TRIVALENT POLYGONAL COMPLEXES OF NONPOSITIVE CURVATURE AND PLATONIC SYMMETRY

Jacek Swiatkowski

Instytut Matematyczny
Uniwersytet Wrockawski
Pl. Grunwaldzki 2/4
50-384 Wrockaw

Poland

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-129

TRIVALENT POLYGONAL COMPLEXES OF NONPOSITIVE CURVATURE AND PLATONIC SYMMETRY

Jacek Świątkowski

Abstract. In this paper we study a natural generalization of Platonic Solids: 2-dimensional simply connected polygonal complexes with flag transitive group of combinatorial automorphisms. Our results give almost complete description of such symmetric complexes with constant valency 3.

The initial local data for assembling such a complex are a regular k-gon and a (highly symmetric) graph L: the link at a vertex. We make the assumption of nonpositive curvature of the complex: this greatly simplifies the question of existence and the real issue is the uniqueness.

The main ingredient of our analysis is the theory of regular graphs, a well developed subject with 50 years of history. Delicate symmetry properties of these graphs yield a variety of local phenomena in complexes and provide the appropriate tool for the study of the uniqueness question.

We should point out that many examples of the complexes we consider have appeared already in literature: most prominently 2-dimensional Bruhat-Tits buildings, but also more recent constructions in [BB], [Be1], [Hae] and [Hag].

We show that many of the symmetric complexes have nondiscrete automorphism groups: buildings are clearly in this class, and some other examples were previously constructed in [Be2]. These automorphism groups resemble p-adic Lie groups and their further study should be worthwile.

1. Introduction.

Polygonal complex is a 2-dimensional polyhedral cell complex. We shall refer to 0-, 1- and 2-cells of a polygonal complex as vertices, edges and faces respectively. Polygonal complex X is *trivalent* if each its edge is adjacent to exactly three faces.

We say that a polygonal complex X is *symmetric*, if the group $Aut(X)$ of all its combinatorial automorphisms acts transitively on *flags* in X, i.e. on incident triples $(vertex, edge, face)$. Obvious examples of symmetric polygonal complexes (not trivalent!) are the (boundaries of) Platonic Solids.

Given a vertex v in a polygonal complex X, the *link* of X at v, denoted $L(v, X)$, is a graph whose vertices and edges represent respectively the edges and faces adjacent to v in X, and incidence relation is induced from that in X.

Each symmetric polygonal complex X determines a natural number $k \geq 3$ and a regular graph L such that all faces of X are k-gonal, and links at all vertices are isomorphic to L. Recall that graph L is *regular* if the group $Aut(L)$ of its automorphisms acts transitively on oriented edges in L. We say that X is *nonpositively curved*, if the following inequality holds:

$$\text{(NPC)} \qquad g(L) \geq \frac{2k}{k-2},$$

where $g(L)$, the *girth* of graph L, is the minimal number of edges in a nontrivial circuit in L. Inequality (NPC) is related to the CAT(0) condition of Alexandrov, see section 4 of [Gro]; it is also related to small cancellation conditions from combinatorial group theory.

In this paper we classify and construct trivalent symmetric polygonal complexes that are nonpositively curved and simply connected.

Examples of symmetric polygonal complexes have already appeared in the literature. Some of them can be obtained by the construction of M. Davis using Coxeter groups (see [Bel] and [Hag]), and by construction of A. Haefliger using complexes of groups (see [Hae]). Unexpectedly, in some cases more than one simply connected symmetric polygonal complex with given local data k and L exists. This phenomenon was discovered by J. Tits for the case of Bruhat-Tits buildings of type $\widetilde{A}_2$, and by F. Haglund for some other cases, see [Tit] and [Hag].

In the above context it is natural to consider the following class of objects. A (k, L)-*complex* is a simply connected polygonal complex all faces of which are k-gonal and links of which at all vertices are isomorphic to graph L. If graph L is regular and a pair (k, L) satisfies the (NPC) condition, then examples of (k, L)-complexes can be constructed inductively using a method described by W. Ballmann and M. Brin in section 1 of [BB], see also section 2 below. This construction however says nothing about symmetry of the resulting complexes. If L is isomorphic to 1-skeleton of a Platonic Solid, then uncountably many nonisomorphic (k, L)-complexes exist, as proved in [BB] and [Hag]. It is clear that most of those complexes are far from being symmetric. In this paper we show that above phenomenon is much more general.

Polygonal complexes are studied as a source of interesting groups. We say that a group Γ of automorphisms of a symmetric polygonal complex X is *rigid*, if its elements are determined by restriction to the star of any vertex; it is *weakly rigid*, if it is not rigid, but its elements are determined by restriction the star of star of any vertex. Recall that *star* of a subcomplex consists of all cells that intersect this subcomplex. If links at vertices of complex X are finite, then it follows from rigidity (or weak rigidity) of group Γ that vertex stabilisers of this group are finite, the group is finitely generated, and it acts properly discontinuously on X. A group Γ is *flexible*, if for any finite subcomplex $P \subset X$ there exists an automorphism F in Γ such that $F|_P = id_P$ and $F \neq id_X$. In such a case stabilisers of vertices are uncountable, and the compact-open topology on Γ is nondiscrete. In this paper we show that all three above phenomena of rigidity, weak rigidity and flexibility appear among automorphism groups of the symmetric complexes that we construct.

Before formulating the Main Theorem of the paper it is necessary to recall briefly some notions and facts from the theory of regular trivalent graphs. Graph L is called *trivalent* if each its vertex is adjacent to exactly three edges. In particular, links at all vertices of a trivalent polygonal complex are trivalent. Given a natural number s, an s-*arc* in a graph is a sequence $(v_0, v_1, \ldots, v_s)$ of its vertices, such that

(i) (v_i, v_{i+1}) is an edge for $i = 0, \dots, s-1$;
(ii) $v_i \neq v_{i+2}$ for $i = 0, \dots, s-2$.

Any group G of automorphisms of a graph clearly acts on the set of all s-arcs in it for any s. We say that G is *s-arc-transitive* if its action on the set of s-arcs in the graph is transitive; it is *s-regular* if this action is simply transitive (i.e. transitive and with trivial stabilisers). Graph L is called *s-regular* if the group $Aut(L)$ of all its automorphisms is s-regular. It was proved by W. Tutte [Tut] that any finite, connected, trivalent and regular graph is s-regular for some $s \in \{1, 2, 3, 4, 5\}$. In the later work [DM] D. Djokovič and G. Miller distinguished the subcases of s'- and s''-regularity for $s = 2$ and $s = 4$, taking the properties of edge stabilisers into account. They proved that each finite, connected, trivalent and regular graph is s-regular for some $s \in \{1, 2', 2'', 3, 4', 4'', 5\}$.

Examples of s-regular graphs for $s \in \{2', 3, 4', 5\}$ were given by W. Tutte in [Tut]. The first example of 1-regular graph was found by R. Frucht in [Fru]. Examples of $2''$- and $4''$-regular graphs has been constructed quite recently by M. Conder and P. Lorimer (see [CL]), and they have 6.652.800 and $29!/48$ vertices respectively! The infinite families of s-regular graphs are known to exist for $s \in \{2', 3, 4', 5\}$, see [Big] and [Cox]. The book [Bou], the R. M. Foster census of trivalent symmetric graphs, contains all known examples up to 512 vertices.

Main Theorem.

Let L be a finite, connected, trivalent and regular graph, $k \geq 3$ a natural number, and assume that they satisfy the nonpositive curvature condition (NPC).

(1) If L is s-regular with $s \in \{3, 4', 4'', 5\}$ and if $k \geq 4$ then there exists a unique (k, L)-complex X, it is symmetric, and the group $Aut(X)$ is flexible.
(2) If one of the following conditions is satisfied:
   (i) L is s-regular with $s \in \{1, 2'\}$;
   (ii) L is $2''$-regular and k is even;

   then there are two distinct symmetric (k, L)-complexes, and in each case the group $Aut(X)$ is rigid.
(3) If L is $2''$-regular and k is odd then no (k, L)-complex is symmetric.
(4) If L is 3-regular and $k = 3$ then there are at least as many distinct (k, L)-complexes as $Aut(L)$-invariant elements in the cohomology group $H^1(L, Z_2)$. The group $Aut(X)$ is in each such case weakly rigid.
(5) If L is 5-regular and $k = 3$ then there are two distinct symmetric (k, L)-complexes, and for each of them the group $Aut(X)$ is flexible.

