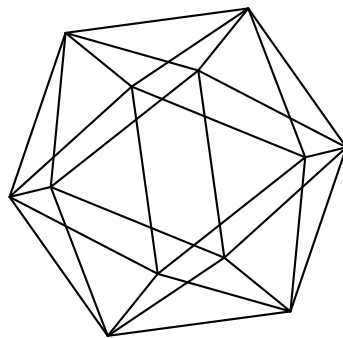# Max-Planck-Institut für Mathematik Bonn

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by

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GROMOV–WITTEN INVARIANTS OF THE RIEMANN SPHERE

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ABSTRACT. A conjectural formula for the k-point generating function of Gromov–Witten invariants of the Riemann sphere for all genera and all degrees was proposed in [11]. In this paper, we give a proof of this formula together with an explicit analytic (as opposed to formal) expression for the corresponding matrix resolvent. We also give a formula for the k-point function as a sum of $(k-1)!$ products of hypergeometric functions of one variable. We show that the k-point generating function coincides with the $\epsilon \to 0$ asymptotics of the analytic k-point function, and also compute three more asymptotics of the analytic function for $\epsilon \to \infty$, $q \to 0$, $q \to \infty$, thus defining new invariants for the Riemann sphere.

CONTENTS

1. STATEMENTS OF THE MAIN RESULTS

1.1. Gromov–Witten invariants of $\mathbb{P}^1$.

Let $\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta)$ be the moduli space of stable maps from algebraic curves of genus g with k distinct marked points to $\mathbb{P}^1$, of degree $\beta \in H_2(\mathbb{P}^1; \mathbb{Z})$

$$\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta) = \left\{ f : (\Sigma_g, p_1, \ldots, p_k) \to \mathbb{P}^1 \mid f_*([\Sigma_g]) = \beta \right\} / \sim .$$

Here, $(\Sigma_g, p_1, \ldots, p_k)$ denotes an algebraic curve of genus g with at most double-point singularities and with the distinct marked points $p_1, \ldots, p_k$, and the equivalence relation $\sim$ is defined by isomorphisms of $\Sigma_g \to \mathbb{P}^1$ identical on $\mathbb{P}^1$ and on the markings. Let $\mathcal{L}_i$ be the ith tautological line bundle on $\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta)$, and $\psi_i := c_1(\mathcal{L}_i)$, $i = 1, \ldots, k$. Denote by $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta) \to \mathbb{P}^1$ the ith evaluation map.

The genus g, degree β Gromov–Witten (GW) invariants of $\mathbb{P}^1$ are integrals of the form

$$\int_{\left[\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta)\right]^{\text{virt}}} \mathrm{ev}_1^*(\phi_{\alpha_1}) \cdots \mathrm{ev}_k^*(\phi_{\alpha_k}) \, \psi_1^{i_1} \cdots \psi_k^{i_k} =: \langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle_{g,d} . \tag{1}$$

Here, $\alpha_1, \ldots, \alpha_k \in \{1, 2\}$, $i_1, \ldots, i_k \geq 0$, $\phi_1 = 1$, $\phi_2 = \omega \in H^2(\mathbb{P}^1; \mathbb{C})$ normalized by $\int_{\mathbb{P}^1} \omega = 1$, and $\left[\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta)\right]^{\text{virt}}$ denotes the virtual fundamental class [24, 1, 2, 25]. In the right-hand-side of equation (1), the "degree" $\beta \in H_2(\mathbb{P}^1; \mathbb{Z})$ has been replaced by an integer d through $d := \int_\beta \omega$.

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The GW invariant $\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle_{g,d}$ vanishes unless the degree–dimension matching holds: $2g - 2 + 2d + 2k = \sum_{\ell=1}^{k} i_\ell + \sum_{\ell=1}^{k} \alpha_\ell$.

For $k \geq 1$ and $i_1, \ldots, i_k \geq 0$, $\alpha_1, \ldots, \alpha_k \in \{1, 2\}$, denote

$$\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle \;=\; \langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle(\epsilon, q) \;:=\; \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \epsilon^{2g-2} q^d \, \langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle_{g,d} \,.$$

We will call $\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle$ the *k-point $\mathbb{P}^1$ correlator*, and $\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle_{g,d}$ the *k-point $\mathbb{P}^1$ correlator of genus g and degree d*. Due to the degree–dimension matching, $\epsilon^2 \langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle(\epsilon, q)$ is a homogeneous polynomial of ϵ^2, q. More precisely,

$$\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle(\epsilon, q) \;=\; \sum_{\substack{g,d \geq 0 \\ 2g+2d-2=\sum_{\ell=1}^{k}(i_\ell + \alpha_\ell - 2)}} \epsilon^{2g-2} q^d \langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle_{g,d} \,.$$

Note that this expression vanishes if $\sum_{\ell=1}^{k}(i_\ell + \alpha_\ell)$ is odd.

Definition 1. The *free energy $\mathcal{F}$* is defined as the following generating series of $\mathbb{P}^1$ correlators

$$\mathcal{F} \;=\; \mathcal{F}(\mathbf{T}; \epsilon, q) \;:=\; \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{1 \leq \alpha_1, \ldots, \alpha_k \leq 2 \\ i_1, \ldots, i_k \geq 0}} T_{i_1}^{\alpha_1} \ldots T_{i_k}^{\alpha_k} \, \langle \tau_{i_1}(\alpha_1) \cdots \tau_{i_k}(\alpha_k) \rangle(\epsilon, q) \qquad (2)$$

where $\mathbf{T} = (T_j^{\alpha})_{\alpha=1,2,\, j \geq 0}$. The *partial k-point correlation functions* are the power series

$$\langle\!\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle\!\rangle(x; \epsilon, q) \;:=\; \left. \frac{\partial^k \mathcal{F}(\mathbf{T}; \epsilon, q)}{\partial T_{i_1}^{\alpha_1} \ldots \partial T_{i_k}^{\alpha_k}} \right|_{T_i^{\alpha} = \delta_{\alpha 1} \, \delta_{i0} \, x} \,.$$

Clearly, $\langle\!\langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle\!\rangle(0; \epsilon, q) = \langle \tau_{i_1}(\phi_{\alpha_1}) \cdots \tau_{i_k}(\phi_{\alpha_k}) \rangle(\epsilon, q)$. In this paper, we consider in particular the partial correlation-functions of the form $\langle\!\langle \tau_{i_1}(\omega) \cdots \tau_{i_k}(\omega) \rangle\!\rangle(x; \epsilon, q)$, and consider the following generating series [11], called the *k-point function*:

$$F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon, q) \;:=\; \epsilon^k \sum_{i_1, \ldots, i_k \geq 0} \frac{(i_1 + 1)! \cdots (i_k + 1)!}{\lambda_1^{i_1+2} \ldots \lambda_k^{i_k+2}} \langle\!\langle \tau_{i_1}(\omega) \cdots \tau_{i_k}(\omega) \rangle\!\rangle(x; \epsilon, q) \qquad (k \geq 1). \quad (3)$$

Here $\lambda_1, \lambda_2, \ldots$ are indeterminates. The dependence on q in $F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon; q)$ can be recovered from $F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon; 1)$ by rescaling:

$$F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon, q) \equiv q^{-k/2} F_k\big(q^{-1/2}\lambda_1, \ldots, q^{-1/2}\lambda_k;\, q^{-1}x;\, q^{-1/2}\epsilon; 1\big) \qquad (k \geq 1). \qquad (4)$$

In particular, $F_k(\lambda_1, \ldots, \lambda_k; 0; \epsilon, q) \equiv q^{-k/2} F_k\big(q^{-1/2}\lambda_1, \ldots, q^{-1/2}\lambda_k; 0;\, q^{-1/2}\epsilon; 1\big)$.

1.2. **The k-point function in terms of matrix resolvents.** The matrix resolvent (MR) approach of computing logarithmic derivatives of tau-functions of continuous integrable systems was introduced in [3, 4, 5]. It was further extended in [10] to discrete integrable systems. The *Toda conjecture* (now a theorem) [8, 19, 20, 29, 7] says that $e^{\mathcal{F}}$ is the tau-function of a particular solution (which will be called the GW solution) to the Toda Lattice Hierarchy. So we can apply the MR approach [10] to the computation of the $\mathbb{P}^1$ correlators.

Definition 2 ([10, 11]). Let $U_n(\lambda; \epsilon) = \begin{pmatrix} \epsilon\, n + \frac{\epsilon}{2} - \lambda & 1 \\ -1 & 0 \end{pmatrix}$. Define the *matrix resolvent* $R_n(\lambda; \epsilon)$ for the GW solution of the Toda Lattice Hierarchy as the unique formal solution to the following problem

$$R_{n+1}(\lambda; \epsilon)\, U_n(\lambda; \epsilon) - U_n(\lambda; \epsilon)\, R_n(\lambda; \epsilon) = 0, \tag{5}$$

$$\operatorname{tr} R_n(\lambda; \epsilon) = 1, \qquad \det R_n(\lambda; \epsilon) = 0, \tag{6}$$

$$R_n(\lambda; \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{O}(\lambda^{-1}), \qquad \lambda \to \infty. \tag{7}$$

This solution $R_n(\lambda; \epsilon)$ belongs to $\mathrm{Mat}_2\big(\mathbb{Z}[n, \epsilon]\big[[\lambda^{-1}]\big]\big)$. Define $\mathcal{R}(\lambda; x; \epsilon) := R_{x/\epsilon}(\lambda; \epsilon)$.

Theorem 1. *The formal series (3) with $k \geq 2$ have the expressions*

$$F_2(\lambda_1, \lambda_2; x; \epsilon, 1) = \frac{\operatorname{tr}\left[\mathcal{R}(\lambda_1; x; \epsilon)\,\mathcal{R}(\lambda_2; x; \epsilon)\right] - 1}{(\lambda_1 - \lambda_2)^2}, \tag{8}$$

$$F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon, 1) = -\sum_{\sigma \in S_k/C_k} \frac{\operatorname{tr}\left[\mathcal{R}(\lambda_{\sigma(1)}; x; \epsilon)\ldots\mathcal{R}(\lambda_{\sigma(k)}; x; \epsilon)\right]}{\prod_{i=1}^{k}(\lambda_{\sigma(i)} - \lambda_{\sigma(i+1)})}, \qquad k \geq 3. \tag{9}$$

Here S_k and C_k are the symmetric group and standard cyclic subgroup, with $\sigma(k+1) = \sigma(1)$ for $\sigma \in S_k$.

The proof, based on the Toda conjecture, uses a simple observation [11] and the MR approach [10]. The idea of the proof has been explained in [11]; we provide the details in Section 2.4 of the current paper.

The following property, proved in Section 3, is related to the concept of *bispectrality* (see e.g. [16]).

Theorem 2. *The matrix-valued formal series $\mathcal{R}(\lambda; x; \epsilon)$ depends only on ϵ and $\lambda - x$.*

In other words, $\mathcal{R}(\lambda; x; \epsilon)$ has the form

$$\mathcal{R}(\lambda; x; \epsilon) = M\Big(\frac{\lambda - x}{\epsilon}; \frac{1}{\epsilon}\Big) \tag{10}$$

for some $M(z; s)$, which is a formal power series in z^{-1}. On the other hand, from its definition, $\mathcal{R}(\lambda; x; \epsilon)$ satisfies

$$\mathcal{R}(\lambda; x + \epsilon; \epsilon) \begin{pmatrix} x + \frac{\epsilon}{2} - \lambda & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} x + \frac{\epsilon}{2} - \lambda & 1 \\ -1 & 0 \end{pmatrix} \mathcal{R}(\lambda; x; \epsilon) = 0$$

which in terms of $M(z; s)$ becomes

$$M(z - 1; s) \begin{pmatrix} z - \frac{1}{2} & -s \\ s & 0 \end{pmatrix} = \begin{pmatrix} z - \frac{1}{2} & -s \\ s & 0 \end{pmatrix} M(z; s). \tag{11}$$

Similarly, from equations (6) and (7) we deduce that $M(z; s)$ also satisfies

$$\operatorname{tr} M(z; s) = 1, \qquad M(\infty; s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12}$$

We call (11) the *topological difference equation*, which is an analogue of the topological ODE [4, 5].

Proposition 1. *There exists a unique element M^* in $M_2\big(\mathbb{C}(s)\big[[z^{-1}]\big]\big)$ satisfying equations (11)–(12). Moreover, M^* belongs to $M_2\big(\mathbb{Q}[s]\big[[z^{-1}]\big]\big)$, and it satisfies $\det M^* = 0$.*

See Section 3 for the proof. Proposition 1 will be used to prove Theorem 2, with $M = M^*$ (see equation (10) above).

The following theorem, which was conjectured in [11], gives an explicit formula for the matrix $M(z; s)$. We will prove it in Section 2.4. (A different proof was given recently by O. Marchal [27].)

