A UNIQUENESS RESULT FOR ORTHOGONAL GROUPS AS 2-COMPACT GROUPS

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Abstract. Two connected compact Lie groups are isomorphic if and only if their maximal torus normalizer are isomorphic. It is conjectured that this result generalizes to p-compact groups. Here, we prove the generalization for orthogonal groups $O(n)$, the special orthogonal groups $SO(2k+1)$ and the spinor groups $Spin(2k+1)$ considered as 2-compact groups.

1. Introduction.

A p-compact group is the homotopy theoretic generalization of a compact Lie group. This concept was introduced by Dwyer and Wilkerson [7]. And it turned out that p-compact groups behave astonishingly similar as compact Lie groups; e.g. there exist maximal tori, Weyl groups and normalizer of maximal tori. Since their existence, the classification problem was one of the main questions. The classification scheme of connected p-compact groups is of the same form as for connected compact Lie groups. Every connected p-compact group has a finite covering which is a product of torus and a simply connected p-compact group [18], and every simply connected p-compact group splits into a product of finitely many simple simply connected p-compact groups [9] [22]. The classification boils down to the case of simple p-compact groups [14]. And in this situation, a case by case checking based on the Clark-Ewing list of rational pseudo reflection groups has solved the problem for odd primes ([2], [3], [4], [10], [16], [17], [19], [23]). For the prime 2, the known results in this direction concern unitary and special unitary groups [19], 2-compact groups with Weyl group isomorphic to an elementary abelian 2-group [10] and the case of the exceptional Lie group G_2 [25].

In this work we consider products of orthogonal groups and characterize them as 2-compact groups by their associated normalizer of the maximal torus. Before we state our main results, we first have to recall some basic notation of the theory of p-compact groups from [7]. For further notation and definition as well as for the theory of p-compact groups see the survey articles [13] or [21] or the references given there.

A loop space X is a pair (X, BX) of spaces, such that such that BX is pointed and such that the loop space ΩBX is equivalent to X. The loop space inherits

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properties from the space X, e.g. X is called connected if the space X is so and finite or $\mathbb{Z}$-finite if the integral cohomology $H^*(X;\mathbb{Z})$ vanishes in large degrees and is in any degree a finitely generated abelian group. The loop space X is called a p-compact group, if X is $\mathbb{F}_p$-finite and if X and BX are p-complete. Every p-compact group X has a maximal torus $T_X \subset X$, a maximal torus normalizer $N_X \subset X$ which is a finite extension of T_X and a Weyl group W_X which acts on the p-adic lattice $L_X := \pi_2(BT_X)$. The maximal torus normalizer fits into a fibration $BT_X \to BN_X \to BW_X$ which also determines the action of W_X on BT_X respectively on L_X. Moreover, the composition $N_X \to X \to \pi_0(X)$ factors through a homomorphism $W_X \to \pi_0(X)$ and is always an epimorphism. The kernel of $N_X \to \pi_0(X) \cong \pi_1(BX)$ respectively the fiber of the map $BN_X \to B\pi_0(X)$ is a maximal torus normalizer of the component X_0 of the unit of X and the kernel of $W_X \to \pi_0(X)$ is the Weyl group of X_0. Examples of p-compact groups are given by the p-adic completion of compact Lie groups. If the group of components $\pi_0(G)$ of a compact Lie group G is a finite p-group, then the pair $(G_p^\wedge, BG_p^\wedge)$ gives rise to a p-compact group, also denoted by G. In particular, the orthogonal groups $O(n)$ give rise to 2-compact groups.

1.1 Definition.
(i) Two loop spaces or p-compact groups X and Y are called isomorphic if the associated classifying spaces BX and BY are homotopy equivalent.
(ii) Two p-compact groups X and Y are called N-isomorphic if the there exist isomorphisms $N_X \xrightarrow{\cong} N_Y$ and $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ such that the diagram

$$\begin{array}{ccc} BN_X & \longrightarrow & BN_Y \\ \downarrow & & \downarrow \\ B\pi_0(X) & \longrightarrow & B\pi_0(Y) \end{array}$$

commutes up to homotopy.

In analogy to compact Lie groups, the classification conjecture reads as

1.2 Conjecture. *Two p-compact groups X and Y are isomorphic if and only if they are N-isomorphic.*

As already mentioned, for odd primes, this conjecture is known to be true. In this case, one even can proof a weaker version, namely, X and Y are isomorphic if and only if N_X and N_Y are isomorphic, and, if in addition X is connected, then it suffices to have isomorphic Weyl groups (see the above mentioned references). But, as a comparison of $O(2)$ and $SO(3)$ and of $SO(2n+1)$ and $Sp(n)$ show, a weaker version cannot be true for the prime 2.

For $p = 2$, we want to prove the conjecture for finite products of orthogonal groups.