Remark. The methods of this paper are not sufficient to handle completely the cases when L is 3- or 4-regular and $k = 3$. In 4-regular case even the existence of symmetric $(3, L)$-complexes is not clear. In 3-regular case symmetric $(3, L)$-complexes are not classified completely. We should point out however that in the former case the group $Aut(X)$ of all automorphisms of any symmetric $(3, L)$-complex is flexible, while in the latter case it is weakly rigid. This follows from the proof of uniquness part of Proposition 6.1 below.

As a by-product of the methods developed to prove Main Theorem we get the following.

Corollary. Let L ba an s-regular graph and $k \geq 3$ be a natural number, and assume that they satisfy condition (NPC). If $s \in \{1, 2', 2''\}$, or if $s \in \{3, 4', 4'', 5\}$ and $k = 3$, then there exist uncountably many pairwise nonisomorphic (k, L)-complexes.

Structure of the paper. In section 2 we recall the general inductive construction of (k, L)-complexes, emphasising the role of label systems — combinatorial data which rule steps of the construction. In section 3 we introduce characteristic functions of label systems and apply them in section 4 to construct combinatorial invariants of small subcomplexes in (k, L)-complexes. In section 5 we study conditions for label systems expressed in terms of the invariants of section 4, and in section 6 we show that these conditions determine (k, L)-complexes uniquely. In section 7 we apply the results of previous sections to prove Main Theorem and Corollary.

Acknowledgments. This paper is a part of my PhD thesis defended in June 1994 at the University of Wrocław. I would like to thank my advisor Tadeusz Januszkiewicz for the inspiration and numerous discussions. The final version was written during my visit in Max Planck Institut in Bonn in 1996, where I found perfect conditions for work and scientific atmosphere.

2. Inductive construction of (k, L)-complexes.

For the rest of this paper we assume that k and L satisfy the assertions of the Main Theorem.

In this section we sketch the inductive construction of (k, L)-complexes — a special case of a more general construction described in section 1 of [BB]. We emphasise the features of this construction which are important for our purposes.

2.1. (k, L)-complexes with convex boundary.

Let K be a polygonal complex with all faces k-gonal. A vertex w of K is called 1-*free* if the link $L(w, K)$ consists of a single edge; it is 2-*free* if $L(w, K)$ is isomorphic to a star of vertex in a 3-valent tree, i.e. it consists of three edges adjacent to a common vertex; it is 3-*free* if $L(w, K)$ is isomorphic to a star of edge in a 3-valent tree. An edge of K is called *free* if it is adjacent to exactly one face of K, and it is *interior* if it is adjacent to three faces of K. A vertex w is *interior* if the link $L(w, K)$ is isomorphic to L.

K is a (k, L)-*complex with convex boundary*, if each its vertex is either interior or m-free, for some $m \in \{1, 2\}$ if $k \geq 4$, and for some $m \in \{1, 2, 3\}$ if $k = 3$. Note that then each edge of K is either interior or free, and define the *boundary* ∂K of K to be the subcomplex of K consisting of all its free edges.

2.2. Initial step of the construction.

Arrange a disjoint collection of k-gonal faces around an initial vertex v, by glueing them along edges according to the pattern provided by L, so that the link of resulting complex at v is isomorphic to L, and only the edges which are adjacent to v have been glued. We will denote the complex obtained in this way by B_1.

2.3. General inductive step.

Note that the complex B_1 constructed above is a (k, L)-complex with convex boundary, and therefore we can apply everything that follows below in this subsection to it, thus initiating a process of induction.

Let K be a finite (k, L)-complex with convex boundary, and assume $\partial K \neq \emptyset$. Denote by $\widetilde{K}$ the complex obtained from K by glueing two new k-gonal faces to each its free edge, without performing any other glueings. For each free (i.e. not interior) vertex w of K consider a *label map* $\lambda_w : L(w, \widetilde{K}) \to L$, which is by definition locally injective, i.e. injective on star of each vertex in $L(w, \widetilde{K})$. Denoting by $V_\partial K$ the set of the boundary vertices of K, call a collection $\Lambda = \{\lambda_w : w \in V_\partial K\}$ of such label maps a *label system* for K.

Note that under the nonpositive curvature condition (NPC), the only cases when a label map λ_w is not injective (globally) are:
(i) w is 2-free and $g(L) \leq 4$;
(ii) w is 1-free and $g(L) = 3$.
A 3-free vertex w, by convexity, appears only when $k = 3$. But then it follows from (NPC) that $g(L) \geq 6$, which implies injectivity of λ_w. In any case, we have

$$\lambda_w|_{L(w,K)} \text{ is injective.} \tag{2.3.1}$$

For each free vertex w of K consider a copy B_1^w of a complex isomorphic to B_1 and an isomorphism $\gamma_w : L(v_w, B_1^w) \to L$, where v_w is the interior vertex of B_1^w. Consider the composition map $(\gamma_w)^{-1} \circ \lambda_w : L(w, \widetilde{K}) \to L(v_w, B_1^w)$, and denote by $\psi_w : st(w, \widetilde{K}) \to B_1^w$ the naturally induced map on stars of the corresponding vertices. Define a complex $\overline{K} = \overline{K}(\Lambda)$ by

$$\overline{K}(\Lambda) = \widetilde{K} \cup_{\psi_{w_1}} B_1^{w_1} \cup \ldots \cup_{\psi_{w_r}} B_1^{w_r}, \tag{2.3.2}$$

where $w_1, \ldots, w_r$ are all the vertices of ∂K.

Note that, due to (2.3.1), K is naturally embedded in $\overline{K}$. Moreover, all the vertices of K become interior in $\overline{K}$, while $\overline{K}$ is again a finite (k, L)-complex with nonempty convex boundary.

To obtain a (k, L)-complex, start with the complex B_1 of 2.2, and apply to it the inductive step of the construction succesively infinitely many times. At each step it is possible to contract $\overline{K}$ to K, by which the resulting complex X is contractible and in particular simply connected. We omit further details related to the construction.

2.4. Reparametrizations in L.

A (k, L)-complex obtained by the inductive construction of 2.2 and 2.3 depends a priori on the choices of label systems at corresponding steps of construction. In the next sections we study this dependence in detail. In the lemma below we make the first observation in this direction.

Let K be a (k, L)-complex with convex boundary. If $w \in \partial K$ and $\lambda_w, \lambda'_w : L(w, \widetilde{K}) \to L$ are label maps, we say that λ'_w is obtained from λ_w by a *reparametrization in* L, if $\lambda'_w = g \circ \lambda_w$ for some $g \in Aut(L)$.

2.4.1. Lemma. If Λ' is obtained from a label system Λ for a (k, L)-complex K with convex boundary by reparametrizations in L of its label maps, then the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ are isomorphic by an isomorphism extending the identity map id_K.

The proof of above Lemma consists of straightforward manipulations with reparametrizations, and we omit it.

2.5. Lemma. Any (k, L)-complex X can be obtained by means of the inductive construction of 2.2 and 2.3, starting from any its vertex, if the label systems at succesive steps of the construction are chosen appropriately.

