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ALGEBRAS AND ALGEBRAIC CURVES ASSOCIATED WITH PDES AND BÄCKLUND TRANSFORMATIONS

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Abstract. Using the language of jet spaces, for any analytic PDE $\mathcal{E}$ we define, in a coordinate-free way, a family of associative algebras $\mathbb{A}(\mathcal{E})$.

In the considered examples, which include the KdV, Krichever-Novikov, nonlinear Schrödinger, Landau-Lifshitz equations, the algebras $\mathbb{A}(\mathcal{E})$ are commutative and are isomorphic to the function field of an algebraic curve of genus 1 or 0. This provides an invariant meaning for algebraic curves related to some PDEs.

Also, the algebras $\mathbb{A}(\mathcal{E})$ help to prove that some pairs of PDEs from the above list are not connected by Bäcklund transformations.

To define $\mathbb{A}(\mathcal{E})$, we use fundamental Lie algebras $\mathbb{F}(\mathcal{E})$ of $\mathcal{E}$ introduced in [15]. Elements of $\mathbb{A}(\mathcal{E})$ are intertwining operators for the adjoint representations of Lie subalgebras of certain quotients of $\mathbb{F}(\mathcal{E})$.

In the last 30 years, it has been relatively well understood how to construct integrable PDEs from some infinite-dimensional Lie algebras and algebraic curves (see, e.g., [1, 4, 6, 8, 7, 9, 10, 12, 16, 20, 21, 27, 31, 33] and references therein).

This preprint belongs to a series of papers, where we study the inverse problem: given a PDE, how to determine whether it is related to Lie algebras and algebraic curves, and how to recover these algebraic structures from the PDE, in a coordinate-independent way?

Our strategy is to define geometric invariants for arbitrary (not necessarily integrable) PDEs such that for integrable PDEs these invariants reproduce infinite-dimensional Lie algebras and algebraic curves. Some invariants in the form of Lie algebras have been introduced in [15]. Using the results of [15], in the present preprint we study how to recover an algebraic curve from a PDE. We use the geometric coordinate-independent approach to PDEs by means of infinite jet spaces.

It is well known that many PDEs possess a zero-curvature representation[1] (ZCR) parametrized by points of an algebraic curve C. However, the invariant meaning of the curve C for a given PDE is not clear, because a PDE may have several ZCRs parametrized by different curves.

For example, the Landau-Lifshitz equation has a ZCR with elliptic parameter [9, 32] and a ZCR with rational (polynomial) parameter [2]. Nevertheless, there are strong indications that, for this PDE, the elliptic curve is 'more important' than the rational curve (e.g., the elliptic curve is used in the construction of solutions for the Landau-Lifshitz equation [4, 22]).

On the other hand, for the nonlinear Schrödinger (NLS) equation, only a ZCR with rational (polynomial) parameter is known [9]. All the experience in the study of this PDE suggests that there is no nontrivial ZCR with elliptic parameter for the NLS equation, but this has never been proved.

Remark 1. A well-known empirical explanation of the difference between the NLS and Landau-Lifshitz equations is the following. Some Riemann-Hilbert problems on the rational curve $\mathbb{C}P^1$ lead to solutions of certain PDEs, including the NLS equation [9]. On the other hand, solutions of the Landau-Lifshitz equation can be obtained from a Riemann-Hilbert problem on an elliptic curve [4, 22].

[1]For PDEs in two independent variables the notion of ZCR is essentially equivalent to that of Lax pair.

However, since there is no canonical way to associate a Riemann-Hilbert problem with a given PDE, this does not provide a fully invariant meaning for the curves.

Similarly, an elliptic curve often occurs in the study of the Krichever-Novikov equation [20, 21, 23, 24].

In this preprint, the following interpretation for such curves is presented. We consider arbitrary analytic PDEs $\mathcal{E}$. In particular, PDEs may have any number of variables. For any $\mathcal{E}$, we define, in a coordinate-free way, a family of associative algebras $\mathbb{A}(\mathcal{E})$ such that the following property holds.

In all studied examples, if $\mathcal{E}$ is an integrable PDE in two independent variables related to an algebraic curve C of genus ≤ 1 then the algebras $\mathbb{A}(\mathcal{E})$ are commutative and are isomorphic to the function field of the curve C.

Example 1. For the KdV and nonlinear Schrödinger equations, the algebras $\mathbb{A}(\mathcal{E})$ are isomorphic to the function field of a rational curve.

For the Landau-Lifshitz and Krichever-Novikov equations, the algebras $\mathbb{A}(\mathcal{E})$ are isomorphic to the function field of an elliptic curve.

