Finite gap theory of the Clifford torus

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Abstract

We construct the spectral curve and the Baker–Akhiezer function for the Dirac operator which corresponds to the Clifford torus via the Weierstrass representation. This torus appears in many conjectures in differential geometry. By constructing this Baker–Akhiezer function we demonstrate a general procedure for constructing Dirac operators and their Baker–Akhiezer functions corresponding to singular spectral curves

1 Introduction and main results

In this paper we construct the spectral curve and the Baker–Akhiezer function for the Dirac operator which form the data of the Weierstrass representation of the Clifford torus. This torus appears in many conjectures in differential geometry (see Section 2).

By constructing this Baker–Akhiezer function we demonstrate a general procedure for constructing Dirac operators and their Baker–Akhiezer functions corresponding to singular spectral curves. This procedure is exposed in Section 3.

The Clifford torus is a torus embedded into $\mathbb{R}^3$. The Dirac operator which corresponds to it via the Weierstrass representation (see [11, 12]) is

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

with the potential

$$U(z,\bar{z}) = \frac{\sin y}{2\sqrt{2}(\sin y - \sqrt{2})}, \quad z = x + iy. \tag{1}$$

We have

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Theorem 1 *The Baker–Akhiezer function of the Dirac operator $\mathcal{D}$ with the potential (1) is a vector function $\psi(z,\bar{z},P)$, where $z \in \mathbb{C}$ and P belongs to the spectral curve Γ of this operator, such that*

1) the spectral curve Γ is a sphere $\mathbb{C}P^1 = \bar{\mathbb{C}}$ with two marked points $\infty_+ = (\lambda = \infty), \infty_- = (\lambda = 0)$ where λ is an affine parameter on $\mathbb{C} \subset \mathbb{C}P^1$ and with two double points obtained by stacking together the points from the following pairs:

$$\left(\frac{1+i}{4}, \frac{-1+i}{4}\right) \quad \text{and} \quad \left(-\frac{1+i}{4}, \frac{1-i}{4}\right);$$

2) the function ψ is meromorphic on $\Gamma \setminus \{\infty_\pm\}$ and has at the marked points ("infinities") the following asymptotics:

$$\psi \approx \begin{pmatrix} e^{k_+ z} \\ 0 \end{pmatrix} \text{ as } k_+ = \lambda \to \infty; \quad \psi \approx \begin{pmatrix} 0 \\ e^{k_- \bar{z}} \end{pmatrix} \text{ as } k_- = -\frac{|u|^2}{\lambda} \to \infty$$

where $k_\pm^{-1}$ are local parameters near $\infty_\pm$ and $u = \frac{1+i}{4}$;

3) ψ has three poles on $\Gamma \setminus \{\infty_\pm\}$ which are independent on z and have the form

$$p_1 = \frac{-1+i+\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_2 = \frac{-1+i-\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_3 = \frac{1}{\sqrt{8}}.$$

Therewith the geometric genus $p_g(\Gamma)$ and the arithmetic genus $p_a(\Gamma)$ of Γ are as follows:

$$p_g(\Gamma) = 0, \qquad p_a(\Gamma) = 2.$$

The Baker–Akhiezer function satisfies the Dirac equation

$$\mathcal{D}\psi = 0$$

at any point of $\Gamma \setminus \{\infty_+, \infty_-, p_1, p_2, p_3\}$ where the potential U of the Dirac operator takes the form (1).

The Clifford torus is constructed via the Weierstrass representation (2) from the function

$$\psi = \psi\left(z, \bar{z}, \frac{1-i}{4}\right).$$

Remark that in the proof of this theorem which will be given in Section 4 it is actually will be showed that the Baker–Akhiezer function ψ takes the form

$$\psi_1(z,\bar z,\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar z}\left(q_1\frac{\lambda}{\lambda-p_1} + q_2\frac{\lambda}{\lambda-p_2} + (1-q_1-q_2)\frac{\lambda}{\lambda-p_3}\right),$$

$$\psi_2(z,\bar z,\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar z}\left(t_1\frac{p_1}{p_1-\lambda} + t_2\frac{p_2}{p_2-\lambda} + (1-t_1-t_2)\frac{p_3}{p_3-\lambda}\right)$$

where $u = \frac{1+i}{4}$ and q_1, q_2, t_1, t_2 are functions of $z, \bar z$ which are uniquely defined by the conditions

$$\psi\left(z,\bar z,\frac{1+i}{4}\right) = \psi\left(z,\bar z,\frac{-1+i}{4}\right),$$

$$\psi\left(z,\bar z,-\frac{1+i}{4}\right) = \psi\left(z,\bar z,\frac{1-i}{4}\right).$$

It appears that q_1, q_2, t_1, t_2 are 2π-periodic functions of y and are independent on x.

We would like to mention one interesting feature:

- the spectral curve Γ admits a holomorphic involution $\sigma(\lambda) = -\lambda$ which preserves the infinities $\infty_\pm$. Although both the Jacoby variety $J(\Gamma/\sigma)$ of the quotient space Γ/σ and the Prym variety of this involution are noncomplete Abelian varieties, the potential is a smooth function.

This is explained by some effect unfamiliar for other operators. It is as follows.