1.3 Theorem. *Let $G = \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group. Then, G and X are isomorphic if and only if they are N-isomorphic.*

The proof of this theorem is based on homotopy uniqueness results for products of orthogonal groups [20] in terms of mod-2 cohomology and

1.4 Theorem. *Let $G = \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group. If G and X are N-isomorphic, then there exist an isomorphism $H^*(BX) \cong H^*(BG)$ of algebras over the Steenrod algebra.*

Proof of Theorem 1.3. As shown in [20], every 2-complete space with the same mod-2 cohomology as BG is homotopy equivalent to the 2-adic completion $BG^\wedge_2$. □

As a corollary we will also see that the groups $Spin(2k+1)$ and $SO(2k+1)$ satisfy the above conjecture.

1.5 Corollary. *Let $G = Spin(2k+1)$ or $SO(2k+1)$ and let X be a 2-compact group. Then X and G are isomorphic if and only if they are N-isomorphic.*

Proof. Let X be a 2-compact group which is N isomorphic to $SO(2k+1)$. Since $N_{O(2k+1)} \cong N_{SO(2k+1)} \times \mathbb{Z}/2$, the 2-compact group $X \times \mathbb{Z}/2$ is N-isomorphic to $O(2k+1)$ and therefore, by Theorem 1.3, isomorphic to $O(2k+1)$. The projection $BX \times B\mathbb{Z}/2 \to B\mathbb{Z}/2$ on the second factor is given by the classifying map of the non trivial 1-dimensional mod-2 cohomology class. The fiber of this map is BX. the same holds for $BO(2k+1)^\wedge_2$. This shows that BX and $BSO(2k+1)^\wedge_2$ are equivalent and that X and $SO(2k+1)$ are isomorphic as 2-compact groups.

Let X be a 2-compact group, which is N-isomorphic to $Spin(2k+1)$. The center of X can be read off from the maximal torus normalizer [8]. Hence, X and $Spin(2k+1)$ have the same center equal to $\mathbb{Z}/2$ and the quotients $\overline{X} := X/\mathbb{Z}/2$ and $SO(2k+1)$ are N-isomorphic and therefore isomorphic. Since X is the simply connected cover of $\overline{X}$ and $Spin(2k+1)$ the simply connected cover of $SO(2k+1)$, the 2-compact groups X and $Spin(2k+1)$ are isomorphic. □

Starting from Corollary 1.5, we plan in further paper to prove uniqueness results for the 2-compact group $DI(4)$ constructed in [6] in terms of the maximal torus normalizer as well as in terms of the mod-2 cohomology of $BDI(4)$.

The paper is organized as follows. In Section 2 we provide some facts for orthogonal groups as 2-compact groups and fix some notation. Section 3, the last section, contains the tedious proof of Theorem 1.4. We also note that cohomology is always taken with $\mathbb{Z}/2$-coefficients; i.e. $H^*(-) = H^*(-;\mathbb{F}_2)$.

We will use the language and theory of p-compact groups all over the places. For references, about p-compact groups we refer the reader to the survey articles [13] and [21] or the references mentioned there.

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2. Products of orthogonal type.

In this section we provide necessary facts about products of orthogonal groups and fix some notation.

2.1 Notation and Remarks. Let $n = 2k + \theta$ with $\theta \in \{0, 1\}$. We use Θ to denote the group $\mathbb{Z}/2^\theta$. The maximal torus normalizer of $O(n)$ is isomorphic to $O(2) \wr \Sigma_k \times \Theta$. The group $O(2)$ contains a maximal elementary abelian 2-subgroup $E_{O(2)}$ of rank 2. We denote by $(\mathbb{Z}/2)^k \cong t_{O(n)} \subset T_{O(n)}$ the maximal elementary abelian 2-subgroup of the maximal torus of $O(n)$. We have a sequence

$$t_{O(n)} \subset (\mathbb{Z}/2)^n \cong E_{O(n)} := E^k_{O(2)} \times \Theta \subset N_{O(n)} \subset O(n)$$

of inclusions.

Every elementary abelian 2-subgroup of $O(n)$ is contained in $E_{O(n)}$. The subgroup $E_{O(n)}$ is self centralizing, i.e. the inclusion $E_{O(n)} \subset C_{O(n)}(E_{O(n))}$ induces an isomorphism of $E_{O(n)}$ with the centralizer of $E_{O(n)}$. The Weyl group $\overline{W}_{O(n)} := W_{O(n)}(E_{O(n)})$ is isomorphic to Σ_n which acts by permutations on $E_{O(n)}$. Actually, the normalizer $N_{O(n)}(E_{O(n)})$ of $E_{O(n)}$ is isomorphic to $\mathbb{Z}/2 \wr \Sigma_n \subset O(n)$. All this follows from classical representation theory. Moreover, $H^*(BO(n);\mathbb{Z}/2) \cong H^*(BE_{O(n)};\mathbb{Z}/2)^{\Sigma_n}$.