Proof: Consider a subcomplex K of X with convex boundary. We shall interpret the star $st(K, X)$ of K in X, as a complex $\overline{K}(\Lambda)$ obtained from K by a step described in 2.3, with use of some label system Λ. Recall that the star $st(K, X)$ is the subcomplex of X consisting of all faces of $\tilde{K}$, and faces adjacent to the boundary vertices of K. We then have the obvious map $\mu : \tilde{K} \to st(K, X)$, extending id_K, well defined up to transpositions at pairs of faces not contained in K, and adjacent to its boundary edges. For $w \in V_\partial K$, let $\mu_w : L(w, \tilde{K}) \to L(w, st(K, X))$ be the induced map on links. Consider any isomorphism $\gamma_w : L(w, st(K, X)) \to L$, and put

$$\lambda_w = \gamma_w \circ \mu_w, \tag{2.5.1}$$

thus getting the label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$. A standart checking shows that the complexes $st(K, X)$ and $\overline{K}(\Lambda)$ are then canonically isomorphic by the isomorphism extending id_K.

The lemma follows by applying above observation succesively to the subcomplexes in X obtained from the star of any vertex by iterating the operation of taking the star.

3. Order systems and characteristic functions.

By Lemma 2.5, each (k, L)-complex can be obtained by a variant of the inductive construction of section 2. The choices of label systems provide freedom in the construction, allowing nonisomorphic complexes to appear. In this section we introduce characteristic functions of label systems — the notion usefull in the study of the uniqueness question for (k, L)-complexes.

3.1. Order systems in L.

Let p be a vertex and e an edge of L.

3.1.1. Let x, y be the vertices adjacent to e. Denote by O_x and O_y the pairs of edges in L, distinct from e, adjacent to x and y respectively, with distinguished order for each pair. We will call them *ordered pairs of peripheral edges at edge e*, and the whole system (e, O_x, O_y) an *order-system at e*.

3.1.2. Let x, y, z be all the vertices of $st(p, L)$ distinct from p. Denote by O_x, O_y and O_z the pairs of edges in L not contained in $st(p, L)$, adjacent to x, y and z respectively, with distinguished order for each pair. We will call them *ordered pairs of peripheral edges* at star of vertex p, and the whole system (p, O_x, O_y, O_z) an *order-system at star of* p.

3.1.3. Let x, y, z and u be all the vertices of $st(e, L)$ not adjacent to e. The system (e, O_x, O_y, O_z, O_u) consisting of ordered pairs of edges in L, adjacent to x, y, z and u respectively, and not contained in $st(e, L)$, will be called an *order-system at star of* e, and its elements *ordered pairs of peripheral edges at star of* e.

3.1.4. Denote by O_p the cyclically ordered triple of edges in L adjacent to p. We will call (p, O_p) a *cyclic order at* p.

All orders at pairs of peripheral edges, or at a triple in the last case, will be called *peripheral orders.*

3.2. Good label systems.

Assume we have fixed order-systems in L of all four types 3.1.1-3.1.4, and let K be a finite (k, L)-complex with convex boundary, as defined in 2.1. Denote by $V_\partial K$ the set of vertices of K contained in the boundary ∂K.

We will say that a label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$ is *good*, if its label maps satisfy the following conditions:

(i) if w is 1-free in K, then $\lambda_w(L(w, K)) = e$, where e is the edge appearing in the fixed order-system of type 3.1.1 in L;
(ii) if w is 2-free in K, then $\lambda_w(L(w, K)) = st(p, L)$, where p is the vertex appearing in the fixed order-system of type 3.1.2 in L;
(iii) if w is 3-free in K, then $\lambda_w(L(w, K)) = st(e, L)$, where e is the edge appearing in the fixed order-system of type 3.1.3 in L.

3.2.1. Remark. Note that, due to transitivity of the group $Aut(L)$ on the vertices and edges of L, each label system can be made good by the appropriate reparametrizations in L of its label maps.

3.3. Characteristic functions of good label systems.

Let $E_\partial K$ denote the set of boundary edges of K. For each edge $d \in E_\partial K$ consider the set of two faces in $\widetilde{K} \setminus K$ adjacent to d and call it the *set of peripheral faces at* d. Let $\Lambda = \{\lambda_w : w \in V_\partial K\}$ be a good label system for K. Then for each vertex $w \in V_\partial K$ the label map λ_w establishes a correspondence between the elements of sets of peripheral faces at edges of $E_\partial K$ adjacet to w, and elements of peripheral pairs of edges in the fixed order system of the corresponding type. Using the above correspondence, for each edge $d \in E_\partial K$ we can induce an order on the set of peripheral faces at d by any of the two label maps λ_w and λ_v, where w and v are the endpoints of d. We define a *characteristic function* $\chi_\Lambda : E_\partial K \to Z_2$ by

$$\chi_\Lambda(d) = \begin{cases} 0 & \text{if the two induced orders as above agree;} \\ 1 & \text{otherwise.} \end{cases}$$

3.4. K-equivalences of label maps and label systems.

Given $w \in V_\partial K$, we say that label maps $\lambda, \lambda' : L(w, \widetilde{K}) \to L$ are *K-equivalent* if $\lambda|_{L(w,K)} = \lambda'|_{L(w,K)}$, i.e. their restrictions to the link of K coincide. Label systems

$\Lambda = \{\lambda_w : w \in V_\partial K\}$ and $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$ are *K-equivalent* if for each $w \in V_\partial K$ the label maps λ_w and λ'_w are K-equivalent.

The significance of the notions of K-equivalence and characteristic function of a good label system becomes apparent due to the following lemma, which will be used later for the classifications. Characteristic functions will be used also to define local invariants of (k, L)-complexes in the next section.

3.5. Lemma. If two good label systems $\Lambda = \{\lambda_w : w \in V_\partial K\}$ and $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$ are K-equivalent, and have equal characteristic functions, then the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ are isomorphic by an isomorphism extending the identity map id_K.

Proof: Consider the systems $\{\psi_w : w \in V_\partial K\}$ and $\{\psi'_w : w \in V_\partial K\}$ of glueing maps used to get $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ respectively, as in (2.3.2). By construction of $\overline{K}(\Lambda)$, each map ψ_w is a restriction to $st(w, \widetilde{K})$ of some isomorphism $\overline{\gamma_w} : st(w, \overline{K}(\Lambda)) \to B_1^w$, and similarly ψ'_w is a restriction of some $\overline{\gamma'_w} : st(w, \overline{K}(\Lambda')) \to B_1^w$. Consider the following conditions for a combinatorial map $f : \overline{K}(\Lambda) \to \overline{K}(\Lambda')$:

$$f|_K = id_K; \tag{3.5.1}$$

$$f|st(w, \overline{K}(\Lambda)) = (\overline{\gamma'_w})^{-1} \circ \overline{\gamma_w} \text{ for } w \in V_\partial K. \tag{3.5.2}$$

Note that, due to K-equivalence of label systems Λ and Λ', each condition of form (3.5.2) is compatible with condition (3.5.1). Furthermore, due to equality of the characteristic functions, conditions of form (3.5.2) are pairwise compatible with each other. It is then clear that (3.5.1) and (3.5.2) determine f uniquely, and that f is an isomorphism. This finishes the proof.

4. Local invariants.

In this section we introduce a collection of invariants for small subcomplexes of (k, L)-complexes.

Let X be a (k, L)-complex. A *1-ball* in X is a star of a vertex in X. By the proof of Lemma 2.5, we can view any star of an edge, of a face or of a 1-ball in X as obtained from the corresponding edge, face or 1-ball in one step as in 2.3, with use of some appropriate label system. We shall use the characteristic function of such a system to define a combinatorial invariant of the corresponding star. The invariant will be defined in different ways in the corresponding cases of s-regularity of L. To show that in each case the invariant is well defined, we will use the following Proposition which extracts what is important for our purposes from Propositions 1-5 of [DM].

4.1. Proposition.

(1) If L is 1-regular then any automorphism of L which fixes a vertex p in L preserves a cyclic order at p;

(2) If L is $2'$-regular then any automorphisms of L which fixes an edge e in L (but not necessarily its endpoints) either preserves both peripheral orders at e, or reverses them both;

(3) If L is $2''$-regular then
   (a) any automorphisms of L fixing edge e as oriented edge (i.e. together with its endpoints) either preserves both peripheral orders at e, or reverses them both;
   (b) any automorphism of L fixing an edge e, but reversing its orientation, reverses exactly one of the peripheral orders.