The potential U is written in the terms of Prym functions (as in the case of the two-dimensional Schrödinger operator and some other operators, see [3]). Although the Prym variety is isomorphic to $\mathbb{C}^*$, the potential depends on one real-valued variable y in a way that it is a restriction of some meromorphic function on the Prym variety onto a compact circle $S^1 \subset \mathbb{C}^*$. This circle is a compact subgroup of $\mathbb{C}^*$.

The correspondence between tori in $\mathbb{R}^3$ and Dirac operators $\mathcal{D}$ with real-valued potentials is established by the Weierstrass representation.

It is based on a local representation of any surface immersed into $\mathbb{R}^3$ by the formulas

$$x^k = x^k(0) + \int\left(x^k_z dz + \overline{x^k_z} d\bar z\right), \quad k = 1,2,3, \tag{2}$$
$$x^1_z = \frac{i}{2}(\bar\psi_2^2 + \psi_1^2), \quad x^2_z = \frac{1}{2}(\bar\psi_2^2 - \psi_1^2), \quad x^3_z = \psi_1\bar\psi_2,$$

where ψ meets the Dirac equation

$$\mathcal{D}\psi = 0.$$

In this event z defines a conformal parameter $z = x+iy$ on the surface, the first fundamental form equals $e^{2\alpha}dzd\bar{z}$ and

$$U = \frac{He^{\alpha}}{2}$$

where H is the mean curvature (see [7, 11]).

After a globalization we obtain for a closed surface in $\mathbb{R}^3$ a representation by these formulas where ψ is a solution of the Dirac equation and the Dirac operator acts on smooth sections of some spinor bundles over a conformally equivalent surface of constant curvature [11, 12].

It appears that the spectral curve of $\mathcal{D}$ on the zero energy level defined initially for the two-dimensional Schrödinger operator in [2] has to have some geometric meaning.

Let us briefly recall the origin of the spectral curves in theory of differential operators with double-periodic coefficients.

The spectral curve Γ is a complex curve which parameterizes the Floquet functions on the zero energy level, i.e. joint eigenfunctions of $\mathcal{D}$ and translations T_γ by periods

$$T_\gamma f(z,\bar{z}) = f(z+\gamma, \bar{z}+\bar{\gamma}), \quad U(z+\gamma,\bar{z}+\bar{\gamma}) = U(z,\bar{z}).$$

Here $D\psi = 0$ (the "eigenvalue" equals zero) and ψ is considered as a formal analytic solution to this equation not necessary belonging to some fixed functional space. Floquet functions are glued together into a meromorphic function $\psi(z,\bar{z},P)$ on the spectral curve. If Γ is of finite genus, it is also completed by two "infinities" at which ψ has exponential asymptotics. This would be the Baker–Akhiezer function. Given a pair of generators γ_1, γ_2 of the period lattice, to every Floquet function $\psi(z,\bar{z},P)$ where corresponds two functions on the spectral curve holomorphic outside the "infinities", the multipliers $\mu_1(P)$ and $\mu_2(P)$, $P \in \Gamma$, such that

$$T_{\gamma_j}\psi(P) = \mu_j \psi(P).$$

We correspond to a torus in $\mathbb{R}^3$ an operator $\mathcal{D}$ via the Weierstrass representation of the torus and the spectral curve of this operator and define the spectral genus of a torus as the geometric genus of the normalization of the spectral curve (see [12]).

We conjectured that

- the spectral curve Γ corresponding to a torus in $\mathbb{R}^3$ and the multipliers $\mu_1, \mu_2 : \Gamma \to \mathbb{C}$ are invariant under conformal transformations of the ambient space $\mathbb{R}^3$ (that was confirmed in [5]);
- given a conformal class of a torus, the Willmore functional attains its minima on tori with the minimal value of the spectral genus.

In geometric problems the spectral curves related to integrable surfaces are not always smooth. This was discussed for minimal tori in S^3 in [6]. For general tori in $\mathbb{R}^3$ we have to define the spectral curves via Baker–Akhiezer functions ψ. We show how this is done for the Clifford torus in Theorem 1.

The Willmore conjecture reads that the global minimum is attained on the Clifford torus for which the spectral genus vanishes (as it is showed by Theorem 1). We think that

- given a conformal class of a torus, the Willmore functional attains its minima on tori with the minimal value of the spectral genus and the minimal value of the arithmetic genus of the spectral curve Γ_ψ defined via the Baker–Akhiezer function.

We shall discuss this conjecture elsewhere together with the expected relation between the spectral curves of tori in S^3 (as they are defined in [14]) and the spectral curves of their stereographic images in $\mathbb{R}^3$. For the Clifford torus the mapping

$$\mathbb{C}^* \to \mathbb{C}^* / \left\{ \frac{1+i}{4} \sim \frac{-1+i}{4}, -\frac{1+i}{4} \sim \frac{1-i}{4} \right\}$$

establishes an isomorphism of the corresponding spectral curves and the equivalence of the multipliers μ_1, μ_2. This was the basic idea of the computations in Section 4, i.e. in the proof of Theorem 1.

We summarize the contents of the paper:

in Section 2 we explain the geometry of Clifford torus in $\mathbb{R}^3$;

in Section 3 we expose the procedure for constructiing Dirac operators and their Baker–Akhiezer functions corresponding to singular spectral curves;

in Section 4 we prove Theorem 1.

