SMOOTH FRÉCHET GLOBALIZATIONS OF HARISH-CHANDRA MODULES

JOSEPH BERNSTEIN AND BERNHARD KRÖTZ

Date: December 14, 2008.

1. **Introduction**

Let G be a linear reductive real Lie G with Lie algebra $\mathfrak{g}$. Let us fix a maximal compact subgroup K of G. The representation theory of G admits an algebraic underpinning encoded in the notion of a *Harish-Chandra module.*

By a Harish-Chandra module we shall understand a finitely generated $(\mathfrak{g}, K)$-module with finite K-multiplicities. Let us denote by $\mathcal{HC}$ the category whose objects are Harish-Chandra modules and whose morphisms are linear $(\mathfrak{g}, K)$-maps. By a *globalization* of a Harish-Chandra module V we understand a representation (π, E) of G such that the K-finite vectors of E are isomorphic to V as a $(\mathfrak{g}, K)$-module.

Let us denote by $\mathcal{SAF}$ the category whose objects are smooth admissible Fréchet representations of G with continuous linear G-maps as morphisms. We consider the functor:

$$\mathcal{F} : \mathcal{SAF} \to \mathcal{HC}, \quad E \mapsto E_K := \{K - \text{finite vectors of E}\}\,.$$

The Casselman-Wallach theorem ([3] or [9], Sect. 11) asserts that $\mathcal{F}$ is an equivalence of categories. To phrase it differently, each Harish-Chandra module V admits a unique smooth Fréchet globalization (π, V^∞). Moreover,

$$V^\infty = \pi(\mathcal{S}(G))V$$

where $\mathcal{S}(G)$ is the Schwartz-algebra of rapidly decreasing functions on G, and $\pi(\mathcal{S}(G))V$ stands for the vector space spanned by $\pi(f)v$ for $f \in \mathcal{S}(G)$, $v \in V$.

One objective of this paper is to give an elementary proof of this fact. Our strategy goes as follows. We first consider spherical principal series representations of G with their canonical Hilbert-globalizations as subspaces of $L^2(K)$. For such representations we define a Dirac-type sequence and establish uniform lower bounds for K-finite matrix coefficients (see Theorem 4.5 below). The Casselman-Wallach theorem for these type of represesentations is an immediate consequence. The case of arbitrary Harish-Chandra modules will be reduced to this case.

We wish to emphasize that our lower bounds are locally uniform in representation parameters which allows us to prove a version of the Casselman-Wallach theorem with holomorphic dependance on representation parameters (see Section 7). This for instance is useful for the theory of Eisenstein series.

Acknowledgment: We wish to thank Birgit Speh and Henrik Schlichtkrull for several very useful discussions.

2. Basic representation theory

In this section we collect some basic notions of representation theory.

2.1. Representations on topological vector spaces

Throughout this whole article topological vector spaces are understood to be Hausdorff, locally convex and complete.

Let G be a Lie group and E a topological vector space. By a *representation* of G on E we understand a homomorphism $\pi : G \to GL(E)$ such that the resulting action $G \times E \to E$ is continuous. For an element $v \in E$ we shall denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v$$

the corresponding orbit map.

If E is a Banach, resp. Hilbertian, space then we speak of a Banach, resp. Hilbertian, representation of G. We call (π, E) a *Fréchet representation* if there exists a countable family of semi-norms $(\rho_n)_{n\in\mathbb{N}}$ which define the topology of E and such that for all $n \in \mathbb{N}$ the action $G \times (E, \rho_n) \to (E, \rho_n)$ is continuous.

Remark 2.1. *(a) If (π, E) is a Fréchet representation, then E is a Fréchet space as E is required to be complete and the the topology defining family $(\rho_n)_{n\in\mathbb{N}}$ is countable.*

(b) In the literature one sometimes encounters the notion Fréchet representation for a continuous action on a Fréchet space. This notion is weaker as our notion. In the Appendix we will show that that our notion of Fréchet representation is equivalent to the notion of moderate growth in [3]*, p. 391.*

If (π, E) is a Fréchet representation, then we call a semi-norm ρ on E a *continuous semi-norm*, if $G \times (E, \rho) \to (E, \rho)$ is continuous.

Let (π, E) be a representation of G. We call a vector $v \in E$ *smooth* if γ_v is a smooth map and denote by E^∞ the space of all smooth vectors. Note that $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G, acts naturally on E^∞. We topologize E^∞ as follows: If $(p_i)_{i\in I}$ denotes a family of semi-norms which define the topology of E and if $u_1, u_2, \ldots$ is a basis for $\mathcal{U}(\mathfrak{g})$, then

$$p_{i,n}(v) := p_i(d\pi(u_n)v) \qquad (i \in I, n \in \mathbb{N}, v \in E^\infty)$$

is a family of semi-norms which turn E^∞ into a locally convex, Hausdorff complete vector space. As a result π induces a representation on E^∞.

We call a representation (π, E) *smooth* if $E = E^\infty$ as topological vector spaces.

Example 2.2. *Suppose that (π, E) is a Banach representation. Then E^∞ is a Fréchet space and (π, E^∞) is a smooth Fréchet representation of G.*

2.2. Integration of representations

We fix a left Haar measure dg on G and write $L^1(G)$ for the Banach space of integrable functions on G. Note that $L^1(G)$ is a Banach algebra with multiplication given by convolution:

$$f * h(x) = \int_G f(g)h(g^{-1}x)\ dg \qquad (x \in G)$$

for $f, h \in L^1(G)$. We write $C_c^\infty(G) < L^1(G)$ for the subalgebra of test function on G.

If (π, E) is representation of G then we denote by by Π the corresponding algebra representation of $C_c^\infty(G)$:

$$\Pi(f)v = \int_G f(g)\pi(g)v\ dg \qquad (f \in C_c^\infty(G), v \in E)\,.$$

Note that the defining vector valued integral actually converges as E is complete.

Depending on the type of the representation (π, E) larger algebras as $C_c^\infty(G)$ might act on E. For instance if (π, E) is a bounded Banach representation, then Π extends to a representation of $L^1(G)$. The natural algebra acting on a Fréchet representation is the algebra of rapidly decreasing functions $\mathcal{R}(G)$ and the natural algebra acting on a smooth Fréchet representation is the Schwartz algebra $\mathcal{S}(G)$.

In order to define $\mathcal{R}(G)$ and $\mathcal{S}(G)$ we need the notion of a norm on G, see [8], Sect. 2.A.2. From now on we assume that G is a linear reductive group. We fix a faithful representation $\iota : G \to \mathrm{Gl}(n, \mathbb{R})$ and define a *norm* on G by:

$$\|g\| := \max\{\|\iota(g)\|, \|\iota(g^{-1})\|\}\,.$$

The norm satisfies the following properties:

- $\|g\| \geq 1$ for all $g \in G$.
- $\|g\| = \|g^{-1}\|$ for all $g \in G$.
- $\|g_1 g_2\| \leq \|g_1\| \cdot \|g_2\|$ for all $g_1, g_2 \in G$.
- $\{g \in G \mid \|g\| \leq r\}$ is a compact subset of G for all $r \geq 0$.

- $\| \exp(tX)\| = \| \exp(X)\|^t$ for all semi-simple elements $X \in \mathfrak{g}$ and $t \geq 0$.

Let us emphasize the dependence of $\| \cdot \|$ on the chosen embedding ι: if $\iota' : G \to \mathrm{Gl}(n', \mathbb{R})$ is another faithful realization and $\| \cdot \|'$ the corresponding norm, then there exists $r_1, r_2 > 0$ such that $\| \cdot \|^{r_1} \leq \| \cdot \|' \leq \| \cdot \|^{r_2}$.

Having the notion of a norm on G we define the space of *rapidly decreasing functions* by

$$\mathcal{R}(G) = \{ f \in C(G) \mid \forall n \in \mathbb{N} \ \sup_{g \in G} \|g\|^n |f(g)| < \infty \} .$$

Let us point out that $\mathcal{R}(G)$ is a Fréchet subalgebra of $L^1(G)$ which is independent of the choice of the particular norm on G.

We write $L \times R$ for the regular representation of $G \times G$ functions on G:

$$(L \times R)(g_1, g_2) f(g) := f(g_1^{-1} g g_2)$$

for $g, g_1, g_2 \in G$ and $f \in C(G)$. For $u \in \mathcal{U}(\mathfrak{g})$ we will abbreviate $L_u := dL(u)$ and likewise R_u for the derived representations.

We note that $(L \times R, \mathcal{R}(G))$ is a Fréchet representation of $G \times G$ whose smooth vectors constitute the *Schwartz space*

$$\mathcal{S}(G) := \{ f \in C^\infty(G) \mid \forall u, v \in \mathcal{U}(\mathfrak{g}), \forall n \in \mathbb{N} \\ \sup_{g \in G} \|g\|^n |L_u R_v f(g)| < \infty \} .$$

It is clear that $\mathcal{S}(G)$ is a Fréchet subalgebra of $\mathcal{R}(G)$ (see [8], Sect. 7.1 for a discussion in a wider context).

Remark 2.3. *For a function $f \in \mathcal{R}(G)$ the following assertions are equivalent: (1) f is $\mathcal{S}(G)$, i.e. f is $L \times R$-smooth; (2) f is R-smooth; (3) f is L-smooth. In fact, a left derivative L_u at a point $g \in G$ is the same as a right derivative $R_{\mathrm{Ad}(g)^{-1} u}$ at g. Now observe that $\| \mathrm{Ad}(g)\| \leq \|g\|^r$ for all $g \in G$ and a fixed $r > 0$.*

Finally let us explain how $\mathcal{R}(G)$ acts on Fréchet representations. First observe that the Banach-Steinhaus theorem implies for a Banach representation (π, E) that $\|\pi(g)\|$ is locally bounded. This, together with the sub-multiplicativity of the norm on G, yields the existence of a constant $r > 0$ such that $\|\pi(g)\| \leq \|g\|^r$ for all $g \in G$ (cf. [8], Lemma 2.A.2.2). As a consequence we obtain that $\mathcal{R}(G)$ acts naturally on all Fréchet representations. Likewise one obtains that $\mathcal{S}(G)$ acts on all smooth Fréchet representations.

Remark 2.4. *(a) If E is a smooth representation, then one has*

$$\Pi(\mathcal{R}(G))E = \Pi(\mathcal{S}(G))E = E\,.$$

In fact, by Dixmier-Malliavin [4] *one has $\Pi(C_c^\infty(G))E = E$. With $\mathcal{S}(G) \supset C_c^\infty(G)$ and $\mathcal{R}(G) * C_c^\infty(G) \subset \mathcal{S}(G)$ the assertions follow.*
(b) Let V be a Harish-Chandra module and (π, E) a Banach globalization. Then

$$\Pi(\mathcal{R}(G))V = \Pi(\mathcal{S}(G))V\,.$$

In order to see that we use a result of Harish-Chandra which asserts that for each $v \in V$ there exists a $K \times K$-finite $h \in C_c^\infty(G)$ such that $\Pi(h)v = v$. As $\mathcal{R}(G) * C_c^\infty(G) \subset \mathcal{S}(G)$ the asserted equality is established.

2.3. **Harish-Chandra modules**

Let us fix a maximal compact subgroup K of G. It is no loss of generality to assume that the norm $\|\cdot\|$ on G is $K\times K$-invariant. Likewise we may request that all continuous semi-norms on the considered G-modules E are K-invariant.

We call a representation (π, E) of G *admissible* if for all irreducible representations (τ, W) of K the multiplicity space $\mathrm{Hom}_K(W, E)$ is finite dimensional.

By a $(\mathfrak{g}, K)$-module V we understand a module for $\mathfrak{g}$ and K such that:

- The actions are compatible, i.e.

  $$k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v$$

  for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V$.
- The K-action is algebraic, i.e. V is a union of finite dimensional algebraic K-modules.

Note that if f (π, E) is an admissible Banach representation of G, then the space of K-finite vectors of E, say E_K, consists of smooth vectors and is stable under $\mathfrak{g}$ – in other words E_K is an admissible $(\mathfrak{g}, K)$-module.

