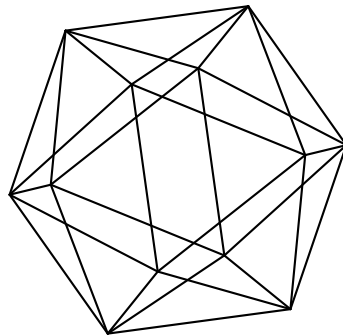# Max-Planck-Institut für Mathematik Bonn

Manin's conjecture vs. Malle's conjecture

by

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MANIN'S CONJECTURE VS. MALLE'S CONJECTURE

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ABSTRACT. By a heuristic argument, we relate two conjectures. One is a version of Manin's conjecture about the distribution of rational points on a Fano variety. We concern specific singular Fano varieties, namely quotients of projective spaces by finite group actions, and their singularities play a key role. The other conjecture is a generalization of Malle's conjecture about the distribution of extensions of a number field. Main tools are several Dirichlet series and previously obtained techniques, especially the untwisting, for the counterpart over a local field.

CONTENTS

1. INTRODUCTION

The aim of this paper is to relate the distribution of rational points on a Fano variety to the distribution of extensions of a number field. Central in the two problems are respectively Manin's conjecture [19] and Malle's conjecture [26, 27]. We consider variants of these conjectures, and see by a heuristic argument that they explain each other.

Let K be a number field and X a Fano variety over K having at worst log terminal singularities. Giving an adelic metric to the anti-canonical divisor $-K_X$

1

defines a height function $H : X(K) \to \mathbb{R}_{>0}$. For a subset $U \subset X(K)$ and for a real number $B > 0$, we let

$$\mathcal{N}_U(B) := \sharp\{x \in U \mid H(x) \le B\}.$$

We are interested in the asymptotic behavior of this number as B tends to infinity. As a subset U, we consider a *cothin subset,* the complement of a thin subset. In turn, a *thin subset* of $X(K)$ is defined as a subset contained in the image of $Y(K)$ for a generically finite morphism $Y \to X$ admitting no rational section.

We consider the following version of Manin's conjecture:

Conjecture 1.1 (Conjecture 5.6)**.** *Suppose that X is a log terminal Fano variety.*

(1) *If X is canonical (has only canonical singularities) and K is sufficiently large, then for a sufficiently small cothin subset $U \subset X(K)$, we have*

$$\mathcal{N}_U(B) \sim CB(\log B)^{\rho(X)+\gamma(X)-1},$$

where C is a positive constant, $\rho(X)$ is the Picard number of X and $\gamma(X)$ is the number of crepant divisors over X.

(2) *If X is not canonical and if K is sufficiently large, then for any cothin subset $U \subset X(K)$, we have*

$$\mathcal{N}_U(B) \sim CB^\alpha(\log B)^\beta$$

with $\alpha > 1$ and $\beta \ge 0$.

The use of cothin subsets was first suggested by Peyre [30] and more seriously discussed by Le Rudulier [32] and Browning–Loughran [10]. This modification of considered subsets is appropriate for quotient varieties which we will consider. It is also explained in relation to the distribution of number field extensions (Remark 12.13). The formula in the first assertion is a special case of a formula considered by Batyrev–Tschinkel [6] and also deduced from a conjecture by Batyrev–Manin [2].

We will relate this version of Manin's conjecture with the distribution of Galois extensions of a number field. Let K still denote a number field and let G be a finite group. We mean by a *G-field (over K)* a finite Galois extension L/K endowed with an isomorphism $\mathrm{Gal}(L/K) \cong G$. We denote the G-equivariant isomorphism classes of G-fields by G-$\mathrm{Fie}(K)$. Let V be a finite-dimensional faithful G-represenatation V over K and S a finite set of places of K containing all infinite places such that V is defined over the S-integer ring $\mathcal{O}_S$. We will define the notion of V-*discrimiant* of G-fields and denote them by D_L^V. This invariant is the global version of counting functions for extensions of a local field which appeared in the study of the wild McKay correspondence [36, 43, 40, 39, 41]. Dummit [16] introduced a probably closely related invariant of ρ-*discriminants* and studied the distribution of G-fields with respect to it. Although not knowing the precise relation between V- and ρ-discriminants, the author was influenced by Dummit's work. For a real number $B > 0$, let $N_{G,V,K}(B)$ be the number of isomorphism classes of G-fields L over K with $D_L^V \le B$. To formulate our conjecture on the distribution of G-fields, we recall that there exists the *age* function age : $G \to \mathbb{Q}_{\ge 0}$ associated to the given G-represenatation V, which often appears in the context of the McKay correspondence (see [20]). Let age(G) be the minimum of age(g), $g \in G \setminus \{1\}$ and $\upsilon(G)$ the number of non-trivial K-conjugacy classes (conjugacy classes of G modulo a $\mathrm{Gal}(\overline{K}/K)$-action) with minimum age. We raise the following conjecture:

Conjecture 1.2 (Conjecture 12.10). *If K is sufficiently large, then*

$$N_{G,V,K}(B) \sim CB^{1/\operatorname{age}(G)}(\log B)^{v(G)}.$$

Except that K is sufficiently large, this contains Malle's conjecture [27] as a special case: if G is a transitive subgroup of the symmetric group S_n and if $V = K^{2n}$ is the direct sum of two copies of the natural permutation representation, then the above conjecture is equivalent to Malle's conjecture. For a technical reason, we also introduce a variant of V-discriminant, the *extended V-discriminant,* denoted $\tilde{D}_L^V$. Since for some constants $C_1, C_2 > 0$, $C_1 D_L^V \le \tilde{D}_L^V \le C_2 D_L^V$, we expect that the similarly defined number $\tilde{N}_{G,V,K}(B)$ for $\tilde{D}_L^V$ would satisfy an asymptotic formula of the same form.

To relate Conjectures 1.1 and 1.2 to each other, we identify the given G-representation V with the affine space $\mathbb{A}_K^d$ and consider the quotient variety $X = \mathbb{A}_K^d/G$ and its compactification $\overline{X} = \mathbb{P}_K^d/G$. If the map $\mathbb{P}_K^d \to \overline{X}$ is étale in codimension one, then $\overline{X}$ is a log terminal Fano variety whose minimal log discrepancy is equal to $\operatorname{age}(G)$. We define *primitive K-points* of X as those K-points of X not coming from K-points of intermediate covers of $\mathbb{A}_K^d \to X$. We propose the set $X_{\mathrm{prim}}(K)$ of primitive K-points as the sufficiently small cothin subset $U \subset \overline{X}(K)$ in Conjecture 1.1. We fix an adelic metric on $-K_{\overline{X}}$, which determines height functions on X and V. We consider the following Dirichlet series: the *height zeta functions*

$$Z_{X_{\mathrm{prim}}(K)}(s) = \sum_{x \in X_{\mathrm{prim}}(K)} H(x)^{-s},$$

$$Z_{V(K)}(s) = \sum_{y \in V(K)} H(y)^{-s},$$

and the *extended V-Discriminant zeta function,*

$$Z^{\mathrm{disc}}(s) = \sum_{L \in G\text{-}\mathrm{Fie}(K)} \left(\tilde{D}_L^V\right)^{-s}.$$

From a heuristic argument, we expect that the product $Z^{\mathrm{disc}}(s)Z_{V(K)}(s)$ would be a good approximation of $Z_{X_{\mathrm{prim}}(K)}(s)$, and they would have the right-most poles of the same place and the same order, assuming suitable meromorphic continuation of these zeta functions beyond the abscissae of convergence. Under the heuristic assumption we will make, Conjectures 1.1 implies a weak version of 1.2 and vice versa. Moreover, when $\operatorname{age}(G) = 1$, then the two conjectures become equivalent. If we look even at the residues of these Dirichlet series at the right-most poles, then we might be able to relate Peyre's refinement [29] of Manin's conjecture in terms of Tamagawa measures (and Batyrev–Tschinkel's generalization [6] of it) with Bhargava's probabilistic heuristics [8] about the distribution of number fields. However this subject is not pursued in this paper.

The mentioned heuristic argument relies on the *untwisting technique.* It was first used by Denef–Loeser [15] in their proof of a version of the McKay correspondence with an explicit construction. Later it was generalized and made more intrinsic by the author [39, 41], which can be easily translated into the case of global fields. For each $L \in G\text{-}\mathrm{Fie}(K)$, we construct the *untwisting variety* $V^{|L|}$, which is again isomorphic to $\mathbb{A}_K^d$, but there exists no canonical isomorphism between V and $V^{|L|}$. There exists a one-to-one correspondence of G-equivariant L-points of V and (non-equivariant) K-points of $V^{|L|}$. It is then straightforward to see that we have a

bijection

$$\bigsqcup_{L \in G\text{-Fie}(K)} V_{\mathrm{ur}}^{|L|}(K)/\mathrm{Aut}(L) \to X_{\mathrm{prim}}(K).$$

Here $V_{\mathrm{ur}}^{|L|}$ is the preimage of X_{ur} in $V^{|L|}$ with $X_{\mathrm{ur}} \subset X$ the unramified locus of $V \to X$. Therefore, if we define heights on $V^{|L|}$ as the pull-backs of the one on X, then we trivially have the equality among height zeta functions,

$$Z_{X_{\mathrm{prim}}(K)}(s) = \frac{1}{\sharp Z(G)} \sum_{L \in G\text{-Fie}(K)} Z_{V_{\mathrm{ur}}^{|L|}(K)}(s),$$

noting $\sharp Z(G) = \sharp \mathrm{Aut}(L)$. Using Peyre's refinement [29] of Manin's conjecture and comparing Tamagawa measures on V and $V^{|L|}$, we heuristically expect that $Z_{V_{\mathrm{ur}}^{|L|}(K)}(s)$ would be approximated by $\left(\tilde{D}_L^V\right)^{-s} Z_{V(K)}(s)$, and hence that $Z_{X_{\mathrm{prim}}(K)}(s)$ would be approximated by $Z^{\mathrm{disc}}(s) Z_{V(K)}(s)$.

To end this introduction, we here mention related works. In the paper [17, page 153] of Ellenberg and Venkatesh, it was mentioned, as a comment by Tschinkel, a similarity between their work on Malle's conjecture and Batyrev's one on rational points on Fano varieties. In another paper of theirs [18, page 732], the relation between Malle's conjecture and Manin's conjecture was more explicitly noted. In the same paper, they use the field of multi-symmetric functions, which is the function field of the above quotient variety X. The approach using X or its function field is regarded as a revisitation of Noether's approach to the inverse Galois problem (see [34]), with more emphasis on the quantitative aspect. As more recent and more similar works, we mention Le Rudulier's papers [31, 32] and the author's unpublished manuscript [38]. The former considers S_n-permutation actions on products of projective lines or planes, and the latter considers permutation actions of more general finite groups on products of projective spaces. Our work can be also considered as the global version of the wild McKay correspondence[1] (see [36, 43, 40, 39, 41]). Mäki [25] studied the distribution of G-fields for abelian G with respect to conductors rather than classically used discriminants and Wood [35] considered more general counting functions, in particular, Artin conductors (see Section 5 of her paper). In the case where V is a *balanced* G-representation, V-discriminants and Artin conductors basically coincide, which was proved in [36] in the case of local fields.

