GL_4-EQUIVARIANT DESINGULARIZATION OF THE UNIVERSAL CURVE OVER THE ELLINGSRUD–PIENE–STRØMME COMPACTIFICATION OF THE SPACE OF TWISTED RATIONAL CUBIC CURVES IN $\mathbb{P}^3$.

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1. Introduction

Let W be 4-dimensional complex vector space. Let $\mathring{\mathfrak{W}}$ be the space of smooth rational twisted cubic curves in the projective space $\mathbb{P}(W)$ and denote by $\mathfrak{W}$ its Ellingsrud–Piene–Strømme compactification via nets of determinantal quadrics (see [2]). Let $\mathring{\mathcal{I}} \subset \mathbb{P}(W) \times \mathring{\mathfrak{W}}$ be the *universal curve*, i.e. the point-curve incidence subspace $\mathring{\mathcal{I}} = \{(x, C) \mid x \in C\}$ and let $\mathcal{I}$ be its closure in the space $\mathbb{P}(W) \times \mathfrak{W}$. It is easy to see that $\mathcal{I}$ is singular. This paper introduces a certain desingularization $\tilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{I}$ of the space $\mathcal{I}$. Moreover, some exact sequences, relating the pull-backs of the universal bundles over $\mathfrak{W}$ and $\mathbb{P}(W)$ to $\widetilde{\mathcal{M}}$ are found.

Now we describe briefly our construction. Both $\mathbb{P}(W)$ and $\mathfrak{W}$ are realized as moduli spaces of stable representations of the quiver $Q_{\text{small}} = \ \bullet \xrightarrow{W} \bullet$ of dimensions $(1,3)$ and $(2,3)$ correspondingly. We consider the quiver $Q = \ \bullet \xrightarrow{\mathbb{C}} \bullet \xrightarrow{W} \bullet$ and the moduli space of its stable $(1,2,3)$-dimensional representations. The latter moduli spaces, as was pointed by A. King in [3], depend on the choice of the character of the group $\mathbf{GL}(1) \times \mathbf{GL}(2) \times \mathbf{GL}(3)$. When the character varies these moduli spaces transform. This phenomenon was investigated by M. Thaddeus in [7].

In particular, for certain characters χ_0 and χ_1 the corresponding moduli spaces are nothing but $\mathbb{P}(W)$ and $\mathfrak{W}$. Now we consider the family of fractional characters $\chi_t = (1-t)\chi_0 + t\chi_1$, $t \in \mathbb{Q}$ and the corresponding moduli spaces $\mathcal{M}_t$ of χ_t-semistable representations of Q. We describe the transformations of $\mathcal{M}_t$ that take place while t varies from $t = 0$ to $t = 1$ and define the space $\widetilde{\mathcal{M}}$ as the fiber product of intermediate moduli spaces (see Corollary 4.6).

The space $\widetilde{\mathcal{M}}$ may be applied to investigation of the derived category of coherent sheaves on the space $\mathfrak{W}$ (see the discussion after the Corollary 5.7).

Also it can be applied to description of the derived category of coherent sheaves on the Fano threefolds V_{22}. The first step in this direction was made in [4] where an exceptional collection of vector bundles on these threefolds was constructed. But it appears to be much more difficult problem to prove that this collection is full, i.e. generates the derived category. However it is possible to do it using the space $\widetilde{\mathcal{M}}$. This will be done in the forthcoming paper.

The paper is organized as follows. In the Section 2 we recall some basic definitions and facts about representations of quivers and their moduli spaces. In Section 3 we describe the projective space $\mathbb{P}(W)$ and the Ellingsrud–Piene–Strømme compactification $\mathfrak{W}$ of the space of twisted cubics in $\mathbb{P}(W)$ as a certain moduli spaces of $(1,2,3)$-dimensional representations of the quiver Q and introduce the family $\mathcal{M}_t$ of moduli spaces such that $\mathcal{M}_0 = \mathbb{P}(W)$ and $\mathcal{M}_1 = \mathfrak{W}$. We also describe the set of critical values t_i in which

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transformations of the moduli spaces $\mathcal{M}_t$ occur, namely we show that the set of critical values consists of $t_0 = 0$, $t_1 = 2/3$ and $t_2 = 1$. In Section 4 we describe the transformations that occur at the points t_0 and t_2.Namely choosing $1/3$ and $5/6$ as typical representatives in the intervals (t_0, t_1) and (t_1, t_2) we show that $\mathcal{M}_{1/3}$ is a $\mathbb{P}^{10}$-fibration over $\mathbb{P}(W) = \mathcal{M}_0$ and $\mathcal{M}_{5/6}$ is a $\mathbb{P}^1$-fibration over $\mathfrak{W} = \mathcal{M}_1$ (see Propositions 4.1 and 4.2). Afterwards we apply the Thaddeus theory to the transformation which occur at the point $t_1 = 2/3$ and get its coarse description. We define here the space $\widetilde{\mathcal{M}}$ as the fiber product $\mathcal{M}_{1/3} \times_{\mathcal{M}_{2/3}} \mathcal{M}_{5/6}$ and construct the desingularization map $\tilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{I}$. In Section 5 we describe the last transformation in details. After that we deal with universal families of representations of Q over $\mathcal{M}_{1/3}$ and $\mathcal{M}_{5/6}$. We construct an exact sequence including the pull-backs to $\widetilde{\mathcal{M}}$ of these universal families. This sequence is important for applications to Fano threefolds V_{22}.

The main results of the paper are Theorem 4.7 and Propositions 5.1, 5.2 describing the space $\widetilde{\mathcal{M}}$.

Notations.

1. Spaces:

W — fixed 4-dimensional vector space;

$\mathring{\mathfrak{W}}$ — space of smooth rational twisted cubic curves in $\mathbb{P}(W)$;

$\mathfrak{W}$ — compactification of $\mathring{\mathfrak{W}}$ via nets of determinantal quadrics (see [2]);

$\mathring{\mathcal{I}} \subset \mathbb{P}(W) \times \mathring{\mathfrak{W}}$ — the *universal curve*;

$\mathcal{I}$ — closure of $\mathring{\mathcal{I}}$ in $\mathbb{P}(W) \times \mathfrak{W}$;

$\mathfrak{R} = \mathfrak{R}(\alpha) = (A_1^* \otimes A_2) \oplus (W^* \otimes A_2^* \otimes A_3)$ — vector space parameterizing α-dimensional representations of Q, see 2.1;

$\mathfrak{R}_t^{ss} \subset \mathfrak{R}$ — subset of χ_t-semistable representations;

$\mathfrak{R}^- = \mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{5/6}^{ss}$, $\mathfrak{R}^+ = \mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{1/3}^{ss}$, $\mathfrak{R}^0 = \mathfrak{R}^+ \cap \mathfrak{R}^-$;

$\mathcal{M}_t = \mathcal{M}_Q((1,2,3), \chi_t) = \mathfrak{R}_t^{ss}//\mathbf{GL}(\alpha)$ — moduli space of α-dimensional χ_t-semistable representations of Q;

$\widetilde{\mathcal{M}} = \mathcal{M}_{1/3} \times_{\mathcal{M}_{2/3}} \mathcal{M}_{5/6}$ — see Corollary 4.6;

$M^- = \mathfrak{R}^-//\mathbf{GL}(\alpha) = \mathbb{P}_{M^0}(S^-) \subset \mathcal{M}_{1/3} = \mathbb{P}(\mathcal{R})$, $M^+ = \mathfrak{R}^+//\mathbf{GL}(\alpha) = \mathbb{P}_{M^0}(S^+) \subset \mathcal{M}_{5/6} = \mathbb{P}(\mathcal{E})$;

$M^0 = \mathfrak{R}^0//\mathbf{GL}(\alpha) = \mathbf{Gr}(2, W) \times \mathbb{P}(W^*) \subset \mathcal{M}_{2/3}$;

$\widetilde{M} = M^- \times_{M^0} M^+ \subset \widetilde{\mathcal{M}}$.

2. Sheaves:

$\mathcal{E}$, $\mathcal{K}$ — universal 2-dimensional and 3-dimensional vector bundles on $\mathfrak{W}$;

$\mathcal{R}$ — 11-dimensional exceptional vector bundle on $\mathbb{P}(W)$, see (9);

$\mathcal{L}_{\mathcal{E}}$, $\mathcal{L}_{\mathcal{R}}$ — tautological linear subbundles on $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{R})$;

U — tautological 2-dimensional subbundle on $\mathbf{Gr}(2, W)$ and its pull-back on $M^0 = \mathbf{Gr}(2, W) \times \mathbb{P}(W^*)$;

η — tautological linear subbundle on $\mathbb{P}(W^*)$ and its pull-back on M^0;

$\mathcal{N}^w$ — weight component of $\mathcal{N}_{G \cdot A/\mathfrak{R}}$ with respect to action of $\hat{G}_A$;

L — linear bundle on $\mathfrak{R}^0$, defined before Corollary 4.6;

$S^{\pm}$ — vector bundles on M^0, defined before Corollary 4.6;

$\mathcal{L}^{\pm}$ — tautological linear subbundles on $M^{\pm} = \mathbb{P}_{M^0}(S^{\pm})$;

$\mathcal{L}' = {p^-}^*(\Lambda^2 U \otimes \eta) \otimes \mathcal{L}^-$ — linear subbundle in ${p^-}^* U$ on M^-;

E^- — 2-dimensional exceptional vector bundle on $\mathbb{P}(\mathcal{R})$, see Remark 7;

E^+, K^+ — 2 and 3-dimensional vector bundles on M^+, defined before Proposition 5.3;

$\mathcal{O}_{\mathcal{X}}$, $\mathcal{T}_{\mathcal{X}}$, $\mathcal{N}_{\mathcal{Y}/\mathcal{X}}$ — structure sheaf of $\mathcal{X}$, tangent sheaf of $\mathcal{X}$ and normal sheaf of $\mathcal{Y} \subset \mathcal{X}$.

3. Maps:

$\pi_0 : \mathbb{P}(\mathcal{R}) \to \mathbb{P}(W)$, $\pi_1 : \mathbb{P}(\mathcal{E}) \to \mathfrak{W}$ — natural projections;

$\pi_0' : \mathbb{P}(W) \times \mathring{\mathfrak{W}} \to \mathbb{P}(W)$, $\pi_1' : \mathbb{P}(W) \times \mathring{\mathfrak{W}} \to \mathring{\mathfrak{W}}$ — natural projections;

$\hat{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}(\mathcal{R})$, $\hat{\pi}_1 : \widetilde{\mathcal{M}} \to \mathbb{P}(\mathcal{E})$ — blow-ups along M^- and M^+ respectively;
$\tilde{\pi}_0 = \pi_0 \cdot \hat{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}(W)$, $\tilde{\pi}_1 = \pi_1 \cdot \hat{\pi}_1 : \widetilde{\mathcal{M}} \to \mathfrak{W}$ — projections;
$\tilde{\pi} = \tilde{\pi}_0 \times \tilde{\pi}_1 : \widetilde{\mathcal{M}} \to \mathcal{I} \subset \mathbb{P}(W) \times \mathfrak{W}$;
$p^- : \mathbb{P}(\mathcal{R}) = \mathcal{M}_{1/3} \to \mathcal{M}_{2/3}$, $p^+ : \mathbb{P}(\mathcal{E}) = \mathcal{M}_{5/6} \to \mathcal{M}_{2/3}$, see 3.6;
$p = p^- \cdot \hat{\pi}_0 = p^+ \cdot \hat{\pi}_1 : \widetilde{M} \to M^0$.