Theorem 3. *The matrix-valued power series $M = M(z; s)$ has the following explicit expression*

$$M = \begin{pmatrix} 1 + \alpha & Q - P \\ Q + P & -\alpha \end{pmatrix}, \tag{13}$$

where $\alpha = \alpha(z; s)$, $P = P(z; s)$, $Q = Q(z; s) \in \mathbb{Q}[s][[z^{-1}]]$ are given by

$$\alpha(z; s) = 2 \sum_{j=0}^{\infty} \frac{1}{z^{2j+2}} \sum_{i=0}^{j} s^{2i+2} \frac{1}{i!(i+1)!} \sum_{\ell=0}^{i} (-1)^{\ell} \left(i - \ell + \tfrac{1}{2}\right)^{2j+1} \binom{2i+1}{\ell}, \tag{14}$$

$$P(z; s) = \sum_{j=0}^{\infty} \frac{1}{z^{2j+1}} \sum_{i=0}^{j} s^{2i+1} \frac{1}{i!^2} \sum_{\ell=0}^{i} (-1)^{\ell} \left(i - \ell + \tfrac{1}{2}\right)^{2j} \left[\binom{2i}{\ell} - \binom{2i}{\ell-1}\right], \tag{15}$$

$$Q(z; s) = -\frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{z^{2j+2}} \sum_{i=0}^{j} s^{2i+1} \frac{2i+1}{i!^2} \sum_{\ell=0}^{i} (-1)^{\ell} \left(i - \ell + \tfrac{1}{2}\right)^{2j} \left[\binom{2i}{\ell} - \binom{2i}{\ell-1}\right]. \tag{16}$$

1.3. **Explicit formulas in terms of hypergeometric functions and Bessel functions.** Define a meromorphic matrix-valued function $B = B(z; s)$ by

$$B(z; s) = \frac{1}{2} \begin{pmatrix} 1 + G(z, s) & \frac{4s}{1-2z} \widetilde{G}(z-1, s) \\ \frac{4s}{1+2z} \widetilde{G}(z, s) & 1 - G(z, s) \end{pmatrix}, \quad z \in \mathbb{C} - \mathbb{Z}_{\text{odd}}, \ s \in \mathbb{C} \tag{17}$$

where $G(z; s)$ and $\widetilde{G}(z; s)$ are the (generalized) hypergeometric functions

$$G(z; s) = {}_1F_2\left(\frac{1}{2}; \frac{1}{2} - z, \frac{1}{2} + z; -4s^2\right) = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{s^{2m}}{(z - m + \frac{1}{2})_{2m}}, \tag{18}$$

$$\widetilde{G}(z; s) = {}_1F_2\left(\frac{1}{2}; \frac{1}{2} - z, \frac{3}{2} + z; -4s^2\right) = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{z + \frac{1}{2}}{(z - m + \frac{1}{2})_{2m+1}} s^{2m}. \tag{19}$$

Here, $( )_k$ is the increasing Pochhammer symbol, i.e. $(x)_k := \Gamma(x+k)/\Gamma(x) = x(x+1)\cdots(x+k-1)$. Note that the series in (18), (19) converge absolutely and locally uniformly away from $z \in \mathbb{Z} + \frac{1}{2}$ so that the product $\cos(\pi z) B(z; s)$ extends to a (matrix-valued) holomorphic function on all of $\mathbb{C}^2$.

Theorem 4. *For fixed $s \in \mathbb{C}$, the asymptotic expansion of $B(z; s)$ in all orders as $z \to \infty$ at a bounded distance from $\mathbb{Z} + \frac{1}{2}$ coincides with the formal power series $M(z; s)$, i.e. $B(z; s) \sim M(z; s)$.*

The proof will be given in Section 3. Here and throughout this paper, we use $\sim$ for a full asymptotic expansion: e.g. "$f(\epsilon) \sim g(\epsilon)$ as $\epsilon \to 0$" means that the asymptotic expansions of f and g agree as power series in ϵ to all orders.

We now define *analytic k-point functions* $H_k(z_1, \ldots, z_k; s)$ ($k \geq 2$; the case $k = 1$ will be treated later) by

$$H_2(z_1, z_2; s) := \frac{\text{tr}\left[B(z_1; s) B(z_2; s)\right] - 1}{(z_1 - z_2)^2} = -\frac{1}{2} \text{tr} \left[\frac{B(z_1; s) - B(z_2; s)}{z_1 - z_2}\right]^2, \tag{20}$$

$$H_k(z_1, \ldots, z_k; s) := -\sum_{\sigma \in S_k / C_k} \frac{\text{tr}\left[B(z_{\sigma(1)}; s) \cdots B(z_{\sigma(k)}; s)\right]}{\prod_{i=1}^{k} (z_{\sigma_i} - z_{\sigma_{i+1}})}. \tag{21}$$

Then equation (89) and Theorem 4 imply that the full asymptotic expansion of $H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{1}{\epsilon}\right)$ coincides with $F_k(\lambda_1, \ldots, \lambda_k; \epsilon)$ as $\lambda_i \to \infty$ at a bounded distance away from $\epsilon \mathbb{Z} + \frac{\epsilon}{2}$, $i = 1, \ldots, k$. Notice that this statement is not entirely trivial, since the presence of poles in (20) and (21) when two

z_i's coincide means that a priori we must order the $|\lambda_i|$'s in order to obtain an asymptotic expansion. However, the following proposition (which will be proved in Section 3) implies that the asymptotics are the same for all orderings of the $|\lambda_i|$'s.

Proposition 2. *The functions $H_k(z_1, \ldots, z_k; s)$, $k \geq 2$ are analytic along the diagonals $z_i = z_j$, $i \neq j$ (as above it is assumed that none of z_1, $\ldots$, z_k is a half-integer).*

The next point is very nice. Normally, even if one knows a closed formula for a collection of 2×2 matrices, it is not easy to compute the trace of their product. Here, however, there is a nice simplification which lets us write the above traces as products. The reason is that, since $\det B(z; s) = 0$ by Proposition 1 and Theorem 4, the matrix $B(z; s)$ must factor as the product of a column vector and a row vector. This factorization, given explicitly in the following proposition, will immediately lead us to a factorization formula for the traces.

Proposition 3. *The matrix valued function B has the following expressions*

$$B(z; s) \;=\; u(z)\, u(-z)^T \;=\; \frac{\pi s}{\cos(\pi z)} V(z)\, V(-z)^T, \tag{22}$$

where

$$u(z) = u(z; s) = \begin{pmatrix} j_z(s^2) \\ \frac{s}{z+\frac{1}{2}} j_{z+1}(s^2) \end{pmatrix}, \qquad V(z) = V(z; s) \;=\; \begin{pmatrix} J_{z-\frac{1}{2}}(2s) \\ J_{z+\frac{1}{2}}(2s) \end{pmatrix} \;=\; \frac{s^{z-\frac{1}{2}}}{\Gamma(z+\frac{1}{2})}\, u(z; s).$$

Here $J_\nu(y)$ denotes the standard Bessel function [33] and $j_a(X)$ a modified Bessel function:

$$j_a(X) := \sum_{n \geq 0} \frac{(-X)^n}{n!^2 \binom{n+a-1/2}{n}}, \qquad J_\nu(y) := \frac{(y/2)^\nu}{\Gamma(\nu+1)} j_{\nu+\frac{1}{2}}(y^2/4). \tag{23}$$

Now define two analytic functions $D(a, b; s)$ and $D^*(a, b; s)$ by

$$\begin{aligned}
D(a, b; s) &\;=\; \frac{u(-a, s)^T u(b, s)}{a - b} \;=\; \frac{j_{-a}(X)\, j_b(X) \;+\; \frac{X}{(\frac{1}{2}-a)(\frac{1}{2}+b)} j_{1-a}(X)\, j_{1+b}(X)}{a - b}, \\[4pt]
D^*(a, b; s) &\;=\; \frac{V(-a, s)^T V(b, s)}{a - b} \;=\; \frac{J_{-a-\frac{1}{2}}(2s)\, J_{b-\frac{1}{2}}(2s) \;+\; J_{\frac{1}{2}-a}(2s)\, J_{\frac{1}{2}+b}(2s)}{a - b} \\[4pt]
&\;=\; \frac{s^{b-a-1}}{\Gamma(\frac{1}{2} - a)\, \Gamma(\frac{1}{2} + b)}\, D(a, b; s),
\end{aligned} \tag{24}$$

where $X = s^2$. Then from Proposition 3 and the fact that $\operatorname{tr}(A_1 B_1 \cdots A_k B_k) = \operatorname{tr}(B_1 A_2 \cdots B_k A_1)$, we find that the trace in (89) factorizes as a product of the one-variable functions D or D^*, and we obtain:

Theorem 5. *The analytic functions H_k, $k \geq 2$ have the expressions*

$$H_k(z_1, \ldots, z_k; s) \;=\; -\sum_{\sigma \in S_k/C_k} \prod_{i=1}^{k} D\big(z_{\sigma(i)}, z_{\sigma(i+1)}; s\big) \;-\; \frac{\delta_{k,2}}{(z_1 - z_2)^2} \tag{25}$$

or alternatively

$$H_k(z_1, \ldots, z_k; s) \;=\; -\frac{\pi^k s^k}{\prod_{i=1}^{k} \cos(\pi z_i)} \sum_{\sigma \in S_k/C_k} \prod_{i=1}^{k} D^*\big(z_{\sigma(i)}, z_{\sigma(i+1)}; s\big) \;-\; \frac{\delta_{k,2}}{(z_1 - z_2)^2}. \tag{26}$$

with $D(a, b; s)$ and $D^*(a, b; s)$ as in (24).

Example 1. *The function H_2 has the expression:*

$$H_2(z_1, z_2; s) \;=\; -D\big(z_1, z_2; s\big) D\big(z_2, z_1; s\big) \;-\; \frac{1}{(z_1 - z_2)^2}. \tag{27}$$

Next, we note that, although the original definition (24) would give a complicated formula for $D(a, b; s)$ as a double infinite sum, in fact it simplifies to a single infinite sum (hypergeometric series):

Proposition 4. *The function $D(a, b; s)$ has the following explicit expression*

$$D(a, b; s) = \sum_{n=0}^{\infty} \frac{(a - b - 2n + 1)_{n-1}}{n! \, (-a + \frac{1}{2})_n \, (b + \frac{1}{2})_n} s^{2n}, \tag{28}$$

where in the first term $(a - b + 1)_{-1} := 1/(a - b)$. Equivalently,

$$D(a, b; s) = \frac{1}{a - b} \, {}_2F_3\Big(\frac{b - a}{2}, \frac{b - a + 1}{2}; \frac{1}{2} - a, \frac{1}{2} + b, b - a + 1; -4s^2\Big).$$

Proof. This follows from a product formula for Bessel functions given on p. 147 of [33]. □

1.4. **One-point functions.** In the above we looked at k-point functions with $k \geq 2$. We now consider the case $k = 1$. Define two meromorphic functions $H_1(z, s)$ and $H_1^*(z, s)$ as modified limiting functions of $D(a, b; s)$ and $D^*(a, b; s)$, namely

$$H_1(z, s) = -\lim_{b \to z}\Big(D(z, b; s) - \tfrac{1}{z-b}\Big), \tag{29}$$

$$H_1^*(z, s) = -\frac{\pi s}{\cos(\pi z)} \lim_{b \to z}\Big(D^*(z, b; s) - \tfrac{\cos(\pi z)}{\pi s \, (z-b)}\Big). \tag{30}$$

From the definition it follows immediately that the functions H_1 and H_1^* are related by

$$H_1^*(z; s) = H_1(z; s) + \log s - \psi\Big(\frac{1}{2} + z\Big), \tag{31}$$

where ψ denotes the digamma function. Using equations (24) and Proposition 4 along with l'Hospital's rule we get the following explicit expressions:

$$H_1(z; s) = \sum_{n \geq 1} \frac{(2n - 1)! \, s^{2n}}{n!^2 (z - n + \frac{1}{2})_{2n}}, \tag{32}$$

$$H_1^*(z; s) = \frac{\pi s}{\cos(\pi z)} \left(J_{-\frac{1}{2}-z}(2s) \frac{\partial J_{-\frac{1}{2}+z}(2s)}{\partial z} + J_{\frac{1}{2}-z}(2s) \frac{\partial J_{\frac{1}{2}+z}(2s)}{\partial z} \right). \tag{33}$$

From (32) one observes that $s \frac{\partial H_1(z;s)}{\partial s} = G(z; s) - 1$.

Theorem 6. *The formal series* (3) *with $k = 1$ has the expression*

$$F_1(\lambda; x; \epsilon, 1) = \frac{1}{\epsilon}\Big(H_1^*\Big(\frac{\lambda - x}{\epsilon}; \frac{1}{\epsilon}\Big) + \log \lambda - \frac{x}{\lambda}\Big), \tag{34}$$

where the right-hand-side is understood as its asymptotic expansion as $\lambda \to \infty$.

Alternatively, using (32) and (31), we can write (34) explicitly as

$$F_1(\lambda; x; \epsilon, 1) = \frac{1}{\epsilon} \sum_{j \geq 2} \frac{x^j}{j \, \lambda^j} + \sum_{j \geq 2} \frac{\epsilon^j}{j \, (\lambda - x)^j} \sum_{i=0}^{\infty} \frac{\epsilon^{-1-2i}}{i!^2} \sum_{\ell=0}^{2i} (-1)^\ell \binom{2i}{l} B_j\big(i - \ell + \tfrac{1}{2}\big), \tag{35}$$

which can also be written as a pure power series in λ^{-1} as

$$F_1(\lambda; x; \epsilon, 1) = \sum_{j \geq 2} \frac{\epsilon^j}{j \, \lambda^j} \sum_{i=0}^{\infty} \frac{\epsilon^{-1-2i}}{i!^2} \sum_{\ell=0}^{2i} (-1)^\ell \binom{2i}{\ell} B_j\big(\tfrac{x}{\epsilon} + i - \ell + \tfrac{1}{2}\big). \tag{36}$$

Here, $B_j(u)$ denotes the jth Bernoulli polynomial (the unique polynomial solution to $\int_v^{v+1} B_j(u) du = v^j$).

Note that the internal sum in (35) or (36) is simply the $(2i)^{\text{th}}$ backward-difference of the polynomial $B_j\left(i + \frac{1}{2}\right)$ or $B_j\left(\frac{x}{\epsilon} + i + \frac{1}{2}\right)$, respectively, and since the nth difference of a polynomial of degree $< n$ vanishes, we can replace the sum $\sum_{i=0}^{\infty}$ by $\sum_{i=0}^{[j/2]}$ in both equations. Also, since $\Delta_{-1}^{2i}\left(B_j\left(\frac{x}{\epsilon} + i + \frac{1}{2}\right)\right) = \Delta_{-1}^{2i-1}\left(\Delta_{-1}\left(B_j\left(\frac{x}{\epsilon} + i + \frac{1}{2}\right)\right)\right) = j\,\Delta_{-1}^{2i-1}\left(\left(\frac{x}{\epsilon} + i - \frac{1}{2}\right)^{j-1}\right)$, we have the following more elementary expressions

$$F_1(\lambda; x; \epsilon, 1) = \sum_{g \geq 1} \frac{\epsilon^{2g-1}}{(\lambda - x)^{2g}} \frac{\left(1 - 2^{2g-1}\right) B_{2g}}{2^{2g}\,g} + \frac{1}{\epsilon} \sum_{j \geq 2} \frac{x^j}{j\,\lambda^j}$$
$$+ \sum_{j \geq 2} \frac{\epsilon^j}{(\lambda - x)^j} \sum_{i=1}^{[j/2]} \frac{\epsilon^{-1-2i}}{i!^2} \sum_{\ell=0}^{2i-1} (-1)^\ell \binom{2i-1}{\ell} \left(i - \ell - \tfrac{1}{2}\right)^{j-1},$$

$$F_1(\lambda; x; \epsilon, 1) = \sum_{j \geq 2} \frac{\epsilon^{j-1}}{j\,\lambda^j} B_j\left(\tfrac{x}{\epsilon} + \tfrac{1}{2}\right) + \sum_{j \geq 2} \frac{\epsilon^j}{\lambda^j} \sum_{i=1}^{[j/2]} \frac{\epsilon^{-1-2i}}{i!^2} \sum_{\ell=0}^{2i-1} (-1)^\ell \binom{2i-1}{\ell} \left(\tfrac{x}{\epsilon} + i - \ell - \tfrac{1}{2}\right)^{j-1},$$

where the first sum in each case corresponds to the digamma term in (31), and $B_j := B_j(0)$ is the j^{jh} Bernoulli number.

