A Root System for the Lyons Group

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0. Introduction

Sims [1973] proved the existence and uniqueness of the sporadic group Ly predicted by Lyons [1972] through the computer-aided construction of a presentation which, unfortunately, is rather cumbersome and does not lead to an insight into the structure of the group.

Meanwhile, much more information on Ly has become available.

Kantor [1981] found a Tits geometry for Ly which is "almost" a building. Meyer, Neutsch and Parker [1985] gave the absolute minimal representation of Ly (111-dimensional over F_5). Later, Wilson [1984,1985] compiled the list of all maximal subgroups in Ly. His investigation uses the minimal representation explicitly, while the verification of the latter depends on Sims' presentation. For that reason, it would be of great interest to have a simpler existence and uniqueness proof.

Inspired by Kantor's results, we were led to the idea of giving a more symmetric presentation for Ly by making use of its beautiful geometry.

Its properties almost immediately follow from simple considerations of several subgroups, such as $G_2(5)$ or $2^\wedge A_{11}$.

Our relations are shown to be fulfilled by certain generators ("roots") of the Lyons group, and most probably they define Ly itself.

The whole reasoning is carried through without invoking any deep theorems or technicalities.

The geometric spirit of our presentation renders this possible. It is a first step towards an understanding of the Lyons group.

1. Relations in a group of Ly type

We say a group Λ is of Ly type if it has the following properties:

(1) Λ is simple;

(2) Λ contains an involution z with $C_\Lambda(z) \cong 2\hat{}A_{11}$.

Lyons [1972] shows that a group fulfilling (1) and (2) is of order

(1.1) $\quad |Ly| = 2^8.3^7.5^6.7.11.31.37.67$

and that it contains a unique conjugacy class of subgroups $\cong G_2(5)$.
Let Λ_o be one of them and B a Borel subgroup of Λ_o, i.e. a 5-Sylow normalizer.
Then B is also a Borel group in Λ.
Furthermore, let $T \cong 4^2$ be a (maximal) torus in B, N_o and N its normalizers in Λ_o and Λ, respectively, and $W_o = N_o/T$ and $W = N/T$ the corresponding Weyl groups.
From the theory of Chevalley groups, cf. e.g. Carter [1972], we deduce

(1.2) $\quad W_o \cong D_{12} \cong S_3 x S_2$

while Kantor [1981] shows

(1.3) $\quad W \cong S_4 x S_3$

A proper subgroup of Λ or Λ_o which contains a Borel group will be called parabolic.
Kantor [1981] has shown that Λ contains exactly three conjugacy classes of maximal parabolic groups. They can be associated with the points P, lines L and planes F of a Tits geometry with the Buekenhout diagram

```
                 o-----------o-----6-----o
(1.4)            5           5           5
                 P           L           F
```

$G_2(5) \qquad 5^{1+4}:4S_6 \qquad 5^3.SL_3(5)$

Two objects (points, lines or planes) are called incident with each other if their intersection (as groups) is parabolic.
The apartment A(T) associated with T is the set of all objects fixed by

T. A(T) is a subgeometry with Buekenhout diagram

```
(1.5)          o-----------o-----6-----o
               1           1           1
```

and may be represented (Kantor [1981]) by a simplicial complex A of dimension 2:

```
                 1c -------- 2a -------- 1b
                 / \         / \         / \
                /   \       /   \       /   \
         4a -------- 3b -------- 4c -------- 3a
         / \         / \         / \         / \
        /   \       /   \       /   \       /   \
  1b -------- 2c -------- 1a -------- 2b -------- 1c
        \   /       \   /       \   /       \   /
         \ /         \ /         \ /         \ /
         3a -------- 4b -------- 3c -------- 4a
            \       /   \       /   \       /
             \     /     \     /     \     /
                 1c -------- 2a -------- 1b
```

Fig. 1

Here the 0-, 1-, 2-simplices correspond to the 12 points, 36 lines, 24 planes of the apartment, respectively.

The Weyl group $W \cong S_4 x S_3$ acts as S_4 on the numbers {1,2,3,4} and as S_3 on the letters {a,b,c}.

In analogy to Chevalley theory we now define the root groups associated with T as the groups X obeying the conditions:

(1) $X \cong (F_5,+) \cong 5$;

(2) $T \lneq N(X)$.

It follows from the known structure of $C_{Ly}(5B)$ (Lyons [1972]) that all root groups must be generated by 5A-elements.

Since $C_{Ly}(5A) \cong 5^{1+4}{:}(2^\wedge A_6)$ does not contain a Klein four group, $C_T(X) \cong 4$.

The Λ-normalizer of a 5A-group is a line. Thus there is a natural bijection between the root groups and the lines in A(T).

The extension of T with the commutator subgroup

(1.6) $\quad W' \cong A_4 x A_3$

of W splits, so there is a unique element k of order 3 in N which corresponds to the Weyl element (abc) and centralizes T. In fact (Lyons [1972]),

(1.7) $\quad C_\Lambda(T) = T \times K$

with

(1.8) $\quad K = \langle k\rangle$

Furthermore, there is a set of 16 complements of T in $T{:}A_4$. These groups are evidently conjugate under T, so we may elect an arbitrary one of them and denote it by Ω.

Ω is generated by 4 elements ω_i ($1 \leq i \leq 4$) which correspond to the 120^o - rotations with centres in the points whose names contain the number i.