The definition of $\mathbb{A}(\mathcal{E})$ uses only the PDE $\mathcal{E}$. In particular, the definition does not use any specific ZCRs or Lax pairs. However, in order to compute $\mathbb{A}(\mathcal{E})$ explicitly, it is helpful to know ZCRs of $\mathcal{E}$.

Since $\mathbb{A}(\mathcal{E})$ is a coordinate-independent invariant of $\mathcal{E}$, one has the following property. Suppose that two PDEs $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic, i.e., $\mathcal{E}_1$ can be obtained from $\mathcal{E}_2$ by a change of variables, and vice versa. Then $\mathbb{A}(\mathcal{E}_1)$ is isomorphic to $\mathbb{A}(\mathcal{E}_2)$. We allow arbitrary (consistent) changes of variables, which may mix independent variables, dependent variables, and derivatives in the PDEs.

To give the definition of $\mathbb{A}(\mathcal{E})$, we need to recall a geometric approach to PDEs by means of jet spaces (see, e.g., [3, 15, 17, 18, 26] and references therein).

Using infinite jet spaces, one can treat a PDE as a geometric object: a manifold $\mathcal{E}$ with an n-dimensional distribution $\mathcal{C}$ called the *Cartan distribution*[2], where n is the number of independent variables in the PDE. If vector fields X, Y belong to the distribution $\mathcal{C}$ then the commutator $[X, Y]$ belongs to $\mathcal{C}$ as well.

The manifold $\mathcal{E}$ is usually infinite-dimensional in the following sense. For each point $a \in \mathcal{E}$ there is a countable system of coordinates v_i, $i \in \mathbb{Z}_+$, on a neighborhood of a. A function f on a neighborhood of a is said to be smooth at a if f depends smoothly on a finite number of the coordinates v_i.

Solutions of the PDE correspond to n-dimensional integral submanifolds of the distribution $\mathcal{C}$. Note that the Frobenius theorem on integral submanifolds of involutive distributions is not applicable, because $\mathcal{E}$ is infinite-dimensional.

In local coordinates, let a system of PDEs for functions $u^i = u^i(x_1, \dots, x_n)$, $i = 1, \dots, d$, be given by equations

$$F_\alpha\Big(x_1, \dots, x_n, u^1, \dots, u^d, \dots, \frac{\partial^k u^j}{\partial x_{i_1} \dots \partial x_{i_k}}, \dots\Big) = 0, \qquad \alpha = 1, \dots, q. \tag{1}$$

Recall that x_i, u^j, and all partial derivatives of u^j play the role of coordinates for the corresponding infinite jet space J^∞. Then $\mathcal{E}$ is the submanifold of J^∞ given by the infinite collection of equations

$$D_{x_{i_1}} \dots D_{x_{i_s}}(F_\alpha) = 0, \qquad i_k = 1, \dots, n, \quad \alpha = 1, \dots, q, \quad s = 0, 1, 2, \dots, \tag{2}$$

[2] An *n-dimensional distribution* on $\mathcal{E}$ is an n-dimensional subbundle of the tangent bundle of $\mathcal{E}$.

where D_{x_i} are the total derivatives operators. In other words, the set of equations (2) consists of all differential consequences of equations (1).

The assumption that the set $\mathcal{E}$ determined by (2) is a nonsingular submanifold of J^∞ requires some non-degeneracy conditions for equations (1). These conditions are valid on an open dense subset of J^∞ for practically all PDEs in applications. For analytic PDEs these conditions are always[3] valid on an open dense subset of J^∞.

The operators $D_{x_1}, \ldots, D_{x_n}$ can be regarded as vector fields on J^∞. They are tangent to the submanifold $\mathcal{E} \subset J^\infty$, and the restrictions of the vector fields $D_{x_1}, \ldots, D_{x_n}$ to $\mathcal{E}$ span the Cartan distribution $\mathcal{C}$ on $\mathcal{E}$.

Points of $\mathcal{E}$ are in one-to-one correspondence with 'formal solutions' of the system of PDEs, i.e., formal Taylor series satisfying the PDEs.

Note that if two systems of PDEs are isomorphic (i.e., are connected by a change of variables) then the corresponding manifolds $\mathcal{E}$ are connected by a diffeomorphism that preserves the Cartan distribution. Therefore, the pair $(\mathcal{E}, \mathcal{C})$ is the right object to study if one is interested in properties that are invariant with respect to changes of variables.

The pair $(\mathcal{E}, \mathcal{C})$ is called the *infinite prolongation* of the initial system of PDEs.