1.5. **Four asymptotics.** We already know that $F_k(\lambda_1, \ldots, \lambda_k; 0; \epsilon, 1)$ contains all GW invariants of $\mathbb{P}^1$ in the stationary sector for all genera and all degrees. (The dependence on q can be recovered by rescalings). This suggests the possibility of studying the $\epsilon \to \infty$ or $q \to \infty$ limit using the analytic k-point functions. Namely, we study the functions $H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right)$, $k \geq 1$ and their four asymptotic behaviours: $\epsilon \to 0$, $\epsilon \to \infty$, $q \to 0$, $q \to \infty$.

For any fixed $k \geq 1$, introduce the grading operator $\text{gr} := \epsilon \frac{\partial}{\partial \epsilon} + 2q \frac{\partial}{\partial q} + \sum_{i=1}^{k} \lambda_i \frac{\partial}{\partial \lambda_i}$. Obviously, $\text{gr}\, H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) = 0$. We begin with the $\epsilon \to 0$ limit. The condition $\lambda_i / 2\sqrt{q} > 1$ in the following theorem may look strange at first. It comes from the fact that the asymptotics of $J_\nu(\nu x)$ as $\nu \to \infty$ with $x > 0$ fixed are different according as $x < 1$, $x = 1$, or $x > 1$. (See [33], p. 225.)

Theorem 7.A. *Fix $k \geq 1$. For $q, \epsilon, \lambda_1, \ldots, \lambda_k$ satisfying $0 < \frac{2\sqrt{q}}{\lambda_i} < 1$, $\frac{\lambda_i}{\epsilon} > 0$, $i = 1, \ldots k$, as $\epsilon \to 0$ (with fixed $\lambda_1, \ldots, \lambda_k, q$), we have expansions of the form*

$$H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) \sim \sum_{g \geq 0} \epsilon^{2g-2+2k} H_k^{[g]}(\lambda_1, \ldots, \lambda_k; q) \qquad (k \geq 2), \tag{37}$$

$$H_1^*\left(\frac{\lambda}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) \sim \log q^{1/2} - \log \lambda + \sum_{g \geq 0} \epsilon^{2g} H_1^{*,[g]}(\lambda; q), \tag{38}$$

where $H_k^{[g]}$ for $k \geq 2$ are rational functions of $q, \lambda_1, \ldots, \lambda_k$ satisfying $\text{gr}\, H_k^{[g]} = (2 - 2g - 2k) H_k^{[g]}$; while for $k = 1$ we have $H_1^{*,[0]} = \log \frac{2\lambda}{\lambda + (\lambda^2 - 4q)^{\frac{1}{2}}}$ and $H_1^{*,[g]}$ $(g \geq 1)$ has the form

$$\lambda^{2g} H_1^{*,[g]}(\lambda; q) = \frac{P_g\left(\frac{q}{\lambda^2}\right)}{\left(1 - \frac{4q}{\lambda^2}\right)^{\frac{6g-1}{2}}}, \qquad P_g(0) = 2 \frac{(2g-1)!\,(1 - 2^{2g-1})\,B_{2g}}{4^g\,(2g)!} \tag{39}$$

with $P_g(x)$ being a polynomial with rational coefficients of degree $2g - 1$. Moreover, the sum

$$\sum_{g \geq 0} \epsilon^{2g-2+k} H_k^{[g]}(\lambda_1, \ldots, \lambda_k; q) \ (k \geq 2) \quad or \quad \sum_{g \geq 0} \epsilon^{2g-1} H_1^{*,[g]}(\lambda; q) \ (k = 1)$$

coincides with $F_k(\lambda_1, \ldots, \lambda_k; 0; \epsilon, q)$ as a formal power series in $\lambda_1^{-1}, \ldots, \lambda_k^{-1}, q$.

The proof will be given in Section 5. Theorem 7.A tells that the $\epsilon \to 0$ limit gives GW invariants of $\mathbb{P}^1$. We remark that even the simple consequence of (37) that $H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) = O(\epsilon^{2k-2})$ as $\epsilon \to 0$ already seems to be quite non-trivial. The first few $H_k^{[g]}$ (or $H_1^{*,[g]}$) are given by

$$H_1^{*,[1]} = -\frac{\lambda\,(\lambda^2 - 16q)}{24\,(\lambda^2 - 4q)^{\frac{5}{2}}}, \qquad H_1^{*,[2]} = \frac{\lambda\,(7\lambda^6 - 94q\lambda^4 + 8256q^2\lambda^2 + 18432q^3)}{960\,(\lambda^2 - 4q)^{\frac{11}{2}}},$$

$$H_2^{[0]} = \frac{\lambda_1\lambda_2 - \sqrt{\lambda_1^2 - 4q}\sqrt{\lambda_2^2 - 4q} - 4q}{2(\lambda_1 - \lambda_2)^2\sqrt{\lambda_1^2 - 4q}\sqrt{\lambda_2^2 - 4q}},$$

$$H_2^{[1]} = \frac{q}{4\left(\lambda_1^2 - 4q\right)^{\frac{7}{2}}\left(\lambda_2^2 - 4q\right)^{\frac{7}{2}}}\Big(\lambda_1^3\lambda_2^3(\lambda_1^2 + \lambda_2^2) + 4q\lambda_1\lambda_2\big(4\lambda_1^4 + 5\lambda_1^3\lambda_2 - \lambda_1^2\lambda_2^2 + 5\lambda_1\lambda_2^3 + 4\lambda_2^4\big)$$
$$- 16q^2\lambda_1\lambda_2\big(10\lambda_1^2 + 17\lambda_1\lambda_2 + 10\lambda_2^2\big) + 64q^3\big(2\lambda_1^2 + 11\lambda_1\lambda_2 + 2\lambda_2^2\big) + 768q^4\Big),$$

$$H_3^{[0]} = q\frac{\lambda_1\lambda_2\lambda_3 + 4q(\lambda_1 + \lambda_2 + \lambda_3)}{(\lambda_1^2 - 4q)^{\frac{3}{2}}(\lambda_2^2 - 4q)^{\frac{3}{2}}(\lambda_3^2 - 4q)^{\frac{3}{2}}}.$$

The next is to look at the $\epsilon \to \infty$ limit of $H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right)$.

Theorem 7.B. *Fix $k \geq 1$. $\forall\, \lambda_1, \ldots, \lambda_k, q \in \mathbb{C}$, the following asymptotic holds true: as $\epsilon \to \infty$,*

$$H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) \sim \sum_{g \geq 0} \epsilon^{-2g} H_{k,[g]}(\lambda_1, \ldots, \lambda_k; q), \tag{40}$$

where $H_{k,[g]} \in \mathbb{Q}[\lambda_1, \ldots, \lambda_k, q]$, and $\operatorname{gr} H_{k,[g]} = 2g\, H_{k,[g]}$.

The sum in the RHS of (40) converges if $|\epsilon| > 2\max\{|\lambda_1|, \ldots, |\lambda_k|\}$.

Thirdly we look at the $q \to 0$ limit.

Theorem 7.C. *Fix $k \geq 1$. For $\lambda_i \notin \epsilon\mathbb{Z} + \frac{\epsilon}{2}$, $i = 1, \ldots, k$, as $q \to 0$,*

$$H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) \sim \sum_{d \geq 0} q^d\, H_{k,d}(\lambda_1, \ldots, \lambda_k; \epsilon), \tag{41}$$

where $H_{k,d}(\lambda_1, \ldots, \lambda_k; \epsilon)$ are rational functions of $\lambda_1, \ldots, \lambda_k, \epsilon$ with poles only at $\lambda_i = m\,\epsilon/2$ with $|m| < 2d$ odd, and $\operatorname{gr} H_{k,d} = -2d\, H_{k,d}$. The $q \to 0$ asymptotic of H_1 has the explicit expression

$$H_1\left(\frac{\lambda}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right) \sim \sum_{d=1}^{\infty} q^d \frac{(2d-1)!}{d!^2\,\prod_{j=1}^d\left(\lambda^2 - \frac{(2j-1)^2}{4}\epsilon^2\right)}. \tag{42}$$

The proof is in Section 5. The first few rational functions $H_{k,d}(\lambda_1, \ldots, \lambda_k, q)$ are listed here:

$$H_{2,1} = \epsilon^2\frac{1}{(\lambda_1^2 - \frac{\epsilon^2}{4})(\lambda_2^2 - \frac{\epsilon^2}{4})}, \qquad H_{2,2} = \epsilon^2\frac{3\lambda_1^2 + 2\lambda_1\lambda_2 + 3\lambda_2^2 - 9\epsilon^2}{(\lambda_1^2 - \frac{\epsilon^2}{4})(\lambda_2^2 - \frac{\epsilon^2}{4})(\lambda_1^2 - \frac{9\epsilon^2}{4})(\lambda_2^2 - \frac{9\epsilon^2}{4})},$$

$$H_{2,3} = \epsilon^2\frac{10\lambda_1^4 + 8\lambda_1^3\lambda_2 + 12\lambda_1^2\lambda_2^2 + 8\lambda_1\lambda_2^3 + 10\lambda_2^4 - \epsilon^2(110\lambda_1^2 + 68\lambda_1\lambda_2 + 110\lambda_2^2) + 325\epsilon^4}{(\lambda_1^2 - \frac{\epsilon^2}{4})(\lambda_2^2 - \frac{\epsilon^2}{4})(\lambda_1^2 - \frac{9\epsilon^2}{4})(\lambda_2^2 - \frac{9\epsilon^2}{4})(\lambda_1^2 - \frac{25\epsilon^2}{4})(\lambda_2^2 - \frac{25\epsilon^2}{4})},$$

$$H_{3,1} = \epsilon^4\frac{1}{(\lambda_1^2 - \frac{\epsilon^2}{4})(\lambda_2^2 - \frac{\epsilon^2}{4})(\lambda_3^2 - \frac{\epsilon^2}{4})},$$

$$H_{3,2} = \epsilon^4\frac{48(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) + 32\lambda_1\lambda_2\lambda_3(\lambda_1 + \lambda_2 + \lambda_3) - 24\epsilon^2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 6(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)) + 351\epsilon^4}{(\lambda_1^2 - \frac{\epsilon^2}{4})(\lambda_2^2 - \frac{\epsilon^2}{4})(\lambda_3^2 - \frac{\epsilon^2}{4})(\lambda_1^2 - \frac{9\epsilon^2}{4})(\lambda_2^2 - \frac{9\epsilon^2}{4})(\lambda_3^2 - \frac{9\epsilon^2}{4})}.$$

Finally we look at the $q \to \infty$ limit.

Theorem 7.D. *Fix $k \geq 2$. As $q \to \infty$, with fixed ϵ, $\lambda_1, \ldots, \lambda_k$ and $\left| \arg\left(q^{1/2}/\epsilon\right) \right| < \pi$,*

$$\left(\prod_{i=1}^{k} \cos\frac{\pi\lambda_i}{\epsilon} \right) H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon} \right) \sim \sum_{d \geq 0} q^{-\frac{d}{2}} H_k^{d,0}(\lambda_1, \ldots, \lambda_k, \epsilon)$$

$$+ \sum_{d \geq 1} q^{-\frac{d}{2}} \sum_{m=1}^{\min\{d,k\}} \left[H_k^{d,m}(\lambda_1, \ldots, \lambda_k, \epsilon) \cos\left(4m\frac{q^{1/2}}{\epsilon} \right) + \widetilde{H}_k^{d,m}(\lambda_1, \ldots, \lambda_k, \epsilon) \sin\left(4m\frac{q^{1/2}}{\epsilon} \right) \right] \quad (43)$$

where $H_k^{d,m}(\lambda_1, \ldots, \lambda_k, \epsilon)$ and $\widetilde{H}_k^{d,m}(\lambda_1, \ldots, \lambda_k, \epsilon)$ are elements in the ring

$$\mathbb{Q}(\lambda_1, \ldots, \lambda_k, \epsilon)\left[\sin\frac{\pi\lambda_1}{\epsilon}, \cos\frac{\pi\lambda_1}{\epsilon}, \ldots, \sin\frac{\pi\lambda_k}{\epsilon}, \cos\frac{\pi\lambda_k}{\epsilon} \right].$$

For $k = 1$ and $\left| \arg\left(q^{1/2}/\epsilon\right) \right| < \pi$, as $q \to \infty$ (with fixed ϵ, λ), the following asymptotic holds

$$\cos\left(\frac{\lambda}{\epsilon}\pi \right) H_1^*\left(\frac{\lambda}{\epsilon}; \frac{q^{1/2}}{\epsilon} \right) \sim -\frac{\pi}{2}\sin\left(\frac{\pi\lambda}{\epsilon} \right) + \sin\left(\frac{\pi\lambda}{\epsilon} \right) \sum_{d \geq 0} \frac{(2d-1)!! \prod_{j=-d}^{d}(\lambda - \epsilon j)}{(2d+1)\, d!\, 2^{3d+1}\, q^{d+\frac{1}{2}}}$$

$$+ \cos\left(4\frac{q^{1/2}}{\epsilon} \right) \sum_{d \geq 1} q^{-\frac{d}{2}} H_1^{*d} + \sin\left(4\frac{q^{1/2}}{\epsilon} \right) \sum_{d \geq 1} q^{-\frac{d}{2}} \widetilde{H}_1^{*d} \quad (44)$$

where $\epsilon^{d-2} H_1^{*d}$, $\epsilon^{d-2}\widetilde{H}_1^{*d}$ are elements in $\mathbb{Q}[\lambda, \epsilon]$, and $\operatorname{gr} H_1^{*d} = d\, H_1^{*d}$, $\operatorname{gr} \widetilde{H}_1^{*d} = d\, \widetilde{H}_1^{*d}$.