(4) If L is 3-regular then any automorphism of L fixing a vertex p either preserves all three peripheral orders at star of p, or reverses them all, for some proper choice of the orders (it is not true for each choice).

(5) If L is s-regular for some $s \in \{4', 4'', 5\}$ then any automorphism of L fixing a vertex p either preserves all three peripheral orders at star of p, or reverses exactly two of them.

(6) The converse of any of above statements is true, i.e. for any change of peripheral orders mentioned in any of the cases (1)-(5) above, there exists a corresponding automorphism of L which results with exactly this change.

4.2. Local invariants. Let L be an s-regular graph and X a (k, L)-complex.

4.2.1. The case $s = 1$. To introduce an invariant in this case, we view the star of an edge in X as obtained from this edge by a step of construction as in 2.3. This requires however some explanation, since a complex consisting of a single edge is not a (k, L)-complex with convex boundary.

Let K consist of a single edge $d = (w_1, w_2)$. Define $\widetilde{K}$ to consist of three k-gonal faces glued to d along some of their edges. Define a label system Λ for K to consist of two label maps $\lambda_i : L(w_i, \widetilde{K}) \to L$. Then one obtains an extension $\overline{K}(\Lambda)$ according to the pattern determined by Λ, as in (2.3.2). It follows by the arguments as in the proof of Lemma 2.5 that the star of any edge in X can be viewed as obtained by the construction as above.

To get an invariant, fix a cyclic order (p, O_p) in L, as defined in 3.1.4. A label system Λ is said to be good, if $\lambda_i(L(w_i, K)) = p$, for $i = 1, 2$. Given such a good label system, induce twice the cyclic order from L to the set of three faces of $\widetilde{K}$, by pulling back with respect to the label maps λ_i, and define characteristic function $\chi_\Lambda : \{d\} \to Z_2$ by

$$\chi_\Lambda(d) = \begin{cases} 0 & \text{if the two induced cyclic orders agree;} \\ 1 & \text{otherwise.} \end{cases}$$

The value of χ_Λ does not depend on reparametrizations in L for which the label maps λ_i remain good since, by Proposition 4.1(1), the stabilizer of vertex p in the group $Aut(L)$ preserves the cyclic order O_p. This allows to define an invariant ε by

$$\varepsilon_X(d) = \chi_\Lambda(d),$$

where Λ is a good label system as above.

4.2.2. The case $s = 2'$. Given a face f in a (k, L)-complex X, put $K = f$, and consider a good label system Λ of form (2.5.1) for K. Since, by Proposition 4.1(2), an element

$[\chi_\Lambda] \in H^1(\partial K, Z_2)$ determined by the characteristic function χ_Λ (viewed as a 1-cocycle in ∂K) does not depend on the choice of Λ, put

$$\xi_X(f) = [\chi_\Lambda].$$

4.2.3. The case $s = 2''$. Consider a variant of the order system of form 3.1.3 consisting of an oriented edge e and two orders at peripheral pairs at e. Let f^+ be an oriented face in a (k, L)-complex X, and w its vertex. Then f^+ determines orientation of the edge d_f corresponding to f in link $L(w, X)$. Put $K = f$. We will say that a label map λ_w for K is *good* if it maps d_f onto e preserving orientations.

Consider a good label system Λ of form (2.5.1) for K (which clearly exists due to the transitivity of $Aut(L)$ on the set of oriented edges in L) and note that, due to Proposition 4.1(3)(a), the cohomology element

$$\xi''_X(f^+) = [\chi_\Lambda]$$

does not depend on reparametrizations in L for which Λ remains good in the above sense.

Note also that, due to Proposition 4.1(3)(b), if $f^\pm$ denote the oppositely oriented faces related to a nonoriented face f then

(4.2.3.1) $\xi''_X(f^+) \neq \xi''_X(f^-)$ if k is odd, and $\xi''_X(f^+) = \xi''_X(f^-)$ if k is even.

4.2.4. The case $s = 3$ and $k = 3$. For a vertex v in X, let $K = st(v, X)$ and let Λ be a good label system of form (2.5.1) for K. Since, by Proposition 4.1(4), an element $[\chi_\Lambda] \in H^1(\partial K, Z_2)$ does not depend on the choice of Λ, put

$$\eta_X(v) = [\chi_\Lambda].$$

4.2.5. The case $s \in \{4', 4'', 5\}$ and $k = 3$. Let v, K and Λ be as in 4.2.4. Then, by Propostion 4.1(5), the number

$$\sigma_X(v) = \sum_{d \in E_\partial K} \chi_\Lambda(d) \in Z_2$$

does not depend on the choice of Λ.

We summarize results of this section in the following.

4.3. Lemma. The quantities ε, ξ, η and σ defined in 4.2 are the invariants of combinatorial isomorhism for the corresponding complexes $st(K, X)$.

In the case of $s = 3$ this requires the following more carefull statement. Let $T : st^2(v, X) \to st^2(w, X)$ be an isomorphism, and put $T_1 = T|_{\partial st(v,X)}$. Then T_1 pulls back $\eta_X(w)$ to $\eta_X(v)$, i.e. $T_1^*(\eta_X(w)) = \eta_X(v)$.

5. Conditions for label systems, related to local invariants.

The general construction of (k, L)-complexes, as presented in section 2, does not provide any control on local invariants of resulting complexes. In this section we describe a method to get such a control, in terms of restrictive conditions for label systems used at corresponding steps of construction. We use letter C as a variable, speaking about condition C whenever refering to a condition as above. We prove that conditions of this type are consistent, i.e. that there exist label systems which satisfy them. We also prove that these conditions determine resulting complexes uniquely.

5.1. Description of conditions.

An *n-ball* is a complex obtained in n steps of the construction as in 2.2 and 2.3. A *center* of an n-ball is the initial vertex of this construction. A (k, L)-*n-ball* is an n-ball obtained by the construction with use of data k and L. Note that each (k, L)-n-ball is a (k, L)-complex with convex boundary.

Let L be an s-regular graph and let K be a (k, L)-n-ball. In all the cases 5.1.1-5.1.5 below $\mathcal{L}_K^C$ denotes the set of all good label systems for K satisfying condition C.

5.1.1. The case $s = 1$. Denote by $E_{out}K$ the set of all *outer* edges in K, i.e. those which have at least one vertex on the boundary ∂K. Consider any function $\varepsilon_C : E_{out}K \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \varepsilon_{\overline{K(\Lambda)}}(e) = \varepsilon_C(e) \text{ for each } e \in E_{out}K.$$

5.1.2. The case $s = 2'$, or $s = 2''$ and k is even. Denote by $F_{out}K$ the set of all *outer* faces of K, i.e. those which contain at least one boundary vertex. Consider any function $\xi : F_{out}K \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \xi_{\overline{K(\Lambda)}}(f) = \xi_C(f) \text{ for each } f \in F_{out}K.$$

In the above equation we use the canonical identification of $H^1(\partial f, Z_2)$ with Z_2.

Note that due to (4.2.3.1), when G is $2''$-regular and k is even, we can view ξ'' as an invariant of nonoriented faces. In this case we define condition C in exactly the same way as above, replacing ξ by ξ''.

5.1.3. The case $s = 2''$ and k is odd. For each face $f \in F_{out}K$ choose an orientation, and denote the set of so oriented faces by $F_{out}^{or}K$ (note that only one of the two oriented faces corresponding to a given face of $F_{out}K$ is contained in $F_{out}^{or}K$). Consider a function $\xi_C'' : F_{out}^{or}K \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \xi_{\overline{K(\Lambda)}}''(f) = \xi_C''(f) \text{ for each } f \in F_{out}^{or}K.$$

5.1.4. The case $s = 3$ and $k = 3$. Denote by K^- the concentric $(n-1)$-ball contained in K. For each $v \in V_\partial K^-$, consider the boundary $\partial st(v, K)$ of the 1-ball in K centered at v, and an element $\eta_C(v) \in H^1(\partial st(v,K), Z_2)$, thus getting a *sheaf* η_C of cohomology elements. Define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \eta_{\overline{K}(\Lambda)}(v) = \eta_C(v) \text{ for each } v \in V_\partial K^-.$$

5.1.5. The case $s \in \{4', 4'', 5\}$ and $k = 3$. Under notation of 5.1.4, consider a function $\sigma_C : V_\partial K^- \to Z_2$ and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \sigma_{\overline{K}(\Lambda)}(v) = \sigma_C(v) \text{ for each } v \in V_\partial K^-.$$

The rest of this section is devoted to proving the following two results.