The outline of the paper is as follows. In Section 2 we set up frequently used notation and convention. After reviewing basic materials on adelic metrics and heights in Section 3 and singularities in Section 4, we recall several versions of Manin's conjecture in Section 5. In Section 6, we briefly recall height zeta functions and a version of Tauberian theorem. In Section 7 we recall Peyre's refinement of Manin's conjecture in the case of projective spaces. Sections 8 to 11 are devoted to the study of quotients of projective spaces by finite group actions in the context of Manin's conjecture. In Section 12 we recall Malle's conjecture and propose a

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[1]The previously obtained *tame* McKay correspondence [3, 15] is of course relevant too. Indeed we mainly concern the tame situation. However, in the tame and *local* case, being somehow trivial, the viewpoint of counting extensions was seemingly missing or only implicit. It should be also related, the McKay correspondence in the context of curve counting theories like Gromov–Witten and Donaldson–Thomas, on which the author lacks expertise.

generalization of it. In the final Section we relate Manin's conjecture and Malle's conjecture as a consequence of materials prepared in earlier sections.

2. Notation and convention

Throughout the paper, we work over a base number field denoted by K. We write the set of places of K as M_K and the subsets of finite and infinite places as $M_{K,f}$ and $M_{K,\infty}$ respectively. For $\mathfrak{p} \in M_K$, $K_\mathfrak{p}$ denotes the corresponding completion of K. The normalized absolute value on $K_\mathfrak{p}$ is denoted by $\|\cdot\|_\mathfrak{p}$: if p is the place of $\mathbb{Q}$ with $\mathfrak{p} \mid p$ and $|\cdot|_p$ denotes the p-adic absolute value, then $\|\cdot\|_\mathfrak{p} = \left|N_{K_\mathfrak{p}/\mathbb{Q}_p}(\cdot)\right|_p$. When $\mathfrak{p}$ is finite, we write its integer ring by $\mathcal{O}_\mathfrak{p}$, the residue field by $\kappa_\mathfrak{p}$ and the cardinality of $\kappa_\mathfrak{p}$ by $N_\mathfrak{p}$. We denote by S a finite subset of M_K containing $M_{K,\infty}$, and by S^c its complement $M_K \setminus S$. We denote the integer ring of K by $\mathcal{O}_K$ and the S-integer ring by $\mathcal{O}_S$.

We sometimes denote an arbitrary field of characteristic zero by F. We denote by $\overline{F}$ an algebraic closure of F. A *variety* over F means a separated integral scheme of finite type over F. A $\mathbb{Q}$-*divisor on* a variety or a scheme means a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor. A *divisor over* a variety X means an equivalence classes of prime divisors on proper birational modifications of X, or equivalently a divisorial valuation on the function field of X having its center on X. *Points* of schemes usually mean morphisms: for instance, for an F-variety X and an F-algebra L, an L-point of X means an F-morphism $\operatorname{Spec} L \to X$. The set of L-points of X is then denoted by $X(L)$. We denote by X_L the base change $X \otimes_F L$.

The symbol L usually denotes an algebra over a field, except that in Section 3, it denotes an invertible sheaf.

Formulas of the form $\sim CB^\alpha(\log B)^\beta$ are always asymptotic and we understand that B tends to the infinity and C is a certain positive real constant independent of B.

3. Adelic metrics and heights

In this section, we briefly recall adelic metrics on invertible sheaves and associated height functions. For details we refer the reader to [29, 30, 11]. We slightly generalize these notions to $\mathbb{Q}$-divisors, which is rather straightforward.

Let X be a projective variety over a number field K and L an invertible sheaf on X. For any field extension F/K and $x \in X(F)$, we denote by $L(x)$ the pull-back of L by $x : \operatorname{Spec} F \to X$. For $\mathfrak{p} \in M_K$, a $\mathfrak{p}$-*adic metric* on L is the data of $\mathfrak{p}$-adic norms on $L(x)$, $x \in X(K_\mathfrak{p})$, satisfying the following continuity: for a section s of L over a Zariski open subset $U \subset X$, the function

$$U(K_\mathfrak{p}) \to \mathbb{R}_{>0}, \, x \mapsto \|s(x)\|_\mathfrak{p}$$

is continuous.

For a finite set $S \subset M_K$ containing all infinite places, a *model* of (X, L) over $\mathcal{O}_S$ means the pair $(\mathcal{X}, \mathcal{L})$ of an integral projective $\mathcal{O}_S$-scheme $\mathcal{X}$ and an invertible sheaf $\mathcal{L}$ on it such that X is the generic fiber of $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_S$ and L is the pullback of $\mathcal{L}$ to X. We also say that $\mathcal{L}$ is a model of L. A model determines a specific choice of a $\mathfrak{p}$-adic metric on L for $\mathfrak{p} \in S^c$ as follows. For $x \in X(K_\mathfrak{p})$, which is identified with an $\mathcal{O}_\mathfrak{p}$-point of $\mathcal{X}$, the $K_\mathfrak{p}$-line $L(x)$ contains the pull-back $\mathcal{L}(x)$ of $\mathcal{L}$ by $x : \operatorname{Spec} \mathcal{O}_\mathfrak{p} \to \mathcal{X}$. Fixing an isomorphism $\alpha : L(x) \xrightarrow{\sim} K_\mathfrak{p}$ mapping $\mathcal{L}(x)$ onto $\mathcal{O}_\mathfrak{p}$, we define a $\mathfrak{p}$-adic metric $\|\cdot\|_\mathfrak{p}$ on $L(x)$ by

$$\|y\|_\mathfrak{p} := \|\alpha(y)\|_\mathfrak{p}.$$

An *(adelic) metric* on L is a collection $(\|\cdot\|_\mathfrak{p})_{\mathfrak{p} \in M_K}$ of $\mathfrak{p}$-adic metrics on L such that there exists a model $(\mathcal{X}, \mathcal{L})$ of (X, L), say over $\mathcal{O}_S$, defining $\|\cdot\|_\mathfrak{p}$ for $\mathfrak{p} \in S^c$. We call an invertible sheaf endowed with a metric a *metrized invertible sheaf*. For a metrized invertible sheaf L, we often fix such a model $(\mathcal{X}, \mathcal{L})$ as above and call $\mathcal{L}$ *the* model of L.

Given a metrized invertible sheaf L, the *height* of $x \in X(K)$ is then defined as

$$H(x) := \prod_{\mathfrak{p} \in M_K} \|y\|_\mathfrak{p}^{-1}$$

for an arbitrary $y \in L(x) \setminus \{0\}$. Thanks to the product formula, this is independent of the choice of y. When we need to specify the sheaf L, we write H_L for H. If L' is another metrized invertible sheaf, then the tensor product $L \otimes L'$ has the induced metric and the associated height function satisfies

$$H_{L \otimes L'}(x) = H_L(x) \cdot H_{L'}(x).$$

For our purpose, it is more convenient to use the divisorial notation and to generalize to $\mathbb{Q}$-divisors. By a $\mathbb{Q}$-*divisor* on a scheme, we mean a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor. We say that a $\mathbb{Q}$-divisor D on X is *metrized* if for a certain integer $r > 0$, rD is Cartier and $\mathcal{O}_X(rD)$ is metrized. For such a metrized $\mathbb{Q}$-divisor D, we define the height function $X(K) \to \mathbb{R}_{>0}$ by

$$H_D(x) := H_{rD}(x)^{1/r}.$$

For two metrized $\mathbb{Q}$-divisors D and D', the sum $D + D'$ is naturally metrized and we have

$$H_{D+D'}(x) = H_D(x) \cdot H_{D'}(x).$$

Let us consider a $\mathbb{Q}$-divisor $\mathcal{D}$ on $\mathcal{X}$ whose restriction to X is D. When the metrized sheaf $\mathcal{O}_X(rD)$ has the model $\mathcal{O}_\mathcal{X}(r\mathcal{D})$, then we call $\mathcal{D}$ the *model* of the metrized divisor D.

If $f : Y \to X$ is a morphism of K-varieties and D is a metrized $\mathbb{Q}$-divisor on X, then the pull-back divisor f^*D is naturally metrized. For $y \in Y(K)$, we have

$$(3.1) \qquad\qquad H_{f^*D}(y) = H_D(f(y)).$$

4. Singularities

In this section, we recall the notion of *discrepancies*, which are important invariants of singularities especially in the birational geometry, and related notions. We refer the reader to [22] for more details.

Let X be a normal variety over an arbitrary field F of characteristic zero with the canonical divisor K_X $\mathbb{Q}$-Cartier. For a proper birational morphism $f : Y \to X$ with Y normal, we can uniquely write

$$(4.1) \qquad K_Y = f^* K_X + \sum_E a(E) \cdot E$$

where $a(E) \in \mathbb{Q}$ and E runs over the exceptional prime divisors. The rational number $a(E)+1$ is called the *log discrepancy* of E ($a(E)$ itself is called the *discrepancy*).

A *divisor over X* means a prime divisor on a normal variety Y proper birational over X. Two divisors over X should be considered as identical if they give the same valuation on the function field of X. Namely a divisor over X is a divisorial valuation of the function field $K(X)$ having its center on X. The log discrepancy is actually an invariant of a divisor over X, that is, independent of the birational model on which the divisor lies.

Definition 4.1. The *minimal log discrepancy* of X is defined as

$$\mathrm{mld}(X) := \inf_E \{a(E) + 1\},$$

where E runs over all divisors over X. We call X (or singularities of X) *terminal (resp. canonical, log terminal)* if $\mathrm{mld}(X) > 1$ (resp. ≥ 1, > 0).

Suppose that X is log terminal. For a log resolution $f : Y \to X$, we have

$$\mathrm{mld}(X) = \min\{a(E) + 1 \mid E \text{ exceptional divisor of } f\}.$$

Definition 4.2. We define a *crepant divisor (resp. minimally discrepant divisor)* over X as a divisor E over X with $a(E) = 0$ (resp. $a(E) + 1 = \mathrm{mld}(X)$).

If X is canonical, then there exist only finitely many crepant divisors over X and they all appear on Y for an arbitrary log resolution $Y \to X$. The same is true for minimally discrepant divisors if X is log terminal.

Notation 4.3. We denote the number of crepant divisors over X by $\gamma(X)$ and the one of minimally discrepant divisors by $\delta(X)$.

We note that the minimal log discrepancy is stable under extensions of the base field. In particular,

$$\mathrm{mld}(X) = \mathrm{mld}(X_{\overline{F}}).$$

However $\delta(X)$ and $\gamma(X)$ can change by extensions of the base field. This happens when relevant divisors over X are not geometrically irreducible. However, if F'/F is a sufficiently large finite extension, then $\delta(X_{F'}) = \delta(X_{\overline{F}})$ and $\gamma(X_{F'}) = \gamma(X_{\overline{F}})$.

5. Rational points on singular Fano varieties

In this section, we discuss the distribution of rational points on (possibly singular) Fano varieties, especially Manin's conjecture [19] and several variants of it.

5.1. **Smooth varieties.** Let X be a projective variety over a number field K and D a metrized $\mathbb{Q}$-divisor on it. For any subset $U \subset X(K)$ and for $B > 0$, let

$$\mathcal{N}_{U,D}(B) := \sharp\{x \in U \mid H_D(x) \leq B\}.$$

When there is no confusion, we omit the subscript D. If $\mathcal{N}_U(B) < \infty$ for all B, it is natural to study its asymptotic behavior as B tends to infinity. If D is ample, then the finiteness of $\mathcal{N}_U(B)$ holds for any U and B. As a slight generalization,

if D is big, then there exists a Zariski open subset $V \subset X$ such that $\mathcal{N}_{V(K)}(B)$ is finite for every $B > 0$.

Manin's conjecture is about the case where X is a smooth Fano variety, that is, the anti-canonical divisor $-K_X$ is ample, and $D = -K_X$. We state a variant of it which uses cothin subsets.

Definition 5.1. For a K-variety X, a subset $A \subset X(K)$ is said to be *thin* if there exists a generically finite morphism $Y \to X$ of K-varieties without admitting a rational section $X \dashrightarrow Y$ such that A is contained in the image of $Y(K)$. A subset of $X(K)$ is said to be *cothin* if it is the complement of a certain thin subset.