Let us emphasize that a K-admissible $(\mathfrak{g}, K)$-module is not necessary finitely generated as a $\mathfrak{g}$-module. For example the tensor product of two infinite dimensional highest weight modules for $\mathfrak{sl}(2,\mathbb{R})$ is admissible but not finitely generated as a $\mathfrak{g}$-module. This brings us to the notion of a *Harish-Chandra module* by which we understand a $(\mathfrak{g}, K)$-module V such that one of the following equivalent conditions hold:

(i) V is finitely generated as a $\mathfrak{g}$-module and K-admissible.

(ii) V is K-admissible and $\mathcal{Z}(\mathfrak{g})$-finite. Here $\mathcal{Z}(\mathfrak{g})$ denotes the center of $\mathcal{U}(\mathfrak{g})$.
(iii) V is finitely generated as an $\mathfrak{n}$-module, where $\mathfrak{n}$ is a maximal unipotent subalgebra of $\mathfrak{g}$.

Given a Harish-Chandra module V we say that a representation (π, E) of G is a *globalization* of V, if the K-finite vectors E_K of E are smooth and isomorphic to V as a $(\mathfrak{g}, K)$-module.

Remark 2.5. *We caution the reader that there exists irreducible Banach representation* (π, E) *of* G *which are not admissible* [6]*. However, if* $(\pi, \mathcal{H})$ *happens to be unitary irreducible representation, then Harish-Chandra has shown that* π *is admissible.*

3. Smooth Fréchet globalizations of Harish-Chandra modules

This section is devoted to a general study of smooth Fréchet globalizations (SF-globalizations for short) of Harish-Chandra modules.

Let us introduce a preliminary notion and call a Harish-Chandra module V *good* if it admits a unique smooth Fréchet globalization. Equivalently V is good if and only if for any SF-globalization (π, E) one has $\Pi(\mathcal{S}(G))V = E$. Eventually it will turn out that all Harish-Chandra modules are good (Casselman-Wallach).

The main objective of this section is to show that Harish-Chandra modules are good if and only if they feature certain lower bounds for matrix coefficients which are uniform in the K-type (see Proposition 3.4 and Lemma 3.5).

In order to discuss good Harish-Chandra modules it is useful to introduce two other preliminary notions, namely *minimal* and *maximal* SF-globalizations.

3.1. Minimal and maximal smooth globalizations

This paragraph is devoted to a general discussion of the extremal SF-topologies on a Harish-Chandra module.

Let us first remark that any Harish-Chandra module V admits an SF-globalization. In fact, one can embed V into a smooth principal series module $I^\infty = C^\infty(G \times_{P_{\min}} U)$ where $P_{\min} = MAN$ is a minimal parabolic subgroup and U is a finite dimensional module for $P_{\min}/N$ (Casselman's Theorem, see [8], Sect. 4). Taking the closure of V in I^∞ yields an SF-globalization of V.

An SF-globalization, say V^∞, of an Harish-Chandra module V will be called *minimal* if the following universal property holds: if (π, E) is an SF-globalization of V, then there exists a continuous G-equivariant map $V^\infty \to E$ which extends the identity morphism $V \to V$.

It is clear that minimal globalizations are unique. Let us show that they actually exist. For that let us fix an SF-globalization (π, E) of V. Let $\mathbf{v} = \{v_1, \ldots, v_k\}$ be a set of generators of V and consider the map

$$\mathcal{S}(G)^k \to E, \quad \mathbf{f} = (f_1, \ldots, f_k) \mapsto \sum_{j=1}^k \Pi(f_j)v_j \, .$$

This map is linear, continuous and G-equivariant (with $\mathcal{S}(G)^k$ considered as a module for G under the left regular representation). Let us write

$$\mathcal{S}(G)_{\mathbf{v}} := \{\mathbf{f} \in \mathcal{S}(G)^k \mid \sum_{j=1}^k \Pi(f_j)v_j = 0\}$$

for the kernel of this linear map. Note that $\mathcal{S}(G)_{\mathbf{v}}$ is a closed G-submodule of $\mathcal{S}(G)^k$ which is independent of the choice of the particular SF-globalization (π, E). Moreover it is clear that $\mathcal{S}(G)/\mathcal{S}(G)_{\mathbf{v}}$ is an SF-module for G and in addition a globalization of V. By construction $\mathcal{S}(G)/\mathcal{S}(G)_{\mathbf{v}}$ is the minimal globalization V^∞.

Lemma 3.1. *Let V be a good Harish-Chandra module and V^∞ its unique SF-globalization. Let $W \subset V$ be a submodule and $U := V/W$. Let $\overline{W}$ be the closure of W in V^∞. Then $U^\infty = V^\infty/\overline{W}$.*

Proof. Let us write $(\pi_U, V^\infty/\overline{W})$ for the quotient representation obtained from (π, V^∞). Then $\Pi(\mathcal{S}(G))V = V^\infty$ implies that $\Pi_U(\mathcal{S}(G))U = V^\infty/\overline{W}$ and hence the assertion. □

Let us call an SF-globalization of V, say $V^\infty_{\max}$, *maximal* if for any SF-globalization (π, E) of V there exist a continuous linear G-map $E \to V^\infty_{\max}$ sitting above the identity morphism $V \to V$.

It is clear that maximal globalizations are unique provided they exists. The existence is obtained by duality. Let us provide the details.

Let V be Harish-Chandra module and V^* the corresponding dual Harish-Chandra module. Then $V = U^*$ with $U = V^*$. Let U^∞ be the minimal SF-globalization of U. If $(U^\infty)^*$ denotes the strong topological dual of U^∞, then we define the maximal SF-globalization of V by

$$V_{\max} := ((U^\infty)^*)^\infty \, .$$

At this point we need to verify that $V_{\max}$ is indeed a Fréchet representation for G. But this is seen as follows: we first view U as a quotient of some minimal principal series module say $U = I/W$. As the G-smooth structure on I^∞ is the same as the K-smooth structure, the same holds for $\overline{U}$. From this one deduces that $V_{\max}$ is indeed an SF-globalization (smooth distributions on a compact manifold are smooth functions).

Let us show that $V_{\max}$ has the universal property: Let (π, E) be an SF-globalization of V. As E^* is a module for $\mathcal{S}(G)$, we obtain a continuous G-morphism $U^\infty \to E^*$. Dualizing yields a continuous G-map $E^{**} \to (U^\infty)^*$. Taking the smooth vectors we get continuous G-morphisms $E \to (E^{**})^\infty \to V_{\max}^\infty$.

With the notion of maximal globalization we readily obtain the dual version of the previous Lemma.

Lemma 3.2. *Let V be a good Harish-Chandra module and V^∞ its unique SF-globalization. Let $W \subset V$ be a submodule and Let $\overline{W}$ be the closure of W in V^∞. Then $\overline{W} = W_{\max}^\infty$.*

Let us note that a Harish-Chandra module V is good if and only if $V^\infty = V_{\max}^\infty$. Further, V is good if and only if V^* is good.

We conclude this paragraph with an observation which will be frequently used later on.

Lemma 3.3. *Let $V_1 \subset V_2 \subset V_3$ be an inclusion chain of Harish-Chandra modules. Suppose that V_2 and V_3/V_1 are good. Then V_2/V_1 is good.*

Proof. Let $\overline{V_3}$ be an SF-globalization of V_3. Let $\overline{V}_1$, $\overline{V_2}$ be the closures of $V_{1,2}$ in $\overline{V}_3$. By our first assumption we have $\overline{V_2} = V_2^\infty$ and thus Lemma 3.1 implies that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty$. Our second assumption gives $(V_3/V_1)^\infty = \overline{V_3}/\overline{V_1}$ and Lemma 3.2 then yields that $\overline{V_2}/\overline{V_1} = (V_2/V_1)_{\max}^\infty$. □

3.2. **Lower bounds for matrix coefficients**

Let us denote by $\hat{K}$ the set of equivalence classes of irreducible unitary representations of K. We often identify an equivalence class $[\tau] \in \hat{K}$ with a representative τ. If V is a K-module, then we denote by $V[\tau]$ its τ-isotypical part.

If $\mathfrak{t} \subseteq \mathfrak{k}$ is a maximal torus, then we often identify τ with its highest weight in $i\mathfrak{t}^*$ (with respect to a fixed positive system). In particular, $|\tau| \geq 0$ will refer to the Cartan-Killing norm of the highest weight of τ.

Proposition 3.4. *Suppose that V is a cyclic good Harish-Chandra module. Let $0 \neq \xi \in V^*$ be a cyclic vector. Then for all continuous norms q on V^∞, there exits constants $c_1, c_2, c_3 > 0$ such that for all $\tau \in \hat{K}$ and $v \in V[\tau]$ there exist a $g_\tau \in G$ such that $\|g_\tau\| \leq (1+|\tau|)^{c_1}$ and*

$$|\xi(\pi(g_\tau)v)| \geq \frac{c_2}{(1+|\tau|)^{c_3}} \cdot q(v) \, .$$

Proof. By assumption there exists an $n \in \mathbb{N}$ and $c > 0$ such that

$$|\xi(\pi(g)v)| \leq c \cdot \|g\|^n q(v)$$

for all $v \in V^\infty$ and $g \in G$. With

$$C_n(G) := \{f \in C(G) \mid |f(g)| << \|g\|^n\}$$

we obtain an embedding

$$V^\infty \to C_n(G), \quad v \mapsto (g \mapsto (\xi(\pi(g)v))) \, .$$

For $N \geq n$ we write E_N for the Banach completion of V^∞ with respect to the norm

$$p_N(v) := \sup_{g \in G} \frac{|\xi(\pi(g)v)|}{\|g\|^N} \, .$$

As V is good, we obtain that

$$V^\infty = E_N^\infty = E_{N'}^\infty \tag{3.1}$$

for all $N, N' \geq n$. The Banach globalizations E_N have the property that a vector $v \in E_N$ is smooth if and only if it is smooth for the representation restricted to K (this is a consequence of Lemmas 8.2 and 8.3 in the appendix). Let us denote by $\Delta_\mathfrak{k}$ the Casimir element of $\mathfrak{k}$ and define for $s \in \mathbb{R}$

$$p_{N,s}(v) := p_N((1+\Delta_\mathfrak{k})^{\frac{s}{2}} v)$$

for all $v \in V^\infty$. For $N' = N + l$ with $l > 0$ and $N \geq n$ we thus conclude from (3.1) the existence of an $s > 0$ such that

$$p_N(v) \leq p_{N',s}(v) \tag{3.2}$$

for all $v \in V^\infty$.

Let us fix $\tau \in \hat{K}$, $v \in V[\tau]$ and $g_\tau \in G$ such that $g \mapsto \frac{|\xi(\pi(g)v)|}{\|g\|^{N'}}$ becomes maximal at g_τ. We conclude from (3.2) that

$$\frac{|\xi(\pi(g_\tau)v)|}{\|g_\tau\|^N} \leq (1+|\tau+\rho_{\mathfrak{k}}|^2-|\rho_{\mathfrak{k}}|^2)^{\frac{s}{2}} \cdot \frac{|\xi(\pi(g_\tau)v)|}{\|g_\tau\|^{N+l}}$$

for all $v \in V[\tau]$, i.e.

$$\|g_\tau\| \leq (1+|\tau+\rho_{\mathfrak{k}}|^2-|\rho_{\mathfrak{k}}|^2)^{\frac{s}{2l}}\,.$$

Here $\rho_{\mathfrak{k}} \in i\mathfrak{t}^*$ is the usual half sum $\rho_{\mathfrak{k}} = \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{k}}$.

On the other hand (3.1) implies likewise that there exists an $s' > 0$ such that

$$q(v) \leq p_{N',s'}(v)$$

for all $v \in V^\infty$. For $v \in V[\tau]$ we then get

$$|\xi(\pi(g_\tau)v)| \geq \frac{\|g_\tau\|^{N'}}{(1+|\tau+\rho_{\mathfrak{k}}|^2-|\rho_{\mathfrak{k}}|^2)^{\frac{s'}{2}}} \cdot q(v)\,.$$

All assertions follow. □

For later reference we record the following converse of the lower bound in the proposition above.

