Cohomology of arithmetic groups and special values of automorphic L-functions on GL_n

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July 26, 2000

Introduction

To any automorphic representation π of the adelic points of the group GL_n one can attach an L-function $L(\pi, s)$ defined by an infinte product of Euler type, convergent for $\mathrm{Re}(s)$ sufficiently large but which can be holomorphically continued to an entire function if π is cuspidal. We want to investigate the properties of the values of these L-functions at critical points l of the representation π. More precisely we want to study the properties of the collection of values $L(\pi \otimes \chi, l)$, where χ runs over all idele class characters of finite order and conductor a power of the prime $\mathfrak{p}$ and l is any non-positive critical value for π. In particular, this fixes the type at infinity of χ to be the trivial character id or the signum character sgn. In view of the functional equation we may restrict to points $l \leq 0$. If n is even, $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$ and π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$, which is of cohomological type with coefficients in some finite dimensional representation (ρ, V) and which satisfies a certain regularity condition (cf. Definition 4.3), we obtain the following result (cf. Theorem 6.7):

For all critical points $l \leq 0$ there is a complex period $\Omega_l(\pi)$ such that for all χ as above we have

$$\frac{L(\pi \otimes \chi, l)}{\Omega_l(\pi)} \in E_\pi(\chi), \tag{0.1}$$

where E_π is the field of definition of π. Moreover there is a finite abelian extension K/E_ρ, where $E_\rho \leq E_\pi$ is the field of definition of ρ such that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ we have

$$\left(G(\chi)^{\frac{n}{2}} \frac{L(\pi \otimes \chi, l)}{\Omega_l(\pi)} \right)^\sigma = G(\chi^\sigma)^{\frac{n}{2}} \frac{L(\pi^\sigma \otimes \chi^\sigma, l)}{\Omega_l(\pi^\sigma)}. \tag{0.2}$$

The proof of this result is by an induction over the rank n of the group GL_n as follows. Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ for arbitrary $n \geq 4$ of the above type. Depending on the component at infinity π_∞ we define a representation $\hat{\pi}_\infty$ of $\mathrm{GL}_{n-2}(\mathbb{R})$ such that for all cuspidal representation $\hat{\pi}$ of $\mathrm{GL}_{n-2}(\mathbb{A})$ with infinity component $\hat{\pi}_\infty$ the following holds (cf. Theorem 6.6)

There is a period $\Omega_l(\pi, \pi') \in \mathbb{C}^*$ *and an integer* $l' \leq 0$ *depending on* l *such that*

$$\frac{L(\pi \otimes \chi, l)}{L(\pi' \otimes \chi, l')} \in \Omega_l(\pi, \pi') E_\pi E_{\pi'}(\chi) \tag{0.3}$$

and for all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$

$$\left(\frac{G(\chi)}{\Omega_l(\pi, \pi')} \frac{L(\pi \otimes \chi, l)}{L(\pi' \otimes \chi, l')}\right)^\sigma = \frac{G(\chi^\sigma)}{\Omega_l(\pi^\sigma, \pi'^\sigma)} \frac{L(\pi^\sigma \otimes \chi^\sigma, l)}{L(\pi'^\sigma \otimes \chi^\sigma, l')}. \tag{0.4}$$

If now n is even and π satisfies the regularity condition we find that the same is true for $\hat{\pi}$ and l' again is a critical point for π'. Hence, we can inductively apply Eq. (0.3) ,(0.4) to reduce to the case of L-functions on $\mathrm{GL}_2(\mathbb{A})$. Thus Eq. (0.1) and (0.2) finally follow from the well known algebraicity properties of the values at critical integers of the L-functions $L(f, s)$ attached to holomorphic elliptic modular forms.

The central point is the compuation of the quotient of quotient of L-functions appearing in (0.3). Since we look to negative points l, we have to make use of the analytic continuation of the Euler product defining $L(\pi, s)$ to compute (0.3). In general there are two quite different approaches to the problem of analytic continuation, the first one being the theory of Mellin transforms of automorphic forms and second one being the theory of the constant term of Eisenstein series as developped by Langlands and generalized to generic Fourier coefficients by Shahidi. We will combine the theory of the Mellin transform as well as the Langlands-Shahidi theory to compute (0.3).

In more detail, we let $P \leq \mathrm{GL}_{n-1}$ be the maximal parabolic subgroup of type $(n-2, 1)$. For any character χ as above the product $\hat{\pi}[\chi] = \hat{\pi} \otimes \chi$ defines a cuspidal representation of the Levi subgroup $M_P(\mathbb{A})$ and we consider the Mellin transform on $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ of automorphic forms, which are contained in the space belonging to the representation

$$\pi \times \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_{n-1}(\mathbb{A})} \hat{\pi} \otimes \chi.$$

Such a Mellin transform for forms on the group $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ is given by the Zeta integral belonging to the Rankin-Selberg convolutions introduced in [J-S]. We shall choose a special pair of automorphic forms ϕ and $E_{\hat{\pi}[\chi]}$ corresponding to π resp. $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_{n-1}(\mathbb{A})} \hat{\pi} \otimes \chi$ for which we compute the Mellin transform. To this end we compute the local factors of the Rankin-Selberg integral at all places, which in particular implies that we have to compute the Whittaker function of the Eisenstein series $E_{\hat{\pi}[\chi]}$. Since the representation $\hat{\pi}[\chi]$ ramifies at $\mathfrak{p}$ we have to extend the Langlands-Shahidi method to certain ramified induced representations as well. Of course this relies on a generalization of the Casselman-Shalika formulas for the spherical Whittaker function.

As a result we obtain a formula for the quotient of L-functions in (0.3) as an integral involving the forms ϕ_π and $E_{\hat{\pi}[\chi]}$.

Let now n be even. Under the assumption that π has non-vanishing cohomology this integral involving ϕ_π and $E_{\pi'}$ can be computed in a purely topological manner: there is a class ω corresponding to ϕ_π and an Eisenstein class $\omega_{\mathrm{Eis},\pi'}$ in the cohomology of the symmetric space of GL_n and GL_{n-1} with coefficient system of highest weight $\mathbf{m}$ and λ respectively such that

$$(0.5)\qquad \frac{L(\pi\otimes\chi,l)}{L(\pi'\otimes\chi,l')} = \text{Lin. comb.}\quad \int_{\substack{\text{fundamental}\\ \text{class}}} r_u^*\omega\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\right)\cup\omega_{\mathrm{Eis},\pi'},$$

where $u\in N_n(F_{\mathfrak{p}})$ runs over a finite set of matrices. Using the theory developped in [Ha 1] and [Sch] we are able to compute the behaviour of the analytically defined class $\omega_{\mathrm{Eis},\pi'}$ under the operation of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ (cf. Theorem 5.4). To establish (0.5) as well as the rationality of the Esienstein class it is decisive that the highest weight λ of the coefficient system satisfies certain properties (cf. $(\mathbf{W.1})$ - $(\mathbf{W.5})$).

Since the integral (0.5) is constructed by topological methods it commutes with the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and we deduce the claims (0.3) and (0.4).

In view of a study of integrality properties, we proceed a little differently than described above. We use the result of the computation of the Mellin transform of ϕ_π and $E_{\pi'}$ to define a p-adic distribution $d\mathbf{P}_{\pi,l}$ on $\mathbb{Z}_p^*$ in section 3 such that its p-adic Mellin transform satisfies for all χ as above

$$(0.6)\qquad \int_{\mathbb{Z}_p^*}\chi\, d\mathbf{P}_{\pi,l} = \frac{L(\pi\otimes\chi,l)}{L(\pi'\otimes\chi,l')}$$

and we then compute $d\mathbf{P}_{\pi,l}$ via an integral like (0.5) (cf. Proposition 6.2). Quite analogous we obtain the rationality of the values $d\mathbf{P}(U)$, $U\leq\mathbb{Z}_p^*$, which via (0.6) implies (0.3) and (0.4). Since it seems to be very difficult to determine the denominators fo the Eisenstein classes we can not say anything about p-adic analytic properties of the vlues of $L(\pi\otimes\chi,l)$; on the other hand the Eisenstein classes obviously being independent of π at least we see, that there is a univeral bound for the denominators of the congruences satisfied by the values of $L(\pi\otimes\chi,l)$.

To ensure that we are not constructing the zero distribution we have to make the

Assumption. A certain integral involving terms at infinity does not vanish

(cf. section 6.2). Finally we remark that instead of the maximal parabolic subgroup P it may be possible to apply the method to more general parabolic subgroups $P\leq\mathrm{GL}_{n-1}$ of type $(n_1,\dots,n_t,1)$ to obtain different formulas for the critical values $L(\pi\otimes\chi\eta,l)$.

In section 1 we select the Eisenstein series $E_{\hat{\pi}[\chi]}$ and compute its Whittaker function. In particular we generalize the Casselman-Shalika formula for the computationof values of Whittaker functions on elements of the split torus to the place $\mathfrak{p}$, where the representation $\hat{\pi}[\chi]$ ramifies. The proof, again, is by induction on the rank n. In section 2 we introduce the Mellin transform and we reduce its computation to the computation of the local Rankin-Selberg convolution at $\mathfrak{p}$. The final formula then appers in Theorem 2.7. In section 3 we define the distribution $d\mathbf{P}_\pi$ and compute its p-adic Mellin transform (cf. Theorem 3.5). In section 4 explain the rationality properties of (regular) algebraic automorphic representations relying on [C]. Section 5 contains the (analytic) construction of the Eisenstein cohomology classes and also the proof of their rationality in the cases, which are relevant to us (cf. Theorem 5.4). Finally in section 6 we apply all this to the computation of (0.3) and in 6.3 we explain the induction step, which finally yields (0.1) and (0.2). To conclude the induction step we separate two different cases into which the representation π can fall (cf. Definition 4.3).

Notations.

We denote by $B_n \leq \mathrm{GL}_n$ the Borel subgroup of upper triangular matrices and we write N_n resp. T_n for the unipotent radical of B_n resp. the maximal torus consisting of diagonal matrices. Let $P \leq \mathrm{GL}_n$ be a parabolic subgroup. We denote by U_P the unipotent radical and by $M_P = P/U_P$ the Levi component of P and we write A_P for the maximal central torus in M_P. By $\mathfrak{m}$, ${}^0\mathfrak{m}$, $\mathfrak{n}$ we understand the Lie algebras of $M_P(\mathbb{R})$, ${}^0M_P(\mathbb{R})$, $N_P(\mathbb{R})$. We set $\mathfrak{a}_P = \mathrm{Lie}(A_P)$ and $\mathfrak{b}_P = \mathfrak{h} \cap {}^0\mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of the standard torus T_{n-1} of diagonal matrices in GL_{n-1}. We let $\Phi_{n-1} = \Phi(\mathrm{GL}_{n-1}, T_{n-1})$ be the set of roots of GL_{n-1} with respect to the standard torus. Our choice of the standard minimal parabolic subgroup $B_{n-1} \leq \mathrm{GL}_{n-1}$ determines a choice of positive roots $\Phi^+(\mathrm{GL}_{n-1}, T_{n-1})$ of GL_{n-1} and we denote by $\Delta_{n-1} = \Delta(\mathrm{GL}_{n-1}, T_{n-1})$ resp. $\Delta(M_P, T_{n-1})$ the set of simple positive roots of GL_{n-1} resp. M_P. with respect to the standard torus in GL_{n-1}. Moreover by $\Delta(P, A_P)$ we denote the simple roots of P with respect to the split component. We set

$$\rho_P = 1/2 \sum_{\alpha \in \Delta(P, A_P)} \alpha$$

and also $\check{\rho} = \rho_{B_n}$, i.e. $\check{\rho}$ is the half sum of positive roots in GL_n. Finally we write δ_P to denote the modulus of the parabolic subgroup P.

For any algebraic group $G/\mathbb{Q}$ we denote by $X(G)$ the set of all $\mathbb{Q}$-morphisms $G \to \mathrm{GL}_1$ and we denote by

$$\langle\ ,\ \rangle : X(A_P) \times A_P \to \mathbb{G}_m$$

the natural pairing. We define

$${}^0G = \bigcap_{\chi \in X_{\mathbb{Q}}(G)} \ker \chi^2.$$

For any parabolic subgroup $P \leq \mathrm{GL}_n$ the connected component of the unit element of the $\mathbb{R}$ rational points $A_P^0 = A_P(\mathbb{R})$ is a split component for P and we have the Langlands decomposition

$$P(\mathbb{R}) = {}^0M_P(\mathbb{R}) A_P^0 U_P(\mathbb{R}).$$

We note that A_P^0 is isomorphic to $(\mathbb{R}_+^*)^t$ via

$$(a_1, \cdots, a_t \mapsto \mathrm{diag}(a_1, \ldots, a_1, \ldots, a_t, \ldots, a_t).$$

We identify characters of A_P^0 with characters of $\mathfrak{a}_P$ via the exponential mapping

$$\exp : \mathfrak{a}_P \to A_P^0.$$

Let F be a number field. We denote by $\mathcal{O}$ resp. $\mathbb{A}$ the ring of integers resp. ring of adeles $\mathbb{A}$ of F and S_∞ is the set of archimedean places of F. We write $\mathcal{O}_v$ for the completion of $\mathcal{O}$ at v and we set $\hat{\mathcal{O}} = \prod_{v \notin S_\infty} \mathcal{O}_v$. For any finite place v we denote by q_v resp. ϖ_v the cardinality of the residue field resp. a local prime element at the place v.

More specific, let $F/\mathbb{Q}$ be a totally real number field. For any finite place v we define the maximal compact subgroup $K_{n,v} = \mathrm{GL}_n(\mathcal{O}_v)$ and if $v \in S_\infty$ corresponds to an embedding $F \hookrightarrow \mathbb{R}$ we set $K_{n,v} = \mathrm{SO}_n(\mathbb{R}) Z_n^0(\mathbb{R})$. We write $K_{n,f}$ for the product of the $K_{n,v}$ over all finite places and $K_{n,\infty}$ for the product of the $K_{n,v}$ over all infinite places. Moreover, by

$$\pi_0 = \prod_{v \in S_\infty} \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$$

we understand the group of connected components of $\mathrm{GL}_n(\prod_{v \in S_\infty} F_v)$. Obviously, $\pi_0 \cong (\mathbb{Z}/2\mathbb{Z})^{[F:\mathbb{Q}]}$.

If $P \leq \mathrm{GL}_n$ is a parabolic subgroup and $K \leq \mathrm{GL}_n(\mathbb{R})$ a compact subgroup we define K^{M_P} to be the image of the intersection $K \cap P(\mathbb{R})$ in $M_P(\mathbb{R})$. In the same way we define $K^{M_P} \leq M_P(\mathbb{A}_f)$ for compact subgroups $K \leq \mathrm{GL}_n(\mathbb{A}_f)$.

By $\mathcal{A}_F(n)$ we understand the space of automorphic forms on $\mathrm{GL}_n(\mathbb{A})$ and $L_0^2(\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}))$ is the space of all cusp forms.

We write $\mathbf{1}_G$ to denote the unit element in G and we abbreviate $\mathbf{1}_n = \mathbf{1}_{\mathrm{GL}_n}$. Finally we denote by $[x]$, $x \in \mathbb{R}$ the largest integer equal to or smaller than x.

Acknowledgements. This work was begun while staying at Université de Paris Sud, Orsay and the main part of it was completed during a stay at Max-PLanck Institut für Mathematik, Bonn. It was written down while staying at University of Heidelberg. I am greatful to these institutions for their invitations.

1 Generic Fourier coefficients of Eisenstein series

Let $F/\mathbb{Q}$ be a finite extension and let $\mathbb{A}$ denote its ring of adeles. To technically simplify the exposition we assume that F has class number equal to one. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, $n \in \mathbb{N}$. We fix a prime $\mathfrak{p}$ of $\mathcal{O}_F$

such that $\pi_{\mathfrak{p}}$ does not ramify and we let $\mathbf{X}_{\mathfrak{p}}$ be the set of all idele class characters $\chi : F^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ of finite order with infinity component $\chi_\infty = \mathrm{id}$ and conductor $f_\chi = \mathfrak{p}^e$ a $\mathfrak{p}$-power.

We want to describe a formula, which for a pair of cuspidal representations π resp. $\hat{\pi}$ on $\mathrm{GL}_n(\mathbb{A})$ resp. $\mathrm{GL}_{n-2}(\mathbb{A})$ computes the behaviour of the ratio

$$\frac{L(\pi\otimes\chi, n)}{L(\hat{\pi}\otimes\chi, n)} \tag{1.1}$$

of principal L-function as the representations vary over twists with all characters χ in $\mathbf{X}_{\mathfrak{p}}$. To derive this formula we shall simultaneously apply the theory of Rankin-Selberg convolutions an the Langlands-Shahidi theory. In this section we explain the way we apply the Langlands-Shahidi method, which relies on the computation of generic Fourier coefficients of Eisenstein series. In particular we extend the method to include ramified primes. This will rely on a extension of the formulas for the values of calss-1 Whittaker functions to certain ramified representations.

We let $\hat{\pi}$ be a unitary, cuspidal representation of $\mathrm{GL}_{n-2}(\mathbb{A})$ and we denote by S' the set of all places, where $\hat{\pi}$ ramifies including the archimedean ones. We assume that $\mathfrak{p}\notin S'$, i.e. $\hat{\pi}_{\mathfrak{p}}$ is of class-1. We let $P \le \mathrm{GL}_{n-1}$ be the maximal parabolic subgroup of type $(n-2,1)$. In particular, the Levi subgroup of M_P of P is isomorphic to $\mathrm{GL}_{n-2}\times\mathrm{GL}_1$ and we obtain an embedding of GL_{n-2} into M_P

$$\begin{array}{ccccc} & & P & \subset & \mathrm{GL}_{n-1} \\ & & \uparrow & & \\ \mathrm{GL}_{n-2} & \hookrightarrow & \mathrm{GL}_{n-2}\times\mathrm{GL}_1 & & \end{array}$$

For any character $\chi\in\mathbf{X}_{\mathfrak{p}}$ we define the cuspidal representation of the Levi subgroup M_P

$$\hat{\pi}[\chi] = \hat{\pi}\otimes\chi$$

and we denote by $V_{\hat{\pi}[\chi]}$ be the space of cusp forms on which $\hat{\pi}[\chi]$ can be realized. We note that $\hat{\pi}[\chi]$ factorizes over $A_P^0\backslash\mathrm{GL}_{n-1}(\mathbb{A})$, hence the restriction of the central character $\omega_{\hat{\pi}[\chi]}$ to A_P^0 vanishes identically.

For any place v we define a local height function

$$H_{P,v} : M_P(F_v) \to \mathfrak{a}_P$$

attached to the parabolic P: we set $e_v = q_v$ if v is finite and $e_v = \exp$ if v is archimedean; $H_{P,v}$ then is the unique function satisfying

$$|\nu(m)|_v = e_v^{\langle\nu, H_{P,v}(m)\rangle}$$

for all rational characters $\nu\in X(M_P)_{\mathbb{Q}}$ and $m\in M_P(F_v)$. Using the decomposition $\mathrm{GL}_{n-1}(F_v) = M_P(F_v)N_P(F_v)K_{n-1,v}$ we may extend $H_{P,v}$ to $\mathrm{GL}_{n-1}(F_v)$. The (global) height function

$$H_P : \mathrm{GL}_{n-1}(\mathbb{A}) \to \mathfrak{a}_P$$

is defined via the equation

$$e^{\langle \nu, H_P(m)\rangle} = \prod_v e_v^{\langle \nu, H_{P,v}(m_v)\rangle}$$

for all $\nu \in X(M_P)_{\mathbb{Q}}$.

Let $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. With the above choice of data we then define the (unitarily) induced representation

$$\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi] = \mathrm{Ind}_{P_{\mathbb{A}}}^{\mathrm{GL}_{n-1}(\mathbb{A})} e^{\langle \Lambda, H_P(\cdot)\rangle} \otimes \hat{\pi}[\chi].$$

We immediately deduce $\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi] = \otimes_v \mathrm{I}_{P,\Lambda,v}\hat{\pi}[\chi]$, where

$$\mathrm{I}_{P,\Lambda,v}\hat{\pi}[\chi] = \mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_{n-1}(F_v)} e_v^{\langle \Lambda, H_{P,v}(\cdot)\rangle} \otimes \hat{\pi}[\chi].$$

We denote by $f = \prod_v f_v$ the conductor of $\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi]$.

The theory of Eisenstein series yields an intertwining operator

$$\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda} : \mathrm{I}_{P,\Lambda}\hat{\pi}[\chi] \to \mathcal{A}_F(n-1).$$

and for the moment we shall assume that this map is well defined. More precisely, let $f \in \mathcal{C}^\infty(N_P(\mathbb{A})A_P^0 P(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}))$ such that for any $g \in \mathrm{GL}_{n-1}(\mathbb{A})$ the map

$$\{m \mapsto f(mg)\} \in V_{\hat{\pi}[\chi]},$$

where $m \in A_P^0\backslash M_P(\mathbb{A})$. Decomposing $g = mnk$ as above we set

$$f_\Lambda(g) = e^{\langle \Lambda + \rho_P, H_P(g)\rangle} f(g).$$

We then assume that the Eisenstein series

$$E(f,\Lambda,g) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_{n-1}(F)} f_\Lambda(\gamma g)$$

is well defined for certain values of Λ . This assumption will be verified later on in those cases, which are important to us. We denote by $V_{\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi]}$ the image of $\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi]$ under Eis.