Let $(\mathcal{E}^1, \mathcal{C}^1)$ and $(\mathcal{E}^2, \mathcal{C}^2)$ be the infinite prolongations of two systems of PDEs. A *morphism* between $(\mathcal{E}^1, \mathcal{C}^1)$ and $(\mathcal{E}^2, \mathcal{C}^2)$ is a smooth map $\varphi\colon \mathcal{E}^1 \to \mathcal{E}^2$ such that for any $a \in \mathcal{E}^1$ one has $\varphi_*(\mathcal{C}^1_a) \subset \mathcal{C}^2_{\varphi(a)}$, where φ_* is the differential of φ and for $b \in \mathcal{E}^i$ the vector subspace $\mathcal{C}^i_b \subset T_b\mathcal{E}^i$ is determined by the distribution $\mathcal{C}^i$ for $i = 1, 2$.

If φ is a surjective submersion and the map

$$\varphi_*\Big|_{\mathcal{C}^1_a} : \mathcal{C}^1_a \to \mathcal{C}^2_{\varphi(a)}$$

is an isomorphism of vector spaces for any $a \in \mathcal{E}^1$ then the morphism φ is said to be a (*differential*) *covering*. Then φ maps integral submanifolds of the distribution $\mathcal{C}^1$ to integral submanifolds of the distribution $\mathcal{C}^2$ and preserves the dimension of integral submanifolds. Therefore, φ maps solutions of the system $\mathcal{E}^1$ to solutions of $\mathcal{E}^2$.

The notion of coverings was introduced by I. S. Krasil′shchik and A. M. Vinogradov [19] in order to give a geometric interpretation for various well-known constructions in the theory of nonlinear PDEs. In particular, Bäcklund transformations, Lax pairs, and Wahlquist-Estabrook prolongation structures from soliton theory are determined by coverings.

Note that even local classification of differential coverings is highly nontrivial due to different possible configurations of the distributions.

Example 2. It is easy to check that if a function $v(x, t)$ is a solution of the modified KdV equation

$$v_t = v_{xxx} - 6v^2 v_x \tag{3}$$

then the function

$$u = v_x - v^2 \tag{4}$$

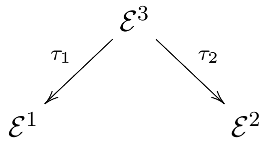
satisfies the KdV equation

$$u_t = u_{xxx} + 6uu_x, \qquad u = u(x, t). \tag{5}$$

This is the well-known Miura transformation. Formula (4) determines a covering from the infinite prolongation of equation (3) to the infinite prolongation of (5).

[3] If system (1) is inconsistent then $\mathcal{E} = \emptyset$, which can also be regarded as a (trivial) submanifold of J^∞.

A *Bäcklund transformation* between two systems $\mathcal{E}^1$ and $\mathcal{E}^2$ is given by another system $\mathcal{E}^3$ and a pair of coverings

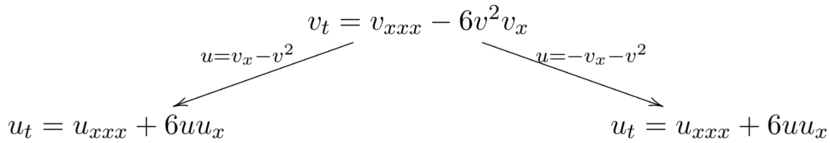


such that for any $a_i \in \mathcal{E}^i$ the fibers $\tau_i^{-1}(a_i) \subset \mathcal{E}^3$ are finite-dimensional for $i = 1, 2$.

Then one can obtain solutions of $\mathcal{E}^2$ from solutions of $\mathcal{E}^1$ (and vice versa) as follows. Take a solution s of $\mathcal{E}^1$, compute its preimage $\tau_1^{-1}(s)$ in $\mathcal{E}^3$, which is a family of solutions of $\mathcal{E}^3$, and map this family to $\mathcal{E}^2$ by τ_2. In local coordinates, in order to compute $\tau_1^{-1}(s)$ for a given solution s of $\mathcal{E}^1$ one needs to solve a system of ordinary differential equations.

If $\mathcal{E}^1 = \mathcal{E}^2 = \mathcal{E}$ then in this way one obtains new solutions for $\mathcal{E}$ from known solutions.

For example, the coverings



determine a well-known Bäcklund auto-transformation for the KdV equation. One can find hundreds of examples of Bäcklund transformations in [25, 29, 30] and references therein.