Lemma 3.5. *Let (π, E) be an SF-globalization of a Harish-Chandra module V. Suppose that there exists a continuous K-invariant Hilbert semi-norm q on E, $\xi \in V^*$ and constants $c_1, c_2 > 0$ such that for all $v \in V[\tau]$ there exists an $g_\tau \in G$ such that $\|g_\tau\| \leq (1+|\tau|)^{c_1}$ and*

$$|\xi(\pi(g_\tau)v)| \geq \frac{1}{(1+|\tau|)^{c_2}} \cdot q(v)\,.$$

Suppose in addition that the same holds for the dual representation (π^*, E^*) with repect to the dual norm q^*, i.e. there exists $\xi^* \in V$, constants $c_1', c_2' > 0$ such that for all $v^* \in V^*[\tau]$ there exists an $g_\tau' \in G$ $\|g_\tau'\| \leq (1+|\tau|)^{c_1'}$ and

$$|\xi^*(\pi^*(g_\tau')v^*)| \geq \frac{1}{(1+|\tau|)^{c_2'}} \cdot q^*(v^*)\,.$$

Then V is good.

Proof. Let $(\tilde{\pi}, \tilde{E})$ be an SF-globalization of V. We have to show that $E \simeq \tilde{E}$.

Let p be a continuous semi-norm on $\tilde{E}$. It is no loss of generality to assume that p is a K-invariant norm and that $\tilde{E}$ consists of the smooth vectors of the Banach completion of V with respect to p.

Write q_l, resp. p_l, for the l-th K-Sobolev norm of q, resp. p (defined as in the proof of the preceding proposition). By assumption, there exists an $k < 0$ such that

$$\sup_{\substack{g\in G\\ \|g\|\leq|\tau|^{c_1}}} |\xi(\pi(g)v)| \geq c_2 \cdot q_k(v) \tag{3.3}$$

vor all $v \in V[\tau]$.

We claim that there exist a constant $C > 0$ and an $s, t \in \mathbb{R}$ such that

$$p_t(v) \geq C \cdot q_s(v) \tag{3.4}$$

for all $v \in V$. It is no loss of generality to assume that p is a K-invariant. Suppose that (3.4) is verified for all $v \in V[\tau]$ with $\tau \in \hat{K}$ for some $t = 0$. Then it holds for all $v \in V$ by rising t.

Write E^* for the topological dual of E and (π^*, E^*) for the corresponding dual representation. Let p^* be the norm dual to p. Note that

$$\xi(\pi(g)v) = \xi(\tilde{\pi}(g)v)$$

for all $g \in G$. By the continuity of p there exists an $N \in \mathbb{N}$ such that

$$|\xi(\tilde{\pi}(g)v)| \leq p^*(\xi)p(\pi(g)v) \leq p^*(\xi) \cdot p(v) \cdot \|g\|^N$$

for all $g \in G$. The claim follows from (3.3).

Applying the above reasoning for the dual representation we arrive at constants $C' > 0$ and $j, l \in \mathbb{R}$ such that

$$p_j^*(v^*) \geq C' \cdot q_l^*(v^*)$$

for all $v^* \in V^*$. Dualizing this inequality then yields

$$C' \cdot q_{-l}(v) \leq p_{-l}(v)$$

for all $v \in V_\lambda$. It follows that we can squeeze p between two Sobolev norms of q. Consequently $E \simeq \tilde{E}$. □

4. **Spherical principal series representation**

This section is devoted to a thorough study of spherical principal series representation of G. We will introduce a Dirac-type sequence for such representations and establish lower bounds for matrix-coefficients which are uniform in the K-types. These lower bounds are essentially sharp, locally uniform in the representation parameter, and stronger than the more abstract estimates in Proposition 3.4.

The lower bounds established give us a constructive method for finding Schwartz-functions representing a given smooth vector.

Let us write $G = NAK$ for an Iwasawa decomposition of G. Accordingly we decompose elements $g \in G$ as

$$g = \tilde{n}(g)\tilde{a}(g)\tilde{k}(g)$$

with $\tilde{n}(g) \in N$, $\tilde{a}(g) \in A$ and $\tilde{k}(g) \in K$. Set $M = Z_K(A)$ and define a minimal parabolic subgroup of G by $P_{\min} = NAM$.

The Lie algebras of A, N and K shall be denoted by $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{k}$. Complexification of Lie-algebras are indicated with a $\mathbb{C}$-subscript, i.e. $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ etc. As usually we define $\rho \in \mathfrak{a}^*$ by $\rho(Y) := \frac{1}{2}\operatorname{tr}(\operatorname{ad}_{\mathfrak{n}} Y)$ for $Y \in \mathfrak{a}$.

The smooth spherical principal series with parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is defined by

$$\begin{aligned}\mathcal{H}_\lambda^\infty := \{f \in C^\infty(G) \mid & (\forall nam \in P_{\min}, \forall g \in G) \\ & f(namg) = a^{\rho+\lambda} f(g)\}\end{aligned}$$

We note that R defines a smooth representation of G on $\mathcal{H}_\lambda^\infty$ which we denote henceforth by π_λ. The restriction map to K defines a K-isomorphism:

$$\operatorname{Res}_K : \mathcal{H}_\lambda^\infty \to C^\infty(K\backslash M), \quad f \mapsto f|_K \,.$$

The resulting action of G on $C^\infty(M\backslash K)$ is given by

$$[\pi_\lambda(g)f](Mk) = f(M\tilde{k}(kg))\tilde{a}(kg)^{\lambda+\rho} \,.$$

This action lifts to a continuous action on the Hilbert completion $\mathcal{H}_\lambda = L^2(M\backslash K)$ of $C^\infty(M\backslash K)$. We note that this representation is unitary provided that $\lambda \in i\mathfrak{a}^*$.

We denote by V_λ the K-finite vectors of π_λ and note that $V_\lambda = \mathbb{C}[M\backslash K]$ as K-module..

4.1. K-expansion of smooth vectors

We recall $\hat{K}$, the set of equivalence classes of irreducible unitary representations of K. If $[\tau] \in \hat{K}$ we let (τ, U_τ) be a representative. Further we write $\hat{K}_M$ for the subset of M-spherical equivalence classes, i.e.

$$[\tau] \in \hat{K}_M \iff U_\tau^M := \{u \in U_\tau \mid \tau(m)u = u \ \forall m \in M\} \neq \{0\} \,.$$

Given a finite dimensional representation (τ, U_τ) of K we denote by (τ^*, U_τ^*) its dual representation. With each $[\tau] \in \hat{K}_M$ comes the realization mapping

$$r_\tau : U_\tau \otimes (U_\tau^*)^M \to L^2(M\backslash K), \quad u \otimes \eta \mapsto (Mk \mapsto \eta(\tau(k)u)) .$$

Let us fix a K-invariant inner product on U_τ. This inner product induces a K-invariant inner product on U_τ^*. We obtain an inner product on $U_\tau \otimes (U_\tau^*)^M$ which is independent of the chosen inner product on U_τ. If we denote by $d(\tau)$ the dimension of U_τ, then Schur-orthogonality implies that

$$\frac{1}{d(\tau)} \|u \otimes \eta\|^2 = \|r_\tau(u \otimes \eta)\|^2_{L^2(M\backslash K)} .$$

Taking all realization maps together we arrive at a K-module isomorphism

$$\mathbb{C}[M\backslash K] = \sum_{\tau \in \hat{K}_M} U_\tau \otimes (U_\tau^*)^M .$$

Let us fix a maximal torus $\mathfrak{t} \subset \mathfrak{k}$ and a positive chamber $\mathcal{C} \subset i\mathfrak{t}^*$. We often identify τ with its highest weight in $\mathcal{C}$ and write $|\tau|$ for the Cartan-Killing norm of the highest weight. As $d(\tau)$ is polynomial in τ we arrive at the following characterization of the smooth functions:

$$C^\infty(M\backslash K) = \Big\{ \sum_{\tau \in \hat{K}_M} c_\tau u_\tau \ | c_\tau \in \mathbb{C}, u_\tau \in U_\tau \otimes (U_\tau^*)^M, \|u_\tau\| = 1 \\ (\forall N \in \mathbb{N}) \sum_{\tau \in \hat{K}_M} |c_\tau|(1 + |\tau|)^N < \infty \Big\} .$$

Let us denote by δ_{Me} the point-evaluation of $C^\infty(M\backslash K)$ at the base point Me. We decompose δ_{Me} into K-types:

$$\delta_{Me} = \sum_{\tau \in \hat{K}_M} \delta_\tau$$

where

$$\delta_\tau = d(\tau) \sum_{i=1}^{l(\tau)} u_i \otimes u_i^*$$

with $u_1, \ldots, u_{l(\tau)}$ any basis of U_τ^M and $u_1^*, \ldots, u_{l(\tau)}^*$ its dual basis. For $1 \leq i \leq l(\tau)$ we set

$$\delta_\tau^i := \sqrt{d(\tau)} u_i \otimes u_i^*$$

and record that $\delta_\tau = \sqrt{d(\tau)} \sum_{i=1}^{l(\tau)} \delta_\tau^i$. Note the following properties of δ_τ and δ_τ^i:

- $\|\delta_\tau^i\|_\infty = \delta_\tau^i(Me) = \sqrt{d(\tau)}$.
- $\|\delta_\tau^i\|_{L^2(M\backslash K)} = 1$.
- $\delta_\tau * \delta_\tau = \delta_\tau$.
- $\delta_\tau * f = f$ for all $f \in L^2(M\backslash K)_\tau := \operatorname{im} r_\tau$.

4.2. **Non-compact model**

We have seen that the restriction map Res_K realizes $\mathcal{H}_\lambda^\infty$ as a function space on $M\backslash K$. Another standard realization will be useful for us. Let us denote by $\overline{N}$ the opposite of N. Here, $\mathfrak{n}$ stands for the Lie algebra of N. As $NAM\overline{N}$ is open and dense in G we obtain a faithful restriction mapping:

$$\mathrm{Res}_{\overline{N}} : \mathcal{H}_\lambda^\infty \to C^\infty(\overline{N}), \quad f \mapsto f|_{\overline{N}}.$$

Note that this map is not onto. The transfer of compact to non-compact model is given by

$$\begin{aligned} \mathrm{Res}_{\overline{N}} \circ \mathrm{Res}_K^{-1} : C^\infty(M\backslash K) \to C^\infty(\overline{N}), \\ f \mapsto F;\ F(\overline{n}) := \tilde{a}(\overline{n})^{\lambda+\rho} f(\tilde{k}(\overline{n})) \end{aligned}$$

The transfer of the Hilbert space structure on $\mathcal{H}_\lambda = L^2(M\backslash K)$ results in the L^2-space $L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\operatorname{Re}\lambda} d\overline{n})$ with $d\overline{n}$ an appropriately normalized Haar measure on $\overline{N}$. In the sequel we also write $\mathcal{H}_\lambda$ for $L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\operatorname{Re}\lambda} d\overline{n})$ in the understood context. The full action of G in the non-compact model is not of relevance to us, however we will often use the A-action which is much more transparent in the non-compact picture:

$$[\pi_\lambda(a) f](\overline{n}) = a^{\lambda+\rho} f(a^{-1}\overline{n}a)$$

for all $a \in A$ and $f \in L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\operatorname{Re}\lambda} d\overline{n})$.

The fact that Res_K is an isomorphism follows from the geometric fact that $P_{\min}\backslash G \simeq M\backslash K$. Now $\overline{N}$ embeds into $P_{\min}\backslash G = M\backslash K$ as an open dense subset. In fact the complement is algebraic and we are going to describe it explicitly.

Let (σ, W) be a finite dimensional faithful irreducible representation of G. We assume that W is K-spherical, i.e. W admits a non-zero K-fixed vector, say v_K. It is known that σ is K-spherical if and only if there is a real line $L \subset W$ which is fixed under $\overline{P}_{\min} = MA\overline{N}$. Let $L = \mathbb{R}v_0$ and $\mu \in \mathfrak{a}^*$ be such that $\sigma(a)v_0 = a^\mu \cdot v_0$ for all $a \in A$, in other words: v_0 is a lowest weight vector of σ and μ is the corresponding lowest weight.