5.2. Proposition. Let C be any of the conditions described in 5.1. Then the set $\mathcal{L}_K^C$ of good label systems satisfying this condition is nonempty.

5.3. Proposition. Any two label systems $\Lambda_1, \Lambda_2 \in \mathcal{L}_K^C$ are K-equivalent, after appropriate reparametrizations in L for the label maps of one of them.

Note that, in view of Lemmas 2.4.1 and 3.5, Proposition 5.3 is the first step towards the proof that each condition C determines the resulting complex $\overline{K}$ uniquely up to an isomorphism extending the identity map id_K. The proof of this fact will be completed in section 6.

Before strarting the proofs of above propositions, we need to establish some facts concerning the notion of an order structure in a trivalent graph.

5.4. Order structures in trivalent graphs.

By a *trivalent graph* we mean a connected graph Q, each vertex of which is adjacent either to one or to three edges. Q is a *tree*, if it is simply connected, i.e. contains no circuit. Let ∂Q be the set of all *boundary vertices* of Q, i.e. those which are adjacent to only one edge. We define three types of order structures in a trivalent graph Q.

Type 1. Let v be an interior (i.e. not boundary) vertex of Q. An *order atlas at v* consists of a cyclic order at v (i.e. a cyclic order for the triple of edges of Q adjacent to v). An *order structure* of this type in Q consists of a collection of fixed order atlasses at all interior edges of Q.

Type 2. Let e be an *interior* edge of Q, i.e. the one whose both endpoints are interior. An *order atlas at e* consists of two order systems at e (as described in 3.1.1), that differ by the simultaneous change of both peripheral orders. An *order structure* in Q of this type consists of a collection of fixed order atlesses at all interior edges of Q.

Type 3. We shall need also a variant of order structure consisting of order atlasses at oriented interior edges of Q. Order atlasses of this structure consist of two order systems that differ by the simultaneous change of both peripheral orders. Moreover, order atlass at oppositely oriented edge consists of order systems that differ from the ones in the original atlas by exactly one peripheral order.

5.4.1. Example: $Aut(L)$-invariant order structures in L. Let graph L be s-regular. In the three cases corresponding to $s = 1$, $s = 2'$ and $s = 2''$ we define invariant order structures in L of types 1, 2 and 3 respectively. In each case we do this simply by taking an orbit of the action of $Aut(L)$ on the set of order systems in L of type 3.1.4, 3.1.1 and 4.2.3 respectively. We arrange order systems from any such orbit into a collection of order atlasses at vertices, edges and oriented edges of L. Order structures as above are well defined due to the corresponding cases of Proposition 4.1.

5.4.2. Lemma. Consider an $Aut(L)$-invariant order structure in L, as in 5.4.1, and a trivalent tree Q equipped with an order structure of the same type. Then there exists a combinatorial immersion $i : Q \to L$ preserving order structures. Moreover, this immersion is unique up to a reparametrization in L.

Proof: Consider an s-arc in Q, where $s = 1$ for type 1, and $s = 2$ for types 2 and 3. (If there is no such arc in Q then there is no order structure and you can immerse Q arbitrarily getting uniqueness by the regularity of L.) Immerse this s-arc into L arbitrarily and note that in each case there exists a unique order preserving extension of any such immersion to the whole Q.

The uniqueness part of Lemma follows from the uniqueness of above extension.

5.5. Proof of Proposition 5.2.

The proof goes separately for the corresponding cases of s-regularity of L.

The case $s = 1$. Consider a vertex $w \in V_\partial K$, and an edge $e = (w, u)$ not contained in ∂K. The vertex q_e, which represents edge e, is then interior in the link $L(w, K)$ (note that this link is a trivalent tree). Since vertex u of e is interior in K, the link $L(u, K)$ is isomorphic to L. Recall that we have fixed a cyclic order system at a vertex p in L, and now we extend it in a unique way to an $Aut(L)$-invariant order structure in L. We induce this order structure to $L(u, K)$, which clearly does not depend on the choice of an isomorphism with L. This determines a cyclic order on the set F_e of three faces of K adjacent to e. Condition C requires that $\varepsilon_{\overline{K}(\Lambda)}(e) = \varepsilon_C(e)$, and so a cyclic order on F_e induced by a label map $\lambda_w \in \Lambda$ has to agree or not agree with the above order according to the value of $\varepsilon_C(e)$. Thus the value of $\varepsilon_C(e)$ determines a cyclic order at vertex q_e in $L(w, K)$.

By the argument as above applied to all outer nonboundary edges of K, condition C determines order structures in links $L(w, K)$ at all boundary vertices w of K. The restrictions $\lambda_w|_{L(w,K)}$ of label maps in any $\Lambda \in \mathcal{L}_K^C$ have to preserve these order structures (with respect to the $Aut(L)$-invariant order structure in L defined above).

To construct a label system $\Lambda \in \mathcal{L}_K^C$, immerse links $L(w, K)$ into L in a way preserving order structures, which is possible by Lemma 5.4.2. Extend these immersions arbitrarily and use reparametrizations in L to get a good label system $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$. Consider the complex $\overline{K}(\Lambda')$, and note that $\varepsilon_{\overline{K}(\Lambda')}(e) = \varepsilon_C(e)$ for all outer nonboundary edges of K. Now, for each boundary edge d of K, compare the values of $\varepsilon_{\overline{K}(\Lambda')}(d)$ and $\varepsilon_C(d)$. Whenever they are different, change the label map λ'_w at one endpoint w of d, at the peripheral pair of edges in $L(w, \widetilde{K})$ corresponding to the peripheral pair of faces in $\widetilde{K}$ adjacent to d.

After these changes we get a label system Λ which clearly belongs to $\mathcal{L}_K^C$, which finishes the proof in this case.

The case $s \in \{2', 2''\}$. Consider the case $s = 2'$. Let f be a face in K, which is outer, but has no edge contained in the boundary ∂K, and denote by w its unique vertex contained in ∂K. Note that the edge e_f representing f is inner in the link $L(w, K)$. Since all the vertices u of f other than w are inner in K, the links $L(u, K)$ are isomorphic to L. We have fixed in L an order system at an edge, as in 3.1.1, and now we extend it in a unique way to an $Aut(L)$-invariant order structure in L of the corresponding type 2. We induce this order structure to links $L(u, K)$, and consider order atlasses at edges representing face f in these links. The equality $\xi_{\overline{K}(\Lambda)}(f) = \xi_C(f)$ determines then an order atlas at the edge e_f in $L(w, K)$ which has to be preserved by any label map λ_w of a label system $\Lambda \in \mathcal{L}_K^C$. Applying this to all faces as f in K, we get that condition C determines order structures in links $L(w, K)$ at all boundary vertices of K.

To construct a label system $\Lambda \in \mathcal{L}_K^C$, choose label preserving immersions of links $L(w, K)$ into L, extend them arbitrarily and reparametrize in L to get a good label system $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$. By construction, we have $\xi_{\overline{K}(\Lambda')}(f) = \xi_C(f)$ for all outer faces f of K with no edge contained in ∂K. Consider an outer face g with at least one edge e contained in ∂K. If $\xi_{\overline{K}(\Lambda')}(g) \neq \xi_C(g)$, choose one endpoint w of e and modify λ_w only at the peripheral pair of edges in $L(w, \widetilde{K})$ representing the peripheral pair of faces in $\widetilde{K}$ adjacent to e. Repeating this for all faces g as above, we get a new label system Λ which now belongs to $\mathcal{L}_K^C$.