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2 The Clifford torus

In fact, in differential geometry two objects are called the Clifford torus:

1) (a torus in S^3) the product of circles of the same radii which lies in the unit 3-sphere $S^3 \subset \mathbb{R}^4$ and therewith is defined by the equations:

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}$$

where $(x_1, \dots, x_4)$ are Euclidean coordinates in $\mathbb{R}^4$;

2) (a torus in $\mathbb{R}^3$) the following torus of revolution: take in the x_1x_3 plane a circle γ of radius $r = 1$ such that the distance between its center and the x_1 axis equals $R = \sqrt{2}$ and obtain a torus of revolution in $\mathbb{R}^3$ by rotating this circle γ around the x_1 axis.

The torus in S^3 is considered up to isometries of S^3 and in symplectic geometry it is considered as a torus in a symplectic 4-space $\mathbb{R}^4$ and is also called by this name.

The torus in $\mathbb{R}^3$ is considered up to conformal transformations of $\mathbb{R}^3$. In this event it is distinguished among tori of revolution by the ratio $R/r = \sqrt{2}$.

These tori are related to many conjectures in geometry:

- the Willmore functional defined on immersed surfaces in $\mathbb{R}^3$ by the formula
$$\mathcal{W}(M) = \int_M (H^2 - K) d\mu,$$
where $d\mu$ is the area form in the induced metric, H and K are the mean curvature and the Gaussian curvature, respectively, for tori is not less than $2\pi^2$ and attains its minimum on the Clifford torus (and its images under conformal transformations of $\mathbb{R}^3$) (the Willmore conjecture);
- the Clifford torus in S^3 is the unique (up to isometries) embedded minimal torus in S^3 (the Lawson conjecture);
- the area of any minimal torus in S^3 is not less than $2\pi^2$ which is the area of the Clifford torus;
- the Clifford torus in $S^3 \subset \mathbb{R}^4$ minimizes the Willmore functional
$$\mathcal{W}(M) = \frac{1}{4} \int_M |\mathbf{H}|^2 d\mu,$$
where $\mathbf{H}$ is the mean curvature vector, for tori in $\mathbb{R}^4$, or at least for Lagrangian tori in $\mathbb{R}^4$ with the standard symplectic structure;

- the Clifford torus in $S^3 \subset \mathbb{R}^4$ minimizes the area in its Hamiltonian isotopy class of Lagrangian tori in $\mathbb{R}^4$, i.e. among tori obtained from the Clifford torus by Hamiltonian deformations (the Oh conjecture).

The relation between two different notions of the Clifford torus and the corresponding conjectures is supplied by the stereographic projection F of S^3 onto $\mathbb{R}^3 \cup \{\infty\} = \bar{\mathbb{R}}^3$. For that project $S^3 \setminus \{(0,0,0,1)\}$ onto $\mathbb{R}^3 = \{x_4 = 0\}$ from the north pole $N = (0,0,0,1) \in S^3$ mapping N into $\infty \in \bar{\mathbb{R}}^3$. This mapping establishes the conformal equivalence of S^3 and $\bar{\mathbb{R}}^3$ and in coordinates takes the form

$$F(x_1,x_2,x_3,x_4) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right), \quad x_1^2+x_2^2+x_3^2+x_4^2 = 1.$$

It is known that for a minimal torus $T^2 \subset S^3$ its area coincides with the value of the Willmore functional on its stereographic image:

$$\mathrm{Area}(T^2) = \mathcal{W}(F(T^2)).$$

Therefore the third conjecture follows from the Willmore conjecture although The opposite is not true (these conjectures are not equivalent). Moreover the Lawson conjecture also implies the third conjecture (for expositions of the conjectures, some related results and their relations for tori in $\mathbb{R}^3$ see, for instance, [1, 11] and for tori in $\mathbb{R}^4$ see [9]).

Let us parameterize the Clifford torus in S^3 as follows:

$$x_1 = \frac{\cos x}{\sqrt{2}}, \ x_2 = \frac{\sin x}{\sqrt{2}}, \ x_3 = \frac{\cos y}{\sqrt{2}}, \ x_4 = \frac{\sin y}{\sqrt{2}}$$

where $0 \leq, x, y \leq 2\pi$. Then the Clifford torus in $\mathbb{R}^3$ is given by the formulas

$$r(x,y) = \left(\frac{\cos x}{\sqrt{2}-\sin y}, \frac{\sin x}{\sqrt{2}-\sin y}, \frac{\cos y}{\sqrt{2}-\sin y}\right). \tag{3}$$

It is easy to compute that x, y define a conformal parameter $z = x+iy$ on the torus in which the induced metric takes the form

$$ds^2 = \frac{dzd\bar{z}}{(\sqrt{2}-\sin y)^2},$$

the normal vector (assuming that the orientation on the surface is defined by the positively oriented frame (r_x, r_y)) equals

$$N = \left(\frac{\cos x(1-\sqrt{2}\sin y)}{\sqrt{2}-\sin y}, \frac{\sin x(1-\sqrt{2}\sin y)}{\sqrt{2}-\sin y}, \frac{-\cos y}{\sqrt{2}-\sin y}\right),$$

the second fundamental form is

$$\frac{(\sqrt{2}\sin y - 1)}{(\sqrt{2}-\sin y)^2}dx^2 + \frac{1}{(\sqrt{2}-\sin y)^2}dy^2,$$

the principal curvatures take very simple forms:

$$\kappa_1 = 1, \quad \kappa_2 = \sqrt{2}\sin y - 1,$$

and therefore the mean curvature equals

$$H = \frac{\sin y}{\sqrt{2}}.$$

The potential of the Weierstrass representation is

$$U = \frac{\sin y}{2\sqrt{2}(\sqrt{2}-\sin y)}. \tag{4}$$

Other important functions related to this representation are

$$\frac{\partial}{\partial z}(x_1 + ix_2) = i\bar{\psi}_2^2, \quad \frac{\partial x_3}{\partial z} = \psi_1\bar{\psi}_2$$

and for the Clifford torus they are equal to

$$\begin{aligned} \frac{\partial}{\partial z}(x_1+ix_2) = \frac{\partial}{\partial z}\frac{\cos x + i\sin x}{\sqrt{2}-\sin y} &= \frac{i}{2}e^{ix}\frac{\sqrt{2}-\cos y - \sin y}{(\sqrt{2}-\sin y)^2}, \\ \frac{\partial x_3}{\partial z} = \frac{\partial}{\partial z}\frac{\cos y}{\sqrt{2}-\sin y} &= \frac{i}{2}\frac{\sqrt{2}\sin y - 1}{(\sqrt{2}-\sin y)^2}. \end{aligned} \tag{5}$$

3 Baker–Akhiezer functions and Dirac operators

Let us recall the definition of the Baker–Akhiezer (vector) function ψ corresponding to the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

By definition it depends on a complex variable $z \in \mathbb{C}$ and on a parameter P on a complex curve Γ of finite arithmetic genus $g = p_a(\Gamma)$ and meets the following conditions:

1) ψ is meromorphic in P outside a couple of marked points $\infty_\pm \in \Gamma$ and has poles at $g+1$ points $P_1+\dots+P_{g+1}$ (the points $\infty_+, \infty_-, P_1, \dots, P_{g+1}$ are nonsingular);

2) ψ has the following asymptotics near $\infty_\pm$:

$$\psi \approx e^{k_+ z}\left[\left(\begin{array}{c} 1 \\ 0 \end{array}\right) + \left(\begin{array}{c} \xi_1^+ \\ \xi_2^+ \end{array}\right) k_+^{-1} + O(k_+^{-2})\right] \quad \text{as } P \to \infty_+,$$

$$\psi \approx e^{k_- \bar{z}}\left[\left(\begin{array}{c} 0 \\ 1 \end{array}\right) + \left(\begin{array}{c} \xi_1^- \\ \xi_2^- \end{array}\right) k_-^{-1} + O(k_-^{-2})\right] \quad \text{as } P \to \infty_-,$$

where $k_\pm^{-1}$ are local parameters near infinities such that

$$k_\pm^{-1}(\infty_\pm) = 0.$$

It follows from the general theory of Baker–Akhiezer functions ([8, 3]) that the such function is unique (for a generic divisor $D = P_1 + \dots + P_{g+1}$) and therefore we can find an operator $\mathcal{D}$ with potentials U and V such that $\mathcal{D}\psi$ is meromorphic with analogous singularities but with the asymptotics

$$\mathcal{D}\psi = e^{k_+ z} O(k_+^{-1}) \ \text{ as } P \to \infty_+, \quad \mathcal{D}\psi = e^{k_- \bar{z}} O(k_-^{-1}) \ \text{ as } P \to \infty_-.$$

The uniqueness of ψ implies that $\mathcal{D}\psi = 0$. We have

Theorem 2 ([12, 13]) *The Baker–Akhiezer function ψ satisfies the Dirac equation*

$$\mathcal{D}\psi = 0 \quad \textit{with} \quad U = -\xi_2^+, \quad V = \xi_1^-.$$

If the curve Γ is singular we assume that it has some special form. Let us now define what it is this form and recall some algebro-geometric properties of such singular curves following [10].

Let $\Gamma_{\rm nm}$ be a nonsingular complex curve. Take on $\Gamma_{\rm nm}$ effective divisors $D_1, \dots, D_n$, i.e. for any $k = 1, \dots, k$ such a divisor D_k is a formal sum of finitely many points on $\Gamma_{\rm nm}$ with positive coefficients:

$$D_k = a_{k1} Q_{k1} + \dots + a_{km_k} Q_{km_k},$$

$$Q_{kj} \in \Gamma_{\rm nm}, a_{kj} > 0, a_{kj} \in \mathbb{Z}, j = 1, \dots, m_k, k = 1, \dots, n.$$

We assume that the supports $\mathrm{supp} D_k = Q_{k1} + \dots + Q_{km_k}$, $k = 1, \dots, n$, of these divisors are pairwise nonintersecting. Then we denote by

$$\Gamma_{D_1, \dots, D_n}$$

the singular curve obtained by contracting all points from each divisor D_k into one singular point $S_k \in \Gamma$. This is done with respect to the multiplicities. This means that a function f which is meromorphic on Γ is, by definition, a meromorphic function $f : \Gamma_{\rm nm} \to \mathbb{C}$ such that it has no poles at $\cup_k {\rm supp} D_k$, and for each divisor

$$D_k = a_{k1}Q_{k1} + \dots + a_{km_k}Q_{km_k}$$

we have

$$f(Q_{k1}) = \dots = f(Q_{km_k}), \quad \partial^m f(Q_{kl}) = 0 \ \text{ for } l = 1, \dots, (a_{kl} - 1),$$

where $k = 1, \dots .n$. Here ∂ is the Cauchy derivation with respect to a complex parameter on Γ.