Let now $\langle\cdot,\cdot\rangle$ be an inner product on W which is θ-covariant: if $g = k\exp(X)$ for $k \in K$ and $X \in \mathfrak{p}$ and $\theta(g) := k\exp(-X)$, then covariance means

$$\langle\sigma(g)v, w\rangle = \langle v, \sigma(\theta(g)^{-1})w\rangle$$

for all $v, w \in W$ and $g \in G$. Such an inner product is unique up to scalar by Schur's Lemma. Henceforth we request that v_0 is normalized and we fix v_K by $\langle v_0, v_K\rangle = 1$. Consider on G the function

$$f_\sigma(g) := \langle\sigma(g)v_0, v_0\rangle\,.$$

The restriction of f_σ to K is also denoted by f_σ.

Let now $\overline{n} \in \overline{N}$ and write $\overline{n} = \tilde{n}(\overline{n})\tilde{a}(\overline{n})\tilde{k}(\overline{n})$ according to the Iwasawa decomposition. Then $\tilde{k}(\overline{n}) = n^*\tilde{a}(\overline{n})^{-1}\overline{n}$ for some $n^* \in N$. Consequently

$$f_\sigma(\tilde{k}(\overline{n})) = \tilde{a}(\overline{n})^{-\mu}\,.$$

If $(\overline{n}_j)_j$ is a sequence in $\overline{N}$ such that $\tilde{k}(\overline{n}_j)$ converges to a point in $M\backslash K - \tilde{k}(\overline{N}) =: M\backslash K - \overline{N}$, then $\tilde{a}(\overline{n}_j)^{-\mu} \to 0$. Hence

$$M\backslash K - \overline{N} \subset \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}\,.$$

As f_σ is non-negative one obtains for all regular σ that equality holds:

$$M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$$

(this reasoning is not new and goes back to Harish-Chandra). Let us fix such a σ now.

We claim that the mapping $\overline{n} \to f_\sigma(\overline{n})$ is the inverse of a polynomial mapping, i.o.w. the map

$$\overline{N} \to \mathbb{R}, \quad \overline{n} \mapsto \tilde{a}(\overline{n})^\mu$$

is a polynomial map. But his follows from

$$\tilde{a}(\overline{n})^\mu = \langle\sigma(\overline{n})v_K, v_0\rangle$$

by means of our normalizations.

In order to make estimates later on we introduce coordinates on $\overline{N}$. For that we first write $\overline{\mathfrak{n}}$ as semi-direct product of $\mathfrak{a}$-root vectors:

$$\overline{\mathfrak{n}} = \mathbb{R}X_1 \ltimes (\mathbb{R}X_2 \ltimes (\ldots \ltimes \mathbb{R}X_n)\ldots)\,.$$

Accordingly we write elements of $\overline{\mathfrak{n}}$ as $X := \sum_{j=1}^n x_j X_j$ with $x_i \in \mathbb{R}$. We note the following two facts:

- The map

  $$\Phi : \overline{\mathfrak{n}} \to \overline{N}, \quad X \mapsto \overline{n}(X) := \exp(x_1 X_1) \cdot \ldots \cdot \exp(x_n X_n)$$

  is a diffeomorphism.
- One can normalize the Haar measure $d\overline{n}$ of $\overline{N}$ in such a way that:

  $$\Phi^*(d\overline{n}) = dx_1 \cdot \ldots \cdot dx_n \, .$$

We introduce a norm on $\overline{\mathfrak{n}}$ by setting

$$\|X\|^2 := \sum_{j=1}^{n} |x_j|^2 \qquad (X \in \overline{\mathfrak{n}}) \, .$$

Finally we set

$$f_\sigma(X) := f_\sigma(\tilde{k}(\overline{n}(X))) = \tilde{a}(\overline{n}(X))^{-\mu}$$

and summarize our discussion.

Lemma 4.1. *Let $m > 0$. Then there exists $C > 0$ and a finite dimensional K-spherical representation (σ, W) of G such that:*

(i) $M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$.
(ii) $|f_\sigma(X)| \leq C \cdot (1 + \|X\|)^{-m}$ *for all* $X \in \overline{\mathfrak{n}}$.

4.3. **Dirac type sequences**

Dirac sequences do not exist for Hilbert representations as they are features of an L^1-theory. However, rescaled they exist for the Hilbert representations we shall consider.

Some additional terminology is of need. We fix an element $Y \in \mathfrak{a}$ such that $\alpha(Y) \geq 1$ for all roots $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$. For $t > 0$ we put

$$a_t := \exp((\log t) Y) \, .$$

Note that for $\eta \in \mathfrak{a}_\mathbb{C}^*$ one has

$$a_t^\eta = t^{\eta(Y)} \, .$$

In the sequel we will often abbreviate and simply write t^η for $t^{\eta(Y)}$.

Recall our function f_σ on K. We let $\xi = \xi_\sigma$ the corresponding function transferred to $\overline{N} \simeq \overline{\mathfrak{n}}$, i.e.

$$\xi(X) := \tilde{a}(\overline{n}(X))^{\rho+\lambda} f_\sigma(\tilde{k}(\overline{n}(X))) = \tilde{a}(\overline{n}(X))^{\rho+\lambda-\mu} \, .$$

Note that ξ is a K-finite vector. We choose μ (and hence $m > 0$) to be large enough so that ξ is integrable and write $\|\xi\|_1$ for the corresponding $L^1(\overline{N})$-norm. Then, for λ real, ξ is a positive function and

$$\left(\frac{a_t^{\rho-\lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)\xi \right)_{t>0}$$

forms a Dirac sequence for $t \to \infty$. If λ is not real, then the ξ is oscillating and we have to be more careful.

We recall that $\xi(X)$ satisfies the inequality

$$\xi(X) \leq C \cdot (1 + \|X\|)^{-m}$$

where we can choose m as large as we wish (provided σ is sufficiently regular and large). Record the normalization $\xi(0) = 1$.

Recall the generating functions $\delta_\tau^i \in L^2(M\backslash K)_\tau$, $1 \leq i \leq l(\tau)$ for $\tau \in \hat{K}_M$. In the sequel we abbreviate and set $d := d(\tau)$, $l := l(\tau)$.

Let $D_\tau^i(\overline{n}) = \tilde{a}(\overline{n})^{\rho+\lambda}\delta_\tau^i(\tilde{k}(\overline{n}))$ the transfer of δ_τ^i to the non-compact model. We also set $D_\tau^i(X) := D_\tau(\overline{n}(X))$ for $X \in \overline{\mathfrak{n}}$. Let us note that $|D_\tau^i(0)| = \sqrt{d}$ and , in case where $\operatorname{Re}\lambda$ is dominant, $\|D_\tau^i\|_\infty = |D_\tau^i(0)|$.

The goal of this section is to control the spread of $\pi_\lambda(a_t)\xi$ over the K-types. For that we set:

$$d^i(\tau, t) := \langle \pi_\lambda(a_t)\xi, D_\tau^i \rangle \, .$$

It is our goal to estimate $d^i(\tau, t)$ from below.

Lemma 4.2. *Fix $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and let $\gamma > 1$. Then there exist a choice of ξ and constants $c_1, c_2, c_3 > 0$, $s \in \mathbb{R}$ such that for $t(\tau) := (c_1 \cdot |\tau|)^\gamma$ and all $1 \leq i \leq l$:*

$$|d^i(\tau, t(\tau))| \geq c_2 \cdot |\tau|^s$$

for all τ with $|\tau| \geq c_3$.

In dependence of λ and γ, the vector ξ and the constants c_1, c_2, c_3, s can be chosen locally constant. Moreover, s is such that

$$|\tau|^s \leq C \cdot \sqrt{d} \cdot a_{t(\tau)}^{-\rho + \operatorname{Re}\lambda}$$

holds for all τ and a fixed constant $C > 0$.

Proof. Fix $t_0 > 0$ and set $t = t_0^{-\gamma}$ for some $\gamma > 1$. In the sequel we let $t \to \infty$. Let $1 \leq i \leq l$. Define

$$d_1^i(\tau, t) := \int_{\{\|X\| \geq t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D_\tau(X)} \cdot \tilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} \, dX \, .$$

In our first step of the proof we wish to estimate $d_1^i(\tau, t)$. For that let $C_1, q_1 > 0$ be such that

$$\tilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda} \leq C_1 \cdot (1 + \|X\|)^{q_1} \, .$$

Likewise, by the definition of D_τ^i we obtain constants $C_2, q_2 > 0$ such that

$$|D_\tau^i(X)| \leq C_2 \cdot \sqrt{d}(1+\|X\|)^{q_2}$$

for all τ and $1 \leq i \leq l$. Set $q := q_1 + q_2$ and $C' := C_1 C_2$.

In the following computations we use the notation

$$dX := dx_1 \cdot \ldots \cdot dx_n$$

for $X = \sum_{j=1}^n x_j X_j$. From the definitions and the inequalities just stated we arrive at our starting point:

$$|d_1^i(\tau,t)| \leq \sqrt{d} \cdot C' \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\| \geq \cdot t_0\}} |\xi(\operatorname{Ad}(a_t)^{-1}X)| \cdot (1+\|X\|)^q \, dX \, .$$

As $|\xi(X)| \leq C'' \cdot (1+\|X\|)^{-m}$ for some constants $C'', m > 0$ we thus get with $C := C'C''$ that

$$|d_1^i(\tau,t)| \leq \sqrt{d} \cdot C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\| \geq \cdot t_0\}} \frac{(1+\|X\|)^q}{(1+\|\operatorname{Ad}(a_t)^{-1}X\|)^m} \, dX \, .$$

By the definition of a_t we get that $\|\operatorname{Ad}(a_t)^{-1}X\| \geq t\|X\|$ and hence

$$|d_1^i(\tau,t)| \leq \sqrt{d} \cdot C \cdot t^{\operatorname{Re}\lambda+\rho} \int_{\{\|X\| \geq \cdot t_0\}} \frac{(1+\|X\|)^q}{(1+t\|X\|)^m} \, dX \, .$$

Now we are in the situation to use polar coordinates for X:

$$\begin{aligned}
|d_1^i(\tau,t)| &\leq C \cdot \sqrt{d} \cdot t^{\operatorname{Re}\lambda+\rho} \int_{t_0}^\infty \frac{r^n(1+r)^q}{(1+tr)^m} \, \frac{dr}{r} \\
&= C \cdot \sqrt{d} \cdot t_0^{n-\gamma(\operatorname{Re}\lambda+\rho)} \int_1^\infty \frac{r^n(1+t_0 r)^q}{(1+tt_0 r)^m} \, \frac{dr}{r} \\
&= C \cdot \sqrt{d} \cdot t_0^{n-\gamma(\operatorname{Re}\lambda+\rho)} \int_1^\infty \frac{r^n(1+t_0 r)^q}{(1+t_0^{1-\gamma} r)^m} \, \frac{dr}{r} \\
&= C \cdot \sqrt{d} \cdot t_0^{n-\gamma(\operatorname{Re}\lambda+\rho)+m(\gamma-1)} \int_1^\infty \frac{r^n(1+t_0 r)^q}{(t_0^{\gamma-1}+r)^m} \, \frac{dr}{r} \\
&\leq C \cdot \sqrt{d} \cdot t_0^{n-\gamma(\operatorname{Re}\lambda+\rho)+m(\gamma-1)} \int_1^\infty r^{n+q-m} \, \frac{dr}{r} \, .
\end{aligned}$$

Henceforth we request that $m > n+q+1$. Further as $\gamma > 1$ we gain a constant C_1, only depending on m, such that:

$$|d_1^i(\tau,t)| \leq C_1 \cdot \sqrt{d} \cdot t_0^{n-\gamma(\operatorname{Re}\lambda+\rho)+m(\gamma-1)} \, .$$

We have to choose t in relationship to $|\tau|$. From the definition of δ_τ and basic finite dimensional representation theory we gain for every

$\epsilon > 0$ a constant $c_1 > 0$ such that for all τ and $\|X\| \leq (c_1 \cdot |\tau|)^{-1}$ the following estimate holds:

$$\frac{1}{\sqrt{d}} \cdot |D^i_\tau(X) - D^i_\tau(0)| \leq \frac{\epsilon}{2}\,. \tag{4.1}$$

This brings us to our choice of t_0, namely

$$t_0(\tau) := (c_1 \cdot |\tau|)^{-1}\,.$$

Then for every $m' > 0$ there exist a choice of ξ and a constant $C_2 > 0$ such that

$$|d^i_1(\tau, t(\tau))| \leq C_2 \cdot |\tau|^{-m'}\,.$$

Write now $d^i(\tau, t) = d^i_0(\tau, t) + d^i_1(\tau, t)$. We have just seen that main contribution to $d^i(\tau, t)$ for $t \to \infty$ will likely come from $d^i_0(t, \tau)$. This is indeed the case.