For $G = \prod_i O(n_i)$, taking products establishes the same type of groups which fit into the diagram of inclusions

$$\begin{array}{ccccccc}
t_G & \longrightarrow & T_G & & & & \\
\downarrow & & \downarrow & & & & \\
E_G & \longrightarrow & N_G & \longrightarrow & N_G & \longrightarrow & G
\end{array}$$

and satisfy the same properties. Among other things we collect these facts in the next proposition. And $\overline{W}_G := W_G(E_G)$ denotes the Weyl group of E_G in G.

By SG we denote the component of the unit of G. Then $SG = \prod_i SO(n_i)$ is a product of special orthogonal groups. We will also use the notation SX for the component of the unit of a p-compact group X.

2.2 Proposition. *Let $G \cong \prod_i O(n_i)$ be a finite product of orthogonal groups. Then, the following holds:*
(i) E_G is self centralizing.
(ii) The mod-2 cohomology of BN_G is detected by elementary abelian 2-subgroups and so is the mod-2 cohomology of BG.
(iii) Up to conjugation, every elementary abelian subgroup of G is contained in E_G.
(iv) $\overline{W}_G := \prod_i \overline{W}_{O(n_i)} \cong \prod_i \Sigma_{n_i}$.
(v) $H^*(BG;\mathbb{Z}/2) \cong H^*(BE_G;\mathbb{Z}/2)^{\overline{W}_G}$.
(vi) For any elementary abelian 2-group E, the inclusion $E_G \to G$ induces a bijection $[BE, BE_G]/\overline{W}_G \cong [BE, BG]$.
(vii) For any elementary abelian 2-subgroup $E \subset G$, the centralizer $C_G(E)$ is again a finite product of orthogonal groups.

Proof. : The proof is obvious and/or a straight forward application of real representation theory. It only might be necessary to make some comments for (ii). As a product of Θ and a wreath product of $O(2)$ with a symmetric group, the mod-2 cohomology of $BN_{O(n)}$ is detected by elementary abelian 2-subgroups [11] and so is the mod-2 cohomology of BN_G. $\square$

3. Proof of Theorem 1.4.

Let $G \cong \prod_i O(n_i)$ be a finite product of orthogonal groups and let X be a 2-compact group which is N-isomorphic to G. In this section we want to prove a version of Proposition 2.2 for X. It will include Theorem 1.4.

Since X and G are N-isomorphic, we have the following diagram of inclusions

$$\begin{array}{ccccccc} t_G & \longrightarrow & T_G & & & & \\ \downarrow & & \downarrow & & & & \\ E_G & \longrightarrow & N_G & \longrightarrow & N_G & \longrightarrow & G \end{array}$$

We denote by $i_X : E_G \to X$ the inclusion given by the bottom line. As already mentioned, SX denotes the component of the unit of X.

3.1 Proposition.
(i) The subgroup $E_G \subset X$ is self centralizing; i.e. $C_X(E_G) = E_G$.
(ii) The mod-2 cohomology of BX is detected by elementary abelian 2-subgroups.
(iii) Every elementary abelian 2-subgroup is subconjugated to E_G.
(iv) For every abelian 2-group E the map $[BE, BE_G] \to [BE, BX]$ factors through the bijection $[BE, BE_G]/\overline{W}_G \xrightarrow{\cong} [BE, BX]$.
(v) $H^*(BX) \cong H^*(BE_G)^{\overline{W}_G}$.

The proof of this proposition splits into several parts and is worked out in the rest of this section. In particular, the parts (iii) and (iv) are proved via inductions, which themselves split into several steps.

3.2 Remark. According to the splitting of $G = \prod_i O(n_i)$, the maximal torus normalizer $N_G = \prod N_{O(n_i)}$ splits into a product. By [22], this implies a splitting $X \cong \prod_i X_i$ such that $N_{O(n_i)} \subset X_i$ is a maximal torus normalizer and such that the diagram

$$\begin{array}{ccc} \prod_i N_{O(n_i)} & \xrightarrow{\cong} & N_G \\ \downarrow & & \downarrow \\ \prod_i X_i & \xrightarrow{\cong} & X \end{array}$$

commutes. In particular, X_i and $O(n_i)$ are N-isomorphic. Moreover, all statements of Proposition 3.1 also split in the same manner. Hence, on the way of proving the proposition, we always can assume that $G = O(n)$ is an orthogonal group. This even is possible for the inductions necessary for the proofs of (iii) and (iv).

3.3 Proof of (i) and (ii): Part (ii) follows from Proposition 2.3, since N_G is the maximal torus normalizer of X and since $H^*(BX) \to H^*(BN_G)$ is a monomorphism [7].