4. Other:
Q, Q_{small} — quivers, see 2.1, 2.4;
A^0, A^1 — canonical subrepresentations in A, see proofs of Propositions 3.1, 3.2;
$\mathbf{GL}(\alpha) = \mathbf{GL}(\alpha_1) \times \mathbf{GL}(\alpha_2) \times \mathbf{GL}(\alpha_3)$, see 2.1;
$\Delta \subset \mathbf{GL}(\alpha)$, $G = \mathbf{GL}(\alpha)/\Delta$ — diagonal subgroup and quotient group;
$\widetilde{G}_A$ and G_A — stabilizers of $A \in \mathfrak{R}^0$ in the groups $\mathbf{GL}(\alpha)$ and G;
$\widehat{G}_A$ — subgroup in $\mathbf{GL}(\alpha)$, see Lemma 4.3;
$\mu_\chi(A)$ — slope of A with respect to χ, see 2.2;
χ_t — family of characters of $\mathbf{GL}(\alpha)$, see (5);
v — generator of $\mathbf{Hom}(G_A, \mathbb{C}^*)$, see Remark 10;
$\mathcal{A}^-$, $\mathcal{A}^+$, — universal families on $\mathcal{M}_{1/3}$, $\mathcal{M}_{5/6}$, see (10) and (11);
$\mathcal{A}'$, $\mathcal{A}''$ — universal families on $\mathcal{M}((1,1,2), \chi_{2/3})$, $\mathcal{M}((0,1,1), \chi_{2/3})$, see Proposition 5.3.

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2. Representations of the quiver and the moduli spaces.

2.1. Let W be 4-dimensional vector space. Let Q be the following quiver

$$\bullet \xrightarrow{\quad \mathbb{C} \quad} \bullet \xrightarrow{\quad \mathrm{W} \quad} \bullet$$

where $\mathbb{C}$ is considered as one-dimensional vector space.

Recall basic definitions of the theory of representations of quivers. An $\alpha = (\alpha_1, \alpha_2, \alpha_3)$-dimensional representation A of the quiver Q is a triple (A_1, A_2, A_3) of vector spaces such that $\mathbf{dim}\, A_i = \alpha_i$, together with homomorphisms

$$a_{12} : \mathbb{C} \to \mathbf{Hom}(A_1, A_2), \quad a_{23} : W \to \mathbf{Hom}(A_2, A_3), \quad a_{13} : W \to \mathbf{Hom}(A_1, A_3)$$

such that

$$a_{13}(w) = a_{23}(w) \cdot a_{12}(1), \qquad (w \in W)$$

Fixing vector spaces A_i we see, that the representation is a tensor A in the vector space $\mathfrak{R}(\alpha) = (A_1^* \otimes A_2) \oplus (W^* \otimes A_2^* \otimes A_3)$.

A homomorphism f from representation A to representation A' is a triple (f_1, f_2, f_3) of linear maps $f_i : A_i \to A'_i$ such that for any $z \in \mathbb{C}$ and $w \in W$ the following diagram commutes:

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_{12}(z)} & A_2 & \xrightarrow{a_{23}(w)} & A_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ A'_1 & \xrightarrow{a'_{12}(z)} & A'_2 & \xrightarrow{a'_{23}(w)} & A'_3 \end{array}$$

A subrepresentation in A is a triple of subspaces $A'_i \subset A_i$ which is closed under the maps $a_{12}(z)$ and $a_{23}(w)$ for all $z \in \mathbb{C}$, $w \in W$.

2.2. Fix some α. Let $\mathbf{GL}(\alpha) = \mathbf{GL}(\alpha_1, \alpha_2, \alpha_3) = \mathbf{GL}(\alpha_1) \times \mathbf{GL}(\alpha_2) \times \mathbf{GL}(\alpha_3)$ and $\mathbb{C}^* = \Delta \subset \mathbf{GL}(\alpha)$ is the diagonal subgroup. The group $\mathbf{GL}(\alpha)$ naturally acts on the space $\mathfrak{R}(\alpha)$ and Δ is the kernel of the action. The classes of isomorphism of α-dimensional representations of Q is nothing but the orbits of $\mathbf{GL}(\alpha)$ in $\mathfrak{R}(\alpha)$.

Let us denote by $\chi = (c_1, c_2, c_3)$ a character of $\mathbf{GL}(\alpha)$ defined as follows

$$\chi(g_1, g_2, g_3) = (\mathbf{det}\, g_1)^{c_1} \cdot (\mathbf{det}\, g_2)^{c_2} \cdot (\mathbf{det}\, g_3)^{c_3}.$$

Note that for $\lambda \in \Delta$ we have $\chi(\lambda) = \lambda^{\langle\chi,\alpha\rangle} = \lambda^{c_1\alpha_1+c_2\alpha_2+c_3\alpha_3}$.

The number $\mu_\chi(A) = \langle\chi, \alpha\rangle$ is called *the slope* of A with respect to χ. If $\chi_{|\Delta} = \mathbf{id}$ then the slope of A is zero. The representation A is called *semistable* (*stable*) with respect to χ if its slope is zero and for all proper subrepresentations $A' \subset A$ we have $\mu_\chi(A') \geq 0$ (> 0 respectively).

The χ-semistable representations A and A' are called S-equivalent if they have the same composition factors in the category of χ-semistable representations of Q (note that simple objects in this category are but χ-stable representations).

Theorem 2.3 (A. King, [3])**.** *There exists a coarse moduli space $\mathcal{M}_Q(\alpha,\chi)$ of χ-semistable α-dimensional representations of Q. The space $\mathcal{M}_Q(\alpha,\chi)$ is a projective variety, its points are in one-one correspondence with S-equivalence classes of χ-semistable representations. The space $\mathcal{M}_Q(\alpha,\chi)$ contains an open smooth subspace $\mathcal{M}^s_Q(\alpha,\chi)$ whose points correspond to isomorphism classes of χ-stable representations. If α is indivisible vector then $\mathcal{M}^s_Q(\alpha,\chi)$ is a fine moduli space for families of χ-stable representations of Q.*

Remark 1. A universal family $\mathcal{A}$ of representations of Q on the fine moduli space $\mathcal{M}^s(\alpha,\chi)$ is not unique (it can be twisted by any invertible sheaf). Hence it is not generally true that for any variety X equipped with a family of α-dimensional χ-stable representations $\mathcal{G}$ there exists a map $f : X \to \mathcal{M}^s(\alpha,\chi)$ such that $\mathcal{G} = f^*\mathcal{A}$. The correct statement is that under the same conditions there exists a map $f : X \to \mathcal{M}^s(\alpha,\chi)$ such that $\mathcal{G} = f^*(\mathcal{A}\otimes\mathcal{L})$ for some invertible sheaf $\mathcal{L}$ over $\mathcal{M}^s(\alpha,\chi)$. In other words, for given $\mathcal{G}$ we can chose universal family so that the desired map f will exist.

However, in the case when α is indivisible vector there is a possibility to avoid indeterminacy in choice of the universal family. Namely, fix a triple of integers (m_1, m_2, m_3), such that $m_1\alpha_1+m_2\alpha_2+m_3\alpha_3 = \pm1$. Then there exists a unique universal family $\mathcal{A}$ over $\mathcal{M}^s(\alpha,\chi)$ such that

$$m_1 \cdot \mathbf{c}_1(\mathcal{A}_1) + m_2 \cdot \mathbf{c}_1(\mathcal{A}_2) + m_3 \cdot \mathbf{c}_1(\mathcal{A}_3) = 0 \tag{1}$$

($\mathbf{c}_1(\mathcal{A}_i)$ is the first Chern class of $\mathcal{A}_i$). If both the family $\mathcal{G}$ over X and the universal family $\mathcal{A}$ over $\mathcal{M}^s(\alpha,\chi)$ satisfy the same condition (1) then there exists a map $f : X \to \mathcal{M}^s(\alpha,\chi)$ such that $\mathcal{G} = f^*\mathcal{A}$.

In the cases when it is not important we do not specify which universal family we use. In other cases, if it is possible (i.e. when $\alpha_2 - \alpha_3 = \pm1$), we will normalize universal families by condition

$$\mathbf{c}_1(\mathcal{A}_2) = \mathbf{c}_1(\mathcal{A}_3). \tag{2}$$

2.4. Consider also the quiver Q_{small}

$$\bullet \xrightarrow{\ \mathrm{W}\ } \bullet$$

All definitions can be repeated for Q_{small} (as well as for any other quiver) hence we can speak about moduli spaces of representations of Q_{small}.

Note also, that there is two embeddings of Q_{small} into Q, namely k_{13}, which sends first and second vertices of Q_{small} to the first and third vertices of Q, and k_{23}, which sends first and second vertices of Q_{small} to the second and third vertices of Q. The corresponding restriction functors F_{13} and F_{23} from the category of representations of Q to the category of representations of Q_{small} induce (generally rational) maps of the corresponding moduli spaces.

3. The moduli spaces of $(1,2,3)$-dimensional representations of Q.

From now on we will consider the case $\alpha = (1,2,3)$. The group $\mathbf{GL}(\alpha)/\Delta$ will be denoted by G.

Let $\chi_0 = (-6,0,2)$ and $\chi_1 = (0,-3,2)$ be the characters of the group $\mathbf{GL}(\alpha)$. For both these characters we have $\langle \chi, \alpha \rangle = 0$, hence we can speak about χ_0 and χ_1-(semi)stability of α-dimensional representations of Q.

Proposition 3.1. $\mathcal{M}(\alpha, \chi_0) \cong \mathbb{P}(W)$.

Proof. Every $(1,2,3)$-dimensional representation $A = (A_1, A_2, A_3)$ of the quiver Q has a canonical $(1,1,3)$-dimensional subrepresentation

$$A^0 = (A_1, a_{12}(A_1), A_3).$$

Its slope $\mu_{\chi_0}(A^0)$ is zero. It is clear, that A^0 is semistable with respect to the character χ_0 considered as character of the group $\mathbf{GL}(1,1,3)$. Moreover, every semistable $(1,1,3)$-dimensional representation A of Q is stable: if A' were its proper α'-dimensional subrepresentation with zero slope, then we should have

$$3\alpha'_1 = \alpha'_3, \qquad 0 \le \alpha'_1 \le 1, \qquad 0 \le \alpha'_3 \le 3$$

and that is possible only if

$$\alpha'_1 = \alpha'_3 = 0 \qquad \text{or} \qquad \alpha'_1 = 1,\ \alpha'_3 = 3.$$

In the first case we should also have $\alpha'_2 = 0$ (otherwise A should have the subrepresentation $(A_1, A_2, 0)$ with negative slope), i.e. $A' = 0$.

In the second case we should have $\alpha'_2 = 1$ (otherwise A should have the subrepresentation $(A_1, 0, 0)$ with negative slope), i.e. $A' = A$.

This implies that $0 \subset A^0 \subset A$ is the Jordan-Hölder composition series of A. The composition factors are A^0 and the trivial $(0,1,0)$-dimensional representation. Therefore α-dimensional representations A and A' are S-equivalent iff $A^0 \cong A'^0$.

This means that there is a canonical map

$$\psi_0 : \mathcal{M}(\alpha, \chi_0) \to \mathcal{M}((1,1,3), \chi_0) = \mathcal{M}^s((1,1,3), \chi_0).$$

and it is an isomorphism.

Consider the restriction functor F_{13} from the category of $(1,1,3)$-dimensional representations of Q to the category of $(1,3)$-dimensional representations of Q_{small}, induced by the embedding k_{13}:

$$F_{13}(A_1, A_2, A_3) = (A_1, A_3).$$

This functor certainly maps the subcategory of χ_0-stable representations of Q to the subcategory of χ_0-stable representations of Q_{small} (now we consider χ_0 as the character of the group $\mathbf{GL}(1) \times \mathbf{GL}(3)$) and is an equivalence of the categories. Hence the induced map

$$\hat{\psi}_0 : \mathcal{M}((1,1,3), \chi_0) \to \mathcal{M}_{Q_{\text{small}}}((1,3), \chi_0)$$

is an isomorphism.