Let us assume for a moment that λ is imaginary. Define $I \in \mathbb{C}$ and $I' > 0$ by

$$I := \int_{\overline{\mathfrak{n}}} \xi(X)\; dX$$
$$I' := \int_{\overline{\mathfrak{n}}} |\xi(X)|\; dX$$

The first obstacle we face is that I might be zero. However as $\xi(X) = \tilde{a}(n(X))^{\rho+\lambda-\mu}$, there are for each μ in the "half line" $\mathbb{N}\mu$ infinitely many lowest weights for which $I \neq 0$ (apply Carleman's theorem, see [7], 3.71). So for any m' we find such a I. Set now $\epsilon := \frac{|I|}{I'} > 0$.

In the following computation we will use the simple identity:

$$\int_{\overline{\mathfrak{n}}} \pi_\lambda(a_t) f(X)\; dX = t^{\lambda-\rho} \int_{\overline{\mathfrak{n}}} f(X)\; dX$$

for all integrable functions f.

$$\begin{aligned}
|d^i(\tau, t(\tau))| &= \left| \int_{\{\|X\| \le t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D_\tau(X)}\; dX \right| + O\left(\frac{1}{|\tau|^{m'}}\right) \\
&\ge \sqrt{d} \left| \int_{\{\|X\| \le t_0\}} (\pi_\lambda(a_t)\xi)(X)\; dX \right| \\
&\quad - \left| \int_{\{\|X\| \le t_0\}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{(D_\tau(X) - D_\tau(0))}\; dX \right| \\
&\quad + O\left(\frac{1}{|\tau|^{m'}}\right) \\
&\ge \sqrt{d} \cdot t^{-\rho} \cdot I - \epsilon/2 \cdot \sqrt{d} \int_{\{\|X\| \le t_0\}} |(\pi_\lambda(a_t)\xi)(X)|\; dX \\
&\quad + O\left(\frac{1}{|\tau|^{m'}}\right) \\
&\ge \sqrt{d} \cdot t^{-\rho} \cdot I - \epsilon/2 \cdot d \cdot t^{-\rho} \cdot I' + O\left(\frac{1}{|\tau|^{m'}}\right) \\
&= \sqrt{d} \cdot t^{-\rho} \cdot I/2 + O\left(\frac{1}{|\tau|^{m'}}\right)
\end{aligned}$$

and the theorem for λ imaginary follows.

The case of arbitrary generic λ is similar to the situation above: We replace dX by $\tilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\; dX$ in the formulas before. Now at the change of variables we have to be slightly more careful as:

$$\begin{aligned}
&\int_{\overline{\mathfrak{n}}} [\pi_\lambda(a_t)f](X)\; \tilde{a}(\overline{n}(X))^{-2\operatorname{Re}\lambda}\; dX = \\
&\quad = a_t^{\lambda-\rho} \int_{\overline{\mathfrak{n}}} f(X)\; \tilde{a}(\overline{n}(\operatorname{Ad}(a_t)X))^{-2\operatorname{Re}\lambda}\; dX\,.
\end{aligned}$$

In order to proceed as above we simply have to observe that

$$\tilde{a}(\overline{n}(\operatorname{Ad}(a_t)X))^{-2\operatorname{Re}\lambda} \to 1$$

uniformly on compacta for $t \to \infty$.

□

For $k \in \mathbb{R}$ we denote by S_k a k-th Sobolev norm for the representation $(\pi_\lambda, \mathcal{H}_\lambda)$.

Corollary 4.3. *Let* $Q \subset \mathfrak{a}_{\mathbb{C}}^*$ *be a compact subset. Then there exists* $\xi \in \mathbb{C}[M\backslash K]$*, constants* $c_1, c_2, c_3 > 0$ *such that*

$$\sup_{\substack{g\in G\\ \|g\|\leq|\tau|^{c_1}}} |\langle \pi_\lambda(g)\xi, v\rangle| \geq c_2 \frac{1}{(|\tau|+1)^{c_3}}\|v\|$$

for all $\lambda \in Q$, $\tau \in \hat{K}_M$ *and* $v \in V_\lambda[\tau]$*. In particular there exist a* $k \in \mathbb{R}$ *such that*

$$\sup_{\substack{g\in G\\ \|g\|\leq|\tau|^{c_1}}} |\langle \pi_\lambda(g)\xi, v\rangle| \geq c_2 S_k(v)$$

for all $\lambda \in Q$, $\tau \in \hat{K}_M$ *and* $v \in V_\lambda[\tau]$*.*

Note $V_\lambda^* \simeq V_{-\lambda}$. Thus Lemma 3.5 in conjunction with the above Corollary yields the Casselman-Wallach Theorem for spherical principal series:

Corollary 4.4. *Let* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *and* V_λ *the Harish-Chandra module of the corresponding spherical principal series. Then* V_λ *admits a unique smooth Fréchet globalization.*

For an element $v \in L^2(M\backslash K)$ and $\tau \in \hat{K}_M$ we write v_τ for the τ-isotypical part.

If we raise $\gamma > 1$ in the Lemma above appropriately, we obtain the following result.

Theorem 4.5. *Let* $Q \subset \mathfrak{a}_{\mathbb{C}}^*$ *be a compact subset and* $N > 0$*. Then there exists* $\xi \in \mathbb{C}[M\backslash K]$ *and constants* $c_1, c_2 > 0$ *such that for all* $\tau \in \hat{K}_M$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$*, there exists* $a_\tau \in A$*, independent of* λ*, with* $\|a_\tau\| \leq (1+|\tau|)^{c_1}$ *and numbers* $d(\lambda,\tau) \in \mathbb{C}$ *such that*

$$\|[\pi_\lambda(a_\tau)\xi]_\tau - d(\lambda,\tau)\delta_\tau\| \leq \frac{1}{(|\tau|+1)^{N+c_2}}$$

and

$$|d(\lambda,\tau)| \geq \frac{1}{(|\tau|+1)^{c_2}}\,.$$

Proof. The proof is a small modification of the proof of Lemma 4.2. Let us emphasize the crucial points. Let $D_\tau^{ij} = \sqrt{d(\tau)}u_j \otimes u_i^*$. Then

$$[\pi_\lambda(a_t)\xi]_\tau = \sum_{ij} d_{ij} D_\tau^{ij} \tag{4.2}$$

For $i \neq j$ we know that $D_\tau^{ij}(0) = 0$. Thus raising γ and employing the analogous estimate as in (4.1) we can obtain on the right hand side of (4.2) any domination of polynomial type of the off diagonal terms against a diagonal term. Finally observe that $D_\tau^{ii}(X) \approx \sqrt{d(\tau)}$ for $X \in \{\|X\| < t_0\}$ with t_0 as in the proof of Lemma 4.2. The assertion follows. □

4.4. **Constructions in the Schwartz algebra**

Let us fix a relatively compact open neighborhood $Q \subset \mathfrak{a}_\mathbb{C}^*$. We choose the K-finite element $\xi \in \mathbb{C}[M\backslash K]$ such that the conclusion of Theorem 4.5 is satisfied.

Lemma 4.6. *Let U be an $\mathrm{Ad}(K)$-invariant neighborhood of $\mathbf{1}$ in G and $\mathcal{F}(U)$ the space of $\mathrm{Ad}(K)$-invariant test functions supported in U. Then there exists a holomorphic map*

$$Q \to \mathcal{F}(U), \quad \lambda \mapsto h_\lambda$$

such that $\Pi_\lambda(h_\lambda)\xi = \xi$.

Proof. Let $V_\xi \subset \mathbb{C}[M\backslash K]$ be the K-module generated by ξ. Let $n := \dim V_\xi$. Let U_0 be a $\mathrm{Ad}(K)$-invariant neighborhood of $\mathbf{1} \in G$ such that $U_0^n \subset U$.

Note that any $h \in \mathcal{F}(U_0)$ induces operators

$$T(\lambda) := \Pi_\lambda(h)|_{V_\xi} \in \mathrm{End}(V_\xi)\,.$$

The compactness of Q allows us to employ uniform Dirac-approximation: we can choose h such that

$$Q \to \mathrm{Gl}(V_\xi), \quad \lambda \mapsto T(\lambda)$$

is defined and holomorphic. Let $n := \dim V_\xi$. By Cayley-Hamilton $T(\lambda)$ is a zero of its characteristic polynomial and hence

$$\mathrm{id}_{V_\xi} = \frac{1}{\det T(\lambda)} \sum_{j=1}^n c_j(\lambda) T(\lambda)^j$$

with $c_j(\lambda)$ holomorphic. Set now

$$h_\lambda := \frac{1}{\det T(\lambda)} \sum_{j=1}^n c_j(\lambda) \underbrace{h_\lambda * \ldots * h_\lambda}_{\text{j-times}}\,.$$

Then $Q \ni \lambda \mapsto h_\lambda \in \mathcal{F}(U)$ is holomorphic and $\Pi_\lambda(h_\lambda)\xi = \xi$. □

For a compactly supported measure ν on G and $f \in \mathcal{S}(G)$ we define $\nu * f \in \mathcal{S}(G)$ by

$$\nu * f(g) = \int_G f(x^{-1}g)\; d\nu(x)\,.$$

For an element $g \in G$ we denote by δ_g the Dirac delta-distribution at g. Further we view δ_τ as a compactly supported measure on G via the correspondence $\delta_\tau \leftrightarrow \delta_\tau(k)\; dk$.

For each $\tau \in \hat{K}_M$ we define $h_{\lambda,\tau} \in \mathcal{S}(G)$ by

$$h_{\lambda,\tau} := \delta_\tau * \delta_{a_{t(\tau)}} * h_\lambda\,. \tag{4.3}$$

Call a sequence $(c_\tau)_{\tau\in\hat{K}_M}$ rapidly decreasing if

$$\sup_\tau |c_\tau|(1+|\tau|)^R < \infty$$

for all $R > 0$.

Lemma 4.7. *Let $(c_\tau)_\tau$ be a rapidly decreasing sequence $(c_\tau)_\tau$ and $h_{\lambda,\tau}$ defined as in (4.3). Then*

$$H_\lambda := \sum_{\tau\in\hat{K}_M} c_\tau \cdot h_{\lambda,\tau}$$

is in $\mathcal{S}(G)$ and the assignment $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is holomorphic.

Proof. Fix $\lambda \in Q$. For simplicity set $H = H_\lambda$, $h_{\lambda,\tau} = h_\tau$.

It is clear that the convergence of H is uniform on compacta and hence $H \in C(G)$. For $u \in \mathcal{U}(\mathfrak{g})$ we record

$$R_u(h_\tau) = \delta_\tau * \delta_{a_{t(\tau)}} * R_u(h)$$

and as a result $H \in C^\infty(G)$. So we do not have to worry about right derivatives. To show that $H \in \mathcal{S}(G)$ we employ Remark 2.3: it remains to show that $H \in \mathcal{R}(G)$, i.e.

$$\sup_{g\in g} \|g\|^r \cdot |H(g)| < \infty \tag{4.4}$$

for all $r > 0$. Fix $r > 0$. Write $g = k_1 a k_2$ for some $a \in A$, $k_1, k_2 \in K$. Then

$$\|g\|^r |h_\tau(g)| \leq \|a\|^r \cdot \sup_{k,k'\in K} |h(a_t^{-1} k a k')|\,.$$

Let $Q \subset A$ be a compact set with $\log Q$ convex and $\mathcal{W}$-invariant and such that $\operatorname{supp} h \subset KQK$. We have to determine those $a \in A$ with

$$a_t^{-1} K a \cap KQK \neq \emptyset \, . \tag{4.5}$$

Define $Q_t \subset A$ through $\log Q_t$ being the convex hull of $\mathcal{W}(\log a_t + \log Q)$. Then (4.5) implies that

$$a \in Q_t \, .$$

But this means that $\|a\| << |\tau|^c$ for some $c > 0$, independent of τ. Hence (4.4) is verified and H is indeed in $\mathcal{S}(G)$.