Since E_G is self centralizing in G, we have $C_{N_G}(E_G) = E_G$, too. By reasons of size, every monomorphism of $E_G \to N_G$ takes image in $E_G \subset N_G$. By [15], there exist a lift $E_G \to N_G$ of $i_X : E_G \to X$ such that $C_N(E_G) \subset C_X(E_G)$ is the maximal torus normalizer. Hence, the maximal torus normalizer of $C_X(E_G)$ is E_G.

This implies that $C_X(E_G)$ is discrete, therefore its own maximal torus normalizer and congruent to E_G. This proves the first part.

The second part is a consequence of Proposition 2.2 (ii) and the fact that for every p-compact group X the map $BN_X \to BX$ induces an monomorphism in mod-p cohomology [7], in particular $H^*(BX) \to H^*(BN_G)$ is a monomorphism. $\square$

Before we continue with the proof we digress for the following lemma:

3.4 Lemma. *Let $E \subset T_G$ be an elementary abelian 2-subgroup. Then $C_G(E)$ and $C_X(E)$ are N-isomorphic.*

Proof. By [8, 7.6], both $C_G := C_G(E)$ and $C_X := C_X(E)$ have the same Weyl groups and therefore the same maximal torus normalizer, namely $C_N := C_{N_G}(E)$. We only have to show that the components of the unit are N-isomorphic.

Since E is a toral subgroup and therefore $E \subset SX$ and since $SC_X(E)$ is contained in SX, we have $SC_X(E) = SC_{SX}(E)$. But the Weyl group of $SC_{SX}(E)$ is completely determined by the Weyl group action of W_{SX} on the maximal torus $T_G = T_X$ [8, 7.6]. Since SG and SX are N-isomorphic, and since the same holds for G, both centralizers, C_G and C_X, have N-isomorphic components of the unit. $\square$

3.5 Claim. *Proposition 3.1 (iii) is true for $E \cong \mathbb{Z}/2$.*

Proof. We prove this via an induction over the order of the Weyl group of G. Because of Remark 3.2 we can assume that $G = O(n)$. If $G \cong O(2)$, then $X = N_G = G$. And for $O(2)$ all of Proposition 3.1 is true. This is the beginning of the induction.

Up to conjugation $E \subset N_G$. Since $N_G = O(2) \wr \Sigma_k \times \Theta$, the generator of E is given by an element $x = ((A_1, ..., A_k; \sigma), a) \in N_G$ with $A_i \in O(2)$, $\sigma \in \Sigma_k$ and $a \in \Theta$. Since $E \cong \mathbb{Z}/2$, the permutation σ is of order two. Without loss of generality we can assume that $\sigma = \prod_{j=1}^{r} \tau_{2j-1,2j}$ is a product of commuting transpositions. Therefore $x \in (O(2) \wr \mathbb{Z}/2)^r \times O(2)^{k-2r} \times \Theta$. If $r = 0$, then E obviously is subconjugated to E_G. If $r \geq 1$ then there exists a $\mathbb{Z}/2 \cong E' \subset (T_G)^E$ which is neither central in N_G nor in G or in X. By Lemma 3.4, $C_G(E')$ and $C_X(E')$ are N-isomorphic. Moreover, they have smaller Weyl groups than G. Since E_G and E are contained in $C_{N_G}(E')$, E is subconjugated to E_G by induction hypothesis. $\square$

3.6 Claim. *Proposition 3.1 (iii) is true for any general elementary abelian 2-group E.*

Proof. This time we make an induction over the order of E. The starting point of the induction is given by Claim 3.5. Again, we can assume that $G = O(n)$. If the dimension of E is greater than 1, the composition $E \subset X \to \pi_0(X)$ has a non trivial kernel and we can split $E \cong E_0 \oplus E_1$ such that $\mathbb{Z}/2 \cong E_0 \subset SX$. Since E_0 is cyclic, it is subconjugated to T_G, in fact to t_G. The centralizers $C_X(E_0)$ and $C_G(E_0)$ are N-isomorphic (Lemma 3.4), and $E_G \subset C_X(E_0)$. By induction hypothesis, E_1 is subconjugated to E_G inside of $C_X(E_0)$, which shows that E is subconjugated to E_G inside X. $\square$

We make the map $BE_G \to BX$ into a fibration and denote the fiber by X/E_G. The inclusion $i_X : E_G \to X$ establishes a proxy action of E_G on X/E_G (see [7]) and the homotopy fixed point set $(X/E_G)^{E_G}$ fits into a fibration

$$(X/E_G)^{E_G} \to map(BE_G, BE_G)_{\text{lifts}} \to map(BE_G, BX)_{Bi_X} \ .$$

The total space is given by all self maps of BE_G which are lifts of Bi_X. Since $E_G \subset X$ is self centralizing (Proposition 3.1 (i)) the fiber is homotopically discrete and the set of components $\overline{W}_X := \pi_0((X/E_G)^{E_G})$ is a group, the Weyl group $W_X(E_G)$ of $E_G \subset X$. Since $\pi_0(map(BE_G, BE_G) = Hom(E_G, E_G)$, every element of $\overline{W}_X$ is represented by an automorphism of E_G. This establishes an action of $\overline{W}_X$ on E_G as well as on BE_G.