The following is a variant of Manin's conjecture:

Conjecture 5.2. *Let X be a smooth Fano variety over K. If K is sufficiently large, then for a sufficiently small cothin subset $U \subset X(K)$ and for an arbitrary adelic metric on $-K_X$, we have*

$$\mathcal{N}_U(B) \sim CB(\log B)^{\rho(X)-1}.$$

Here $\rho(X)$ is the Picard number, the rank of the Néron–Severi group.

The precise meaning of "if K is sufficiently large" is that there exists a finite extension K_0/K such that for all finite extension L/K_0, the relevant statement is true after the base change to L. This condition hopefully assures that $X(K)$ is not too small (at least not empty) and $\rho(X)$ is equal to $\rho(X_{\overline{K}})$ so that the subtleness of the exponent of the log factor would be diminished.

Originally the K-point set of a Zariski open subset was considered rather than a cothin subset (see [19, 2]). However that version has a counter-example found by Batyrev–Tschinkel [4]. Peyre [30] then suggested to consider cothin subsets as one of possible ways to remedy the pathological situation. Le Rudulier [32] and Browning–Loughran [10] more seriously studied this modification of the conjecture.

Next we consider a generalization by Batyrev–Manin [2]. Let X be a smooth projective variety over K, $\mathrm{NS}(X)$ its Néron-Severi group, $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes \mathbb{R}$ and $\mathrm{PEf}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ the pseudo-effective cone, the closure of the cone generated by the classes of effective divisors. We suppose that $K_X \notin \mathrm{PEf}(X)$, which is equivalent to that X is uniruled [9]. Let D be a big $\mathbb{Q}$-divisor, that is, its class is in the interior of $\mathrm{PEf}(X)$ (see [23, p. 147]).

Definition 5.3. We define $\alpha(D) \in \mathbb{R}$ by the condition

$$\alpha(D) \cdot [D] + [K_X] \in \partial\mathrm{PEf}(X),$$

where $\partial\mathrm{PEf}(X)$ denotes the boundary of $\mathrm{PEf}(X)$. We denote the point $\alpha(D) \cdot [D] + [K_X]$ by $\partial(D)$.

Batyrev–Manin [2] conjectures that the cone $\mathrm{PEf}(X)$ is polyhedral around $\partial(D)$.

Definition 5.4. Under this conjecture, we let $\beta(D)$ be the codimension in $\mathrm{NS}(X)_{\mathbb{R}}$ of the minimal face of $\mathrm{PEf}(X)$ containing $\partial(D)$.

The following is a variant of their conjecture:

Conjecture 5.5. *Let X be a smooth projective variety over a number field K with $K_X \notin \mathrm{PEf}(X)$ and D a big divisor on X. If K is sufficiently large, then for a sufficiently small cothin subset $U \subset X(K)$, giving an arbitrary adelic metric to D, we have*

$$\mathcal{N}_{U,D}(B) \sim CB^{\alpha(D)}(\log B)^{\beta(D)-1}.$$

If X is a Fano variety and $D = -K_X$, we recover Conjecture 5.2.

5.2. **Log terminal Fano varieties.** When the given Fano variety has singularities, we cannot generally expect the asymptotic formula of the same form and need to take contribution of singularities into account. Batyrev–Tschinkel [6] explored Fano varieties having canonical singularities. The following conjecture is a variant of a special case of their conjectural formula (page 323, *loc.cit.*).

Conjecture 5.6. *Let X be a log terminal Fano variety and $\gamma(X)$ be the number of crepant divisors over X. If X is canonical and K is sufficiently large, then for a sufficiently small cothin $U \subset X(K)$,*

$$\mathcal{N}_U(B) \sim CB(\log B)^{\rho(X)+\gamma(X)-1}.$$

If X is not canonical and K is sufficiently large K, then for any cothin $U \subset X(K)$, we have

$$\mathcal{N}_U(B) \sim CB^\alpha(\log B)^\beta$$

with $\alpha > 1$ and $\beta \geq 0$.

Actually one can deduce this conjecture (at least with U sufficiently small for the second assertion) from Conjecture 5.5 and the following lemma.

Lemma 5.7. *Let $f : Y \to X$ be a log resolution of a log terminal Fano variety X. We have that*

$$\alpha(-f^*K_X) \begin{cases} = 1 & (X\text{: canonical}) \\ > 1 & (otherwise) \end{cases}.$$

Moreover, if X is canonical, K is sufficiently large and $\mathrm{PEf}(X)$ is polyhedral around $\partial(-f^*K_X)$, then the minimal face of $\mathrm{PEf}(X)$ containing $\partial(-f^*K_X)$ has codimension $\rho(X) + \gamma(X)$.

Proof. Let $f : Y \to X$ be a log resolution and write

$$K_Y - f^*K_X = \sum_{i=1}^{l} a_i E_i$$

where E_i are exceptional prime divisors and $a_i \in \mathbb{Q}$. We first show the first assertion in the case where X is canonical (that is, for every i, $a_i \geq 0$). Since K_Y and $-f^*K_X$ are both big, the invariant $\alpha(-f^*K_X)$ is defined. The class

$$B := [K_Y] - [f^*K_X] = \sum a_i[E_i]$$

clearly belongs to $\mathrm{PEf}(X)$. To see that it lies in the boundary $\partial \mathrm{PEf}(X)$, it suffices to show that for an ample class A and for any $\epsilon > 0$, $B - \epsilon A$ does not belong to $\mathrm{PEf}(Y)$. Let $U \subset X$ be the locus where f is an isomorphism. Its complement $X \setminus U$ has codimension ≥ 2. If $C \subset X$ is a smooth curve obtained as the intersection of $\dim X - 2$ general hyperplane sections, then C is contained in U and its lift $\tilde{C}$ to Y does not meet the exceptional locus of f. Therefore, the intersection number, $B \cdot \tilde{C}$, is negative. However, $\tilde{C}$ is a member of a family of curves covering general points of Y, any class in $\mathrm{PEf}(Y)$ should intersect $\tilde{C}$ with a non-negative intersection number. We have proved the first assertion in this case.

Next consider the case where X is not canonical. At least one of a_i is negative. The intersection of $\mathrm{PEf}(Y)$ and the linear space spanned by $[E_i]$, $i = 1, \ldots, l$ is the cone spanned by $[E_i]$, $i = 1, \ldots, l$. This shows that the divisor class B is not in $\mathrm{PEf}(Y)$.

As for the second assertion, from [12, Lem. 4.1], each ray $\mathbb{R}_{\geq 0}[E_i]$ is an extremal ray of $\mathrm{PEf}(Y)$. Moreover they are linearly independent. It follows that the minimal face of $\mathrm{PEf}(Y)$ containing B is the cone generated by those classes $[E_i]$ with $a_i > 0$, whose codimension is $\rho(X) + \gamma(X)$. $\qquad\square$

5.3. **Toric Fano varieties.** When the given Fano variety is not canonical, then it seems generally difficult to determine the exponents of B and $\log B$ in the expected asymptotic formula. In the case of toric varieties, however, we can obtain a natural upper bound. We mean by a toric variety an equivariant compactification of the torus $T = (\mathbb{G}_{m,K})^d$.[2]

Lemma 5.8. *Let X be a log terminal toric Fano variety which is not canonical (that is, $0 < \mathrm{mld}(X) < 1$) and $f : Y \to X$ a toric resolution. Then*

$$\alpha(-f^*K_X) \leq \frac{1}{\mathrm{mld}(X)}.$$

Moreover, if the equality holds, then the minimal face of $\mathrm{PEf}(Y)$ containing $\partial(-f^*K_X)$ has codimension $\leq \delta(X)$.

Proof. Let D_i, $i \in I$ be the torus invariant prime divisors on X and $\tilde{D}_i$ their respective strict transforms on Y. Let E_j, $j \in J$ be the exceptional prime divisors of f. Canonical divisors of X and Y have the natural expressions,

$$K_X = -\sum_{i \in I} D_i \text{ and } K_Y = -\sum_{i \in I} \tilde{D}_i - \sum_{j \in J} E_j.$$

We write

$$K_Y = f^*K_X + \sum_{j \in J} a_j E_j$$

with $a_j > -1$. For a real number c, we have

$$(5.1) \qquad -c(f^*K_X) + K_Y = \sum_i (c-1)\tilde{D}_i + \sum_j \left(c(1 + a_j) - 1\right) E_j.$$

For $c = 1/\mathrm{mld}(X) > 1$, this divisor is clearly effective, which shows the first assertion.

Suppose that the equality, $\alpha(-f^*K_X) = 1/\mathrm{mld}(X)$, holds. Let

$$J_0 := \{j \mid a_j + 1 = \mathrm{mld}(X)\},$$

whose cardinality is by definition $\delta(X)$. In the right hand side of (5.1), D_i, $i \in I$ and E_j, $J \setminus J_0$ have positive coefficients and E_j, $j \in J_0$ have zero coefficients. We note that $\mathrm{PEf}(Y)$ is generated by $[D_i]$, $i \in I$ and $[E_j]$, $j \in J$. It follows that the minimal face containing $\partial(-f^*K_X)$ is the cone generated by $[D_i]$, $i \in I$ and $[E_j]$, $j \in J \setminus J_0$. Its codimension is clearly at most $\sharp J_0 = \delta(X)$. $\qquad\square$

Remark 5.9. The above proof shows that if we use the group of the torus invariant divisors (NOT modulo linear equivalence) rather than the Néron–Severi group and the cone of the effective torus invariant divisors, then the equalities in the inequalities of the lemma hold. The numbers, $1/\mathrm{mld}(X)$ and $\delta(X)$, are the exponents of an asymptotic formula expected from our heuristics (Section 13) and a generalization of Malle's conjecture (Conjecture 12.10). On the other hand, these numbers are

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[2]Batyrev and Tschinkel [1, 5] considers a more general notion of algebraic tori, which become isomorphic to $(\mathbb{G}_{m,K})^d$ after a finite extension of the base field K.

generally wrong ones from the viewpoint of the Batyrev–Manin conjecture as we see below. Thus, there seems to be some subtleness or obstruction so that our heuristics do not work when X is not canonical.

Corollary 5.10. *Suppose that X is a toric Fano variety of Picard number one with $0 < \mathrm{mld}(X) < 1$. Suppose that there exists a cone of the defining fan which contains all singular cones of the fan as its faces. Then, for some real numbers a and b with $1 < a < 1/\mathrm{mld}(X)$ and $b \geq 0$, we have*

$$\mathcal{N}_{T(K), -K_X}(B) \sim CB^a(\log B)^b$$

with T the maximal torus.

Proof. Since Conjecture 5.5 holds for smooth toric varieties with $U = T(K)$, we need to show that the strict inequality holds in Lemma 5.8. We follows the notation of Lemma 5.8 and its proof. The assumption on the Picard number shows that there are exactly $d + 1$ torus invariant prime divisors on X, say $D_0, \ldots D_d$ with $d = \dim X$. Without loss of generality, we may suppose that the cone spanned by the rays corresponding to $D_1, \ldots, D_d$ contains all singular cones, and that the chosen toric resolution $f : Y \to X$ subdivides only this cone and its faces. Then D_0 is linearly equivalent to $\sum_{i=1}^{d} a_i \tilde{D}_i + \sum_{j \in J} b_j E_j$ with $a_i, b_j > 0$. Therefore the class of D_0 is in the interior of $\mathrm{PEf}(X)$. Therefore, for $c = 1/\mathrm{mld}(X)$, the divisor (5.1) is not in the boundary of $\mathrm{PEf}(X)$. This shows the desired inequality. □

6. Height zeta functions

The height zeta function is an analytic tool to study the distribution of rational points. Suppose that a K-variety X is given a height function $H : X(K) \to \mathbb{R}_{>0}$. For $U \subset X(K)$, we write again

$$\mathcal{N}_U(B) = \sharp\{x \in U \mid H(x) \leq B\}.$$

The *height zeta function* $Z_U(s)$ is defined as the (generalized) Dirichlet series

$$Z_U(s) := \sum_{x \in U} H(x)^{-s}.$$

If the height function is associated to a metrized ample divisor, then $\mathcal{N}_U(B)$ has the polynomial growth and hence $Z_U(s)$ is convergent for some half-plane $\Re(s) > a$. The infimum of such a is called the *abscissa of convergence*.