The name 'coverings' for such morphisms is used because they generalize usual topological coverings of finite-dimensional manifolds M. Indeed, let M^1, M^2 be finite-dimensional manifolds and $\varphi\colon M^1 \to M^2$ be a smooth map that is a topological covering. Then φ becomes a differential covering if we consider the distribution $\mathcal{C}^i$ equal to the entire tangent bundle of M^i for $i = 1, 2$.

Recall that topological coverings of M correspond to actions of the fundamental group $\pi_1(M, b)$ for $b \in M$.

From now on we suppose that all manifolds and maps of manifolds are complex-analytic. Let $(\mathcal{E}, \mathcal{C})$ be the infinite prolongation of an analytic PDE, and $a \in \mathcal{E}$. In [15] we have defined, in a coordinate-independent way, the fundamental Lie algebra $\mathbb{F}(\mathcal{E}, a)$, which plays the role of 'fundamental group' for differential coverings. The algebra $\mathbb{F}(\mathcal{E}, a)$ generalizes Wahlquist-Estabrook prolongation algebras [13, 35]. Note that Wahlquist-Estabrook algebras are defined only for some narrow classes of PDEs and do not have any coordinate-independent meaning.

According to [15], the Lie algebra $\mathbb{F}(\mathcal{E}, a)$ has a natural topology. For any Lie algebra L, a homomorphism $\psi\colon \mathbb{F}(\mathcal{E}, a) \to L$ is said to be *admissible* if ψ is continuous, where L is endowed with discrete topology.

An element $w \in \mathbb{F}(\mathcal{E}, a)$ is said to be *solvable* if for any Lie algebra L and any admissible homomorphism $\psi\colon \mathbb{F}(\mathcal{E}, a) \to L$ the element $\psi(w)$ lies in a solvable ideal of the algebra $\psi\big(\mathbb{F}(\mathcal{E}, a)\big)$. Clearly, solvable elements form an ideal $I(\mathcal{E}, a) \subset \mathbb{F}(\mathcal{E}, a)$.

Studied examples suggest that the ideal $I(\mathcal{E}, a)$ is not important for main applications to Bäcklund transformations. Therefore, it makes sense to consider the quotient Lie algebra $\mathbb{RF}(\mathcal{E}, a) = \mathbb{F}(\mathcal{E}, a)/I(\mathcal{E}, a)$. We do not impose any topology on $\mathbb{RF}(\mathcal{E}, a)$.

Remark 2. Below we will use the following standard construction. Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{A}$ be a commutative associative algebra over $\mathbb{C}$. Then the tensor product $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ has the Lie algebra structure

$$[g_1 \otimes a_1,\, g_2 \otimes a_2] = [g_1, g_2] \otimes a_1 a_2, \qquad g_1,\, g_2 \in \mathfrak{g}, \qquad a_1,\, a_2 \in \mathcal{A}. \tag{6}$$

Example 3. Consider the infinite-dimensional Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda],$$

where $\mathbb{C}[\lambda]$ is the algebra of polynomials in λ. If we regard λ as a coordinate on the rational curve $C = \mathbb{C}$, the algebra $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ becomes equal to the algebra of $\mathfrak{sl}_2(\mathbb{C})$-valued functions on the curve C.

Let $\mathcal{E}$ be the infinite prolongation of the KdV equation. From the description of $\mathbb{F}(\mathcal{E}, a)$ in [15] it follows that

$$\mathbb{RF}(\mathcal{E}, a) \cong \mathfrak{sl}_2(\mathbb{C}[\lambda]) \qquad \forall\, a \in \mathcal{E}. \tag{7}$$

The same result is valid also for the nonlinear Schrödinger equation.

Example 4. The Landau-Lifshitz equation reads (see, e.g, [9] and references therein)

$$\mathrm{LL}(e_1, e_2, e_3) = \left\{ S_t = S \times S_{xx} + S \times (\mathbf{J} \cdot S), \quad S = \big(s^1(x,t),\, s^2(x,t),\, s^3(x,t)\big), \quad \sum_{i=1}^{3} (s^i)^2 = 1 \right\}, \tag{8}$$

where $\mathbf{J} = \mathrm{diag}(e_1, e_2, e_3)$ is a constant diagonal (3×3)-matrix with $e_1, e_2, e_3 \in \mathbb{C}$ and $\times$ is the usual vector product. We consider the *fully anisotropic* case $e_1 \neq e_2 \neq e_3 \neq e_1$.