We fix a non-trivial character $\tau = \otimes_v \tau_v : F\backslash \mathbb{A} \to \mathbb{C}^*$ such that for all finite places the conductor of τ_v equals $\mathcal{O}_v$. We also denote by τ the induced (generic) character on $N_n(\mathbb{A})$: $n \mapsto \sum_i \tau(n_{i,i+1}),\ n = (n_{ij})$ and we denote by $W(\pi,\tau)$ the Whittaker model of π . The isomorphism mapping π into its Whittaker model is given by

$$\begin{array}{ccc} V_\pi & \to & W(\pi,\tau) \\ \phi & \mapsto & \phi^\tau(g), \end{array} \tag{1.2}$$

where

$$\phi^\tau(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(ng)\,\bar{\tau}(n)\,dn.$$

We want to extend this isomorphism to induced automorphic representations. We denote by w_P the longest element in the Weyl group of M_P and we abbreviate $w_{n,l} = w_{\mathrm{GL}_n}$. Hence, we have

$$w_P = \mathrm{diag}(w_{n-2,l}, 1).$$

The Whittaker model of $\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi]$ is then given as

$$W(\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi], \tau) = W(\hat{\pi}, \tau) \otimes \mathbb{C}_\chi,$$

where by $\mathbb{C}_\chi$ we understand the 1-dimensional space, on which $\mathrm{GL}_1(\mathbb{A})$ acts via χ. We write $f(g;h)$, $f \in \mathrm{I}_{P,\Lambda}V_{\hat{\pi}[\chi]}$ to denote the value of the automorphic function $f(g)$ at $h \in M_P(\mathbb{A})$ and we use an analogous notation for representations induced from spaces of Whittaker functions. We let $Q \leq \mathrm{GL}_{n-1}$ be any parabolic subgroup containing P, i.e. $Q = P$ or $Q = \mathrm{GL}_{n-1}$. $P \cap M_Q \leq M_Q$ is then a parabolic subgroup with Levi subgroup equal to M_P. Hence, we may define

$$\mathrm{I}^{M_Q}_{P\cap M_Q,\Lambda}\hat{\pi}[\chi] = \mathrm{Ind}^{M_Q(\mathbb{A})}_{(P\cap M_Q)(\mathbb{A})}\, e^{\langle\Lambda, H_P(\cdot)\rangle} \otimes \hat{\pi}[\chi];$$

analogously we define $\mathrm{I}^Q_{P,\Lambda,v}\hat{\pi}[\chi]$. We note that this definition works for all pairs of parabolic subgroups P, Q in GL_{n-1} satisfying $P \leq Q$. If $f \in \mathrm{I}^Q_{P,\Lambda}V_{\hat{\pi}[\chi]}$ is any section we write $f(g;h)$ to denote the value of the automorphic form $f(g)$ at $h \in M_P(\mathbb{A})$; we use an analogous notation for representations, which are induced from spaces of Whittaker functions. The isomorphism (1.2) then induces a map

$$\mathcal{P}_Q : \mathrm{I}_{P,\Lambda}V_{\hat{\pi}[\chi]} \xrightarrow{\sim} \mathrm{Ind}^{\mathrm{GL}_{n-1,\mathbb{A}}}_{Q(\mathbb{A})}(W(\mathrm{I}^{M_Q}_{P\cap M_Q,\Lambda}\hat{\pi}[\chi], \tau)),$$

which sends a section f to

$$\mathcal{P}_Q f(g; \mathbf{1}_{M_Q}) = \int_{U_{M_Q}(F)\backslash U_{M_Q}(\mathbb{A})} f(ug; \mathbf{1})\bar{\tau}(u)\, du,$$

where $U_{M_Q} = N_{n-1} \cap M_Q$ is the unipotent radical of the Levi component M_Q. This defines $\mathcal{P}_Q f$ in a unique way. If $Q = \mathrm{GL}_{n-1}$ we obtain an isomorphism

$$\mathcal{P}_{\mathrm{GL}_{n-1}} : \mathrm{I}_{P,\Lambda}\hat{\pi}[\chi] \to W(\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi], \tau).$$

We abbreviate $\mathrm{I}^\star_{P,\Lambda}\hat{\pi}[\chi] = \mathrm{I}_{P,\Lambda}W(\hat{\pi}[\chi], \tau)$ and $\mathcal{P}^* = \mathcal{P}_P$. We want to relate $\mathcal{P}^*$ to the computation of Fourier coefficients of Eisenstein series. For the moment we let $P, Q \leq \mathrm{GL}_{n-1}$ be any parabolic subgroups satisfying $P \leq Q$ and $\hat{\pi}[\chi]$ a cuspidal representation of $M_P(\mathbb{A})$. For any place v we define a map

$$\mathcal{P}^Q_{P,v} : \mathrm{Ind}^{\mathrm{GL}_{n-1,F_v}}_{P(F_v)} e^{\langle\Lambda, H_P(\cdot)\rangle} \otimes W(\hat{\pi}_{\chi,v}, \tau_v) \to \mathrm{Ind}^{\mathrm{GL}_{n-1}(F_v)}_{Q(F_v)} W(\mathrm{I}^{M_Q}_{P\cap M_Q,\Lambda,v}\hat{\pi}[\chi], \tau_v)$$

by sending a section f_v to

$$\mathcal{P}^{Q}_{P,v} f(g) = \int_{w_P w_Q (M_Q \cap U_P) w_Q w_P (F_v)} f_v(w_P w_Q u g; \mathbf{1}) \bar{\tau}_v(u)\, du.$$

We set $\mathcal{P}^Q_P = \otimes_v \mathcal{P}^Q_{P,v}$ and $\mathcal{P}^Q_{P,v}$ is given by the same integral as above over the adelic points of $w_P w_Q (U_{M_Q} \cap M_P) w_Q w_P$. We return to our situation and let P be the parabolic subgroup as chosen above and $Q = \mathrm{GL}_{n-1}$. We obtain an isomorphism

$$\mathcal{P}^{\mathrm{GL}_{n-1}}_P : I^*_{P,\Lambda} \hat{\pi}[\chi] \to W(\mathrm{I}_{P,\Lambda} \hat{\pi}[\chi], \tau).$$

Explicitely, in this case we have

$$w_P w_Q (U_{M_Q} \cap M_P) w_Q w_P = U_{1,n-2}, \tag{1.3}$$

where $U_{1,n-2}$ is the unipotent radical of the parabolic subgroup of type $(1, n-2)$.

Together with the Eisenstein operator and the assignment $\phi \mapsto E^\tau$ we obtain a diagram

$$\begin{array}{ccc} \mathrm{I}_{P,\Lambda} V_{\hat{\pi}[\chi]} & \overset{\mathcal{P}^\star}{\to} & \mathrm{Ind}_{P_{\mathbb{A}}}^{\mathrm{GL}_{n-1,\mathbb{A}}} W(\hat{\pi}[\chi], \tau) \\ {\scriptstyle \mathrm{Eis}} \downarrow & & \downarrow \mathcal{P}^{\mathrm{GL}_{n-1}}_{P,v} \\ V_{\mathrm{I}_{P,\Lambda} \hat{\pi}[\chi]} & \overset{E \mapsto E^\tau}{\to} & W(\mathrm{I}_{P,\Lambda} \hat{\pi}[\chi], \tau). \end{array} \tag{1.4}$$

Lemma 2.1 in [Sha 1] then states that (1.3) commutes. As an immediate consequence we see that the local components of $E^{\bar{\tau}}(f, \Lambda, g)$ of $E(f, \Lambda, g)$ are given by

$$E^{\bar{\tau}}_v(f, \Lambda, g) = \int_{U_{1,n-1}(F_v)} \mathcal{P}^* f_{\Lambda,v}(w_P w_{l,n-1} n g) \tau_v(n)\, dn, \tag{1.5}$$

where we assume that $f = \otimes_v f_v$. We want to apply (1.5) to explicitely compute $E^{\bar{\tau}}(f, \Lambda, g)$ and to this end we have to fix a section f and hence an Eisenstein series $E(f, \Lambda)$. Using the above realizations of $\mathrm{I}_{P,\Lambda} \hat{\pi}[\chi]$ we define a section $f^\star_{\hat{\pi}[\chi],\Lambda} = \otimes_v f^\star_{\hat{\pi}[\chi],\Lambda,v}$ in $\mathrm{I}^\star_{P,\Lambda} \hat{\pi}[\chi]$ as follows.

- For any finite place $v \notin S' \cup \{\mathfrak{p}\}$ the representation $\mathrm{I}_{P,\Lambda} \hat{\pi}[\chi]$ is unramified and we define $f^\star_{\hat{\pi}[\chi],\Lambda,v}$ to be the spherical function in $\mathrm{I}^\star_{P,\Lambda,v} \hat{\pi}[\chi]$. Thus, writing $g = mnk$, $p \in P(F_v)$, $n \in U_P(F_v)$, $k \in K_{n-1,v}$ the section is given as

$$f^\star_{\hat{\pi}[\chi],\Lambda,v}(g) = e_v^{\langle \Lambda + \check{\rho}, H_{P,v}(p) \rangle}\, \hat{\pi}[\chi](m)\, w^0_{\hat{\pi},v} \otimes 1,$$

where $w^0_{\hat{\pi},v}$ is the unique spherical function in $W(\hat{\pi}_v, \tau_v)$ satisfying $w^0_{\hat{\pi},v}(e) = 1$.

- At all finite places $v \in S'$ we define $f^\star_{\hat{\pi}[\chi],\Lambda,v}$ as follows. We set

$$w = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}$$

and we denote by $w^0_{\hat\pi,v}$ the essential vector in $W(\hat\pi_v,\tau_v)$ and by f_v the conductor of $\hat\pi_v$. We then define

$$f^\star_{\hat\pi[\chi],\Lambda,v}(g)=\begin{cases}\hat\pi[\chi](m)w^0_{\hat\pi,v}\otimes 1\omega_{\hat\pi}(k_{n-1,n-1}) & \text{if} \quad g=mnwk\in M_P(F_v)N_P(F_v)K_0(n-1,f_v)\\ 0 & \text{else.}\end{cases}$$

It is easily verified that $f^\star_{\hat\pi[\chi],\Lambda,v}$ is well defined, $K_1(n-1,f_v)$-invariant on the right and that $f^\star_{\hat\pi[\chi],\Lambda,v}$ spans the space $\left(\mathrm{Ind}_{P_{F_v}}^{\mathrm{GL}_{n-1,F_v}}(W(\hat\pi[\chi]_v,\tau_v))\right)^{K_1(n-1,f_v)}$ of $K_1(n-1,f_v)$-invariant vectors.

- For $v=\mathfrak{p}$ we decompose $g=mnk\in M_P(F_v)U_P(F_v)K_{n-1,v}$ and we set

$$f^\star_{\hat\pi[\chi],\Lambda,v}(g)=\chi_v(k_{n-1,n-1})\,e_v^{\langle\Lambda+\check\rho,H_{P,v}(m)\rangle}\hat\pi[\chi,v](m)w^0_{\hat\pi,v}\otimes 1, \tag{1.6}$$

if $k=(k_{ij})\in K_0(n-1,\mathfrak{p}^e)$ and $f^\star_{\hat\pi[\chi],\Lambda,v}(g)=0$ else. Here, $w^0_{\hat\pi,\mathfrak{p}}$ is the spherical Whittaker function in $W(\hat\pi_{\mathfrak{p}},\tau_{\mathfrak{p}})$ satisfying $w^0_{\hat\pi,\mathfrak{p}}(e)=1$. Obviously, $f^\star_{\hat\pi[\chi],\Lambda,v}$ is invariant on the right under $K_1(n-1,\mathfrak{p}^e)$ and it is easy to see that

$$\left(\mathrm{Ind}_{P_{F_{\mathfrak{p}}}}^{\mathrm{GL}_{n-1,F_{\mathfrak{p}}}}(W(\hat\pi[\chi]_{\mathfrak{p}},\tau_{\mathfrak{p}}))\right)^{K_1(n-1,f_{\mathfrak{p}})}=\langle f^\star_{\hat\pi[\chi],\Lambda,\mathfrak{p}}\rangle_{\mathbb{C}}.$$

- If v is archimedean $\mathrm{I}_{P,\Lambda,v}\hat\pi[\chi]$ does not depend on χ_v and we let $f^\star_{\hat\pi[\chi],\Lambda,v}$ be an arbitrary but fixed section in $\mathrm{I}_{P,\Lambda,v}\hat\pi[\chi]$, which does not depend on χ.

We set $f_{\hat\pi[\chi],\Lambda}={\mathcal{P}^\star}^{-1}(f^\star_{\hat\pi[\chi],\Lambda})$ and $E(f_{\hat\pi[\chi],\Lambda},g)=\mathrm{Eis}_{P,\hat\pi[\chi],\Lambda}(f_{\hat\pi[\chi],\Lambda})$ and we denote by $v_{\hat\pi[\chi],\Lambda}=E^{\bar\tau}(f_{\hat\pi[\chi],\Lambda})$ the Whittaker function of $E(f_{\hat\pi[\chi],\Lambda})$.

With this choice of an Eisenstein series we now turn to the computation of the finite part of its Whittaker function $E^{\bar\tau}(f_{\hat\pi[\chi],\Lambda})$. We shall use the decomposition $E^{\bar\tau}_{\mathrm{fin}}(f_{\hat\pi[\chi],\Lambda})=\otimes_{v\notin S_\infty}E^{\bar\tau}_v(f_{\hat\pi[\chi],\Lambda})$ and compute the local Whitaker functions at all finite places.

1.1. The (ramified) places $v\in S'-S_\infty$.

Here, the following qualitative result will be enough for our purposes.

Lemma 1.1. *For all $v\in S'-S_\infty$ there is an element $g^0_v\in\mathrm{GL}_{n-1}(F_v)$ such that $E^{\bar\tau}_v(f_{\hat\pi[\chi],\Lambda})(g^0_v)$ does not vanish for almost all (i.e. all but finitely many) unitary characters $\chi:\ F^*\backslash\mathbb{A}^*\to\mathbb{C}^*$ with conductor prime to S'.*

Proof. Since $f^\star_{\hat\pi[\chi],\Lambda,v}$ is not equal to the zero section there is a $g^0_v\in\mathrm{GL}_{n-1}(F_v)$ such that $f^\star_{\hat\pi[\chi],\Lambda,v}(g^0_v)\neq 0$. We denote by $X^{\mathrm{nr}}_v=(F^*_v/\mathcal{O}^*_v)\hat{}$ the set of all unitary, unramified characters of F_v. For all $\chi_v\in X^{\mathrm{nr}}_v$ there is a bijection

$$\begin{array}{ccc}\mathrm{I}_{P,\Lambda,v}\pi[1]_v & \to & \mathrm{I}_{P,\Lambda,v}\hat\pi_v[\chi_v]\\ f(g) & \mapsto & \chi_v(m_2)f(g),\end{array}$$

where $g=mnk$ with $m=\mathrm{diag}(m_1,m_2)$. By its construction the sections $f^\star_{\hat\pi[\chi],\Lambda,v}\in\mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_{n-1}(F_v)}(W(\hat\pi_v,\tau_v)\otimes\mathbb{C}_{\chi_v})$ correspond to each other under this bijection and Proposition 3.1 in [Sha 2] then implies that the assignment

$$\chi_v\mapsto v_{\hat\pi[\chi],\Lambda,v}(g^0_v),\quad \chi_v\in X^{\mathrm{nr}}_v,$$

with $v_{\hat{\pi}[\chi],\Lambda,v}$ defined by (1.4) is a holomorpic map. On the other hand, since $F_v^*/\mathcal{O}_v^*$ is discrete, X_v^{nr} is compact and any infinite subset $X \subset X_v^{\mathrm{nr}}$ has a culmination point in X_v^{nr}. But $v_{\hat{\pi}[\chi],\Lambda,v}(g_v^0)$ does not vanish identically and hence, the identity theorem for holomorphic functions implies that $v_{\hat{\pi}[\chi],\Lambda,v}(g_v^0)$ can not vanish for infinitely many χ_v in X_v^{nr}. This proves the Lemma.

For later purpose for all $v \in S'$ we fix a compact open subgroup $K_{n-1,v}^* \leq K_{n-1,v}$ such that $g_v^0 \in K_{n-1,v}^*$ and $v_{\hat{\pi}[\chi],\Lambda,v}$ is invariant under $K_{n-1,v}^*$ on the right. We set $g_{S'}^0 = \otimes_{v\in S'} g_v^0$

1.2. The (unramified) places $v \notin S' \cup \{\mathfrak{p}\}$.

At places $v \notin S' \cup \{\mathfrak{p}\}$ we need to know the values of $E_v^{\bar{\tau}}(f_{\hat{\pi}[\chi],\Lambda})$ at e. To formulate the result we note that the Lie algebra $\mathfrak{a}_{P,\mathbb{C}}$ is naturally isomorphic to $\mathbb{C}^2$. This induces an identification $\mathbb{C}^2 \xrightarrow{\sim} \mathfrak{a}_{P,\mathbb{C}}^*$, which we write as $(s_1, s_2) \mapsto \Lambda_{s_1,s_2}$. Explicitely,

$$\Lambda_{s_1,s_2,\mathbb{A}} = e^{\langle \Lambda_{s_1,s_2}, H_P(\cdot)\rangle} : \begin{pmatrix} m_1 & * \\ & m_2 \end{pmatrix} \mapsto |\det m_1|^{\frac{s_1}{n-2}} |m_2|^{s_2}.$$

For later pupose we set

$$\Lambda_{\mathbb{A}}(g) = e^{\langle \Lambda, H_P(g)\rangle},\ \Lambda_v(g) = e_v^{\langle \Lambda, H_{P,v}(g)\rangle} \tag{1.7}$$

and we also set $\Lambda_{\mathrm{fin}} = \otimes_{v\neq\infty}\Lambda_v$. In particular, we have $\Lambda_{\mathbb{A}}|_{A_P^0} = \Lambda$.

Lemma 1.2. *For all places v, where $\hat{\pi}$ and χ do not ramify, we have*

$$E_v^{\bar{\tau}}(f_{\hat{\pi}[\chi],\Lambda}, e) = L(\hat{\pi}_v \times \chi_v^{-1}, 1 + s_1 - s_2)^{-1}.$$

Proof. Since v is not contained in S', there are unramified characters $\nu_1, \ldots, \nu_{n-2} : F_v^* \to \mathbb{C}^*$ such that

$$\hat{\pi}_v \cong \mathrm{Ind}(\nu_1, \ldots \nu_{n-2}).$$

We set $\nu_{n-1} = \chi_v$ and $\underline{\nu} = \nu_1 \otimes \cdots \otimes \nu_{n-1} : B_{n-1}(F_v) \to \mathbb{C}^*$. We denote by f_1^0 the spherical function in $|\cdot|_v^{s_1} \otimes \mathrm{Ind}_{B_{n-1}(F_v)}^{\mathrm{GL}_{n-2,F_v}}(\nu_1 \otimes \ldots \otimes \nu_{n-2})$ satisfying $f_1^0(e) = 1$; the spherical function in $\mathrm{Ind}_{B_{n-1}(F_v)}^{\mathrm{GL}_{n-1}(F_v)}(e_v^{\langle \Lambda, H_{P,v}(\cdot)\rangle} \otimes \underline{\nu})$ then reads

$$f^0(g) = f_1^0(m_1)\chi_v(m_2)|m_2|_v^{s_2}$$

for $g = mnk \in M_P(F_v)N_P(F_v)K_{n-1,v}$ and $m = \mathrm{diag}(m_1, m_2)$. Let $Q \leq \mathrm{GL}_{n-1}$ be any parabolic sibgroup containing B_{n-1}. Attached to the pair (B_{n-1}, Q) there is the intertwining map

$$\begin{aligned} \mathcal{P}_{B_{n-1}}^Q : \quad & \mathrm{Ind}_{B_{n-1,F_v}}^{\mathrm{GL}_{n-1,F_v}}(e_v^{\langle \Lambda, H_{P,v}(\cdot)\rangle} \otimes \underline{\nu}) \\ & \to \mathrm{Ind}_{Q(F_v)}^{\mathrm{GL}_{n-1,F_v}}(W(\mathrm{Ind}_{B_{n-1}\cap M_Q(F_v)}^{M_Q(F_v)} e_v^{\langle \Lambda, H_{P,v}(\cdot)\rangle} \otimes \underline{\nu}, \tau_v)) \end{aligned}$$

which sends an element f, which we view as $\mathbb{C}$-valued function, to the section $\mathcal{P}^{Q}_{B_{n-1}}f$, which is uniquely determined by the equation

$$\mathcal{P}^{Q}_{B_{n-1}}f(g;\mathbf{1}_{M_Q(F_v)})=\int_{U_{M_Q}(F_v)}f(w_Q ng)\bar{\tau}_v(n)\,dn.$$

We note that in case $Q=\mathrm{GL}_{n-1}$ the right hand side is just $W(\mathrm{I}_{P,\Lambda,v}\hat{\pi}[\chi],\tau_v)$. Using a straightforward calculation we obtain

- $\mathcal{P}^{P}_{B_{n-1},v}(f^0)(e;\mathbf{1})=\int_{N_{n-2}(F_v)}f^0_1(w_{l,n-2}n)\tau_v(n)\,dn.$
- The diagram

$$\begin{array}{ccc}\mathrm{Ind}_{B_{n-1}(F_v)}^{\mathrm{GL}_{n-1}(F_v)}(e_v^{\langle\Lambda,H_{P,v}(\cdot)\rangle}\otimes\underline{\nu}) & \stackrel{\mathcal{P}^{Q}_{B_{n-1}}}{\rightarrow} & \mathrm{Ind}_{Q(F_v)}^{\mathrm{GL}_{n-1}(F_v)}(W(e_v^{\langle\Lambda,H_{P,v}(\cdot)\rangle}\otimes\underline{\nu}))\\ \mathcal{P}^{P}_{B_{n-1}} & \searrow & \downarrow\mathcal{P}^{\mathrm{GL}_{n-1}}_{P}\\ & & I^{*}_{P,\Lambda,v}\hat{\pi}[\chi]\end{array}$$

  commutes.

Uniqueness of the essential vector up to scalar multiples together with the first property implies that

$$f^{\star}_{\hat{\pi}[\chi],\Lambda,v}(g;\mathbf{1})=\mathcal{P}^{P}_{B_{n-1},v}(f^0)(g;\mathbf{1}_{M_P(F_v)})\left(\int_{N_{n-2}(F_v)}f^0_1(w_{l,n-2}n)\bar{\tau}(n)\,dn\right)^{-1}.$$

Using the commutativity of (1.3) and applying the above transitivity we obtain

$$\begin{aligned}E^{\bar{\tau}}_v(f_{\hat{\pi}[\chi],\Lambda},e) &= \mathcal{P}^{\mathrm{GL}_{n-1}}_{P,v}(f^{\star}_{\hat{\pi}[\chi],\Lambda,v})(e)\\ &= \frac{\mathcal{P}^{\mathrm{GL}_{n-1}}_{B_n,v}f^0_{\hat{\pi}[\chi],\Lambda,v}(e)}{\mathcal{P}^{\mathrm{GL}_{n-2}}_{B_{n-2},v}f^0_1(e)}.\end{aligned}$$

We may now apply the formula of Casselman-Shalika to compute the factors appearing in the nominator and in the denominator. We denote by t_v the matrix

$$t_v=\begin{pmatrix}\nu_1(\varpi_v) & & \\ & \ddots & \\ & & \nu_{n-1}(\varpi_v)\end{pmatrix}\in\mathrm{GL}_{n-1}(\mathbb{C}).$$

Moreover, for any pair $i<j$ we define the coroot

$$H_{ij}=\mathrm{diag}(\dots,\varpi_v,\dots,\varpi_v^{-1},\dots),$$

where ϖ_v resp. ϖ_v^{-1} appear in the i-th resp. j-th position and all other entries are equal to 1. Applying Theorem 5.4 in [C-S] we then obtain

$$E_v^{\bar{\tau}}(f_{\hat{\pi}[\chi],\Lambda}, e) = \frac{\prod_{i<j}(1 - q_v^{-1}|H_{ij}|_{\mathfrak{p}}^{\Lambda}\, t_{v,i}t_{v,j}{}^{-1})}{\prod_{i<j<n-1}(1 - q_v^{-1}|H_{ij}|_{\mathfrak{p}}^{\Lambda}\, t_{v,i}t_{v,j}{}^{-1})}.$$

Taking into account that ${}^L(\mathrm{GL}_{n-2} \times \mathrm{GL}_1) = \mathrm{GL}_{n-2}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, this may be rewritten as

$$\begin{aligned} E_v^{\bar{\tau}}(f_{\hat{\pi}[\chi],\Lambda}, e) &= L(\hat{\pi}_v \times \chi_v, r_{n-2} \otimes \tilde{r}_1, 1 + s_1 - s_2)^{-1} \\ &= L(\hat{\pi}_v \times \tilde{\chi}_v, 1 + s_1 - s_2)^{-1}, \end{aligned}$$

where $r_i : \mathrm{GL}_i(\mathbb{C}) \to \mathrm{GL}_i(\mathbb{C})$ is the standard representation and $\tilde{r}_i$ denotes the contragredient. Hence, the Lemma is proven.

Our assumptions on the conductor of $\hat{\pi}_{\mathfrak{p}}$ imply that $L(\hat{\pi}_{\mathfrak{p}} \otimes \chi, s) = 1$. Moreover replacing the representation $\hat{\pi}$ by a twist $\tilde{\eta} \otimes \hat{\pi}$, where $\tilde{\eta}$ is an idele class character of finite order with sufficiently high conductor $f_{\tilde{\eta}}$ we may assume that

$$L(\hat{\pi}_v \otimes \chi_v, s) = 1 \tag{1.8.C}$$

for all places $v \in S'$. Using Lemma 1.2 we then obtain

$$v_{\hat{\pi}[\chi],\Lambda}^{S' \cup \{\mathfrak{p}\}}(e) = L(\hat{\pi} \otimes \chi^{-1}, 1 + s_1 - s_2)^{-1}, \tag{1.9}$$

where $v_{\hat{\pi}[\chi],\Lambda}^{S'} = \otimes_{v \notin S'} v_{\hat{\pi}[\chi],\Lambda,v}$

1.3. The place $\mathfrak{p}$.

In this subsection, to simplify the notation we set $r = n - 1$ and we use the notation $v_{\hat{\pi}[\chi],\Lambda,v}$ instead of $E_v^{\bar{\tau}}(f_{\hat{\pi}[\chi],\Lambda})$. We want to compute the values of $v_{\hat{\pi}[\chi],\mathfrak{p}}$ on elements of the torus $T_r(F_{\mathfrak{p}})$. In view of the existence of the central character of $\mathrm{I}_{P,\Lambda,\mathfrak{p}}\hat{\pi}[\chi]$, it is enough to to this for diagonal matrices of the form

$$t = \mathrm{diag}(1, t_1, \ldots, t_{r-1}).$$

We abbreviate $\mathbf{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_{r-1})$, where $\mathrm{v}_i = v_{\mathfrak{p}}(t_i)$. An easy exercise proves that $v_{\hat{\pi}[\chi],\mathfrak{p}}(t)$ vanishes unless

$$-\mathrm{v}_1 \geq 0, \mathrm{v}_2 - \mathrm{v}_1 \geq 0 \ldots, \mathrm{v}_{r-1} - \mathrm{v}_{r-2} \geq 0, \tag{1.10}$$

which we shall assume henceforth. Since the diagram (1.4) commutes we obtain

$$v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}(t) = \mathcal{P}^P_{\mathrm{GL}_{n-1}}(f^\star_{\hat{\pi}[\chi],\mathfrak{p}})(t) = \int_{U_{1,r-1}(F_{\mathfrak{p}})} f^\star_{\hat{\pi}[\chi],\mathfrak{p}}(w_P w_{r,l} ut; \mathbf{1}_{r-1}, 1)\, \tau_{\mathfrak{p}}(u)\, du.$$

Here, $U_{1,r-1}$ is the unipotent radical of the parabolic subgroup of type $(1,r-1)$ and

$$w_P w_{r,l} = \begin{pmatrix} & \mathbf{1}_{n-2} \\ 1 & \end{pmatrix}.$$

In the following we compute this integral.