Now it is remained to note that the latter space is nothing else then $\mathbb{P}(W)$ equipped with the universal family of representations

$$(\mathcal{O}_{\mathbb{P}(W)}(1), \mathcal{T}_{\mathbb{P}(W)}).$$

and the structure of representation is given by the canonical map $W \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \to \mathcal{T}_{\mathbb{P}(W)}$. $\square$

Remark 2. Since $(\mathcal{O}_{\mathbb{P}(W)}(1), \mathcal{T}_{\mathbb{P}(W)})$ is universal family over $\mathbb{P}(W)$ we have

$$\hat{\psi}_0^* \mathcal{O}_{\mathbb{P}(W)}(1) = \mathcal{A}_1^0, \qquad \hat{\psi}_0^* \mathcal{T}_{\mathbb{P}(W)} = \mathcal{A}_3^0, \tag{3}$$

where $\mathcal{A}^0 = (\mathcal{A}_1^0, \mathcal{A}_2^0, \mathcal{A}_3^0)$ is a certain universal family over $\mathcal{M}((1,1,3), \chi_0)$.

Let $\overset{\circ}{\mathfrak{W}}$ be the space of smooth rational twisted cubic curves in $\mathbb{P}(W)$ and $\mathfrak{W}$ be its Ellingsrud-Piene-Strømme compactification (see [2]). Recall that $\mathfrak{W}$ comes equipped with a pair of vector bundles $\mathcal{E}$ and $\mathcal{K}$ such that

$$\mathbf{rank}\,\mathcal{E} = 2,\ \ \mathbf{rank}\,\mathcal{K} = 3,\ \ \mathbf{det}\,\mathcal{E}^* = \mathbf{det}\,\mathcal{K}^* \text{ is the ample generator of } \mathbf{Pic}\,\mathfrak{W} = \mathbb{Z}$$

and a canonical map $W \otimes \mathcal{E} \to \mathcal{K}$.

Proposition 3.2. *$\mathcal{M}(\alpha, \chi_1) \cong \mathfrak{W}$.*

The proof goes parallel with the proof of the Proposition 3.1.

Proof. Every $(1,2,3)$-dimensional representation $A = (A_1, A_2, A_3)$ of the quiver Q has a canonical $(0,2,3)$-dimensional subrepresentation $A^1 = (0, A_2, A_3)$. Its slope $\mu_{\chi_1}(A^1)$ is zero. It is clear that A^1 is semistable with respect to the character χ_1 considered as character of the group $\mathbf{GL}(0,2,3) = \mathbf{GL}(2) \times \mathbf{GL}(3)$. Moreover, every χ_1-semistable $(0,2,3)$-dimensional representation A of Q is stable: if A' were its α'-dimensional subrepresentation with zero slope, then we should have

$$3\alpha'_2 = 2\alpha'_3, \qquad 0 \le \alpha'_2 \le 2, \qquad 0 \le \alpha'_3 \le 3.$$

but that is possible only if

$$\alpha'_2 = \alpha'_3 = 0 \quad \text{or} \quad \alpha'_2 = 2,\ \alpha'_3 = 3$$

i.e. if $A' = 0$ or $A' = A$.

This means that $0 \subset A^1 \subset A$ is the Jordan-Hölder composition series of A. The composition factors are A^1 and the trivial $(1,0,0)$-dimensional representation. Therefore α-dimensional representations A and A' are S-equivalent iff $A^1 \cong A'^1$.

This means that there is a canonical map

$$\psi_1 : \mathcal{M}(\alpha, \chi_1) \to \mathcal{M}((0,2,3), \chi_1) = \mathcal{M}^s((0,2,3), \chi_1).$$

and it is an isomorphism.

The same arguments as in the proof of the Proposition 3.1 show that the map

$$\hat{\psi}_1 : \mathcal{M}((0,2,3), \chi_1) \to \mathcal{M}_{Q_{\mathrm{small}}}((2,3), \chi_1),$$

induced by the restriction functor $F_{23} : (0, A_2, A_3) \mapsto (A_2, A_3)$ is an isomorphism.

But the latter moduli space is nothing but $\mathfrak{W}$ (see [2]) and the pair $(\mathcal{E}, \mathcal{K})$ of universal bundles over $\mathfrak{W}$ form universal family of representations over $\mathfrak{W}$. □

Remark 3. Since $(\mathcal{E}, \mathcal{K})$ is universal family over $\mathfrak{W}$ we have

$$\hat{\psi}_1^* \mathcal{E} = \mathcal{A}_2^1, \qquad \hat{\psi}_1^* \mathcal{K} = \mathcal{A}_3^1, \tag{4}$$

where $\mathcal{A}^1 = (\mathcal{A}_1^1, \mathcal{A}_2^1, \mathcal{A}_3^1)$ is a certain universal family over $\mathcal{M}((0,2,3), \chi_1)$. Note that the family $\mathcal{A}^1$ satisfies the norming condition (2).

Consider the following linear family of fractional characters of the group $\mathbf{GL}(\alpha)$

$$\chi_t = (1-t)\chi_0 + t\chi_1 = (6t - 6, -3t, 2), \quad t \in \mathbb{Q}. \tag{5}$$

Note that for all t we have $\langle \chi_t, \alpha \rangle = 0$, hence we can speak about χ_t-(semi)stability of α-dimensional representations of the quiver Q.

Let $\mathfrak{R}_t^{ss} \subset \mathfrak{R}(\alpha)$ denote the set of all points which correspond to α-dimensional χ_t-semistable representations of Q and let $\mathcal{M}_t = \mathfrak{R}_t^{ss} // \mathbf{GL}(\alpha) = \mathcal{M}(\alpha, \chi_t)$ be the corresponding moduli space.

Theorem 3.3 (Thaddeus, [7]). *There exists a finite number of rationals* $t_0 < t_1 < \cdots < t_N$ *such that*
1) *For* $t < t_0$ *or* $t > t_N$ *we have* $\mathfrak{R}_t^{ss} = \mathcal{M}_t = \emptyset$ *and for* $t_0 \le t \le t_N$ *we have* $\mathfrak{R}_t^{ss} \neq \emptyset$.
2) *For all* $t_k < t, t' < t_{k+1}$ *we have* $\mathfrak{R}_t^{ss} = \mathfrak{R}_{t'}^{ss}$ *and* $\mathcal{M}_t \cong \mathcal{M}_{t'}$, $(k = 0, \dots, N-1)$.
3) *Let us denote by* $t_k + 0$ $(t_k - 0)$ *any* t *such that* $t_k < t < t_{k+1}$ *(resp.* $t_{k-1} < t < t_k$*). Then we have an embeddings* $\mathfrak{R}_{t_k \pm 0}^{ss} \subset \mathfrak{R}_{t_k}^{ss}$ *inducing proper maps* $p_{t_k}^{\pm} : \mathcal{M}_{t_k \pm 0} \to \mathcal{M}_{t_k}$ $(k = 0, \dots, N)$.
4) *If* $k = 1, \dots, N-1$ *then maps* $p_{t_k}^{\pm}$ *are birational.*

Lemma 3.4. *In our case* $N = 2$, $t_0 = 0$, $t_1 = 2/3$, $t_2 = 1$.

Proof. Let A be an α-dimensional representation of Q. As was pointed in the proofs of the Propositions 3.1 and 3.2 it contains canonical subrepresentations A^0 ($(1,1,3)$-dimensional) and A^1 ($(0,2,3)$-dimensional). Since

$$\mu_{\chi_t}(A^0) = \langle \chi_t, (1,1,3) \rangle = 3t, \quad \mu_{\chi_t}(A^1) = \langle \chi_t, (0,2,3) \rangle = 6 - 6t$$

it follows that the representation A can be χ_t-semistable only if $0 \le t \le 1$. Since $\mathcal{M}_0$ and $\mathcal{M}_1$ are nonempty (see Propositions 3.1 and 3.2) we have $t_0 = 0$ and $t_N = 1$.

Now we need to determine all values $0 < t < 1$ such that $\mathfrak{R}_t^{ss} \neq \mathfrak{R}_{t+0}^{ss}$ or $\mathfrak{R}_t^{ss} \neq \mathfrak{R}_{t-0}^{ss}$. If one of these holds then there exist a representation A and its proper subrepresentation A', such that A is χ_t-semistable, but $\mu_{\chi_{t\pm 0}}(A') < 0$. Since $\mu_{\chi_t}(A')$ is continuous function of t this means that

$$\mu_{\chi_t}(A') = 0. \tag{6}$$

Let $\mathbf{dim}\, A' = \alpha' = (\alpha'_1, \alpha'_2, \alpha'_3)$. Then (6) is equivalent to

$$2\alpha'_3 - 6\alpha'_1 = 3t(\alpha'_2 - 2\alpha'_1).$$

This equation under constraints

$$0 \le \alpha'_1 \le 1, \quad 0 \le \alpha'_2 \le 2, \quad 0 \le \alpha'_3 \le 3, \quad \alpha' \neq (0,0,0), \quad \alpha' \neq (1,2,3), \quad 0 < t < 1$$

has only the following solutions

$$\begin{array}{rl} t = 1/3 & \alpha' = (1,0,2) \\ & \alpha' = (0,2,1) \\ t = 2/3 & \alpha' = (1,0,1) \\ & \alpha' = (0,2,2) \\ & \alpha' = (1,1,2) \\ & \alpha' = (0,1,1) \end{array}$$

The solutions from first to forth must be put aside for the following reason. If A has $(1,0,1)$-dimensional or $(1,0,2)$-dimensional subrepresentation then it also has $(1,0,0)$-dimensional subrepresentation, but $\langle \chi_t, (1,0,0) \rangle = 6t - 6 < 0$ for all $0 < t < 1$, i.e. A is unstable. Similarly, if A has $(0,2,1)$-dimensional or $(0,2,2)$-dimensional subrepresentation then it also has $(1,2,2)$-dimensional subrepresentation, but $\langle \chi_t, (1,2,2) \rangle = -2$, i.e. A is unstable.

This means that in the interval $(0,1)$ there is unique critical value $t_1 = 2/3$ and the corresponding character is $\chi_{2/3} = (-2,-2,2)$. □

Corollary 3.5. *The set* $\mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{2/3+0}^{ss}$ *consists of all representations* A *containing* $(0,1,1)$*-dimensional subrepresentation* A'. *The set* $\mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{2/3-0}^{ss}$ *consists of all representations* A *containing* $(1,1,2)$*-dimensional subrepresentation* A''. *The set* $\mathfrak{R}_{2/3}^{ss} \setminus (\mathfrak{R}_{2/3-0}^{ss} \cup \mathfrak{R}_{2/3+0}^{ss})$ *consists of all representations* A *which are the direct sum*

$$A = A' \oplus A'' \tag{7}$$

of $(0,1,1)$*-dimensional and* $(1,1,2)$*-dimensional subrepresentations. The decomposition* (7) *is unique.*

Proof. The first and second parts follow immediately from the calculations of Lemma 3.4. In the third part we need to prove that if $\chi_{2/3}$-semistable representation A contains $(0,1,1)$ and $(1,1,2)$-dimensional subrepresentations A' and A'' then $A = A' \oplus A''$. To this end consider exact sequence

$$0 \to A' \to A \to A/A' \to 0 \tag{8}$$

induced by the embedding $A' \to A$. Since both A' and A'' are $\chi_{2/3}$-stable and non-isomorphic we have $\mathbf{Hom}(A'', A') = 0$. Hence the composition of the embedding $A'' \to A$ and of the projection $A \to A/A'$ can not be zero and therefore it is an isomorphism (because A'' is $\chi_{2/3}$-stable and $\mathbf{dim}\, A/A' = \mathbf{dim}\, A''$). This means that exact sequence (8) splits and $A = A' \oplus A''$.