Consider the ideal $\mathcal{I} \subset \mathbb{C}[v_1, v_2, v_3]$ generated by the polynomials

$$v_i^2 - v_j^2 + (e_i - e_j), \qquad i,\, j = 1, 2, 3. \tag{9}$$

Set

$$E = \mathbb{C}[v_1, v_2, v_3]/\mathcal{I}. \tag{10}$$

Then E is the algebra of regular functions on the elliptic curve $C \subset \mathbb{C}^3$ defined by polynomials (9). Let $\bar{v}_j \in E$ be the image of $v_j \in \mathbb{C}[v_1, v_2, v_3]$. Consider also a basis α_1, α_2, α_3 of the Lie algebra $\mathfrak{so}_3(\mathbb{C})$ with the relations

$$[\alpha_1, \alpha_2] = \alpha_3, \qquad [\alpha_2, \alpha_3] = \alpha_1, \qquad [\alpha_3, \alpha_1] = \alpha_2 \tag{11}$$

and endow the space $\mathfrak{so}_3(\mathbb{C}) \otimes_{\mathbb{C}} E$ with the Lie algebra structure described in (6).

Denote by $\mathfrak{R}(e_1, e_2, e_3)$ the Lie subalgebra generated by the elements

$$\alpha_1 \otimes \bar{v}_1,\ \alpha_2 \otimes \bar{v}_2,\ \alpha_3 \otimes \bar{v}_3 \ \in \ \mathfrak{so}_3(\mathbb{C}) \otimes E.$$

The Lie algebra $\mathfrak{R}(e_1, e_2, e_3)$ is infinite-dimensional, it was studied in [28] and is isomorphic to a subalgebra of algebras studied in [4, 11].

Let $\mathcal{E}$ be the infinite prolongation of the Landau-Lifshitz equation. From the description of $\mathbb{F}(\mathcal{E}, a)$ in [15] it follows that

$$\mathbb{RF}(\mathcal{E}, a) \cong \mathfrak{R}(e_1, e_2, e_3) \qquad \forall\, a \in \mathcal{E}. \tag{12}$$

For constants e_1, e_2, $e_3 \in \mathbb{C}$ with $e_1 \neq e_2 \neq e_3 \neq e_1$, consider the Krichever-Novikov equation [21, 34]

$$\mathrm{KN}(e_1, e_2, e_3) = \left\{ u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \qquad u = u(x,t) \right\}. \tag{13}$$

According to [15], for the infinite prolongation $\mathcal{E}$ of this PDE, there is an isomorphism (12) as well.

That is, $\mathbb{RF}(\mathcal{E}, a)$ in the above-mentioned examples is isomorphic to an infinite-dimensional Lie algebra of certain matrix-valued functions on an algebraic curve C, where C is rational in Example 3 and is elliptic in Example 4.

However, this description of $\mathbb{RF}(\mathcal{E}, a)$ does not yet give an invariant meaning of the curve C, because isomorphisms (7), (12) are not canonical. In order to recover an algebraic curve from $\mathbb{RF}(\mathcal{E}, a)$ in an invariant way, we use the following algebraic constructions.

Let L be a Lie algebra. Recall that a linear map $g\colon L \to L$ is called an *intertwining operator* if $g([p_1, p_2]) = [g(p_1), p_2] = [p_1, g(p_2)]$ for any $p_1, p_2 \in L$. Equivalently, $g \circ \operatorname{ad}(p_1) = \operatorname{ad}(p_1) \circ g$, that is, the map g is an intertwining operator for the adjoint representation of L.

Such operators are often used in the study of integrable PDEs with Lax pairs (e.g., for construction of Poisson structures [27] and symmetry recursion operators [5]). Relations of our results with that of [5] are discussed in Remark 3 below.

We need the following version of intertwining operators. Consider a pair (h, H), where $H \subset L$ is a Lie subalgebra of finite codimension and $h\colon H \to L$ is a linear map satisfying $h([p_1, p_2]) = [h(p_1), p_2] = [p_1, h(p_2)]$ for any $p_1, p_2 \in H$.

Two such pairs (h, H) and $(\tilde{h}, \tilde{H})$ are called equivalent if there is a subalgebra $U \subset H \cap \tilde{H}$ of finite codimension such that $h\big|_U = \tilde{h}\big|_U$. Denote by $[(h, H)]$ the corresponding equivalence class, and let $\mathbb{IT}(L)$ be the set of equivalence classes. Then $\mathbb{IT}(L)$ has a natural structure of associative algebra, where the sum and the product are defined as follows.

$$[(h_1, H_1)] + [(h_2, H_2)] = [(h_1 + h_2,\ H_1 \cap H_2)], \qquad [(h_1, H_1)] \cdot [(h_2, H_2)] = [(h_1 \circ h_2,\ \hat{H})],$$
$$\hat{H} = \big\{ w \in H_1 \cap H_2 \ \big|\ h_2(w) \in H_1 \big\}.$$

Clearly, if $\dim L < \infty$ then $\mathbb{IT}(L) = 0$.