Let $u=(u_1,\dots,u_{r-1})$ as in (2.1) with v_i replaced by u_i; by the change of integration variables $u_i \mapsto t_i^{-1}u_i$, $i=1,\dots,r-1$ we obtain for $v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}(t)$ the expression

$$|t|_{\mathfrak{p}}^{-1}\int_{F_{\mathfrak{p}}\times\cdots\times F_{\mathfrak{p}}} f^{\star}_{\hat{\pi}[\chi],\mathfrak{p}}\left(\begin{pmatrix} 0 & t_1 & & \\ & \ddots & \ddots & \\ & & 0 & t_{r-1} \\ 1 & u_1 & \cdots & u_{r-1}\end{pmatrix}; \mathbf{1}_{r-1}, 1\right)\ \tau_{\mathfrak{p}}(t_1^{-1}u_1)\ du_1\dots du_{r-1},$$

where $|t|_{\mathfrak{p}} = \prod_{i=1}^{r-1}|t_i|_{\mathfrak{p}}$. We write for short

$$\tilde{t} = \mathrm{diag}(t_1,\dots,t_{r-1},1)$$

and

$$u = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & u_1 & \cdots & u_{r-1}\end{pmatrix}.$$

Let $\mathbf{c}=(c_1,\dots,c_{r-1})\in\mathbb{Z}^{r-1}$ be a finite sequence of integers. We set

$$U_{\mathbf{c}} = \{(u_1,\dots,u_{r-1})\in(F_{\mathfrak{p}})^{r-1} : v_{\mathfrak{p}}(u_i)=c_i,\ i=1,\dots,r-1\}.$$

and since $\mathrm{meas}(F_{\mathfrak{p}}^{r-1}-(F_{\mathfrak{p}}^*)^{r-1})=0$ the integral decomposes

$$v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}(t) = \sum_{\mathbf{c}\in\mathbb{Z}^{r-1}} I(t,\mathbf{c}), \tag{1.11}$$

where

$$I(t,\mathbf{c}) = |t|_{\mathfrak{p}}^{-1}\int_{U_{\mathbf{c}}} f^{\star}_{\hat{\pi}[\chi],\mathfrak{p}}\left(\tilde{t}u; \mathbf{1}_{r-1},1\right)\ \tau_{\mathfrak{p}}(t_1^{-1}u_1)\ du_1\dots du_{r-1}.$$

We want to calculate $I(D,\mathbf{c})$. To any $\mathbf{c}=(c_1,\dots,c_{r-1})\in\mathbb{Z}^{r-1}$ we shall attach a subset $\mathcal{J}(\mathbf{c})=\{i_1<\dots<i_k\}$ of $\{1,\dots,r-1\}$. The evaluation of $I(t,\mathbf{c})$ will then essentially depend on the value of $\mathcal{J}(\mathbf{c})$. We first assume that $c_i\le 0$, $i=1,\dots,r-1$; in this case we inductively define $\mathcal{J}(\mathbf{c})$ as follows: we set $i_0:=r$ and assuming that i_h has already been defined we set

$$i_{h+1} = \min\{1\le i<i_h : v_{\mathfrak{p}}(u_i)\le v_{\mathfrak{p}}(u_j) \text{ for all } 1\le j<i_h\}$$

as long as the set on the right hand side is non-empty. Obviously the last element of this sequence is $i_k = 1$ with $k \leq r-1$. In the general case we first define the sequence $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_{r-1})$ by setting

$$\tilde{c_i} = \begin{cases} c_i & \text{if } c_i \leq 0 \\ 0 & \text{if } c_i > 0, \end{cases}$$

and we then define $\mathcal{J}(\mathbf{c}) = \mathcal{J}(\tilde{\mathbf{c}})$.

Using the description (1.6) of the essential vector it is immediate that $f^{\star}_{\tilde{\pi}[\chi],\mathfrak{p}}\left(\tilde{t}u; \mathbf{1}_{r-1}, 1\right) \neq 0$ only if

$$v_{\mathfrak{p}}(c_{r-1}) \leq -e + \min\{0, v_{\mathfrak{p}}(c_1), \ldots, v_{\mathfrak{p}}(c_{r-2})\}. \tag{1.12}$$

In particular, only those $\mathbf{c}$ contribute non-trivially to (1.11), which satisfy $r-1 \in \mathcal{J}(\mathbf{c})$.

Lemma 1.3. *Let $u_1, \ldots, u_{r-1} \in F_{\mathfrak{p}}$ and denote by $\mathcal{J}(\mathbf{c}) = \{\mathbf{i_1} > \ldots > \mathbf{i_k}\}$ the sequence of integers attached to $\mathbf{c} = (v_{\mathfrak{p}}(u_1), \ldots, v_{\mathfrak{p}}(u_{r-1}))$ in the above sense. Then the following decomposition holds*

$$\begin{pmatrix} 0 & t_1 & & \\ & \ddots & \ddots & \\ & & 0 & t_{r-1} \\ 1 & u_1 & \cdots & u_{r-1} \end{pmatrix} = \delta^* \cdot \begin{pmatrix} \frac{1}{u_{i_k}} & v_{i_k} & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \ddots & & & & & \\ & & \frac{u_{i_k}}{u_{i_{k-1}}} & v_{i_{k-1}} & \cdots & \cdots & \cdots \\ & & & \ddots & & & \\ & & & & \frac{u_{i_2}}{u_{i_1}} & v_{i_1} & \cdots \\ & & & & & \ddots & \\ & & & & & & u_{i_1} \end{pmatrix} \cdot k,$$

where $k \in \mathrm{GL}_r(\mathcal{O}_{\mathfrak{p}})$ and δ^* is the diagonal matrix

$$\delta^* = \mathrm{diag}((-1)^k t_1, t_2, \ldots, t_{i_h-1}, (-1)^h t_{i_h}, t_{i_h+1}, \ldots, t_{r-1}, 1),$$

with the entry $(-1)^h t_{i_h}$, $h = 1, \ldots, k$ appearing in the i_h-th position. Furthermore in the above matrix all entries, which are left out are equal to zero, the entries on the diagonal, which are indicated by dots, are equal to 1, the entries in the i_h-th row, which are indicated by dots are arbitrary and

$$v_{i_h} = \begin{cases} \frac{u_{i_h+1}}{u_{i_h}} & \text{if } i_h < i_{h-1} - 1 \\ 1 & \text{if } i_h = i_{h-1} - 1 \end{cases}$$

appears in the i_h-th column ($i_0 = r$).

Proof. We first deal with the case $v_{\mathfrak{p}}(u_i) \leq 0$ for $i = 1, \ldots, r-1$. Multiplying on the left with the diagonal matrix

$$\mathrm{diag}(t_1^{-1}, \ldots, t_{r-1}^{-1}, 1)$$

we may assume that $d_1 = \ldots = d_{r-1} = 1$. We then proceed by induction on $r \in \mathbb{N}$. The case $r = 1$ being trivial we therefore may assume that the assertion of the Lemma is true for $r-1 \times r-1$ -matrices and we have to prove it in the case of $r \times r$ -matrices. Our starting point is the following decomposition

$$
(1.13) \qquad \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & u_1 & \cdots & u_{r-1} \end{pmatrix} = M \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \frac{1}{u_{i_1}} & \cdots & \frac{u_{i_1-1}}{u_{i_1}} & 1 & \frac{u_{i_1+1}}{u_{i_1}} & \cdots & \frac{u_r}{u_{i_1}} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},
$$

where

$$
M = \begin{pmatrix} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ -\frac{1}{u_{i_1}} & \cdots & \cdots & -\frac{u_{i_1-1}}{u_{i_1}} & 1 & -\frac{u_{i_1+1}}{u_{i_1}} & \cdots & -\frac{u_r}{u_{i_1}} \\ & & & & 0 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & & & 0 & 1 \\ & & & & u_{i_1} & & & 0 \end{pmatrix}.
$$

Since $v_{\mathfrak{p}}(u_{i_1}) \leq v_{\mathfrak{p}}(u_j)$ for $1 \leq j \leq r$, we see that the second matrix on the right hand side of (1.13) is contained in $\mathrm{GL}_r(\mathcal{O}_{\mathfrak{p}})$. We distinguish cases and first assume that $i_h < i_{h-1} - 1$. The first matrix on the right hand side further decomposes

$$
M = \begin{pmatrix} 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 0 & 1 & & & & & \\ -\frac{1}{u_{i_1}} & \cdots & \cdots & -\frac{u_{i_1-1}}{u_{i_1}} & -\frac{u_{i_1+1}}{u_{i_1}} & \cdots & -\frac{u_r}{u_{i_1}} & 1 \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & u_{i_1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{i_1} & & & & & \\ & 0 & \cdots & & \cdots & 1 \\ & 1 & 0 & & & \vdots \\ & & 1 & & & \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & 0 \end{pmatrix}.
$$

The permutation matrix is obviously contained in $\mathrm{GL}_r(\mathcal{O}_{\mathfrak{p}})$ and the matrix appearing in the upper left corner of size $i_1 \times i_1$ of the first matrix on the right hand side reads

$$
(1.14) \qquad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \frac{-1}{u_{i_1}} \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & u_1 & \cdots & u_{i_1-1} \end{pmatrix}.
$$

On the other hand the sequence of integers $\mathcal{J}(\{u_1,\dots,u_{i_1-1}\})$ attached to $\{u_1,\dots,u_{i_1-1}\}$ is just $(i_2,\dots,i_k)$. Hence, applying the induction hypothesis to the matrix (1.14) we obtain the desired decomposition. In case that $i_1 = i_0 - 1$ we see that M equals

$$\begin{pmatrix} 0 & 1 & & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ -\frac{1}{u_{r-1}} & \cdots & \cdots & -\frac{u_{r-2}}{u_{r-1}} & & 1 \\ & & & & & u_{r-1} \end{pmatrix}$$

Again, applying the induction hypothesis to the upper left corner of size $r-1\times r-1$ we obtain the desired decomposition in the case of $r\times r$-matrices. The general case now follows immediately: using column operations it is easy to see that there is a matrix $n \in N_r(\mathcal{O}_\mathfrak{p})$ such that

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & u_1 & \cdots & u_{r-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & \tilde{u}_1 & \cdots & \tilde{u}_{r-1} \end{pmatrix} \cdot n,$$

and $\tilde{u}_i$ equals u_i if $v_\mathfrak{p}(u_i) \leq 0$ and equals 1 if $v_\mathfrak{p}(u_i) > 0$. Hence, we may apply our result to the first matrix on the right hand side and taking into account that by definition $\mathcal{J}(v_\mathfrak{p}(u_1),\dots,v_\mathfrak{p}(u_{r-1})) = \mathcal{J}(v_\mathfrak{p}(\tilde{u}_1),\dots,v_\mathfrak{p}(\tilde{u}_{r-1}))$ we obtain the desired decomposition in the general case. Thus the lemma is proven.

We note that, by an easy calculation, condition (2.13) is seen to imply that the matrix k appearing in the Lemma is contained in $K_1(r,\mathfrak{p}^e)$.

Let $\mathcal{J} = \{i_1 = r-1 > \dots > i_{k-1} > i_k = 1\}$ be any subset of $\{1,\dots,r-1\}$. We denote by $\check{\mathcal{J}}$ the subset of $\mathcal{J}$ consisting of all $i \in \mathcal{J}$ such that $i-1 \in \mathcal{J}$. In particular $\check{\mathcal{J}}$ does not contain $i_k = 1$. Using the Lemma we then obtain the decomposition

$$\tilde{t}u = n\delta\delta^* k.$$

Here, δ is the diagonal matrix

$$\delta = \operatorname{diag}\left(\frac{1}{u_{i_k}}, 1,\dots,1, \frac{u_{i_k}}{u_{i_{k-1}}}, 1,\dots,1, \frac{u_{i_2}}{u_{i_1}}, 1,\dots,u_{i_1}\right),$$

where $\frac{u_{i_{h+1}}}{u_{i_h}}$ appears in the i_h-th position for $h \in \mathcal{J}(\mathbf{c})$ and

$$n = \begin{pmatrix} 1 & \star_2 & & & * \\ & \ddots & \ddots & & \\ & & 1 & \star_{r-1} & \\ & & & 1 & 1/u_{i_1} \\ & & & & 1 \end{pmatrix},$$

where

$$\star_i = \begin{cases} -\frac{t_{i_h-1}}{t_{i_h}}\frac{u_{i_h}}{u_{i_{h+1}}} & \text{if } i = i_h \in \check{\mathcal{J}}(\mathbf{c}) \\ (-1)^{i_h}\frac{t_{i_h}}{t_{i_h+1}}\frac{u_{i_h+1}}{u_{i_h}} & \text{if } i \notin \mathcal{J}(\mathbf{c}), \text{ but } i-1 = i_h \in \mathcal{J}(\mathbf{c}) \\ 0 & \text{else.} \end{cases}$$

We denote by $n_{r-1}, \delta_{r-1}, \delta^*_{r-1}$ the upper left corner of size $r-1 \times r-1$ of n, δ, δ^*. As mentioned above (1.12) implies that k is contained in $K_1(r, \mathfrak{p}^e)$ and we obtain

$$\begin{aligned} f^\star_{\hat{\pi}[\chi],\mathfrak{p}}(\tilde{t}u) &= f^\star_{\hat{\pi}[\chi],\mathfrak{p}}(k; n_{r-1}\delta_{r-1}\delta^*_{r-1}, u_{i_1}) \\ &= \Lambda_\mathfrak{p}(\delta\delta^*)\,\tau_\mathfrak{p}(n_{r-1}) w^0_{\hat{\pi},\mathfrak{p}}(\delta_{r-1}\delta^*_{r-1})\,\chi_\mathfrak{p}(u_{i_1}). \end{aligned}$$

Hence we have to calculate

$$I(t,\mathbf{c}) = |t|^{-1}_\mathfrak{p} \int_{T_{r-1}(\mathcal{O}_\mathfrak{p})} \tau_\mathfrak{p}(n_{r-1})\,\tau_\mathfrak{p}(t_1^{-1}u_1)\, w^0_{\hat{\pi},\mathfrak{p}}(\delta_{r-1}\delta^*_{r-1})\,\chi_\mathfrak{p}(u_{i_1})\Lambda_\mathfrak{p}(\delta\delta^*)\, du_1 \ldots du_{r-1}.$$

Taking into account that $\frac{t_{i_h}}{t_{i_h+1}} \in \mathcal{O}_\mathfrak{p}$ and $\frac{u_{i_h+1}}{u_{i_h}} \in \mathcal{O}_\mathfrak{p}$ for $i \in \mathcal{J}(\mathbf{c}) - \check{\mathcal{J}}(\mathbf{c})$ we see that

$$\tau_\mathfrak{p}(n_{r-1}) = \prod_{i_h \in \check{\mathcal{J}}(\mathbf{c})} \tau_\mathfrak{p}\left(-\frac{t_{i_h-1}}{t_{i_h}}\frac{u_{i_h}}{u_{i_{h+1}}}\right).$$

We denote by $\varpi^\mathbf{c} = \mathrm{diag}(\varpi^{c_1}, \ldots, \varpi^{c_{r-1}})$ the diagonal matrix of length $r-1$. Since $w^0_{\hat{\pi},\mathfrak{p}}$ is $\mathrm{GL}_{r-1}(\mathcal{O}_\mathfrak{p})$-invariant on the right we have $w^0_{\hat{\pi},\mathfrak{p}}(\delta_{r-1}) = w^0_{\hat{\pi},\mathfrak{p}}(\Delta(\mathbf{c}))$, where

$$\Delta(\mathbf{c}) = \mathrm{diag}(\Delta_1, \ldots, \Delta_{r-1})$$

is the diagonal matrix of length $r-1$ with entries $\Delta_i = \varpi^{c_{i_{h+1}} - c_{i_h}}$ if $i = i_h \in \mathcal{J}(\mathbf{c})$ and $\Delta_i = 1$ if $i \notin \mathcal{J}(\mathbf{c})$ (by the symbol '$c_{i_{k+1}}$' we mean 0). Thus, writing $u_i = \varpi^{c_i}\varepsilon_i$, $i = 1, \ldots, r-1$ we obtain

$$\begin{aligned} I(t,\mathbf{c}) &= |t|^{-1}_\mathfrak{p}\chi_\mathfrak{p}(\varpi^{c_{i_1}}) \int_{T_{r-1}(\mathcal{O}_\mathfrak{p})} \prod_{i_h \in \check{\mathcal{J}}(\mathbf{c})} \tau_\mathfrak{p}\left(-\frac{t_{i_h-1}}{t_{i_h}}\Delta_{i_h}\frac{\varepsilon_{i_h}}{\varepsilon_{i_{h+1}}}\right) \tau_\mathfrak{p}(t_1^{-1}\varpi^{c_1}\varepsilon_1) \\ &\quad \times w^0_{\hat{\pi},\mathfrak{p}}(\varpi^\mathbf{v}\Delta(\mathbf{c}))\,\chi_\mathfrak{p}(\varepsilon_{i_1})\Lambda_\mathfrak{p}(\mathrm{diag}(\varpi^\mathbf{v}\Delta(\mathbf{c}), c_{i_1}))\, d\varepsilon_1, \ldots, d\varepsilon_{r-1}. \end{aligned}$$

The change of integration variables

$$\varepsilon_{i_h} \mapsto (-1)^{k-h} \prod_{r=h,\ldots,k} \varepsilon_{i_r}, \quad h = k, \ldots, 1$$

leaves $T_{r-1}(\mathcal{O}_\mathfrak{p})$ stable and yields

$$\begin{aligned} I(t,\mathbf{c}) &= |t|^{-1}_\mathfrak{p}\,\chi_\mathfrak{p}(\varpi^{c_{r-1}})\, w^0_{\hat{\pi},\mathfrak{p}}(\varpi^\mathbf{v}\Delta(\mathbf{c}))\,\chi_\mathfrak{p}^{k-1}(-1)\,\Lambda_\mathfrak{p}(\mathrm{diag}(\varpi^\mathbf{v}\Delta(\mathbf{c}), c_{i_1})) \\ &\quad \times \int_{T_{r-1}(\mathcal{O}_\mathfrak{p})} \prod_{i \in \check{\mathcal{J}}(\mathbf{c})} \tau_\mathfrak{p}\left(\frac{t_{i-1}}{t_i}\Delta_i\varepsilon_i\right) \tau_\mathfrak{p}(t_1^{-1}\varpi^{c_1}\varepsilon_1) \prod_{i \in \mathcal{J}(\mathbf{c})} \chi_\mathfrak{p}(\varepsilon_i)\, d\varepsilon_1, \ldots, d\varepsilon_{r-1}. \end{aligned}$$

We put $t_0 := 1$. Recalling that $i_k = 1 \notin \check{\mathcal{J}}(\mathbf{c})$ we see that the above integral decomposes into the product

$$\prod_{i\in\check{\mathcal{J}}(\mathbf{c})\cup\{1\}} \int_{\mathcal{O}_\mathfrak{p}^*} \tau_\mathfrak{p}\left(\frac{t_{i-1}}{t_i}\Delta_i\varepsilon_i\right)\chi_\mathfrak{p}(\varepsilon_i)\,d\varepsilon_i \;\times \prod_{i\in\mathcal{J}(\mathbf{c})-(\check{\mathcal{J}}(\mathbf{c})\cup\{1\})} \int_{\mathcal{O}_\mathfrak{p}^*} \chi_\mathfrak{p}(\varepsilon_i)\,d\varepsilon_i.$$

It follows immediately that $I(t,\mathbf{c})$ vanishes unless $\mathcal{J}(\mathbf{c}) = \check{\mathcal{J}}(\mathbf{c}) \cup \{1\}$. Since $\mathcal{J}$ contains $r-1$, this equivalent to $\mathcal{J} = \{1,\dots,r-1\}$, i.e. $i_h = r-h,\ h = 1,\dots,r-1$ and in this case we obtain

$$\begin{aligned} I(t,\mathbf{c}) \;&=\; |t|_\mathfrak{p}^{s_1-1}|\Delta(\mathbf{c})|_\mathfrak{p}^{s_1}|c_{i_1}|_\mathfrak{p}^{s_2}\,\chi_\mathfrak{p}((-1)^r\varpi^{c_{r-1}})\,w^0_{\hat{\pi},\mathfrak{p}}(\varpi^{\mathbf{v}}\Delta(\mathbf{c})) \\ &\qquad\qquad \prod_{i=1,\dots,r-1}\int_{\mathcal{O}_\mathfrak{p}^*}\tau_\mathfrak{p}\left(\frac{t_{i-1}\varpi^{c_i}}{t_i\varpi^{c_{i-1}}}\varepsilon_i\right)\chi_\mathfrak{p}(\varepsilon_i)\,d\varepsilon_i. \end{aligned}$$

On the other hand

$$\int_{\mathcal{O}_\mathfrak{p}^*}\tau_\mathfrak{p}(\varpi^{-c}\varepsilon)\chi_\mathfrak{p}(\varepsilon)\,d^*\varepsilon = \begin{cases} \chi_\mathfrak{p}(\varpi^c)\,G(\chi_\mathfrak{p}) & \text{if } c = e \\ 0 & \text{else}\end{cases}$$

and we further deduce that $I(t,\mathbf{c})$ does not vanish only if in addition

$$-v_\mathfrak{p}(t_1) + c_1 = e, \quad v_\mathfrak{p}(t_{i-1}) + c_i - v_\mathfrak{p}(t_i) - c_{i-1} = e, \quad i = 2,\dots,r-1.$$

Hence, $\mathbf{c} = (ie + v_\mathfrak{p}(t_i))_{i=1,\dots,r-1}$ is the only vector, which contributes to $v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}$. In particular $\Delta(\mathbf{c}) = \varpi^{-e+v_i-v_{i-1}}$, $i = 1,\dots,r-1$ and taking into account that

$$\prod_{i=1}^{r-1}\chi_\mathfrak{p}^{-1}\left(\frac{t_{i-1}}{t_i}\frac{\varpi^{c_i}}{\varpi^{c_{i-1}}}\right) = \chi_\mathfrak{p}(t_{r-1})\chi_\mathfrak{p}^{-1}(\varpi^{c_{r-1}})$$

we finally obtain:

Proposition 1.4. *Let $v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}$ be the Whittaker function in $W(\mathrm{I}_{P,\Lambda,\mathfrak{p}}\hat{\pi}[\chi],\tau_\mathfrak{p})$ attached to the essential vector $f^\star_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}$ defined in (1.5) and put $t = \mathrm{diag}(1,t_1,\dots,t_{r-1})$. Then, $v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}(t)$ vanishes unless condition (2.11) is fullfilled and in this case we have*

$$v_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}(t) = \chi_\mathfrak{p}((-1)^r)\,G(\chi_\mathfrak{p})^{r-1}\chi_\mathfrak{p}(t_{r-1})\,|t|_\mathfrak{p}^{s_1-1}|t_{r-1}|_\mathfrak{p}^{s_2-s_1}|\varpi|_\mathfrak{p}^{e(r-1)(s_2-s_1)}\,w^0_{\hat{\pi},\mathfrak{p}}(\varpi^{\mathbf{v}+\mathbf{e}}),$$

where

$$\mathbf{e} = (e, 2e, \dots, (r-1)e).$$

This formula now solves the problem of computing the values of the essential function on torus elements in that it expresses these values using Whittaker functions on groups in lower rank. In particular, applying it in the case $e = 0$ it should lead to a new proof of the Casselman-Shalika formula for GL_n.