Finally the fact that the assignment $\lambda \mapsto H_\lambda$ is holomorphic follows from the previous Lemma. □

Theorem 4.8. *Let $Q \subset \mathfrak{a}_\mathbb{C}^*$ be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(M\backslash K) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_\lambda(f(\lambda, v))\xi = v \, .$$

In particular, $\Pi_\lambda(\mathcal{S}(G))V_\lambda = \mathcal{H}_\lambda^\infty$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

Proof. Let $v \in \mathcal{H}_\lambda^\infty$. Then $v = \sum_\tau c_\tau v_\tau$ with v_τ normalized and $(c_\tau)_\tau$ rapidly decreasing. As $\mathcal{S}(G)$ is stable under left convolution with $C^{-\infty}(K)$ we readily reduce to the case where $v_\tau = \frac{1}{\sqrt{d(\tau)}}\delta_\tau$.

In order to explain the idea of the proof let us first treat the case where the Harish-Chandra module is a multiplicity free K-module. This is precisely the case when G is locally a product of real rank one groups.

Recall the definition of $d(\lambda, \tau)$ and define

$$H_\lambda := \sum_\tau \frac{c_\tau}{d(\lambda, \tau)} h_{\lambda,\tau} \, .$$

It follows from Lemma 4.2 and the Lemma above that $Q \ni \lambda \to H_\lambda \in \mathcal{S}(G)$ is defined and holomorphic. By multiplicity one we get that

$$\Pi_\lambda(H_\lambda) = \sum_\tau c_\tau \delta_\tau \, .$$

and the assertion follows for the multiplicity free case.

Let us move to the general case. For that we employ the more general approximation in Theorem 4.5 and set

$$H'_\lambda = \sum_{\tau \in \hat{K}_M} \frac{c_\tau}{d(\lambda, \tau)} h_{\lambda,\tau} \, .$$

Then

$$\Pi_\lambda(H'_\lambda) = \sum_{\tau \in \hat{K}_M} c_\tau \delta_\tau + R$$

where, given $k > 0$, we can assume that $\|R_\tau\| \le |c_\tau| \cdot (|\tau| + 1)^{-k}$ for all τ (choose N in Theorem 4.5 big enough). Finally we remove the remainder R_τ by left convolution with $C^{-\infty}(K)$. □

5. Reduction steps I: extensions, tensoring and induction

In this section we will show that "good" is preserved by induction, tensoring with finite dimensional representations and as well by extensions. We wish to emphasize that these results are not new can be found for instance in [9], Sect. 11.7.

5.1. Extensions

Lemma 5.1. *Let*

$$0 \to U \to L \to V \to 0$$

be an exact sequence of Harish-Chandra modules. If U and V are good, then L is good.

Proof. Let $(\pi, \overline{L})$ be a smooth Fréchet globalization of L. Define a smooth Fréchet globalization $(\pi_U, \overline{U})$ of U by taking the closure of U in $\overline{L}$. Likewise we define a smooth Fréchet globalization $(\pi_V, \overline{V})$ of $V = L/U$ by $\overline{V} := \overline{L}/\overline{U}$. By assumption we have $\overline{U} = \Pi_U(\mathcal{S}(G))U$ and $\overline{V} = \Pi_V(\mathcal{S}(G))V$. As $0 \to \overline{U} \to \overline{L} \to \overline{V} \to 0$ is exact, we deduce that $\Pi(\mathcal{S}(G))L = \overline{L}$, i.e. L is good. □

As Harish-Chandra modules admit finite composition series we conclude:

Corollary 5.2. *In order to show that all Harish-Chandra modules are good it is sufficient to establish that all irreducible Harish-Chandra modules are good.*

5.2. Tensoring with finite dimensional representations

This subsection is devoted to tensoring a Harish-Chandra module with a finite dimensional representation.

Let V be a Harish-Chandra module and V^∞ its minimal globalization. Let (σ, W) be a finite dimensional representation of G. Set

$$\mathbf{V} := V \otimes W$$

and note that $\mathbf{V}$ is a Harish-Chandra module as well. It is our goal to show that the minimal globalization of $\mathbf{V}$ is given by $V^\infty \otimes W$.

Let us fix a covariant inner product $\langle\cdot,\cdot\rangle$ on W. Let $w_1, \ldots, w_k$ be a corresponding orthonormal basis of W. With that we define the $C^\infty(G)$-valued $k \times k$-matrix

$$\mathfrak{S} := (\langle \sigma(g) w_i, w_j\rangle)_{1\leq i,j\leq k}$$

and record the following:

Lemma 5.3. *With the notation introduced above, the following assertions hold:*

(i) *The map*

$$\mathcal{S}(G)^k \to \mathcal{S}(G)^k,\ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})$$

is a linear isomorphism.

(ii) *The map*

$$[C_c^\infty(G)]^k \to [C_c^\infty(G)]^k,\ \mathbf{f} = (f_1, \ldots, f_k) \mapsto \mathfrak{S}(\mathbf{f})\,.$$

is a linear isomorphism.

Proof. First, we observe that the determinant of $\mathfrak{S}$ is 1 and hence $\mathfrak{S}$ is invertible. Second, all coefficients of $\mathfrak{S}$ and $\mathfrak{S}^{-1}$ are of moderate growth, i.e. dominated by a power of $\|g\|$. Both assertions follow. □

Lemma 5.4. *Let V be a Harish-Chandra module and (σ, W) be a finite dimensional representation of G. Then*

$$\mathbf{V}^\infty = V^\infty \otimes W\,.$$

Proof. We denote by $\tilde\pi = \pi \otimes \sigma$ the tensor representation of G on $V^\infty \otimes W$. It is sufficient to show that $v \otimes w_j$ lies in $\tilde\Pi(\mathcal{S}(G))\mathbf{V}$ for all $v \in V^\infty$ and $1 \leq j \leq k$.

Fix $v \in V^\infty$. It is no loss of generality to assume that $j = 1$. By assumption we find $\xi \in V$ and $f \in \mathcal{S}(G)$ such that $\Pi(f)\xi = v$.

We use the previous lemma and obtain an $\mathbf{f} = (f_1, \ldots, f_k) \in \mathcal{S}(G)^k$ such that

$$\mathfrak{S}^t(\mathbf{f}) = (f, 0, \ldots, 0)\,.$$

We claim that

$$\sum_{j=1}^k \tilde\Pi(f_j)(\xi \otimes w_j) = v \otimes w_1\,.$$

In fact, contracting the left hand side with $w_i^* = \langle\cdot, w_i\rangle$ we get that

$$(\mathrm{id}\otimes w_i^*)\left(\sum_{j=1}^k \tilde\Pi(f_j)(\xi\otimes w_j)\right) = \sum_{j=1}^k \int_G f_j(g)\langle\sigma(g)w_j, w_i\rangle \pi(g)\xi\ dg$$
$$= \delta_{1i}\cdot \int_G f(g)\pi(g)\xi\ dg = \delta_{1i}\cdot v$$

and the proof is complete. □

5.3. **Induction**

Let $P \supset P_{\min}$ be a parabolic subgroup with Langlands decomposition

$$P = N_P A_P M_P\,.$$

Note that $A_P < A$, $M_P A_P = Z_G(A_P)$ and $N = N_P \rtimes (M_P \cap N)$. For computational purposes it is useful to recall that parabolics P above $P_{\min}$ are parameterized by subsets F of the simple roots Π in $\Sigma(\mathfrak{a},\mathfrak{n})$. We then often write P_F instead of P, A_F instead of A_P etc. The correspondence $F \leftrightarrow P_F$ is such that

$$A_F = \{a\in A \mid (\forall\alpha\in F)\ a^\alpha = 1\}\,.$$

We make an emphasis on the two extreme cases for F, namely: $P_\emptyset = P_{\min}$ and $P_\Pi = G$.

In the sequel we write $\mathfrak{a}_P$, $\mathfrak{n}_P$ for the Lie algebras of A_P and N_P and denote by $\rho_P\in\mathfrak{a}_P^*$ the usual half sum. Note that $K_P := K\cap M_P$ is a maximal compact subgroup of M_P. Let V_σ be a Harish-Chandra module for M_P and $(\sigma, V_\sigma^\infty)$ its minimal SF-globalization.

For $\lambda \in (\mathfrak{a}_P)_\mathbb{C}^*$ we define as before the smooth principal series with parameter (σ,λ) by

$$E_{\sigma,\lambda} = \{f\in C^\infty(G, V_\sigma^\infty)) \mid (\forall\ nam\in P\ \forall\ g\in G)$$
$$f(namg) = a^{\rho_P+\lambda}\sigma(m)f(g)\}\,.$$

and representation $\pi_{\sigma,\lambda}$ by right translations in the arguments of functions in $E_{\sigma,\lambda}$.

In this context we record:

Proposition 5.5. *Let $P\supseteq P_{\min}$ be a parabolic subgroup with Langlands decomposition $P = N_PA_PM_P$. Let V_σ be a good Harish-Chandra module for M_P. Then for all $\lambda\in(\mathfrak{a}_P)_\mathbb{C}^*$ the induced Harish-Chandra module $V_{\sigma,\lambda}$ is good. In particular, $V_{\sigma,\lambda}^\infty = E_{\sigma,\lambda}$.*

Note that $V_{\sigma,\lambda}^* = V_{\sigma^*,-\lambda}$ is induced from the good module V_σ^*. Hence it is sufficient to establish lower bounds (Lemma 3.5). We will do this later in greater generality with dependence on parameters in Theorem 7.6 below.

6. Reduction steps II: deformations and discrete series

The goal of this section is to show that all Harish-Chandra modules are good. Note that all Harish-Chandra modules V can be written as a quotient U/H where U is good. Suppose that H is in fact a kernel of an intertwiner $I : U \to W$ with W good. Suppose in addition that we can deform $I : U \to W$ holomorphically (as to be made precise in the following section). Then, provided U and W are good we will show that $\operatorname{im} I \simeq U/H$ is good. Finally we will show that every irreducible Harish-Chandra module V is a direct summand of $\operatorname{im} I$ where $I : U \to W$ is a deformable intertwiner of good modules.

6.1. Deformations

For a complex manifold D and a Harish-Chandra module U we write $\mathcal{O}(D, U)$ for the space of maps $f : D \to U$ such that for all $\xi \in U^*$ the contraction $\xi \circ f$ is holomorphic. Henceforth we will use D exclusively for the open unit disc.

By a holomorphic family of Harish-Chandra modules (parameterized by D) we understand a family of Harish-Chandra modules $(U_s)_{s\in D}$ such that:

(i) For all $s \in D$ one has $U_s = U_0 =: U$ as K-modules.
(ii) For all $X \in \mathfrak{g}$, $v \in U$ and $\xi \in U^*$ the map $s \mapsto \xi(X_s \cdot v)$ is holomorphic. Here we use X_s for the action of X in U_s.

Given a holomorphic family $(U_s)_{s\in D}$ we form $\mathcal{U} := \mathcal{O}(D, U)$ and endow it with the following $(\mathfrak{g}, K)$-structure: for $X \in \mathfrak{g}$ and $f \in \mathcal{U}$ we set

$$(X \cdot f)(s) := X_s \cdot f(s)\,.$$

Of particular interest are the Harish-Chandra modules $\mathbf{U}_k := \mathcal{U}/s^k\mathcal{U}$ for $k \in \mathbb{N}$. To get a feeling for this objects let us discuss a few examples for small k.