Each ω_i is uniquely determined by the choice of Ω and the corresponding Weyl permutation, namely

(1.9) $\quad \omega_1$ --> (234); ω_2 --> (143); ω_3 --> (124); ω_4 --> (132) in W

The group

(1.10) $\quad \langle\omega_i, k\rangle = \Omega \times K \cong A_4 \times A_3$

(one of just 16 complements of T in $N' = T{:}W' \cong 4^2{:}(A_4 x A_3)$) is represented as a regular permutation group on the root groups X_L.

This allows to specify a set of 36 generators ("roots") for each of the 36 X_L.

We are free to take any generator for one of them, e.g. X(1a,2b). Call it x(1a,2b). Then apply $\Omega \times K$ to this root to define the remaining ones. A complete system of 36 root elements generated in this way will be called a standard (root) system.

Without restriction of generality we may assume that the following relations hold (the exact exponents depend on the choice of $x(1a,2b)$, but this clearly does not matter, since all allowed possibilities are equivalent because they lead to the same group):
It will be convenient to define an orientation of the lines in A according to the rules

(1.11) $\qquad a \dashrightarrow b, \qquad b \dashrightarrow c, \qquad c \dashrightarrow a$

Now we consider a point P in A.
The 6 lines incident with P form a complete set of long roots for the stabilizer A(P) of P, isomorphic to the Chevalley group $G_2(5)$, while the short roots are given by the sides of the (small) hexagon with centre P spanned by the long roots.
We denote the long and short roots by L_i and K_i ($i \in F_7^x$), respectively, in the following manner:

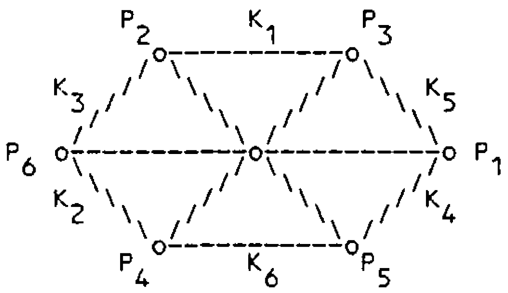

Fig. 2

where L_i points from P to P_i for i = 1,2,4 (squares in F_7) and from P_i to P for i = 3,5,6 (non-squares).
The 12 roots in fig. 2 follow each other in the same order as they do in the standard G_2 root system.
Then the nontrivial Chevalley relations are

(1.12) $\qquad [L_i, L_{2i}] = L_{3i}^4$

(1.13) $\qquad [K_i, K_{3i}] = L_{2i}^3$

$$(1.14)\qquad [K_i,K_{2i}] = L_{2i}^3 K_{3i}^3 L_{6i}^2$$

$$(1.15)\qquad [K_i,L_{4i}] = L_{2i}^4 K_{3i} L_{6i}^4 K_{2i}$$

$$(1.16)\qquad [L_i,K_{3i}] = K_{5i} L_{3i} K_i^4 L_{2i}^4$$

combined with the information that for all $i \in \{1,2,4\}$ the mappings

$$(1.17)\qquad K_i \implies \begin{vmatrix} 1 & 1 \\ . & 1 \end{vmatrix} \qquad\qquad K_{-i} \implies \begin{vmatrix} 1 & . \\ 4 & 1 \end{vmatrix}$$

and

$$(1.18)\qquad L_i \implies \begin{vmatrix} 1 & 1 \\ . & 1 \end{vmatrix} \qquad\qquad L_{-i} \implies \begin{vmatrix} 1 & . \\ 3 & 1 \end{vmatrix}$$

are isomorphisms from $\langle K_i,K_{-i}\rangle$ and $\langle L_i,L_{-i}\rangle$ onto $SL_2(5)$.
It should be noted that our relations differ slightly from those described, e.g., in Humphreys [1975]. This is due to our more symmetric choice of the roots which is more convenient in the context of the Lyons group.
For later reference, we construct an explicit $G_2(5)$ root system in the 7-dimensional minimal representation over F_5 ("reduced octave" algebra = "septime" algebra with the skew-symmetric product given by

$$(1.19)\qquad e_{i+1}\cdot e_{i+2} = e_{i+4}$$

and the cyclically permuted formulas).
Up to conjugacy in $G_2(5)$, our matrices are uniquely determined:

$$(1.20)\quad L_1 = \begin{vmatrix} 1&.&.&.&.&.&. \\ .&1&.&.&.&.&. \\ .&.&1&.&3&1&. \\ .&.&.&1&.&.&. \\ .&.&2&.&1&.&1 \\ .&.&4&.&.&1&2 \\ .&.&.&.&4&3&1 \end{vmatrix} \qquad L_2 = \begin{vmatrix} 1&.&.&.&.&.&. \\ .&1&.&.&2&1&. \\ .&.&1&.&.&.&. \\ .&.&.&1&4&2&. \\ .&3&.&1&1&.&. \\ .&4&.&3&.&1&. \\ .&.&.&.&.&.&1 \end{vmatrix} \qquad L_4 = \begin{vmatrix} 1&.&.&.&.&.&. \\ .&1&3&.&.&.&1 \\ .&2&1&1&.&.&. \\ .&.&4&1&.&.&3 \\ .&.&.&.&1&.&. \\ .&.&.&.&.&1&. \\ .&4&.&2&.&.&1 \end{vmatrix}$$