1.6. **Organization of the paper.** In Section 2 we review the matrix resolvent approach and prove Theorem 1. In Section 3 we prove Proposition 1, Theorems 2–4. In Section 4 we prove Propositions 2–4 and Theorems 5–6. In Section 5 we prove Theorems 7.A–7.D. Further remarks are in Section 6.

2. MATRIX RESOLVENT APPROACH TO THE TODA LATTICE HIERARCHY

The matrix resolvent approach for computing tau-functions of the Toda Lattice Hierarchy was developed in [10]. Let us give a short review. Let L denote the following difference operator

$$L = \Delta + v_n + w_n \Delta^{-1} \quad (45)$$

where Δ denotes the shift operator, i.e. $\Delta : \psi_n \mapsto \psi_{n+1}$. The Toda Lattice Hierarchy is defined by

$$\frac{\partial L}{\partial t_i} = \frac{1}{(i+1)!}\left[A_i, L \right], \qquad i \geq 0, \quad (46)$$

$$A_i = \left(L^{i+1} \right)_+. \quad (47)$$

One observes that the normalization used here for the time variables $t_0, t_1, t_2, \ldots$ is not the standard one [10], but the one suitable for the study of GW invariants of $\mathbb{P}^1$.

2.1. **Toda conjecture.** We begin with a brief recall of the *Toda conjecture*, now a theorem:

Theorem ([29], [15]). *Denote* $\mathcal{F}^{\mathrm{s}} = \mathcal{F}^{\mathrm{s}}(x, \mathbf{t}; \epsilon) := \mathcal{F}\big(T_j^1 = x\,\delta_{j,0},\, T_j^2 = t_j,\, j = 0, 1, \ldots; \epsilon; q = 1\big)$ *with* $\mathbf{t} = (t_0, t_1, t_2, \ldots)$. *Let* $Z := e^{\mathcal{F}^{\mathrm{s}}}$. *Define* u, v *by*

$$v = v(x, \mathbf{t}; \epsilon) := \epsilon \frac{\partial}{\partial t_0} \log \frac{Z(x + \epsilon, \mathbf{t}; \epsilon)}{Z(x, \mathbf{t}; \epsilon)}, \tag{48}$$

$$u = u(x, \mathbf{t}; \epsilon) := \log \frac{Z(x + \epsilon, \mathbf{t}; \epsilon)\, Z(x - \epsilon, \mathbf{t}; \epsilon)}{Z^2(x, \mathbf{t}; \epsilon)}. \tag{49}$$

Then u, v *satisfy the Toda Lattice Hierarchy with the first equation being*

$$\frac{\partial v(x, \mathbf{t}; \epsilon)}{\partial t_0} = \frac{1}{\epsilon}\left(e^{u(x+\epsilon, \mathbf{t}; \epsilon)} - e^{u(x, \mathbf{t}; \epsilon)}\right), \tag{50}$$

$$\frac{\partial u(x, \mathbf{t}; \epsilon)}{\partial t_0} = \frac{1}{\epsilon}\left(v(x, \mathbf{t}; \epsilon) - v(x - \epsilon, \mathbf{t}; \epsilon)\right). \tag{51}$$

The Toda conjecture was formulated in [8, 19, 20], and was later proved by Okounkov–Pandharipande [29]; an extension [15] of this conjecture to the full generating function (2) requires an introduction of the extended Toda hierarchy [7] in terms of a suitably defined logarithm of the difference operator L (45); see also [30]. A slightly stronger version of this conjecture was also confirmed in the above proofs, namely, Z is a particular **tau-function** (in the sense of [15, 14, 10]) of the Toda Lattice hierarchy. This property along with the *string equation*

$$\sum_{i=1}^{\infty} t_i \frac{\partial Z}{\partial t_{i-1}} + \frac{x\, t_0}{\epsilon^2} Z = \frac{\partial Z}{\partial x} \tag{52}$$

uniquely determines Z up to a constant factor (independent of ϵ!) only.

2.2. **Matrix resolvent.** Denote by $\mathbb{Z}[\mathbf{v}, \mathbf{w}]$ the ring of polynomials with integer coefficients in the infinite set of variables $\mathbf{v} = (v_n)$, $\mathbf{w} = (w_n)$, $n \in \mathbb{Z}$. The (basic) matrix resolvent $R_n(\lambda)$ associated with L is defined as the unique solution to the following problem [10]

$$R_{n+1}(\lambda)\, U_n(\lambda) - U_n(\lambda) R_n(\lambda) = 0, \tag{53}$$

$$\operatorname{tr} R_n(\lambda) = 1, \qquad \det R_n(\lambda) = 0, \tag{54}$$

$$R_n(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{O}\big(\lambda^{-1}\big) \in \operatorname{Mat}\big(2, \mathbb{Z}[\mathbf{v}, \mathbf{w}][[\lambda^{-1}]]\big) \tag{55}$$

where $U_n(\lambda) := \begin{pmatrix} v_n - \lambda & w_n \\ -1 & 0 \end{pmatrix}$. Write

$$R_n(\lambda) = \begin{pmatrix} 1 + \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}, \qquad \alpha_n(\lambda),\, \beta_n(\lambda),\, \gamma_n(\lambda) \in \mathrm{O}\big(\lambda^{-1}\big).$$

Then the above equations (53)–(54) become a series of recursive relations for $\alpha_n, \beta_n, \gamma_n$:

$$\beta_n = -w_n\, \gamma_{n+1}, \tag{56}$$

$$\alpha_{n+1} + \alpha_n + 1 = (\lambda - v_n)\, \gamma_{n+1}, \tag{57}$$

$$(\lambda - v_n)\,(\alpha_n - \alpha_{n+1}) = w_n\, \gamma_n - w_{n+1}\, \gamma_{n+2}, \tag{58}$$

$$\alpha_n + \alpha_n^2 + \beta_n\, \gamma_n = 0. \tag{59}$$

Along with the initial values (55) one can find $\alpha_n, \beta_n, \gamma_n$ in an algebraic way [10]. Indeed, write

$$\gamma_n = \sum_{j \geq 0} \frac{c_{n,j}}{\lambda^{j+1}}, \qquad \alpha_n = \sum_{j \geq 0} \frac{a_{n,j}}{\lambda^{j+1}}.$$

Substituting these expressions into (56)–(59) we obtain

$$c_{n,j+1} \;=\; v_{n-1}\,c_{n,j} \;+\; a_{n,j} \;+\; a_{n-1,j}\,, \tag{60}$$

$$a_{n,j+1} \;-\; a_{n+1,j+1} \;+\; v_n\,[a_{n+1,j} - a_{n,j}] \;+\; w_{n+1}\,c_{n+2,j} \;-\; w_n\,c_{n,j} \;=\; 0\,, \tag{61}$$

$$a_{n,\ell} \;=\; \sum_{i+j=\ell-1} \left[w_n\,c_{n,i}\,c_{n+1,j} - a_{n,i}\,a_{n,j} \right]\,, \tag{62}$$

$$a_{n,0} = 0\,, \qquad c_{n,0} = 1\,. \tag{63}$$

We shall call (53)–(55), or (56)–(59), or (60)–(63) the *matrix-resolvent recursive relations for the Toda Lattice Hierarchy*. It should be noted that these recursive relations are valid for an arbitrary solution $\big(v_n(\mathbf{t}), w_n(\mathbf{t})\big)$ to the Toda Lattice Hierarchy; moreover, the form of matrix-resolvent recursive relations as well as equation (63) do not depend on the solution.

2.3. **From matrix resolvent to tau-function.** For an arbitrary solution $\big(v_n(\mathbf{t}), w_n(\mathbf{t})\big)$ of the Toda Lattice Hierarchy, there exists a unique (up to multiplying by exponential of an arbitrary linear function in $n, t_0, t_1, \dots$) function $\tau_n(\mathbf{t})$ satisfying [10]

$$\sum_{i,j\geq 0} \frac{\partial^2 \log \tau_n(\mathbf{t})}{\partial t_i\, \partial t_j} \frac{(i+1)!(j+1)!}{\lambda^{i+2}\mu^{j+2}} \;=\; \frac{\operatorname{tr} R_n(\lambda;\mathbf{t}) R_n(\mu;\mathbf{t}) - 1}{(\lambda-\mu)^2}\,, \tag{64}$$

$$\frac{1}{\lambda} \;+\; \sum_{i\geq 0} \frac{\partial}{\partial t_i} \log \frac{\tau_{n+1}(\mathbf{t})}{\tau_n(\mathbf{t})} \frac{(i+1)!}{\lambda^{i+2}} \;=\; \gamma_{n+1}(\lambda;\mathbf{t})\,, \tag{65}$$

$$\frac{\tau_{n+1}(\mathbf{t})\tau_{n-1}(\mathbf{t})}{\tau_n^2(\mathbf{t})} \;=\; w_n\,. \tag{66}$$

Here, $R_n(\lambda;\mathbf{t}) := R_n(\lambda)|_{v_n=v_n(\mathbf{t}),\, w_n=w_n(\mathbf{t})}$. We call $\tau_n(\mathbf{t})$ the *tau-function* of the solution $\big(v_n(\mathbf{t}), w_n(\mathbf{t})\big)$. Indeed, by interpolating using $x = n\epsilon$, we know that the Toda Lattice Hierarchy is a tau-symmetric integrable system of Hamiltonian PDEs within the normal form of [14], and the identification between $\tau_n(\mathbf{t})$ and the tau-function of [14, 7, 15] is made in [10]. By a straightforward residue computation (comparing coefficients of μ^{-2} in (64)), we obtain

$$\sum_{i\geq 0} \frac{(i+1)!}{\lambda^{i+2}} \frac{\partial^2 \log \tau_n(\mathbf{t})}{\partial t_0 \partial t_i} \;=\; \alpha_n(\lambda;\mathbf{t})\,. \tag{67}$$

Theorem A ([10]). *Generating series of logarithmic derivatives of $\tau_n(\mathbf{t})$ have the following expressions*

$$\sum_{i_1,\dots,i_k\geq 0} \frac{\partial^k \log \tau_n(\mathbf{t})}{\partial t_{i_1}\cdots \partial t_{i_k}} \prod_{\ell=1}^{n} \frac{(i_\ell+1)!}{\lambda_\ell^{i_\ell+2}} \;=\; -\sum_{\sigma \in S_k/C_k} \frac{\operatorname{tr}\left[R_n(\lambda_{\sigma_1};\mathbf{t})\cdots R_n(\lambda_{\sigma_k};\mathbf{t}) \right]}{\prod_{i=1}^{k}(\lambda_{\sigma_i} - \lambda_{\sigma_{i+1}})}\,, \quad \forall\, k \geq 3\,. \tag{68}$$

2.4. **Proof of Theorem 1.** The first step is to give the initial value of the GW solution.

Lemma 1. *The initial value of the solution* (48), (49) *of the Toda Lattice Hierarchy is given by*

$$u(x, \mathbf{t} = \mathbf{0}; \epsilon) \;=\; 0\,,$$

$$v(x, \mathbf{t} = \mathbf{0}; \epsilon) \;=\; x + \frac{\epsilon}{2}\,.$$

Proof The string equation (52) can be written equivalently as

$$\sum_{i=1}^{\infty} t_i \frac{\partial \mathcal{F}^{\mathrm{s}}}{\partial t_{i-1}} \;+\; \frac{x\, t_0}{\epsilon^2} \;=\; \frac{\partial \mathcal{F}^{\mathrm{s}}}{\partial x}\,. \tag{69}$$

Differentiating both sides of (69) w.r.t. t_0 we obtain $\sum_{i=1}^{\infty} t_i \frac{\partial^2 \mathcal{F}^{\mathrm{s}}}{\partial t_{i-1} \partial t_0} + \frac{x}{\epsilon^2} = \frac{\partial^2 \mathcal{F}^{\mathrm{s}}}{\partial x \partial t_0}$. Taking $t_1 = t_2 = \cdots = 0$ in this equation gives $\frac{\partial^2 \mathcal{F}^{\mathrm{s}}}{\partial x \partial t_0}(x, t_0, 0, 0, \ldots; \epsilon) \equiv \frac{x}{\epsilon^2}$. In particular, $\frac{\partial^2 \mathcal{F}^{\mathrm{s}}}{\partial x \partial t_0}(x, 0, 0, \ldots; \epsilon) = \frac{x}{\epsilon^2}$. Therefore using (48) we have

$$v(x, 0, 0, \ldots; \epsilon) = \frac{e^{\epsilon \partial_x} - 1}{\epsilon \partial_x} \epsilon^2 \frac{\partial^2 \mathcal{F}^{\mathrm{s}}}{\partial x \partial t_0}(x, 0, 0, \ldots; \epsilon) = x + \frac{\epsilon}{2}.$$

We now look at the initial value of u. Since

$$u(x, 0, 0, \ldots; \epsilon) = \mathcal{F}^{\mathrm{s}}(x + \epsilon, 0, 0, \ldots; \epsilon) + \mathcal{F}^{\mathrm{s}}(x - \epsilon, 0, 0, \ldots; \epsilon) - 2\mathcal{F}^{\mathrm{s}}(x, 0, 0, \ldots; \epsilon),$$

we only need to find coefficients of x^n in the Taylor expansion of $\mathcal{F}^{\mathrm{s}}(x, 0, 0, \ldots; \epsilon)$. The degree-dimension matching implies $2g - 2 + 2d + n = 0$. So the only possible choices are $(g, d, n) = (0, 0, 2), (0, 1, 0), (1, 0, 0)$. The constant terms do not contribute to $u(x, 0, 0, \ldots; \epsilon)$. The quadratic term cannot appear because of the well-known expression of the genus zero primary potential (the potential of the corresponding Frobenius manifold) is

$$F = \left(\epsilon^2 \mathcal{F}^{\mathrm{s}} \right)_{\epsilon=0} = \frac{1}{2}(v^1)^2 v^2 + e^{v^2}, \quad \text{with } v^1 = v|_{\epsilon=0}, \; v^2 = u|_{\epsilon=0}.$$

Clearly, after restricting to $v^2 = 0$ and $v^1 = x$, there is no x^2 term. $\qquad \square$

We now proceed to the proof of Theorem 1. Recall the interpolation formula $x = n\epsilon$. Then the above Lemma 1 implies that, for the particular solution (48)–(49) to the Toda Lattice Hierarchy

$$u_n(\mathbf{t} = \mathbf{0}; \epsilon) = 0, \tag{70}$$

$$v_n(\mathbf{t} = \mathbf{0}; \epsilon) = \epsilon n + \frac{\epsilon}{2}. \tag{71}$$