For heuristic arguments, it is convenient to assume meromorphic continuation beyond the abscissa of convergence, although it is generally difficult to prove it. For the sake of generality (we will consider the same condition for extended V-discriminant zeta functions), we consider the Dirichlet series

$$f(s) := \sum_{x \in I} F(x)^{-s}$$

associated to an arbitrary function $F : I \to \mathbb{R}_{>0}$ on a countable set I.

Condition 6.1. The series $f(s)$ converges for some half-plane $\Re(s) > a$. If the abscissa of convergence, say α, is not $-\infty$, then $f(s)$ extends to a meromorphic function along the line $\Re(s) = \alpha$ which is holomorphic except at $s = \alpha$.

Supposing that $f(s)$ satisfies the condition, let β be the order of the pole at $s = \alpha$ and

$$C := \frac{1}{\Gamma(\beta)\alpha} \lim_{s \to \alpha} \frac{f(s)}{(s - \alpha)^\beta}.$$

Then a version of Tauberian theorem shows

$$\sharp\{x \in I \mid F(x) \leq B\} \sim CB^\alpha (\log B)^{\beta-1} \quad (B \to \infty).$$

7. Peyre's constants for projective spaces

Peyre [29] refined Manin's conjecture by specifying the constant factor C in terms of Tamagawa measures. We only need it for projective spaces. In this case, his refinement is actually a theorem and generalizes the earlier result of Schanuel [33] as the case of a special metric on the anti-canonical divisor. It is essential in our work to consider various metrics.

Let $X = \mathbb{P}^d_K$ be the projective space over a number field K. We suppose that the anti-canonical divisor $-K_X$ is metrized. For each $\mathfrak{p}$, the metric defines a measure $\mu_\mathfrak{p}$ on the $K_\mathfrak{p}$-analytic manifold $X(K_\mathfrak{p})$ as follows. For local coordinates $x_1, \ldots, x_d$ on $X(K_\mathfrak{p})$, consider the differential form

$$\omega = \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_d} \right\|_\mathfrak{p} dx_1 \wedge \cdots \wedge dx_d.$$

The measure is locally defined by

$$\mu_\mathfrak{p}(A) := \int_A \omega.$$

Here the integral is defined with respect to the normalized Haar measure on $(K_\mathfrak{p})^d$. The local measures glue together and give a measure on the whole space $X(K_\mathfrak{p})$. If the fixed model of $-K_X$ is the anti-canonical divisor $-K_{\mathbb{P}^d_{\mathcal{O}_S}/\mathcal{O}_S}$ of $\mathbb{P}^d_{\mathcal{O}_S}$ over $\mathcal{O}_S$, then for $\mathfrak{p} \in S^c$,

$$\mu_\mathfrak{p}(X(K_\mathfrak{p})) = \frac{\mathbb{P}^d_{\mathcal{O}_\mathfrak{p}}(\kappa_\mathfrak{p})}{N_\mathfrak{p}^d} = \sum_{i=0}^d N_\mathfrak{p}^{-i},$$

where $\kappa_\mathfrak{p}$ is the residue field of $K_\mathfrak{p}$ and $N_\mathfrak{p} := \sharp\kappa_\mathfrak{p}$. More generally, suppose that the model of $-K_X$ is

$$-K_{\mathbb{P}^d_{\mathcal{O}_S}/\mathcal{O}_S} + \sum_{\mathfrak{p} \in S^c} a_\mathfrak{p} \mathbb{P}^d_{\kappa_\mathfrak{p}},$$

where $a_\mathfrak{p}$ are rational numbers and almost all of them are zero, and $\mathbb{P}^d_{\kappa_\mathfrak{p}}$ are regarded as prime divisors on $\mathbb{P}^d_{\mathcal{O}_S}$. Then, for $\mathfrak{p} \in S^c$, the $\mathfrak{p}$-adic metric of $-K_X$ is simply $N_\mathfrak{p}^{-a_\mathfrak{p}}$ times the $\mathfrak{p}$-adic metric induced from the model $-K_{\mathbb{P}^d_{\mathcal{O}_S}/\mathcal{O}_S}$ and

$$\mu_\mathfrak{p}(X(K_\mathfrak{p})) = N^{-a_\mathfrak{p}} \sum_{i=0}^d N_\mathfrak{p}^{-i}.$$

Theorem 7.1 ([29, Cor. 6.2.17, 6.2.18]. See also [31].)**.** *Let c_K be the residue of the Dedekind zeta function $\zeta_K(s)$ at $s = 1$, let Disc_K be the absolute Discriminant of K and let*

$$C_{\mathrm{Peyre}} := \frac{c_K}{|\mathrm{Disc}_K|^{d/2}} \prod_{\mathfrak{p} \in M_{K,f}} \left(1 - \frac{1}{N_\mathfrak{p}}\right) \mu_\mathfrak{p}(X(K_\mathfrak{p})) \times \prod_{\mathfrak{p} \in M_{K,\infty}} \mu_\mathfrak{p}(X(K_\mathfrak{p})).$$

For any Zariski open subset $U \subset X$, we have

$$\mathcal{N}_{U(K)}(B) \sim C_{\mathrm{Peyre}}B.$$

Remark 7.2. That we can take any Zariski open subset U is a consequence of the property (E_V) in [29].

8. Quotients of projective spaces

In this section, we consider a special class of log terminal Fano varieties. Let G be a finite group and $V = \mathbb{A}_K^d$ an affine space endowed with a linear faithful G-action. The action extends to the projective space $\overline{V} = \mathbb{P}_K^d$. We write the associated quotient varieties as

$$X := V/G \text{ and } \overline{X} := \overline{V}/G.$$

Suppose that the quotient map $\overline{V} \to \overline{X}$ is étale in codimension one. It is equivalent to the condition that $G \subset \mathrm{GL}(V)$ has no pseudo-reflection and no element of $G \backslash \{1\}$ acts on V by a scalar multiplication. Then the pull-back of $-K_{\overline{X}}$ to $\overline{V}$ is $-K_{\overline{V}}$ and hence $\overline{X}$ is a log terminal Fano variety. If G is abelian and K is so large to have enough roots of unity, then $\overline{X}$ is a toric variety. The Picard number of $\overline{X}$ is one. Moreover its defining fan satisfies the condition in Corollary 5.10.

Remark 8.1. If some element of $G \setminus \{1\}$ acts by a scalar multiplication, then we can reduced to the situation where this does not happen, simply replacing $V = \mathbb{A}_K^d$ with $\mathbb{A}_K^{d+1}$ by adding an additional variable on which G acts trivially.

We recall a representation-theoretic formula for the minimal log discrepancies of $X = V/G$ and $\overline{X} = \mathbb{P}_K^d/G$. Since the minimal log discrepancy is stable under base field extensions, we consider the induced G-action on $V_{\overline{K}}$. We fix a primitive $\sharp G$-th root of unity, $\zeta \in \overline{K}$. We can diagonalize each $g \in G \subset \mathrm{GL}(V_{\overline{K}})$ and write

$$g = \mathrm{diag}(\zeta^{a_1}, \ldots, \zeta^{a_d}) \quad (0 \le a_i < \sharp G).$$

Definition 8.2 ([20])**.** We define the *age* of g, denoted by $\mathrm{age}(g)$, to be

$$\frac{1}{\sharp G} \sum_{i=1}^{d} a_i.$$

We then define the *age* of G, denoted $\mathrm{age}(G)$, as the minimum of $\mathrm{age}(g)$, $g \in G \backslash \{1\}$.

Note that $\mathrm{age}(g)$ depends on the choice of ζ, but $\mathrm{age}(G)$ does not. The following proposition gives a simple representation theoretic description of the minimal log discrepancy of $X = V/G$ and the one of $\overline{X} = \mathbb{P}_K^d/G$.

Proposition 8.3. *We have*

$$\mathrm{mld}(\overline{X}) = \mathrm{mld}(X) = \mathrm{age}(G).$$

Proof. The right equality should be well-known for specialists. See [42] for instance. We prove the left one. Obviously $\mathrm{mld}(\overline{X}) \ge \mathrm{mld}(X)$. To see the converse, we consider the embedding

$$V = \mathbb{A}_K^d \hookrightarrow \tilde{V} = \mathbb{A}_K^{d+1}, \ (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 1)$$

and the induced embedding $\mathrm{GL}(V) \hookrightarrow \mathrm{GL}(\tilde{V})$. We denote the image of G in $\mathrm{GL}(\tilde{V})$ by $\tilde{G}$. This subgroup $\tilde{G}$ does not have pseudo-reflection either and has the same age as G. Now $\overline{X}$ is the quotient of $(\tilde{V} \setminus \{0\})/\tilde{G}$ by the natural $\mathbb{G}_m$-action. Therefore

$$\begin{aligned}
\mathrm{mld}(\overline{X}) &= \mathrm{mld}(\tilde{V} \setminus \{0\}/\tilde{G}) \\
&\leq \mathrm{mld}(\tilde{V}/\tilde{G}) \\
&= \mathrm{age}(\tilde{G}) \\
&= \mathrm{age}(G) \\
&= \mathrm{mld}(X).
\end{aligned}$$

$\square$

Let $\mathrm{Conj}(G)$ be the set of conjugacy classes of G. We can make $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on $\mathrm{Conj}(G)$ via the cyclotomic character [27].

Definition 8.4. We define the *set of K-conjugacy classes* as the quotient of $\mathrm{Conj}(G)$ by the action of $\mathrm{Gal}(\overline{K}/K) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and denote it by $K\text{-}\mathrm{Conj}(G)$. A K-*conjugacy class* of K then consists of elements of G giving the same element of $K\text{-}\mathrm{Conj}(G)$.

If K is sufficiently large, then the action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Conj}(G)$ is trivial and $\mathrm{Conj}(G) = K\text{-}\mathrm{Conj}(G)$. Therefore the notion of K-conjugacy does not play an important role in our formulation of conjectures. However, the following proposition provides one more evidence for the close relation between Manin's and Malle's conjectures.

Proposition 8.5. *For $g \in G \setminus \{1\}$ with $\mathrm{age}(g) = \mathrm{age}(G)$, if $[g] = [g']$ in $K\text{-}\mathrm{Conj}(G)$, then $\mathrm{age}(g) = \mathrm{age}(g')$. Moreover the number of K-conjugacy classes of G with minimal age is equal to the number of minimally discrepant divisors, $\delta(X)$.*

Notation 8.6. We call an element of G or a $(K\text{-})$conjugacy class of G the *youngest* if its age is equal to the one of G. We denote the number of the youngest K-conjugacy classes of G by $\upsilon(G)$.

The second assertion of the proposition reads

$$\delta(X) = \upsilon(G).$$

Proof of the proposition. We first sketch the outline of the proof. A version of the McKay correspondence says that there exists a one-to-one correspondence between minimally discrepant divisors over $X_{\overline{K}}$ and the youngest conjugacy classes of G (this already proves the proposition when K is sufficiently large). We can make the correspondence compatible with the $\mathrm{Gal}(\overline{K}/K)$-actions and the proposition follows.