3.7 Claim.
(i) The Weyl group $\overline{W}_X$ is finite and acts faithfully on E_G.
(ii) $H^*(BX) \cong H^*(BE_G)^{\overline{W}_X}$.

Proof. The Weyl group $\overline{W}_X$ is finite since $(X/E_G)^{E_G}$ is $\mathbb{F}_2$-finite [7]. The action is obviously faithful.

By Proposition 3.1 (ii) and (iii), $H^*(BX) \to H^*(BE_G)$ is the Adams-Wilkerson embedding [1]. As a consequence of [26, Theorem II], [5, 3.6] and properties of Lannes' T-functor [12] we get $H^*(BX) \cong H^*(BC_X(E_G))^{\overline{W}_X}$, which is Part (ii). □

3.8 Claim. *The Weyl group $\overline{W}_X$ contains $\overline{W}_G$.*

Proof. Again we can assume that $G = O(n)$. That is that $\overline{W}_G = \Sigma_n$ For $n \geq 3$ the two subgroups $\Sigma_2 \times \Sigma_{n-2}$ and $\Sigma_{n-2} \times \Sigma_2$ generate Σ_n. The inclusions of both subgroups are induced by the Weyl group inclusion $\overline{W}_{C_G(E)} \subset \overline{W}_G$ for an elementary abelian 2-subgroup $\mathbb{Z}/2 \cong E \subset t_G$. By Lemma 3.4, $C_G(E)$ and $C_X(E)$ are N-isomorphic. Now, the claim follows by an induction over the order of the Weyl group. □

The next step is a proof of Proposition 3.1 (iv). We need some further lemmas.

3.9 Lemma. *Let $G = O(n)$ and $\Delta \subset E_G$ denote the diagonal. Then, Δ is a central subgroup of X.*

Proof. If n is even, $\Delta \subset SX$ is subconjugated to T_G and the statement follows from Lemma 3.4. If $n = 2k + 1$ is odd, $N_{O(n)} \cong N_{SO(n)} \times \mathbb{Z}/2$. By [22], X splits into the product $SX \times \mathbb{Z}/2$ and SX and $SO(n)$ are N-isomorphic. Since the center of a connected p-compact group can be read off from the maximal torus normalizer [8], the center of SX is trivial and the center of X equals $\mathbb{Z}/2$.

Because Lannes' T-functor commutes with taking invariants [24] and because $H^*(BX) \cong H^*(BC_X(E_G))^{\overline{W}_X}$, a central elementary abelian 2-subgroup $E \subset E_G \subset X$ induces a map $H^*(BE_G) \to H^*(BE)$ which is fixed under the action of $\overline{W}_X$. Since $\Sigma_n \subset \overline{W}_X$, only the diagonal $\Delta \subset E_G$ can have this property. Thus, $\Delta \subset E_G \subset X$ has to be central. □

Let X be a p-compact group with maximal torus normalizer $N_X \subset X$. A elementary abelian p-subgroup $E \subset N_X$ is called special if $C_{N_X}(E) \subset C_X(E)$ is a maximal torus normalizer. As shown in [15], for every elementary abelian $E \subset X$, there exists a special monomorphism $E \to N_X$, such that the composition $E \to N_X \subset X$

and $E \subset X$ are conjugate. In [15], this monomorphism is called a preferred lift. For $G = O(n)$, we denote by $\mathbb{Z}/2 \cong E(l) \subset E_G \cong (\mathbb{Z}/2)^n$ the subgroup given by the diagonal of the first l-components and by $E'(l)$ the diagonal of the last l components. If $x \in E(l)$ is a generator, then $-x$ is a generator of $E'(n-l)$ (we write E_G respectively $\mathbb{Z}/2$ multiplicatively). Moreover, since $\Delta \subset E_G$ is central in X, both $E(l)$ and $E'(n-l)$ have the same centralizer.

3.10 Lemma. *Let $G = O(n)$ and let $n = 2k + \theta$.*
(i) If $\theta = 0$ (n is even) and if a subgroup $\mathbb{Z}/2 \cong E \subset N_G$ is special for X, then there exists $1 \le l \le n$ such that, as subgroups of N_G, E and $E(l)$ are conjugate.
(ii) Every subgroup $E(l) \subset E_G$ is special as a subgroup of X.
(iii) $C_G(E(l))$ and $C_X(E(l))$ are N-isomorphic.