We recall that $w^0_{\hat\pi,\mathfrak{p}}$ is the spherical function satisfying $w^0_{\hat\pi,\mathfrak{p}}(e) = 1$. In particular, [C-S], Theorem 5.4 implies that $w^0_{\hat\pi[\chi],\mathfrak{p}}(t)$ does not vanish if (1.10) holds. Taking into account that $c\ell(F) = 1$ we see that $\chi_\mathfrak{p}(\varpi) = 1$ and we finally deduce in the case that $s_1, s_2 \in \mathbb{Z}$

$$v_{\hat\pi[\chi],\mathfrak{p}}(\varpi^{\mathbf{v}}) = c\, G(\chi_\mathfrak{p})^{n-2} w^0_{\hat\pi,\mathfrak{p}}(\varpi^{\mathbf{v}+\mathbf{e}}). \tag{1.15}$$

Here, $c \in \mathbb{Q}^*$ is a non-vanishing rational number independent of χ.

Collecting our results we finally obtain in the case that $s_1, s_2 \in \mathbb{Z}$

$$\begin{aligned}(1.16)\qquad & E^{\bar\tau}_{\mathrm{fin}}(f_{\hat\pi[\chi],\Lambda}, e \times \varpi^{\mathbf{v}} \times g^0_{S'}) = \\ & \qquad E^{\bar\tau}_{S'-S_\infty}(f_{\hat\pi[\chi],\Lambda}, g^0_v)\, c\, G(\chi_\mathfrak{p})^{n-2} w^0_{\hat\pi,\mathfrak{p}}(\varpi^{\mathbf{v}+\mathbf{e}})\, L(\hat\pi[\chi]\otimes\chi^{-1}, 1 + s_1 - s_2),\end{aligned}$$

where $E^{\bar\tau}_{S'-S_\infty}(f_{\hat\pi[\chi],\Lambda}) = \otimes_{v\in S'-S_\infty} E^{\bar\tau}_v(f_{\hat\pi[\chi],\Lambda})$ and $c \in \mathbb{Q}^*$ do not vanish.

2 The Rankin-Selberg Convolution

In this section we introduce the global integral which we use to represent the automorphic L-function. Attached to the pair of representations

$$\pi \times \mathrm{I}_{P,\Lambda}\,\hat\pi[\chi]$$

there is a Rankin-Selberg convolution. We choose a special vector $\phi \in V_\pi$, where V_π denotes the space of cusp forms on which π can be realized. We shall then evaluate the Zeta integral of the Rankin-Selberg convolution for the pair of automorphic forms $(\phi, E_{f_{\pi_\chi,\Lambda}})$. This integral decomposes into a product

$$E^{\bar\tau}_{\mathrm{fin}}(f_{\hat\pi[\chi],\Lambda}, e \times \varpi^{\mathbf{v}} \times g^0_{S'}) \prod_{v\notin S_\infty} I_v(\phi^\tau_v), \tag{2.1}$$

where $I_v(\phi^\tau_v)$ is a certain local integral involving the Whittaker function of ϕ. Using the caclulations from the last section we obtain the denominator in (2.1). Using the local zeta integral we compute the $I_v(\phi^\tau_v)$'s and obtain the nominator in (2.1).

We begin with the definition of $\phi \in V_\pi$ and to this end we choose a Whittaker function $w = \otimes_v w_v \in W(\pi, \tau)$.

- For any finite place v not equal to $\mathfrak{p}$ and not contained in S' we let w_v be the essential vector in $W(\pi_v, \tau_v)$.
- If v is contained in S' we let w_v be the (unique) function whose restriction to $\mathrm{GL}_{n-1}(F_\mathfrak{p})$ via the embedding $g \mapsto \mathrm{diag}(g, 1)$ is given by

$$w_v|_{K^*_{n-1,v}} = \mathrm{ch}_{K^*_{n-1,v}},$$

where $\mathrm{ch}_{K^*_{n-1,v}}$ denotes the characteristic function of $K^*_{n-1,v}$. The existence of such a Whittaker function is an immediate consequence of Theorem F in [Gel-Kaz].

We set $w_{S'} = \prod_v w_v$, $v_{\hat{\pi}[\chi],\Lambda,S'} = \prod_v v_{\hat{\pi}[\chi],\Lambda,v}$ and $g^0_{S'} = \prod_v g^0_v$, where in each case the product runs over all finite places in S'.

• At the place $v = \mathfrak{p}$ the representation $\pi_\mathfrak{p}$ is unramified and hence there are unramified characters $\alpha_1, \dots, \alpha_n : F_\mathfrak{p} \to \mathbb{C}^*$ such that

$$\pi_\mathfrak{p} \cong \mathrm{Ind}_{B_n(F_\mathfrak{p})}^{\mathrm{GL}_n(F_\mathfrak{p})}(\alpha_1, \dots, \alpha_n).$$

We abbreviate $\alpha(b) = \prod_{i=1}^n \alpha_i(b_{ii})$ for $b = (b_{ij}) \in B_n(F_\mathfrak{p})$. We denote by $\mathcal{I}(n) \leq \mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ the Iwahori subgroup consisting of elements $k \in K_\mathfrak{p}(n)$, which are congruent to upper triangular matrices modulo $\mathfrak{p}$. Let $\psi^1_\mathfrak{p} \in \mathrm{Ind}_{B_n(F_\mathfrak{p})}^{\mathrm{GL}_n(F_\mathfrak{p})}(\alpha)$ be the following $\mathcal{I}(n)$-invariant vector

$$\psi^1_\mathfrak{p}(g) := \begin{cases} \alpha\delta_{B_n}^{1/2}(b) & \text{for} \quad g = bw_{n,l}\beta,\, b \in B(n, F_\mathfrak{p}), \beta \in \mathcal{I}(n) \\ 0 & \text{else.} \end{cases}$$

We then define $w_\mathfrak{p} \in W(\pi_\mathfrak{p}, \tau_\mathfrak{p})$ to be the image of $\psi^1_\mathfrak{p}$ under the intertwining operator

$$\mathrm{Ind}_{B_n(F_\mathfrak{p})}^{\mathrm{GL}_n(F_\mathfrak{p})}(\alpha_1, \dots, \alpha_n) \to W(\pi_\mathfrak{p}, \tau_\mathfrak{p}),$$

which maps an element ψ to the function

$$w_\psi(g) := \int_{N_n(F_\mathfrak{p})} \psi(w_{n,l}ng)\bar{\tau}_\mathfrak{p}(n)\, dn.$$

(cf. also [J-S 2], (3.2) Proposition).
• For any place $v \in S_\infty$, we let w_v be an arbitrary but fixed Whittaker function in $W(\pi_v, \tau_v)$.

We define $\phi \in V_\pi$ to be the cusp form attached to w.

We shall use the following *conventions* without further mentioning: we identify any matrix $h \in \mathrm{GL}_n(F_\mathfrak{p})$ with its image in $\mathrm{GL}_n(\mathbb{A})$ given by embedding it into the $\mathfrak{p}$-component of the adeles. In particular for any matrix $h \in \mathrm{GL}_n(F_\mathfrak{p})$ and function f on $\mathrm{GL}_n(\mathbb{A})$ we denote $h \cdot f(g) := f(gh)$ the right translate of f by h. For $m' > m$ we identify GL_m with its image in $\mathrm{GL}_{m'}$ under the embedding

$$g \mapsto \begin{pmatrix} g & \\ & \mathbf{1}_{m'-m} \end{pmatrix}.$$

We denote by $H \in \mathrm{GL}_{n-1}(F_\mathfrak{p})$ the matrix

$$H = \mathrm{diag}(\varpi, \varpi^2, \dots, \varpi^{n-1})$$

and we put $u^H = HuH^{-1}$. From now on we fix an idele class character

$$\eta:\ F^*\backslash\mathbb{A}^* \to \mathbb{C}^*$$

of finite order and conductor $f_\eta = \mathfrak{p}\mathfrak{m}$, where $\mathfrak{m}$ is prime to $\mathfrak{p}$. Using the Whittaker function w defined above, we then set

$$w_\eta(g) := \sum_u \prod_{t=1}^{n-2} \eta_{\mathfrak{p}}^t(u_{t,t+1})\ w\left(g\begin{pmatrix} u^H & \\ & 1\end{pmatrix}\right),$$

where the matrix $u = (u_{ij})$ runs over all elements of a set of representatives of $N_{n-1}(F_v)$ modulo $N_{n-1}(F_v)^{H^{-1}}$, which satisfy $u_{t,t+1} \in (\mathcal{O}_{\mathfrak{p}}/\varpi\mathcal{O}_{\mathfrak{p}})^*$. We let ϕ_η be the cusp form belonging to w_η .

We denote by $H_x := \mathrm{diag}(1,\ldots,1,x)$, $x \in F_{\mathfrak{p}}^*$ the diagonal matrix of length $n-1$ and we set $X_{n-1}^{\mathbb{A}} := \mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A})$. The integral, which we want to calculate, then reads

$$\mathbf{J}_{\hat\pi[\chi],\Lambda,\eta}(s) = \sum_v \chi_{\mathfrak{p}}\eta_{\mathfrak{p}}^{n-1}(v_{n-1}) \int_{X_{n-1}^{\mathbb{A}}} \phi_\eta\,(gv)\,\eta\otimes E(f_{\hat\pi[\chi],\Lambda}, gH_{\varpi^{a-e}})\,|\det g|^{s-1/2}\,dg.$$

In the above expression a is a natural number and $v = (v_1,\ldots,v_{n-1})$ runs over $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^a\times\ldots\times\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^a\times(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^e)^*$. Also, we identify v with its image under the embedding

$$\begin{array}{ccc} F_{\mathfrak{p}}^{n-1} & \to & N_n(F_{\mathfrak{p}}) \\ v = (v_1,\ldots,v_{n-1}) & \mapsto & \begin{pmatrix} 1 & & & & v_1/\varpi^a \\ & \ddots & & & \vdots \\ & & 1 & & v_{n-2}/\varpi^a \\ & & & 1 & v_{n-1}/\varpi^e \\ & & & & 1 \end{pmatrix}. \end{array}$$

Here, the vector $(\varpi^a,\ldots,\varpi^a,\varpi^e)$ is understood by the range of summation for v . For brevity, we usually write $\mathbf{J}_{\hat\pi[\chi],\Lambda}$ instead of $\mathbf{J}_{\hat\pi[\chi],\Lambda,\eta}(s)$.

Our starting point for the computation of this integral is its decomposition into a product of local integrals, which is valid for $\mathrm{Re}(s) >> 0$ and in our case takes the form

$$\mathbf{J}_{\hat\pi[\chi],\Lambda} = \prod_{v\neq\mathfrak{p}} I(w_{\eta,v}, \eta_v\otimes v_{\hat\pi[\chi],\Lambda,v}, s) \sum_{v_1,\ldots,v_{n-1}} \chi_{\mathfrak{p}}\eta_{\mathfrak{p}}^{n-1}(v_{n-1})\, I(H_{\varpi^{e-a}}v\cdot w_{\eta,\mathfrak{p}}, \eta_{\mathfrak{p}}\otimes v_{\hat\pi[\chi],\Lambda,\mathfrak{p}}, s).$$

In the above equation

$$I(w_v, v_v, s) = \int_{N_{n-1}(F_v)\backslash\mathrm{GL}_{n-1}(F_v)} w_v\begin{pmatrix} g & \\ & 1\end{pmatrix} v_v(g)|\det g|_v^{s-1/2}\,dg$$

is the local zeta integral of the Rankin-Selberg convolution. This reduces the computation of the global integral to the calculation of the local ones appearing in the above decomposition.

• At all places v prime to $\mathfrak{p}$ and S', the Whittaker function $w_{\eta,v} = w_v$ is the essential vector and $\eta_v \otimes v_{\hat{\pi}[\chi],\Lambda,v}$ is by definition $K_{n-1,v}$-invariant. As an immediate consequence of (4.1) Théorème in [J-P-S 2] we obtain

$$I(w_{\eta,v}, \eta_v \otimes v_{\hat{\pi}[\chi],\Lambda,v}, s) = v_{\hat{\pi}[\chi],\Lambda,v}(e)\, L(\pi_v \times \eta_v \otimes \hat{\pi}_v, s+s_1)\, L(\pi_v \otimes \chi_v\eta_v, s+s_2).$$

• Next, we look to the places $v \in S'$. From now on we shall assume that for all $v \in S'$ the v-component $f_{\eta,v}$ of the conductor of η is sufficiently large so that

$$L(\pi_v \times \eta_v \otimes \chi_v, s) = 1. \tag{2.2.C}$$

On the other hand the definition of $v_{\hat{\pi}[\chi],\Lambda,v}$ and $w_{\eta,v} = w_v$ immediately yields that

$$I(w_v, \eta_v \otimes v_{\chi,\Lambda,v}, s) = (\mathrm{vol}(K_v^*))^{-1}\, w_v v_{\hat{\pi}[\chi],\Lambda,v}(g_v^0).$$

• At all archimedean places v we deduce from [J-P-S 3] that there is a polynomial $P = P^{\eta}_{w_v, v_{\hat{\pi}[\chi],\lambda,v}} \in \mathbb{C}[T]$ such that

$$\begin{aligned} I(w_v, \eta_v \otimes v_{\hat{\pi}[\chi],\Lambda,v}, s) &= P(s) L(\pi_v \times \eta_v \otimes \mathrm{I}_{P,\Lambda,v}\hat{\pi}[\chi], s) \\ &= P(s) \prod_{i=1}^{t-1} L(\pi_v \times \eta_v \otimes \hat{\pi}_v, s) L(\pi_v \otimes \eta_v\chi_v, s). \end{aligned}$$

We set $P^{\eta}_{w_\infty, v_{\hat{\pi}[\chi],\Lambda,\infty}} = \prod_{v\in S_\infty} P^{\eta}_{w_v, v_{\hat{\pi}[\chi],\Lambda,v}}$.

• At the place $\mathfrak{p}$ we assume that $f_{\chi_\mathfrak{p}} = \mathfrak{p}^e$ with $e \geq 2$. The character $\chi_\mathfrak{p}\eta_\mathfrak{p}$ is then ramified and since $\pi_\mathfrak{p}$ is unramified, we see that

$$L(\pi_\mathfrak{p} \otimes \chi_\mathfrak{p}\eta_\mathfrak{p}, s) = 1.$$

Collecting our results so far we arrive at the equation

$$\mathbf{J}_{\hat{\pi}[\chi],\Lambda} = P(s) v_{\hat{\pi}[\chi],\Lambda,S'}(g^0_{S'}) v^{S'\cup\{\mathfrak{p}\}}_{\hat{\pi}[\chi],\Lambda}(e)\, L_{S'\cup\{\mathfrak{p}\}}(\pi \times \eta \otimes \hat{\pi}, s+s_1) L(\pi \otimes \eta\chi, s+s_2)\, \mathbf{J}_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}},$$

where

$$\mathbf{J}_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}} = \sum_{v} \chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1}(v_{n-1})\, I(H_{\varpi^{e-a}} v \cdot w_{\eta,\mathfrak{p}}, v_{\hat{\pi}[\chi],\mathfrak{p}}, s)$$

and

$$v^{S'\cup\{\mathfrak{p}\}}_{\hat{\pi}[\chi],\Lambda} = \otimes_{v\notin S'\cup\{\mathfrak{p}\}} v_{\hat{\pi}[\chi],\Lambda,v}.$$

On the other hand since $\mathfrak{p} \notin S \cup S'$ we have $L(\pi_\mathfrak{p} \times \eta_\mathfrak{p} \otimes \hat{\pi}_\mathfrak{p}, s) = 1$. Again, twisting $\hat{\pi}$ with some character we even may assume

$$L(\pi_v \times \eta_v \otimes \hat{\pi}_v, s) = 1 \tag{2.3.C}$$

for all $v \in S'$ and replacing $E_v^{\bar\tau}(f_{\hat\pi[\chi],\Lambda}, e)$ by the expression in Eq. (1.9) we then obtain

$$(2.4)\quad \mathbf{J}_{\hat\pi[\chi],\Lambda} = P(s) v_{\hat\pi[\chi],\Lambda,S}(g^0_{S'})\, L(\pi \times \eta \otimes \hat\pi, s+s_1) \frac{L(\pi \otimes \eta\chi, s+s_2)}{L(\hat\pi \otimes \chi^{-1}, 1+s_1-s_2)} \mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}},$$

where $P = P_{w_\infty, v_{\hat\pi[\chi],\Lambda,\infty}}$. Hence, to complete the calculation of the integral it remains to compute $\mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}}$ on the p-adic group $\mathrm{GL}_{n-1}(F_\mathfrak{p})$. This will be done in the next subsection.

For later purpose we will assume that (1.8.C) holds for all $v \in S \cup S'$. Thus, concerning η we assume that

$$(\mathbf{C.1})\qquad f_{\eta,\mathfrak{p}} = \mathfrak{p} \quad\text{and}\quad L(\pi_v \times \eta_v\chi_v, s) = 1 \quad\text{for}\quad v \in S'$$

and concerning $\hat\pi$ we assume that

$$(\mathbf{C.2})\quad L(\pi_v \times \eta_v \otimes \hat\pi_v, s) = 1 \quad\text{for}\quad v \in S \cup S' \quad\text{and}\quad L(\hat\pi_v \otimes \chi_v^{-1}, s) = 1 \quad\text{for}\quad v \in S'$$

The assumptions can always be achieved by enlargening the conductor of η and or by twisting $\hat\pi$ with a ramified character. Since π and $\hat\pi$ are unramified at $\mathfrak{p}$ $(\mathbf{C.1})$ and $(\mathbf{C.2})$ also hold for $v = \mathfrak{p}$ as soon as $f_\chi = \mathfrak{p}^e$ with $e \geq 2$. Finally, we note that the conditions on η only depend on π.

2.1 Analysis of the local integral $\mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}}$.

We reduce the calculation of the $\mathfrak{p}$-component $\mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}}$ to the computation of the values of Whittaker functions on elements of the standard torus in $\mathrm{GL}_n(F_\mathfrak{p})$. The idea is to inductively compute $\mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}}$ by applying certain operators to it, whose effect we determine explicitely.

Let $\mu : F_\mathfrak{p}^* \to \mathbb{C}^*$ be a character with conductor $f_\mu = \mathfrak{p}^e$ with $e \geq 1$ and fix an integer $m \in \mathbb{N}$, $m < n-1$. For any vector $\mathbf{a} = (a_1, \ldots, a_{m-1}) \in \mathbb{N}^{m-1}$ we define the operator

$$\mathcal{R}_{\mathbf{a},\mu} : W(\pi_\mathfrak{p}, \tau_\mathfrak{p}) \to W(\pi_\mathfrak{p}, \tau_\mathfrak{p})$$

by setting

$$\mathcal{R}_{\mathbf{a},\mu}(w)(g) = \sum_v \mu(v_m) w(gv),$$

where $v = (v_1, \ldots, v_{m-1})$ runs over a system of representatives of $\mathcal{O}_\mathfrak{p}/\varpi^{a_1}\mathcal{O}_\mathfrak{p} \times \ldots \times \mathcal{O}_\mathfrak{p}/\varpi^{a_{m-1}}\mathcal{O}_\mathfrak{p} \times (\mathcal{O}_\mathfrak{p}/\varpi^e\mathcal{O}_\mathfrak{p})^*$ and we identify v in the argument of w with its image under

$$(2.5)\qquad \begin{array}{ccc} F_\mathfrak{p}^{m-1} & \to & N_m(F_\mathfrak{p}) \\ v = (v_1, \ldots, v_{m-1}) & \mapsto & \begin{pmatrix} 1 & & & & v_1/\varpi^{a_1} \\ & \ddots & & & \vdots \\ & & 1 & & v_{m-1}/\varpi^{a_{m-1}} \\ & & & 1 & v_m/\varpi^e \\ & & & & 1 \end{pmatrix}. \end{array}$$

Again, the vector $(\varpi^{a_1}, \dots, \varpi^e)$ is understood by the range of summation for v.

Using the characters η and χ introduced in section 1, we define for $m \in \mathbb{N}$

$$\mathcal{R}_{m,\eta_\mathfrak{p}} = \mathcal{R}_{(m,\dots,3,2),\eta_\mathfrak{p}^m}.$$

In the case $m = 1$ we understand that $\mathcal{R}_{1,\eta_\mathfrak{p}} = \mathcal{R}_{\eta_\mathfrak{p}}$. For arbitrary $w \in W(\pi_\mathfrak{p}, \tau_\mathfrak{p})$ we then put

$$w_{\eta_\mathfrak{p},m} = \mathcal{R}_{m,\eta_\mathfrak{p}} \circ \dots \circ \mathcal{R}_{1,\eta_\mathfrak{p}}\, w$$

and

$$w_{\chi_\mathfrak{p}} = \mathcal{R}_{(a,\dots,a),\chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1}}\, w_{\eta_\mathfrak{p},n-2},$$

where the vector $(a, \dots, a)$ has length $n-2$. Since

$$\begin{pmatrix} 1 & & & u_{1,n-1}/\varpi^{n-2} \\ & \ddots & & \vdots \\ & & 1 & u_{n-2,n-1}/\varpi \\ & & & 1 \end{pmatrix} \cdot \dots \cdot \begin{pmatrix} 1 & u_{1,2}/\varpi \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_{1,2} & & u_{1,n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & u_{n-2,n-1} \\ & & & 1 \end{pmatrix}^H$$

we see, that if we specialize w to be the $\mathfrak{p}$-component of the Whittaker function, which was defined in section 1, we obtain

$$w_{\eta_\mathfrak{p},n-2} = w_{\eta,\mathfrak{p}}. \tag{2.6}$$

Remark 2.1. We note that if w is invariant on the right under $\mathcal{I}(n)$, then the definition of $w_{\eta_\mathfrak{p},m}$ and $w_{\chi_\mathfrak{p}}$ does not depend on the choice of the systems of representatives of $\mathcal{O}_\mathfrak{p}/\varpi^{a_i}\mathcal{O}_\mathfrak{p}$, which we use in the definition of the operators $\mathcal{R}_{\mu,\mathbf{a}}$.

Using the remark we obtain

$$w_{\chi_\mathfrak{p}}(g) = \sum_v \chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1}(v_{n-1})\, w_{\eta,\mathfrak{p}}(gv), \tag{2.7}$$

where the vector $(v_1, \dots, v_{n-1})$ runs over $\mathcal{O}_\mathfrak{p}/\mathfrak{p}^a \times \dots \times \mathcal{O}_\mathfrak{p}/\mathfrak{p}^a \times (\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e)^*$. Thus, we find

$$\begin{aligned} \mathbf{J}_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}} &= \int_{N_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} w_{\chi_\mathfrak{p}}(g)\, v_{\hat{\pi}[\chi],\mathfrak{p}}(gH_{\varpi^{a-e}})\, dg. \\ &= \int_{N_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} \mathcal{R}_{(a,\dots,a),\chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1}} \circ \dots \circ \mathcal{R}_{1,\eta_\mathfrak{p}} w_\mathfrak{p}(g)\, v_{\hat{\pi}[\chi],\mathfrak{p}}(gH_{\varpi^{a-e}})\, dg \end{aligned}$$

and it remains to determine the effect of $\mathcal{R}_{\mathbf{a},\mu}$ on Whittaker functions.