$$(1.21)\quad L_6 = \begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \bar{1} & 3 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot & \bar{1} & \cdot & 3 \\ \cdot & \cdot & 2 & \cdot & \cdot & \bar{1} & 4 \\ \cdot & \cdot & \cdot & \cdot & 2 & \bar{1} & 1 \end{vmatrix} \qquad L_5 = \begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \cdot & \cdot & 4 & 3 & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{1} & 2 & 4 & \cdot \\ \cdot & \bar{1} & \cdot & 3 & \bar{1} & \cdot & \cdot \\ \cdot & 2 & \cdot & \bar{1} & \cdot & \bar{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{1} \end{vmatrix} \qquad L_3 = \begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bar{1} & \bar{1} & \cdot & \cdot & \cdot & 3 \\ \cdot & 4 & \bar{1} & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \bar{1} & \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \bar{1} & \cdot \\ \cdot & 2 & \cdot & 4 & \cdot & \cdot & \bar{1} \end{vmatrix}$$

$$(1.22)\quad K_1 = \begin{vmatrix} 4^1 & 3^1 & \cdot & 4^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \bar{1} & 3 & \cdot \\ 3 & 4 & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot & \bar{1} & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot & \cdot & \bar{1} & 1 \\ \cdot & \cdot & \cdot & \cdot & 3 & 4 & 1 \end{vmatrix} \qquad K_2 = \begin{vmatrix} 1 & 1 & 1 & \cdot & 4 & 2 & 2 \\ 4 & \cdot & 3 & \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & \bar{1} & 2 & \bar{1} & \cdot \\ \cdot & \bar{1} & \cdot & 3 & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & 4 & \cdot & \bar{1} & \cdot \\ 3 & \cdot & 4 & \cdot & \cdot & \cdot & 4 \end{vmatrix} \qquad K_4 = \begin{vmatrix} 1 & 1 & 1 & \cdot & 1 & 2 & 3 \\ \cdot & 4 & \bar{1} & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 & 1 & \cdot & \cdot & 4 \\ 4 & \cdot & \cdot & \cdot & 3 & 4 & \cdot \\ 3 & \cdot & \cdot & \cdot & 4 & 4 & \cdot \\ \cdot & 2 & \cdot & \bar{1} & \cdot & \cdot & \bar{1} \end{vmatrix}$$

$$(1.23)\quad K_6 = \begin{vmatrix} 4^1 & 3^1 & \cdot & \bar{1}^3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & 4 & 3 & \cdot \\ 2 & \bar{1} & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{1} & \cdot & \bar{1} & \cdot & 2 \\ \cdot & \cdot & 2 & \cdot & \cdot & \bar{1} & 4 \\ \cdot & \cdot & \cdot & \cdot & 3 & \bar{1} & 1 \end{vmatrix} \qquad K_5 = \begin{vmatrix} 1 & 1 & 1 & \cdot & 1 & 2 & 3 \\ 4 & \cdot & 3 & \cdot & \cdot & \cdot & \bar{1} \\ \cdot & \cdot & \cdot & \bar{1} & 2 & 4 & \cdot \\ \cdot & 4 & \cdot & 3 & 1 & \cdot & \cdot \\ \cdot & 3 & \cdot & 1 & \cdot & \bar{1} & \cdot \\ 2 & \cdot & \bar{1} & \cdot & \cdot & \cdot & 4 \end{vmatrix} \qquad K_3 = \begin{vmatrix} 1 & 1 & 4 & \cdot & 1 & 3 & 3 \\ \cdot & \bar{1} & 1 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 & 1 & \cdot & \cdot & \bar{1} \\ 4 & \cdot & \cdot & \cdot & 3 & \bar{1} & \cdot \\ 2 & \cdot & \cdot & \cdot & 1 & 4 & \cdot \\ \cdot & 2 & \cdot & 4 & \cdot & \cdot & \bar{1} \end{vmatrix}$$

Conjugation with $\Omega \times K$ merely permutes the roots (without exponents). Because of this fact, all standard systems are equivalent and lead to the same set of relations.

The lines in $A(T)$ form 3 parallel classes of 12 lines each. Every parallel class splits into 2 connected components called (great) circles.

A special line pair is a pair (L,L') of lines which are contained in a great circle and either have one point in common ("long" pair) or are mutual antipodes ("short" pair). The reason for this notation is that short and long special pairs form opposite pairs of short and long roots, respectively, in a certain $G_2(5)$ subgroup.

Since two opposite long root groups in $G_2(5)$ have the same centralizer in T, this must also be true for all 6 lines in a great circle.

The centralizer of an arbitrary T-involution z in the $A(T)$-point P contains 4 roots in each parallel class. Thus the group $\langle\Pi\rangle$ generated by a parallel class Π is a subgroup of $H = C_\Lambda(z) \cong 2^{\wedge}A_{11}$.

Let $\bar{H} = H/\langle z\rangle \cong A_{11}$.

As all subgroups isomorphic to 4^2 are conjugate in H we may assume without restriction that

$$(1.24)\qquad \bar{T} = T/\langle z\rangle = \langle(1234)(5678),(1234)(8765)\rangle$$

In H or $\bar{H}$ exactly 12 groups $\cong 5$ exist which are normalized by T or $\bar{T}$. Except for a permutation of the letters $\{1,2,3,4,5,6,7,8,9,X,E\}$ normalizing T only the following correspondence between the roots and the permutations in $\bar{H}$ is allowed by the Chevalley relations for the points:

```
                o --(E5867)-- o --(98576)-- o

         o --(94132)-- o --(X1423)-- o --(E4132)-- o

  o --(X5867)-- o --(E8576)-- o --(95867)-- o --(X8576)-- o

         o --(91423)-- o --(X4132)-- o --(E1423)-- o

                o --(E5867)-- o --(98576)-- o
```

Fig. 3

It is obvious that these roots generate $2^\wedge A_{11}$.