The argument for the case $s = 2''$ is the same.

The case $s = 3$ and $k = 3$. Choose an arbitrary good label system $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$, and for each vertex $v \in V_\partial K^-$, compare the cohomology elements $\eta_{\overline{K}(\Lambda')}(v)$ and $\eta_C(v)$. If they are different, consider a subgraph $A(v)$ in $\partial st(v, K)$ consisting of all edges not contained in ∂K. Note that $A(v)$ is isomorphic to $L(v, \widetilde{K^-})$, and hence it is a tree and contains not all edges of $\partial st(v, K)$.

Claim. Let $\eta \in H^1(\partial st(v, K), Z_2)$ and let h be a Z_2-valued function defined on the set of edges of $A(v)$. Then h can be extended to a 1-cocycle in $\partial st(v, K)$ representing the cohomology element η.

Proof of Claim: Let ϕ be any 1-cocycle representing η. Define a set of edges $\delta(h, \phi) := \{e \in A(v) : h(e) \neq \phi(e)\}$ and assume that its cardinality satisfies $|\delta(h, \phi)| \neq 0$ (otherwise there is nothing to prove). Choose any edge $e \in \delta(h, \phi)$ and note that, since $A(v)$ is a tree, the removal of e splits $A(v)$ into two components. Choose any of those components and denote by V the set of its vertices (among them one vertex adjacent to e). Let $M := \{m_v : v \in V\}$ be the set of modifications, each consisting of changing the values at all edges in $\partial st(v, K)$ adjacent to v. Applying all modifications from M to the 1-cocycle ϕ, we get a cohomological to it cocycle ϕ' with $|\delta(h, \phi')| = |\delta(h, \phi)| - 1$. Then the Claim follows by induction.

It follows from the Claim that we can modify any cocycle representing $\eta_{\overline{K}(\Lambda')}(v)$ into a cocycle representing $\eta_C(v)$, by changing its values only at edges outside $A(v)$, that is at edges contained in ∂K. For each such edge e (contained in ∂K) at which we want to change the value of a corresponding cocycle, choose a vertex w adjacent to e and modify λ'_w at the pair of peripheral edges in $L(w, \widetilde{K})$ corresponding to the pair of peripheral faces in $\widetilde{K}$ adjacent to e. By the fact that each boundary edge of K is contained in exactly one 1-ball $st(v, K)$ with $v \in \partial K^-$, modifications as above do not influence one another, and produce a new label system Λ which belongs to $\mathcal{L}_K^C$. This finishes the proof of Proposition 5.2 in this case.

The case $s \in \{4', 4'', 5\}$ **and** $k = 3$. Choose an arbitrary good label system $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$ and, for each vertex $v \in V_\partial K^-$, compare the numbers $\sigma_{\overline{K}(\Lambda)}(v)$ and $\sigma_C(v)$. Whenever they are different, choose a boundary edge e of K belonging to $\partial st(v, K)$. Such edge always exists, as it was argued in the proof of the previous case. Choose one endpoint w of e and modify the label map λ'_w at the peripheral pair of edges in $L(w, \widetilde{K})$ corresponding to the peripheral pair of faces in $\widetilde{K}$ adjacent to e. For the same reason as in the previous case, modifications as above do not influence one another, and so the new label system Λ belongs to $\mathcal{L}_K^C$.

This finishes the proof of Proposition 5.2.

5.6. Proof of Proposition 5.3.

In the cases when $s \in \{1, 2', 2''\}$ K-equivalence up to reparametrization in L follows from the uniqueness part in Lemma 5.4.2 applied to restrictions $\lambda_w|_{L(w,K)}$ of label maps, which are order preserving as in the proof of Proposition 5.2. The cases when $s \in \{3, 4', 4'', 5\}$ are even easier. The group $Aut(L)$ is then so big that, by convexity of the boundary of K, any two label maps are K-equivalent up to reparametrization in L. This finishes the proof.

6. Local conditions determine (k, L)-complexes uniquely.

This section is devoted to the proof of the following.

6.1. Proposition. Let K be a (k, L)-n-ball as defined at the beginning of 5.1, and let L be an s-regular graph. Assume that either $s \notin \{4', 4''\}$ or $k \geq 4$. Let C be a condition of the corresponding form described in 5.1, or just the empty condition if $s \in \{3, 4', 4'', 5\}$ and $k \geq 4$. Then for any two label systems $\Lambda, \Lambda' \in \mathcal{L}_K^C$ there exists an isomorphism between the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ which extends the identity map id_K. Moreover, if $s \in \{1, 2', 2''\}$, or if $s = 3$, $k = 3$ and $n \geq 2$, then this isomorphism is unique, while there is more than one such isomorphism in the other cases.

Remark. In the case when $s \in \{3, 4', 4'', 5\}$ and $k \geq 4$ we consider the empty condition C (no restrictions for label systems), so that the space $\mathcal{L}_K^C$ consists of all good label systems for K. The Proposition says then that any two $(n+1)$-balls obtained from K are isomorphic by an isomorphism extending id_K.

Before proving Proposition 6.1 we introduce the notion of modifications of characteristic functions and give two preparatory lemmas.

6.2. Modifications of characteristic functions.

Consider the following *elementary modifications* of a function $\chi : E_\partial K \to \{0,1\}$:

(i) change the values of χ at both edges of ∂K adjacent to a 1-free vertex of K;

(ii) change the value of χ at any edge of ∂K adjacent to a 1-free vertex of K;

(iii) change the values of χ at all three edges of ∂K adjacent to a 2-free vertex of K;

(iv) change the values of χ at any two of the three edges of ∂K adjacent to a 2-free vertex of K;

(v) change the values of χ at all four edges of ∂K adjacent to a 3-free vertex of K;

(vi) change the values of χ at any two of the four edges of ∂K adjacent to a 3-free vertex w of K such that the vertices in $L(w,K)$ corresponding to those two edges are at polygonal distance 2 in $L(w,K)$.

A *modification* is a finite sequence of elementary modifications of the forms as below:

(1) when $s=1$, no modifications;

(2) when $s=2'$ or $s=2''$, form (i) only;

(3) when $s=3$, forms (ii) and (iii);

(4) when $s=4'$ or $s=4''$, forms (ii), (iv) and (v);

(5) when $s=5$, forms (ii), (iv) and (vi).

6.3. Lemma. Let $\chi' : E_\partial K \to \{0,1\}$ be a function obtained from a characteristic function $\chi = \chi_\Lambda$ of a label system Λ by a modification (suitable for the corresponding regularity type of L). Then there exists a K-equivalent to Λ label system Λ', obtained from Λ by reparametrizations in L, such that $\chi' = \chi_{\Lambda'}$.

Proof: The Lemma follows from Propositions 1-5 of [DM], where the pointwise stabilizers of subgraphs in L corresponding to the links of K at the boundary vertices are described (compare Proposition 4.1, which contains however not enough information).

6.4. Lemma. Under the notation of Proposition 6.1, characteristic functions of any two K-equivalent label systems from $\mathcal{L}_K^C$ differ at most by a modification (suitable for the corresponding case of regularity of L).

In the proof of Lemma 6.4 we shall use the following.

6.4.1. Sublemma. Let G be a connected finite graph and let E_G denote the set of its edges. Let $h_1, h_2 : E_G \to Z_2$ be two functions such that

$$\sum_{e\in E_G} h_1(e) = \sum_{e\in E_G} h_2(e) \in Z_2. \tag{6.4.2}$$

Then h_2 can be obtained from h_1 by a finite sequence of modifications consisting of changing the values of the function at some two edges adjacent to a common vertex in G.