We denote by $\check{U}_{\mathbf{a}}(m)$ the subgroup of $\mathrm{GL}_m(F_{\mathfrak{p}})$ consisting of all matrices of the form

$$k = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ y_1 & \cdots & y_{m-1} & y_m \end{pmatrix} \tag{2.8}$$

satisfying $y_i \in \varpi^{a_i}\mathcal{O}_{\mathfrak{p}}, i = 1, \ldots, m-1$ and $y_m \in \mathcal{O}_{\mathfrak{p}}^*$.

Lemma 2.2. *Let $w \in W(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ and assume that there is a character $\mu' : F_{\mathfrak{p}}^* \to \mathbb{C}^*$, such that for any $k \in \check{U}_{\mathbf{a}}(m)$ as above we have*

$$w(gk) = w(g)\,\mu'(y_m).$$

Then, the support of $\mathrm{res}_{\mathrm{GL}_m} \mathcal{R}_{\mathbf{a},\mu}(w)$ is contained in

$$N_m(F_{\mathfrak{p}})\mathrm{GL}_{m-1}(F_{\mathfrak{p}})\check{U}_{\mathbf{a}}(m),$$

and for any $g = ng'k$, $n \in N_m(F_{\mathfrak{p}})$, $g' \in \mathrm{GL}_{m-1}(F_{\mathfrak{p}})$ and $k \in \check{U}_{\mathbf{a}}(m)$ we find

$$\mathcal{R}_{\mathbf{a},\mu}(w)(g) = \tau_{\mathfrak{p}}(n) \prod_{i=1}^{m-1} q^{a_i} G(\mu)\mu'\mu^{-1}(y_m)\,(w)(g'),$$

where $G(\mu) = \sum_{u \in (\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_wp)^*} \mu(u)\tau_{\mathfrak{p}}(u/\varpi^e)$ denotes the Gaussian sum.

Proof. Let $g = (g_{ij}) \in \mathrm{GL}_m(F_{\mathfrak{p}})$. Using the commutation rule

$$\begin{pmatrix} g & \\ & 1 \end{pmatrix} v = v' \begin{pmatrix} g & \\ & 1 \end{pmatrix},$$

where

$$v' = \left(*, \ldots, *, \frac{g_{m,m}v_m}{\varpi^e} + \sum_{i=1}^{m-1} \frac{g_{m,i}v_i}{\varpi^{a_i}} \right)$$

is the matrix as in (2.5) we find

$$\mathcal{R}_{\mathbf{a},\mu}\, w(g) = \sum_{v_m} \mu(v_m)\tau_{\mathfrak{p}}\left(\frac{g_{m,m}v_m}{\varpi^e}\right) \cdot \prod_{i=1}^{m-1} \sum_{v_i} \tau_{\mathfrak{p}}\left(\frac{g_{m,i}v_i}{\varpi^{a_i}}\right) w(g).$$

Using

$$\sum_{v_m \in (\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_wp)^*} \mu(v_m)\tau_{\mathfrak{p}}\left(\frac{g_{m,m}v_m}{\varpi^e}\right) = \begin{cases} \mu^{-1}(g_{m,m})G(\mu) & g_{m,m} \in \mathcal{O}_{\mathfrak{p}}^* \\ 0 & g_{m,m} \notin \mathcal{O}_{\mathfrak{p}}^* \end{cases}$$

as well as the character relation

$$\sum_{v_i \in \mathcal{O}_{\mathfrak{p}}/\varpi^{a_i}\mathcal{O}_{\mathfrak{p}}} \tau_{\mathfrak{p}}\left(\frac{g_{m,i}v_i}{\varpi^{a_i}}\right) = \begin{cases} q^{a_i} & g_{m,i} \in \varpi^{a_i}\mathcal{O}_{\mathfrak{p}} \\ 0 & g_{m,i} \notin \varpi^{a_i}\mathcal{O}_{\mathfrak{p}}, \end{cases}$$

we see, that $\mathcal{R}_{\mathbf{a},\mu}\, w(g) \neq 0$ implies, that $g_{m,i} \in \varpi^{a_i}\mathcal{O}_{\mathfrak{p}}$ for $i = 1, \dots, m-1$ and $g_{m,m} \in \mathcal{O}_{\mathfrak{p}}^*$. Together with the decomposition

$$g = \begin{pmatrix} 1 & & & \frac{g_{1,m}}{g_{m,m}} \\ & \ddots & & \vdots \\ & & 1 & \frac{g_{m-1,m}}{g_{m,m}} \\ & & & 1 \end{pmatrix} \begin{pmatrix} g' & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ g_{m,1} & \cdots & g_{m,m-1} & g_{m,m} \end{pmatrix},$$

where $g' = (g'_{ij}) \in \mathrm{GL}_{m-1}(F_{\mathfrak{p}})$ with entries $g'_{ij} = g_{ij} - \frac{g_{i,m}}{g_{m,m}} g_{m,j}$ this proves that g is contained in $N_m(F_{\mathfrak{p}})\mathrm{GL}_{m-1}(F_{\mathfrak{p}})\check{U}_{\mathbf{a}}(m)$. To prove the second part of the claim let $g = (g_{ij}) \in \mathrm{GL}_m(F_{\mathfrak{p}})$ and assume that

$$g = n\, \mathrm{diag}(g', 1)\, k,$$

where $n \in N_m(F_{\mathfrak{p}})$, $g' \in \mathrm{GL}_{m-1}(F_{\mathfrak{p}})$ and $k \in \check{U}_{\mathbf{a}}(m)$ as in the lemma. Then $g_{m,i} = y_i$, $i = 1, \dots, m$ and we obtain

$$\mathcal{R}_{\mathbf{a},\mu}\, w(g) = \mu^{-1}(y_m) G(\mu) \prod_{i=1}^{m-1} q^{a_i}\, w(ng'_k).$$

Thus the assumption on the behaviour of w on the right implies the claim, which proves the lemma.

Remark 2.3. In the case $m = 1$ we use the conventions $N_1(F_{\mathfrak{p}}) = \{1\}, \mathrm{GL}_0 = \{1\}, \check{U}_0(1) = \mathcal{O}_{\mathfrak{p}}^*$ and $\mathrm{res}_{\mathrm{GL}_0}\, w = w(\mathbf{1}_n)$. In particular, the assertions then read

$$\mathrm{supp}\,\mathrm{res}_{\mathrm{GL}_1} \mathcal{R}_\mu w \subset \mathcal{O}_{\mathfrak{p}}^*$$

and for $\varepsilon \in \mathcal{O}_{\mathfrak{p}}^*$

$$\mathcal{R}_\mu w(\varepsilon) = G(\mu)\mu'\mu^{-1}(\varepsilon) w(\mathbf{1}_n).$$

Taking into account the recursive definition of the Whittaker functions $w_{\eta_{\mathfrak{p}},m}$ and $w_{\chi_{\mathfrak{p}}}$ Lemma 2.2 makes it possible to completely determine $w_{\eta_{\mathfrak{p}},m}$ and $w_{\chi_{\mathfrak{p}}}$. In a first step we compute $w_{\eta_{\mathfrak{p}},m}$; we prove:

Let $w \in W(\pi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ and assume that w is invariant on the right under $\mathcal{I}(m)$. Then the support of $\mathrm{res}_{\mathrm{GL}_m} w_{\eta_{\mathfrak{p}},m}$ is contained in

$$N_m(F_{\mathfrak{p}})\check{U}_*(m, \mathfrak{p}), \tag{2.9}$$

where $\check{U}_*(m, \mathfrak{p}) \leq \mathcal{I}(m)$ consists of all matrices

$$k = \begin{pmatrix} k_{1,1} & & 0 \\ \vdots & \ddots & \\ k_{m,1} & \cdots & k_{m,m} \end{pmatrix}$$

satisfying $k_{ij} \in \mathfrak{p}^{i-j+1}$ *for* $i > j$ *and for any such* $k = (k_{ij}) \in \check{U}_*(m, \mathfrak{p})$ *we have*

$$w_{\eta_\mathfrak{p},m}(k) = \eta_\mathfrak{p}^{-1}(\det k) \prod_{i=1}^{m} q^{i(i+1)/2-1} G(\eta_\mathfrak{p}^i)\, w(\mathbf{1}_n). \tag{2.10}$$

We will simultaneously prove both parts of the claim using induction on m. The case $m = 1$ being trivial we immediately assume that the lemma holds for $m-1$. We have $w_{\eta_\mathfrak{p},m} = \mathcal{R}_{(m,\dots,2),\eta_\mathfrak{p}^m}\, w_{\eta_\mathfrak{p},m-1}$ and Lemma 2.2 yields

$$\operatorname{supp} \operatorname{res}_{\mathrm{GL}_m} w_{\eta_\mathfrak{p},m} \subset N_m(F_\mathfrak{p}) \mathrm{GL}_{n-1}(F_\mathfrak{p}) \bar{U}_{(m,\dots,2)}(m).$$

We want to apply Lemma 2.2 to calculate the values of $w_{\eta_\mathfrak{p},m}$ on this set. Therefore we need to know the behaviour of $w_{\eta_\mathfrak{p},m-1}$ on the right under $\bar{U}_{(m,\dots,2)}(m)$.

Lemma 2.4. *Let* $w \in W(\pi_\mathfrak{p}, \tau_\mathfrak{p})$ *and assume that* w *is invariant on the right under* $\mathcal{I}(m)$. *Then, for any* $k \in \bar{U}_{(m,\dots,2)}(m)$ *as in (2.4) we have*

$$w_{\eta_\mathfrak{p},m-1}(gk) = \eta_\mathfrak{p}^{m-1}(y_m)\, w_{\eta_\mathfrak{p},m-1}(g).$$

We postpone the proof of this lemma and continue the proof of (2.9) and (2.10). Let $g = g'_k$, where $g' \in \mathrm{GL}_{m-1}(F_\mathfrak{p})$ and $k \in \bar{U}_{(m,\dots,2)}(m)$. Then the above lemma together with Lemma 2.2 yields

$$w_{\eta_\mathfrak{p},m}(g) = \prod_{i=1}^{m-1} q^{i+1} G(\eta_\mathfrak{p}^m) \eta_\mathfrak{p}^{-1}(y_m)\, w_{\eta_\mathfrak{p},m-1}(g'). \tag{2.11}$$

But

$$\bar{U}_*(m-1, \mathfrak{p}) \cdot \bar{U}_{(m,\dots,2)}(m) = \bar{U}_*(m, \mathfrak{p})$$

and we may therefore apply the induction hypothesis on $\operatorname{supp}\operatorname{res}_{\mathrm{GL}_{m-1}} w_{\eta_\mathfrak{p},m-1}$ to obtain

$$\operatorname{supp} \operatorname{res}_{\mathrm{GL}_m} w_{\eta_\mathfrak{p},m} \subset N_m(F_\mathfrak{p}) \bar{U}_*(m, \mathfrak{p}).$$

To prove Eq. (2.10) we note that any $k = (k_{ij}) \in \bar{U}_*(m, \mathfrak{p})$ decomposes as

$$k = \begin{pmatrix} k' & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{m-1} & \mathbf{0} \\ k_{m,1} \cdots & k_{m,m} \end{pmatrix},$$

where $k' = (k'_{ij}) \in \mathrm{GL}_{m-1}(F_\mathfrak{p})$ and $k'_{ij} = k_{ij}$. Hence, (2.11) becomes

$$w_{\eta_\mathfrak{p},m}(k) = q^{m(m+1)/2-1} G(\eta_\mathfrak{p}^m) \eta_\mathfrak{p}^{-1}(y_m)\, w_{\eta_\mathfrak{p},m-1}(k),$$

which together with the induction hypothesis on the values of $\operatorname{res}_{\mathrm{GL}_{m-1}} w_{\eta_\mathfrak{p},m-1}$ proves Eq. (2.10)(note that $\eta_\mathfrak{p}^{-1}(y_m)\eta_\mathfrak{p}^{-1}(\det k') = \eta_\mathfrak{p}^{-1}(\det k)$).

Proof of Lemma 2.4. We say that the matrix $A \in \mathrm{GL}_m(F_\mathfrak{p})$ is of type (t_m), if the entries satisfy

$$\begin{aligned} &a_{ij} \in \varpi^{-(j-i)}\mathcal{O}_\mathfrak{p} \quad \text{if} \quad j > i, \\ &a_{ij} \in \varpi^{i-j+1}\mathcal{O}_\mathfrak{p} \quad \text{if} \quad j < i, \\ &a_{i,i} \in \mathcal{O}_\mathfrak{p}^*. \end{aligned}$$

(2.12) Claim. Let $A \in \mathrm{GL}_m(F_\mathfrak{p})$ be of type (t_m). Then there is a matrix $\beta_A \in \bar{N}(m,\mathfrak{p})$ such that

$$A = \begin{pmatrix} \varepsilon_1 & \cdots & u_{ij} \\ & \ddots & \vdots \\ & & \varepsilon_m \end{pmatrix} \beta_A$$

and $\varepsilon_i \equiv a_{ii} \pmod{\mathfrak{p}}$, $i = 1, \dots, m$ and $u_{ij} \equiv a_{ij}$ $(\mathrm{mod}\ \varpi^{i-j+1})\mathcal{O}_\mathfrak{p}$, $j > i$. Moreover if A runs over a set of matrices of type (t_m) such that the entries $a_{ij}, j > i$ run over a system of representatives of $\varpi^{-(j-i)}\mathcal{O}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}$ and $a_{i,i+1}$ run over $\varpi^{-1}\mathcal{O}_\mathfrak{p}/\mathcal{O}_\mathfrak{p} - \mathcal{O}_\mathfrak{p}$, then the same is true for the entries u_{ij}.

Using the claim we will prove the lemma. Let $k \in \bar{U}_{(m,\dots,2)}(m)$ as in (2.8). Then we have

$$k \begin{pmatrix} 1 & \cdots & u_{ij}/\varpi^{j-i} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & u_{ij}/\varpi^{j-i} \\ & \ddots & \vdots \\ y_1' & \cdots & y_m' \end{pmatrix},$$

where $y_i' = y_i + \sum_{h=1}^{i-1} y_h \frac{u_{ih}}{\varpi^{h-i}}$. The assumption $k \in \bar{U}_{(m,\dots,2)}(m)$ then implies that the above matrix is of type (t_m). Therefore (2.6) together with the (2.12) yields

$$w_{\eta_\mathfrak{p},m-1}\left(g\begin{pmatrix} k & \\ & \mathbf{1}_{n-m}\end{pmatrix}\right) = \sum_{u_{ij}} \prod_{t=1}^{m-1} \eta_\mathfrak{p}^t(u_{t,t+1})\, w\left(g\begin{pmatrix} \varepsilon_1 & \cdots & u_{ij}/\varpi^{j-i} & \\ & \ddots & \vdots & \\ & & \varepsilon_m & \\ & & & \mathbf{1}_{n-m}\end{pmatrix}\right),$$

where the summation variables u_{ij} runs over $u_{ij} \in \mathcal{O}_\mathfrak{p}/\varpi^{j-i}\mathcal{O}_\mathfrak{p}$ for $j > i+1$ and $u_{i,i+1} \in (\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*$ (note that w is invariant under $\mathcal{I}(m)$). Taking into account the right invarinace of w under $\mathrm{diag}(\varepsilon_1, \dots, \varepsilon_m)$ as well as $\varepsilon_t \equiv 1 \pmod{\mathfrak{p}}$ for $t < m$ and $\varepsilon \equiv y_m' \equiv y_m \pmod{\mathfrak{p}}$ and changing the summation variables $u_{ij} \mapsto \varepsilon_j u_{ij}$ we find

$$w_{\eta_\mathfrak{p},m-1}\left(g\begin{pmatrix} k & \\ & \mathbf{1}_{n-m}\end{pmatrix}\right) = \eta_\mathfrak{p}^{m-1}(y_m) \sum_{u_{ij}} \prod_{t=1}^{m-1} \eta_\mathfrak{p}^t(u_{t,t+1})\, w\left(g\begin{pmatrix} 1 & \cdots & u_{ij}/\varpi^{j-i} \\ & \ddots & \vdots \\ & & 1\end{pmatrix}\right).$$

This finishes the proof of the lemma.

Proof of (2.12) Claim. We use induction on m. The case $m = 2$ as well as the induction step $m-1 \to m$ are direct consequences of the following decomposition:

any matrix $A \in \mathrm{GL}_m(F_\mathfrak{p})$ decomposes as

$$A = \begin{pmatrix} & A' & a_{1,m} \\ & & \vdots \\ 0 & \cdots \quad 0 & a_{m,m} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ \frac{a_{m,1}}{a_{m,m}} & \cdots & \frac{a_{m,1}}{a_{m,m}} & 1 \end{pmatrix},$$

where the matrix $A' = (a'_{ij}) \in \mathrm{GL}_{m-1}(F_\mathfrak{p})$ has entries $a'_{ij} = a_{ij} - a_{i,m}\frac{a_{m,j}}{a_{m,m}}$ and if A is of type (t_m) then A' is of type (t_{m-1}).

We call an integer a sufficiently large, if it satisfies the conditions

(2.13.C) $\qquad a > n \quad \text{and} \quad w_{\eta_\mathfrak{p},n-2} \quad \text{is invariant under} \quad K(n,\mathfrak{p}^a).$

Applying once more Lemma 2.2 and reasoning as in the proof of (2.9) and (2.10) we find:

Proposition 2.5. *For $a \in \mathbb{N}$ sufficiently large, we find*

$$\operatorname{supp}\operatorname{res}_{\mathrm{GL}_{n-1}} w_{\chi_\mathfrak{p}} \subset N(n-1,F_\mathfrak{p})\bar{U}_{*0}(n-1,\mathfrak{p}^a),$$

where $\bar{U}_{*0}(n-1,\mathfrak{p}^a) \leq \mathcal{I}(n-1)$ is the subgroup of matrices

$$k = \begin{pmatrix} & k' & \mathbf{0} \\ y_1 & \cdots & y_{n-1} \end{pmatrix},$$

satisfying $k' \in \bar{U}_*(n-2,\mathfrak{p})$, $y_1,\ldots,y_{n-2} \in \mathfrak{p}^a$ and $y_{n-1} \in \mathcal{O}_\mathfrak{p}^*$. Moreover, for $k \in \bar{U}_{*0}(n-1,\mathfrak{p}^a)$ as above we have

$$w_{\chi_\mathfrak{p}}(k) = \chi_\mathfrak{p}^{-1}(y_{n-1})\eta_\mathfrak{p}^{-1}(\det k)\prod_{i=1}^{n-2} q^{i(i+1)/2-1}G(\eta_\mathfrak{p}^i)\, q^{(n-2)a}G(\chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1})\cdot w(\mathbf{1}_n).$$

Using the Proposition we may now evaluate $\mathbf{J}_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}}$ for integers a, which are sufficiently large: since

$$H_{\varpi^{e-a}}\bar{U}_{*0}(n-1,\mathfrak{p}^a)H_{\varpi^{a-e}} \subset K_0(n-1,\mathfrak{p}^e)$$

we deduce that

$$v_{\hat{\pi}[\chi],\mathfrak{p}}(gkH_{\varpi^{a-e}}) = v_{\hat{\pi}[\chi],\mathfrak{p}}(gH_{\varpi^{a-e}})\chi_\mathfrak{p}(k_{n-1,n-1})\eta_\mathfrak{p}(\det k)$$

for $k = (k_{ij}) \in \bar{U}_{*0}(n-1,\mathfrak{p}^a)$. Hence, using Proposition 2.5 we obtain
(2.14)

$$\mathbf{J}_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}} = \int_{\bar{U}_{*0}(n-1,\mathfrak{p}^a)} dg\, \prod_{i=1}^{n-2} q^{i(i+1)/2-1}G(\eta_\mathfrak{p}^i)q^{(n-2)a}\, G(\chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1})\, w_\mathfrak{p}(e)v_{\hat{\pi}[\chi],\mathfrak{p}}(H_{\varpi^{a-e}}).$$

In order to complete the computation of $\mathbf{J}_{\hat\pi[\chi],\Lambda,\mathfrak{p}}$ we are left with determining the restriction of the Whittaker functions to the standard torus in GL_n .

2.2. Calculation of the values of $w_{\mathfrak{p}}$ *.*

Lemma 2.6. *Let* α^w *denote the conjugate of the character* α *under the longest element* $w = w_{n,l}$ *of the Weyl group of* GL_n *. For* $t = \mathrm{diag}(t_1, \ldots, t_n) \in T_n(F_{\mathfrak{p}})$ *we have*

$$w_{\mathfrak{p}}(t) = \begin{cases} \alpha^w \delta_{B_n}^{1/2}(t) & \text{if } \ v_{\mathfrak{p}}(t_1) \geq \ldots \geq v_{\mathfrak{p}}(t_n) \\ 0 & \text{else.} \end{cases}$$

In particular, $w_{\mathfrak{p}}(e) = 1$ *.*

Proof. We have to compute the integral

$$w_{\mathfrak{p}}(t) = \int_{N_n(F_{\mathfrak{p}})} \psi_{\mathfrak{p}}^1(w_{n,l} n\, t)\, \bar\tau_{\mathfrak{p}}(n)\, dn \tag{2.15}$$

and we start by determining the set of matrices $n = (n_{ij}) \in N_n(F_{\mathfrak{p}})$ such that $w_{n,l} n t \in \mathrm{supp}(\psi_{\mathfrak{p}}^1)$. Taking into account the desription of the support of $\psi_{\mathfrak{p}}^1$ (see section 1) this is equivalent to

$$w_{n,l} n t = b^{-1} w_{n,l} \beta$$

for some $b = (b_{ij}) \in B(n, F_{\mathfrak{p}})$ and $\beta = (\beta_{ij}) \in \mathcal{I}(n)$. This may be rewritten as

$$\begin{pmatrix} t_n b_{1,1} & \cdots & t_{n-j+1} b_{ij} \\ & \ddots & \vdots \\ & & t_1 b_{nn} \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \vdots \\ 1 & \cdots & \frac{t_j}{t_{n-i+1}} n_{(n-i+1),j} \end{pmatrix} = \begin{pmatrix} \beta_{n,1} & \cdots & \beta_{n,n} \\ \vdots & & \vdots \\ \beta_{1,1} & \cdots & \beta_{1,n} \end{pmatrix}$$

and an easy induction on $i = n, \ldots, 1$ proves that this is equivalent to $b_{ij} \in t_{n-j+1}^{-1} \varpi \mathcal{O}_{\mathfrak{p}}$ for $j > i$, $v_{\mathfrak{p}}(b_{ii}) = v_{\mathfrak{p}}(t_{n-i+1}^{-1})$ for $i = 1, \ldots, n$ and $n_{ij} \in \frac{t_i}{t_j} \mathcal{O}_{\mathfrak{p}}$. Consequently, the support of the integral (2.15) is contained in

$$\{(n_{ij}) \in N(n, F_{\mathfrak{p}}) :\, n_{ij} \in t_i/t_j \mathcal{O}_{\mathfrak{p}}\} = t N_n(\mathcal{O}_{\mathfrak{p}}) t^{-1}$$

and since α is unramified the integral reads

$$w_{\mathfrak{p}}(t) = \int_{t N_n(\mathcal{O}_{\mathfrak{p}}) t^{-1}} \alpha \delta_{B_n}^{1/2}(w_{n,l} t w_{n,l}) \sum_{i=1}^{n-1} \bar\tau_{\mathfrak{p}}(n_{i,i+1}) \prod_{j>i} dn_{ij}.$$

Using character relations for $\tau_{\mathfrak{p}}$ we see that the integral vanishes unless $v_{\mathfrak{p}}(t_i) \geq v_{\mathfrak{p}}(t_j)$ for $i < j$. On the other hand if this is true then $\tau_{\mathfrak{p}}(n_{i,i+1}) = 1$ and we obtain

$$w_{\mathfrak{p}}(t) = \mathrm{vol}(t N_n(\mathcal{O}_{\mathfrak{p}}) t^{-1})\, \alpha \delta_{B_n}^{1/2}(w_{n,l} t w_{n,l}).$$

The volume reads $\mathrm{vol}(t N_n(\mathcal{O}_{\mathfrak{p}}) t^{-1}) = \delta_{B_n}(t)$. Together with the identity

$$\delta_{B_n}(w_{n,l} t w_{n,l}) = \delta_{B_n}^{-1}(t)$$

this completes the proof of the proposition.