Substituting (70)–(71) into (56)–(59) we obtain the following recursive relations for the entries of the *initial* (basic) matrix resolvent

$$\alpha_{n+1} + \alpha_n + 1 = \left(\lambda - \epsilon n - \frac{\epsilon}{2} \right) \gamma_{n+1}, \tag{72}$$

$$\left(\lambda - \epsilon n - \frac{\epsilon}{2} \right)(\alpha_n - \alpha_{n+1}) = \gamma_n - \gamma_{n+2}, \tag{73}$$

$$\alpha_n + \alpha_n^2 - \gamma_n \gamma_{n+1} = 0. \tag{74}$$

Here, "initial" means at $\mathbf{t} = 0$. The theorem is then proved by taking $\mathbf{t} = 0$ in (68) and (64). $\qquad \square$

As before, write

$$\gamma_n = \gamma_n(\lambda; \epsilon) = \sum_{j \geq 0} \frac{c_{n,j}}{\lambda^{j+1}}, \qquad \alpha_n = \alpha_n(\lambda; \epsilon) = \sum_{j \geq 0} \frac{a_{n,j}}{\lambda^{j+1}}. \tag{75}$$

Then equations (72)–(74) become

$$c_{n,j} = \epsilon \left(n - \frac{1}{2} \right) c_{n,j-1} + a_{n,j-1} + a_{n-1,j-1}, \tag{76}$$

$$a_{n,j} - a_{n+1,j} + \left(\epsilon n + \frac{\epsilon}{2} \right)(a_{n+1,j-1} - a_{n,j-1}) + c_{n+2,j-1} - c_{n,j-1} = 0, \tag{77}$$

$$a_{n,j} = \sum_{i=0}^{j-1} \left(c_{n,i} c_{n+1,j-1-i} - a_{n,i} a_{n,j-1-i} \right) \tag{78}$$

together with the initial data for the recursion

$$a_{n,0} = 0, \quad c_{n,0} = 1. \tag{79}$$

The first several terms of α_n, γ_n are given by

$$\alpha_n = \frac{1}{\lambda^2} + \frac{2n\epsilon}{\lambda^3} + \frac{3n^2\epsilon^2 + \frac{\epsilon^2}{4} + 3}{\lambda^4} + \frac{4n^3\epsilon^3 + n(\epsilon^3 + 12\epsilon)}{\lambda^5} + \cdots, \tag{80}$$

$$\gamma_n = \frac{1}{\lambda} + \frac{n\epsilon - \frac{\epsilon}{2}}{\lambda^2} + \frac{n^2\epsilon^2 - n\epsilon^2 + \frac{\epsilon^2}{4} + 2}{\lambda^3} + \frac{n^3\epsilon^3 - \frac{3}{2}n^2\epsilon^3 + n(\frac{3\epsilon^3}{4} + 6\epsilon) - \frac{\epsilon^3}{8} - 3\epsilon}{\lambda^4} + \cdots. \tag{81}$$

3. SOLVING THE MATRIX-RESOLVENT RECURSIVE RELATIONS OF $\mathbb{P}^1$

The goal of this section is to solve equations (5)–(7). We start with proving Proposition 1.

3.1. Proof of Proposition 1.
Write $M(z,s) = \begin{pmatrix} 1 + a(z,s) & b(z,s) \\ c(z,s) & -a(z,s) \end{pmatrix}$. The topological difference equation (11), i.e. $M(z-1;s)\begin{pmatrix} z-1/2 & -s \\ s & 0 \end{pmatrix} = \begin{pmatrix} z-1/2 & -s \\ s & 0 \end{pmatrix} M(z;s)$, written in terms of a, b, c reads as follows:

$$b(z) + c(z-1) = 0,$$

$$\left(z - \frac{1}{2}\right) b(z) + s\left(1 + a(z-1) + a(z)\right) = 0,$$

$$s\,b(z-1) + s\,c(z) + \left(z - \frac{1}{2}\right)\left(a(z-1) - a(z)\right) = 0.$$

Here, $a(z), b(z), c(z)$ are short notations for $a(z,s), b(z,s), c(z,s)$; below we keep using these notations when no confusion will occur. It follows from these equations that

$$c(z) = s\,\frac{1 + a(z) + a(z+1)}{z + \frac{1}{2}}, \qquad b(z) = -s\,\frac{1 + a(z-1) + a(z)}{z - \frac{1}{2}}. \tag{82}$$

Moreover, the topological difference equation is reduced to the following 3rd order linear difference equation (with a parameter s) for a:

$$s^2\left(\frac{1 + a(z) + a(z+1)}{z + \frac{1}{2}} - \frac{1 + a(z-2) + a(z-1)}{z - \frac{3}{2}}\right) + \left(z - \frac{1}{2}\right)\left(a(z-1) - a(z)\right) = 0. \tag{83}$$

Write $a(z,s) = \sum_{k \geq 0} A_k\, z^{-k-1}$. Then equation (83) is equivalent to the following equations:

$$-8(k+2)A_{k+1} = -16\,s^2\,\delta_{k,0} - 3\left(1 + 4s^2\right)A_{k-1} + 2\left(1 + 4s^2\right)A_k$$

$$+ \sum_{\substack{k_1,n\geq 0 \\ k_1+n+1=k}} A_{k_1}\left((-1)^{n+1}12\,s^2 - 2^{n+2}s^2 + (3 - 4s^2)\right)\binom{1+k_1}{n}$$

$$+ \sum_{\substack{k_1,n\geq 0 \\ k_1+n=k}} A_{k_1}\left((-1)^n 8\,s^2 - 2^{n+3}s^2 - 2(1 + 4s^2)\right)\binom{1+k_1}{n}$$

$$- 12 \sum_{\substack{k_1\geq 0,n\geq 1 \\ k_1+n=k+1}} A_{k_1}\binom{1+k_1}{n} + 8 \sum_{\substack{k_1\geq 0,n\geq 2 \\ k_1+n=k+2}} A_{k_1}\binom{1+k_1}{n}, \qquad k \geq -1. \tag{84}$$

Here it is understood as $A_{-2} = A_{-1} = 0$. Together with (82), this recursion proves the existence and uniqueness of a solution M^* of the form $M^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{k\geq 1} M_k^*\, z^{-k}$. Moreover, the fact that each

entry of M_k^* is a polynomial of s can be seen easily from this recursion (for A_k). Finally, taking the determinants of both sides of (11) we have

$$s^2 \det M^*(z-1) \;=\; s^2 \det M^*(z) \quad \Rightarrow \quad \det M^*(z-1) \;=\; \det M^*(z)\,.$$

It is easy to see that $\det M^*(z) \in z^{-1} Q[s][[z^{-1}]]$. Noting that $A_0 = 0$ we find that the coefficient of z^{-1} in $\det M^*(z)$ also vanishes. Therefore, $\det M^*(z)$ vanishes. The proposition is proved. $\qquad\square$

3.2. Proof of Theorem 2.

In this subsection, we prove Theorem 2. The proof is similar with the one given in [5] (see the "Key Lemma" i.e. Lemma 4.2.3 therein).

Proof of Theorem 2. Define $R_n^*(\lambda; \epsilon) = M^*\left(\frac{\lambda}{\epsilon} - n; \frac{1}{\epsilon}\right)$. It is easy to check that $R_n^*(\lambda; \epsilon)$ satisfies (5)–(7). Since $R_n(\lambda; \epsilon)$ is the unique solution to (5)–(7), we have $R_n(\lambda; \epsilon) = R_n^*(\lambda; \epsilon)$. By definition, $\mathcal{R}(\lambda; x; \epsilon) = R_{x/\epsilon}(\lambda; \epsilon)$. Hence $\mathcal{R}(\lambda; x; \epsilon)$ only depends on $\lambda - x$ and ϵ. The theorem is proved. $\qquad\square$

3.3. Proof of Theorem 3 and Theorem 4.

To prove Theorems 3 and 4, we must show two things:

i) Prove that the entries of the matrix-valued meromorphic function $B(z; s)$ defined by equations (17)–(19) have asymptotic expansions as power series in z^{-1} (for $|z| \to \infty$ at a bounded distance away from half integers) given by the RHS of (13).
ii) Show that the function $B(z; s)$ satisfies the properties (11)–(12) with M replaced by B.

For step i), we must look at the asymptotics of $G(z; s)$ and $\widetilde{G}(z; s)$ as $|z| \to \infty$ with s bounded, say $|s| \le S$. We consider only the case of G, since the case of $\widetilde{G}$ is exactly similar. We claim first that

$$G(z; s) \;:=\; \sum_{m=0}^{\infty} \binom{2m}{m} \frac{s^{2m}}{\left(z - m + \frac{1}{2}\right)_{2m}} \;=\; \sum_{m=0}^{N-1} \binom{2m}{m} \frac{s^{2m}}{\left(z - m + \frac{1}{2}\right)_{2m}} \;+\; \mathrm{O}\!\left(z^{-2N}\right) \qquad (85)$$

for any fixed $N \in \mathbb{N}$ as $z \to \infty$ at a bounded distance from $\mathbb{Z} + \frac{1}{2}$. Indeed, the terms with $N \le m \le \frac{1}{2}|z|$ in (85) are individually bounded by $\frac{2^{2m} S^{2m}}{(|z|/2)^{2m}}$ (because each factor in the Pochhammer symbol in the denominator has absolute value $\ge \frac{1}{2}|z|$), so their sum is $\le \sum_{m=N}^{\infty} \left(\frac{4S}{|z|}\right)^{2m} = \mathrm{O}(z^{-2N})$. The terms with $m > \frac{1}{2}|z|$ are individually bounded by $\frac{2^{2m} s^{2m}}{\delta\, m!\,(m-1)!}$, where δ is the distance from z to $\mathbb{Z} + \frac{1}{2}$, so their sum is smaller than any fixed negative powers of $|z|$ as $|z| \to \infty$ with δ fixed. Now using a partial fraction development in each summand in (85), we find

$$G(z; s) \;=\; 1 + 2 \sum_{m=1}^{N-1} \frac{s^{2m}}{m!\,(m-1)!} \sum_{\ell=0}^{2m-1} \frac{(-1)^\ell \binom{2m-1}{\ell}}{z - m + \ell + \frac{1}{2}} \;+\; \mathrm{O}\!\left(z^{-2N}\right)$$

$$= \; 1 + 2 \sum_{r=1}^{2N-1} \frac{1}{z^r} \sum_{0 \le \ell < 2m \le r} \frac{(-1)^\ell s^{2m}}{m!(m-1)!} \binom{2m-1}{\ell} \left(m - \ell - \tfrac{1}{2}\right)^{r-1} \;+\; \mathrm{O}\!\left(z^{-2N}\right), \qquad (86)$$

where we have removed the terms with $2m > r$ because the $(2m-1)$st (backwards) difference of a polynomial of degree $r - 1$ vanishes identically if $2m > r$. We also note that the terms with r odd give zero (replace ℓ by $2m - 1 - \ell$), so we can set $r = 2j + 2$, $m = i + 1$ to recover the expression given in (14), proving that $G(z; s) \sim 1 + 2\alpha$ as claimed.

Now we do step ii). The explicit expression for M given in the statement of Theorem 3 clearly has the form $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{k \ge 1} M_k\, z^{-k}$. Therefore using Proposition 1 we only need to show that $M(z, s)$ satisfies (11)–(12). Then due to Proposition 4 it suffices to show that $B(z, s)$ satisfies (11)–(12)

for $z \in \mathbb{C} - \mathbb{Z}_{\mathrm{odd}}$. Identity (12) is obvious for $B(z, s)$. Identity (11) is equivalent to

$$\widetilde{G}(z; s) \;=\; \frac{G(z; s) + G(z+1, s)}{2}, \tag{87}$$

$$\frac{\widetilde{G}(z+\frac{1}{2}, s)}{z+1} - \frac{\widetilde{G}(z-\frac{3}{2}, s)}{z-1} \;=\; \frac{z}{2s^2}\left[G\left(z+\frac{1}{2}, s\right) - G\left(z-\frac{1}{2}, s\right)\right]. \tag{88}$$

Identity (87) is true since $G(z; s) + G(z+1, s) = \sum_{i=0}^{\infty} \binom{2i}{i}\left[\frac{s^{2i}}{(z-i+\frac{1}{2})_{2i}} + \frac{s^{2i}}{(z-i+\frac{3}{2})_{2i}}\right] = 2\,\widetilde{G}(z; s)$. Similarly, we find that identity (88) is true. □

Note that the k-point function $F_k(\lambda_1, \ldots, \lambda_k; 0; \epsilon, 1)$ $(k \geq 2)$ can be expressed in terms of M by

$$F_k(\lambda_1, \ldots, \lambda_k; x; \epsilon, 1) \;=\; -\sum_{\sigma \in S_k/C_k} \frac{\mathrm{tr}\left[M\left(\frac{\lambda_{\sigma(1)}-x}{\epsilon}; \frac{1}{\epsilon}\right) \cdots M\left(\frac{\lambda_{\sigma(k)}-x}{\epsilon}; \frac{1}{\epsilon}\right)\right]}{\prod_{i=1}^{k}\left(\lambda_{\sigma(i)} - \lambda_{\sigma(i+1)}\right)} - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{89}$$

The validity of this identity is understood in the formal power series ring $\mathbb{Q}\left[\left[x, \lambda_1^{-1}, \ldots, \lambda_k^{-1}\right]\right]$.

Proposition 5. *For any $k \geq 2$, the following formula holds true*

$$\epsilon^k \sum_{i_1, \ldots, i_k \geq 0} \frac{(i_1 + 1)! \cdots (i_k + 1)!}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} \sum_{\substack{m, g, d \geq 0 \\ 2g+2d-2+2m = \sum i_\ell}} \frac{q^d}{m!}\epsilon^{2g-2}\left\langle \tau_{i_1}(\omega) \cdots \tau_{i_k}(\omega)\tau_0(1)^m \right\rangle_{g,d}$$

$$= -\frac{1}{k}\sum_{\sigma \in S_k} \frac{\mathrm{tr}\left[\mathcal{R}\left(q^{-1/2}\lambda_{\sigma_1}; q^{-1}; q^{-1/2}\epsilon\right) \ldots \mathcal{R}\left(q^{-1/2}\lambda_{\sigma_k}; q^{-1}; q^{-1/2}\epsilon\right)\right]}{\prod_{i=1}^{k}(\lambda_{\sigma_i} - \lambda_{\sigma_{i+1}})} - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{90}$$

Proof. Use Proposition 1, (10), as well as (4). □

Remark. R. Pandharipande [32] proves that the numbers $\left\langle \tau_1(\omega)^{2g-2+d} \right\rangle_{g,d}$ coincide with the classical Hurwitz numbers $H_{g,d}$ defined by Hurwitz [23]. A polynomial algorithm of computing these numbers has been obtained very recently [12] based on Pandharipande's equation [32, 12]. Although the formula (89) for F_k with $k = 2g - 2 + d$ contains the numbers $H_{g,d}$, the algorithm designed from (89) is not of polynomial-time (note that however (89) contains much more information than $H_{g,d}$).

