the commutative diagram above that $\tilde{g}_1^*(y_{30})$ is non-trivial.

7.9 Lemma. *For every map $g: S^{30} \to D_2(X)$ the induced homomorphism on reduced cohomology with $\mathbb{Z}/2$-coefficients is trivial.*

Clearly, the statements of Lemma 7.8 and 7.9 contradict each other. This shows that there can be no decomposition (2.1) of the sphere with $(m_1, m_2) = (8, 11)$.

7.10 Remark. It might be instructive to discuss why the above arguments don't work to exclude the possibilities $(m_1, m_2) = (2, 2), (4, 5)$ (we recall from the introduction that there are isoparametric hypersurfaces with these multiplicities). In these cases it is *not* true that the normal bundle of M_1 is stably trivial, and so Lemma 7.5 does not hold; if it did, we could prove non-existence of $(2, 2)$ and $(4, 5)$ the same way as we do for $(8, 11)$.

Proof. We need to describe the $\mathbb{Z}/2$-cohomology of the space $D_2(X)$ (cf. [Mil, p. 40], [Jo, p. 256]). For the moment, let X be a CW complex, and $x, y \in H^*(X)$ be elements of degree p and q, respectively. Then there are elements in the cohomology of $D_2(X)$ denoted

$$e^t \otimes x \otimes x \in H^{t+2p}(D_2(X)),\ t \geq 0 \qquad \text{and} \qquad [x, y] \in H^{p+q}(D_2(X)).$$

Moreover, if $\{x_1, x_2, \dots\}$ is a basis of homogeneous elements of $H^*(X)$, then a basis for $H^*(D_2(X))$ is provided by the following elements:

$$e^t \otimes x_i \otimes x_i,\ t \geq 0 \qquad [x_i, x_j],\ i \neq j.$$

Furthermore, the action of the Steenrod algebra on $H^*(D_2(X))$ is determined by the following formulae for $x \in H^n(X)$, $y \in H^m(X)$, $k \geq 1$:

$$Sq^t(e^k \otimes x \otimes x) = \sum_{0 \leq i < t/2} \binom{k+n-i}{t-2i} e^{k+t-2i} \otimes Sq^i x \otimes Sq^i x$$

$$Sq^t(e^0 \otimes x \otimes x) = \sum_{0 \leq i < t/2} \binom{n-i}{t-2i} e^{t-2i} \otimes Sq^i x \otimes Sq^i x + \sum_{0 \leq i < t/2} [Sq^{t-i}x, Sq^i x]$$

$$Sq^t[x, y] = \sum_{0 \leq i \leq t} [Sq^{t-i}x, Sq^i y].$$

Now we specialize to $X = M_1^{(m)} = S^{11} \cup_{f_1} e^{19}$. A basis of $H^*(X)$ is $\{x_{11}, x_{19}\}$, where x_{11}, x_{19} are elements of degree 11 and 19, respectively. Hence $H^{30}(D_2(X))$ is generated by the elements $[x_{11}, x_{19}]$ and $e^8 \otimes x_{11} \otimes x_{11}$. Since $Sq^8(x_{11}) = x_{19}$ (cf. (7.4)), the above formulae imply

$$\begin{aligned} Sq^8(e^0 \otimes x_{11} \otimes x_{11}) &= e^8 \otimes x_{11} \otimes x_{11} + [x_{19}, x_{11}] \\ Sq^4(e^4 \otimes x_{11} \otimes x_{11}) &= e^8 \otimes x_{11} \otimes x_{11} \end{aligned}$$

This shows that every element of $H^{30}(D_2(X))$ can be written as a linear combination of the elements $Sq^8(e^0 \otimes x_{11} \otimes x_{11})$ and $Sq^4(e^4 \otimes x_{11} \otimes x_{11})$. We note that the homomorphism induced by any map $g\colon S^{30} \to D_2(X)$ sends these elements to zero; e.g.,

$$g^*(Sq^8(e^0 \otimes x_{11} \otimes x_{11}) = Sq^8(g^*(e^0 \otimes x_{11} \otimes x_{11})) = 0$$

since $g^*(e^0 \otimes x_{11} \otimes x_{11}) \in H^{22}(S^{30}) = 0$. This proves Lemma 7.9. $\square$

References

[Ad] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[Ab] U. Abresch, *Isoparametric hypersurfaces with four or six distinct principal curvatures*, Math. Ann. **264** (1983), 283–302.

[BM] M. Barratt and M. Mahowald, *The metastable homotopy of $O(n)$*, Bull. AMS **70** (1964), 758–760.

[Br] E. H. Brown, *Generalisations of the Kervaire invariant*, Ann. of Math. **95** (1972), 268–383.

[Ca] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espace spheriques*, Math. Z. **45** (1939), 335 –337.

[CR] T. E. Cecil, P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Math., vol. 107, Pitman, London, 1985.

[Fa] F. Fang, *Multiplicities of principal curvatures of isoparametric hypersurfaces*, preprint (1996).

[FKM] D. Ferus, H. Karcher, H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502.

[GH] K. Grove and S. Halperin, *Dupin hypersurfaces, group actions and the double mapping cylinder*, J. Diff. Geom. **26** (1987), 429–459.

[HL] W. Y. Hsiang, H. B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Diff. Geometry **5** (1971), 1–38.

[Hu] D. Husemoller, *Fibre bundles*, Lecture Notes in Mathematics, vol. 368, Springer Verlag, 1974.

[Jo] J. D. S. Jones, *The Kervaire invariant of extended power manifolds*, Topology **17** (1974), 249–266.

[Ma] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, vol. 271, Springer Verlag, 1972.

[Mil] R. J. Milgram, *Unstable homotopy from the stable point of view*, Lecture Notes in Mathematics, vol. 368, Springer Verlag, 1974.

[Miy] R. Miyaoka, *Compact Dupin hypersurfaces with three principal curvatures*, Math. Z. **187** (1984), 433–452.

[MO] R. Miyaoka, T. Ozawa, *Construction of taut embeddings and Cecil-Ryan conjecture*, Collection: Geometry of manifolds (Matsumoto, 1988), Perspect. Math. 8, Academic Press, Boston, MA, 1989, pp. 181–189.

[Mü1] H.-F. Münzner, *Isoparametrische Hyperflächen in Sphären (geometrischer Teil)*, Math. Ann. **251** (1980), 57–71.

[Mü2] ______, *Über die Zerlegung der Sphäre in Ballbündel. II.*, Math. Ann. **256** (1981), 215–232.

[OT] H. Ozeki, M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres. I.*, Tohoku Math. J. **27** (1975), 515–559; *II* **28** (1976), 7–55.

[PT] U. Pinkall, G. Thorbergsson, *Deformations of Dupin hypersurfaces*, Proc. of AMS **107** (1989), 1037–1043.

[Sn] V. P. Snaith, *A stable decomposition for* $\Omega^n \Sigma^n X$, J. London Math. Soc. **7** (1974), 577–583.

[Th] G. Thorbergsson, *Dupin hypersurfaces*, Bull. London Math. Soc. **15** (1983), 493–498.

[TT] R. Takagi, T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry (S. Kobayashi, M. Obata and V. Takahashi, eds.), in honor of K. Yano, Kinokuniya, Tokio, 1972, pp. 469–481.

[To] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, vol. 49, Princeton University Press, Princeton, New Jersey, 1962.

[Wh] G. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer, New York, Heidelberg, Berlin.

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA and Max–Planck–Institut füer Mathematik, Gottfried–Claren-Str. 26, 53225 Bonn, Germany