2.3. The final formula.

Collecting all our results, we finally obtain

Theorem 2.7. *Assume that* $f_\chi = \mathfrak{p}^e$ *with* $e \geq 2$ *and* $\chi_\infty = \mathrm{id}$ *. Then we have*

$$\frac{L(\pi \otimes \chi\eta, s+s_2)}{L(\hat{\pi} \otimes \chi^{-1}, 1+s_2-s_1)} = \clubsuit\, P(s)^{-1}\, L(\pi \times \eta \otimes \hat{\pi}, s+s_1)^{-1}\, \mathbf{J}_{\hat{\pi}[\chi],\Lambda,\eta}(s),$$

where

$$\clubsuit = c\, G(\chi_{\mathfrak{p}}\eta_{\mathfrak{p}}^{n-1})^{-1} G(\chi_{\mathfrak{p}})^{2-n} |f_\chi|_{\mathfrak{p}}^{-1} v_{\hat{\pi}[\chi],S}(g_S^0)^{-1} w^0_{\hat{\pi},\mathfrak{p}}(\varpi^{\mathbf{v}'})^{-1}.$$

with $c \in \mathbb{Q}^*$ *. If we want to indicate the dependence on* $\hat{\pi}[\chi]$ *we write* $\clubsuit_{\hat{\pi},\chi}$

Remark 2.8. The formula in Theorem 2.7 computes the values $L(\pi \otimes \chi\eta, s)$ of the automorphic L -function on GL_n using certain period integrals attached to automorphic forms on GL_n as well as values of L -functions on groups GL_m of *lower* rank. Hence, the complete calculation will be by induction on the rank of the group.

In the next section we shall evaluate the period integral $\mathbf{J}_{\hat{\pi}[\chi],\Lambda}(s)$ for $s = 1/2$ in a different way.

3 The period Integral

In this section we compute the Fourier transform of the twisted values $L(\pi \otimes \chi\eta, 1/2)$. As a result we find a distribution $d\mathbf{P}_\Lambda$, whose p -adic Mellin transform computes the values of the L -function.

The computation will be based on a certain period integral $\mathbf{P}_{\Lambda,\eta}(\varepsilon; \mathfrak{p}^e)$, which we introduce first. We denote by $\mathbf{X}_e$ the set of all characters $\mu :\, F^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ satisfying the properties: $\mu_v = \mathrm{id}$ for all $v \in S_\infty$ and the conductor f_μ divides $\mathfrak{p}^e$. This set is in bijection

$$\mathbf{X}_e \leftrightarrow \{\mu_{\mathfrak{p}} : (\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_{\mathfrak{p}})^* \to \mathbb{C}^* \,:\, \mu_{\mathfrak{p}}(\mathcal{O}^*) = 1\}$$

with the set of $\mathcal{O}_F^*$ -invariant characters on $(\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_{\mathfrak{p}})^*$, the bijection being given by sending a character μ to its $\mathfrak{p}$ -component $\mu_{\mathfrak{p}}$ and restricting to $\mathcal{O}_{\mathfrak{p}}^*$. In particular we may choose a local prime $\varpi = \varpi_{\mathfrak{p}}$ such that $\eta_{\mathfrak{p}}(\varpi) = 1$. We use the following notation: if G is a finite group and V a vector space, we denote by G^V the set of all maps $f : G \to V$. Also we write $\mathcal{A}(n-1) = \mathcal{A}(\mathrm{GL}_{n-1}(\mathbb{Q})/\mathrm{GL}_{n-1}(\mathbb{A}))$. The duality of $\mathbf{X}_e$ and $\mathbf{O}_e := \mathcal{O}_{\mathfrak{p}}^*/((1+\varpi^e\mathcal{O}_{\mathfrak{p}})\mathcal{O}_F^*)$ then defines in the usual way a Fourier transform

$$\mathcal{F} :\, \mathbf{X}_e^{\mathcal{A}(n-1)} \to \mathbf{O}_e^{\mathcal{A}(n-1)},$$

sending $f : G \to \mathcal{A}(n-1)$ to

$$\mathcal{F}_x(f) = \sum_{\mu \in \mathbf{X}_e} \mu^{-1}(x) f_\mu.$$

We denote by D the diagonal matrix of length $n-1$

$$D = \begin{pmatrix} \varpi & & & \\ & \ddots & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} \in T_{n-1}(F_{\mathfrak{p}}),$$

and we fix a complex number κ. We set $\hat{\pi}[\mu] = \hat{\pi} \otimes \mu$. In section 1 we defined for primitive characters μ in $\mathbf{X}_e$ automorphic forms $E(f_{\hat{\pi}\otimes\mu,\Lambda}, g)$ and we now extend this definition to all characters $\mu \in \mathbf{X}_e$ with conductor $f_\mu = \mathfrak{p}^{e_\mu}$ setting

$$E^{\kappa}_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}(g) = E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}(g) = \kappa^{-(e-e_\mu)} E(f_{\hat{\pi}[\mu],\Lambda}, gD^{-(e-e_\mu)}). \tag{3.1}$$

The family $\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}$ then defines an element of $\mathbf{X}_e^{\mathcal{A}(n-1)}$. Accordingly, the transform $\mathcal{F}_x\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}$ is an element of $\mathbf{O}_e^{\mathcal{A}(n-1)}$, which we also view in the obvious way as element of $((\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_{\mathfrak{p}})^*)^{\mathcal{A}(n-1)}$. The following properties are immediate. For $\delta \in \mathcal{O}_{\mathfrak{p}}^*$ we have

$$\mathcal{F}_x\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}(gH_\delta) = \mathcal{F}_{\delta^{-1}x}\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}(g). \tag{3.2}$$

For every $\chi \in \mathbf{X}_e$ the inverse transform reads

$$\begin{aligned} E_{\hat{\pi}[\chi],\Lambda,\mathfrak{p}^e} &= \mathcal{F}_\chi\{\mathcal{F}_x\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}\} \\ &= \frac{1}{|(\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_{\mathfrak{p}})^*|} \sum_{x\in(\mathcal{O}_{\mathfrak{p}}/\varpi^e\mathcal{O}_{\mathfrak{p}})^*} \chi_{\mathfrak{p}}(x)\, \mathcal{F}_x\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}. \end{aligned} \tag{3.3}$$

Concerning the first property we observe that

$$D^{e-e_\mu} K_0(n-1,\mathfrak{p}^e) D^{-(e-e_\mu)} \subset K_0(n-1,\mathfrak{p}^{e_\mu}),$$

and since $E_{\hat{\pi}[\mu],\Lambda}$ is by definition the essential vector in $\mathrm{I}_{P,\Lambda,\mathfrak{p}}\hat{\pi}[\mu]$ we obtain for $k = (k_{ij}) \in K_0(n-1,\mathfrak{p}^e)$

$$E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}(gk) = E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}(g)\, \mu_{\mathfrak{p}}(k_{n-1,n-1}).$$

The definition of the Fourier transform then immediately implies the first claim.

For any vector $v = (v_1, \ldots, v_{n-1})$ such that $v_1, \ldots, v_{n-2} \in \mathcal{O}_{\mathfrak{p}}$ and $v_{n-1} \in \mathcal{O}_{\mathfrak{p}}^*$ and any $\varepsilon \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^e\mathcal{O}_{\mathfrak{p}})^*$ we introduce the period integral

$$\mathbf{P}_{\Lambda,\eta}(v,\varepsilon;\mathfrak{p}^e) := \int_{X_{n-1}^{\mathbb{A}}} \phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix} v\right) \eta\otimes\mathcal{F}_{\varepsilon+\varpi^e\mathcal{O}_{\mathfrak{p}}}\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}(g)\, dg.$$

Here, in accordance with (2.5) we identify v with the unipotent matrix

$$v = \begin{pmatrix} 1 & & & & v_1/\varpi^a \\ & \ddots & & & \vdots \\ & & \ddots & & v_{n-2}/\varpi^a \\ & & & 1 & v_{n-1}/\varpi^e \\ & & & & 1 \end{pmatrix}$$

To simplify the notation we mostly drop the index η and we write $E_{\varepsilon,\Lambda,\mathfrak{p}^e} = \mathcal{F}_{\varepsilon+\varpi^e\mathcal{O}_\mathfrak{p}}\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}$.

Lemma 3.1. *i) Assume that $e \geq a$. Then for any matrix*

$$u = \begin{pmatrix} 1 & & & o_1 \\ & \ddots & & \vdots \\ & & 1 & o_{n-2} \\ & & & 1 \end{pmatrix} \in N_{n-1}(\mathcal{O}_\mathfrak{p})$$

the following relation holds:

$$\mathbf{P}_{\Lambda,\eta}(v,\varepsilon;\mathfrak{p}^e) = \mathbf{P}_{\Lambda,\eta}(uv,\varepsilon;\mathfrak{p}^e).$$

ii) For any $\delta \in \mathcal{O}_\mathfrak{p}^*$ we have

$$\eta_\mathfrak{p}^{n-1}(\delta)\,\mathbf{P}_{\Lambda,\eta}(H_\delta\, v,\varepsilon;\mathfrak{p}^e) = \mathbf{P}_{\Lambda,\eta}(v,\delta\varepsilon;\mathfrak{p}^e).$$

Here, we view $v = (v_1,\ldots,v_{n-1})$ as column vector.

Proof. i) We set $u' := H_{\varpi^{-(e-a)}} u H_{\varpi^{e-a}}$. Since $E_{\varepsilon,\mathfrak{p}^e}$ is invariant under $N_{n-1}(\mathcal{O}_\mathfrak{p})$ we obtain by the change of variable $g \mapsto gu$

$$\mathbf{P}_\Lambda(v;\mathfrak{p}^e) = \int_{X_{n-1}^\mathbb{A}} \phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix} u'v\right)\eta\otimes E_{1,\mathfrak{p}^e}(g)\,dg.$$

On the other hand the expression in (2.6) for $w_{\eta_\mathfrak{p},n-2}$ together with the independence of this expression of the choice of representatives in $\mathcal{O}_\mathfrak{p}/\varpi^e\mathcal{O}_\mathfrak{p}$ (cf. Remark 2.1) proves, that $w_{\eta_\mathfrak{p},n-2}$ is invariant under all matrices $\mathrm{diag}(\tilde{u},1)$, where

$$\tilde{u} = \begin{pmatrix} 1 & & & o_1 \\ & \ddots & & \vdots \\ & & 1 & o_{n-2} \\ & & & 1 \end{pmatrix} \in N(n-1,\mathcal{O}_\mathfrak{p}).$$

Therefore, the same holds for ϕ_η and together with the identity

$$\begin{pmatrix} u' & \\ & 1\end{pmatrix} v = \begin{pmatrix} 1 & & & & (v_1+o_1v_{n-1})/\varpi^a \\ & \ddots & & & \vdots \\ & & 1 & & (v_{n-2}+o_{n-2}v_{n-1})/\varpi^a \\ & & & 1 & v_{n-1}/\varpi^e \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & o_1\varpi^{e-a} & 0 \\ & \ddots & & \vdots & \vdots \\ & & \ddots & o_{n-2}\varpi^{e-a} & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

we obtain

$$\mathbf{P}_\Lambda(v;\mathfrak{p}^e) = \mathbf{P}_\Lambda((v_1+o_1v_{n-1},\dots,v_{n-2}+o_{n-2}v_{n-1},v_{n-1});\mathfrak{p}^e),$$

which is the claim.

ii) Recalling the definition of the period integral we find

$$\begin{aligned}&\mathbf{P}_{\Lambda,\eta}(H_\delta v,\varepsilon;\mathfrak{p}^e)\\ =&\int_{X_{n-1}^{\mathbb{A}}}\phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & & & v_1/\varpi^a\\ & \ddots & & \vdots\\ & & \ddots & v_{n-2}/\varpi^e\\ & & 1 & \delta v_{n-1}/\varpi^e\\ & & & 1\end{pmatrix}\right)\eta\otimes E_{\varepsilon,\mathfrak{p}^e}(g)\,dg.\end{aligned}$$

On the other hand Eq. (2.10) implies

$$\phi_\eta\left(g\begin{pmatrix} H_\delta & \\ & 1\end{pmatrix}\right) = \phi_\eta(g)\eta_{\mathfrak{p}}^{n-2}(\delta). \tag{3.4}$$

Together with the change of integration variable $g\mapsto gH_{\delta^{-1}}$ we obtain

$$\mathbf{P}_{\Lambda,\eta}(H_\delta v,\varepsilon;\mathfrak{p}^e) = \eta_{\mathfrak{p}}^{2-n}(\delta)\int_{X_{n-1}^{\mathbb{A}}}\phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix}v\right)\eta\otimes E_{\varepsilon,\mathfrak{p}^e}(gH_{\delta^{-1}})\,dg.$$

The assertion then follows using Eq. (3.2). Thus, the Lemma is proven.

Corollary 3.2 *For $e\geq a$ the product $\eta_{\mathfrak{p}}^{n-1}(v_{n-1})\,\mathbf{P}_{\Lambda,\eta}(v,\varepsilon;\mathfrak{p}^e)$ does not depend on v.*

From now on we shall assume that $e\geq a$. For any $\varepsilon\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^e)^*$ we therefore may set

$$\mathbf{P}_\Lambda(\varepsilon;\mathfrak{p}^e) := \eta_{\mathfrak{p}}^{n-1}(v_{n-1})\,\mathbf{P}_\Lambda(v,\varepsilon;\mathfrak{p}^e)$$

and we define

$$d\mathbf{P}_\Lambda(\varepsilon+\mathfrak{p}^e\mathcal{O}_{\mathfrak{p}}) := (\kappa q^{n-1}\gamma)^e\mathbf{P}_\Lambda(\varepsilon;\mathfrak{p}^e), \tag{3.5}$$

where

$$\gamma=\gamma_\pi = \alpha\delta_{B_n}^{1/2}(\mathrm{diag}(1,\varpi^{-1},\dots,\varpi^{-1})). \tag{3.6}$$

It is then immediate by the following distribution relation for the periods $\mathbf{P}_\Lambda(\varepsilon;\mathfrak{p}^e)$ that $d\mathbf{P}_{\Lambda,\eta}$ defines a distribution on $\mathcal{O}_{\mathfrak{p}}^*$.

Proposition 3.3. *The periods $\mathbf{P}_{\Lambda,\eta}(\varepsilon;\mathfrak{p}^e)$ satisfy the distribution relation*

$$\mathbf{P}_{\Lambda,\eta}(\varepsilon;\mathfrak{p}^e) = \kappa q^{n-1}\gamma\sum_{\substack{\varepsilon'\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^{e+1}\mathcal{O}_{\mathfrak{p}})^*\\ \varepsilon'\equiv\varepsilon\pmod{\mathfrak{p}^e}}}\mathbf{P}_{\Lambda,\eta}(\varepsilon';\mathfrak{p}^{e+1}).$$

Proof. The distribution relation for the period integral follows from the distribution relations, which are satisfied by the Eisenstein series $E_{\varepsilon,\mathfrak{p}^e} = \mathcal{F}_{\varepsilon+\varpi^e\mathcal{O}_\mathfrak{p}}\{E_{\hat{\pi}[\mu],\Lambda,\mathfrak{p}^e}\}$ and the cusp form ϕ_η :

- The sections $E_{\varepsilon,\mathfrak{p}^e}$ satisfy the distribution relation

$$E_{\varepsilon,\mathfrak{p}^e}(gD^{-1}) = \kappa \sum_{\varepsilon'} E_{\varepsilon',\mathfrak{p}^{e+1}}(g), \tag{3.7}$$

where the sum runs over all elements $\varepsilon' \in (\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{e+1}\mathcal{O}_\mathfrak{p})^*$ which are congruent to ε modulo $\varpi^e\mathcal{O}_\mathfrak{p}$. Since $\eta_\mathfrak{p}(\varpi) = 1$ the same relation holds for $\eta \otimes E_{\varepsilon,\mathfrak{p}^e}$

- We have

$$\phi(g) = \gamma \sum_{w_1,\dots,w_{n-1}} \phi(gD^*w), \tag{3.8}$$

where $D^* = \mathrm{diag}(\varpi,\dots,\varpi,1)$ is the diagonal matrix of length n , w is the matrix

$$w = \begin{pmatrix} 1 & & & w_1/\varpi \\ & \ddots & & \vdots \\ & & 1 & w_{n-1}/\varpi \\ & & & 1 \end{pmatrix}. \tag{3.9}$$

and $w_1,\dots,w_{n-1}$ run over $\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}$.

To prove the first distribution relation one simply evaluates the left hand side by plugging in the definition of $E_{\varepsilon',e+1}(g)$ and using character relations. The second one is an easy consequence of the relation

$$\psi^1_\mathfrak{p}(g) = \alpha\delta_{B_n}^{1/2}(\mathrm{diag}(1,\varpi^{-1},\dots,\varpi^{-1})) \sum_{w_1,\dots,w_{n-1}} \psi^1_\mathfrak{p}(gD'w)$$

for the section $\psi^1_\mathfrak{p}$, which we chose in section 1.

We may now prove the proposition. Let $v \in N(n,F_\mathfrak{p})$ be as in the definition of the periods. Using the expression for $w_{\eta,n-2}$ and the distribution relation for ϕ we find

$$\begin{aligned} \phi_\eta(gv) &= \sum_{u_{ij},v_k} \prod_{t=1}^{n-2} \eta^t_\mathfrak{p}(u_{t,t+1})\chi_\mathfrak{p}\eta^{n-1}_\mathfrak{p}(v_{n-1})\phi(gv') \\ &= \gamma \sum_{u_{ij},v_k} \prod_{t=1}^{n-2} \eta^t_\mathfrak{p}(u_{t,t+1})\chi_\mathfrak{p}\eta^{n-1}_\mathfrak{p}(v_{n-1}) \sum_{w_1,\dots,w_{n-1}} \phi(gv'D^*w), \end{aligned}$$

where

$$v' = \begin{pmatrix} 1 & u_{1,2}/\varpi & \cdots & u_{1,n-1}/\varpi^{n-2} & v_1/\varpi^a \\ & \ddots & & \vdots & \vdots \\ & & 1 & u_{n-2,n-1}/\varpi & v_{n-2}/\varpi^a \\ & & & 1 & v_{n-1}/\varpi^e \\ & & & & 1 \end{pmatrix}$$

and w and $w_1, \ldots, w_{n-1}$ are as in (3.9). A small computation proves

$$v' D^* w = D^* \begin{pmatrix} 1 & \cdots & \cdots & u_{ij}/\varpi^{j-i} & x_1 \\ & \ddots & & \vdots & \vdots \\ & & 1 & \vdots & x_{n-2} \\ & & & 1 & x_{n-1} \\ & & & & 1 \end{pmatrix}$$

where $x_i = \frac{v_i}{\varpi^{a+1}} + \sum_{h=i+1}^{n-1} \frac{w_h}{\varpi} \frac{u_{ih}}{\varpi^{h-i}} + \frac{w_i}{\varpi}$ and $x_{n-1} = \frac{v_{n-1} + \varpi^e w_{n-1}}{\varpi^{e+1}}$. Thus we get

$$\phi_\eta(gv) = \gamma \sum_{w_1, \ldots, w_{n-1}} \phi_\eta(g D^* w'),$$

where

$$w' = \begin{pmatrix} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_{n-1} \\ & & & 1 \end{pmatrix}.$$

We turn to the calculation of $\mathbf{P}_\Lambda(\varepsilon; \mathfrak{p}^e)$ and we choose v as follows: $v_1 = \ldots = v_{n-2} = 0$ and $v_{n-1} = 1$. Then, replacing $\phi_\eta(g H_{\varpi^{e-a}})$ by the above expression and changing the variable $g \mapsto g D^{-1}$ we find

$$\mathbf{P}_\Lambda(\varepsilon; \mathfrak{p}^e) = \gamma \sum_{w_1, \ldots, w_{n-1}} \int_{X_{n-1}^{\mathbb{A}}} \phi_\eta \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} H_{\varpi^{e+1-a}} & \\ & 1 \end{pmatrix} w' \right) \eta \otimes E_{\varepsilon, \mathfrak{p}^e}(g D^{-1}) \, dg.$$

Next we use the distribution relation (3.7) as well as (3.2) to obtain

$$\mathbf{P}_\Lambda(\varepsilon; \mathfrak{p}^e) = \gamma\kappa \sum_{w_1, \ldots, w_{n-1}} \sum_{\varepsilon' \equiv \varepsilon \pmod{\mathfrak{p}^e}} \int_{X_{n-1}^{\mathbb{A}}} \phi_\eta \left(g H_{\varpi^{e+1-a}} w' \right) \eta \otimes E_{\varepsilon', \mathfrak{p}^{e+1}}(g) \, dg.$$

Using the periods this may be rewritten

$$\mathbf{P}_\Lambda(\varepsilon; \mathfrak{p}^e) = \gamma\kappa \sum_{w_1, \ldots, w_{n-1}} \sum_{\varepsilon' \equiv \varepsilon \pmod{\mathfrak{p}^e}} \mathbf{P}_\Lambda((\varpi^a x_1, \ldots, \varpi^a x_{n-2}, \varpi^{e+1} x_{n-1}), \varepsilon'; \mathfrak{p}^{e+1}).$$

In the above equation, $\varpi^a x_1, \dots, \varpi^a x_{n-2} \in \mathcal{O}_\mathfrak{p}$ and $\varpi^{e+1}x_{n-1} = \varepsilon + \varpi^e w_{n-1} \in 1 * \mathfrak{p}^e\mathcal{O}_\mathfrak{p}$. Therefore using Lemma 3.1 and taking into account that $\eta_\mathfrak{p}|_{1+\mathfrak{p}\mathcal{O}_\mathfrak{p}} = 1$ we arrive at

$$\mathbf{P}_\Lambda(\varepsilon;\mathfrak{p}^e) = \gamma\kappa q^{n-1} \sum_{w_{n-1}\in\mathcal{O}_\mathfrak{p}/\varpi\mathcal{O}_\mathfrak{p}} \mathbf{P}_\Lambda((0,\dots,0,1),\varepsilon'(\varepsilon+\varpi^e w_{n-1})^{-1};\mathfrak{p}^{e+1}).$$

Changing the summation variable ε in the obvious way the assertion of the proposition follows.

Like the periods $\mathbf{P}_{\Lambda,\eta}(v;\mathfrak{p}^e)$ the distribution $d\mathbf{P}_{\Lambda,\eta}$ depends on the choice of the components at infinity $w_\infty = \otimes_{v\in S_\infty} w_v$ and $v_{\hat\pi[\chi],\Lambda,\infty} = \otimes_{v\in S_\infty} v_{\hat\pi[\chi],\Lambda,v}$ and if we want to indicate this dependence we write $d\mathbf{P}_{\Lambda,\eta} = d\mathbf{P}^{\Lambda,\eta}_{w_\infty,v_{\hat\pi[\chi],\Lambda,\infty}}$.