It is remained to prove the uniqueness of the decomposition (7). Assume that $A = A' \oplus A'' = A''' \oplus A''''$. Since the composition of the embedding $A' \to A$ and of the projection $A \to A''''$ must be zero it follows that $A' = A'''$. By the same reasons $A'' = A''''$. □

Remark 4. The proof of the Lemma 3.4 implies also that for any $0 < t < 2/3$ and $2/3 < t < 1$ all χ_t-semistable representations are χ_t-stable and therefore all varieties $\mathcal{M}_t$ are fine moduli spaces. We will denote by $\mathcal{A}^\pm = (\mathcal{A}_1^\pm, \mathcal{A}_2^\pm, \mathcal{A}_3^\pm)$ the universal families of representations normalized by the condition (2) over $\mathcal{M}_t$, $(t \in (2/3, 1))$ and $\mathcal{M}_t$, $(t \in (0, 2/3))$ respectively, by

$$a_{12} : \mathcal{A}_1^\pm \to \mathcal{A}_2^\pm, \quad a_{13} : W \otimes \mathcal{A}_1^\pm \to \mathcal{A}_3^\pm \quad \text{and} \quad a_{23} : W \otimes \mathcal{A}_2^\pm \to \mathcal{A}_3^\pm$$

the structural maps and by

$$a_{13}^T : \mathcal{A}_1^\pm \to W^* \otimes \mathcal{A}_3^\pm, \qquad a_{23}^T : \mathcal{A}_2^\pm \to W^* \otimes \mathcal{A}_3^\pm$$

the transposed maps.

3.6. By the Theorem 3.3 we have the following picture:

$$\begin{tikzcd}[column sep=small]
& \mathcal{M}_{1/3} \arrow[dl, "p_0"'] \arrow[dr, "p^-"] & & \mathcal{M}_{5/6} \arrow[dl, "p^+"'] \arrow[dr, "p_1"] & \\
\mathbb{P}(W) = \mathcal{M}_0 & & \mathcal{M}_{2/3} & & \mathcal{M}_1 = \mathfrak{W}
\end{tikzcd}$$

Here the values $t = 1/3$ and $t = 5/6$ are used as typical representatives in the intervals $(0, 2/3)$ and $(2/3, 1)$.

Maps $p^\pm$, p_0, p_1 are proper, and maps $p^\pm$ are birational. Below we will describe these maps more precisely.

4. The transformations of $\mathcal{M}_t$ and the space $\widetilde{\mathcal{M}}$

Consider the natural morphism $\mathcal{O}_{\mathbb{P}(W)}(1) \to W^* \otimes \mathcal{T}_{\mathbb{P}(W)}$ of vector bundles over $\mathbb{P}(W)$. This is an embedding of vector bundles, therefore its cokernel is a vector bundle over $\mathbb{P}(W)$. Let us denote it by $\mathcal{R}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(W)}(1) \longrightarrow W^* \otimes \mathcal{T}_{\mathbb{P}(W)} \longrightarrow \mathcal{R} \longrightarrow 0. \tag{9}$$

Remark 5. The above exact sequence is nothing but the right mutation of the exceptional vector bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$ through the exceptional vector bundle $\mathcal{T}_{\mathbb{P}(W)}$. Therefore the bundle $\mathcal{R}$ is also exceptional.

Proposition 4.1. *Let $\mathbb{P}(\mathcal{R})$ be the projectivization of $\mathcal{R}$ and $\pi_0 : \mathbb{P}(\mathcal{R}) \to \mathbb{P}(W)$ be the natural projection. There exists an isomorphism $f : \mathcal{M}_{1/3} \to \mathbb{P}(\mathcal{R})$ making the below diagram commutative*

$$\begin{tikzcd}
\mathcal{M}_{1/3} \arrow[r, "\sim", "f"'] \arrow[d, "p_0"'] & \mathbb{P}(\mathcal{R}) \arrow[d, "\pi_0"] \\
\mathcal{M}_0 \arrow[r, "\sim", "\hat{\psi}_0 \cdot \psi_0"'] & \mathbb{P}(W)
\end{tikzcd}$$

Proof. Both $\mathcal{M}_{1/3}$ and $\mathcal{M}((1,1,3),\chi_0)$ are fine moduli spaces. Let $\mathcal{A}^- = (\mathcal{A}_1^-, \mathcal{A}_2^-, \mathcal{A}_3^-)$ and $\mathcal{A}^0 = (\mathcal{A}_1^0, \mathcal{A}_2^0, \mathcal{A}_3^0)$ be the corresponding universal families. By the arguments of the proof of Proposition 3.1 the family $\mathcal{A}^-$ contains subfamily $(\mathcal{A}_1^-, a_{12}(\mathcal{A}_1^-), \mathcal{A}_3^-)$, which is a family of χ_0-stable $(1,1,3)$-dimensional representations. Hence there exists a map $\psi_{1/3} : \mathcal{M}_{1/3} \to \mathcal{M}((1,1,3),\chi_0)$ such that $\psi_{1/3}^*\mathcal{A}_1^0 = \mathcal{A}_1^-$, $\psi_{1/3}^*\mathcal{A}_3^0 = \mathcal{A}_3^-$. Combining this with (3) we get

$$\psi_{1/3}^*\hat{\psi}_0^*\mathcal{O}_{\mathbb{P}(W)}(1) = \mathcal{A}_1^-, \qquad \psi_{1/3}^*\hat{\psi}_0^*\mathcal{T}_{\mathbb{P}(W)} = \mathcal{A}_3^-.$$

On the other hand, it is evident that $\psi_0 \cdot p_0 = \psi_{1/3}$, therefore $\hat{\psi}_0 \cdot \psi_0 \cdot p_0 = \hat{\psi}_0 \cdot \psi_{1/3}$. Let us denote the latter map by p (this is the composition of the left and the bottom arrows of the above diagram). We have

$$p^*\mathcal{O}_{\mathbb{P}(W)}(1) = \mathcal{A}_1^-, \qquad p^*\mathcal{T}_{\mathbb{P}(W)} = \mathcal{A}_3^-.$$

Applying p^* to (9) we get

$$0 \longrightarrow \mathcal{A}_1^- \xrightarrow{a_{13}^T} W^* \otimes \mathcal{A}_3^- \longrightarrow p^*\mathcal{R} \longrightarrow 0.$$

The map a_{13}^T factors as $\mathcal{A}_1^- \xrightarrow{a_{12}} \mathcal{A}_2^- \xrightarrow{a_{23}^T} W^* \otimes \mathcal{A}_3^-$ and a_{12} is an embedding of vector bundles, therefore $p^*\mathcal{R}$ contains $\mathcal{A}_2^-/\mathcal{A}_1^-$ as a linear subbundle. This means that there exists a map $f : \mathcal{M}_{1/3} \to \mathbb{P}(\mathcal{R})$ such that $\pi_0 \cdot f = p$ and $\mathcal{A}_2^-/\mathcal{A}_1^- = f^*\mathcal{L}_{\mathcal{R}}$ is the pull-back of the tautological linear subbundle $\mathcal{L}_{\mathcal{R}} \subset \pi_0^*\mathcal{R}$ over $\mathbb{P}(\mathcal{R})$.

On the other hand the embedding $\mathcal{L}_{\mathcal{R}} \subset \pi_0^*\mathcal{R}$ and the pull-back by π_0 of (9) give rise to the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_0^*\mathcal{O}_{\mathbb{P}(W)}(1) & \longrightarrow & E^- & \longrightarrow & \mathcal{L}_{\mathcal{R}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_0^*\mathcal{O}_{\mathbb{P}(W)}(1) & \longrightarrow & W^* \otimes \pi_0^*\mathcal{T}_{\mathbb{P}(W)} & \longrightarrow & \pi_0^*\mathcal{R} & \longrightarrow & 0
\end{array}$$

where E^- is a 2-dimensional vector bundle over $\mathbb{P}(\mathcal{R})$. This diagram allows to define a structure of $(1,2,3)$-dimensional representation of Q in the family $(\pi_0^*\mathcal{O}_{\mathbb{P}(W)}(1), E^-, \pi_0^*\mathcal{T}_{\mathbb{P}(W)})$. It is easy to check that all representations in this family are $\chi_{1/3}$-stable. Therefore there is unique map $g : \mathbb{P}(\mathcal{R}) \to \mathcal{M}_{1/3}$ such that this family is the pull-back of the universal family over $\mathcal{M}_{1/3}$.

The map g is certainly inverse to f, hence both of them are isomorphisms. □

Remark 6. Since for the universal families we have chosen the normalization (2) (see Remark 1) we should twist the previous family by $\mathcal{L}_{\mathcal{R}} \otimes \mathcal{O}_{\mathbb{P}(W)}(-3)$. Below by the universal family on $\mathcal{M}_{1/3} = \mathbb{P}(\mathcal{R})$ we will always mean the family

$$\mathcal{A}^- = (\mathcal{L}_{\mathcal{R}} \otimes \pi_0^*\mathcal{O}_{\mathbb{P}(W)}(-2), E^- \otimes \mathcal{L}_{\mathcal{R}} \otimes \pi_0^*\mathcal{O}_{\mathbb{P}(W)}(-3), \mathcal{L}_{\mathcal{R}} \otimes \pi_0^*\mathcal{T}_{\mathbb{P}(W)}(-3)). \tag{10}$$

Remark 7. The extension

$$0 \to \pi_0^*\mathcal{O}_{\mathbb{P}(W)}(1) \to E^- \to \mathcal{L}_{\mathcal{R}} \to 0$$

defining E^- can be interpreted as *left mutation* of the exceptional bundle $\pi_0^*\mathcal{O}_{\mathbb{P}(W)}(1)$ through the exceptional vector bundle $\mathcal{L}_{\mathcal{R}}$. Hence E^- is a 2-dimensional exceptional vector bundle over $\mathbb{P}(\mathcal{R})$.

Proposition 4.2. *Let $\mathbb{P}(\mathcal{E})$ be the projectivization of the universal 2-dimensional vector bundle $\mathcal{E}$ on $\mathfrak{W}$ and $\pi_1 : \mathbb{P}(\mathcal{E}) \to \mathfrak{W}$ be the natural projection. There exists an isomorphism $f : \mathcal{M}_{5/6} \to \mathbb{P}(\mathcal{E})$ making the below diagram commutative*

$$\begin{array}{ccc}
\mathcal{M}_{5/6} & \xrightarrow[f]{\sim} & \mathbb{P}(\mathcal{E}) \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle \pi_1}\downarrow \\
\mathcal{M}_1 & \xrightarrow[\hat{\psi}_1 \cdot \psi_1]{\sim} & \mathfrak{W}
\end{array}$$

Proof. The proof goes parallel with the proof of the Proposition 4.1 . Both $\mathcal{M}_{5/6}$ and $\mathcal{M}((0,2,3),\chi_1)$ are fine moduli spaces. Let $\mathcal{A}^+ = (\mathcal{A}_1^+, \mathcal{A}_2^+, \mathcal{A}_3^+)$ and $\mathcal{A}^1 = (0, \mathcal{A}_2^1, \mathcal{A}_3^1)$ be the corresponding universal families. By the arguments of the proof of Proposition 3.2 the family $\mathcal{A}^+$ contains the subfamily $(0, \mathcal{A}_2^+, \mathcal{A}_3^+)$, which is a family of χ_1-stable $(0,2,3)$-dimensional representations. Hence there exists a map $\psi_{5/6} : \mathcal{M}_{5/6} \to \mathcal{M}((0,2,3),\chi_1)$ such that $\psi_{5/6}^*\mathcal{A}_2^1 = \mathcal{A}_2^+$, $\psi_{5/6}^*\mathcal{A}_3^1 = \mathcal{A}_3^+$. Combining this with (4) we get

$$\psi_{5/6}^*\hat{\psi}_1^*\mathcal{E} = \mathcal{A}_2^+, \qquad \psi_{5/6}^*\hat{\psi}_1^*\mathcal{K} = \mathcal{A}_3^+.$$

On the other hand, it is evident that $\psi_1 \cdot p_1 = \psi_{5/6}$, therefore $\hat{\psi}_1 \cdot \psi_1 \cdot p_1 = \hat{\psi}_1 \cdot \psi_{5/6}$. Let us denote the latter map by p (this is the composition of the left and bottom arrows of the above diagram). We have

$$p^*\mathcal{E} = \mathcal{A}_2^+.$$

Therefore the bundle $p^*\mathcal{E}$ contains $\mathcal{A}_1^+$ as a linear subbundle. This means that there exists a map $f : \mathcal{M}_{5/6} \to \mathbb{P}(\mathcal{E})$ such that $\pi_1 \cdot f = p$ and $\mathcal{A}_1^+ = f^*\mathcal{L}_{\mathcal{E}}$ is the pull-back of the tautological linear subbundle $\mathcal{L}_{\mathcal{E}} \subset \pi_1^*\mathcal{E}$ over $\mathbb{P}(\mathcal{E})$.