To make arguments more precise, we use results from [42]. However the reader may skip this part, since it is not related to the rest of the paper. Let X_{sing} be the singular locus of X, $f : Y \to X$ a log resolution which is an isomorphism over $X \setminus X_{\mathrm{sing}}$. Let $J_\infty X$ and $J_\infty Y$ be the arc spaces of X and Y and let $J'_\infty Y$ be the preimage of X_{sing} by $J_\infty Y \to Y \to X$. Let $\mathcal{X} := [V/G]$ be the quotient stack and $\mathcal{J}_\infty \mathcal{X}$ its twisted arc space and $\mathcal{J}'_\infty \mathcal{X}$ the preimage of X_{sing} by $\mathcal{J}_\infty \mathcal{X} \to \mathcal{X} \to X$. There exists a canonical one-to-one correspondence of the points sets,

$$(8.1) \qquad\qquad |J'_\infty Y| \leftrightarrow |\mathcal{J}'_\infty \mathcal{X}|,$$

outside measure zero subsets with respect to the motivic measures. The same is true for $\overline{K}$-point sets of $J'_\infty Y$ and $\mathcal{J}'_\infty \mathcal{X}$, and the correspondence is $\mathrm{Gal}(\overline{K}/K)$-equivariant.

Once we fix a primitive $\sharp G$-th root of unity $\zeta \in \overline{K}$, there exists a one-to-one correspondence between connected components of $|(\mathcal{J}'_\infty \mathcal{X})_{\overline{K}}|$ with respect to the Zariski topology and $\mathrm{Conj}(G)$, which respects $\mathrm{Gal}(\overline{K}/K)$-actions. Therefore there exists a one-to-one correspondence between connected components of $|\mathcal{J}'_\infty \mathcal{X}|$ and K-$\mathrm{Conj}(G)$. We denote by $C_1, \ldots, C_l$ the connected component of $|\mathcal{J}'_\infty \mathcal{X}|$ corresponding to the youngest K-conjugacy classes.

Let $E_1, \ldots, E_m$ be the minimally discrepant divisors on Y and $\tilde{E}_1, \ldots, \tilde{E}_m$ their preimages in $J'_\infty Y$. The correspondence (8.1) respects suitably normalized motivic measures, which take values in a semiring $\mathfrak{R}^{1/r}$ having the dimension function $\dim : \mathfrak{R}^{1/r} \to \mathbb{Q}$. If we denote these normalized measures by μ^ν_Y and $\mu^\nu_\mathcal{X}$, then we have

$$\mu^\nu_Y(J'_\infty Y) \equiv \sum_i \mu^\nu_Y(\tilde{E}_i) \equiv \mu^\nu_\mathcal{X}(\mathcal{J}'_\infty \mathcal{X}) \equiv \sum_j \mu^\nu_\mathcal{X}(C_j),$$

where the congruence are taken modulo lower dimensional elements. This shows that for each i, the generic point of $\tilde{E}_i$ corresponds to the generic point of some C_j. This completes the proof. $\square$

9. A SIMPLE CORRESPONDENCE OF POINTS

In this section, V is an arbitrary variety over a base field F of characteristic zero with a faithful G-action and $X = V/G$ is the associated quotient variety.

Definition 9.1. When $\mathrm{Spec}\, L$ is an étale G-torsor over $\mathrm{Spec}\, F$, we call the K-algebra L with a G-action a G-*ring (over F)*. An *isomorphism* of two G-rings is a G-equivariant F-isomorphism. When the underlying ring of a G-ring L is a field, we call L a G-*field*. We denote the set of G-rings modulo isomorphism by G-$\mathrm{Rin}(F)$ and the one of G-fields by G-$\mathrm{Fie}(F)$. We sometimes call a G-field a *large G-field*, distinguishing it from *small G-fields* introduced later.

Definition 9.2. For $L \in G$-$\mathrm{Rin}(F)$ and a K-variety Y with G-action, a G-*equivariant L-point* of Y is simply a G-equivariant morphism $\mathrm{Spec}\, L \to Y$. We denote the set of G-equivariant L-points of Y by $Y(L)^G$.

Let V be an F-variety with a faithful G-action and let $X := V/G$. For $L \in G$-$\mathrm{Rin}(F)$, taking the G-quotients of the source and the target defines a natural map

$$V(L)^G \to X(F).$$

The group $\mathrm{Aut}(L)$ of automorphisms L (as a G-ring) acts on $V(L)^G$ and the above map factors through the quotient set $V(L)^G/\mathrm{Aut}(L)$. Let V_{ur} and X_{ur} be the unramified loci of $V \to X$ in V and X respectively. We obtain the map

$$V_{\mathrm{ur}}(L)^G/\mathrm{Aut}(L) \to X_{\mathrm{ur}}(F),$$

which is easily seen to be injective. Conversely, given $x \in X_{\mathrm{ur}}(F)$, the morphism

$$\mathrm{Spec}\, L := \mathrm{Spec}\, F \times_X V \to V$$

defines a G-equivariant L-point. We thus obtain a bijection

$$\bigsqcup_{L \in G\text{-}\mathrm{Rin}(F)} V_{\mathrm{ur}}(L)^G/\mathrm{Aut}(L) \to X_{\mathrm{ur}}(F).$$

Lemma 9.3. *A point $x \in X_{\mathrm{ur}}(F)$ is in the image of $V_{\mathrm{ur}}(L)^G/\mathrm{Aut}(L)$ for $L \in G\text{-}\mathrm{Fie}(F)$ if and only if x is not in the image of $(V/H)(F)$ for any proper subgroup $H \subsetneq G$.*

Proof. If $x \in X_{\mathrm{ur}}(F)$ is the image of $(V/H)(F)$ for $H \subsetneq G$, then $\mathrm{Spec}\, F \times_X (V/H)$ is the disjoint union of $\sharp(G/H)$ copies of $\mathrm{Spec}\, F$ and hence

$$\mathrm{Spec}\, F \times_X V = (\mathrm{Spec}\, F \times_X (V/H)) \times_{V/H} V$$

is not connected. Conversely, if $\mathrm{Spec}\, L = \mathrm{Spec}\, F \times_X V$ is not connected, then let H be the stabilizer of a component. Taking the H-quotients, we get an F-point of V/H which maps to x. $\qquad\square$

Definition 9.4. When $x \in X_{\mathrm{ur}}(F)$ satisfies one of the equivalent conditions in the last lemma, we say that x is a *primitive F-point* of X. We denote the set of primitive F-points by $X_{\mathrm{prim}}(F)$.

When F is a number field, the subset $X_{\mathrm{prim}}(F) \subset X(F)$ is the cothin subset obtained by removing all "natural" thin subsets. This is why we will below think of $X_{\mathrm{prim}}(F)$ as a candidate of a *sufficiently small* cothin subset below.

By construction, we have a natural bijection

$$\tag{9.1} \bigsqcup_{L \in G\text{-}\mathrm{Fie}(F)} V(L)^G/\mathrm{Aut}(L) \to X_{\mathrm{prim}}(F).$$

We need the following fact later.

Lemma 9.5. *For $L \in G\text{-}\mathrm{Fie}(F)$, the $\mathrm{Aut}(L)$-action on $V_{\mathrm{ur}}(L)^G$ is free.*

Proof. Let $g \in G$, $y \in V_{\mathrm{ur}}(L)^G$ and $x := \pi(y) \in X(K)$. If $g(y) = y$, then $y : \mathrm{Spec}\, L \to V_{\mathrm{ur}}$ factors through $\mathrm{Spec}\, L^{\langle g \rangle}$, where $L^{\langle g \rangle}$ is the invariant subfield by the cyclic group $\langle g \rangle$. There exists a morphism

$$\mathrm{Spec}\, L^{\langle g \rangle} \to \mathrm{Spec}\, L \cong \mathrm{Spec}\, F \times_{x,X} V_{\mathrm{ur}}$$

which is a section of the natural morphism $\mathrm{Spec}\, L \to \mathrm{Spec}\, L^{\langle g \rangle}$. Therefore $\mathrm{Spec}\, L$ is the disjoint union of $\sharp \langle g \rangle$ copies of $\mathrm{Spec}\, L^{\langle g \rangle}$, which contradicts the assumption $L \in G\text{-}\mathrm{Fie}(F)$. $\qquad\square$

10. Untwisting

The *untwisting* technique allows us to reduce the study of G-equivariant L-points on affine spaces for $L \in G\text{-}\mathrm{Fie}(K)$ to the study of ordinary K-points on affine spaces. It was introduced by Denef–Loeser [15] over the ring of power series $k[[t]]$ with k a field of characteristic zero containing enough roots of unity in an explicit way. The author [40, 39, 41] generalized it to the integer ring of every local field in a more intrinsic way, which admits a literal translation to a more general situation. We basically follows the presentation in [41].

10.1. **Untwisting over a general base.** Let $\mathcal{O}$ be an integrally closed noetherian domain of dimension ≤ 1 with the fraction field $F = \mathrm{frac}(\mathcal{O})$ of characteristic zero. Suppose that the affine space $W = \mathbb{A}_{\mathcal{O}}^d$ has an $\mathcal{O}$-linear G-action. We write $W = \mathrm{Spec}\, S^{\bullet}\mathcal{F}$, where $\mathcal{F}$ is a free $\mathcal{O}$-module of rank d and $S^{\bullet}\mathcal{F}$ is its symmetric algebra over $\mathcal{O}$. For $L \in G\text{-}\mathrm{Rin}(F)$, let $\mathcal{O}_L$ be the integral closure of $\mathcal{O}$ in L. The natural G-action on L restricts to $\mathcal{O}_L$.

Definition 10.1. The associated *tuning module* is defined by
$$\Xi_L := \mathcal{H}om_{\mathcal{O}}^G(\mathcal{F}, \mathcal{O}_L),$$
the sheaf of G-equivariant $\mathcal{O}$-linear maps.

It turns out that Ξ_L is a locally free $\mathcal{O}$-module of rank d (see [36, 39]). Let
$$\Theta_L := \mathcal{H}om_{\mathcal{O}_T}(\Xi_L, \mathcal{O}).$$

Definition 10.2. The *untwisting variety* and the *pre-untwisting variety* of W with respect to L are respectively defined as
$$W^{|L|} := \operatorname{Spec} S^{\bullet}\Theta,$$
$$W^{\langle L \rangle} := W \otimes_{\mathcal{O}} \mathcal{O}_L.$$

The untwisting variety $W^{|L|}$ is a vector bundle of rank d over $\operatorname{Spec}\mathcal{O}$. If $\mathcal{O}$ is a principal ideal domain, then $W^{|L|}$ is $\mathcal{O}$-isomorphic to the original affine space W. We exploit this fact in later sections.

There exists a commutative diagram of natural morphisms:

(10.1)
$$
\begin{array}{ccc}
 & W^{\langle L \rangle} & \\
 \swarrow & & \searrow \\
W & & W^{|L|} \\
 \searrow & & \swarrow \\
 & W/G &
\end{array}
$$

The construction of this diagram is compatible with the base change by a flat morphism $\mathcal{O}' \to \mathcal{O}$ with $\mathcal{O}'$ another integrally closed noetherian domain of dimension ≤ 1. The two upper arrows are étale over the generic point $\operatorname{Spec} F$ of $\operatorname{Spec}\mathcal{O}$.

The diagram induces a one-to-one correspondence between G-equivariant $\mathcal{O}_L$-points of W and $\mathcal{O}$-points of $W^{|L|}$:
$$W(\mathcal{O}_L)^G \leftrightarrow W^{|L|}(\mathcal{O})$$
At the cost of a "little" twist of the target space W, this correspondence allows us to reduce the study of equivariant points to the one of ordinary ones. There exist natural $\operatorname{Aut}(L)$-actions on the sets $W(\mathcal{O}_L)^G$ and $W^{|L|}(\mathcal{O})$ and the above correspondence is compatible with these actions. The correspondence is also compatible with the natural maps $W(\mathcal{O}_L)^G \to (W/G)(\mathcal{O})$ and $W^{|L|}(\mathcal{O}) \to (W/G)(\mathcal{O})$.