Proposition 3.4. *For all characters χ with conductor $f_\chi = \mathfrak{p}^e$, ($e > a$), infinity component $\chi_v = \mathrm{id}$ for all $v \in S_\infty$ and satisfying $\chi(x) = 1$ for all $x \in \mathcal{V}_F$ we have*

$$\int_{\mathcal{O}_\mathfrak{p}^*} \chi_\mathfrak{p}\, d\mathbf{P}_{\Lambda,\eta} = q^{a(n-2)}(\gamma\kappa q^{n-1})^e\, \mathbf{J}_{\hat\pi[\chi],\Lambda,\eta}(1/2).$$

Proof. Using (3.3) $\mathbf{J}_{\hat\pi[\chi],\Lambda}(1/2)$ may be written

$$\begin{aligned}\mathbf{J}_{\hat\pi[\chi],\Lambda}(1/2) &= \sum_{v_1,\dots,v_{n-1}} \sum_{\varepsilon\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e)^*} \chi_\mathfrak{p}\eta_\mathfrak{p}^{n-1}(v_{n-1})\,\chi_\mathfrak{p}(\varepsilon) \\ &\quad\times \int_{X^\mathbb{A}_{n-1}} \phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix} v\right)\eta\otimes E_{\varepsilon,\mathfrak{p}^e}(g)\,dg.\end{aligned}$$

Changing the summation variable $g \mapsto gH_\varepsilon$ and using Lemma 3.1, the right hand side becomes

$$\begin{aligned}\mathbf{J}_{\hat\pi[\chi],\Lambda}(1/2) = &\sum_{v_1,\dots,v_{n-1}} \sum_{\varepsilon\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e)^*} \chi_\mathfrak{p}(\varepsilon v_{n-1}) \\ &\times \int_{X^\mathbb{A}_{n-1}} \phi_\eta\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} H_{\varpi^{e-a}} & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ & & & 1 & 1/\varpi^e \\ & & & & 1\end{pmatrix}\right)\eta\otimes E_{\varepsilon v_{n-1},\mathfrak{p}^e}(g)\,dg.\end{aligned}$$

The change of the summation variable $\varepsilon \mapsto \varepsilon v_{n-1}^{-1}$ then yields

$$\begin{aligned}\mathbf{J}_{\hat\pi[\chi],\Lambda}(1/2) &= \sum_{v_1,\dots,v_{n-1}} \sum_{\varepsilon\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e\mathcal{O}_\mathfrak{p})^*} \chi_\mathfrak{p}(\varepsilon)\,\mathbf{P}_\Lambda((0,\dots,0,1),\varepsilon;\mathfrak{p}^e) \\ &= \sum_{v_1,\dots,v_{n-1}} \chi_\mathfrak{p}(\varepsilon)\,\mathbf{P}_\Lambda(\varepsilon;\mathfrak{p}^e) \\ &= (\kappa\gamma q^{n-1})^{-e} q^{a(n-2)} \sum_{\varepsilon\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^e\mathcal{O}_\mathfrak{p})^*} \chi_\mathfrak{p}(\varepsilon) d\mathbf{P}_\Lambda(\varepsilon+\mathfrak{p}^e\mathcal{O}_\mathfrak{p}).\end{aligned}$$

Hence, the assertion and the proposition is proven.

Together with Theorem 2.7 we immediately obtain the representation of the values of the L-function as a p-adic Mellin transform.

Theorem 3.5. *Assume that the conductor f_η is sufficiently large as in $(\mathbf{C.1})$. Let $\chi: F^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ be any character with conductor $f_\chi = \mathfrak{p}^e$ for $e \geq a$ and infinity component $\chi_\infty = \mathrm{id}$. Then the following holds*

$$(\gamma\kappa q^{n-1})^e \clubsuit \int_{\mathcal{O}_\mathfrak{p}^*} \chi_\mathfrak{p}\, d\mathbf{P}_{\Lambda,\eta} = P_{w_\infty, v_{\hat{\pi}[\chi],\Lambda,\infty}}(1/2)\, \frac{L(\pi \otimes \chi\eta, 1/2 + s_2)}{L(\hat{\pi} \otimes \chi^{-1}, 1 + s_1 - s_2)}.$$

We recall that in the formulation of the Theorem $\Lambda = (s_1, s_2)$.

Remark 3.6. The distribution $d\mathbf{P}_{\Lambda,\eta}$ is $\mathcal{O}_F^*$-invariant: for all $o \in \mathcal{O}_F^*$ we have

$$d\mathbf{P}_{\Lambda,\eta}(o\varepsilon + \mathfrak{p}^e) = d\mathbf{P}_{\Lambda,\eta}(\varepsilon + \mathfrak{p}^e).$$

In particular, $d\mathbf{P}_{\Lambda,\eta}$ is even.

4 Algebraic automorphic representations.

To simplify the exposition from now on we shall assume $F = \mathbb{Q}$, i.e. $\mathbb{A}$ denotes the ring of rational adeles. We introduce the type of cuspidal represention of $\mathrm{GL}_n(\mathbb{A})$ for which we shall examine the values of the belonging L-function. These are algebraic automorphic representations, which in addition satisfy a regularity condition. Using results of Clozel it turns out that these representations have realizations as submodules of the cohomology of the symmetric space with coefficients in a finite dimensional representation of $\mathrm{GL}_n(\mathbb{C})$.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, whose infinity component is of the following type. If $n = 2m$ is even we assume that there is a $w \in \mathbb{Z}$ and integers $l_1 > l_2 > \ldots > l_m > 0$ satisfying $l_i \equiv l_j \pmod 2$ such that

$$\pi_\infty \cong \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_\mathbb{R}^{w/2} \otimes D_{l_1}, \ldots, |\cdot|_\mathbb{R}^{w/2} \otimes D_{l_m}), \tag{4.1}$$

where $P_n \leq \mathrm{GL}_n$ is the parabolic subgroup of type $(2, 2, \ldots, 2)$. If $n = 2m + 1$ is odd there is a $w \in \mathbb{Z}$ and integers $l_1 > l_2 > \cdots > l_m > 0$ satisfying $l_i \equiv l_j \pmod 2$ such that

$$\pi_\infty \cong \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_\mathbb{R}^{w/2} \otimes D_{l_1}, \ldots, |\cdot|_\mathbb{R}^{w/2} \otimes D_{l_m}, |\cdot|_\mathbb{R}^{w}) \tag{4.2}$$

where $P_n \leq \mathrm{GL}_n$ is the parabolic subgroup of type $(2, 2, \ldots, 2, 1)$.

We diagonally embed $\mathbb{C} \hookrightarrow \mathbb{C}^n$ and we identify $\mathbb{C}$ with its image. We denote by $\sim$ the equivalence relation on $\mathbb{Z}^n$, which is induced by the standard action of the Weyl group $W_{\mathrm{GL}_n} = S_n$ of GL_n. We set

$$\mathbf{l} = (l_1, l_2, \dots, -l_2, -l_1) \tag{4.3}$$

if n is even and

$$\mathbf{l} = (l_1, l_2, \dots, 0, \dots, -l_2, -l_1) \tag{4.4}$$

if n is odd. The type at infinity $\mathbf{p} = (p_i)$ of the representation π (cf. [C], p. 107) then reads

$$\mathbf{p} = \big(w/2 + 1/2\mathbf{l} - (n-1)/2\big)_{\sim}.$$

In particular, we deduce that π is algebraic in the sense of [C], Définition 1.8, p. 89 if and only if

$$\mathbf{l} + w \equiv 1 \pmod 2$$

if n is even and

$$\mathbf{l} \equiv 0 \pmod 2 \quad \text{and} \quad w \equiv 0 \pmod 2.$$

if n is odd. Moreover, since $l_i \neq l_j$ for $i \neq j$ we see that in this case π is automatically regular in the sense of [C], Définition 3.12, p. 111. We set

$$\mathbf{m}_\pi = w/2 + 1/2\mathbf{l} - \check{\rho},$$

where $\check{\rho}$ denotes the half sum of positive simple roots of GL_n with respect to T_n. If there is no danger of confusion we drop the index and simply write $\mathbf{m} = (m_i)$. Explicitely, we have

$$\mathbf{m} = \left(\frac{w}{2} + \frac{l_1}{2} - \frac{n-1}{2}, \frac{w}{2} + \frac{l_2}{2} - \frac{n-3}{2}, \dots, \frac{w}{2} - \frac{l_1}{2} + \frac{n-1}{2}\right)$$

if n is even and

$$\mathbf{m} = \left(\frac{w}{2} + \frac{l_1}{2} - \frac{n-1}{2}, \frac{w}{2} + \frac{l_2}{2} - \frac{n-3}{2}, \dots, \frac{w}{2}, \dots, \frac{w}{2} - \frac{l_1}{2} + \frac{n-1}{2}\right)$$

if n is odd. In particular, if π is algebraic we deduce that $\mathbf{m}_\pi$ is an integral and dominant weight (not necessarily regular). Moreover, $\mathbf{m} = (m_i) \in \mathbb{Z}^n$ is pure of weight $w \in \mathbb{Z}$, which means that there is a $w \in \mathbb{Z}$ such that

$$m_i + m_{n+1-i} = w, \quad i = 1, \dots, n/2$$

if n is even and

$$m_i + m_{n+1-i} = w, \quad i = 1, \dots, (n-1)/2 \quad \text{and} \quad m_{(n+1)/2} = w/2$$

if n is odd.

On the other hand, if $\mathbf{m} \in \mathbb{Z}^n$ is dominant and satisfies the purity condition then there is a representation π_∞ of $\mathrm{GL}_n(\mathbb{R})$ such that $\mathbf{m} = \mathbf{m}_\pi$: for any vector $\mathbf{l} = (l_i)$ of the form (4.3) resp. (4.4) and $w \in \mathbb{Z}$ we write

$$\pi_\infty(w, \mathbf{l}) \tag{4.5}$$

to denote the representation (4.1) resp. (4.2). Let w be the weight of $\mathbf{m}$ and define $\mathbf{l} = (l_i)$ by

$$\mathbf{l} = 2\mathbf{m} + 2\check{\rho} - w; \tag{4.6}$$

we obtain $\mathbf{m} = \mathbf{m}_\pi$ if $\pi_\infty = \pi_\infty(w, \mathbf{l})$ (Explicitely $l_i = 2m_i - 2(i-1) + n - 1 - w$, $i = 1, \dots, m = n$). Of course, π is necessarily algebraic and regular.

From now on we shall assume that π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ with infinity component as follows.
• If $n = 2m$ is even there are integers $l_1 > l_2 > \cdots > l_m > 0$ satisfying $l_i \equiv l_j \pmod 2$ such that

$$\pi_\infty \cong |\cdot|^{w/2} \otimes \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(D_{l_1}, D_{l_2}, \dots, D_{l_m}) \tag{\textbf{Coh}.\textbf{a}}$$

where $P_n \leq \mathrm{GL}_n$ is the parabolic subgroup of type $(2, 2, \dots, 2)$ and $w = 0$ if the l_i are odd and $w = -1$ if the l_i are even.
• If $n = 2m+1$ is odd there are even integers $l_1 > l_2 > \cdots > l_m > 0$ such that

$$\pi_\infty \cong \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(D_{l_1}, D_{l_2}, \dots, D_{l_m}, \mathrm{id}) \tag{\textbf{Coh}.\textbf{b}}$$

where $P_n \leq \mathrm{GL}_n$ is the parabolic subgroup of type $(2, 2, \dots, 2, 1)$ (cf. [C], p. 114).

The representation π is algebraic and regular and hence, $\mathbf{m} = \mathbf{m}_\pi \in \mathbb{Z}^n$ is dominant; we denote by ρ the finite dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ acting on the vector space $M_\mathbf{m}$ with highest weight $\mathbf{m}$ with respect to the standard torus. We define

$$\tilde{\mathbf{m}} = -w_{n,l}\mathbf{m} = -w + \mathbf{m}$$

and $\tilde{\mathbf{m}} = (\tilde{m}_i)$ is the highest weight of the contragredient representation $(\tilde{\rho}, \tilde{M}_\mathbf{m})$. Lemme 3.14 in [C] then implies that

$$H^\bullet(\mathfrak{g}_n, \mathrm{O}_n(\mathbb{R})Z_n^0(\mathbb{R}), \pi_\infty \otimes \epsilon_\infty \otimes \tilde{\rho}) \neq 0 \tag{4.7}$$

for $b_n \leq \bullet \leq t_n$. Here,

$$b_n = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \end{cases}$$

$$t_n = \begin{cases} \frac{(n+1)^2-1}{4} - 1 & \text{if } n \text{ is even} \\ \frac{(n+1)^2}{4} - 1 & \text{if } n \text{ is odd} \end{cases}$$

and

$$\epsilon_\infty = \begin{cases} \mathrm{id} & n \text{ even} \\ \mathrm{id} & n \equiv 1 \pmod 4 \\ \mathrm{sgn} & n \equiv 3 \pmod 4. \end{cases}$$

Since $\mathrm{O}_n(\mathbb{R})$ normalizes $\mathrm{SO}_n(\mathbb{R})$ the group of connected components $\pi_0 = \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ acts on $H^\bullet(\mathfrak{g}_n, \mathrm{SO}_n(\mathbb{R})Z_n^0(\mathbb{R}), \pi_\infty \otimes \tilde{\rho})$ via right translation. Obviously, $\pi_0 \cong \mathbb{Z}/2\mathbb{Z}$ and we denote by id resp. sgn the trivial resp. the non-trivial representation of π_0. Eq. (4.7) then implies that π_0 acts on the relative Lie algebra cohomology via ϵ_∞, i.e. we obtain

$$H^{b_n}(\mathfrak{g}_n, \mathrm{SO}_n(\mathbb{R})Z_n^0(\mathbb{R}), \pi_\infty \otimes \tilde{\rho}) \cong \begin{cases} \mathrm{sgn} \oplus \mathrm{id} & \text{if } n \equiv 0 \pmod 2) \\ \mathrm{id} & \text{if } n \equiv 1 \pmod 4) \\ \mathrm{sgn} & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

The claim in the case n odd follows immediately from [C], Lemme 3.14, while in the case n even one has to take into account that the restriction of π to the group of matrices with positive determinant decomposes

$$\pi|_{\mathrm{GL}_n^+(\mathbb{R})} = \pi^+ \oplus \pi^-$$

and $\pi^+ \cong \pi^-$ as $\mathrm{GL}_n^+(\mathbb{R})$-modules. In particular, the cohomology groups are two resp. one dimensional. The non-vanishing of the cohomology at infinity yields an embedding of the finite part of π into the cohomology as follows. We choose a basis $\{\omega_1', \dots, \omega_{\dim X_n^1}'\}$ of the dual of the tangent space $(\mathfrak{g}_n/\mathfrak{k}_n\mathfrak{z}_n)^*$ of $X_n^1 = \mathrm{GL}_n(\mathbb{R})/K_{n,\infty}$ as well as a basis $\{a\}$ of $M_{\mathbf{m}}$. We let

$$\xi_{\pi,\infty} = \sum_{\mathbf{j}=(j_1,\dots,j_{b_n})} \sum_a \omega_{\mathbf{j}}' \, w_{\infty,\mathbf{j},a} \otimes a$$

be a generator of

$$H^{b_n}(\mathfrak{g}_n, K_{n,\infty}Z_n^0(\mathbb{R}), W(\pi_\infty, \tau_\infty) \otimes \rho)$$

if n is odd and of

$$H^{b_n}(\mathfrak{g}_n, K_{n,\infty}Z_n^0(\mathbb{R}), W(\pi_\infty, \tau_\infty) \otimes \rho)^+$$

if n is even, where the superscript $+$ indicates the $+1$-eigenspace for the action of π_0. For any compact open subgroup $K \leq \mathrm{GL}_n(\hat{\mathbb{Z}})$ we introduce the manifolds

$$S_n(K) = \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/KK_{n,\infty}.$$

The finite dimensional representation $(\tilde{\rho}, \tilde{M}_{\mathbf{m}})$ defines a locally constant sheaf $\tilde{\mathcal{M}}_{\mathbf{m}} = \tilde{\mathcal{M}}_{\mathbf{m},K}$ on $S_n(K)$ and there are cohomology groups

$$H_?^\bullet(S_n(K), \tilde{\mathcal{M}}_{\mathbf{m},K}), \quad ? = \text{sing, c, cusp}.$$

Passing to the limit we define the cohomology groups

$$H^\bullet_?(\tilde{S}_n, \tilde{\mathcal{M}}_{\mathbf{m}}) = \operatorname{inj}\lim_K H^\bullet_?(S_n(K), \tilde{\mathcal{M}}_{\mathbf{m},K}), \quad ? = \text{sing, c, cusp}.$$

These groups are modules under the canonical action of $\mathrm{GL}_n(\mathbb{A}_f) \times \pi_0$.

For any Whittaker function $w_f \in W(\pi_f, \tau_f)$ the product $w_f \xi_{\pi,\infty}$ now defines an element of $H^{b_n}(\mathfrak{g}_n, K_{n,\infty} Z_n^0(\mathbb{R}), W(\pi,\tau) \otimes \rho)$ and together with the embedding $\pi \hookrightarrow L_0^2(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}))$ this yields an injection

$$\begin{array}{rccc} \xi_{\pi,\infty}: & W(\pi_f, \tau_f) & \hookrightarrow & H^{b_n}_{\mathrm{cusp}}(\tilde{S}_n, \tilde{\mathcal{V}}) \\ & w_f & \mapsto & \omega. \end{array}$$

In coordinates, the image of w_f reads

$$\omega = \sum_{\mathbf{j}=(j_1,\dots,j_{b_n})} \sum_a \omega'_{\mathbf{j}} \, \phi_{\mathbf{j},a} \otimes a,$$

where $\phi_{\mathbf{j},a}$ is the cusp form corresponding to $w_f w_{\infty,\mathbf{j},a}$. We remark that the cuspidal cohomology defines a subspace of the cohomology with compact supports

$$H^{b_n}_{\mathrm{cusp}}(\tilde{S}_n, \mathcal{M}_{\mathbf{m}}) \leq H^{b_n}_{\mathrm{c}}(\tilde{S}_n, \tilde{\mathcal{M}}_{\mathbf{m}})$$

(cf. [C], p 123).

The critical points. – We finish this section with the determination of the critical points of the representation $\pi \otimes \eta$. Here, $\eta : \mathbb{Q}^* \backslash \mathbb{A}^* \to \mathbb{C}^*$ is an idele class character of finite order and we set $\flat = 0$ if $\eta_\infty = \mathrm{id}$ and $\flat = 1$ if $\eta_\infty = \mathrm{sgn}$.

Lemma 4.1. *If n is even, the critical points l for $\pi \otimes \eta$ are*

$$l = -\frac{l_m + w}{2} + 1, -\frac{l_m + w}{2} + 2, \dots, \frac{l_m - w}{2} - 1, \frac{l_m - w}{2} \in \frac{1}{2} + \mathbb{Z},$$

independent of η. Here, w was defined to be 0 if the l_i are odd and -1 if the l_i are even. If n is odd, the critical points for $\eta \otimes \pi$ are all integers

$$l = -\frac{l_m}{2} + 1, -\frac{l_m}{2} + 2, \dots, \frac{l_m}{2} - 1, \frac{l_m}{2} \in \mathbb{Z},$$

which satisfy the congruence

$$l \equiv \begin{cases} \flat & \pmod 2 \quad \text{if} \quad l \geq 1 \\ \flat + 1 & \pmod 2 \quad \text{if} \quad l \leq 0. \end{cases}$$

Proof. We first assume $n = 2m$ to be even. Since $D_l \otimes \mathrm{sgn} \cong D_l$ we see that $\pi_\infty \otimes \mathrm{sgn} \cong \pi_\infty$ and we may assume $\eta_\infty = \mathrm{id}$. We denote by $W_\mathbb{R}$ the Weil group of

$\mathbb{R}$. Using the notation of [Kn], ch. 3.6 the representation π_∞^{w} of $W_{\mathbb{R}}$ attached to π_∞ via local Langlands correpondence reads

$$\pi_\infty^{\mathrm{w}} = (l_1, w/2) \oplus (l_2, w/2) \oplus \cdots \oplus (l_m, w/2).$$

Taking into account that $L(\pi_\infty, s) = L(\pi_\infty^{\mathrm{w}}, s)$ we deduce from the formulas in [Kn], ch. 3.6 that

$$L(\pi_\infty, s) = \prod_{i=1}^{m} 2(2\boldsymbol{\pi})^{-(s+w/2+l_i/2)}\Gamma\left(s + \frac{l_i + w}{2}\right).$$

Since the Γ-Function has its poles at $s = 0, -1, -2, \dots$ the claim follows. If $n = 2m+1$ is odd we find for the representation of the Weil group corresponding to $\pi_\infty \otimes \eta_\infty$

$$(\pi_\infty \otimes \eta_\infty)^{\mathrm{w}} = (l_1, 0) \oplus (l_2, 0) \oplus \cdots \oplus (l_m, 0) \oplus (\flat, 0).$$

and hence, the local factor of the L-function reads

$$L(\pi_\infty \otimes \eta_\infty, s) = \prod_{i=1}^{m} 2(2\boldsymbol{\pi})^{-(s+l_i)}\Gamma\left(s + \frac{l_i}{2}\right)\boldsymbol{\pi}^{(s+1)/2}\Gamma\left(\frac{s+\flat}{2}\right).$$

Since the l_i are even if n is odd, the statement about the poles of the Γ-function again yields the claim. This completes the proof of the lemma.

Remark 4.2. 1) In terms of the weight $\mathbf{m}_\pi = (m_i)$ attached to π the set of critical points reads as follows. If n is even the critical points are

$$-m_{[n/2]} + 1/2, \dots, 1/2, \dots, m_{[n/2]} + 1/2 - w \in 1/2 + \mathbb{Z},$$

if n is odd these are all integers in the range

$$-m_{[n/2]}, \dots, 0, 1, \dots, m_{[n/2]} + 1,$$

which satisfy the same congruence as in Lemma 4.1.

2) The functional equation implies

$$L(\pi \otimes \chi\eta, l) = L(|\cdot|^w \otimes \tilde{\pi} \otimes \chi^{-1}\eta^{-1}, 1 - l - w)$$

and $|\cdot|^w \otimes \tilde{\pi}$ again is of the form described in $(\mathbf{Coh.a})$ resp. $(\mathbf{Coh.b})$. Hence it is sufficient to examine the values at critical points $l \leq (1-w)/2$. In all cases this implies $l \leq 1/2$.

We say that (l, η) is a critical pair for π if l is one of the critical points for $\pi \otimes \eta$ as described in Lemma 4.1.

In our further treatment of the L-function attached to twists of the representation π we shall have to distinguish the following cases.

Definition 4.3. Let $\pi = \pi(w, \mathbf{l})$ be a representaion of $\mathrm{GL}_n(\mathbb{A})$ as in $(\mathbf{Coh})$ and η an idele class character of finite order. We say that

$$\begin{array}{llllll} \pi \quad \text{is of} & \text{type I} & \text{if} \quad n \equiv 0 \pmod 2 & \text{and} & \mathbf{l} \equiv 0 & \pmod 2 \\ & \text{type II} & \quad\;\; n \equiv 0 \pmod 2 & & \mathbf{l} \equiv 1 & \pmod 2 \\ & \text{type III} & \quad\;\; n \equiv 1 \pmod 2 & & \mathbf{l} \equiv 0 & \pmod 2. \end{array}$$

From now on we shall assume that $n = 2m$ is even and π is as in $(\mathbf{Coh})$. For later purpose we assume that in addition to $(\mathbf{Coh})$ the vector $\mathbf{m}_\pi = (m_i)$ attached to π satisfies the following condition; we set $\Delta_i = m_i - m_{i+1}$, $i = 1, \dots, n/2 - 1$.

$$(\mathbf{Reg}) \qquad \begin{array}{llll} \Delta_i > 0,\ i < n/2 - 1 & \text{if} & \pi \ \text{is of type} & \text{I} \\ \Delta_i > 2,\ i < n/2 - 2, \quad \Delta_{n/2-1} \geq n & & \pi & \text{II.} \end{array}$$

Thus, in case I we assume that $\mathbf{m}$ is regular, while in case II we make a slightly stronger assumption.

5 Eisenstein series and the cohomology at infinity

We shall describe a topological realization of the induced automorphic representation $\mathrm{I}_{P,\Lambda}\hat{\pi}[\chi]$ as certain subspace of the cohomology of the symmetric space attached to GL_{n-1} consisting of classes, which do not vanish at infinity. This will be based on the theory of Eisenstein cohomology and hence the construction of such a space starts from a Hecke invariant subspace in the cohomology of the boundary of the symmetric space. As a result we find classes $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda}$ in the cohomology of the symmetric space of GL_{n-1}, which turn out to be rational.

5.1. A Hecke-invariant subspace in the cohomology of the boundary.

We want to describe the cohomology of the boundary as module under the Hecke algebra in terms of the (cuspidal) cohomology of groups of lower rank.