Example 6.1. *(a) For $k = 1$ the constant term map*

$$\mathbf{U}_1 \to U, \quad f + s\mathcal{U} \mapsto f(0)$$

is an isomorphism of $(\mathfrak{g}, K)$-modules.

(b) For $k = 2$ we observe that the map

$$\mathbf{U}_2 \to U \oplus U, \quad f + s^2\mathcal{U} \mapsto (f(0), f'(0))$$

provides an isomorphism of K-modules. The resulting $\mathfrak{g}$-action on the right hand side is twisted and given by

$$X \cdot (u_1, u_2) = (Xu_1, Xu_2 + X'u_1)$$

where

$$X'u := \frac{d}{ds}\Big|_{s=0} X_s \cdot u\,.$$

Let us remark that $X' = 0$ for all $X \in \mathfrak{k}$.

We notice that $\mathbf{U}_2$ features the submodule $s\mathcal{U}/s^2\mathcal{U}$ which corresponds to $\{0\} \oplus U$ in the above trivialization. The corresponding quotient $(\mathcal{U}/s^2\mathcal{U})/(s\mathcal{U}/s^2\mathcal{U})$ identifies with $U \oplus U/\{0\} \oplus U \simeq U$. In particular $\mathcal{U}/s^2\mathcal{U}$ is good if U is good by the extension Lemma 5.1.

From the previous discussion it follows that $\mathbf{U}_k$ is good for all $k \in \mathbb{N}_0$ provided that U is good.

Let now W be another Harish-Chandra module and $\mathcal{W}$ a holomorphic deformation of W as above. By a morphism $\mathcal{I} : \mathcal{U} \to \mathcal{W}$ we understand a family of $(\mathfrak{g}, K)$-maps $I_s : U_s \to W_s$ such that for all $u \in U$ and $\xi \in W^*$ the assignments $s \mapsto \xi(I_s(u))$ are holomorphic. Let us write I for I_0 set $I' := \frac{d}{ds}\Big|_{s=0} I_s$ etc. We set $H := \ker I$.

We now make two additional assumptions on our holomorphic family of intertwiners:

- I_s is invertible for all $s \neq 0$.
- There exists a $k \in \mathbb{N}_0$ such that $J(s) := s^k I_s^{-1}$ is holomorphic on D.

If these conditions are satisfied, then we call $I : U \to W$ *holomorphically deformable*.

For all $m \in \mathbb{N}$ we write $\mathbf{I}_m : \mathbf{U}_m \to \mathbf{W}_m$ for the intertwiner induced by $\mathcal{I}$. Likewise we define $\mathbf{J}_m$.

Example 6.2. *In order to get a feeling for the intertwiners $\mathbf{I}_m$ let us consider the example $\mathbf{I}_2 : \mathbf{U}_2 \to \mathbf{W}_2$. In triviliazing coordinates this map is given by*

$$\mathbf{I}_2(u_1, u_2) = (I(u_1), I(u_2) + I'(u_1))\,.$$

We set $\mathbf{H}_m := \ker \mathbf{I}_m \subset \mathbf{U}_m$. For $m < n$ we view $\mathbf{U}_m$ as a K-submodule of $\mathbf{U}_n$ via the inclusion map

$$\mathbf{U}_m \to \mathbf{U}_n, \quad f + s^m\mathcal{U} \mapsto \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} s^j + s^n\mathcal{U}\,.$$

We write $p_{n,m} : \mathbf{U}_n \to \mathbf{U}_m$ for the reverting projection (which are $(\mathfrak{g}, K)$-morphisms).

The following Lemma is related to an observation of Casselman as recorded in [9], 11.7.9.

Lemma 6.3. *The morphism*

$$\mathbf{I}_{2k}|_{\mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}} : \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U} \to s^k\mathcal{W}/s^{2k}\mathcal{W}$$

is onto. Moreover, its kernel is given by $s^k\mathbf{H}_k \subset s^k\mathcal{U}/s^{2k}\mathcal{U}$*.*

Proof. Clearly, $\mathbf{I}_{2k}^{-1}(s^k\mathcal{W}/s^{2k}\mathcal{W}) \subset \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$ and hence the map is defined. Let us check that it is onto. Let $[w] \in s^k\mathcal{W}/s^{2k}\mathcal{W}$ and $w \in s^k\mathcal{W}$ be a representative. Note that $s^{-k}\mathcal{J}|_{s^k\mathcal{W}} : s^k\mathcal{W} \to \mathcal{U}$ is defined. Set $u := s^{-k}\mathcal{J}(w)$ and write $[u]$ for its equivalence class in $\mathbf{U}_{2k}$. Then $\mathbf{I}_{2k}([u]) = [w]$ and the map is onto.

A simple verification shows that $s^k\mathbf{H}_k$ lies in the kernel. Hence by considering the surjective map K-type by K-type we arrive that it equals the kernel by dimension count. □

If we set $V_3 := \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$, $V_2 := s^k\mathcal{U}/s^{2k}\mathcal{U}$ and $V_1 := s^k\mathbf{H}_k$, the previous Lemma implies an inclusion chain

$$V_1 \subset V_2 \subset V_3$$

with

$$V_2/V_1 \simeq \mathbf{U}_k/\mathbf{H}_k, \qquad V_2 \simeq \mathbf{U}_k \qquad \text{and} \qquad V_3/V_1 \simeq \mathbf{W}_k\,.$$

Hence in combination with the squeezing Lemma 3.3 we obtain that $\mathbf{U}_k/\mathbf{H}_k$ is good if U and W are good.

We wish to show that U/H is good. This follows now by iteration and it is enough to consider the case $k = 2$ in more detail. Write $H_{2,1} := p_{2,1}(\mathbf{H}_2)$ for the projection of $\mathbf{H}_2$ to $\mathbf{U}_1 \simeq U$. Note that $H_{2,1}$ is a submodule of H. We arrive at the exact sequence

$$0 \to U/H \simeq sU/sH \to \mathbf{U}_2/\mathbf{H}_2 \to U/H_{2,1} \to 0\,.$$

But U/H is a quotient of $U/H_{2,1}$. Thus putting an SF-topology on U we get one on H, $\mathbf{U}_2$, $\mathbf{H}_2$, $\mathbf{U}_2/\mathbf{H}_2$ and $U/H_{2,1}$. As a result the induced topology on U/H is both a sub and a quotient of the good topology on $\mathbf{U}_2/\mathbf{H}_2$. Hence U/H is good.

We summarize our discussion.

Proposition 6.4. *Suppose that* $I : U \to W$ *is an intertwiner of good Harish-Chandra modules which allows holomorphic deformations* $\mathcal{I} : \mathcal{U} \to \mathcal{W}$*. Then* $\operatorname{im} I$ *is good.*

6.2. **All Harish-Chandra modules are good**

In this subsection we will prove that all Harish-Chandra modules are good. In view of the deformation result (Prop. 6.4) and the Langlands classification we are readily reduced to the case of discrete series representation. Our present proof uses certain upper bound for matrix coefficients as found in [8].

Lemma 6.5. *All Harish-Chandra modules are good.*

Proof. Let V be a Harish-Chandra module. We have to show that V is good. In view of Corollary 5.2, we may assume that V is irreducible. Next we use Langland's classification (see [5],Ch. VIII, Th. 8.54) and commbine it with our Proposition 6.4 on deformation. This reduces to the case where V is tempered. However, the case of tempered readily reduces to square integrable ([8], Ch. 5, Prop. 5.2.5). Fortunately the case of square integrable is handled in [9], 11.7.4 under the use of the upper bound 4.3.5 in [8]. □

Remark 6.6. *We intend to return to the subject of this subsection via the theory of Jacquet modules.*

7. Applications

7.1. **Lifting $(\mathfrak{g}, K)$-morphisms**

Let (π, E) be an SF-representation of G. Note that we do not assume that E is admissible. Typical examples we have in mind are smooth functions of moderate growth on certain homogeneous spaces. Let us mention a few.

Example 7.1. *(a) Let $\Gamma < G$ be a lattice, that is a discrete subgroup with cofinite volume. Reduction theory (Siegel sets) allows us to control "infinity" of the quotient $Y := \Gamma\backslash G$ and leads to a natural notion of moderate growth. The smooth functions of moderate growth $C^\infty_{\rm mod}(Y)$ become an SF-module for the right regular action of G. The space of K and $\mathcal{Z}(\mathfrak{g})$-finite elements in $C^\infty_{\rm mod}(Y)$ is referred to as the space of automorphic forms on Y.*
(b) Let $H < G$ be a symmetric subgroup, i.e. an open subgroup of the fixed point set of an involutive automorphism of G. We refer to $X := H\backslash G$ as a semisimple symmetric space. The Cartan-decomposition of X allows us to control growth on X and yields the SF-module $C^\infty_{\rm mod}(X)$ of smooth functions with moderate growth.

If (π_1, E_1), (π_2, E_2) are two SF-represesentations, then we denote by $\mathrm{Hom}_G(E_1, E_2)$ for the space of continuous G-equivariant linear maps from E_1 to E_2.

Proposition 7.2. *Let V be a Harish-Chandra module and V^∞ its unique SF-globalization. Then for any SF-representation (π, E) of G the linear map*

$$\mathrm{Hom}_G(V^\infty, E) \to \mathrm{Hom}_{(\mathfrak{g},K)}(V, E_K), \quad T^\infty \mapsto T := T^\infty|_V$$

is a linear isomorphism.

Proof. It is clear that the map is injective and move on to onto-ness. Let us write λ for the representation of G on V^∞. Let $v \in V^\infty$. Then we find $f \in \mathcal{S}(G)$ such that $v = \Lambda(f)w$ for some $w \in V$. We claim that

$$T^\infty(v) := \Pi(f)T(w)$$

defines a linear operator. In order to show that this definition makes sense we have to show that $T^\infty(v) = 0$ provided that $\Lambda(f)w = 0$. Let $\xi \in (E^*)_K$ and $\mu := \xi \circ T \in V^*$. We consider two distributions on G, namely

$$\Theta_1(\phi) := \xi(\Pi(\phi)T(w)) \quad \text{and} \quad \Theta_2(\phi) := \mu(\Lambda(\phi)w) \qquad (\phi \in C_c^\infty(G))\,.$$

We claim that $\Theta_1 = \Theta_2$. In fact, both distributions are $\mathcal{Z}(\mathfrak{g})-$ and $K \times K$-finite. Hence they are represented by analytic functions on G and thus uniquely determined by their derivatives on K. The claim follows.

It remains to show that T is continuous. We recall the construction of the minimal SF-globalization of V, namely $V^\infty = \mathcal{S}(G)/\mathcal{S}(G)_{\mathbf{v}}$ within the notation of Subsection 3.1. Continuity becomes clear. □

7.2. **Automatic continuity**

If V is a Harish-Chandra module, then we write V_{alg}^* for its algebraic dual. Note that V_{alg}^* is naturally a module for $\mathfrak{g}$.

If $\mathfrak{h} < \mathfrak{g}$ is a subalgebra, then we write $(V_{\mathrm{alg}}^*)^{\mathfrak{h}}$, resp. $(V_{\mathrm{alg}}^*)^{\mathfrak{h}-\mathrm{fin}}$, for the space of $\mathfrak{h}$-fixed, resp. $\mathfrak{h}$-finite, algebraic linear functionals on V.

We call a sublagebra $\mathfrak{h} < \mathfrak{g}$ a *(strong) automatic continuity* subalgebra ((S)AC-subalgebra for short) if for all Harish-Chandra modules V one has

$$(V_{\mathrm{alg}}^*)^{\mathfrak{h}} \subset (V^\infty)^* \quad \text{resp.} \quad (V_{\mathrm{alg}}^*)^{\mathfrak{h}-\mathrm{fin}} \subset (V^\infty)^*\,.$$

Problem 7.3. *(a) Is it true that* $\mathfrak{h}$ *is AC if and only if* $\langle \exp \mathfrak{h} \rangle < G$ *has an open orbit on* $G/P_{\min}$.
(b) Is it true that $\mathfrak{h}$ *is SAC if* $[\mathfrak{h}, \mathfrak{h}]$ *is AC?*

The following examples of (S)AC-subalgebras are known:

- $\mathfrak{n}$, the Lie algebra of an Iwasawa N-subgroup, is AC and $\mathfrak{a}+\mathfrak{n}$, the Lie algebra of an Iwasasaw AN-subgroup, is SAC. (Casselman).
- Symmetric subalgebras, i.e. fixed point sets of involutive automorphisms of $\mathfrak{g}$, are AC (Brylinski, Delorme, van den Ban; cf. [2], [1]).