Proof. First we proof (i). As shown in the proof of Claim 3.5, up to conjugation, $E \subset (O(2) \wr \mathbb{Z}/2)^r \times O(2)^{k-2r}$. That is that the generator $x \in E$ has coordinates $A_j \in O(2)$ and $B_i \in O(2) \wr \mathbb{Z}/2$, where $B_i = (C_i, C_i^{-1}; \sigma)$ with $C_i \in O(2)$ and $1 \ne \sigma \in \mathbb{Z}/2$ (we write $\mathbb{Z}/2$ multiplicatively). Conjugation by $(C_i, I; 0)$ shows that B_i is conjugate to $(I, I; \sigma)$ where $I \in O(2)$ denotes the unit. We can assume that $C_i = I$ for all i. We also can transform the A_j's on a normal form, namely $A_j = I$, $A_j = -I$ or A_j is conjugate to the matrix $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. A straight forward calculation shows that the centralizer $C_N(E)$ is isomorphic to

$$(O(2) \times \mathbb{Z}/2) \wr \Sigma_r \times O(2) \wr \Sigma_q \times O(2) \wr \Sigma_t \times (\mathbb{Z}/2 \times \mathbb{Z}/2) \wr \Sigma_s \ ,$$

where q is the number of A_j's equal to I, t the number of A_j's equal $-I$ and s the number of A_j's conjugate to S. Moreover, $2r + q + t + s = k$.

On the other hand, as a subgroup of X, E is subconjugated to E_G (Proposition 3.1 (iii)), and, since $\Sigma_{2k} \subset \overline{W}_X$ (Claim 3.8), E is conjugate to a subgroup $E(l) \subset E_G$. Let $l = 2m + \nu$ with $\nu \in \{0, 1\}$. Then, $C_N(E(l)) \cong O(2) \wr \Sigma_m \times (\mathbb{Z}/2 \times \mathbb{Z}/2)^\nu \times O(2) \wr \Sigma_{k-m-\nu} \times \Theta$. By inspection, the maximal torus of $C_N(E(l))$ has dimension $k - \nu$ and the maximal torus of $C_N(E)$ dimension $r + q + t = k - r - s$ The parity of s shows whether $E \subset SX$ and the parity of ν does this for $E(l)$. Hence, $\nu \equiv s \mod 2$. Since $E \subset N_G$ was special, $k - r - s \ge k - \nu$ which implies that $s = \nu$ and $r = 0$. This shows, that, up to conjugation by elements of Σ_k, $E = E(l')$ and proves Part (i).

Now, we prove Part (ii). If $n = 2k + 1$, then either l or $n - l$ is even and either $E(l)$ or $E'(n-l)$ is contained in SX and therefore subconjugate to T_G. Hence, in this case, the claim follows from Lemma 3.4 and the above remark. If n and l are even, the same argument applies.

Finally, we have to consider the case $n = 2k$ and $l = 2m+1$. By (i), there exists a special subgroup $E(p) \subset E_G$ conjugate to $E(l)$. Since p is odd if and only if l is odd, we can assume that $p = 2q + 1$. The centralizers $C_{N_G}(E(l) = O(2) \wr \Sigma_m \times (\mathbb{Z}/2)^2 \times O(2) \wr \Sigma_{k-l}$ and $C_{N_G}(E(p) = O(2) \wr \Sigma_q \times (\mathbb{Z}/2)^2 \times O(2) \wr \Sigma_{k-q}$ have isomorphic components of the unit, i.e. are of the same rank. Therefore, $\pi_0(C_{N_G}(E(l)) = \mathbb{Z}/2 \wr \Sigma_m \times (\mathbb{Z}/2)^2 \times \mathbb{Z}/2 \wr \Sigma_{k-l}$ is subconjugated to $W_X \cong \pi_0(C_{N_G}(E(q))$. This implies that $m = q$ or $m = k - q$, that $C_{N_G}(E(l)) \cong C_{N_G}(E(p)$ and, in particular, that $E(l) \subset E_G \subset N_G$ is special, which finishes the proof of (ii).

By the above calculations and by Part (ii), $C_G(E(l))$ and $C_X(E(l))$ have isomorphic maximal torus normalizer. We have to show that both are N-isomorphic or, which is sufficient, have isomorphic groups of components.

If a quotient of $\mathbb{Z}/2 \wr \Sigma_r$ is a 2-group then it is an elementary abelian 2-group. And the same holds for the Weyl group of $C_X(E(l))$. This implies that

$$\begin{aligned}\pi_0(C_X(E(l)) &\cong H^1(BC_X(E(l))) \\ &\cong H^1(BE_G)^{W'} \subset H^1(BE_G)^{\Sigma_l \times \Sigma_{n-l}} \cong (\mathbb{Z}/2)^2 ,\end{aligned}$$

where $W' := Iso_{\overline{W}_X}(E(l))$ is the isotropy group of $E(l)$ in $\overline{W}_X$.