We define $\mathbb{A}(\mathcal{E})$ as follows. For $a \in \mathcal{E}$, set

$$\mathbb{A}(\mathcal{E}, a) = \mathbb{IT}\Big(\mathbb{RF}(\mathcal{E}, a)\Big). \tag{14}$$

Then $\mathbb{A}(\mathcal{E})$ is the family of associative algebras $\mathbb{A}(\mathcal{E}, a)$, $a \in \mathcal{E}$.

Since $e_1 \neq e_2 \neq e_3 \neq e_1$, it is easily seen that the ring E defined in (10) is an integral domain. Consider the fraction field F of E.

Clearly, $\bar{v}_1^2 + e_1 = \bar{v}_2^2 + e_2 = \bar{v}_3^2 + e_3$ in E. Let $Q(e_1, e_2, e_3) \subset F$ be the subfield generated by the elements $z = \bar{v}_i^2 + e_i$, $y = \bar{v}_1 \bar{v}_2 \bar{v}_3$. Since $y^2 = (z - e_1)(z - e_2)(z - e_3)$, the field $Q(e_1, e_2, e_3)$ is isomorphic to the function field of the elliptic curve

$$C(e_1, e_2, e_3) = \big\{ (\tilde{z}, \tilde{y}) \in \mathbb{C}^2 \ \big|\ \tilde{y}^2 = (\tilde{z} - e_1)(\tilde{z} - e_2)(\tilde{z} - e_3) \big\} \subset \mathbb{C}^2. \tag{15}$$

Theorem 1. *For any Lie subalgebra $L \subset \mathfrak{R}(e_1, e_2, e_3)$ of finite codimension, the associative algebra $\mathbb{IT}(L)$ is commutative and is isomorphic to $Q(e_1, e_2, e_3)$.*

Proof. The space $\mathfrak{so}_3(\mathbb{C}) \otimes F$ has the F-module structure given by

$$f_1 \cdot \big(w \otimes f_2\big) = w \otimes f_1 f_2, \qquad w \in \mathfrak{so}_3(\mathbb{C}), \quad f_1, f_2 \in F.$$

Since $E \subset F$, one has the natural inclusions of Lie algebras

$$\mathfrak{R}(e_1, e_2, e_3) \ \subset\ \mathfrak{so}_3(\mathbb{C}) \otimes E \ \subset\ \mathfrak{so}_3(\mathbb{C}) \otimes F.$$

For each $f \in F$ consider the map

$$G_f\colon \mathfrak{R}(e_1, e_2, e_3) \to \mathfrak{so}_3(\mathbb{C}) \otimes F, \qquad G_f(p) = f \cdot p, \qquad p \in \mathfrak{R}(e_1, e_2, e_3).$$

Obviously,

$$G_f([p_1, p_2]) = [G_f(p_1), p_2] = [p_1, G_f(p_2)] \qquad \forall\, p_1,\, p_2. \tag{16}$$

Recall that

$$z = \bar{v}_1^2 + e_1 = \bar{v}_2^2 + e_2 = \bar{v}_3^2 + e_3, \qquad y = \bar{v}_1 \bar{v}_2 \bar{v}_3. \tag{17}$$

According to [28], the following elements form a basis of $\mathfrak{R}(e_1, e_2, e_3)$

$$\alpha_i \otimes \bar{v}_i z^l, \quad \alpha_i \otimes \bar{v}_j \bar{v}_k z^l, \quad i, j, k = 1, 2, 3, \quad j < k, \quad j \neq i \neq k, \quad l = 0, 1, 2, \dots \tag{18}$$

Let $d_1(y, z)$ be a polynomial in y, z and $d_2(z) \neq 0$ be a polynomial in z. Using basis (18), one gets that

$$G_{d_1(y,z)}\big(\mathfrak{R}(e_1, e_2, e_3)\big) \subset \mathfrak{R}(e_1, e_2, e_3), \qquad G_{d_2(z)}\big(\mathfrak{R}(e_1, e_2, e_3)\big) \subset \mathfrak{R}(e_1, e_2, e_3),$$

and the space $G_{d_2(z)}\big(\mathfrak{R}(e_1, e_2, e_3)\big)$ is of finite codimension in $\mathfrak{R}(e_1, e_2, e_3)$. Using this property and the assumption $\operatorname{codim} L < \infty$, we obtain that

$$\text{the subspace } \tilde{L} = \big\{ w \in L \;\big|\; G_{d_1(y,z)}(w) \in G_{d_2(z)}(L) \big\} \text{ is of finite codimension in } L. \tag{19}$$