The natural projection

$$\pi : \Gamma_{\rm nm} \to \Gamma$$

is the normalization mapping.

If Γ is singular we assume that it takes the form

$$\Gamma = \Gamma_{D_1,\dots,D_n}.$$

Let $T \subset \Gamma_{D_1,\dots,D_n}$ and T is empty or contains only nonsingular points. Recall that a one-form ω is called a regular form on $\Gamma_{D_1,\dots,D_n} \backslash T$ if ω is a meromorphic form on $\Gamma_{\rm nm} \backslash \pi^{-1}(T)$ which may have poles only at the supports of $D_1, \dots, D_k$ and at each point Q_{kl} the degree of the pole is not greater than a_{kl} and moreover for every k the inequality

$$\sum_l {\rm Res} f\omega(Q_{kl}) = 0$$

which holds for all meromorphic functions on Γ. For a nonsingular curve Γ the notions of regular and holomorphic forms coincide.

For nonsingular curves the values of the arithmetic genus p_a and the geometric genus p_g coincide. For a singular curve $\Gamma = \Gamma_{D_1,\dots,D_k}$ we have

$$p_g(\Gamma) = p_g(\Gamma_{\rm nm}), \quad p_a(\Gamma) = p_g(\Gamma) + \sum_k (\deg D_k - 1)$$

where the degree of the divisor D_k equals to

$$\deg D_k = \deg(a_{k1}Q_{k1} + \dots + a_{km_k}Q_{km_k}) = a_{k1} + \dots + a_{km_k}.$$

We say that $\sigma : \Gamma \to \Gamma$ is an involution of Γ if $\sigma^1 = 1$ and its pullback defines an involution on $\Gamma_{\rm nm}$ such that the singularity divisors $D_1, \dots, D_n$ falls into two groups:

1) divisors which are preserved: $D_j = \sigma(D_j)$;

2) divisors which are interchanges with others: $D_j = \sigma(D_k), j \neq k$.

The first group corresponds to fixed points $S_j = \pi(D_j)$ of an involution on Γ although the points from D_j could be permuted.

Theorem 3 ([12, 13]) *1) Let σ be a holomorphic involution $\sigma : \Gamma \to \Gamma$ such that*

$$\sigma(\infty_\pm) = \infty_\pm, \quad \sigma(k_\pm) = -k_\pm$$

under which the singular points $S_1 = \pi(D_1), \dots, S_k = \pi(D_k)$ are fixed and other singular points are no fixed.

If there exists a meromorphic differential ω on $\Gamma_{2D_1,\dots,2D_k,D_{k+1},\dots,D_n}$ such that ω has two poles at $\infty_\pm$ with the principal parts $\pm k_\pm^2(1 + O(k_\pm^{-1}))dk_\pm^{-1}$, it is regular outside $\infty_\pm$ and it has zeros at $D + \sigma(D)$ then the potentials U and V coincide: $U = V$.

2) Let τ be an antiholomorphic involution $\tau : \Gamma \to \Gamma$ such that

$$\tau(\infty_\pm) = \infty_\mp, \quad \tau(k_\pm) = -\bar{k}_\mp$$

under which the singular points $S'_1 = \pi(D'_1), \dots, S'_l = \pi(D'_l)$ are fixed and other singular points are no fixed (here $(D'_1, \dots, D'_k)$ is obtained by a permutation of the set $(D_1, \dots, D_n)$).

If there exists a meromorphic differential ω' on $\Gamma_{2D'_1,\dots,2D'_l,D'_{l+1},\dots,D'_n}$ such that it has two poles at $\infty_\pm$ with the principal parts $k_\pm^2(1 + O(k_\pm^{-1}))dk_\pm^{-1}$, it is regular outside $\infty_\pm$ and it has zeros at $D + \tau(D)$, then the potentials U and V are real-valued: $U = \bar{U},\ V = \bar{V}$.

Proof. We proved this theorem for smooth curves in [12] and for singular curves for which all singular points are fixed by involution in [13] (here we correct some inaccuracy and typos in the conditions on the antiholomorphic involution). In general, the proof is the same for all these cases: we have to compute the residues of the following differentials with poles only in ∞_+ and ∞_- and remember that their sum taken over all poles vanishes:

1) the sum of the residues of the differential $\psi_1(P)\psi_2(\sigma(P))\omega$ equals $-2\pi i(\xi_2^+ + \xi_1^-) = 0$ which implies that $U = V$;

2) the sums of the residues of the differentials $\psi_1(P)\overline{\psi(\tau(P))}\omega'$ and $\psi_2\overline{\psi_2(\tau(P))}\omega'$ are equal to $2\pi i(\xi_1^- - \bar{\xi}_1^-)$ and $2\pi i(\xi_2^+ - \bar{\xi}_2^+$ respectively which implies that $U = \bar{U}$ and $V = \bar{V}$.

This proves the theorem.

There is a general procedure for constructing Baker–Akhiezer functions corresponding to different operators and smooth curves Γ [8]. In [12] we applied it to the Dirac operator and derived the explicit formulas for ψ and the potentials U and V in terms of the theta function of Γ.