2. Definition and simple geometric properties of the group Γ

The results of section 1 lead us to the definition of a group Γ as follows:

Γ is generated by 36 elements x_L, bijectively associated with the lines L in A (fig. 1). The defining relations of Γ are

(1) C(P)-relations for every point P in A;

(2) S(Π)-relations for each parallel class Π in A.

Here C(P) is the set of Chevalley relations of $G_2(5)$ (cf. section 1) while S(Π) is an arbitrary system of defining relations for $2^\wedge A_{11}$. Best suited for our purpose are the Schur relations:

$$t_i^3 = 1 \qquad (1 \leq i \leq 9) \tag{2.1}$$

$$(t_i . t_j)^2 = z \qquad (1 \leq i,j \leq 9 ;\ i \neq j) \tag{2.2}$$

$$z^2 = 1 \tag{2.3}$$

where the generator t_i corresponds to the permutation (iXE).
We are now able to translate between the two sets of generators of $2^\wedge A_{11}$ (here x(P,Q) is the root element which belongs to the line connecting the points P and Q):

$$t_9 = x(3c,4a).x(4b,3c)^{-1}.x(3a,4b).x(4a,3b)^{-1}.x(3a,4b).x(4b,3c).x(3c,4a)^{-1} \tag{2.4}$$

$$t_1 = x(3a,4b)^{-1}.t_9.x(3a,4b) \qquad t_2 = x(3a,4b)^2.t_9.x(3a,4b)^{-2} \tag{2.5}$$

$$t_3 = x(3a,4b).t_9.x(3a,4b)^{-1} \qquad t_4 = x(3a,4b)^{-2}.t_9.x(3a,4b)^2 \tag{2.6}$$

$$t_5 = x(1a,2b)^{-1}.t_9.x(1a,2b) \qquad t_6 = x(1a,2b)^2.t_9.x(1a,2b)^{-2} \tag{2.7}$$

$$t_7 = x(1a,2b).t_9.x(1a,2b)^{-1} \qquad t_8 = x(1a,2b)^{-2}.t_9.x(1a,2b)^2 \tag{2.8}$$

The reverse transformations are:

$$x(3a,4b) = t_9^{-1}.t_1^{-1}.t_4.t_2^{-1}.t_3.t_1.t_9 \qquad x(1a,2b) = t_9^{-1}.t_5^{-1}.t_8.t_6^{-1}.t_7.t_5.t_9 \tag{2.9}$$

(2.10) $x(4b,3c) = t_2.t_4.t_1^{-1}.t_3.t_2^{-1}$ $\qquad$ $x(2b,1c) = t_6.t_8.t_5^{-1}.t_7.t_6^{-1}$

(2.11) $x(3c,4a) = t_1^{-1}.t_4.t_2^{-1}.t_3.t_1$ $\qquad$ $x(1c,2a) = t_5^{-1}.t_8.t_6^{-1}.t_7.t_5$

(2.12) $x(4a,3b) = t_9^{-1}.t_4^{-1}.t_1.t_3^{-1}.t_2.t_4.t_9$ $\qquad$ $x(2a,1b) = t_9^{-1}.t_8^{-1}.t_5.t_7^{-1}.t_6.t_8.t_9$

(2.13) $x(3b,4c) = t_3.t_1.t_4^{-1}.t_2.t_3^{-1}$ $\qquad$ $x(1b,2c) = t_7.t_5.t_8^{-1}.t_6.t_7^{-1}$

(2.14) $x(4c,3a) = t_4^{-1}.t_1.t_3^{-1}.t_2.t_4$ $\qquad$ $x(2c,1a) = t_8^{-1}.t_5.t_7^{-1}.t_6.t_8$

By the main result of Meyer / Neutsch / Parker [1985] the Lyons group possesses a 111-dimensional irreducible representation over F_5. In this we can easily identify 36 elements which generate Ly and satisfy all relations defining Γ.
Hence we have

<u>Lemma 1:</u>

(a) The Lyons group is a homomorphic image of Γ;

(b) Γ has a 111-dimensional nontrivial representation over F_5.

Let us now consider the following subconfigurations of A in fig. 1:

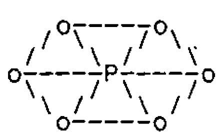

Fig. 4.a

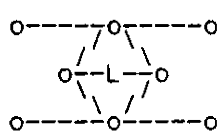

Fig. 4.b

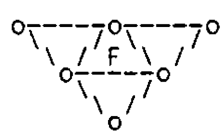

Fig. 4.c

The subgroups of Γ generated by the lines in fig. 4.a; 4.b; 4.c (respectively) are called $\Gamma(P)$; $\Gamma(L)$; $\Gamma(F)$.
For any set $\{O,O',\ldots\}$ of objects we define $\Gamma(O,O',\ldots)$ as the intersec-

tion of the groups $\Gamma(O)$, $\Gamma(O')$, ...
With the above notation we have

<u>Theorem 1:</u>

(a) $\Gamma(P) \cong G_2(5)$;

(b) $\Gamma(L) \cong 5^{1+4}:(2^\wedge A_6)$;

(c) $\Gamma(F) \cong 5^3.SL_3(5)$.