Since $y^2 = (z - e_1)(z - e_2)(z - e_3)$, any element $f \in Q(e_1, e_2, e_3)$ can be presented as a fraction of such polynomials $f = \dfrac{d_1(y, z)}{d_2(z)}$. Then from property (19) it follows that the subspace

$$L_f = \big\{ w \in L \;\big|\; G_f(w) \in L \big\}$$

is of finite codimension in L. Relation (16) implies that L_f is a Lie subalgebra of L. Therefore, the pair (G_f, L_f) determines an element of $\mathbb{IT}(L)$, and we obtain the embedding

$$\Psi \colon Q(e_1, e_2, e_3) \hookrightarrow \mathbb{IT}(L), \qquad \Psi(f) = [(G_f, L_f)].$$

It remains to show that the map Ψ is surjective.

Let $[(h, H)] \in \mathbb{IT}(L)$, where $H \subset L$ is a subalgebra of finite codimension and

$$h \colon H \to L, \qquad h([p_1, p_2]) = [h(p_1), p_2] = [p_1, h(p_2)] \qquad \forall\, p_1,\, p_2 \in H. \tag{20}$$

Let $\mathfrak{R}^i \subset \mathfrak{R}(e_1, e_2, e_3)$ be the subspace spanned by elements (18) for fixed $i = 1, 2, 3$. Then $\mathfrak{R}(e_1, e_2, e_3) = \mathfrak{R}^1 \oplus \mathfrak{R}^2 \oplus \mathfrak{R}^3$ as vector spaces, and

$$\forall\, w \in \mathfrak{R}^i \quad \text{there is a unique } f \in Q(e_1, e_2, e_3) \text{ such that } w = \alpha_i \otimes \bar{v}_i f. \tag{21}$$

Set $H^i = \mathfrak{R}^i \cap H$. Due to properties (11), (21), the space $\tilde{H} = H^1 + H^2 + H^3$ is a Lie subalgebra of H. Since H is of finite codimension in $\mathfrak{R}(e_1, e_2, e_3)$, the subalgebra $\tilde{H}$ is of finite codimension in H.

Let $w_i \in H^i,\ w_i \neq 0,\ i = 1, 2, 3$. Then $[h(w_i), w_i] = h([w_i, w_i]) = 0$. From (11), (21) it follows that $h(w_i) = f_i \cdot w_i$ for some $f_i \in Q(e_1, e_2, e_3)$. Then

$$h([w_1, w_2]) = [h(w_1), w_2] = [w_1, h(w_2)] = f_1 \cdot [w_1, w_2] = f_2 \cdot [w_1, w_2]. \tag{22}$$

Since, by properties (11), (21), one has $[w_1, w_2] \neq 0$, relation (22) implies $f_1 = f_2$. Similarly, one shows that $f_1 = f_2 = f_3$.

Therefore, for any other nonzero elements $w'_i \in H^i$, we also get $h(w'_i) = f' \cdot w'_i$ for some $f' \in Q(e_1, e_2, e_3)$. Similarly to (22), one obtains $h([w_1, w'_2]) = f_1 \cdot [w_1, w'_2] = f' \cdot [w_1, w'_2]$, which implies $f' = f_1$.

Thus there is a unique $f' \in Q(e_1, e_2, e_3)$ such that $h\big|_{\tilde{H}} = G_{f'}\big|_{\tilde{H}}$. Therefore, $[(h, H)] = [(G_{f'}, \tilde{H})]$ in $\mathbb{IT}(L)$, that is, $[(h, H)] = \Psi(f')$.

□

Similarly to Theorem 1, one proves the following result.

Theorem 2. *For any Lie subalgebra $L \subset \mathfrak{sl}_2(\mathbb{C}[\lambda])$ of finite codimension, the associative algebra $\mathbb{IT}(L)$ is commutative and is isomorphic to the field of rational functions in λ.*

Combining Theorems 1, 2 with isomorphisms (7), (12), we obtain the following description of $\mathbb{A}(\mathcal{E},a)$ for the above-mentioned PDEs.

Theorem 3. *For the KdV and nonlinear Schrödinger equations, the algebra $\mathbb{A}(\mathcal{E},a)$ is isomorphic to the field of rational functions in λ.*

For the Landau-Lifshitz (8) *and Krichever-Novikov equations* (13), *the algebra $\mathbb{A}(\mathcal{E},a)$ is isomorphic to the field $Q(e_1,e_2,e_3)$.*

The next theorem shows that $\mathbb{F}(\mathcal{E},a)$ and $\mathbb{A}(\mathcal{E},a)$ allow in many cases to obtain non-existence results for Bäcklund transformations (BT).