Proof of Sublemma: Consider the set $\delta(h_1,h_2) := \{e \in E_G : h_1(e) \neq h_2(e)\}$ and note that, by equality (6.4.2), it consists of even number of edges. Assume that the cardinality $|\delta(h_1,h_2)| \neq 0$ (otherwise there is nothing to prove) and choose any two distinct edges $e_1, e_2 \in \delta(h_1,h_2)$. By connectivity of G, there is an arc $(v_0,\dots,v_n)$ in G such that $(v_0,v_1) = e_1$ and $(v_{n-1},v_n) = e_2$. Let m_i be a modification which changes values at edges (v_{i-1}, v_i) and (v_i, v_{i+1}). Denote by h_1' a function obtained from h_1 by the sequence $m_1, \ldots, m_{n-1}$ of modifications. Then clearly $|\delta(h_1', h_2)| = |\delta(h_1, h_2)| - 2$, and the Sublemma follows by induction.

Proof of Lemma 6.4: Let $\Lambda = \{\lambda_w : w \in V_\partial K\}$ and $\Lambda' = \{\lambda_w' : w \in V_\partial K\}$ be two K-equivalent label systems for K, both satisfying condition C.

The case $s = 1$. Let $e = (u, w)$ be a boundary edge of K. Assume that λ_w and λ_w' differ at the peripheral pair of edges in $L(w, \widetilde{K})$ corresponding to the peripheral pair of faces in $\widetilde{K}$ adjacent to e. Since $\varepsilon_{K(\Lambda)}(e) = \varepsilon_{K(\Lambda')}(e)$, it follows that λ_u and λ_u' differ at the peripheral pair of edges in $L(u, K)$ corresponding to the peripheral pair of faces as above. But this means that $\chi_\Lambda(e) = \chi_{\Lambda'}(e)$. By repeating this argument we get $\chi_\Lambda = \chi_{\Lambda'}$, which finishes the proof in this case.

The case $s \in \{2', 2''\}$. Denote by $V_\partial^{2,3}K$ the set of all 2-free and 3-free edges of ∂K. Let $\Omega_{\partial K}$ be the family of connected components of $|\partial K| \setminus V_\partial^{2,3}K$ (where $|\partial K|$ denotes the topological space underlying graph ∂K). Components $a \in \Omega_{\partial K}$ are clearly polygonal arcs without endpoints, and for each such arc a denote by E_a the set of its edges (consisting of $k - 2$ elements). Let f_a be the face of K containing a. Note that we can view the stars of f_a in $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ as obtained from f_a by a step of construction as in 2.3 (compare the proof of Lemma 2.5). Label systems for f_a used in the corresponding steps can be assumed to consist of the same label maps at vertices of f_a not in ∂K. Similarly, at vertices of f_a contained in ∂K, above label systems for f_a can be assumed to consist of (restrictions of) the corresponding label maps for K. Then, it follows from equalities $\xi_{\overline{K}(\Lambda)}(f_a) = \xi_{\overline{K}(\Lambda')}(f_a)$ and from K-equivalence of Λ with Λ' that for each $a \in \Omega_{\partial K}$

$$\sum_{e \in E_a} \chi_\Lambda(e) = \sum_{e \in E_a} \chi_{\Lambda'}(e) \in Z_2.$$

But then, by Sublemma 6.4.1, χ_Λ can be modified to $\chi_{\Lambda'}$ separately and independantly at disjoint sets E_a.

The argument in the case $s = 2''$ is the same.

The case $s = 3$ **and** $k = 3$. Recall that given a vertex $v \in \partial K^-$, $A(v)$ is a subgraph in $\partial st(v, K)$ consisting of all edges which are nonboundary in K. Moreover, $A(v)$ is a tree and contains not all edges of $\partial st(v, K)$. Donote by $\partial st(v, K)/A(v)$ the *quotient graph* obtained from $\partial st(v, K)$ by shrinking $A(v)$ to a vertex. (This quotient graph may contain both loops and more than one edge between two vertices.) By the fact that $A(v)$ is a tree, we get the following.

Claim. Let ϕ_1, ϕ_2 be Z_2-valued 1-cocycles in $\partial st(v, K)$, and let $\phi_1|_{A(v)} = \phi_2|_{A(v)}$. Denote by $\phi_i/A(v)$ the 1-cocycle in $\partial st(v, K)/A(v)$ obtained from ϕ_i by forgetting the values at edges of $A(v)$. Then $[\phi_1] = [\phi_2]$ iff $[\phi_1/A(v)] = [\phi_2/A(v)]$.

Proof of Claim: By contractibility of $A(v)$, the natural homomorphism $H^1\partial st(v, K) \to H^1\partial st(v, K)/A(v)$ is an isomorphism. Its inverse can be defined on 1-cocycles level by extension with values zero at the edges of $A(v)$. From this the Claim follows easily.

Recall that for all vertices $v \in \partial K^-$ we can view the stars of the 1-balls $st(v, K)$ in $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ respectively, as obtained by a step of construction as in 2.3. We can assume that label systems for $st(v, K)$ used in those steps consist of pairwise equal label maps at vertices of $\partial st(v, K)$ not in ∂K, and that their label maps at vertices contained in ∂K coincide with the (restrictions of) the label maps from Λ and Λ' respectively. By K-equivalence of Λ with Λ', characteristic functions of the label systems as above for $st(v, K)$ coincide at edges of $A(v)$. By equalities $\eta_{\overline{K}(\Lambda)}(v) = \eta_{\overline{K}(\Lambda')}(v)$, in view of above Claim, it follows that $[\chi_\Lambda/A(v)] = [\chi_{\Lambda'}/A(v)]$ in the cohomology group $H^1(\partial st(v, K)/A(v), Z_2)$, where $\chi_\Lambda/A(v)$ is a 1-cocycle in $\partial st(v, K)/A(v)$ obtained by restricting χ_Λ to the edges contained in $\partial st(v, K)$ only.

Observe that each vertex of $\partial st(v, K)$ which is not in $A(v)$ is a 2-free vertex of K, and that all edges of ∂K adjacent to it are contained in $\partial st(v, K)$. It follows that we can obtain $\chi_{\Lambda'}/A(v)$ from $\chi_\Lambda/A(v)$ by elementary modifications of form (iii) at such vertices. It is also clear that we can do this independantly for all 1-balls $st(v, K)$ with $v \in \partial K^-$. This gives a modification of Λ to Λ' and the Lemma follows in this case.

The case $s = 5$ and $k = 3$. For a vertex $v \in \partial K^-$ put $\partial_v K := \partial K \cap st(v, K)$. By reasoning as in the previous case we get

$$\sum_{e \subset \partial_v K} \chi_\Lambda(e) = \sum_{e \subset \partial_v K} \chi_{\Lambda'}(e) \in Z_2$$

for all $v \in \partial K^-$. We want to apply Sublemma 6.4.1 to the graph $\partial_v K$ and to the restricted characteristic functions $\chi_\Lambda|_{\partial_v K}$ and $\chi_{\Lambda'}|_{\partial_v K}$.

First, observe the following two properties which follow easily from the construction 2.3:

(i) each vertex of $\partial_v K$ not contained in $A(v)$ is 2-free in K and all edges of ∂K adjacent to it are contained in $\partial_v K$;

(ii) each vertex w of $\partial_v K \cap A(v)$ is 3-free in K and two edges of ∂K adjacent to w are contained in the same graph $\partial_v K$ iff the polygonal distance between the vertices representing them in the link $L(w, K)$ is 2.

By above properties, elementary modifications of form (iv) and (vi) correspond to modifications as in Sublemma in the corresponding graphs $\partial_v K$. We also have the following.

Claim. Each graph $\partial_v K$ is connected.

Proof of Claim: Consider the vertices of $\partial_v K \cap A(v)$ and note that, by connectivity of $\partial st(v, K)$, any vertex of $\partial_v K$ can be joined by an arc in $\partial_v K$ with one of them. Note also that we can split these vertices uniquely into two sets U and W so that any two vertices $u \in U$ and $w \in W$ can be joined by a 5-arc in $A(v)$. Clearly, it is enough to show that any two vertices u, w as above can be also joined by an arc in $\partial_v K$.