On the other hand, consider the following family of $(1,2,3)$-dimensional representations of Q over $\mathbb{P}(\mathcal{E})$:

$$\mathcal{A}^+ = (\mathcal{L}_{\mathcal{E}}, \pi_1^*\mathcal{E}, \pi_1^*\mathcal{K}). \tag{11}$$

It is easy to check that all representations in this family are $\chi_{5/6}$-stable. Therefore there is unique map $g : \mathbb{P}(\mathcal{E}) \to \mathcal{M}_{5/6}$ such that this family is the pull-back of universal family over $\mathcal{M}_{5/6}$.

The map g is certainly inverse to f, hence both of them are isomorphisms. □

Remark 8. The family (11) satisfies the norming condition (2).

It is left to describe the birational transform $\mathcal{M}_{1/3} \dashrightarrow \mathcal{M}_{5/6}$.

Accordingly to Theorem 3.3 we have embeddings $\mathfrak{R}_{1/3}^{ss} \subset \mathfrak{R}_{2/3}^{ss} \supset \mathfrak{R}_{5/6}^{ss}$. Let

$$\mathfrak{R}^+ = \mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{1/3}^{ss}, \qquad \mathfrak{R}^- = \mathfrak{R}_{2/3}^{ss} \setminus \mathfrak{R}_{5/6}^{ss}, \qquad \mathfrak{R}^0 = \mathfrak{R}^- \cap \mathfrak{R}^+.$$

Remark 9. By the Corollary 3.5 the set $\mathfrak{R}^+$ consists of all representations containing $(1,1,2)$-dimensional subrepresentation, $\mathfrak{R}^-$ consists of all representations containing $(0,1,1)$-dimensional subrepresentation and $\mathfrak{R}^0$ consists of all representations which are the direct sum of $(0,1,1)$-dimensional and $(1,1,2)$-dimensional representations.

Lemma 4.3. *The stabilizer G_A of $A \in \mathfrak{R}^0$ in the group $G = \mathbf{GL}(\alpha)/\Delta$ is isomorphic to $\mathbb{C}^*$. In an appropriate bases (e_i^j) of the spaces A_i $(i = 1,2,3)$ we have*

$$G_A = \widehat{G}_A/\pm 1, \qquad \widehat{G}_A = \{(\lambda, \mathbf{diag}(\lambda^{-1}, \lambda), \mathbf{diag}(\lambda^{-1}, \lambda, \lambda))\}, \quad \lambda \in \mathbb{C}^*.$$

Proof. First of all we will describe the stabilizer $\widetilde{G}_A$ of A in $\mathbf{GL}(\alpha)$. Let $A = A' \oplus A''$ be the decomposition into the sum of $(0,1,1)$ and $(1,1,2)$-dimensional representations. Then

$$A_1 = A_1'', \qquad A_2 = A_2' \oplus A_2'', \qquad A_3 = A_3' \oplus A_3''.$$

Let $(0, (e_2^1), (e_3^1))$ be a basis in A' and $((e_1^1), (e_2^2), (e_3^2, e_3^3))$ be a basis in A''. Let g be an element in G_A. Then $A = gA = gA' \oplus gA''$ is a decomposition of A. By the Corollary 3.5 we have $gA' = A'$, $gA'' = A''$, i.e. $g = g' \oplus g''$, where $g' \in \mathbf{GL}(0,1,1)$ and $g'' \in \mathbf{GL}(1,1,2)$. Since all automorphisms of the simple object over the algebraically closed field are dilations we have $g' = (0, \lambda', \lambda')$ and $g'' = (\lambda'', \lambda'', \mathbf{diag}(\lambda'', \lambda''))$. This means that

$$\widetilde{G}_A = \{(\lambda'', \mathbf{diag}(\lambda', \lambda''), \mathbf{diag}(\lambda', \lambda'', \lambda''))\}, \quad \lambda', \lambda'' \in \mathbb{C}^*.$$

Now the Lemma follows from $G_A = \widetilde{G}_A/\Delta$. □

Remark 10. The group $\mathbf{Hom}(G_A, \mathbb{C}^*)$ of characters of G_A is a free cyclic group generated by

$$v((\lambda, \mathbf{diag}(\lambda^{-1}, \lambda), \mathbf{diag}(\lambda^{-1}, \lambda, \lambda)) = \lambda^2.$$

Remark 11. It is easy to see that $\chi_{1|G_A} = v$. This implies that the restriction map $\mathbf{Hom}(G, \mathbb{C}^*) \to \mathbf{Hom}(G_A, \mathbb{C}^*)$ is surjective.

Let M^+ be the image of $\mathfrak{R}^+ \setminus \mathfrak{R}^-$ in $\mathcal{M}_{5/6}$, M^- be the image of $\mathfrak{R}^- \setminus \mathfrak{R}^+$ in $\mathcal{M}_{1/3}$ and M^0 be the image of $\mathfrak{R}^0$ in $\mathcal{M}_{2/3}$.

Remark 12. The Remark 9 implies that every element of $\mathfrak{R}^+$ is S-equivalent (with respect to $\chi_{2/3}$-stability) to some element of $\mathfrak{R}^0$. This means that $p^+(M^+) = M^0$. Similarly $p^-(M^-) = M^0$. Moreover, sets $\mathfrak{R}^{ss}_{1/3} \setminus \mathfrak{R}^-$ and $\mathfrak{R}^{ss}_{5/6} \setminus \mathfrak{R}^+$ contain no strictly semistable points, therefore the maps $p^+ : \mathbb{P}(\mathcal{R}) = \mathcal{M}_{1/3} \to \mathcal{M}_{2/3}$ and $p^- : \mathbb{P}(\mathcal{E}) = \mathcal{M}_{5/6} \to \mathcal{M}_{2/3}$ are isomorphisms on the complements of $p^\pm : M^\pm \to M^0$ (compare [7, 1.4]).

The Lemma 4.3 and Remark 11 allow to apply the theory of Thaddeus (see [7], Theorem 4.7). This implies the following.

Corollary 4.4. *The restriction $p^\pm_{|M^\pm} : M^\pm \to M^0$ is a locally trivial fibration with fiber at the point $A \in M^0$ the weighted projective space $\mathbb{P}(|w_i^\pm|)$ in the sense of Bialynicki-Birula [1], where $w_i^\pm$ are positive (resp. negative) weights of the action G_A in the fiber of the normal bundle $\mathcal{N}_{G\cdot A/\mathfrak{R}}$.*

The following Proposition describes the weight decomposition of the normal bundle.

Proposition 4.5. *Let $A \in \mathfrak{R}^0$. The weight decomposition of the fiber of the normal bundle $\mathcal{N}_{G\cdot A/\mathfrak{R}}$ with respect to the action of $\widehat{G}_A$ is*

$$\mathcal{N}_{G\cdot A/\mathfrak{R}} = \mathcal{N}^{-2} \oplus \mathcal{N}^0 \oplus \mathcal{N}^{+2}, \qquad \mathbf{dim}\,\mathcal{N}^{-2} = 2, \quad \mathbf{dim}\,\mathcal{N}^0 = 7, \quad \mathbf{dim}\,\mathcal{N}^{+2} = 5.$$

Proof. Let $\mathfrak{gl}(\alpha)$ and $\tilde{\mathfrak{g}}_A$ be the tangent Lie algebras of the groups $\mathbf{GL}(\alpha)$ and $\widetilde{G}_A$ respectively. We have natural exact sequence of $\widehat{G}_A$-modules

$$0 \longrightarrow \tilde{\mathfrak{g}}_A \longrightarrow \mathfrak{gl}(\alpha) \longrightarrow \mathfrak{R} \longrightarrow (\mathcal{N}_{G\cdot A/\mathfrak{R}})_A \longrightarrow 0.$$

Let $A = A' \oplus A''$ be the decomposition of A into the sum of $(1,1,2)$ and $(0,1,1)$-dimensional representations and recall that $\mathfrak{R} = (A_1^* \otimes A_2) \oplus (A_2^* \otimes A_3 \otimes W^*)$. The action of $\widehat{G}_A$ splits the above exact sequence into the direct sum of exact sequences of weight spaces:

$$0 \longrightarrow \tilde{\mathfrak{g}}_A^w \longrightarrow \mathfrak{gl}(\alpha)^w \longrightarrow \mathfrak{R}^w \longrightarrow \mathcal{N}^w \longrightarrow 0. \tag{12}$$

The Lemma 4.3 implies that the weight spaces are following:

w	$\tilde{\mathfrak{g}}_A^w$	$\mathfrak{gl}(\alpha)^w$	$\mathfrak{R}^w$
-2	0	$({A'_2}^* \otimes A''_2) \oplus ({A'_3}^* \otimes A''_3)$	$({A'_1}^* \otimes A''_2) \oplus ({A'_2}^* \otimes A''_3 \otimes W^*)$
0	$\mathbb{C} \oplus \mathbb{C}$	$\bigoplus_{i=1}^{3} ({A'_i}^* \otimes A'_i) \oplus \bigoplus_{i=2}^{3} ({A''_i}^* \otimes A''_i)$	$({A'_1}^* \otimes A'_2) \oplus ({A'_2}^* \otimes A'_3 \otimes W^*) \oplus ({A''_2}^* \otimes A''_3 \otimes W^*)$
2	0	$({A''_2}^* \otimes A'_2) \oplus ({A''_3}^* \otimes A'_3)$	${A''_2}^* \otimes A'_3 \otimes W^*$
others	0	0	0

and Proposition follows. $\square$

Remark 13. The weight 2 of the group $\widehat{G}_A$ corresponds to the weight 1 of the group G_A (see Remark 10), hence the group G_A acts on $\mathcal{N}^{\pm 2}$ with weights equal to ± 1.

Consider the linear bundle $L = {A_2'}^* \otimes A_2''$ over $\mathfrak{R}^0$. It is easy to see that the group G_A acts on L with weight -1, hence the bundles $\mathcal{N}^{\pm 2} \otimes L^{\pm 1}$ are acted upon trivially by G_A. Therefore they descent to some vector bundles $S^{\pm}$ over M^0. Another Theorem of Thaddeus (loc. cit., Theorem 4.8) implies the following.

Corollary 4.6 (And the definition of $\widetilde{\mathcal{M}}$ and $\widetilde{M}$)**.** *a) The spaces $M^{\pm}$ are naturally isomorphic to the projective bundles $\mathbb{P}_{M^0}(S^{\pm})$ and the restrictions of the maps $p^{\pm}$ (see 3.6) on $M^{\pm}$ are but the natural projections $p^{\pm} : \mathbb{P}_{M^0}(S^{\pm}) \to M^0$.*

b) The normal bundles of $M^{\pm}$ in $\mathcal{M}_{5/6}$ and $\mathcal{M}_{1/3}$ respectively are naturally isomorphic to ${p^{\pm}}^* S^{\mp} \otimes \mathcal{L}^{\pm}$, where $\mathcal{L}^{\pm} \subset {p^{\pm}}^* S^{\pm}$ are the tautological linear subbundles on $M^{\pm}$.

c) The blow-ups of $\mathcal{M}_{1/3}$ at M^-, of $\mathcal{M}_{5/6}$ at M^+ and of $\mathcal{M}_{2/3}$ at M^0 are naturally isomorphic to the fiber product $\widetilde{\mathcal{M}} = \mathcal{M}_{1/3} \times_{\mathcal{M}_{2/3}} \mathcal{M}_{5/6}$. The fiber product $\widetilde{M} = M^- \times_{M^0} M^+ \subset \mathcal{M}_{1/3} \times_{\mathcal{M}_{2/3}} \mathcal{M}_{5/6}$ is the common exceptional divisor of all these blow-ups. Its normal bundle is naturally isomorphic to $\mathcal{L}^- \boxtimes \mathcal{L}^+$.