Here we only wish to discuss Casselman's result. We recall the definition of the Casselman-Jacquet module $j(V) = \bigcup_{k\in\mathbb{N}_0}(V/\mathfrak{n}^k V)^*$ and note that $j(V) = (V^*_{\rm alg})^{\mathfrak{a}+\mathfrak{n}-{\rm fin}}$.

Theorem 7.4. (Casselman) *Let* $\mathfrak{n}$ *be the Lie algebra of an Iwasawa* N*-subgroup of* G *and* $\mathfrak{a}+\mathfrak{n}$ *the Lie algebra of an Iwasawa* AN*-subgroup. Then* $\mathfrak{n}$ *is an AC and* $\mathfrak{a}+\mathfrak{n}$ *is SAC. In particular, for all Harish-Chandra modules* V *one has* $j(V) \subset (V^\infty)^*$.

Proof. Let us first prove that $\mathfrak{a}+\mathfrak{n}$ is SAC. Let V be a Harish-Chandra module and $0 \neq \lambda \in j(V)$. By definition there exists a $k \in \mathbb{N}$ such that $\lambda \in (V/\mathfrak{n}^k V)^*$. Write (σ, U_σ) for the finite dimensional representation of $P_{\min}$ on $V/\mathfrak{n}^k V$ and denote by I_σ the corresponding induced Harish-Chandra module. Note that $I_\sigma^\infty = C^\infty(G \times_{P_{\min}} U_\sigma)$.

Applying Frobenius reciprocity to the identity morphism $V/\mathfrak{n}^k V \to U$ yields a non-trivial $(\mathfrak{g}, K)$-morphism $T : V \to I_\sigma$ (cf. [8], 4.2.2). Now T lifts to a continuous G-map $T^\infty : V^\infty \to I_\sigma^\infty$ by Proposition 7.2. If ${\rm ev} : I_\sigma^\infty \to U_\sigma$ denotes the evaluation map at the identity, then $\lambda^\infty := \lambda \circ {\rm ev} \circ T^\infty$ provides a continuous extension of λ to V^∞.

The proof that $\mathfrak{n}$ is AC goes along the same lines. □

7.3. **Lifting of holomorphic families of** $(\mathfrak{g}, K)$**-maps**

We wish to give a version of lifting (cf. Proposition 7.2) which depends holomorphically on parameters.

Theorem 7.5. *Let* $P = N_P A_P M_P$ *be a parabolic subgroup and* V_σ *a Harish-Chandra module of an SAF-representation of* M_P*. Let* (π, E) *be an SF-representation. Suppose that there is a family of* $(\mathfrak{g}, K)$*-intertwiners* $T_\lambda : V_{\sigma,\lambda} \to E_K$ *such that for all* $v \in \mathbb{C}[K \times_{K_P} V_\sigma]$ *and* $\xi \in E^*$ *the assignments* $\lambda \mapsto \xi(T_\lambda(v))$ *are holomorphic. Then for all*

$v \in C^\infty(K \times_{K_P} V_\sigma^\infty)$ and $\xi \in E^*$, the maps

$$(\mathfrak{a}_P)^*_{\mathbb{C}} \ni \lambda \mapsto \xi(T^\infty_\lambda(v)) \in \mathbb{C}$$

are holomorphic.

The proof is an immediate consequence of the analogue to Theorem 4.5 and Theorem 4.8 for the induced family considered.

In the sequel we will use the notation introduced in Subsection 4.5. We let $P = N_P A_P M_P$ be a parabolic above $P_{\min}$. We fix an SAF-representation $(\sigma, V_\sigma^\infty)$ of M_P and write V_σ for the corresponding Harish-Chandra module.

As K-modules we identify all $V_{\sigma,\lambda}$ with $V := \mathbb{C}[K \times_{K_P} V_\sigma]$. Note that V_σ is a K_P-quotient of some $\mathbb{C}[M\backslash K_P]^m$, $m \in \mathbb{N}$. Double induction gives an identification of V as a K-quotient of $\mathbb{C}[M\backslash K]^m$. Note that $C^\infty(M\backslash K)^m$ induces the unique SF-topology on V^∞. For each τ we write $D_{\sigma,\tau}$ for the orthogonal projection of $\underbrace{(\delta_\tau, \ldots, \delta_\tau)}_{m-times}$ to $V[\tau]$, the τ-isotypical part of V.

Theorem 7.6. *Let $P = N_P A_P M_P$ be a parabolic subgroup and V_σ an irreducible unitarizable Harish-Chandra module for M_P. Let $Q \subset (\mathfrak{a}_P)^*_{\mathbb{C}}$ be a compact subset and $N > 0$. Then there exists $\xi \in \mathbb{C}[V_\sigma \times_{K_P} K]$ and constants $c_1, c_2 > 0$ such that for all $\tau \in \hat{K}$, $\lambda \in (\mathfrak{a}_P)^*_{\mathbb{C}}$, there exists $a_\tau \in A$, independent of λ, with $\|a_\tau\| \leq (1 + |\tau|)^{c_1}$ and numbers $d_\sigma(\lambda, \tau) \in \mathbb{C}$ such that*

$$\|[\pi_{\sigma,\lambda}(a_\tau)\xi]_\tau - d_\sigma(\lambda,\tau) D_{\sigma,\tau}\|_2 \leq \frac{1}{(|\tau\| + 1)^{N+c_2}}$$

and

$$|d_\sigma(\lambda,\tau)| \geq \frac{1}{(|\tau| + 1)^{c_2}}\,.$$

Here $\|\cdot\|_2$ refers to the continuous norm on V induced by the realization of V as a quotient of $C[K]^m \subset L^2(K)^m$.

Proof. Let us first discuss the case where $P = P_{\min}$ and σ is finite dimensional. Here the assertion is a simple modification of Theorem 4.5.

As for the general case we identify V_σ as a quotient of a minimal principal series for M_P. Using double induction we can write the $V_{\sigma,\lambda}$'s consistently as quotients of such minimal principal series. The assertion follows. □

As a consequence we get an extension of Theorem 4.8.

Theorem 7.7. *Let $Q \subset (\mathfrak{a}_P)^*_{\mathbb{C}}$ be a compact subset. Then there exist a continuous map*

$$Q \times C^\infty(K \times_{K_P} V_\sigma^\infty) \to \mathcal{S}(G), \quad (\lambda, v) \mapsto f(\lambda, v)$$

which is holomorphic in the first variable, linear in the second and such that

$$\Pi_{\sigma,\lambda}(f(\lambda, v))\xi = v\,.$$

8. **Appendix**

8.1. **Representations of moderate growth**

Let us first recall Casselman's definition of a moderate growth representation, [3] p. 391. A representation (π, E) is called of *moderate growth*, if

- E is a Fréchet space.
- For every semi-norm p on E there exists a semi-norm q on E and an integer $N > 0$ such that

  $$p(\pi(g)v) \leq \|g\|^N q(v)$$

  for all $g \in G$.

Lemma 8.1. *A representation (π, E) is Fréchet if and only if it is of moderate growth.*

Proof. Recall for Banach-representations the following fact: if (π, F) is a Banach representation, then there there exista a constant $r > 0$ such that $\|\pi(g)\| \leq \|g\|^r$ for all $g \in G$ (cf. [8], Lemma 2.A.2.2). Hence a Fréchet representation is of moderate growth.

Conversely, assume that (π, E) is of moderate growth and let p, q and $N > 0$ be as in the definition above. Then

$$\tilde{p}(v) := \sup_{g\in G} \frac{p(\pi(g)v)}{\|g\|^N}$$

defines a semi-norm on E such that

- $p \leq \tilde{p} \leq q$.
- $\tilde{p}(\pi(g)v) \leq \|g\|^N \tilde{p}(v)$ for all $g \in G$.

The first bulleted item implies that the $\tilde{p}$ define the topology on E and the second bulleted item yields that $G \times (E, \tilde{p}) \to (E, \tilde{p})$ is continuous. □

8.2. **Weighted L^2-spaces**

This paragraph is concerned with embeddings of Fréchet representations in weighted L^2-spaces.

For $n > 0$ we define the weighted Hilbert-space $L^2(G)_n := L^2(G, \|g\|^n dg)$ and the weighted Banach-space $C(G)_n := \{f \in C(G) \mid p_n(f) := \sup_{g\in G} \frac{|f(g)|}{\|g\|^n} < \infty\}$. We view both as modules for G under the right regular representation.

Let $k > 0$ be such that $\int_G \|g\|^{-k}\, dg < \infty$. Then for all $n > 0$ one obtains a continuous embedding

$$C(G)_{(n+k)/2} \to L^2(G)_n\,. \tag{8.1}$$

Lemma 8.2. *For all $N > 0$ one has*

$$\bigcap_{n\geq N} C(G)_n^\infty = \bigcap_{n\geq N} L^2(G)_n^\infty\,.$$

Proof. The inclusion $'' \subseteq''$ follows from (8.1).

For the converse one notes that we can control the derivatives of a smoothenig of $\|\cdot\|$ and the reverse inclusion is implied by the Sobolev-Lemma. □

Let now (π, E) be an SF-globalization of a Harish-Chandra module V. Let q be a continuous semi-norm on E and $N > 0$ be such that

$$q(\pi(g)v) \leq \|g\|^N q(v)$$

for all $v \in E$.

For $\xi \in V^*$ and $v \in V$ we define the matrix coefficient

$$m_{v,\xi}(g) := \xi(\pi(g)v) \qquad (g \in G)\,.$$

By assumption, we have

$$|m_{v,\xi}(g)| \leq q^*(\xi)\cdot q(v)\cdot \|g\|^N\,.$$

Let us assume now that $\xi \in V^*$ is cyclic. Then, for $n \leq N$, the map

$$\phi_\xi : E \to C(G)_n, \quad v \mapsto m_{v,\xi}$$

defines a G-equivariant continuous embedding. Likewise if $(n+k)/2 \leq N$ then ϕ_ξ induces a continuous G-equivariant embedding $\psi_\xi : E \to L^2(G)_n$. Write E_n for the closure of $\phi_\xi(E)$ in $C(G)_n$ and $\mathcal{H}_n$ for the closure $\psi_\xi(E)$ in $L^2(G)_n$. It is straightforward that E_n, resp. $\mathcal{H}_n$, is a Banach (resp. Hilbertian) globalization of V.

Lemma 8.3. *Within the notation introduced above, the smooth vectors in $\mathcal{H}_n$ coincide with the K-smooth vectors.*

Proof. The proof is not hard and can be found in (5) on p. 91 of [9]. It will not be repeated here.

□

References

[1] E. van den Ban and P. Delorme, *Quelques propriétés des représentations sphériques pour les espaces symétriques réductifs*, J. Funct. Anal. **80** (1988), no. **2**, 284–307.
[2] J.-L. Brylinski and P. Delorme, *Vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein*, Invent. Math. **109** (1992), no. **3**, 619–664.
[3] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), no. **3**, 385–438.
[4] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **(2) 102** (1978), no. **4**, 307–330.
[5] A.W. Knapp, *Representation theory of semisimple groups. An overview based on examples*, Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.
[6] W. Soergel, *An irreducible not admissible Banach representation of* $\mathrm{SL}(2, R)$, Proc. Amer. Math. Soc. **104** (1988), no. **4**, 1322–1324.
[7] E.C. Titchmarsh, *The theory of functions*, Oxford Univ. Press, 1986
[8] N. Wallach, *Real reductive groups I*, Academic Press 1988
[9] N. Wallach, *Real reductive groups II*, Academic Press 1992

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel,, email:bernstei@math.tau.ac.il

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, email: kroetz@mpim-bonn.mpg.de