Remark 10.3. The twist of the target space (how different W and $W^{|L|}$ are) causes a subtle and hard problem in our study relating the distribution of rational points and the one of number fields. That is why we need to appeal to a heuristic argument below. When $F = K$ is a number field, even if $\mathcal{O} = \mathcal{O}_S$ is a principally ideal domain and so $W \cong W^{|L|}$, the $\mathfrak{p}$-adic metric for $\mathfrak{p} \in S$ is "twisted" in general. Therefore the subtleness remains also in this case.

We can projectivise the whole construction as follows. We put $\overline{\mathcal{F}} := \mathcal{F} \oplus \mathcal{O} \cdot z$ with z a dummy variable and extend the G-action on $\mathcal{F}$ to $\overline{\mathcal{F}}$ by the trivial action on $\mathcal{O} \cdot z$. Applying the construction of Ξ_L and Θ_L to $\overline{\mathcal{F}}$, we obtain $\overline{\Xi_L} = \Xi_L \oplus \mathcal{O}$ and $\overline{\Theta_L} = \Theta_L \oplus \mathcal{O} \cdot z$. We let
$$\overline{W} := \operatorname{Proj} S^{\bullet}\overline{\mathcal{F}} \text{ and } \overline{W}^{|L|} := \operatorname{Proj} S^{\bullet}\overline{\Theta_L}.$$

Let $\overline{W}^{\langle L \rangle} := W^{|L|} \otimes_{\mathcal{O}} \mathcal{O}_L$. We obtain the commutative diagram:

$$
\begin{array}{ccc}
& \overline{W}^{\langle L \rangle} & \\
\swarrow & & \searrow \\
\overline{W} & & \overline{W}^{|L|} \\
\searrow & & \swarrow \\
& \overline{W}/G &
\end{array}
$$

Each entry of the diagram contains the corresponding entry in the preceding one (10.1) as an open dense subscheme.

10.2. **Over a number field.** We apply the untwisting to the case where $T = \operatorname{Spec} K$ with K the given number field and $V = W = \mathbb{A}_K^d$. Let $X := V/G$ as above. For each $L \in G\text{-Rin}(K)$, we have a one-to-one correspondence

$$V(L)^G/\operatorname{Aut}(L) \leftrightarrow V^{|L|}(K)/\operatorname{Aut}(L).$$

Let $V_{\mathrm{ur}}^{|L|}$ be the preimage of X_{ur}. We obtain bijections

$$\bigsqcup_{L \in G\text{-Rin}(K)} V_{\mathrm{ur}}^{|L|}(K)/\operatorname{Aut}(L) \to X_{\mathrm{ur}}(K)$$

and

$$(10.2) \qquad \bigsqcup_{L \in G\text{-Fie}(K)} V_{\mathrm{ur}}^{|L|}(K)/\operatorname{Aut}(L) \to X_{\mathrm{prim}}(K).$$

Let us suppose that $\overline{V} \to \overline{X}$ is étale in codimension one. Then so is $\overline{V}^{|L|} \to \overline{X}$, which we denote by π_L. We have $K_{\overline{V}^{|L|}} = \pi_L^* K_{\overline{X}}$. We give an adelic metric to $-K_{\overline{X}}$ and the induced one to $-K_{\overline{V}^{|L|}}$. We consider the associated heights on $\overline{X}$ and $\overline{V}^{|L|}$ and the height zeta functions $Z_{X_{\mathrm{prim}}(K)}(s)$ and $Z_{V_{\mathrm{ur}}^{|L|}(K)}(s)$ with respect to these heights. From (3.1), the height function on $\overline{X}(K)$ is stable under the $\operatorname{Aut}(L)$-action and the map (10.2) preserves heights. Since the $\operatorname{Aut}(L)$-action on $V_{\mathrm{ur}}^{|L|}(K)$ is free for every $L \in G\text{-Fie}(K)$ (Lemma 9.5), we obtain the following consequence:

Proposition 10.4. *We have*

$$(10.3) \qquad Z_{X_{\mathrm{prim}}(K)}(s) = \frac{1}{\sharp Z(G)} \sum_{L \in G\text{-Fie}(K)} Z_{V_{\mathrm{ur}}^{|L|}(K)}(s).$$

10.3. **The distribution of K-points of $V_{\mathrm{ur}}^{|L|}$.** Let us now choose a finite set of places, $S \subset M_K$, with $M_{K,\infty} \subset S$ such that the G-action on V extends to $\mathcal{V} = \mathbb{A}_{\mathcal{O}_S}^d$. Let $\overline{\mathcal{X}} := \overline{\mathcal{V}}/G$. We suppose that the metrized $-K_{\overline{X}}$ has the model $-K_{\overline{\mathcal{X}}/\mathcal{O}_S}$. Then $-K_{\overline{V}^{|L|}}$ with the induced metric has the model

$$-\pi_L^* K_{\overline{\mathcal{X}}/\mathcal{O}_S} = -K_{\overline{\mathcal{V}}^{|L|}/\mathcal{O}_S} + K_{\overline{\mathcal{V}}^{|L|}/\overline{\mathcal{X}}}.$$

Let us see how the relative canonical divisor $K_{\overline{\mathcal{V}}^{|L|}/\overline{\mathcal{X}}}$ is determined. Firstly, since $\overline{V}^{|L|} \to \overline{V}$ is étale in codimension one, $K_{\overline{\mathcal{V}}^{|L|}/\overline{\mathcal{X}}}$ is supported on the union of

finitely many closed fibers of $\overline{\mathcal{V}}^{|L|} \to \operatorname{Spec} \mathcal{O}_S$. For $\mathfrak{p} \in S^c$, let $F_\mathfrak{p}$ be the fiber of $\mathcal{V}^{|L|} \to \operatorname{Spec} \mathcal{O}_S$ over $\mathfrak{p}$ and $\overline{F_\mathfrak{p}}$ its closure in $\overline{\mathcal{V}}^{|L|}$. If we write

$$K_{\overline{\mathcal{V}}^{|L|}/\overline{\mathcal{X}}} = \sum_{\mathfrak{p} \in S^c} a_\mathfrak{p} \cdot \overline{F_\mathfrak{p}},$$

then

$$K_{\mathcal{V}^{|L|}/\mathcal{X}} = \sum_{\mathfrak{p} \in S^c} a_\mathfrak{p} \cdot F_\mathfrak{p}.$$

To determine the coefficients $a_\mathfrak{p}$, let $L_\mathfrak{p} = L \otimes_K K_\mathfrak{p}$ be the G-ring over $K_\mathfrak{p}$ induced from L. Since the untwisting is compatible with the base change from $\mathcal{O}_S$ to $\mathcal{O}_\mathfrak{p}$, we have

$$K_{(\mathcal{V}_{\mathcal{O}_\mathfrak{p}})^{|L_\mathfrak{p}|}/\mathcal{X}_{\mathcal{O}_\mathfrak{p}}} = a_\mathfrak{p} \cdot F_\mathfrak{p},$$

which was computed in [41]. Let us write $\mathcal{V}_{\mathcal{O}_\mathfrak{p}} = \operatorname{Spec} S^\bullet \mathcal{F}$ for a free $\mathcal{O}_\mathfrak{p}$-module $\mathcal{F}$. Then $\Xi_{L,\mathfrak{p}} := \Xi_L \otimes_{\mathcal{O}_S} \mathcal{O}_\mathfrak{p}$ is identical to

$$\operatorname{Hom}^G_{\mathcal{O}_\mathfrak{p}}(\mathcal{F}, \mathcal{O}_{L_\mathfrak{p}}),$$

which is an $\mathcal{O}_\mathfrak{p}$-submodule of the free $\mathcal{O}_{L_\mathfrak{p}}$-module $\operatorname{Hom}_{\mathcal{O}_\mathfrak{p}}(\mathcal{F}, \mathcal{O}_{L_\mathfrak{p}})$.

Proposition 10.5 ([41, Lem. 6.5])**.** *We have*

$$a_\mathfrak{p} = \frac{1}{\sharp G} \cdot \operatorname{length} \frac{\operatorname{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{O}_{L_\mathfrak{p}})}{\mathcal{O}_{L_\mathfrak{p}} \cdot \Xi_{L,\mathfrak{p}}}.$$

For $L \in G\text{-Fie}(K)$, let $C_{\operatorname{Peyre},L}$ be Peyre's constant from Theorem 7.1 for the projective space $\overline{V}^{|L|}$ and the induced metric on its anti-canonical divisor. For any Zariski open dense subset $U \subset \overline{V}^{|L|}$,

$$N_{U(K)}(B) \sim C_{\operatorname{Peyre},L} B.$$

10.4. **V-discriminants.** We now introduce invariants of a G-ring L over K which measures its ramification as well as the difference of the measures on $\overline{V}$ and $\overline{V}^{|L|}$.

Definition 10.6. For $L \in G\text{-Fie}(K)$, we define the *V-discriminant* D_L^V and the *extended V-discriminant* $\tilde{D}_L^V$ of $L \in G\text{-Fie}(K)$ as

$$D_L^V := \prod_{\mathfrak{p} \in S^c} \frac{\mu_\mathfrak{p}(V(K_\mathfrak{p}))}{\mu_\mathfrak{p}^{|L|}(\overline{V}^{|L|}(K_\mathfrak{p}))} = \prod_{\mathfrak{p} \in S^c} N_\mathfrak{p}^{a_\mathfrak{p}},$$

$$\tilde{D}_L^V := \prod_{\mathfrak{p} \in M_K} \frac{\mu_\mathfrak{p}(V(K_\mathfrak{p}))}{\mu_\mathfrak{p}^{|L|}(\overline{V}^{|L|}(K_\mathfrak{p}))} = \prod_{\mathfrak{p} \in S^c} N_\mathfrak{p}^{a_\mathfrak{p}} \times \prod_{\mathfrak{p} \in S} \frac{\mu_\mathfrak{p}(V(K_\mathfrak{p}))}{\mu_\mathfrak{p}^{|L|}(\overline{V}^{|L|}(K_\mathfrak{p}))}.$$

We often omit the superscript V if it causes no confusion.

The (extended) V-discriminant is a global version of $N_\mathfrak{p}^{a_\mathfrak{p}}$, which was studied by Wood and the author [36] and whose motivic relative was previously considered by the author [39]. Dummit [16] introduced the notion of *ρ-discriminants*, also using the tuning module. It seems that V-discriminants and ρ-discriminants are closely related and differ only by some constant factor, although the author does not know how to relate them precisely.

There exists constants $C_1, C_2 > 0$ independent of L such that

(10.4) $$C_1 D_L \leq \tilde{D}_L \leq C_2 D_L.$$

This is true because the $\mathfrak{p}$-factor of $\tilde{D}_L$ depends only on the G-ring $L \otimes_K K_{\mathfrak{p}}$ over $K_{\mathfrak{p}}$. For each $\mathfrak{p}$, there exist only finitely many such G-rings up to isomorphism. Therefore

$$\frac{\tilde{D}_L}{D_L} = \prod_{\mathfrak{p} \in S} \frac{\mu_{\mathfrak{p}}(V(K_{\mathfrak{p}}))}{\mu_{\mathfrak{p}}^{|L|}(\overline{V}^{|L|}(K_{\mathfrak{p}}))}$$

can take only finitely many distinct values. Thus, which we consider D_L or $\tilde{D}_L$ does not cause much difference, when concerning the distribution of G-fields, which is discussed in later sections.