If l is even, N-isomorphism follows from Lemma 3.4. If l and n are odd, we can replace $E(l)$ by $E'(n-l)$ and, again, N-isomorphism follows from Lemma 3.4. If l is odd and n is even, the maximal torus normalizer contains a direct factor $(\mathbb{Z}/2)^2$ which is contained in $\pi_0(C_X(E(l)))$. Hence $\pi_0(C_X(E(l))) \cong (\mathbb{Z}/2)^2 \cong \pi_0(C_G(E(l)))$. This shows that $C_X(E(l))$ and $C_G(E(l))$ are N-isomorphic and finishes the proof. □

3.11 Claim. *For every elementary abelian 2-group E, the inclusion $E_G \to X$ induces an isomorphism $[BE, BE_G]/\overline{W}_X \xrightarrow{\cong} [BE, BX]$.*

Proof. This is a consequence of the isomorphisms $H^*(BX) \cong H^*(BE_G)^{\overline{W}_X}$ and the facts that, for finite groups, Lannes' T-functor commutes with taking invariants (e.g. see [24]) and that the $[BE, BX] = Hom_{\mathcal{A}_2}((H^*(BX), H^*(BE))$, where $Hom_{\mathcal{A}_2}(-,-)$ denotes the set of maps of algebras over the Steenrod algebra $\mathcal{A}_2$ [12]. □

3.12 Claim. *Part (iv) is true for $E \cong \mathbb{Z}/2$.*

Proof. Again we can assume that $G = O(n)$. Let $\alpha,\ \beta : E \to E_G$ be two homomorphism such that $i_X\alpha, i_X\beta : E_G \to X$ are conjugate. If α is the constant homomorphism then so is β. Hence, we can assume that both are monomorphisms. Up to the action of elements of $\Sigma_n \cong \overline{W}_G \subset \overline{W}_X$, the image of α is $E(l)$ and the image of β is $E(p)$. If $w \in \overline{W}_X$ conjugates $E(l)$ into $E(p)$, then, since the diagonal $\Delta \subset E_G \subset X$ is central, w also conjugates $E'(n-l)$ into $E'(n-p)$.

Let $l = 2m + \nu$ and $p = 2q + \nu'$. The number l is even if and only if $E(l) \subset SX$ if and only if $E(p) \subset SX$ if and only if p is even. And if this is the case, then $E(l)$ and $E(p)$ are conjugate by an element of $W_X = W_G \subset \Sigma_n \subset \overline{W}_X$ and $l = p$. This proves the claim in this case. Moreover, the argument shows that, in general, $\nu = \nu'$ and that, for any $w \in W_X$ and any $2k$, the subgroup $w(E(2k)) \subset E_G$ is conjugate to $E(2k)$ by an element of Σ_n.

By Claim 3.11, there exists $w \in W_X$ such that $w(E(l)) = E(p)$. Let $\sigma \in \Sigma_n$ such that $\sigma w(E(2)) = E(2)$. Then, $\sigma w \in \overline{W}_{C_X(E(2))}$ and $E(l)$ and $\sigma(E(p))$ are conjugate in $C_X := C_X(E(2))$ which is N-isomorphic to $O(2) \times O(n-2)$ (Lemma 3.4). By induction, we can assume that $\overline{W}_{C_X} = \Sigma_2 \times \Sigma_{n-2} \subset \Sigma_n$. Hence, $E(l)$ and $\sigma(E(p))$ are conjugate by an element of Σ_n (Claim 3.11) as well as $E(l)$ and $E(p)$. This shows that $l = p$ and proves the claim.

For $G = O(2)$, which is the beginning of the induction, $X \cong G$, and the claim is obvious. □

3.13 Claim. *Part (iv) is true for any elementary abelian* 2*-group.*

Proof. Again we prove this via an induction over the order of the Weyl group W_G of G. We can assume that $G = O(n)$. Let E be an elementary abelian 2-group and $\alpha, \beta : E \to E_G$ such that the compositions $i_X\alpha, i_X\beta : E \to X$ are conjugate. We can assume that both are monomorphism.