<u>Theorem 2:</u>

The lines in the following configurations (omitting the dotted lines):

(a):

```
o-----o
```

(b):

```
   /o\
  /   \
 o-----o
```

(c):

```
   /o\
  /   \
 o-----o
  \   /
   \o/
```

(d):

```
o-----o-----o
 .  .   .  .
  o:.....:o
 .  .   .  .
o-----o-----o
```

(e):

```
o-----o-----o
 .  .   .  .
  o-------o
 .  .   .  .
o-----o-----o
```

(f): great circle

(g): parallel class

generate groups which are isomorphic to:

(a): 5;

(b): 5^3;

(c): 5^{1+4};

(d): $2^\wedge A_6$;

(e): $5\times(2^\wedge A_6)$;

(f): $2^\wedge A_7$;

(g): $2^\wedge A_{11}$.

<u>Proof of theorems 1 and 2:</u>

(1a): Due to the C(P)-relations, $\Gamma(P)$ is a homomorphic image of $G_2(5)$ (using a theorem of Steinberg, cf. Carter [1972], theorem 12.1.1), so $\Gamma(P)$ is $\cong G_2(5)$ or $\cong 1$.

In the latter case a root group in $\Gamma(P)$ and hence also in $\Gamma(P')$ for a neighbouring point P′ of P would be trivial, so $\Gamma(P') = 1$, too. This leads to $\Gamma = 1$, contradicting Lemma 1.

(2g): Because of the $S(\Pi)$-relations, $\langle\Pi\rangle$ is a homomorphic image of $2^\wedge A_{11}$, so is $\cong 2^\wedge A_{11}$, A_{11} or 1. Only the first possibility is in conformity with (1a), since a long special line pair generates $SL_2(5) \cong 2^\wedge A_5$.

The remaining statements in theorem 2 now follow immediately from (1a) and (2g).

(1b): Since (2c) and (2d) hold, we need only show that 5^{1+4} is normalized by $2^\wedge A_6$. This follows from (1a), applied to the two points incident with L.

(1c): Analogously to (1b), we conclude with the help of (2b) that the three lines incident with F generate a normal subgroup $\Gamma_o(F) \cong 5^3$ of $\Gamma(F)$.

The images in $\Gamma(F)/\Gamma_o(F)$ of the root subgroups in $\Gamma(F)$ fulfill all of the Chevalley relations for the group $SL_3(5)$ (which is defined by these relations) if we map them as follows:

```
o\ --a-- /o\ --A-- /o                     o\      /o
  \C   / /F\ \   b/                         \C  b/
   \ /        \ /       ===>       o --a-- /o\ --A-- o
    o\--------/o                           B/  \c
      \c    B/                            o/     \o
        \o /
```

Fig. 5

$SL_3(5)$ is simple, and according to (2c) not all of the images can be trivial; thus $\Gamma(F) \cong 5^3.SL_3(5)$. This extension does not split, since $\Gamma(F)$ contains a 5-Sylow subgroup of $G_2(5)$ and therefore elements of order 25. This establishes theorems 1 and 2.

We now define for an arbitrary (long or short) special line pair L,L' the groups $T_{LL'}$ and $Q_{LL'}$ as follows:
Let $T_{LL'}$ be the common normalizer of the root groups L and L' and $Q_{LL'}$ the set normalizer of $\{L,L'\}$ in $\langle L,L'\rangle \cong SL_2(5)$.
Furthermore, for each great circle K and each parallel system Π we introduce the abbreviations

(2.15) $\quad T_K = \langle T_{LL'} : L,L' \text{ special line pair in } K \rangle$

(2.16) $\quad Q_K = \langle Q_{LL'} : L,L' \text{ special line pair in } K \rangle$

(2.17) $\quad T_\Pi = \langle T_{LL'} : L,L' \text{ special line pair in } \Pi \rangle$

(2.18) $\quad Q_\Pi = \langle Q_{LL'} : L,L' \text{ special line pair in } \Pi \rangle$

as well as

(2.19) $\quad T = \langle T_{LL'} : L,L' \text{ special line pair} \rangle$

(2.20) $\quad Q = \langle Q_{LL'} : L,L' \text{ special line pair} \rangle$

and for any point P:

(2.21) $\quad T(P) = \langle T_{LL'} : L,L' \text{ special line pair in } \Gamma(P) \rangle$

Of course, $T(P)$ is the standard torus in $\Gamma(P) \cong G_2(5)$.
We then have

Theorem 3:

(a) For each special line pair L,L' in the great circle K is $T_{LL'} = T_K \cong 4$;

(b) for every parallel class Π is $T_\Pi \cong 4x2$;

(c) for all points P is $T(P) = T \cong 4^2$.

Proof:

(a) and (b) follow from an easy calculation in $<\Pi> \cong 2\hat{}A_{11}$. Trivially, we have $T(P) \leq T$. With (a) we deduce for every great circle K with an arbitrary but fixed P that $T_K \leq T(P)$. Since $<T_K> = T$, we get (c).

Theorem 4:

(a) For all special line pairs L,L': $Q_{LL'} = N_{<L,L'>}(T_{LL'}) \cong Q_8$, the quaternion group of order 8; the intersection of T with $Q_{LL'}$ is $T_{LL'}$;
(b) T is a normal subgroup of Q;
(c) each element q of Q permutes the lines of A, inducing an automorphism of A as a simplicial complex;
(d) the image of this action is the full automorphism group $S_4 x S_3$ of A.