Let $C_{\mathrm{Peyre},0}$ be Peyre's constant for the projective space V and the given metric on its anti-canonical divisor, and $C_{\mathrm{Peyre},L}$ the one for $V^{|L|}$ as above. The extended V-discriminant is characterized by

(10.5) $$C_{\mathrm{Peyre},L} = \tilde{D}_L^{-1} C_{\mathrm{Peyre},0}.$$

The following result justifies the statement that V-discriminants measure ramification.

Lemma 10.7. *Suppose that the G-action on $\mathcal{V}_{\kappa_{\mathfrak{p}}}$ is faithful. For $\mathfrak{p} \in S^c$, $a_{\mathfrak{p}} > 0$ if and only if L/K is ramified at $\mathfrak{p}$.*

Proof. To compute $a_{\mathfrak{p}}$, we can consider the untwisting with the base $\mathcal{O}_{\mathfrak{p}}$. Furthermore, from [36], we can replace the base with the completion of the maximal unramified extension of $\mathcal{O}_{\mathfrak{p}}$; we denote it by $\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}$ and its function field by $K_{\mathfrak{p}}^{\mathrm{ur}}$. We denote the G-ring $L \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}}$ over $K_{\mathfrak{p}}^{\mathrm{ur}}$ by $L_{\mathfrak{p}}^{\mathrm{ur}}$. We get the following diagram.

Let H be the stabilizer of a connected component of $\mathrm{Spec}\, L_{\mathfrak{p}}^{\mathrm{ur}}$ and $\left(\mathcal{V}_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}}^{\langle L_{\mathfrak{p}}^{\mathrm{ur}}\rangle}\right)_0$ the corresponding connected component of $\mathcal{V}_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}}^{\langle L_{\mathfrak{p}}^{\mathrm{ur}}\rangle}$. We note that $H = 1$ if and only if L/K is unramified at $\mathfrak{p}$. From [41, Lem. 5.7], $\sharp H \cdot a_{\mathfrak{p}}$ is the multiplicity of the Jacobian ideal of the morphism

(10.6) $$\left(\mathcal{V}_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}}^{\langle L_{\mathfrak{p}}^{\mathrm{ur}}\rangle}\right)_0 \to \mathcal{V}_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}} \mathcal{O}_{L_{\mathfrak{p}}^{\mathrm{ur}}}$$

along the closed fiber

$$\left(\mathcal{V}_{\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}}^{\langle L_{\mathfrak{p}}^{\mathrm{ur}}\rangle}\right)_0 \otimes_{\tilde{\mathcal{O}}_{\mathfrak{p}}^{\mathrm{ur}}} \overline{\kappa_{\mathfrak{p}}}.$$

If $H \neq 1$, then the closed fiber maps onto the H-fixed point locus in $\mathcal{V}_{\overline{\kappa_{\mathfrak{p}}}}$, which is by assumption a proper subset of the closed fiber. Therefore the Jacobian ideal has positive multiplicity and hence $a_{\mathfrak{p}} > 0$. If $H = 1$, then (10.6) is an isomorphism and the Jacobian ideal is trivial. Hence $a_{\mathfrak{p}} = 0$. $\square$

Proposition 10.8. *For every real number $B > 0$, there exist at most finitely many $L \in G\text{-}\mathrm{Rin}(K)$ with $D_L \leq B$. Similarly for $\tilde{D}_L$.*

Proof. From (10.4), it suffices to consider the case of D_L. Replacing S if necessary, we may suppose that for every $\mathfrak{p} \in S^c$, the G-action on $\mathcal{V}_{\kappa_{\mathfrak{p}}}$ is faithful. If L/K is ramified at $\mathfrak{p} \in S^c$, then the lemma above shows $D_L \geq N_{\mathfrak{p}}^{1/\sharp G}$. For the inequality $D_L \leq B$ holding, it is necessary that L/K is unramified over those $\mathfrak{p} \in S^c$ with $N_{\mathfrak{p}}^{1/\sharp G} > B$, in particular, over almost all $\mathfrak{p} \in S$. From [28, p. 203], there exist only finitely many such G-rings L. $\square$

11. MODIFIED HEIGHTS

From (10.5) one might naively expect that heights of K-points of $V^{|L|}$ is roughly $\tilde{D}_L^{-1}$ times the ones of the corresponding points of V through a certain isomorphism $V \cong V^{|L|}$. Since there exists no canonical isomorphism between V and $V^{|L|}$, and even if we make good choices of isomorphisms, since the changes of $\mathfrak{p}$-adic metrics for $\mathfrak{p} \in S$ seem to be hard to trace (see Remark 10.3), it must be difficult to know a precise relation between heights on V and $V^{|L|}$. However, if we assume the naive expectation to be true, then we can explain Conjecture 5.6 in terms of the distribution of G-rings over K and vice versa.

We keep the notation from Sections 10.3 and 10.4. In particular, we fix a metric on $-K_{\overline{X}}$ given by a model $-K_{\overline{\mathcal{X}}/\mathcal{O}_S}$ and the induced height function H on $\overline{X}(K)$. We also fix the induced height functions on $\overline{V}(K)$ and $\overline{V}^{|L|}(K)$, which we denote again by H. For each $L \in G\text{-Fie}(K)$, we fix a K-linear isomorphism

$$(11.1) \qquad \psi_L : V \xrightarrow{\sim} V^{|L|}.$$

Definition 11.1. We define the *modified height* of $y \in V^{|L|}(K)$ as

$$H^{\mathrm{modif}}(y) = \frac{H(\psi_L^{-1}(y))}{\tilde{D}_L}.$$

We define the *modified height zeta function* $Z_{V^{|L|}(K)}^{\mathrm{modif}}(s)$ of $V^{|L|}(K)$ by

$$\sum_{x \in V^{|L|}(K)} H^{\mathrm{modif}}(x)^{-s}.$$

By definition,

$$Z_{V^{|L|}(K)}^{\mathrm{modif}}(s) = \tilde{D}_L^{-s} \cdot Z_{V(K)}(s).$$

Let

$$Z_{V^{|L|}(K)}(s) = \sum_{x \in V^{|L|}(K)} H(x)^{-s}$$

be the height zeta function of $V^{|L|}(K)$ with respect to the given (not modified) height. If we assume meromorphic continuation of these functions as in Condition 6.1, then they have the right-most pole at the same place, of the same order (order one) and of the same residue. Therefore we think of $Z_{V^{|L|}(K)}^{\mathrm{modif}}(s)$ as an approximation of $Z_{V^{|L|}(K)}(s)$.

Definition 11.2. We define the *extended V-discriminant zeta function of G-fields* by

$$Z_{V,G,K}^{\mathrm{disc}}(s) = Z^{\mathrm{disc}}(s) := \frac{1}{\sharp Z(G)} \cdot \sum_{L \in G\text{-Fie}(K)} \tilde{D}_L^{-s}.$$

	place	order
$\alpha < 1$	1	1
$\alpha = 1$	1	$\beta + 1$
$\alpha > 1$	α	β

TABLE 1. the right-most pole of $\mathcal{Z}(s)$

We then think of the function,

$$(11.2) \qquad \mathcal{Z}(s) := \frac{1}{\sharp Z(G)} \cdot \sum_{L \in G\text{-Fie}(K)} Z_{V^{|L|}(K)}^{\mathrm{modif}}(s) = Z^{\mathrm{disc}}(s) Z_{V(K)}(s),$$

as an approximation of $Z_{X_{\mathrm{prim}}(K)}(s)$, and heuristically expect that $\mathcal{Z}(s)$ and $Z_{X_{\mathrm{prim}}(K)}(s)$ have the right-most poles at the same place and of the same order.

Let us suppose that the functions $Z_{V(K)}(s)$ and $Z^{\mathrm{disc}}(s)$ admits meromorphic continuation as in Condition 6.1. Suppose that the right-most pole of $Z^{\mathrm{disc}}(s)$ is at $s = \alpha$ and has order β. The place and the order of the right-most pole of $\mathcal{Z}(s)$ are then given in Table 1.

12. DISTRIBUTION OF NUMBER FIELDS

In this section we first recall Malle's conjecture [26, 27] on the distribution of extensions of a number field. We then propose a generalization of it, which we will relate with a version of Manin's conjecture, Conjecture 5.6, in the next section.

12.1. **Malle's conjecture.** Let G be a transitive finite subgroup of the symmetric group S_n, acting on $[n] := \{1, \dots, n\}$.

Definition 12.1. A degree n field extension L of K is called a *small G-field (over K)* if its Galois closure $\hat{L}/K$ has a Galois group which is permutation isomorphic to G. Two small G-fields are said to be *isomorphic* if there is a K-isomorphism between them. The set of small G-fields over K up to isomorphism is denoted by G-fie(K).

Small G-fields and large G-fields are related by the map

$$G\text{-Fie}(K) \to G\text{-fie}(K), \, L \mapsto L^{G_1},$$

where $G_1 \subset G$ is the stabilizer subgroup of $1 \in [n]$.

Lemma 12.2. *The map above is a $\frac{\sharp N_{S_n}(G) \sharp Z(G)}{\sharp C_{S_n}(G) \sharp G}$-to-one surjection.*

Proof. The map is clearly surjective. Let $S_{G,K}$ be the set of continuous surjections of $\mathrm{Gal}(\bar{K}/K)$ to G. The natural map

$$S_{G,K} \to G\text{-Fie}(K)$$

can be identified with the quotient map associated to the G-action on $S_{G,K}$ by conjugation. Therefore this map is a $\frac{\sharp G}{\sharp Z(G)}$-to-one surjection. On the other hand, the natural map

$$(12.1) \qquad S_{G,K} \to G\text{-fie}(K)$$

is identified with the restriction of the natural map

$$T_{S_n,K} \to n\text{-fie}(K),$$

where $T_{S_n,K}$ is the set of continuous homomorphisms $\mathrm{Gal}(\overline{K}/K) \to S_n$ whose images are transitive subgroups and $n\text{-fie}(K)$ is the set of isomorphism classes of degree n field extensions of K. The last map is, in turn, identified with the quotient map associated to the S_n-conjugation. Hence map (12.1) is a $\frac{\sharp N_{S_n}(G)}{\sharp C_{S_n}(G)}$-to-one surjection. We have proved the lemma. $\qquad\qquad\square$

Notation 12.3. For a finite extension L/K, let $\mathrm{Disc}_{L/K}$ be its discriminant and let $d_L := |N_{K/\mathbb{Q}}(\mathrm{Disc}_{L/K})|$. For a real number $B > 0$, we consider the number of small G-fields with d_L bounded:

$$n(G, B) := \sharp\{L \in G\text{-fie}(K) \mid d_L \leq B\}.$$

Malle [26, 27] raised a conjecture concerning the asymptotic behavior of $n(G, B)$ as B tends to the infinity. To state it, we define some invariants.

Definition 12.4. For $g \in G$, we define its *index*, denoted $\mathrm{ind}(g)$, to be

$$n - \sharp\{g\text{-orbits in } [n]\}.$$

We define the *index* of G, $\mathrm{ind}(G)$, to be the minimum of $\mathrm{ind}(g)$, $g \in G \setminus \{1\}$.

Since the index of an element of G depends only on its K-conjugacy class, the index of a K-conjugacy class makes sense. Let $\beta(G)$ denote the number of K-conjugacy classes having index equal to $\mathrm{ind}(G)$.

Conjecture 12.5 (Malle's conjecture [26, 27])**.** *If K is sufficiently large, then we have*

$$n(G, B) \sim CB^{1/\mathrm{ind}(G)}(\log B)^{\beta(G)-1}.$$

In fact, the condition "if K is sufficiently large" was not assumed in Malle's original formulation. That version, which is stronger than ours, holds for abelian groups [24, 37]. For non-abelian groups, we only refer the reader to surveys [7, 13, 14]. However there exist counter-examples for the original version [21].