By Claim 3.11, there exists $w \in W_X$ such that $w\alpha = \beta$. We split $E \cong E_0 \oplus E_1$ such that $E_0 \cong \mathbb{Z}/2$ As in the proof of Claim 3.12, we can assume that $\alpha(E_0) = E(l)$. Claim 3.12 also shows that, changing w by an element of Σ_n, $\beta(E_0) = E(l)$. That is that w fixes $E(l)$, that $w \in \overline{W}_{C_X(E(l))}$ and that α and β are conjugate in $C_X := C_X(E(l))$. Since C_X and $C_G := C_G(E(l)) \cong O(l) \times O(n-l)$ are N-isomorphic (Lemma 3.10) and have smaller Weyl group than G, there exists $\sigma \in \Sigma_l \times \Sigma_{n-l} \cong \overline{W}_{C_X} \subset \Sigma_n$, such that $\sigma\alpha = \beta$. This finishes the induction step. The starting point is given by the case $G = O(2)$ where nothing is to show. □

By Claim 3.11 the next claim, which is the last one, proves Part (v) and finishes the proof of Proposition 3.1.

3.14 Claim. $\overline{W}_X = \overline{W}_G$.

Proof. By Claim 3.8, $\overline{W}_G \subset \overline{W}_X$. Since $\overline{W}_X$ acts faithfully on E_G (Claim 3.7), it is the subgroup of $Gl(E_G)$ of all elements w such that i_Xw and i_X are conjugate. By Proposition 3.1 (iv), this subgroup is $\overline{W}_G$. □

References.

[1] J.F. Adams & C.W. Wilkerson, *Finite H–spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95-143.

[2] K. Anderson, J. Grødal, J.M. Møller, and A. Viruel, *N-determinism of the E-family,*, in preparation.

[3] C. Broto & A. Viruel, *Homotopy uniqueness of $BPU(3)$*, Group representations: cohomology, group actions, and topology (Seattle, WA, 1996), Amer. Math. Soc., Providence, RI, (1998), 85-93.

[4] C. Broto & A. Viruel, *projective unitary groups are totally N-determined p-compact groups*, preprint.

[5] W.G. Dwyer, H.Miller, & C.W. Wilkerson, *The homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29-45.

[6] W.G. Dwyer & C.W. Wilkerson, *A new finite loop space at the prime two*, J. AMS **6** (1993), 37-63.

[7] W.G. Dwyer & C.W. Wilkerson, *Homotopy fixed–point methods for Lie groups and finite loop spaces*, Ann. Math. (2) **139** (1994), 395-442.

[8] W.G. Dwyer & C.W. Wilkerson, *The center of a p-compact group*, The Cech Contennial. Contemporary Math. **181** (1995), 119-157.

[9] W.G. Dwyer & C.W. Wilkerson, *Product splittings of p–compact groups*, Fund. Math. **147** (1995), 279-300.

[10] W.G. Dwyer & C.W. Wilkerson, *p-compact groups with abelian Weyl groups*, Preprint.

[11] J.H. Gunawardena, J. Lannes, and S. Zarati, *Cohomlogie des groupes symmétrique et application de Quillen*, L.M.S Lectures Notes Math. **139**.

[12] J. Lannes, *Sur les espaces fonctionelles dont la source est la classifiant d'un p–groupe abélien élemementaires*, IHES **75** (1992), 135-244.

[13] J.M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), 413-428.

[14] J.M. Møller, *Deterministic p-compact groups*, Stable and unstable homotopy theory, (Toronto, ON, 1996), Amer. math. Soc., Providence, RI (1998), 255-278.

[15] J.M. Møller, *Normalizers of maximal tori*, Math. Z. **231** (1999), 51-74.

[16] J.M. Møller, *N-determined p-compact groups*, Preprint.

[17] J.M. Møller, *N-determinism of the p-compact groups of the A-family*, Preprint.

[18] J.M. Møller & D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. reine u. angew. Math. **456** (1994), 99-133.

[19] J.M. Møller & D. Notbohm, *Connected finite loop spaces with maximal tori*, Trans. AMS **350** (1998), 3483-3504.

[20] *Homotopieeindeutigkeit von Produkten orthogonaler Gruppen*, Thesis, Göttingen 1996 or ftp://hopf.math.purdue.edu/pub/morgenroth/BOn.dvi.

[21] D. Notbohm, *Classifying spaces of compact lie groups and finite loop spaces*, in: Handbook of Algebraic topology, Ed.: I. James, Elsevier (1995), 1049-1094.

[22] D. Notbohm, *Unstable splittings of classifying spaces of p–compact groups*, to appear Quart. J. Math. Oxford.

[23] D. Notbohm, *Spaces with polynomial cohomology*, Math. Proc. Camb. Phil. Soc. **9** (1999), 55-99.

[24] *unstable modules over the Steenrod algebra and Sullivan's fixed-point set conjecture, University of Chicago Press* (1994).

[25] A. Viruel, *Homotopy uniqueness of BG_2*, Manuscr. Math. **95** (1998), 471-497.

[26] Rings of invariants and insepaprable forms of algebras over the Steenrod algebra, Preprint.

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