The Corollary 4.6 together with the Propositions 4.1 and 4.2 shows that we have the following Theorem.

Theorem 4.7. *There exists a smooth algebraic variety $\widetilde{\mathcal{M}}$ equipped with a pair of proper birational maps $\hat{\pi}_0 : \widetilde{\mathcal{M}} \to \mathbb{P}(\mathcal{R})$ and $\hat{\pi}_1 : \widetilde{\mathcal{M}} \to \mathbb{P}(\mathcal{E})$ such that*
1) the map $\hat{\pi}_0$ is the blow-up of $\mathbb{P}(\mathcal{R})$ along the smooth subvariety $M^- \subset \mathbb{P}(\mathcal{R})$;
2) the map $\hat{\pi}_1$ is the blow-up of $\mathbb{P}(\mathcal{E})$ along the smooth subvariety $M^+ \subset \mathbb{P}(\mathcal{E})$;
3) $M^{\pm} \cong \mathbb{P}_{M^0}(S^{\pm})$, where M^0 is smooth and $S^{\pm}$ are vector bundles over M^0;
4) the blow-ups $\hat{\pi}_0$ and $\hat{\pi}_1$ have common exceptional divisor $\widetilde{M} \overset{i}{\hookrightarrow} \widetilde{\mathcal{M}}$;
5) $\widetilde{M} \cong M^- \times_{M^0} M^+$ and the restrictions of $\hat{\pi}_0$ (resp. $\hat{\pi}_1$) on $\widetilde{M}$ are nothing but projections on the first (resp. second) factors.

The following picture illustrates the Theorem.

$$\begin{tikzcd}
 & \widetilde{\mathcal{M}} \arrow[dl, "\hat{\pi}_0"'] \arrow[dr, "\hat{\pi}_1"] & & & & \widetilde{M} = M^- \times_{M^0} M^+ \arrow[dl, "\hat{\pi}_0"'] \arrow[dr, "\hat{\pi}_1"] & \\
\mathbb{P}(\mathcal{R}) \arrow[d, "\pi_0"'] & & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi_1"] & \overset{i}{\hookleftarrow} & M^- = \mathbb{P}_{M^0}(S^-) \arrow[dr, "p^-"] & & \mathbb{P}_{M^0}(S^+) = M^+ \arrow[dl, "p^+"'] \\
\mathbb{P}(W) & & \mathfrak{W} & & & M^0 &
\end{tikzcd}$$

The complete description of the variety M^0 and of the vector bundles $S^{\pm}$ will be given in Propositions 5.1 and 5.2.

Let $\tilde{\pi}_0 = \pi_0 \cdot \hat{\pi}_0$ and $\tilde{\pi}_1 = \pi_1 \cdot \hat{\pi}_1$ be the projections from $\widetilde{\mathcal{M}}$ to $\mathbb{P}(W)$ and $\mathfrak{W}$ respectively. Let $\tilde{\pi} = \tilde{\pi}_0 \times \tilde{\pi}_1 : \widetilde{\mathcal{M}} \to \mathbb{P}(W) \times \mathfrak{W}$. Let $\mathring{\mathfrak{W}} \subset \mathfrak{W}$ be the locus of smooth curves, $\mathring{\mathcal{I}} = \{(x, C) \mid x \in C\} \subset \mathbb{P}(W) \times \mathring{\mathfrak{W}}$ be the point-curve incidence subspace, π_0', π_1' projections of $\mathbb{P}(W) \times \mathring{\mathfrak{W}}$ on $\mathbb{P}(W)$ and $\mathring{\mathfrak{W}}$ respectively and let $\mathcal{I}$ be the closure of $\mathring{\mathcal{I}}$ in $\mathbb{P}(W) \times \mathfrak{W}$.

Proposition 4.8. *The image $\tilde{\pi}(\widetilde{\mathcal{M}})$ lies in $\mathcal{I}$. The map $\tilde{\pi} : \widetilde{\mathcal{M}} \to \mathcal{I}$ is birational.*

Proof. Over $\mathbb{P}(W) \times \mathring{\mathfrak{W}}$ we have the following resolvent of $\mathring{\mathcal{I}}$:

$$0 \to {\pi_1'}^* \mathcal{E} \otimes {\pi_0'}^* \mathcal{O}_{\mathbb{P}(W)}(-3) \to {\pi_1'}^* \mathcal{K} \otimes {\pi_0'}^* \mathcal{O}_{\mathbb{P}(W)}(-2) \to \mathcal{O}_{\mathbb{P}(W) \times \mathring{\mathfrak{W}}} \to \mathcal{O}_{\mathring{\mathcal{I}}} \to 0.$$

This is nothing but the universal extension of the usual resolvent of the structure sheaf of the twisted cubic curve $C \subset \mathbb{P}(W)$:

$$0 \to \mathcal{O}_{\mathbb{P}(W)}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}(W)}(-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_C \to 0.$$

Multiplying the first resolvent on $\mathcal{O}_{\overset{\circ}{\mathcal{I}}}$ and twisting by ${\pi'_0}^*\mathcal{O}_{\mathbb{P}(W)}(3)$ we see that the bundle ${\pi'_1}^*\mathcal{E}$ has a subsheaf $\mathbf{Tor}^2(\mathcal{O}_{\overset{\circ}{\mathcal{I}}}, \mathcal{O}_{\overset{\circ}{\mathcal{I}}}) \otimes {\pi'_0}^*\mathcal{O}_{\mathbb{P}(W)}(3)$. Since $\overset{\circ}{\mathcal{I}}$ is a smooth subvariety in $\mathbb{P}(W) \times \overset{\circ}{\mathfrak{W}}$ of codimension 2 this subsheaf is locally free of rank 1 and gives rise to a map $f : \overset{\circ}{\mathcal{I}} \to \mathbb{P}_{\overset{\circ}{\mathfrak{W}}}(\mathcal{E}) = \pi_1^{-1}(\overset{\circ}{\mathfrak{W}}) \subset \mathbb{P}_{\mathfrak{W}}(\mathcal{E})$. We claim that this map is an isomorphism. In order to prove this it suffices to check that the degree of the restriction of our linear bundle on the fibers of $\pi'_1 : \overset{\circ}{\mathcal{I}} \to \overset{\circ}{\mathfrak{W}}$ is equal to -1 for all points $c \in \overset{\circ}{\mathfrak{W}}$.

Let $C \subset \mathbb{P}(W)$ be a smooth rational twisted cubic curve and $c \in \overset{\circ}{\mathfrak{W}}$ the corresponding point. Then $(\pi'_1)^{-1}(c) = C$ and

$$\mathbf{Tor}^2(\mathcal{O}_C, \mathcal{O}_C) \otimes \mathcal{O}_{\mathbb{P}(W)}(3) = \Lambda^2 \mathcal{N}^*_{C/\mathbb{P}(W)} \otimes \mathcal{O}_C(9) = \mathcal{O}_C(-10) \otimes \mathcal{O}_C(9) = \mathcal{O}_C(-1).$$

But $\left(\mathbf{Tor}^2(\mathcal{O}_{\overset{\circ}{\mathcal{I}}}, \mathcal{O}_{\overset{\circ}{\mathcal{I}}}) \otimes \mathcal{O}_{\mathbb{P}(W)}(3)\right)_{|(\pi'_1)^{-1}(c)} = \mathbf{Tor}^2(\mathcal{O}_C, \mathcal{O}_C) \otimes \mathcal{O}_{\mathbb{P}(W)}(3)$, hence the degree of the restriction is equal to -1 and the claim follows.

Thus we see, that the open subset $\tilde{\pi}_1^{-1}(\overset{\circ}{\mathfrak{W}}) \subset \widetilde{\mathcal{M}}$ maps isomorphically on $\overset{\circ}{\mathcal{I}}$. Hence $\widetilde{\mathcal{M}}$ maps to its closure $\mathcal{I}$ birationally. □

The Theorem 4.7 and the Proposition 4.8 show that $\widetilde{\mathcal{M}}$ is a desingularization of $\mathcal{I}$. This desingularization is evidently $\mathbf{GL}(W)$-equivariant where the group $\mathbf{GL}(W)$ acts upon moduli spaces via automorphisms of the quiver Q.

5. The exceptional divisor and universal families

We will give here an explicit description of the exceptional divisor $\widetilde{M} \subset \widetilde{\mathcal{M}}$ and of the connection between the pull-backs of universal families on $\widetilde{\mathcal{M}}$. Let $\mathcal{A}^-$ be the universal family (10) over $\mathcal{M}_{1/3} = \mathbb{P}(\mathcal{R})$ and $\mathcal{A}^+$ be the universal family (11) over $\mathcal{M}_{5/6} = \mathbb{P}(\mathcal{E})$. Both of them satisfy the norming condition (2).

Proposition 5.1. *The space M^0 is naturally isomorphic to*

$$M^0 \cong \mathcal{M}((1,1,2), \chi_{2/3}) \times \mathcal{M}((0,1,1), \chi_{2/3}) \cong \mathbf{Gr}(2, W) \times \mathbb{P}(W^*).$$

Proof. The first isomorphism follows immediately from Remark 9. It remains to prove the isomorphisms $\mathcal{M}((1,1,2), \chi_{2/3}) \cong \mathbf{Gr}(2, W)$ and $\mathcal{M}((0,1,1), \chi_{2/3}) \cong \mathbb{P}(W^*)$.

It is easy to see that for $(1,1,2)$-dimensional and $(0,1,1)$-dimensional representations of Q the stability and semistability with respect to $\chi_{2/3}$ coincide, therefore both $\mathcal{M}((1,1,2), \chi_{2/3})$ and $\mathcal{M}((0,1,1), \chi_{2/3})$ are fine moduli spaces. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the universal families over them.

Since family $\mathcal{A}'$ consists of stable representations the map $a_{23} : W \otimes \mathcal{A}'_2 \to \mathcal{A}'_3$ is surjective, hence there exists a map $f : \mathcal{M}((1,1,2), \chi_{2/3}) \to \mathbf{Gr}(2, W)$ such that

$$f^*(W/U) = {\mathcal{A}'_2}^* \otimes \mathcal{A}'_3, \qquad f^*U = {\mathcal{A}'_2}^* \otimes \mathbf{Ker}\, a_{23}, \tag{13}$$

where U denotes the tautological subbundle over $\mathbf{Gr}(2, W)$.

Vice versa, the family

$$(\mathcal{O}_{\mathbf{Gr}(2,W)}, \mathcal{O}_{\mathbf{Gr}(2,W)}, W/U) \tag{14}$$

over $\mathbf{Gr}(2, W)$ is naturally the family of $\chi_{2/3}$-stable representations of Q, hence there exists a map $g : \mathbf{Gr}(2, W) \to \mathcal{M}((1,1,2), \chi_{2/3})$ such that the family (14) is the pull-back of the universal family over $\mathcal{M}((1,1,2), \chi_{2/3})$. Certainly, the maps f and g are mutually inverse.

Similarly, the map $a_{23} : W \otimes \mathcal{A}''_2 \to \mathcal{A}''_3$ is surjective, hence there exists a map $f : \mathcal{M}((0,1,1),\chi_{2/3}) \to \mathbb{P}(W^*)$ such that

$$f^*\eta^* = \mathcal{A}''^*_2 \otimes \mathcal{A}''_3, \qquad f^*(W^*/\eta)^* = \mathcal{A}''^*_2 \otimes \mathbf{Ker}\, a_{23}, \tag{15}$$

where η stands for the tautological linear subbundle in the trivial bundle $W^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}$ over $\mathbb{P}(W^*)$.

Vice versa, the family

$$(0, \eta, \mathcal{O}_{\mathbb{P}(W^*)}) \tag{16}$$

over $\mathbb{P}(W^*)$ gives rise to a map $g : \mathbb{P}(W^*) \to \mathcal{M}((0,1,1),\chi_{2/3})$ such that the family (16) is the pull-back of the universal family over $\mathcal{M}((0,1,1),\chi_{2/3})$. Certainly, the maps f and g are mutually inverse. $\square$

Below we will identify the space M^0 with $\mathbf{Gr}(2,W) \times \mathbb{P}(W^*)$ and consider bundles U and η as bundles over M^0.