3.4. **Proof of Proposition 2.** The main observation is that for any $k \geq 3$, we have

$$H_k(z_1, \ldots, z_k) \;=\; -\sum_{\sigma \in S_k/C_k} \frac{\mathrm{tr}\left[B(z_{\sigma(1)}, s) \ldots B(z_{\sigma(k)}, s)\right]}{\prod_{i=1}^{k}(z_{\sigma_i} - z_{\sigma_{i+1}})}$$

$$= -\sum_{j=1}^{k-1} \sum_{\sigma \in S_{k-1}/C_{k-1}} \frac{\mathrm{tr}\left[B(z_{\sigma(1)}, s) \ldots [B(z_k, s), B(z_{\sigma(j)}, s)] \ldots B(z_{\sigma(k-1)}, s)\right]}{(z_k - z_{\sigma(j)})\prod_{i=1}^{k-1}(z_{\sigma_i} - z_{\sigma_{i+1}})}.$$

So H_k is analytic along $z_k = z_i$ for $i \neq k$ away from the half-integer points. Note that $H_k(z_1, \ldots, z_k)$ is totally symmetric w.r.t. permutations of $z_1, \ldots, z_k$. Therefore H_k is also analytic along $z_j = z_i$ for $i \neq j$ (for any j). The case $k = 2$ follows immediately from the second equality in (20). □

4. PROOF OF THE FACTORIZATION FORMULAS

We begin by giving the proof of Proposition 3 of Section 1.3, giving an explicit factorization of the rank 1 matrix $B(z, s)$ as the product of a column vector and a row vector.

Proof of Proposition 3. We have to prove the following three identities for hypergeometric $_1F_2$-functions as sums of products of Bessel functions:

$$\frac{1 + G(z; s)}{2} = \frac{\pi s}{\cos(\pi z)} J_{z-\frac{1}{2}}(2s) J_{-z-\frac{1}{2}}(2s), \tag{91}$$

$$\frac{1 - G(z; s)}{2} = \frac{\pi s}{\cos(\pi z)} J_{z+\frac{1}{2}}(2s) J_{-z+\frac{1}{2}}(2s), \tag{92}$$

$$\frac{s}{z + \frac{1}{2}} \widetilde{G}(z; s) = \frac{\pi s}{\cos(\pi z)} J_{z+\frac{1}{2}}(2s) J_{-z-\frac{1}{2}}(2s). \tag{93}$$

Indeed,

$$\text{RHS of (93)} = \frac{\pi s}{\cos(\pi z)} \frac{1}{\Gamma(z + \frac{3}{2})\Gamma(-z + \frac{1}{2})} \sum_{n \geq 0} (-1)^n \frac{s^{2n}}{n!^2 \binom{n+z+\frac{1}{2}}{n}} \sum_{n \geq 0} (-1)^n \frac{s^{2n}}{n!^2 \binom{n-z-\frac{1}{2}}{n}}$$

$$= \frac{s}{z + \frac{1}{2}} \sum_{n \geq 0} s^{2n} \sum_{n_1 + n_2 = n} \frac{(-1)^n}{(n_1!)^2 (n_2!)^2 \binom{n_1+z+\frac{1}{2}}{n_1}\binom{n_2-z-\frac{1}{2}}{n_2}} = \text{LHS of (93)}.$$

Similarly one proves (91), (92). The factorization $B = u(z) u(-z)^T$ can also be verified directly. □

Proof of Theorem 5. For $k \geq 2$ we have

$$\text{tr}(B(z_1) \ldots B(z_k)) = \text{tr}\left(u(z_1)u(-z_1)^T u(z_2)u(-z_2)^T \ldots u(z_k) u(-z_k)^T\right)$$

$$= \text{tr}\left(u(-z_1)^T u(z_2) \ldots u(-z_k)^T u(z_1)\right) = \prod_{i=1}^{k} \left(u(-z_i), u(z_{i+1})\right).$$

(indices modulo k). Hence each summand in the trace-product formulas (20), (21) has the form

$$\frac{\text{tr}(B(z_1, s) \ldots B(z_k, s))}{\prod_{i=1}^{k}(z_i - z_{i+1})} = \epsilon^k \prod_{i=1}^{k} D(z_i, z_{i+1}; s)$$

where we recall that $D(a, b; s) = u(-a, s)^T u(b, s)/(a - b)$, as claimed. □

Formula (28) implies the following asymptotic formula for $a, b \notin \mathbb{Z} + \frac{1}{2}$, as $a, b \to \infty$:

$$D(a, b; s) - \frac{1}{a - b} \sim \sum_{p,q \geq 0} \frac{(-1)^{q+1}}{a^{p+1}b^{q+1}} \sum_{n \geq 1} \frac{s^{2n}}{n!} \sum_{\substack{1 \leq i,j \leq n \\ i+j \leq n-1}} (-1)^{i+j} \frac{(i+j-2n)_{n-1}(i - \frac{1}{2})^p(j - \frac{1}{2})^q}{(i-1)!(j-1)!(n-i)!(n-j)!}. \tag{94}$$

Proposition 2 can be alternatively proved by using this formula.

Proof of Theorem 6. Differentiating both sides of (69) w.r.t. t_j $(j \geq 1)$ we obtain

$$\frac{\partial \mathcal{F}^{\text{s}}}{\partial t_{j-1}} + \sum_{i=1}^{\infty} t_i \frac{\partial^2 \mathcal{F}^{\text{s}}}{\partial t_{i-1}\partial t_j} = \frac{\partial^2 \mathcal{F}^{\text{s}}}{\partial x \partial t_j}.$$

Setting $t_0 = t_1 = \cdots = 0$ in this equation yields $\langle\langle \tau_{j-1}(\omega) \rangle\rangle(x; \epsilon, 1) = \langle\langle \tau_0(1)\tau_j(\omega) \rangle\rangle(x; \epsilon, 1)$. Note that equation (65) implies

$$\frac{1}{\lambda} + \epsilon^2 \sum_{i=0}^{\infty} \frac{(i+1)!}{\lambda^{i+2}} \frac{e^{\epsilon \partial_x} - 1}{\epsilon \partial_x} \langle\langle \tau_0(1)\tau_i(\omega) \rangle\rangle(x; \epsilon, 1) = \gamma_{n+1}, \qquad x = n\epsilon.$$

So $\frac{1}{\lambda} + \frac{\epsilon^2}{\lambda^2}\left(\frac{x}{\epsilon^2} + \frac{1}{2\epsilon}\right) + \epsilon^2\sum_{i=1}^{\infty}\frac{(i+1)!}{\lambda^{i+2}}\frac{e^{\epsilon\partial_x}-1}{\epsilon\partial_x}\langle\langle\tau_{i-1}(\omega)\rangle\rangle(x;\epsilon,1) = \gamma_{n+1}$. Therefore,

$$-\epsilon\frac{e^{\epsilon\partial_x}-1}{\epsilon\partial x}\frac{\partial F_1(\lambda;x;\epsilon,1)}{\partial\lambda} = \gamma_{n+1} - \frac{\epsilon^2}{\lambda^2}\left(\frac{x}{\epsilon^2}+\frac{1}{2\epsilon}\right) - \frac{1}{\lambda}\,.$$

Hence we have

$$-\frac{\partial F_1(\lambda;x;\epsilon,1)}{\partial\lambda} = \sum_{j\geq 2}\frac{1}{\lambda^{j+1}}\sum_{i=0}^{\infty}\frac{\epsilon^{-1-2i}}{i!^2}\sum_{l=0}^{2i}(-1)^l\binom{2i}{l}\sum_{m=0}^{j}\binom{j}{m}B_m\,\epsilon^m\left(x+\epsilon\,i+\frac{\epsilon}{2}-\epsilon\,l\right)^{j-m}.$$

Identity (36) follows immediately. The equivalence between this identity and the statement of the theorem has already been explained in Section 1.4. $\qquad\square$

5. Four asymptotics

We have studied several analytic properties of $H_k(z_1,\ldots,z_k;s)$. Motivated by the GW theory, in this section, we will investigate further the functions $H_k\left(\frac{\lambda_1}{\epsilon},\ldots,\frac{\lambda_k}{\epsilon};\frac{q^{1/2}}{\epsilon}\right)$. It should be noted that the function $W(\lambda,\epsilon,q) := B\left(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\right)$ satisfies the following set of equations:

$$W(\lambda+\epsilon,\epsilon,q)\begin{pmatrix}\lambda-\frac{\epsilon}{2} & -\sqrt{q}\\ \sqrt{q} & 0\end{pmatrix} = \begin{pmatrix}\lambda-\frac{\epsilon}{2} & -\sqrt{q}\\ \sqrt{q} & 0\end{pmatrix}W(\lambda,\epsilon,q)\,, \tag{95}$$

$$\operatorname{tr}W(\lambda,\epsilon,q) = 1, \qquad \det W(\lambda,\epsilon,q) = 0\,, \tag{96}$$

$$W(\lambda,\epsilon,q) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + O\left(\lambda^{-1}\right), \quad \lambda\to\infty\,. \tag{97}$$

5.1. The $\epsilon\to 0$ asymptotic. Proof of Theorem 7.A. First we consider $k\geq 2$.

Lemma 2. *For $0 < \frac{2\sqrt{q}}{\lambda} < 1$ and $\frac{\lambda}{\epsilon} > 0$, the following asymptotic formula holds true as $|\epsilon|\to 0$:*

$$\frac{1}{2}G\left(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\right) - \frac{1}{2} \sim \sum_{m=0}^{\infty}\epsilon^m a_m(\lambda;q)\,,$$

$$\frac{\sqrt{q}}{\lambda+\frac{\epsilon}{2}}\widetilde{G}\left(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\right) \sim \sum_{m=0}^{\infty}\epsilon^m c_m(\lambda;q)$$

where a_m, c_m are algebraic functions of $\lambda,\sqrt{q}$. Moreover, for $k,m\geq 0$, the functions $a_{2k+1}(\lambda;q)$ vanish, and a_{2k}, c_m satisfy the homogeneity conditions: $\operatorname{gr}a_{2k} = -2k\,a_{2k}$, $\operatorname{gr}c_m = -m\,c_m$.

The first several a_{2k}, c_m are given explicitly by

$$a_0 = \frac{\lambda}{2(\lambda^2-4q)^{\frac{1}{2}}} - \frac{1}{2}\,, \quad a_2 = \frac{q\lambda(\lambda^2+16q)}{4(\lambda^2-4q)^{\frac{7}{2}}}\,, \quad a_4 = \frac{q\lambda(\lambda^6+247q\lambda^4+2848q^2\lambda^2+3072q^3)}{16(\lambda^2-4q)^{\frac{13}{2}}}\,,$$

$$c_0 = \frac{\sqrt{q}}{(\lambda^2-4q)^{\frac{1}{2}}}\,, \quad c_1 = -\frac{\sqrt{q}\,\lambda}{2(\lambda^2-4q)^{\frac{3}{2}}}\,, \quad c_2 = \frac{\sqrt{q}\,(\lambda^4+6q\lambda^2)}{4(\lambda^2-4q)^{\frac{7}{2}}}\,, \quad c_3 = \frac{\sqrt{q}\lambda(\lambda^4+42q\lambda^2+96q^2)}{8(\lambda^2-4q)^{\frac{9}{2}}}\,.$$

Lemma 2 can be proved either by studying the analytic functions (18)–(19), or by the following lemma regarding the large-order asymptotics of the Bessel functions [33].

Lemma 3. *For any fixed valued $\zeta\in(0,1)$, the following asymptotic holds true: as $\nu\to+\infty$,*

$$J_{\nu-\frac{1}{2}}(\nu\,\zeta) \sim \frac{\left(\nu-\frac{1}{2}\right)^{\nu-\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)}e^V\,, \quad V = \nu\,V_0 + V_1 + \frac{V_2}{\nu} + \frac{V_3}{\nu^2} + \ldots\,, \tag{98}$$

where V_m, $m \geq 0$ are functions of ζ with the first few given by

$$
\begin{aligned}
V_0 &= -1 + \sqrt{1 - \zeta^2} + \log \zeta - \log\left(1 + \sqrt{1 - \zeta^2}\right), \\
V_1 &= \frac{1}{2} + \frac{1}{2}\log\left(1 + \sqrt{1 - \zeta^2}\right) - \frac{1}{2}\log \zeta - \frac{1}{4}\log(1 - \zeta^2), \\
V_2 &= -\frac{1}{6} + \frac{1}{4}\frac{1}{(1 - \zeta^2)} - \frac{5}{24}\frac{1}{(1 - \zeta^2)^{\frac{3}{2}}}, \\
V_3 &= -\frac{1}{48} + \frac{1}{4}\frac{1}{(1 - \zeta^2)^{\frac{3}{2}}} - \frac{1}{4}\frac{1}{(1 - \zeta^2)^2} - \frac{5}{16}\frac{1}{(1 - \zeta^2)^{\frac{5}{2}}} + \frac{5}{16}\frac{1}{(1 - \zeta^2)^3}.
\end{aligned}
$$

For $m \geq 2$, V_m belongs to $\mathbb{Q}\left[\left(1 - \zeta^2\right)^{-\frac{1}{2}}\right]$ having degree $3m - 3$.