For $\Gamma = \Gamma_{D_1,\dots,D_n}$ we have to do the following:

- take a divisor $D = P_1 + \dots + P_s$ of degree $s = p_a(\Gamma) + 1 = p_g(\Gamma_{\mathrm{nm}}) + \sum_k (\deg D_k - 1) + 1$;
- for each divisor $D'_j = P_1 + \dots + P_g + P_{g+j}$ where $g = p_g(\Gamma) = p_a(\Gamma_{\mathrm{nm}})$ and $j = 1, \dots, r = \sum_k (\deg D_k - 1)$ construct the Baker-Akhiezer φ_j on Γ_{nm} with D'_j as the divisor of poles (as it was done in [12]);
- construct the function ψ in the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} q_1\varphi_{1,1} + \dots + q_r\varphi_{r,1} \\ t_1\varphi_{1,2} + \dots + t_r\varphi_{r,2} \end{pmatrix}$$

  by solving the following equations on q_i, t_j:

$$\psi_\alpha(Q_{k1}) = \dots = \psi_\alpha(Q_{km_k}), \quad \partial^m \psi_j(Q_{kl}) = 0,$$

$$\sum_i q_i = \sum_j t_j = 1$$

  where $k = 1, \dots, n$, $\alpha = 1, 2$, $l = 1, \dots, m_k$, and $m = 1, \dots, (a_{kl} - 1)$.

Notice that we have exactly $2r$ equations on the same number parameters. The first two systems means that ψ is defined on Γ and the latter two equations implies the correct asymptotics of ψ at the infinities. The solutions q_i, t_j depend on the spatial parameter z as well as on the parameter on Γ.

Remark that complex curves with more general singularities were considered in [4] as the spectral curves of the operator $L = i\partial_y - \partial_x^2 + u$. However we do not know do such singularities appear on the spectral curve of a Dirac operator and if they do what have to be analogs of the conditions on the involutions σ and τ. In our case such conditions involve not only the spectral curve $\Gamma_{D_1,\dots,D_n}$ but also another curve $\Gamma_{2D_1,\dots,2D_k,D_{k+1},\dots,D_n}$.

4 The spectral curve of the Clifford torus

In this section we prove Theorem 1.

Let $\psi(z,\bar z,\lambda)$ be a function on $\mathbb{C}P^1$ which is meromorphic in $\lambda \in \mathbb{C} \subset \mathbb{C}P^1$ on $\mathbb{C}P \setminus \{0,\infty\}$, has a pole at the point $\lambda = p$ and has the following asymptotics at two "infinities":

$$\psi \approx \begin{pmatrix} e^{k_+ z} \\ 0 \end{pmatrix} \text{ as } k_+ = \lambda \to \infty; \quad \psi \approx \begin{pmatrix} 0 \\ e^{k_- \bar z} \end{pmatrix} \text{ as } k_- = -\frac{|u|^2}{\lambda} \to \infty.$$

This means that it is a Baker–Akhiezer (vector) function corresponding to the data

$$\Gamma_u = \mathbb{C}P^1, D = p, \infty_\pm, k_\pm.$$

It is known from the general theory that such a function is unique and we easily compute it obtaining the following formula:

$$\psi = \frac{\lambda}{\lambda - p}\, e^{\lambda z - \frac{|u|^2}{\lambda}\bar z} \begin{pmatrix} 1 \\ -\frac{p}{\lambda} \end{pmatrix}.$$

The function ψ satisfies the Dirac equation

$$\mathcal{D}\psi = 0$$

where the potentials of the Dirac operator $\mathcal{D}$ are

$$U = p, \quad V = \frac{|u|^2}{p}.$$

If $p = u$ we have the operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & \bar u \end{pmatrix}.$$

We have two involutions on Γ_u: a holomorphic involution σ and an antiholomorphic involution τ which act as follows

$$\sigma(\lambda) = \lambda, \quad \tau(\lambda) = \frac{|u|^2}{\bar\lambda}.$$

We shall look for the spectral data of the Clifford torus in the following form:

- the normalized spectral curve Γ_{nm} is $\mathbb{C}P^1$ with two infinities $\infty_\pm$;

- the actions of σ and τ descend to involutions on Γ;
- $u = \frac{1+i}{4}$.

Consider the following singular curve:

$$\Gamma = \mathbb{C}P^1/\{\pm u \sim \mp \bar{u}\}, \quad u = \frac{1+i}{4},$$

i.e. it is obtained from $\mathbb{C}P^1$ by gluing u with $-\bar{u}$ and $-u$ with $\bar{u}$. We see that Γ is a sphere with a pair of double points. We have

$$p_g(\Gamma) = 0, \quad p_a(\Gamma) = 2.$$

Therefore we look for the Baker–Akhiezer function on Γ in the form

$$\psi_1(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}} \left(q_1 \frac{\lambda}{\lambda - p_1} + q_2 \frac{\lambda}{\lambda - p_2} + (1 - q_1 - q_2)\frac{\lambda}{\lambda - p_3} \right),$$

$$\psi_2(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}} \left(t_1 \frac{p_1}{p_1 - \lambda} + t_2 \frac{p_2}{p_2 - \lambda} + (1 - t_1 - t_2)\frac{p_3}{p_3 - \lambda} \right)$$

where p_1, p_2, p_3 are the poles of ψ and q_1, q_2, t_1, and t_2 are functions of $z, \bar{z}$ obtained from the conditions:

$$\psi(z,\bar{z},u) = \psi(z,\bar{z},-\bar{u}), \quad \psi(z,\bar{z},-u) = \psi(z,\bar{z},\bar{u}).$$