Proof:

The first part of (a) is immediate since $<L,L'> \cong SL_2(5)$. The second part can be verified in $\Gamma(P)$ for an appropriate point P.

In this $\Gamma(P)$ we also see that $Q_{LL'}$ normalizes $T(P) = T$, thus the same holds true for $Q = <Q_{LL'}>$. Furthermore, each $T_{LL'}$ is contained in $Q_{LL'}$, hence in Q; so $T = <T_{LL'}>$ is a subgroup of Q. This proves (b).

Let q be an element of $Q_{LL'}$. If q is contained in $T_{LL'} < T$, (c) holds trivially. If q is in $Q_{LL'} \backslash T_{LL'}$, q induces a permutation of the groups of order 5 which are normalized by T in each of the groups $\Gamma(P)$ and $<\Pi>$ where P is any point with $L,L' < \Gamma(P)$ and Π the parallel system containing L and L'. But all these groups of order 5 are root groups.

From the $C(P')$-relations for appropriate points P' we find that the 16 remaining roots are also permuted. Inspection of the permutations generated by Q easily leads to (c) and (d).

3. Some geometric subgroups of Γ

Let Π be a parallel class and P a point in A. The group $H = \langle\Pi\rangle$ is $\cong 2^{\wedge}A_{11}$ by theorem 2.g. We denote the unique involution in Z(H) by z.

We now prove

Theorem 5:

The intersection of H and $\Gamma(P)$ is $C_{\Gamma(P)}(z) \cong (1/2).2^{\wedge}(S_5 x S_5)$.

Proof:

Since all pairs (Π,P) are equivalent under Q (theorem 4.d), we may restrict ourselves to the case P = 1a and Π = parallel system of fig. 3. Then H and $\Gamma(P)$ obviously contain the 4 roots x(1a,2b), x(2c,1a), x(3b,4c), x(4b,3c) which generate a group $SL_2(5) \mathsf{Y} SL_2(5) \cong 2^{\wedge}(A_5 x A_5)$ of index 2 in $C_{\Gamma(P)}(z) \cong (1/2).2^{\wedge}(S_5 x S_5)$. This group is enlarged by T – T < H and T < $\Gamma(P)$ because of theorem 3.c – to the full centralizer of z in $\Gamma(P)$. As z is in the centre of H, the intersection of H and $\Gamma(P)$ is a subgroup of $C_{\Gamma(P)}(z)$; hence the proposition.

We want to consider several groups which are defined symmetrically with respect to the apartment A(T).
Let

(3.1) $U_1 = \Gamma(1a,1b,1c)$ $U_2 = \Gamma(2a,2b,2c)$

(3.2) $U_3 = \Gamma(3a,3b,3c)$ $U_4 = \Gamma(4a,4b,4c)$

and

(3.3) $U = \langle U_1, U_2, U_3, U_4\rangle$

It will be convenient to have a systematic notation for the circles, parallel systems and corresponding $2^{\wedge}A_{11}$-subgroups in Γ:

We denote the circle containing the points with numbers i and j by K_{ij}

and the parallel system consisting of K_{ij} and K_{kl} by $\Pi_{ij.kl}$.
The corresponding T-involution will be called $z_{ij.kl}$, and we set $H_{ij.kl} = \langle\Pi_{ij.kl}\rangle$.
Hence the torus elements $z_{12.34}$, $z_{13.24}$, $z_{14.23}$ are canonically associated with the double transpositions in the symmetric group S_4, while the circles K_{12}, K_{13}, K_{14}, K_{23}, K_{24}, K_{34} belong to the transpositions of S_4.

Let us now investigate the groups U_i ($1 \leq i \leq 4$) and U:

<u>Theorem 6:</u>

(a) $U_1 \cong U_2 \cong U_3 \cong U_4 \cong U_3(3)$;

(b) $U' = U$; $U/Z(U) \cong U_4(3) \cong O_6^-(3)$; $Z(U) \leq 4x3^2$.

<u>Proof:</u>

We define

(3.4)	$a = x(4b,3c)^4 x(3b,4c)^2$	==>	(132)	in	$H_{12.34}$
(3.5)	$b = x(4b,3c)^1 x(3b,4c)^3$	==>	(143)	in	$H_{12.34}$
(3.6)	$c = x(4b,2c)^1 x(2b,4c)^3$	==>	(124)	in	$H_{13.24}$
(3.7)	$r = x(1a,2b)^1 x(2a,1b)^3$	==>	(568)	in	$H_{12.34}$

$\Gamma(1a)$, $\Gamma(1b)$, $\Gamma(1c)$ contain the $2^\wedge A_5$-groups $\langle x(3b,4c),x(4b,3c)\rangle$, $\langle x(3c,4a),x(4c,3a)\rangle$, $\langle x(3a,4b),x(4a,3b)\rangle$ of $H_{12.34}$, acting on the sets {1,2,3,4,X}, {1,2,3,4,E}, {1,2,3,4,9}, respectively.
Their intersection, the $2^\wedge A_4$-group on {1,2,3,4}, is thus contained in $\Gamma(1a,1b,1c) = U_1$.
Obviously, analogous results for $H_{13.24}$ and $H_{14.23}$ hold.
Hence, by (3.4),(3.5),(3.6),

(3.8) $\quad \langle a,b,c\rangle \leq U_1 = \Gamma(1a,1b,1c) \leq \Gamma(1a)$

In $\Gamma(1a)$ we easily verify - see (1.20),...,(1.23) - that

(3.9) $a^3 = b^3 = c^3 = 1$

(3.10) $aba = bab,\ aca = cac,\ bcb = cbc$

(3.11) $a^b c^{-1} a^b = c^{-1} a^b c^{-1};\ b^a c^{-1} b^a = c^{-1} b^a c^{-1}$

These relations form a presentation of the finite simple group $U_3(3)$, cf. Aschbacher and Hall [1973].