Proposition 5.2. *There are natural isomorphisms of vector bundles over M^0:*

$$S^- \cong \eta^* \otimes U^*, \qquad S^+ \cong \frac{W/U \otimes W^*/\eta}{\mathcal{O}_{M^0}}.$$

Proof. The exact sequence (12) corresponding to $w = -2$ can be included into the commutative diagram

$$\begin{tikzcd}
 & \mathcal{A}'^*_3 \otimes \mathcal{A}''_3 \arrow[r, "a_{23}^T\otimes\mathbf{id}"] \arrow[d] & \mathcal{A}'^*_2 \otimes \mathcal{A}''_3 \otimes W^* \arrow[d] & & \\
0 \arrow[r] & \mathfrak{gl}(\alpha)^{-2} \arrow[r] \arrow[d] & \mathfrak{R}^{-2} \arrow[r] \arrow[d] & \mathcal{N}^{-2} \arrow[r] & 0 \\
 & \mathcal{A}'^*_2 \otimes \mathcal{A}''_2 \arrow[r, "a_{12}^T\otimes\mathbf{id}"] & \mathcal{A}'^*_1 \otimes \mathcal{A}''_2 & &
\end{tikzcd}$$

with exact columns. Since a_{12} is an isomorphism and $\mathbf{Ker}\, a_{23} = \mathcal{A}'_2 \otimes U$ (see (13)) we have

$$\mathcal{N}^{-2} = \mathcal{A}'^*_2 \otimes \mathcal{A}''_3 \otimes U^*.$$

Twisting on $L^{-1} = \mathcal{A}'_2 \otimes \mathcal{A}''^*_2$ we get

$$S^- = \mathcal{A}''^*_2 \otimes \mathcal{A}''_3 \otimes U^*.$$

The first isomorphism now follows from (15).

The cokernel of the morphism

$$(\mathcal{A}''^*_2 \otimes \mathcal{A}'_2) \oplus (\mathcal{A}''^*_3 \otimes \mathcal{A}'_3) \xrightarrow{(\mathbf{id}\otimes a_{23}^T)\oplus(a_{23}^*\otimes\mathbf{id})} \mathcal{A}''^*_2 \otimes \mathcal{A}'_3 \otimes W^*$$

from the sequence (12) corresponding to $w = 2$ is isomorphic to the cokernel of the induced morphism from $\mathcal{A}''^*_2 \otimes \mathcal{A}'_2$ to the cokernel of the morphism

$$\mathcal{A}''^*_3 \otimes \mathcal{A}'_3 \xrightarrow{a_{23}^*\otimes\mathbf{id}} \mathcal{A}''^*_2 \otimes \mathcal{A}'_3 \otimes W^*$$

which by (15) is isomorphic to $\mathcal{A}''^*_2 \otimes \mathcal{A}'_3 \otimes W^*/\eta$. This means that we have an exact sequence

$$0 \to \mathcal{A}''^*_2 \otimes \mathcal{A}'_2 \to \mathcal{A}''^*_2 \otimes \mathcal{A}'_3 \otimes W^*/\eta \to \mathcal{N}^{+2} \to 0.$$

Twisting on L we get

$$0 \to \mathcal{O}_{M^0} \to \mathcal{A}'^*_2 \otimes \mathcal{A}'_3 \otimes W^*/\eta \to S^+ \to 0.$$

The second isomorphism now follows from (13). $\square$

Now we will describe the restriction of the universal family $\mathcal{A}^+$ on the subvariety M^+. Consider the following commutative diagram over M^0:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W\otimes\eta & \longrightarrow & W\otimes\eta\otimes(W/U\otimes W^*/\eta) & \longrightarrow & W\otimes\eta\otimes S^+ & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W/U\otimes\eta & \longrightarrow & \dfrac{W/U\otimes W^*}{\mathcal{O}} & \longrightarrow & S^+ & \longrightarrow & 0
\end{array}$$

Here the top row is the definition of S^+ multiplied on $W\otimes\eta$, the bottom row is evident, the vertical arrows are following: the left arrow is induced by projection $W\to W/U$; the middle arrow is induced by the embedding $\eta\otimes W^*/\eta\hookrightarrow\Lambda^2W^*\otimes\mathcal{O}_{\mathbb{P}(W^*)}$ and by pairing $W\otimes\Lambda^2W^*\to W^*$; the right arrow is induced by the pairing $W\otimes\eta\to\mathcal{O}_{\mathbb{P}(W^*)}$. One can easily check that everything here commutes and the diagram is correct.

The pull-back via p^+ and the embedding $\mathcal{L}^+\subset {p^+}^*S^+$ transform this diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W\otimes{p^+}^*\eta & \longrightarrow & W\otimes{p^+}^*\eta\otimes E^+ & \longrightarrow & W\otimes{p^+}^*\eta\otimes\mathcal{L}^+ & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {p^+}^*(W/U\otimes\eta) & \longrightarrow & K^+ & \longrightarrow & \mathcal{L}^+ & \longrightarrow & 0
\end{array}\tag{17}$$

Here E^+ and K^+ are certain 2-dimensional and 3-dimensional vector bundles (defined by these sequences) on $\mathbb{P}_{M^0}(S^+)$.

The maps ${p^+}^*\eta\to{p^+}^*\eta\otimes E^+$ and $W\otimes{p^+}^*\eta\otimes E^+\to K^+$ define the structure of $(1,2,3)$-dimensional representation of Q in the family

$$({p^+}^*(\Lambda^2U\otimes\eta),{p^+}^*(\Lambda^2U\otimes\eta)\otimes E^+,{p^+}^*\Lambda^2U\otimes K^+).\tag{18}$$

Remark 14. The additional twisting on ${p^+}^*\Lambda^2U$ is needed to satisfy the norming condition (2).

All representations in the family (18) are $\chi_{5/6}$-stable, hence there exists a map $f:\mathbb{P}_{M^0}(S^+)\to\mathcal{M}_{5/6}$ such that this family is the pull-back of the universal family over $\mathcal{M}_{5/6}$. Since the family (18) has $(1,1,2)$-dimensional subfamily

$$({p^+}^*(\Lambda^2U\otimes\eta),{p^+}^*(\Lambda^2U\otimes\eta),{p^+}^*(\Lambda^2U\otimes\eta\otimes W/U))$$

(see (17)) the image of the map f is contained in M^+. Since the above subfamily as well as the corresponding quotient family

$$(0,{p^+}^*(\Lambda^2U\otimes\eta)\otimes\mathcal{L}^+,{p^+}^*\Lambda^2U\otimes\mathcal{L}^+)$$

are nothing but the universal families (14) and (16) over the moduli spaces $\mathcal{M}((1,1,2),\chi_{2/3})$ and $\mathcal{M}((0,1,1),\chi_{2/3})$ twisted by ${p^+}^*(\Lambda^2U\otimes\eta)$ and ${p^+}^*\Lambda^2U\otimes\mathcal{L}^+$ respectively, we have

$$p^+\cdot f=\mathbf{id}_{M^0}\,.$$

This means that $f:\mathbb{P}(S^+)\to M^+$ is an isomorphism. We have proved the following.

Proposition 5.3. *The restriction of the universal family $\mathcal{A}^+$ on M^+ is the family* (18)*. There exists an exact sequence of families over M^+:*

$$0\to{p^+}^*\mathcal{A}'\to\mathcal{A}^+_{|M^+}\to{p^+}^*\mathcal{A}''\otimes\mathcal{L}^+\to 0,$$

where

$$\mathcal{A}'=(\Lambda^2U\otimes\eta,\Lambda^2U\otimes\eta,\Lambda^2U\otimes\eta\otimes W/U),\qquad\mathcal{A}''=(0,\Lambda^2U\otimes\eta,\Lambda^2U).$$

are universal families over $\mathcal{M}((1,1,2),\chi_{2/3})$ and $\mathcal{M}((0,1,1),\chi_{2/3})$ respectively.

Now we will describe the restriction of the universal family $\mathcal{A}^-$ on the subvariety M^-. For convenience we will consider M^- not as $\mathbb{P}(S^-)$ but as $\mathbb{P}(\Lambda^2 U \otimes \eta \otimes S^-) = \mathbb{P}(U)$. Let $\mathcal{L}' = {p^-}^*(\Lambda^2 U \otimes \eta) \otimes \mathcal{L}^-$ be the tautological linear subbundle in ${p^-}^* U$. Consider the commutative diagram of vector bundles over $\mathbb{P}(U)$:

$$\begin{array}{ccccccccc}
0 & \to & W \otimes \left({p^-}^*\eta \otimes \dfrac{{p^-}^*U}{\mathcal{L}'}\right) & \to & W \otimes \left({p^-}^*\eta \otimes \dfrac{{p^-}^*U}{\mathcal{L}'}\right) \oplus W & \to & W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \dfrac{{p^-}^*U}{\mathcal{L}'} & \to & \dfrac{W}{\mathcal{L}'} & \to & {p^-}^*W/U & \to & 0
\end{array}$$

and note that ${p^-}^*\eta \otimes \dfrac{{p^-}^*U}{\mathcal{L}'} \cong {\mathcal{L}^-}^*$. The isomorphism on the second summand $\mathcal{O} \longrightarrow {\mathcal{L}^-}^* \oplus \mathcal{O}$ and the map $W \otimes \left({\mathcal{L}^-}^* \oplus \mathcal{O}\right) \longrightarrow \dfrac{W}{\mathcal{L}'}$ from the above diagram define the structure of $(1,2,3)$-dimensional representation in the family

$$\left({p^-}^*(\Lambda^2 U \otimes \eta), {p^-}^*(\Lambda^2 U \otimes \eta) \otimes \left({\mathcal{L}^-}^* \oplus \mathcal{O}\right), {p^-}^*(\Lambda^2 U \otimes \eta) \otimes \frac{W}{\mathcal{L}'}\right) \tag{19}$$

Remark 15. The additional twisting on ${p^-}^*(\Lambda^2 U \otimes \eta)$ is needed to satisfy the norming condition (2).

All representations in the family (19) are $\chi_{1/3}$-stable, hence there exists a map $f : \mathbb{P}_{M^0}(S^-) \to \mathcal{M}_{1/3}$ such that this family is the pull-back of the universal family $\mathcal{A}^-$ over $\mathcal{M}_{1/3}$. Since family (19) has $(0,1,1)$-dimensional subfamily

$$(0, {p^-}^*(\Lambda^2 U \otimes \eta) \otimes {\mathcal{L}^-}^*, {p^-}^* \Lambda^2 U \otimes {\mathcal{L}^-}^*)$$

the image of the map f is contained in M^-. Since the above subfamily as well as the corresponding quotient family

$$({p^-}^*(\Lambda^2 U \otimes \eta), {p^-}^*(\Lambda^2 U \otimes \eta), {p^-}^*(\Lambda^2 U \otimes \eta) \otimes {p^-}^*(W/U))$$

are nothing but the universal families over the moduli spaces $\mathcal{M}((1,1,2), \chi_{2/3})$ and $\mathcal{M}((0,1,1), \chi_{2/3})$ twisted by ${p^-}^* \Lambda^2 U \otimes {\mathcal{L}^-}^*$ and ${p^-}^*(\Lambda^2 U \otimes \eta)$ respectively, we have

$$p^- \cdot f = \mathbf{id}_{M^0}.$$

This means that $f : \mathbb{P}(S^-) \to M^-$ is an isomorphism. We have proved the following.

Proposition 5.4. *The restriction of the universal family $\mathcal{A}^-$ on M^- is the family* (19). *There is an exact sequence of families over M^-:*

$$0 \to {p^-}^* \mathcal{A}'' \otimes {\mathcal{L}^-}^* \to \mathcal{A}^-_{|M^-} \to {p^-}^* \mathcal{A}' \to 0,$$

where $\mathcal{A}'$ and $\mathcal{A}''$ are the same as in Proposition 5.3.