This leads to the following equations for q_1 and q_2:

$$q_1 \left[e^{uz - \bar{u}\bar{z}} u \left(\frac{1}{u - p_1} - \frac{1}{u - p_3} \right) - e^{u\bar{z} - \bar{u}z} \bar{u} \left(\frac{1}{\bar{u} + p_1} - \frac{1}{\bar{u} + p_3} \right) \right] +$$
$$+ q_2 \left[e^{uz - \bar{u}\bar{z}} u \left(\frac{1}{u - p_2} - \frac{1}{u - p_3} \right) - e^{u\bar{z} - \bar{u}z} \bar{u} \left(\frac{1}{\bar{u} + p_2} - \frac{1}{\bar{u} + p_3} \right) \right] =$$
$$= -e^{uz - \bar{u}\bar{z}} \frac{u}{u - p_3} + e^{u\bar{z} - \bar{u}z} \frac{\bar{u}}{\bar{u} + p_3},$$
$$q_1 \left[e^{\bar{u}z - u\bar{z}} \bar{u} \left(\frac{1}{\bar{u} - p_1} - \frac{1}{\bar{u} - p_3} \right) - e^{\bar{u}\bar{z} - uz} u \left(\frac{1}{u + p_1} - \frac{1}{u + p_3} \right) \right] +$$
$$+ q_2 \left[e^{\bar{u}z - u\bar{z}} \bar{u} \left(\frac{1}{\bar{u} - p_2} - \frac{1}{\bar{u} - p_3} \right) - e^{\bar{u}\bar{z} - uz} u \left(\frac{1}{u + p_2} - \frac{1}{u + p_3} \right) \right] =$$
$$= e^{\bar{u}\bar{z} - uz} \frac{u}{u + p_3} - e^{\bar{u}z - u\bar{z}} \frac{\bar{u}}{\bar{u} - p_3}.$$

The formula for the potential $V = \xi_1^-$ takes the form

$$V = |u|^2 \left(\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{1 - q_1 - q_2}{p_3} \right).$$

We can also compute t_1 and t_2 solving analogous equations. The formula for the potential U is

$$U = t_1 p_1 + t_2 p_2 + (1 - t_1 - t_2) p_3. \tag{6}$$

In both cases we have to know the points p_1, p_2, p_3 to make computations. It follows from Theorem 3 that if these points are chosen in a rather convenient way then the potentials U and V are real-valued and moreover coincide $U = V$. This is the case when there are meromorphic differentials ω and ω' with properties exposed in Theorem 3.

Let us look for such differentials.

The general form of a differential ω on Γ which has poles at $\infty_\pm$ with the principal parts $\pm k_\pm^2 (1 + O(k_\pm^{-1})) dk_\pm^{-1}$ and is regular outside these points is

$$\omega = -\left[\frac{\lambda^2 - |u|^2}{\lambda^2} + a \left(\frac{1}{\lambda - u} - \frac{1}{\lambda + \bar{u}} \right) + b \left(\frac{1}{\lambda + u} - \frac{1}{\lambda - \bar{u}} \right) \right] d\lambda$$

and the general form of a differential ω' on Γ which has poles at $\infty_\pm$ with the principal parts $k_\pm^2 (1 + O(k_\pm^{-1})) dk_\pm^{-1}$ and is regular outside these points is

$$\omega' = -\left[\frac{\lambda^2 + |u|^2}{\lambda^2} + c \left(\frac{1}{\lambda - u} - \frac{1}{\lambda + \bar{u}} \right) + d \left(\frac{1}{\lambda + u} - \frac{1}{\lambda - \bar{u}} \right) \right] d\lambda.$$

We have to find differentials ω and ω' such that their zero sets would be of the form $D + \sigma(D)$ and $D + \tau(D)$ respectively.

Notice that the zero sets of ω and ω' are the zero sets of the polynomials $Q(\lambda)$ and $Q'(\lambda)$ respectively where

$$Q(\lambda) = (\lambda^2 - |u|^2)(\lambda^2 - u^2)(\lambda^2 - \bar{u}^2) +$$

$$+ \lambda^2 (u + \bar{u})[(a - b)(\lambda^2 - |u|^2) + (a + b)(u - \bar{u})\lambda],$$

$$Q'(\lambda) = (\lambda^2 + |u|^2)(\lambda^2 - u^2)(\lambda^2 - \bar{u}^2) +$$

$$+ \lambda^2 (u + \bar{u})[(c - d)(\lambda^2 - |u|^2) + (c + d)(u - \bar{u})\lambda].$$

Since we look for a differential ω with the zero set of the form $D+\sigma(D)$ the polynomial $Q(\lambda)$ has to have only terms with even powers. This implies that $b=-a$ and we have

$$Q(\lambda)=(\lambda^2-|u|^2)[(\lambda^2-u^2)(\lambda^2-\bar{u}^2)+2a(u+\bar{u})\lambda^2].$$

We see that $Q(|u|)=0$ and put

$$p_3=|u|.$$

Other poles p_1 and p_2 satisfy the equation

$$(\lambda^2-u^2)(\lambda^2-\bar{u}^2)+2a(u+\bar{u})\lambda^2=0$$

together with $-p_1,-p_2$. Since $\tau(p_3)=p_3$, the point $p_3=|u|$ has to be at least a double root of the polynomial $Q'(\lambda)$.