Since $\langle a,b,c\rangle$ is nontrivial, we deduce

(3.12) $U_3(3) \cong \langle a,b,c\rangle \lneq U_1 \lneq \Gamma(1a) \cong G_2(5)$

By inspection of the maximal subgroups of $G_2(5)$ we are left with three candidates for U_1, namely $\langle a,b,c\rangle \cong U_3(3)$, $N_{\Gamma(1a)}(\langle a,b,c\rangle) \cong G_2(2)$ and $\Gamma(1a) \cong G_2(5)$.

But, by theorem 5, $C_{U_1}(z_{12.34})$ is the intersection of U_1 and $H_{12.34}$, hence $C_{U_1}(z_{12.34}) = \langle a,b,T\rangle \cong 4S_4$.

$G_2(2)$ and $G_2(5)$ do not contain involution centralizers of this form, so

(3.13) $U_1 = \langle a,b,c\rangle \cong U_3(3)$

Since U_1, U_2, U_3, U_4 are conjugate to each other under Q, (a) follows. r and c are both contained in $\Gamma(3a)$ where we immediately establish the relations

(3.14) $r^3 = 1, \qquad rcr = crc$

while in $H_{12.34} \cong 2^\wedge A_{11}$ the elements a and b evidently commute with r:

(3.15) $ra = ar, \qquad rb = br$

By a result of Aschbacher and Hall [1973] the relations (3.9),(3.10), (3.11),(3.14),(3.15) form a presentation of the full Schur cover of the finite simple group $U_4(3) \cong O_6^-(3)$, so with the abbreviation

(3.16) $U_o = \langle a,b,c,r\rangle$

we get (because $U_4(3)$ is simple and $U_0 \neq 1$)

$$(3.17) \qquad U_0' = U_0, \qquad U_0/Z(U_0) \cong U_4(3)$$

and $Z(U_0)$ is a factor of the Schur multiplier $12\text{x}3 \cong 4\text{x}3^2$ of $U_4(3)$.
To complete the proof of our theorem it remains to show that $U_0 = U$.
First we have $a,b,c \in U_1 \leq U$ and $r \in U_3 \leq U$, so $U_0 \leq U$.
The reverse inequality amounts to $U_i \leq U_0$ for all $i \in \{1,2,3,4\}$.
Clearly this is true for $U_1 = \langle a,b,c\rangle$.
The intersection of U_1 and U_3 contains the torus T as well as c.
Since $\langle c,T\rangle \cong 4S_4$ is maximal in U_3 and centralizes $z_{13.24}$, while $r \in U_3$ does not, we get

$$(3.18) \qquad U_3 = \langle c,T,r\rangle \leq U_0$$

Let now $i = 2$ or 4. The intersections of U_i with U_1 and U_3 are different maximal subgroups ($\cong 4S_4$) of $U_i \cong U_3(3)$ and therefore they together generate U_i. Since they are contained in $\langle U_1,U_3\rangle \leq U_0$, this completes the proof of the required equality

$$(3.19) \qquad U = \langle a,b,c,r\rangle$$

at the same time establishing the theorem.
Having chosen a suitable unitary basis, the matrices in $SU_4(3)$ corresponding to the elements a,b,c,r are found to be

$$(3.20) \qquad a = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-i & -1+i \\ 0 & 0 & 1+i & 1+i \end{array}\right| \qquad b = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-i & 1-i \\ 0 & 0 & -1-i & 1+i \end{array}\right|$$

$$(3.21) \qquad c = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1-i & 0 & -1+i \\ 0 & 0 & 1 & 0 \\ 0 & 1+i & 0 & 1+i \end{array}\right| \qquad r = \left|\begin{array}{cccc} 1+i & -1-i & 0 & 0 \\ 1-i & 1-i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right|$$

where i is a square root of -1.
The matrices in $SO_6^-(3)$ are given by

$$(3.22) \qquad a = \left|\begin{array}{cccccc} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & 1 & 2 & 2 \\ . & . & 2 & 1 & 1 & 2 \\ . & . & 1 & 2 & 1 & 2 \\ . & . & 1 & 1 & 1 & 1 \end{array}\right| \qquad b = \left|\begin{array}{cccccc} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & 1 & 1 & 1 \\ . & . & 2 & 1 & 2 & 1 \\ . & . & 2 & 1 & 1 & 2 \\ . & . & 2 & 2 & 1 & 1 \end{array}\right|$$

$$(3.23) \qquad c = \left|\begin{array}{cccccc} 1 & 2 & . & . & 1 & 1 \\ 1 & 1 & . & . & 1 & 2 \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ 2 & 2 & . & . & 1 & 2 \\ 2 & 1 & . & . & 1 & 1 \end{array}\right| \qquad r = \left|\begin{array}{cccccc} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & 2 & 2 & 2 \\ . & . & 1 & 1 & 2 & 1 \\ . & . & 1 & 1 & 1 & 2 \\ . & . & 1 & 2 & 1 & 1 \end{array}\right|$$