Note that Lemma 3 implies that as $\nu \to +\infty$,

$$
j_\nu\left(\frac{\nu^2 \zeta^2}{4}\right) \sim \left(2 - \tfrac{1}{\nu}\right)^{\nu - \frac{1}{2}} e^U, \quad U = \nu U_0 + U_1 + \frac{U_2}{\nu} + \frac{U_3}{\nu^2} + \dots, \tag{99}
$$

where $\zeta \in (-1, 1)$ is fixed, and

$$
U_0 = -1 + \sqrt{1 - \zeta^2} - \log\left(1 + \sqrt{1 - \zeta^2}\right), \quad U_1 = \frac{1}{2} + \frac{1}{2}\log\left(1 + \sqrt{1 - \zeta^2}\right) - \frac{1}{4}\log(1 - \zeta^2),
$$

$$
U_m = V_m \ (m \geq 2).
$$

It is also easy to see that for $\ell \geq 1$, $U_{2\ell+1} + \frac{1}{\ell(2\ell+1)2^{2\ell+2}}$ belongs to $\left(1 - \zeta^2\right)^{-\frac{2\ell+1}{2}} \cdot \mathbb{Q}\left[\left(1 - \zeta^2\right)^{-\frac{1}{2}}\right]$. We omit further details of the proof of Lemma 2.

Due to Lemma 2, $a_m(\lambda, q)$, $c_m(\lambda, q)$ can be identified with their formal expansions in $\sqrt{q}$. (Indeed, these series are convergent for $2|\sqrt{q}| < |\lambda|$). Therefore, the large λ asymptotic of $M(\frac{\lambda}{\epsilon}; \frac{\sqrt{q}}{\epsilon})$ could be identified with the $\epsilon \to 0$ (double scaling) asymptotic (identification between elements in $\mathbb{Q}[[\lambda^{-1}, \epsilon, \sqrt{q}]]$). Theorem 7.A then follows from Lemma 2. In particular, the identity $\sum_{g \geq 0} \epsilon^{2g-2+k} H_k^{[g]}(\lambda_1, \dots, \lambda_k; q) = F_k(\lambda_1, \dots, \lambda_k; \epsilon, q)$ is understood as an equality between formal power series in $\epsilon, \sqrt{q}, \lambda_1^{-1}, \dots, \lambda_k^{-1}$.

To show the statement for $k = 1$, observe that

$$
H_1^*(z, s) = \frac{1 + G(z; s)}{2} \frac{\partial \log J_{z-\frac{1}{2}}(2s)}{\partial z} + \frac{1 - G(z; s)}{2} \frac{\partial \log J_{\frac{1}{2}+z}(2s)}{\partial z}
$$

where we have used (91)–(92). Then the theorem follows from Lemmata 3 and 2. $\qquad \square$

We remark that, due to Lemma 3 the asymptotic formula (94) can also be viewed as an $\epsilon \to 0$ limit, with $a = \frac{\lambda_1}{\epsilon}$, $b = \frac{\lambda_2}{\epsilon}$ and $s = \frac{\sqrt{q}}{\epsilon}$.

5.2. The $\epsilon \to \infty$ asymptotic. Proof of Theorem 7.B. First consider $k \geq 2$. For $|\epsilon| > 2|\lambda|$ we have

$$
\frac{1}{2}G\left(\frac{\lambda}{\epsilon}; \frac{\sqrt{q}}{\epsilon}\right) - \frac{1}{2} = \sum_{k=0}^{\infty} \mathcal{A}_{2k}(\lambda, q)\epsilon^{-2k}, \tag{100}
$$

$$
\frac{\sqrt{q}}{\lambda + \frac{\epsilon}{2}} \widetilde{G}\left(\frac{\lambda}{\epsilon}; \frac{\sqrt{q}}{\epsilon}\right) = \sqrt{q} \sum_{m=0}^{\infty} \mathcal{C}_m(\lambda, q) \epsilon^{-m}. \tag{101}
$$

where $\mathcal{A}_{2k}, \mathcal{C}_m$ are polynomials in q, λ. Note that the right hand sides are also the $\epsilon \to \infty$ asymptotics of the left hand sides. The polynomials $\mathcal{A}_{2k}, \mathcal{C}_m$ satisfy the following homogeneity conditions:

$$
\mathrm{gr}\,\mathcal{A}_{2k} = 2k\,\mathcal{A}_{2k}, \quad \mathrm{gr}\,\mathcal{C}_m = (m - 1)\,\mathcal{C}_m.
$$

The first several of these polynomials can be read off from

$$\frac{1}{2}G\Big(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\Big) - \frac{1}{2} = -\frac{4q}{\epsilon^2} - \frac{16q\big(\lambda^2 - \frac{1}{3}q\big)}{\epsilon^4} - \frac{64q\big(\lambda^4 - \frac{10}{27}\lambda^2 q + \frac{2}{45}q^2\big)}{\epsilon^6} + \dots,$$

$$\frac{1}{\lambda + \frac{\epsilon}{2}}\widetilde{G}\Big(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\Big) = \frac{2}{\epsilon} - \frac{4\lambda}{\epsilon^2} + \frac{8(\lambda^2 - \frac{2}{3}q)}{\epsilon^3} - \frac{16(\lambda^3 - \frac{2}{9}\lambda q)}{\epsilon^4}$$

$$+ \frac{32(\lambda^4 - \frac{20}{27}\lambda^2 q + \frac{2}{15}q^2)}{\epsilon^5} - \frac{64(\lambda^5 - \frac{20}{81}\lambda^3 q + \frac{2}{75}\lambda q^2)}{\epsilon^6} + \dots$$

where $|\epsilon| > 2|\lambda|$ is assumed. Theorem 7.B follows from (100)–(101) and the definition of H_k. The $k = 1$ statement easily follows from (32) and (31). $\qquad\square$

5.3. The $q \to 0$ asymptotic. Proof Theorem 7.C. By definition,

$$\frac{1}{2}G\Big(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\Big) - \frac{1}{2} = \frac{1}{2}\sum_{i=0}^{\infty}\binom{2i}{i}\frac{q^i}{\prod_{\ell=0}^{2i-1}\big[\lambda + (\ell - i + 1/2)\epsilon\big]} - \frac{1}{2},$$

$$\frac{\sqrt{q}}{\lambda + \frac{\epsilon}{2}}\widetilde{G}\Big(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon}\Big) = \sqrt{q}\sum_{i=0}^{\infty}\binom{2i}{i}\frac{q^i}{\prod_{\ell=0}^{2i}\big[\lambda + (\ell - i + 1/2)\epsilon\big]}.$$

So the definition itself gives the $q \to 0$ asymptotic of the entries of $M(\frac{\lambda}{\epsilon};\frac{\sqrt{q}}{\epsilon})$; the coefficients are clearly rational functions of λ, ϵ. Theorem 7.C then follows from the definition of H_k, $k \geq 2$. For the case $k = 1$, the definition of H_1 automatically gives the $q \to 0$ asymptotic, which simplified to (42). $\qquad\square$

Corollary 1. $\forall\, k \geq 2$, the following formulas hold true

$$\sum_{d\geq 0}q^d H_{k,d}(\lambda_1,\dots,\lambda_k;\epsilon) = \sum_{g\geq 0}\epsilon^{2g-2+2k}H_k^{[g]}(\lambda_1,\dots,\lambda_k;q),$$

$$\sum_{d\geq 0}q^d H_{k,d}(\lambda_1,\dots,\lambda_k;\epsilon) = \sum_{g\geq 0}\epsilon^{-2g}H_{k,[g]}(\lambda_1,\dots,\lambda_k;q).$$

Moreover, the following two identities hold true in the corresponding formal series rings:

$$\sum_{d=1}^{\infty}q^d\frac{(2d-1)!}{d!^2\prod_{j=1}^{d}\big(\lambda^2 - \frac{(2j-1)^2}{4}\epsilon^2\big)} = \sum_{g\geq 0}\epsilon^{2g}H_1^{[g]}(\lambda;q),$$

$$\sum_{d=1}^{\infty}q^d\frac{(2d-1)!}{d!^2\prod_{j=1}^{d}\big(\lambda^2 - \frac{(2j-1)^2}{4}\epsilon^2\big)} = \sum_{g\geq 0}\epsilon^{-2g}H_{1,[g]}(\lambda;q).$$

This corollary indicates that the $q \to 0$ limit connects the $\epsilon \to 0$ and the $\epsilon \to \infty$ limits.

5.4. The $q \to \infty$ asymptotic. Proof of Theorem 7.D. Recall from [33] that for any fixed value of ν, as $|y| \to \infty$ in a sector $|\arg y| \leq \pi - \delta$, the following asymptotic holds true:

$$J_\nu(y) \sim \sqrt{\frac{2}{\pi y}}\left(\cos\big(y - \frac{\pi}{2}\nu - \frac{\pi}{4}\big)\sum_{m=0}^{\infty}\frac{(-1)^m\,(\nu - 2m + \frac{1}{2})_{4m}}{(2m)!\,(2y)^{2m}}\right.$$

$$\left. - \sin\big(y - \frac{\pi}{2}\nu - \frac{\pi}{4}\big)\sum_{m=0}^{\infty}\frac{(-1)^m\,(\nu - 2m - \frac{1}{2})_{4m+2}}{(2m+1)!\,(2y)^{2m+1}}\right). \tag{102}$$

For $k \geq 2$, using (102) and (91) we have

Lemma 4. *For z fixed and $|s| \to \infty$ with $\left|\arg s\right| \leq \pi - \delta$, we have the asymptotic expansions*

$$G(z;s) \sim \frac{\cos(4s)}{\cos(\pi z)} \sum_{r=0}^{\infty} \frac{d_{2r}(z)}{s^{2r}} + \frac{\sin(4s)}{\cos(\pi z)} \sum_{r=0}^{\infty} \frac{d_{2r+1}(z)}{s^{2r+1}} - \tan(\pi z) \sum_{r=0}^{\infty} \frac{\binom{2r}{r} \prod_{j=-r}^{r}(z+j)}{2^{4r+1} \, s^{2r+1}},$$

$$\frac{s}{z+\frac{1}{2}} \widetilde{G}(z;s) \sim \frac{\sin(4s)}{\cos(\pi z)} \sum_{r=0}^{\infty} \frac{e_{2r}(z)}{s^{2r}} + \frac{\cos(4s)}{\cos(\pi z)} \sum_{r=0}^{\infty} \frac{e_{2r+1}(z)}{s^{2r+1}} - \tan(\pi z) \sum_{r=0}^{\infty} \frac{\binom{2r}{r} \prod_{j=1-r}^{r}(z+j)}{2^{4r+1} \, s^{2r}},$$

with explicitly known polynomials $d_r(z) \in \mathbb{Q}[z^2]$, $e_r(z) \in \mathbb{Q}[z(z+1)]$.

The rest of Theorem 7.D follows from the definition of H_k as well as elementary trigonometric identities. In a similar way, one proves statement for $k = 1$. □

Remark. We would like to mention the following formal solution W to equations (95)–(96):

$$W = \begin{pmatrix} \frac{1}{2} - \sqrt{-1}\,w_1(\lambda, \epsilon, q) & \sqrt{-1}\,w_2(\lambda - \epsilon, \epsilon, q) \\ -\sqrt{-1}\,w_2(\lambda, \epsilon, q) & \frac{1}{2} + \sqrt{-1}\,w_1(\lambda, \epsilon, q) \end{pmatrix} \tag{103}$$

where $w_1(\lambda, \epsilon, q) = \sum_{m=0}^{\infty} \frac{(2m-1)!! \prod_{j=-m}^{m}(\lambda+\epsilon j)}{2^{3m+2}\, m!\, q^{m+1/2}}$, $w_2(\lambda, \epsilon, q) = \sum_{m=0}^{\infty} \frac{(2m-1)!! \prod_{j=-(m-1)}^{m}(\lambda+\epsilon j)}{2^{3m+1}\, m!\, q^{m}}$. Clearly, W belongs to $\mathbb{Q}[\lambda, \epsilon][[q^{-1/2}]]$. However, analytic aspect of this formal solution is unclear to us. For example, we do not know if there exists an analytic solution satisfying (97) and with the large q asymptotic given by W. We will consider this problem in a subsequent publication.

6. Further remarks

6.1. **Bispectrality.** Bispectrality is an interesting and rare phenomenon in the theory of integrable systems. Let $\big(u_n(\mathbf{t}), v_n(\mathbf{t})\big)$ be a solution to the Toda Lattice Hierarchy, and $R_n = R_n(\lambda; \mathbf{t})$ its matrix resolvent. Denote $\mathcal{R}(\lambda; x, \mathbf{t}; \epsilon) = R_{x/\epsilon}(\lambda; \mathbf{t})$, also called the matrix resolvent. We say that the matrix resolvent has *bispectrality* if there exists a non-zero scalar function $g(\lambda; \epsilon)$ and an invertible matrix-valued function $A(\lambda; \epsilon)$ such that $g(\lambda; \epsilon)\, A(\lambda; \epsilon)\, \mathcal{R}(\lambda; x, \mathbf{0}; \epsilon)\, A(\lambda; \epsilon)^{-1}$ is a function of $h(\lambda, x)$ and ϵ only for some scalar function h. This type of bispectrality can be defined analogously to other integrable system (where in most cases ϵ can be taken to be 1 for simplicity). For the Toda Lattice Hierarchy, one might guess that the GUE [10] and the $\mathbb{P}^1$ cases are essentially (modulo some group actions) two only possible cases possessing bispectrality of the above type, but there are not enough evidences for supporting this guess. So classifying this type of bispectrality for the Toda Lattice Hierarchy seems still to be an open question. Also, bispectrality looks still mysterious. Indeed, we do not know its origin. We call the solution has the *type-I bispectrality* if the function $h(\lambda, x) = \lambda - x$. Conjecturally, the so-called "topological" solution to an integrable system always has the type-I bispectrality. We hope to study criterion of bispectrality beyond type-I in a future publication (the method given in [16] might be helpful).

Conjecture. *Let M be a semisimple (calibrated) Frobenius manifold. Assume that the integrable hierarchy of topological type of M [8, 14, 22] admits a Lax pair formalism. Then, a solution of this integrable hierarchy is topological iff its matrix resolvent possesses bispectrality of Type I. The same statement is valid for the Hodge hierarchy [9] of M.*

Note that validity of the Main Conjecture for GW invariants of $\mathbb{P}^1$ is confirmed in this paper.