Remark 12.6. Klüners' counter-examples [21] are based on the following phenomenon: in some case, there exist intermediate extensions $K(\zeta)/K$ such that ζ is a root of unity and the value of β differs for K and $K(\zeta)$. If we suppose that K is sufficiently large as in our version of the conjecture, then this problem does not occur. Another way to avoid the problem is to exclude extensions containing such an intermediate extension $K(\zeta)$ from the counting. Klüners already discussed it, but called it "not very natural" (the last section, *loc.cit.*). However, according to the correspondences (9.1) and (10.2), such an exclusion corresponds to removing an additional accumulating thin subset from $X_{\mathrm{prim}}(K)$ (see the following lemma) and getting a smaller cothin subset. It might be thus natural from the viewpoint of the "cothin version" of Manin's conjecture.

Lemma 12.7. *Let V be a quasi-projective geometrically irreducible F-variety with a faithful G-action and consider bijection (9.1). Let E/F be a non-trivial finite field extension and $G\text{-Fie}(F)_E$ the subset of $G\text{-Fie}(F)$ consisting of those G-fields L such that there exists a K-embedding $E \hookrightarrow L$. The image of*

$$\bigsqcup_{L \in G\text{-Fie}(F)_E} V(L)^G/\mathrm{Aut}(L)$$

in $X(F)$ is thin.

Proof. Replacing it with the Galois closure, we may suppose that E/F is Galois. Let H denote its Galois group. For $L \in G\text{-Fie}(F)_E$, an embedding $E \hookrightarrow L$ gives a surjective homomorphism $\psi : G \twoheadrightarrow H$ and we have a natural map

$$V(L)^G \to (V/\mathrm{Ker}(\psi))(E)^H,$$

which factors the natural map $V(L)^G \to X(K)$. Therefore, the image of the problem is contained in the image of

$$\bigsqcup_{\psi : G \twoheadrightarrow H} (V/\mathrm{Ker}(\psi))(E)^H.$$

It is now enough to show the following claim:

Claim. Let V be as in the lemma and $L \in G\text{-Fie}(F)$. Suppose $G \neq 1$. Then the image of $V(L)^G \to X(F)$ is thin.

To show this, we may suppose that V is an affine variety, and can use the untwisting technique for an arbitrary affine variety in [41]. We can construct the untwisting variety $V^{|L|}$ of V as follows. Firstly there exists a G-equivariant closed embedding of V into an affine space $W = \mathbb{A}_F^d$ with a linear faithful G-action. Then $V^{|L|}$ is defined as the preimage of $X = V/G \subset W/G$ in $W^{|L|}$. Note that since we are working in characteristic zero, we do not need to take the normalization as did in [41]. There exists a natural generically finite morphism $V^{|L|} \to X$ of degree $\sharp G$ such that the image of $V_{\mathrm{ur}}(L)^G$ in $X(F)$ is contained in the image of $V^{|L|}(F)$. The construction shows $V^{|L|} \otimes_F L \cong_L V \otimes_F L$. It follows that $V^{|L|}$ is irreducible. The claim, and hence the lemma follow. $\square$

12.2. V-discriminants vs. ordinary discriminants.

Local counterparts of V-discriminants naturally appear as counting weights of G-rings over a local field, especially in the context of the McKay correspondence [36, 43, 40, 39, 41]. It is natural to use (extended) V-discriminats when counting G-rings/fields over a number field as well, and relate them to relevant quotient varieties. Recently Dummit [16] studied the distribution of G-fields with respect to his ρ-discriminant, which is seemingly close (essentially the same) to our V-discriminant.

The relation between V-discriminants and ordinary ones is as follows. Let G be a transitive subgroup of S_n and let $V = \mathbb{A}_K^{2n}$ be the direct sum of two copies of the associated n-dimensional permutation G-representation. Putting $S = M_{K,\infty}$, we define $\mathfrak{p}$-adic metrics on $-K_{\overline{V}}$ given by the model $-K_{\overline{\mathcal{V}}/\mathcal{O}_K}$. Accordingly we associate the V-discriminant D_L^V to each $L \in G\text{-Fie}(K)$.

Proposition 12.8. *We have $D_L^V = d_{L^{G_1}}$.*

Proof. The corresponding assertion over a local field was proved in [36]. The proposition is a direct consequence of it. $\square$

For $g \in G$, let us define its index by regarding G as a subgroup of S_n and its age by regarding G as a subgroup of $\mathrm{GL}(V)$ with V as above.

Proposition 12.9. *We have $\mathrm{ind}(g) = \mathrm{age}(g)$.*

Proof. From the additivity of age and index, we may suppose that g is the cyclic permutation, $1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1$. Then a diagonalization of g as an element of $\mathrm{GL}_{2n}(\overline{K})$ is

$$\mathrm{diag}(1, \zeta, \ldots, \zeta^{n-1})^{\oplus 2}$$

with ζ a primitive n-th root of unity. We have

$$\operatorname{age}(g) = \frac{2}{n} \sum_{i=0}^{n-1} i = n - 1 = \operatorname{ind}(g).$$

$\square$

12.3. **General actions.** Next we consider an arbitrary faithful linear G-action on $V = \mathbb{A}_K^d$ such that the associated map $\overline{V} \to \overline{X}$ is étale in codimension one. We fix an adelic metric on the anti-canonical divisor of $\overline{X} = \overline{V}/G$ and the one induced on the anti-canonical divisor of $\overline{V}$. We also fix S such that the action extends to $\mathbb{A}_{\mathcal{O}_S}^d$. In this setting, we define (extended) V-discriminants for $L \in G$-Fie(K). For $B > 0$, we put

$$\tilde{N}(G, V, B) := \sharp\{L \in G\text{-Fie}(K) \mid \tilde{D}_L \leq B\}$$
$$N(G, V, B) := \sharp\{L \in G\text{-Fie}(K) \mid D_L \leq B\},$$

which are finite thanks to Proposition 10.8. We would like to propose the following generalization of Malle's conjecture:

Conjecture 12.10. *If K is sufficiently large, we have*

$$N(G, V, B) \sim CB^{1/\operatorname{age}(G)}(\log B)^{v(G)-1},$$

with $v(G)$ the number of the youngest K-conjugacy classes.

Thanks to Propositions 12.8 and 12.9, Conjecture 12.10 is indeed a generalization of Conjecture 12.5. From inequalities (10.4), we expect that the same asymptotic formula holds for $\tilde{N}(G, V, B)$, the constant factor C possibly changing.

Example 12.11. Let $G \subset S_n$ be a transitive subgroup and consider the natural G-action on $V = \mathbb{A}_K^d$. For an integer $n > 0$, let $V^{\oplus n}$ denote the direct sum G-representation of n copies of V. We have

$$D_L^{V^{\oplus n}} = (D_L^V)^n.$$

Therefore Conjecture 12.5 for G implies Conjecture 12.10 for $V^{\oplus n}$.

Example 12.12. Suppose that $G = \langle g \rangle$ is a cyclic group of order p with p an odd prime number, that K contains a primitive p-th root ζ of unity and that $V = \mathbb{A}_K^{2n}$ has the G-action determined by

$$g = \operatorname{diag}(\zeta^{a_1}, \zeta^{-a_1}, \zeta^{a_2}, \zeta^{-a_2}, \ldots, \zeta^{a_n}, \zeta^{-a_n}) \in \operatorname{GL}(V).$$

The G-representation V is balanced in the sense of [36]. Moreover the function

$$G \to \mathbb{Q}, \, g \mapsto \operatorname{age}(g)$$

is fair in the sense of [35]. From [35, Th. 3.1], Conjecture 12.10 is true in this case.

Remark 12.13. The reason why we consider the distribution of G-fields rather than G-rings can be explained as follows. Let us consider the case $G = S_n$. There exists a one-to-one correspondence between S_n-rings over K and degree n K-algebras. In general, K-algebras decomposable as the product of several fields outnumber indecomposable ones. In the context of Manin's conjecture, this corresponds to the expected phenomenon that non-primitive rational points of a quotient variety $X = V/G$ outnumber primitive ones. Thus non-primitive rational points constitute an *accumulating thin subset,* which should be removed to have a nice formula.

13. Manin vs. Malle

In this last section, we see how Conjectures 5.6 and 12.10 are related as a consequence of materials prepared in earlier sections.

We suppose that $V = \mathbb{A}_K^d$ is given a faithful linear G-action and that $\overline{V} \to \overline{X}$ is étale in codimension one. Let $S \subset M_K$ be a finite subset containing $M_{K,\infty}$ such that the G-action extends to $\mathbb{A}_{\mathcal{O}_S}^d$. Accordingly we fix an adelic metric on $-K_{\overline{X}}$ and the induced one on $-K_{\overline{V}}$. We assume that the Dirichlet series $Z^{\mathrm{disc}}(s)$, $Z_{V(K)}(s)$ and $Z_{X_{\mathrm{prim}(K)}}(s)$ admit meromorphic continuation as in Condition 6.1. We also assume that $\mathcal{Z}(s) = Z^{\mathrm{disc}}(s) Z_{V(K)}(s)$ and $Z_{X_{\mathrm{prim}(K)}}(s)$ have the right-most poles at the same place and of the same order, and assume that K is sufficiently large.

We discuss three cases separately.

13.1. **The case** $\mathrm{age}(G) > 1$. In this case, Conjecture 12.10 shows that the Dirichlet series $Z^{\mathrm{disc}}(s)$ is convergent at $s = 1$. This and the assumptions show that $\mathcal{Z}(s)$ and hence $Z_{X_{\mathrm{prim}(K)}}(s)$ have simple poles at $s = 1$. Thus Conjecture 5.6 holds for $U = X_{\mathrm{prim}}(K)$.

Conversely, if Conjecture 5.6 holds for $U = X_{\mathrm{prim}}(K)$, then $Z_{X_{\mathrm{prim}(K)}}(s)$ and $\mathcal{Z}(s)$ have the right-most poles at $s = 1$, which are simple poles. Therefore $Z^{\mathrm{disc}}(s)$ is convergent at $s = 1$ and the right-most pole of $Z^{\mathrm{disc}}(s)$ is in $s > 1$. We thus have only a weaker estimation than Conjecture 12.10.

13.2. **The case** $\mathrm{age}(G) = 1$. This case is where Conjectures 5.6 and 12.10 are tied to each other the best. Conjecture 12.10 says that $Z^{\mathrm{disc}}(s)$ has the right-most pole at $s = 1$, which has order $\upsilon(G) = \gamma(X)$ (Proposition 8.5). Our assumptions show that $\mathcal{Z}(s)$ and hence $Z_{X_{\mathrm{prim}(K)}}(s)$ have right-most poles at $s = 1$, which have order $\gamma(X) + 1$. Thus Conjecture 12.10 implies Conjecture 5.6 for $U = X_{\mathrm{prim}}(K)$ under the assumptions. We can similarly see the converse.

13.3. **The case** $\mathrm{age}(G) < 1$. In this case, Conjecture 12.10 says that $Z^{\mathrm{disc}}(s)$ has the abscissa of convergence at $s = 1/\mathrm{age}(G) > 1$. This shows Conjectures 5.6. However, the expected abscissa of $Z_{X_{\mathrm{prim}(B)}}(s)$ (if $X_{\mathrm{prim}}(B)$ is sufficiently small) is not at $s = 1/\mathrm{age}(G)$ at least when G is abelian (see Corollary 5.10).

If Conjecture 5.6 is true, then the abscissa of $Z^{\mathrm{disc}}(s)$ is greater than one.

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