We skip some simple calculations which show that if we assume that $c=d$ and that $(\lambda-|u|)^2$ divides $Q'(\lambda)$ then we have

$$Q'(\lambda)=(\lambda-|u|)^2\left[\lambda^4+2|u|\lambda^3+(4|u|^2-(u^2+\bar{u}^2))\lambda^2+2|u|^3\lambda+|u|^4\right]$$

and

$$c=d=\frac{(u^2+\bar{u}^2)|u|-2|u|^3}{u^2-\bar{u}^2}.$$

We have

$$Q'(\lambda)=(\lambda-|u|)^2|u|^4P\left(\frac{\lambda}{|u|}\right),$$

where

$$P(\mu)=(\mu^2-A\mu+1)(\mu^2-\bar{A}\mu+1),$$
$$A+\bar{A}=-2,\quad |A|^2=2-\frac{u^2+\bar{u}^2}{|u|^2}.$$

and

$$A=t+t^{-1},\quad \bar{A}=\bar{t}+\bar{t}^{-1},$$

where $t,\bar{t},t^{-1}$ and $\bar{t}^{-1}$ are the roots of the polynomial $P(\mu)$.

Until now we did all computations for a general value of u. let us consider the special case when

$$u=\frac{1+i}{4},\quad u^2+\bar{u}^2=0.$$

The polynomial $P(\mu)$ takes a very simple form and we can easily find all its roots:

$$P(\mu)=\mu^4+2\mu^3+4\mu^2+2\mu+1,$$

$$\mu_{1,2} = \frac{-(1-i) \pm \sqrt{-2i-4}}{2}, \quad \mu_{3,4} = \bar{\mu}_{1,2} = \frac{-(1+i) \pm \sqrt{-2i-4}}{2}$$

and we obtain four other zeros of the differential ω':

$$s_1 = \frac{-1+i+\sqrt{-2i-4}}{4\sqrt{2}}, \quad s_2 = \frac{-1+i-\sqrt{-2i-4}}{4\sqrt{2}},$$

$$\tau(s_1) = \frac{-1-i-\sqrt{2i-4}}{4\sqrt{2}}, \quad \tau(s_2) = \frac{-1-i-\sqrt{2i-4}}{4\sqrt{2}}.$$

Therefore for $c = d = \frac{i}{\sqrt{8}}$ the differential ω' has zeroes exactly at $D + \tau(D)$ where $D = s_1 + s_2 + p_3$ where $p_3 = |u|$ and $\tau(p_3) = p_3$.

Notice that the zeros of ω have to be invariant with respect to the involution σ and their product equals $|u|^6$. This implies that we can put either $p_1 = s_1, p_2 = s_2$ either $p_1 = \tau(s_1), p_2 = \tau(s_2)$.

Let us make our choice as follows:

$$p_1 = \frac{-1+i+\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_2 = \frac{-1+i-\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_3 = |u| = \frac{1}{\sqrt{8}}.$$

In this event $a = -b = \frac{1+i}{\sqrt{8}}$.

Substituting these values of $p_k, k = 1, 2, 3$, into the equations for q_1 and q_2 we compute them as functions of x, y and by that obtain the explicit formula for ψ_1 and V. Now the functions ψ_2 and U can be found by solving analogous equations for t_1, t_2 or from the following formulas which follows from Theorem 3:

$$U = V, \quad \psi_2 = \frac{\bar{\partial}\psi_1}{V}.$$

We omit the explicit formulas for q_1, q_2, t_1, t_2 and just mention that these functions appear to be functions of y which are 2π-periodic. The ψ-function at the point $\lambda = u$ equals:

$$\psi_1(z, \bar{z}, \bar{u}) = \frac{1+i}{4} e^{-\frac{ix}{2}} \frac{2e^{\frac{iy}{2}} + \sqrt{2}(1-i)e^{-\frac{iy}{2}}}{\sin y - \sqrt{2}},$$

$$\psi_2(z, \bar{z}, \bar{u}) = -\frac{\sqrt{2}}{4} e^{-\frac{ix}{2}} \frac{2e^{\frac{iy}{2}} - \sqrt{2}(1+i)e^{-\frac{iy}{2}}}{\sin y - \sqrt{2}}.$$

We see that

$$i\bar{\psi}_2^2 = -\frac{(1+i)e^{ix}}{2} \frac{\sqrt{2} - \cos y - \sin y}{(\sin y - \sqrt{2})^2},$$

$$\psi_1 \bar{\psi}_2 = \frac{\sqrt{2}i}{2} \frac{1-\sqrt{2}\sin y}{(\sin y - \sqrt{2})^2}$$

and comparing these formulas with the formulas (5) we conclude that a surface constructed from the function $\frac{1}{\sqrt[4]{2}}\psi(z,\bar{z},\bar{u})$ by the Weierstrass representation is a torus which is mapped to the Clifford torus defined by the formulas (3) by the orthogonal transformation

$$T = \left(\begin{array}{ccc} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{array} \right).$$

This transformation changes the orientation of $\mathbb{R}^3$ which results in the change of the sign of the potential U: the formula (6) gives us

$$U = \frac{\sin y}{2\sqrt{2}(\sin y - \sqrt{2})}$$

and this expression differs by the sign from (4).

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