Proposition 6. *The Conjecture is true for ADE singularities.*

Proof The necessity part is precisely the Lemma 4.2.3 of [5] where it is called the Key Lemma. The sufficiency part follows from the uniqueness theorem of topological ODEs [10], i.e. the space of solutions regular at infinity is equal to the rank of the simple Lie algebra. □

6.2. **Dual topological ODE.** The topological difference equation for $\mathbb{P}^1$ can be written as

$$M(z-1;s)\,A \;-\; A\,M(z;s) \;=\; z\,M(z-1;s)\,B \;-\; z\,B\,M(z;s) \tag{104}$$

with

$$A \;=\; \begin{pmatrix} \frac{1}{2} & s \\ -s & 0 \end{pmatrix}, \qquad B \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{105}$$

Definition 3. The dual topological ODE for the Toda Lattice Hierarchy associated with the solution corresponding to the GW invariants of $\mathbb{P}^1$ (within the stationary sector) is defined by

$$e^y\,\widetilde{M}\,A \;-\; A\,\widetilde{M} \;=\; e^y\Big(\widetilde{M} + \frac{d\widetilde{M}}{dy}\Big)\,B \;-\; B\,\frac{d\widetilde{M}}{dy} \tag{106}$$

where $\widetilde{M} = \widetilde{M}(y;s)$ is a matrix-valued function in y, and s is an arbitrary parameter.

Topological and dual topological equations (11), (106) are related via a Laplace type transform, i.e.

$$\widetilde{M}(y;s) \;=\; \frac{1}{2\pi i}\int_\gamma e^{z\,y}\,M(z;s)\,dz$$

where γ is an appropriate contour on the complex z plane.

6.3. **Analytic invariants of $\mathbb{P}^1$.** We have already seen that the formal series $\epsilon\,F_k(\lambda_1,\dots,\lambda_k;0,\epsilon,q)$, defined as the generating series of the GW invariants $\langle\tau_{i_1}(\phi_{\alpha_1})\cdots\tau_{i_k}(\phi_{\alpha_k})\rangle_{g,d}$ of $\mathbb{P}^1$ (in *full genera* and of *all degrees*) is not convergent as a series of ϵ, or as a (multi-)series of $\lambda_1^{-1},\dots,\lambda_k^{-1}$. However, as a power series of q, it does converge, which gives the motivation of defining the analytic k-point functions $H_k\big(\frac{\lambda_1}{\epsilon},\dots,\frac{\lambda_k}{\epsilon};\frac{q^{1/2}}{\epsilon}\big)$, such that the GW invariants are the coefficients in the full asymptotic of the double scaling limit $\epsilon\to 0$ (or of the $q\to 0$ limit) of $H_k\big(\frac{\lambda_1}{\epsilon},\dots,\frac{\lambda_k}{\epsilon};\frac{q^{1/2}}{\epsilon}\big)$. The definition of H_k is certainly natural, and provides the non-perturbative version of topological quantum field theory for $\mathbb{P}^1$. We refer to the coefficients in the $\epsilon\to\infty$ asymptotics (or again in the $q\to 0$ asymptotic but with $|\epsilon|\gg|\lambda_i|$, $i=1,\dots,k$), and in the $q\to\infty$ asymptotics of $H_k\big(\frac{\lambda_1}{\epsilon},\dots,\frac{\lambda_k}{\epsilon};\frac{q^{1/2}}{\epsilon}\big)$ as *analytic invariants* of $\mathbb{P}^1$. These invariants are counterparts of the GW invariants. For example, the first few $H_{k,[g]}$ are

$$H_{1,[0]} \;=\; 0\,, \quad H_{1,[1]} \;=\; -4q\,, \quad H_{1,[2]} \;=\; -16q\lambda^2 + \frac{8}{3}q^2\,, \quad H_{1,[3]} \;=\; -64q\lambda^4 + \frac{320}{27}q^2\lambda^2 - \frac{128}{135}q^3\,,$$

$$H_{2,[1]} \;=\; 16q\,, \quad H_{2,[2]} \;=\; 64q\big(\lambda_1^2 + \lambda_2^2\big) - \frac{256}{9}q^2\,,$$

$$H_{2,[3]} \;=\; 256q\big(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4\big) - \frac{256}{81}q^2\big(37\lambda_1^2 - 2\lambda_1\lambda_2 + 37\lambda_2^2\big) + \frac{53248}{2025}q^3\,,$$

$$H_{3,[1]} \;=\; -64q\,, \quad H_{3,[2]} \;=\; -256q\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\big) + \frac{6656}{27}q^2\,,$$

$$H_{3,[3]} \;=\; -1024q\,\big(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\big),$$

$$\qquad +\; \frac{4096}{243}q^2\Big(59(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\Big) - \frac{13027328}{30375}q^3\,,$$

where we recall that $H_{k,[g]}$ are defined in the expansion (40). These are the counterparts for $\epsilon\to\infty$.

We also list the first several $H_k^{d,m}$ and $\widetilde{H}_k^{d,m}$ for $k \geq 2$:

$$H_2^{0,0} = = \epsilon^2 \frac{\frac{1}{2} - C_1 C_2 - S_1 S_2}{(\lambda_1 - \lambda_2)^2}, \quad H_2^{1,0} = 0, \quad H_2^{1,1} = -\epsilon^2 \frac{S_1 - S_2}{4(\lambda_1 - \lambda_2)}, \quad \widetilde{H}_2^{1,1} = 0,$$

$$H_2^{2,0} = \frac{1}{32}\left[\epsilon^2 - 2\epsilon^2 S_1 S_2 - 2(\lambda_1 + \lambda_2)^2\right], \quad H_2^{2,1} = 0, \quad \widetilde{H}_2^{2,1} = \epsilon \frac{(\epsilon^2 - 2\lambda_1^2)S_2 - (\epsilon^2 - 2\lambda_2^2)S_1}{16(\lambda_1 - \lambda_2)},$$

$$H_2^{2,2} = \frac{\epsilon^2}{32}, \quad H_2^{3,0} = 0, \quad H_2^{3,1} = \frac{(\epsilon^4 - \epsilon^2(\lambda_1^2 + 2\lambda_2^2) + \lambda_2^4)S_1 - (\epsilon^4 - \epsilon^2(2\lambda_1^2 + \lambda_2^2) + \lambda_1^4)S_2}{32(\lambda_1 - \lambda_2)},$$

$$\widetilde{H}_2^{3,1} = 0, \quad H_2^{3,2} = 0, \quad \widetilde{H}_2^{3,2} = \epsilon \frac{\epsilon^2 - (\lambda_1^2 + \lambda_2^2)}{64},$$

$$H_3^{0,0} = 0, \quad H_3^{1,0} = -\epsilon^2 \frac{(\lambda_2^2 - \lambda_3^2)S_1 + (\lambda_3^2 - \lambda_1^2)S_2 + (\lambda_1^2 - \lambda_2^2)S_3}{4(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)},$$

$$H_3^{1,1} = 0, \quad \widetilde{H}_3^{1,1} = \epsilon^3 \frac{(\lambda_1 - \lambda_2)S_1 S_2 + (\lambda_2 - \lambda_3)S_2 S_3 + (\lambda_3 - \lambda_1)S_3 S_1}{4(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}.$$

Here, $C_1 = \cos\left(\frac{\pi\lambda_1}{\epsilon}\right)$, $C_2 = \cos\left(\frac{\pi\lambda_2}{\epsilon}\right)$, $S_1 = \sin\left(\frac{\pi\lambda_1}{\epsilon}\right)$, $S_2 = \sin\left(\frac{\pi\lambda_2}{\epsilon}\right)$, $S_3 = \sin\left(\frac{\pi\lambda_3}{\epsilon}\right)$. For $k = 1$, we have

$$H_1^{*,1} = 0, \quad H_1^{*,2} = \frac{2\lambda^2 - \epsilon^2}{16}, \quad H_1^{*,3} = 0, \quad H_1^{*,4} = -\frac{2\lambda^6 - 16\epsilon^2\lambda^4 + 32\epsilon^4\lambda^2 - 9\epsilon^6}{384\epsilon^2},$$

$$\widetilde{H}_1^{*,1} = \frac{\epsilon}{4}, \quad \widetilde{H}_1^{*,2} = 0, \quad \widetilde{H}_1^{*,3} = -\frac{\lambda^4 - 3\epsilon^2\lambda^2 + \epsilon^4}{32\epsilon}, \quad \widetilde{H}_1^{*,4} = 0.$$

Here, $H_k^{d,m}$ and $H_1^{*\,d}$ are defined in (43) and (43), respectively, which give the counterparts for $q \to \infty$. It will be interesting to study the Stokes phenomenon of the GW invariants by investigating the asymptotic of $H_k\left(\frac{\lambda_1}{\epsilon}, \ldots, \frac{\lambda_k}{\epsilon}; \frac{q^{1/2}}{\epsilon}\right)$ as ϵ goes to 0 within different sectors.

References

[1] Behrend, K. (1997). Gromov–Witten invariants in algebraic geometry. Inventiones Mathematicae, **127** (3), 601–617.

[2] Behrend, K., Fantechi, B. (1997). The intrinsic normal cone. Inventiones Mathematicae, **128** (1), 45–88.

[3] Bertola, M., Dubrovin, B., Yang, D. (2016). Correlation functions of the KdV hierarchy and applications to intersection numbers over $\overline{\mathcal{M}}_{g,n}$. Physica D: Nonlinear Phenomena, **327**, 30–57.

[4] Bertola, M., Dubrovin, B., Yang, D. (2016). Simple Lie algebras and topological ODEs. IMRN (2016) rnw285.

[5] Bertola, M., Dubrovin, B., Yang, D. (2016). Simple Lie algebras, Drinfeld–Sokolov hierarchies, and multi-point correlation functions. Preprint arXiv: 1610.07534.

[6] Brézin, E., Hikami, S. (2017). *Random matrix theory with an external source* (Vol. **19**). Springer.

[7] Carlet, G., Dubrovin, B., Zhang, Y. (2004). The extended Toda hierarchy. Mosc. Math. J, **4** (2), 313–332.

[8] Dubrovin, B. (1996). Geometry of 2D topological field theories. In "Integrable Systems and Quantum Groups" (Montecatini Terme, 1993). Editors: Francaviglia, M., Greco, S. Springer Lecture Notes in Math. **1620**, 120–348.

[9] Dubrovin, B., Liu, S.-Q., Yang, D., Zhang, Y. (2016). Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs. Advances in Mathematics, **293**, 382–435.

[10] Dubrovin, B., Yang, D. (2017). Generating series for GUE correlators. Letters in Mathematical Physics, **107** (11), 1971–2012.

[11] Dubrovin, B., Yang, D. (2017). On Gromov–Witten invariants of $\mathbb{P}^1$. Preprint arXiv: 1702.01669.

[12] Dubrovin, B., Yang, D., Zagier, D. (2017). Classical Hurwitz numbers and related combinatorics. Moscow Mathematical Journal, **17** (4), 601–633.

[13] Dubrovin, B., Yang, D., Zagier, D. On tau-functions of the KdV hierarchy. *to appear*.

[14] Dubrovin, B., Zhang, Y. (2001). Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. Preprint arXiv: math.DG/0108160.

[15] Dubrovin, B., Zhang, Y. (2004). Virasoro symmetries of the extended Toda hierarchy. Comm. Math. Phys., **250** (1), 161–193.

[16] Duistermaat, J.J., Grünbaum, F.A. (1986). Differential equations in the spectral parameter. Comm. Math. Phys. **69**, 177–240.

[17] Dunin-Barkowski, P., Mulase, M., Norbury, P., Popolitov, A., Shadrin, S. (2014). Quantum spectral curve for the Gromov–Witten theory of the complex projective line. Journal für die reine und angewandte Mathematik (Crelles Journal). doi 10.1515/crelle-2014-0097.

[18] Eguchi, T., Hori, K., Yang, S.-K. (1995). Topological σ-Models and Large-N Matrix Integral. International Journal of Modern Physics A, **10**, 4203–4224.

[19] Eguchi, T., Yang, S.-K. (1994). The topological CP^1 model and the large-N matrix integral. Modern Physics Letters A, **9** (31), 2893–2902.

[20] Getzler, E. (2001). The Toda conjecture. In: Symplectic Geometry and Mirror Symmetry (KIAS, Seoul, 2000). Singapore: World Scientific, pp. 51–79.

[21] Getzler, E., Okounkov, A., Pandharipande, R. (2002). Multipoint series of Gromov–Witten invariants of CP^1. Letters in Mathematical Physics, **62** (2), 159–170.

[22] Givental, A. B. (2001). Gromov–Witten invariants and quantization of quadratic Hamiltonians. Moscow Mathematical Journal, **1** (4), 551–568.

[23] Hurwitz, A. (1891). Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. Mathematische Annalen, **39** (1), 1–60.

[24] Kontsevich M., Manin, Yu. (1994). Gromov–Witten classes, quantum cohomology, and enumerative geometry. Comm. Math. Phys. **164**, 525–562.

[25] Li, J., Tian, G. (1998). Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. Journal of the American Mathematical Society, **11** (1), 119–174.

[26] Manin, Yu. (1996). *Frobenius manifolds, quantum cohomology, and moduli spaces* (Vol. **47**). AMS.

[27] Marchal, O. (2017). WKB solutions of difference equations and reconstruction by the topological recursion. Nonlinearity, **31** (1), 226–262.

[28] Norbury, P., Scott, N. (2014). Gromov–Witten invariants of $\mathbb{P}^1$ and Eynard–Orantin invariants. Geometry & Topology, **18** (4), 1865–1910.

[29] Okounkov, A., Pandharipande, R. (2006). Gromov–Witten theory, Hurwitz theory, and completed cycles. Annals of Mathematics, **163** (2), 517–560.

[30] Okounkov, A., Pandharipande, R. (2006). The Equivariant Gromov–Witten Theory of $\mathbf{P}^1$. Annals of Mathematics, **163** (2), 561–605.

[31] Okounkov, A., Pandharipande, R. (2009). Gromov–Witten theory, Hurwitz numbers, and matrix models. In Proceedings of Symposia Pure Mathematics (Vol. **80**, pp. 325–414). Editors: D. Abramovich et. al. AMS.

[32] Pandharipande, R. (2000). The Toda equations and the Gromov–Witten theory of the Riemann sphere. Letters in Mathematical Physics, **53** (1), 59–74.

[33] Watson, G. N. (1944). *A treatise on the theory of Bessel functions*. 2nd edition. Cambridge University Press.

[34] Witten, E. (1991). Two-dimensional gravity and intersection theory on moduli space. Surveys in Differential Geometry (Cambridge, MA, 1990), (pp. 243–310), Lehigh Univ., Bethlehem, PA.

[35] Zhang, Y. (2002). On the CP^1 topological sigma model and the Toda lattice hierarchy. Journal of Geometry and Physics, **40** (3), 215–232.

[36] Zhou, J. (2015). Emergent geometry and mirror symmetry of a point. Preprint arXiv: 1507.01679.

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