Let $p := p^- \cdot \hat{\pi}_0 = p^+ \cdot \hat{\pi}_1 : \widetilde{M} \to M^0$ be the natural projection.

Proposition 5.5. *On the space $\widetilde{\mathcal{M}}$ there is an exact sequence of coherent sheaves*

$$0 \to \hat{\pi}_0^* \mathcal{A}^- \to \hat{\pi}_1^* \mathcal{A}^+ \to i_*(p^* \mathcal{A}'' \otimes \hat{\pi}_1^* \mathcal{L}^+) \to 0. \tag{20}$$

The restriction of this sequence on $\widetilde{M}$ factors as

$$p^* \mathcal{A}'' \otimes \hat{\pi}_0^* {\mathcal{L}^-}^* \hookrightarrow (\hat{\pi}_0^* \mathcal{A}^-)_{|\widetilde{M}} \twoheadrightarrow p^* \mathcal{A}' \hookrightarrow (\hat{\pi}_1^* \mathcal{A}^+)_{|\widetilde{M}} \twoheadrightarrow p^* \mathcal{A}'' \otimes \hat{\pi}_1^* \mathcal{L}^+.$$

Proof. Morphism $\mathcal{A}^+_{|M^+} \to p^{+*}\mathcal{A}'' \otimes \mathcal{L}^+$ from Proposition 5.3 induces the surjective morphism $\hat{\pi}_1^*\mathcal{A}^+ \twoheadrightarrow i_*(p^*\mathcal{A}'' \otimes \hat{\pi}_1^*\mathcal{L}^+)$. Denote its kernel by $\mathcal{G}$. By reasons of homological dimension $\mathcal{G}$ is locally free. We claim that actually $\mathcal{G}$ is a pull-back from $\mathcal{M}_{1/3}$. In order to prove this we should check that the restriction of $\mathcal{G}$ on all fibers of $\hat{\pi}_0$ is trivial. Since $\hat{\pi}_0$ is an isomorphism on the complement of $\widetilde{M}$ we may restrict ourselves to the latter divisor. The definition of $\mathcal{G}$ and Proposition 5.3 implies that the restriction of $\mathcal{G}$ on $\widetilde{M}$ admits the following exact sequence:

$$0 \to p^*\mathcal{A}'' \otimes \hat{\pi}_1^*\mathcal{L}^+ \otimes \mathcal{N}^*_{\widetilde{M}/\widetilde{\mathcal{M}}} \to \mathcal{G}_{|\widetilde{M}} \to p^*\mathcal{A}' \to 0.$$

Since by Corollary 4.6 we have $\mathcal{N}_{\widetilde{M}/\widetilde{\mathcal{M}}} = \hat{\pi}_0^*\mathcal{L}^- \otimes \hat{\pi}_0^*\mathcal{L}^+$ we get

$$0 \to \hat{\pi}_0^*(p^{-*}\mathcal{A}'' \otimes \mathcal{L}^{-*}) \to \mathcal{G}_{|\widetilde{M}} \to \hat{\pi}_0^* p^{-*}\mathcal{A}' \to 0$$

and the claim follows, i.e. $\mathcal{G} = \hat{\pi}_0^*\mathcal{G}'$ with $\mathcal{G}'$ locally free. Identifying the complements of M^0, M^-, M^+ and $\widetilde{M}$ in $\mathcal{M}_{2/3}$, $\mathcal{M}_{1/3}$, $\mathcal{M}_{5/6}$ and $\widetilde{\mathcal{M}}$ respectively via the maps $\hat{\pi}_0$, $\hat{\pi}_1$ and $p^{\pm}$ we get the following isomorphisms:

$$\mathcal{G}'_{|\mathcal{M}_{1/3}-M^-} \cong \mathcal{G}_{|\widetilde{\mathcal{M}}-\widetilde{M}} \cong \hat{\pi}_1^*\mathcal{A}^-_{|\widetilde{\mathcal{M}}-\widetilde{M}} \cong \mathcal{A}^-_{|\mathcal{M}_{5/6}-M^+} \cong \mathcal{A}^0 \cong \mathcal{A}^+_{|\mathcal{M}_{1/3}-M^-},$$

where by $\mathcal{A}^0$ we denote the universal family over $\mathcal{M}^s_{2/3} = \mathcal{M}_{2/3} - M^0$ satisfying the norming condition (2). The first and the last terms in the above sequence of isomorphisms are locally free sheaves on $\mathcal{M}_{1/3}$ and isomorphism is defined on the complement of the subvariety M^- of big codimension (namely $\mathbf{codim}_{\mathcal{M}_{1/3}} M^- = 5$) hence it can be continued to all $\mathcal{M}_{1/3}$ (Harthog's Theorem). This means that $\mathcal{G} = \hat{\pi}_0^*\mathcal{A}^-$ and the Proposition follows. □

Exact sequence (20) is an exact sequence of representations of Q. It consists of three components corresponding to three vertices of Q. They relate the pull-backs of universal bundles $\mathcal{E}$ and $\mathcal{K}$ with pull-backs of bundles over $\mathbb{P}(W)$. Now we will write them down:

Corollary 5.6. *Over the space $\widetilde{\mathcal{M}}$ there exist the following exact sequences of coherent sheaves:*

$$\begin{array}{llll}
0 \to \hat{\pi}_0^*\mathcal{L}_{\mathcal{R}} \otimes \tilde{\pi}_0^*\mathcal{O}_{\mathbb{P}(W)}(-2) & \to \hat{\pi}_1^*\mathcal{L}_{\mathcal{E}} & \to 0, & \\
0 \to \hat{\pi}_0^*(E^- \otimes \mathcal{L}_{\mathcal{R}}) \otimes \tilde{\pi}_0^*\mathcal{O}_{\mathbb{P}(W)}(-3) & \to \tilde{\pi}_1^*\mathcal{E} & \to i_*(p^*(\Lambda^2 U \otimes \eta) \otimes \hat{\pi}_1^*\mathcal{L}^+) & \to 0, \\
0 \to \hat{\pi}_0^*\mathcal{L}_{\mathcal{R}} \otimes \tilde{\pi}_0^*\mathcal{T}_{\mathbb{P}(W)}(-3) & \to \tilde{\pi}_1^*\mathcal{K} & \to i_*(p^*\Lambda^2 U \otimes \hat{\pi}_1^*\mathcal{L}^+) & \to 0.
\end{array}$$

Now we will write down the sequences describing restrictions of pull-backs of $\mathcal{L}_{\mathcal{E}}$, $\mathcal{E}$ and $\mathcal{K}$ on $\widetilde{M}$ (that is just the pull-backs of the components of the exact sequence of Proposition 5.3).

Corollary 5.7. *Over the divisor $\widetilde{M}$ there exist the following exact sequences of coherent sheaves:*

$$\begin{array}{llll}
0 \to p^*(\Lambda^2 U \otimes \eta) & \to (\hat{\pi}_1^*\mathcal{L}_{\mathcal{E}})_{|\widetilde{M}} & \to 0, & \\
0 \to p^*(\Lambda^2 U \otimes \eta) & \to (\tilde{\pi}_1^*\mathcal{E})_{|\widetilde{M}} & \to p^*(\Lambda^2 U \otimes \eta) \otimes \hat{\pi}_1^*\mathcal{L}^+ & \to 0, \\
0 \to p^*(\Lambda^2 U \otimes \eta \otimes W/U) & \to (\tilde{\pi}_1^*\mathcal{K})_{|\widetilde{M}} & \to p^*\Lambda^2 U \otimes \hat{\pi}_1^*\mathcal{L}^+ & \to 0.
\end{array}$$

Here we gather some information about the Picard group of $\widetilde{\mathcal{M}}$. Let $h = \mathbf{c}_1(\mathcal{O}_{\mathbb{P}(W)}(1)) \in \mathbf{Pic}\,\mathbb{P}(W)$ be the class of hyperplane on $\mathbb{P}(W)$, $H = \mathbf{c}_1(\mathcal{E}^*) = \mathbf{c}_1(\mathcal{K}^*) \in \mathbf{Pic}\,\mathfrak{W}$, $r = \mathbf{c}_1(\mathcal{L}^*_{\mathcal{R}}) \in \mathbf{Pic}\,\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})$, $e = \mathbf{c}_1(\mathcal{L}^*_{\mathcal{E}}) \in \mathbf{Pic}\,\mathbb{P}_{\mathfrak{W}}(\mathcal{E})$ and $d = [\widetilde{M}] \in \mathbf{Pic}\,\widetilde{\mathcal{M}}$. The natural embeddings of these Picard groups into $\mathbf{Pic}\,\widetilde{\mathcal{M}}$ allows to consider all above classes as elements of $\mathbf{Pic}\,\widetilde{\mathcal{M}}$.

Corollary 5.8. *We have the following relations in* $\mathbf{Pic}\,\widetilde{\mathcal{M}}$

$$e = r + 2h, \quad H = 3r + 5h - d, \qquad r = 2H - 5e + 2d, \quad h = 3e - H - d. \tag{21}$$

Proof. The first relation follows from the first sequence of Corollary 5.6. The second relation follows from the third sequence of Corollary 5.6. The third and the fourth relations follows from the first and the second. □

Corollary 5.9. *The following formulas express canonical classes in terms of the above generators of the Picard groups.*

$$K_{\mathbb{P}(W)} = -4h, \qquad K_{\mathfrak{W}} = -4H$$
$$K_{\mathbb{P}_{\mathbb{P}(W)}(\mathcal{R})} = -19h - 11r, \qquad K_{\mathbb{P}_{\mathfrak{W}}(\mathcal{E})} = -3H - 2e,$$
$$K_{\widetilde{\mathcal{M}}} = -19h - 11r + 4d = -3H - 2e + d.$$

Proof. Evident. □

Let $s^- = \mathbf{c}_1(\mathcal{L}^{-*}) \in \mathbf{Pic}\, M^-$ and $s^+ = \mathbf{c}_1(\mathcal{L}^{+*}) \in \mathbf{Pic}\, M^+$.

Corollary 5.10. *We have the following equalities in* $\mathbf{Pic}\, \widetilde{M}$

$$i^*h = s^- + \dots, \quad i^*r = -2s^- + \dots, \qquad i^*H = s^+ + \dots, \qquad i^*e = \dots.$$

where $\dots$ *stands for the pull-back of some class from* M^0*.*

Proof. Follows from the Propositions 5.3 and 5.4. □

The Corollary 5.6 and Corollary 5.7 give a way to compute the cohomology of any vector bundle over $\mathfrak{W}$ which can be produced from the universal bundles $\mathcal{E}$ and $\mathcal{K}$ by tensor operations and of its restriction on $\widetilde{M}$. In particular, if someone could invent, how from these bundles can be constructed an exceptional collection on $\mathfrak{W}$, then its exceptionality could be verified by such computations. Also the fullness of this collection could be deduced. One only need to produce from this collection an exceptional collection on $\widetilde{\mathcal{M}}$ via Orlov's procedure (see [6]) and compare it with another collection, produced from the standard exceptional collection on $\mathbb{P}(W)$.

Unfortunately, we have no idea of how to construct an exceptional collection on $\mathfrak{W}$. But the method works for Fano threefolds V_{22}. Such threefold X can be embedded in the space $\mathfrak{W}$ (see [5]) as zero locus of a certain section of the vector bundle $\mathcal{K}^* \oplus \mathcal{K}^* \oplus \mathcal{K}^*$. But the most amazing, that taking zero locus of the pull-back of this bundle on the space $\widetilde{\mathcal{M}}$ we will get for the threefold X almost the same picture as we have for $\mathfrak{W}$, namely the blow-up of the projectivization of 2-dimensional vector bundle on X is isomorphic to the blow-up of the projectivization of 2-dimensional vector bundle over $\mathbb{P}(W)$. The main difference is that in the case of threefold X the exceptional collection can be constructed and we can get the description of the derived category of coherent sheaves on X. This will be done in the forthcoming paper.

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