By 5-regularity of $\partial st(v, K)$, any 5-arc is contained in a circuit of length g, where g is the girth of $\partial st(v, K)$. Consider such a circuit c for the 5-arc a joining u and w in $A(v)$, and note that the complement of a in c joins u and w in $\partial_v K$. This finishes the proof of Claim.

In view of above properties (i) and (ii) and the Claim, applying Sublemma 6.4.1, we can modify χ_Λ to $\chi_{\Lambda'}$ separately and independantly on graphs $\partial_v K$.

This finishes the proof of Lemma 6.4.

6.5. Proof of Proposition 6.1.

Let Λ and Λ' be two label systems for K satisfying condition C. By Lemma 2.4.1, the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ do not depend, up to an isomorphism extending id_K, on reparametrizations in L for label maps in Λ and Λ'. Thus, due to Proposition 5.3, we can assume that Λ and Λ' are K-equivalent. Then, by Lemma 6.4, characteristic functions χ_Λ and $\chi_{\Lambda'}$ differ by a modification, and using Lemma 6.3 we again modify, say Λ, by reparametrizations in L, so that now Λ and Λ' are K-equivalent and have equal characteristic functions. This allows to apply Lemma 3.5, which finishes the proof of the existence part of Proposition 6.1.

To see the uniqueness in the cases when $s \in \{1, 2', 2''\}$, consider isomorphisms $T_1, T_2 : \overline{K}(\Lambda) \to \overline{K}(\Lambda')$, both extending the identity automorphism id_K. Note that in these cases no nontrivial automorphism of L fixes pointwise a star of vertex or a star of edge in L. This implies that if w is a 2-free or a 3-free vertex of K then $T_1|_{st(w,\overline{K}(\Lambda))} = T_2|_{st(w,\overline{K}(\Lambda))}$ and hence the isomorphisms T_1 and T_2 coincide on the subcomplex

$$M = K \cup \bigcup \{st(w, \overline{K}(\Lambda)) : w \text{ is 2-free or 3-free in } K\}.$$

But then, the links $L(v, M)$ at those 1-free vertices v of K, which have a 2-free or a 3-free neighbour, are big enough to imply the coincidence of T_1 and T_2 on the stars $st(v, \overline{K}(\Lambda))$. Since ∂K is connected, repeating this argument we get $T_1 = T_2$.

The similar argument as above applies to the case when $s = 3$, $k = 3$ and $n \geq 2$, with the following changes. In this case no nontrivial automorphism of L fixes pointwise a star of edge in L. Thus isomorphisms T_1 and T_2 coincide on stars of 3-free vertices of K. The following Claim allows then to extend this coincidence to the whole $\overline{K}(\Lambda)$, by the argument of the previous case.

Claim. If $k = 3$ and K is an n-ball with $n \geq 2$, then at least one vertex in K is 3-free, and no one is 1-free.

Proof of Claim: Constructing an n-ball from an $(n-1)$-ball, we glue 1-balls to it, according to the pattern provided by a label system. The boundary of this n-ball consists then of parts of boundaries of the glued 1-balls. If $k = 3$, no boundary vertex of the 1-ball is 1-free, and so this is also true for K. On the other hand, if $k = 3$ then all the vertices of $\widetilde{B}_{n-1} \setminus B_{n-1}$ become 3-free in B_n, and the Claim follows.

Now, we proceed to prove that there is more than one isomorphism between $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ extending id_K in the following three remaining cases:

(1) $s = 3$, $k = 3$ and K is a 1-ball;
(2) $s = 3$ and $k \geq 4$;
(3) $s \in \{4', 4'', 5\}$.

We start with the observation that in the case (1) only 2-free vertices appear at ∂K, and in the case (2) only 2-free and 1-free ones. In any of these two cases there exists an automorphism of L fixing pointwise an edge (respectively a star of vertex) in L, and transposing all pairs of its peripheral edges; in the case (3) the same is true also for a star of edge in L (see [DM], Propositions 1-5).

Using these facts we can construct in any of the cases (1)-(3) an isomorphism between $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ that differs from a given one at all pairs of peripheral faces in $\widetilde{K}$. This finishes the proof of Proposition 6.1.

7. Proofs of the main results.

Proof of Main Theorem.

In any case except when $s = 3$ and $k = 3$ a necessary condition for a (k, L)-complex X to be symmetric is that the appropriate local invariant (viewed as a Z_2-valued function on the set of vertices, edges, faces or oriented faces of X) is constant. In the remaining case when $s = 3$ and $k = 3$ a necessary condition is that the cohomology element $\eta_X(w)$ is invariant under some 1-arc-transitive subgroup Γ in the group $Aut(\partial st(w, X))$, but the methods of this paper are not sufficient to handle this case completely. Therefore we shall consider a stronger condition that $\eta_X(w)$ is $Aut(\partial st(w, X))$-invariant for each vertex $w \in X$, and that for any two vertices w and u of X the elements $\eta_X(w)$ and $\eta_X(u)$ coincide via some (and hence any) isomorphism of the (boundaries of) corresponding stars of vertices. In such a case we shall say that the invariant η is *constant* on X.

By (4.2.3.1) the invariant ξ'' cannot be constant on a (k, L)-complex X if L is $2''$-regular and k is odd. In all other cases (k, L)-complexes with constant local invariants exist due to Proposition 5.2. The uniqueness in each case (except $s \in \{4', 4''\}$ and $k = 3$) follows from Proposition 6.1, since this proposition allows to construct inductively an isomorphism between any two (k, L)-complexes with constant and equal local invariants. The symmetry follows by the same argument, and the properties of automorphism groups $Aut(X)$ follow then from the uniqueness (or nonuniqueness) part of Proposition 6.1.

Proof of Corollary.

Consider a local invariant of 4.2 suitable for the local data (k, L), and note that in any case it can take at least two distinct values. On the other hand, by Proposition 5.2, we have an absolute freedom in determining the values of the invariant during the inductive construction of a (k, L)-complex. Playing with this we easily get the conclusion of Corollary.

References

[BB] W. Ballmann, M. Brin, *Polygonal complexes and combinatorial group theory*, Geometriae Dedicata 50 (1994), 165–191.

[Be1] N. Benakli, *Polyedre a geometrie locale donne*, C. R. Acad. Sci. Paris 313, Serie I (1991), 561–564.

[Be2] N. Benakli, *Polyedre hyperbolique a groupe d'automorphismes non discret*, C. R. Acad. Sci. Paris 313, Serie I (1991), 667–669.

[Big] N. L. Biggs, *Constructing 5-arc-transitive cubic graphs*, J. London Math. Soc. 26 (1982), 193–200.

[Bou] I. Z. Bouwer (ed.), *The Foster cenzus*, Winnipeg, 1988.

[CL] M. Conder, P. Lorimer, *Automorphism groups of symmetric graphs of valency 3*, J. Comb. Th. (B) 47 (1989), 60–72.

[Cox] H. S. M. Coxeter, *Self-dual configurations and regular graphs*, Bull. Amer. Mat. Soc. 56 (1950), 413–455.

[DM] D. Djokovič, G. Miller, *Regular groups of automorphisms of cubic graphs*, J. Comb. Th. (B) **29** (1980), 195–230.

[Fru] R. Frucht, *A one-regular graph of degree 3*, Canad. J. Math. 4 (1952), 240–247.

[Gro] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. Gersten, ed.), Springer Verlag, 1987, 75–265.

[Hae] A. Haefliger, *Complexes of groups and orbihedra*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky, eds.), World Scientific, 1991, 504–540.

[Hag] F. Haglund, *Les polyedres de Gromov*, C. R. Acad. Sci. Paris 313, Serie I (1991), 603–606.

[Tit] J. Tits, *Spheres of radius 2 in triangle buildings*, Finite geometries, building and related topics (W. Kantor, R. Liebler, S. Payne, E. Shult, eds.), Clarendon Press, 1990, 17–28.

[Tut] W. T. Tutte, *A family of cubical graphs*, Proc. Cambridge Philos. Soc. 43 (1947), 459–474.

Jacek Świątkowski, Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland, swiatkow@math.uni.wroc.pl