Theorem 4. *There is no BT between KdV and Krichever-Novikov, between nonlinear Schrödinger and Landau-Lifshitz.*

For $i=1,2,3$, let $e_i,e'_i\in\mathbb{C}$ be such that $e_1\neq e_2\neq e_3\neq e_1$ and $e'_1\neq e'_2\neq e'_3\neq e'_1$. Consider the corresponding Landau-Lifshitz (8), *Krichever-Novikov equations* (13), *and elliptic curves* (15). *Suppose that the curve $C(e_1,e_2,e_3)$ is not birationally equivalent to $C(e'_1,e'_2,e'_3)$ (that is, $Q(e_1,e_2,e_3)$ is not isomorphic to $Q(e'_1,e'_2,e'_3)$). Then there is no BT between* $\mathrm{KN}(e_1,e_2,e_3)$ *and* $\mathrm{KN}(e'_1,e'_2,e'_3)$*, between* $\mathrm{LL}(e_1,e_2,e_3)$ *and* $\mathrm{LL}(e'_1,e'_2,e'_3)$.

Proof. Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be the infinite prolongations of some of the PDEs mentioned in the theorem. According to [15], if there is a BT between (some open subsets of) $\mathcal{E}^1$ and $\mathcal{E}^2$ then there are open subsets $\hat{\mathcal{E}}^1\subset\mathcal{E}^1$, $\hat{\mathcal{E}}^2\subset\mathcal{E}^2$, coverings

$$\hat{\mathcal{E}}^1 \xleftarrow{\ \tau_1\ } \hat{\mathcal{E}}^3 \xrightarrow{\ \tau_2\ } \hat{\mathcal{E}}^2$$

and points $a_i\in\hat{\mathcal{E}}^i$, $i=1,2,3$, such that $\tau_1(a_3)=a_1$, $\tau_2(a_3)=a_2$, and

(23) $\qquad \mathbb{F}(\hat{\mathcal{E}}^3,a_3)$ is isomorphic to a subalgebra of $\mathbb{F}(\mathcal{E}^i,a_i)$ of finite codimension, $i=1,2$.

From the description of $\mathbb{F}(\mathcal{E},a)$ given in [15] for the above-mentioned PDEs, it follows that property (23) in the considered cases implies

(24) $\qquad \mathbb{RF}(\hat{\mathcal{E}}^3,a_3)$ is isomorphic to a subalgebra of $\mathbb{RF}(\mathcal{E}^i,a_i)$ of finite codimension, $i=1,2$.

Combining (24) with Theorems 1, 2, isomorphisms (7), (12), and definition (14), we obtain that if there is a BT between $\mathcal{E}^1$ and $\mathcal{E}^2$ then $\mathbb{A}(\hat{\mathcal{E}}^3,a_3)\cong\mathbb{A}(\mathcal{E}^1,a_1)\cong\mathbb{A}(\mathcal{E}^2,a_2)$.

Then the required statement follows from the description of $\mathbb{A}(\mathcal{E},a)$ given in Theorem 3. □

Remark 3. Recently, D. K. Demskoi and V. V. Sokolov [5] obtained the following interesting results on the role of elliptic curves for the Landau-Lifshitz and Krichever-Novikov equations. They constructed some algebra $\mathcal{A}$ of symmetry recursion operators for these PDEs such that $\mathcal{A}$ is isomorphic to the algebra of polynomial functions on the elliptic curve (15).

Note that this does not provide a coordinate-independent interpretation for the curve, because the definition of symmetry recursion operators in [5] uses the symbol D_x^{-1} for a specific coordinate x. Also, there is no proof that the algebra $\mathcal{A}$ contains all symmetry recurion operators of the above-mentioned PDEs.

In order to construct recursion operators for the Landau-Lifshitz equation, Demskoi and Sokolov use the fact that this PDE has a Lax pair with values in a Lie algebra isomorphic to $\mathfrak{R}(e_1,e_2,e_3)$. Also, they noticed that, for functions z, y given by (17), one has

$$z\cdot\mathfrak{R}(e_1,e_2,e_3)\subset\mathfrak{R}(e_1,e_2,e_3),\qquad y\cdot\mathfrak{R}(e_1,e_2,e_3)\subset\mathfrak{R}(e_1,e_2,e_3).$$

This observation from [5] gave us the idea to recover the curves by means of intertwining operators of $\mathfrak{R}(e_1, e_2, e_3)$.

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