4. The Lyons group as a homomorphic image of Γ

We now define elements a,b,c,d,x in Γ by

(4.1) $a = \tau . x(3b,4c)^4 . x(4b,3c)^2$

(4.2) $b = x(3b,4c)^2$

(4.3) $c = x(4b,2c)^3$

(4.4) $d = x(1a,2b) . x(2c,1a)^3 . x(1a,2b)$

(4.5) $x = x(3b,4c) . x(4b,3c) . x(3b,4c) . \tau' . x(4a,3b) . x(3b,4c)^3 . x(4a,3b)$

where τ and τ' are the torus elements

(4.6) $\tau = x(4b,2c) . x(2b,4c)^4 . x(4b,2c)^2 . x(2b,4c)^2$

(4.7) $\tau' = x(1a,4b)^4 . x(4c,1a) . x(1a,4b)^2 . x(4c,1a)^2$

The images $\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{x}$ of a,b,c,d,x in the 111-dimensional representation (cf. Lemma 1) obey all of the relations of Sims [1973], and hence they generate the Lyons group.
Furthermore,

(4.8) $\langle \tilde{a},\tilde{b},\tilde{c},\tilde{d} \rangle \cong G_2(5)$

while

(4.9) $\langle a,b,c,d \rangle \leq \Gamma(1a) \cong G_2(5)$

Therefore

(4.10) $\langle a,b,c,d \rangle = \Gamma(1a)$

$x \in Q$ by Theorem 4.c permutes the 36 root groups and corresponds to the automorphism (12)(34).(ac) of the apartment.
$\langle a,b,c,d,x \rangle$ contains the 12 root groups in $\Gamma(1a)$ and, e. g.,

(4.11) $X(1a,2b)^{x} = X(1b,2c)$

Because of the Chevalley relations these 13 root groups generate Γ. This shows the validity of

Theorem 7:

(a) $\langle a,b,c,d,x\rangle = \Gamma$;

(b) Ly is a homomorphic image of Γ.

Remark:

To prove a relation in any subgroup Δ of Γ which is isomorphic to its image $\bar{\Delta}$ in the representation, it is sufficient to check this relation for the appropriate 111-dimensional F_5-matrices.
In particular this holds true for the Sims relations which are expressed in elements of Δ alone.
We may apply this to the following three subgroups:

(4.12) $\Delta_x = \Gamma(1a) \cong G_2(5)$

(4.13) $\Delta_c = H_{12.34} \cong 2\hat{}A_{11}$

(4.14) $\Delta_d = \langle \Gamma(2c,1a), T \rangle \cong 5^{1+4}:4S_6$

The isomorphisms $\Delta_x \cong \bar{\Delta}_x$ and $\Delta_c \cong \bar{\Delta}_c$ have been verified in Theorems 1.a and 2.c, respectively.
$\Delta_d \cong \bar{\Delta}_d$ follows immediately from Theorem 2.e and the fact that all root groups are normalized by T.
These arguments suffice to prove the validity of all Sims relations except three.
We believe that the remaining relations also follow from our presentation, but we have not yet been able to show this.

5. Summary

The goal of this paper is to construct a root system for the Lyons group Ly in analogy to those of the Chevalley groups.

We make ample use of geometric properties of Ly.

It is shown that the construction can be carried out in a fashion nearly identical to the methods of Chevalley theory employed to study the Tits buildings of the groups of Lie type ($G_2(5) < Ly$ should be considered as a prototype).

We are confident that similar ideas can be applied to other (all ?) sporadic groups as well, perhaps in the long run leading to an understanding of these peculiar structures.

Concerning the geometry of the Lyons group itself, more information may be gained by a careful study of the 111-dimensional minimal representation over F_5.

Some initial results in that direction have been obtained.

We hope to present them - together with a proof of the isomorphism of the group Γ (defined in sec. 2) with Ly - in the near future.

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References

M. Aschbacher, M. Hall [1973]:
Groups generated by a class of elements of order 3
J. Alg. 24, 591-612

R. Carter [1972]:
Simple Groups of Lie Type
Wiley Interscience, New York

J. E. Humphreys [1975]:
Linear Algebraic Groups
(Graduate Texts in Mathematics, vol. 21)
Springer-Verlag, Berlin / Heidelberg / New York

W. Kantor [1981]:
Some geometries that are almost buildings
European J. Combin. 2, 239-247

R. Lyons [1972]:
Evidence for a new finite simple group
J. Alg. 20, 540-569

W. Meyer, W. Neutsch [1984]:
Über 5-Darstellungen der Lyonsgruppe
Math. Ann. 267, 519-535

W. Meyer, W. Neutsch, R. Parker [1985]:
The minimal 5-representation of Lyons' sporadic group
Math. Ann. 272, 29-39

C. C. Sims [1973]:
The existence and uniqueness of Lyons' group
in: Finite Groups '72 (Gainesville Conference), 138-141
North Holland Publishing Company, Amsterdam

R. A. Wilson [1984]:
The subgroup structure of the Lyons group
Math. Proc. Cambr. Philos. Soc. 95, 403-409

R. A. Wilson [1985]:
The maximal subgroups of the Lyons group
Math. Proc. Cambr. Philos. Soc. 97, 433-436