We let $\bar{S}_{n-1}(K)$ be the *Borel-Serre* compactification of $S_{n-1}(K)$, which is a manifold with boundary and corners. Using the above notations we may describe the boundary $\partial\bar{S}_{n-1}(K)$ as follows. We let $\mathcal{P}$ be the set of all parabolic subgroups $P \leq \mathrm{GL}_{n-1}$, which are defined over $\mathbb{Q}$. The group GL_{n-1} acts on $\mathcal{P}$ via conjugation and $\partial\bar{S}_{n-1}(K)$ then is the disjoint union of faces

$$\partial\bar{S}_{n-1}(K) = \bigcup_P e(P, K),$$

where P runs over a set of representatives of $\mathcal{P}$ modulo $\mathrm{GL}_{n-1}(\mathbb{Q})$. We denote by $\rho_\lambda : \mathrm{GL}_{n-1} \to \mathrm{GL}(M_\lambda)$ the irreducible representation of GL_{n-1} of highest weight λ acting on a $\mathbb{C}$-vector space M_λ. Using [C] p. 122 we deduce that there is a finite extension E_λ of $\mathbb{Q}$ and a E_λ-vector space M_{E_λ} such that $M_\lambda = M_{E_\lambda} \otimes \mathbb{C}$ and M_{E_λ} is stable under ρ_λ. For any field E/E_λ we set $M_E = M_{E_\lambda} \otimes E$ and we denote by $\mathcal{M}_E$ resp. $\mathcal{M}$ the local system on $S_{n-1}(K)$ corresponding to M_E resp. $M_{\bar{\mathbb{Q}}}$. $\mathcal{M}_\lambda$ extends to a sheaf on the compactification $\bar{S}_{n-1}(K)$ and hence also on the boundary $\partial \bar{S}_{n-1}(K)$, which we also denote by $\mathcal{M}_\lambda$.

The face $e(P,K),\ P \in \mathcal{P}$ is homotopy equivalent to the space

$$P(\mathbb{Q})A_P^0\backslash \mathrm{GL}_{n-1}(\mathbb{A})/K_{n-1,\infty}K \tag{5.1}$$

(cf. [Sch], p.39, 44 non-adelically). We want to describe this quotient more explicitely: we let Ξ be the finite set of double cosets $P(\mathbb{A}_f)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_f)/K$ and for any compact subgroup $K^{M_P} \leq M_P(\mathbb{A}_f)$ we set

$$^0S_{n-1}^{M_P}(K^{M_P}) = M_P(\mathbb{Q})A_P^0\backslash M_P(\mathbb{A})/K^{M_P}K_{n-1,\infty}^{M_P}.$$

Furthermore, for any $\xi \in \Xi$ we denote by $K^{M_P}(\xi)$ the image of

$$\xi K \xi^{-1} \cap P(\mathbb{A}_f)$$

in $M_P(\mathbb{A}_f)$. The face $e(P,K)$ then is isomorphic to a disjoint union of manifolds

$$e(P,K) = \bigcup_{\xi \in \Xi} e_\xi(P,K)$$

such that each of the manifolds $e_\xi(P,K)$ has a locally trivial, canonical fibration

$$e_\xi(P,K) \to {}^0S_{n-1}^{M_P}(K^{M_P}(\xi))$$

with fibre isomorphic to $N_P(\mathbb{Q}) \cap K^{M_P}(\xi)\backslash N_P(\mathbb{R})$. We denote by $\mathcal{H}^\bullet(\mathfrak{n}_P, M_\lambda)$ the locally constant sheaf on ${}^0S_{n-1}^{M_P}(K^{M_P})$ attached to the algebraic M_P-module $H^\bullet(\mathfrak{n}_P, M_\lambda)$. Right translation defines a $M_P(\mathbb{A}_f)$-module structure on

$$H^i({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}^j(\mathfrak{n}_P, M_\lambda)),$$

and passing to the limit over all $K \leq \mathrm{GL}_{n-1}(\mathbb{A}_f)$, the above fibration induces an isomorphism of $\mathrm{GL}_{n-1}(\mathbb{A}_f)$-modules

$$H^\bullet(\tilde{e}(P), \mathcal{M}_\lambda) \cong {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} H^\bullet({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}^\bullet(\mathfrak{n}_P, M_\lambda))$$

(cf. [Ha 1], p. 88). We denote by $W_{n-1} = W_{\mathrm{GL}_{n-1}}$ resp. W_{M_P} the Weyl group of GL_{n-1} resp. M_P and we define the relative Weyl group

$$W^P = \{w \in W_{n-1} : w^{-1}(\alpha) \in \Phi_{n-1}^+ \quad \text{for all} \quad \alpha \in \Delta(M_P, T_{n-1})\}.$$

The set W^P is a canonical choice of a system of representatives for the quotient

$$W^P \leftrightarrow W_{M_P}\backslash W_{n-1}.$$

The structure of $H^\bullet(\mathfrak{n}_P, M_\lambda)$ as M_P-module is then given by the formula of Kostant:

$$H^i(\mathfrak{n}_P, M_\lambda) = \bigoplus_{\substack{w\in W^P\\ \ell(w)=i}} H^i(\mathfrak{n}_P, M_\lambda)(w\cdot\lambda),$$

where $w\cdot\lambda = w(\lambda+\check{\rho})-\check{\rho}$ denotes the twisted action of the Weyl group on weights and $H^i(\mathfrak{n}_P, M_\lambda)(w\cdot\lambda)$ is isomorphic to the unique M_P-module with highest weight $w\cdot\lambda$. For brevity we write $H_{w\cdot\lambda} = H^{\ell(w)}(\mathfrak{n}_P, M_\lambda)(w\cdot\lambda)$ and obtain

$$H^r(\tilde{e}(P), \mathcal{M}_\lambda) = \bigoplus_{\substack{w\in W^P\\ \ell(w)+p=r}} {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} H^p({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda}).$$

The same decomposition holds after tensoring with $\mathbb{C}$ and we define

$$H^r_{\mathrm{cusp}}(\tilde{e}(P), \mathcal{M}_{\lambda,\mathbb{C}}) = \bigoplus_{\substack{w\in W^P\\ \ell(w)+p=r}} {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} H^p_{\mathrm{cusp}}({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}}).$$

The cuspidal cohomology decomposes under the action of the Hecke algebra into a direct sum

$$H^\bullet_{\mathrm{cusp}}({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}}) = \bigoplus_{\pi'} H^p_{\mathrm{cusp}}({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}})(\pi')_{\mathbb{C}},$$

where $\pi' \subset L^2_0(M_P(\mathbb{Q})A_P^0\backslash M_P(\mathbb{A}))$ runs over all unitary, cuspidal representations and $H^p_{\mathrm{cusp}}({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}})(\pi')_{\mathbb{C}}$ is the π'_f-isotypical component. We finally obtain

$$H^r_{\mathrm{cusp}}(\tilde{e}(P), \mathcal{M}_{\lambda,\mathbb{C}}) = \bigoplus_{\substack{w\in W^P\\ \ell(w)+p=r}} \bigoplus_{\pi'} {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} H^p({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}})(\pi')_{\mathbb{C}}. \tag{5.2}$$

We set

$$H^r_{\mathrm{cusp}}(\tilde{e}(P), \mathcal{M}_{\lambda,\mathbb{C}})[w,\pi']_{\mathbb{C}} = {}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} H^{r-\ell(w)}_{\mathrm{cusp}}({}^0\tilde{S}_{n-1}^{M_P}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}})(\pi')_{\mathbb{C}}.$$

We want to extend classes in the cohomology of the face $e(P,K)$ to a neighbourhood of this face in $S_{n-1}(K)$. Such a neighbourhood is given by the space

$$e(P,K)\times A_P^0 = P(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})/K\mathrm{SO}_{n-1}(\mathbb{R})$$

and we describe a lifting of cohomology classes

$$\Phi:\ H^\bullet(e(P,K), \mathcal{M}_\lambda) \to H^\bullet(e(P,K)\times A_P^0, \mathcal{M}_\lambda).$$

For $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we set

$$\omega_\Lambda(g) = e^{\langle \Lambda+\check{\rho}, H_P(g)\rangle}\omega(g).$$

We know that $\omega \in H^\bullet_{\text{cusp}}({}^0\tilde{S}^{M_P}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}})$ is contained in some $H^\bullet({}^0\tilde{S}^{M_P}_{n-1}, \mathcal{H}_{w\cdot\lambda,\mathbb{C}})(\pi')_{\mathbb{C}}$ and Korollar 3.5 in [Sch] then implies that if we choose for Λ the linear form

$$\Lambda_w = -w(\lambda + \check{\rho})|_{A^0_P}$$

we obtain a closed form

$$\omega_{\Lambda_w} \in H^{b_{n-2}+\ell(w)}(\tilde{e}(P) \times A^0_P, \mathcal{M}_\lambda).$$

The assignment $\omega \mapsto \omega_{\Lambda_w}$ defines the map Φ . We set

$$H^r_{\text{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[w,\pi']_{\mathbb{C}} = \Phi(H^r_{\text{cusp}}(\tilde{e}(P), \mathcal{M}_{\lambda,\mathbb{C}})[w,\pi']_{\mathbb{C}}).$$

This is a Hecke-invariant subspace of the cohomology of $\tilde{e}(P) \times A^0_P$.

Indeed, following [Sch], ch. 3.6 we see that

$$H^r_{\text{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[w,\pi']_{\mathbb{C}} \cong H^{r-\ell(w)}(\mathfrak{g}_{n-1}, K_{n-1,\infty}, \mathrm{I}^a_{\mathbb{A}}(\mathbb{C}_{(w\cdot\Lambda+\check{\rho})_{\mathbb{A}}} \otimes V(\pi')) \otimes M_\lambda).$$

Here, to be short, we have used the following notations: we abbreviate $\mathrm{I}_{\mathbb{A}_f} = \mathrm{Ind}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)}_{P(\mathbb{A}_f)}$ and we also set $\mathrm{I}_{\mathbb{R}} = \mathrm{Ind}^{\mathrm{GL}_{n-1}(\mathbb{R})}_{P(\mathbb{R})}$ and $\mathrm{I}_{\mathbb{A}} = \mathrm{Ind}^{\mathrm{GL}_{n-1}(\mathbb{A})}_{P(\mathbb{A})}$. We use an analogous notation for the non-unitary induction. Moreover $\Lambda_{\mathbb{A}}$ is define in (1.7) and by $\mathbb{C}_{\Lambda_{\mathbb{A}}}$ we understand the 1 -dimensional space on which $A_P(\mathbb{A})$ acts via $\Lambda_{\mathbb{A}}$. Analogously we define $\mathbb{C}_{\Lambda_f}$ and $\mathbb{C}_{\Lambda}$. In particular, since the cuspidal spectrum of $M_P(\mathbb{A})$ satisfies multiplicity-1 we deduce

$$H^r_{\text{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[w,\pi']_{\mathbb{C}} \cong \mathrm{mult}_{\pi',\infty}\, \mathrm{I}_{P,\Lambda_w,f}\pi'_f \tag{5.3}$$

where

$$\mathrm{mult}_{\pi',\infty} = \dim H^r(\mathfrak{g}_{n-1}, K_{n-1,\infty}Z^0_{n-1}(\mathbb{R}), \mathrm{I}_{\mathbb{R}}(\mathbb{C}_{w\cdot\Lambda} \otimes \pi'_\infty) \otimes M_\lambda)$$

(note that $\check{\rho}|_{A^0_P} = \rho_P$).

Thus, to any choice of data $P \in \mathcal{P}$, $\pi' \in L^2_0(M_P(\mathbb{Q})A^0_P\backslash M_P(\mathbb{A}))$ and $w \in W^P$ we can assign a subspace in the cohomology of $\tilde{e}(P) \times A^0_P$. We want to apply this to construt non-trivial subspaces in $H^r_{\text{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})$ for a special weights λ attached to the critical points of cuspidal representations on $\mathrm{GL}_n(\mathbb{A})$: we let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ for n even, satisfying (**Coh**) and (**Reg**) and η and idele class character of finite order and $l \leq 1/2$ any critical point for $\eta \otimes \pi$. Thus, in case n is even η does not matter. We denote by $\mathbf{m} = (m_i)$ and w the quantities attached to π as in section 4; depending on $\mathbf{m}$ and l we then choose a weight $\lambda = (\lambda_i) \in \mathbb{Z}^{n-1}$ such that the following conditions hold:

(W.1) λ is integral and dominant.

(W.2) The entries of λ satisfy

$$\begin{array}{llll} \lambda_{n-i} = -\lambda_i & \text{for} \quad i = 1, \ldots, n/2-1 & \text{if} \quad \pi & \text{is of type} \quad \mathrm{I} \\ \lambda_{n-i} = -\lambda_i - 2 & \phantom{\text{for}} \quad i = 1, \ldots, n/2-1 & \phantom{\text{if}} \quad \pi & \phantom{\text{is of type}} \quad \mathrm{II}. \end{array}$$

(W.3) $m_1 \geq \lambda_1 \geq m_2 \geq \lambda_2 \geq \cdots \geq m_{n-1} \geq \lambda_{n-1} \geq m_n$,

(W.4)
$$\lambda_{n/2} = -l + 1/2$$

We want to verify that there is always a weight $\lambda = (\lambda_i)$ satisfying $(\mathbf{W.1})$ - $(\mathbf{W.4})$.

- π is of type I. We define $\lambda = (\lambda_i)$ by

$$\lambda_i = \begin{cases} -l + 1/2 & \text{if} \quad i = n/2 \\ m_{i+1} + 1 & \text{if} \quad i = 1, \ldots, n/2 - 1 \\ m_i & \text{if} \quad i = n/2 + 1, \ldots, n - 1. \end{cases}$$

Trivially $(\mathbf{W.4})$ holds. Since $\mathbf{m}$ is of weight -1 λ satisfies $(\mathbf{W.2})$. Obviously, $(\mathbf{W.1})$ holds for all λ_i , $i \geq n/2$ and in view of $(\mathbf{W.2})$ it holds for all i . It remains to verify $(\mathbf{W.3})$. Obviously, $\lambda_i \geq m_{i+1}$ for $i = 1, \ldots, n/2 - 1$ and since $\mathbf{m}$ is regular we also have $m_i \geq \lambda_i$ for $i = 1, \ldots, n/2 - 1$. Furthermore, Remark 4.2 implies $m_{n/2} \geq \lambda_{n/2} \geq 0$. $(\mathbf{W.2})$ together with the fact that $\mathbf{m}$ has weight -1 and $m_i - m_{i+1} \geq 1$ then shows that $m_i \geq \lambda_i \geq m_{i+1}$ for all $i = 1, \ldots, n-1$, i.e. $(\mathbf{W.3})$ holds.

- π is of type II. We define λ by

$$\lambda_i = \begin{cases} -l + 1/2 & \text{if} \quad i = n/2 \\ m_{n/2} + n - 2 & \text{if} \quad i = n/2 - 1 \\ m_{i+1} + 1 & \text{if} \quad i = 1, \ldots, n/2 - 2 \\ -\lambda_{n-i} - 2 & \text{if} \quad i = n/2 + 1, \ldots, n - 1. \end{cases}$$

Taking into account that $\mathbf{m}$ has weight 0 , $m_i - m_{i+1} \geq 4$ and $m_{n/2-1} - m_{n/2} \geq n$ it is easily verified that $(\mathbf{W.1})$ – $(\mathbf{W.4})$ hold.

In this section only the first two properties of λ (and partly the fourth one) will play a role. Indeed, relying on $(\mathbf{W.1})$ and $(\mathbf{W.2})$ and following the construction described above we shall prove that for a suitable choice of $\hat{w} \in W_P$ the spaces

$$H^\bullet(\tilde{e}(P) \times A_P^0, \mathcal{M}_\lambda)[\hat{\pi}[\chi], \hat{w}]$$

do not vanish. Here, $\hat{\pi}[\chi]$ is the representation defined in section 1 and $\hat{\pi}$ is assumed to be of a special type at infinity. In the next subsection we shall exhibit certain classes $\omega_{\lambda,\pi',\hat{w}} \in H^{b_{n-2}}(\tilde{e}(P), \mathcal{M}_\lambda)[\pi', \hat{w}]$. The Eisenstein classes attached to these classes turn out to be vital for the calculation of the critical values of the automorphic L -function in the next section. This then will rest on $(\mathbf{W.3})$ and $(\mathbf{W.4})$.

In detail, we let

- $P \leq \mathrm{GL}_{n-1}$

be the parabolic subgroup of section 1, i.e. P is the maximal parabolic subgroup of type $(n-2,1)$. The relative Weyl group consists of the elements $W^P = \{W_M \cdot w_i, \quad i = 1,\dots,n-1\}$, where w_i is the cycle

$$w_i = \begin{pmatrix} \mathbf{1}_{i-1} & & & & \\ & 0 & 1 & & \\ & \vdots & \ddots & \ddots & \\ & 0 & \cdots & 0 & 1 \\ & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

We let

- $\hat{w} = w_{\frac{n}{2}}$.

Using $(\mathbf{W}.\mathbf{2})$ we obtain for the operation of $\hat{w}$ on the weight λ

(5.4.I) $$\hat{w} \cdot \lambda = (\lambda_1, \dots, \lambda_{\frac{n}{2}-1}, -\lambda_{\frac{n}{2}-1} - 1, \dots, -\lambda_1 - 1, \lambda_{\frac{n}{2}} + \frac{n-2}{2})$$

if π is of type I and

(5.4.II) $$\hat{w} \cdot \lambda = (\lambda_1, \dots, \lambda_{\frac{n}{2}-1}, -\lambda_{\frac{n}{2}-1} - 3, \dots, -\lambda_1 - 3, \lambda_{\frac{n}{2}} + \frac{n-2}{2})$$

if π is of type II.

We denote by $\lambda|_1 = (\lambda_1, \dots, \lambda_{n-2})$ and $\lambda|_2 = \lambda_{n-1}$ the restrictions of λ. The above expressions show that $\hat{w} \cdot \lambda|_1$ satisfies the purity condition with weight

$$w = \begin{cases} -1 & \text{if } \pi \text{ is of type I} \\ -3 & \phantom{\text{if }} \pi \phantom{\text{ is of type }} \text{II.} \end{cases}$$

Hence, according to section 4 there is a cuspidal representation $\hat{\pi}$ with non-trivial cohomology with coefficient system of weight $w \cdot \lambda|_1$. In detail, as in section 4 we set

$$\mathbf{l}' = 2(\hat{w} \cdot \lambda|_1 + \check{\rho}) - w \in \mathbb{Z}^{n-2}.$$

We remark that $\mathbf{l}' = (l'_i)$ where

(5.5) $$l'_i \in 2\mathbb{Z}.$$

We choose a cuspidal automorphic representation $\hat{\pi}$ of $Z^0_{n-2}(\mathbb{R}) \backslash \mathrm{GL}_{n-2}(\mathbb{A})$ such that $\hat{\pi}$ satisfies $(\mathbf{C}.\mathbf{1})$ and

(5.6) $$\hat{\pi}_\infty \cong \hat{\epsilon}_\infty \otimes \pi_\infty(w, \mathbf{l}')$$

where $\hat{\epsilon}_\infty = \mathrm{sgn}^k$ with

$$k \equiv l - \frac{n-1}{2} \pmod 2.$$

Using (4.6) this is equivalent to

$$\mathbf{m}_{\hat{\epsilon}_\infty \otimes \hat{\pi}} = \hat{w} \cdot \lambda|_1, \tag{5.7}$$

where w is the weight of $\hat{w}\cdot\lambda|_1$. In particular $\hat{\pi}$ has non-vanishing cohomology. Since the group GL_{n-2} admits an involution of Cartan type, the (extension to non-trivial coefficients of the) results in [B-L-S] imply the existence of a cuspidal representation satisfying (5.6) (cf. [B-L-S], p. 34).

As in section 1 we define

- $\hat{\pi}[\chi] = \hat{\pi} \otimes \chi$.

As a consequence we mention for later use that the weights, which we contructed to prove the existence of weights satisfying $(\mathbf{W.1}) - (\mathbf{W.4})$ also satisfy

$$(\mathbf{W.5}) \qquad \hat{w} \cdot \lambda|_1 \quad \text{is regular.}$$

Again, $\pi_0 = \mathrm{O}_n(RR)/\mathrm{SO}_n(\mathbb{R})$ acts on the cohomology of the boundary via right translationand hence, $H^\bullet_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})$ has the structure of an $\mathrm{GL}_n(\mathbb{A}_f) \times \pi_0$ -module. We write sgn to denote the non-trivial character of $\pi_0 \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma 5.1. *We have*

$$H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{\mathbb{C}} \cong \mathrm{sgn}^k \otimes \mathrm{I}_{P,\Lambda_{\hat{w}},f}\, \hat{\pi}[\chi],$$

where $k \equiv -l + (n-1)/2 \pmod 2$.

Proof. In view of Eq. (5.3) to prove the Lemma we have to determine $\mathrm{mult}_{\hat{\pi}[\chi],\infty}$. We first determine the $({}^0\mathfrak{m}, \mathrm{O}_{n-1}(\mathbb{R})^{M_P})$ -cohomology of $\hat{\pi}[\chi]_\infty$.

The discussion in section 4 implies that $\mathbf{m}_{\pi(w,\mathbf{l}')} = \hat{w} \cdot \lambda|_1$ and we deduce from [C], Lemme 3.14 that $\pi(w, \mathbf{l}')$ has non-vanishing cohomology with coefficients in $M_{\hat{w}\cdot\lambda|_1}$; in particular since $\hat{\pi}_\infty = \hat{\epsilon}_\infty \otimes \pi_\infty(w, \mathbf{l}')$ we obtain

$$H^i(\mathfrak{sl}_{n-2}, \mathrm{O}_{n-2}(\mathbb{R}), \hat{\pi}_\infty \otimes \epsilon'_{1,\infty} \otimes M_{\hat{w}\cdot\lambda|_1}) = \begin{cases} \mathbb{C} & \text{if} \quad i = b_{n-2} \\ 0 & \text{if} \quad i < b_{n-2} \end{cases}$$

where $\epsilon'_{1,\infty} = \mathrm{sgn}^k$ with

$$k \equiv l - \frac{n-1}{2} \pmod 2$$

and b_n is the bottom degree, in which cohomology of a cuspidal representation on GL_n can occur (cf. section 4). Similarly,

$$\hat{w} \cdot \lambda|_2 = -l + (n-1)/2$$

and hence,

$$H^i(\mathbf{0}, \pm 1, \chi_\infty \otimes \epsilon'_{2,\infty} \otimes x^{\hat{w}\cdot\lambda|_2}) = \begin{cases} \mathbb{C} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i > 0, \end{cases}$$

where $\epsilon'_{2,\infty} = \mathrm{sgn}^k$ with k satisfying

$$k \equiv -l + (n-1)/2 \pmod 2. \tag{5.8}$$

We set $\epsilon'_\infty = \epsilon'_{1,\infty} \otimes \epsilon'_{2,\infty}$, which we view as character of $M_P(\mathbb{R})$. If n is even we choose $\epsilon'_{1,\infty} = \epsilon'_{2,\infty}$. We then obtain $\epsilon'_\infty = \mathrm{sgn}^k$, where k is as above.

The Lie algebra ${}^0\mathfrak{m}$ resp. the group $\mathrm{O}_{n-1}(\mathbb{R})^M$ decompose ${}^0\mathfrak{m} = \mathfrak{sl}_{n-2} \times \mathbf{0}$ and $\mathrm{O}_{n-1}(\mathbb{R})^M = \mathrm{O}_{n-2}(\mathbb{R}) \times \{\pm 1\}$ and applying the Kunneth rule yields an isomorphism

$$H^\bullet({}^0\mathfrak{m}, \mathrm{O}_{n-1}(\mathbb{R})^M, \hat{\pi}[\chi]_\infty \otimes \epsilon'_\infty \otimes M_{\hat{w}\cdot\lambda}) \cong H^\bullet(\mathfrak{sl}_{n-2}, \mathrm{O}_{n-2}(\mathbb{R}), \hat{\pi}_\infty \otimes \epsilon'_\infty \otimes M_{\hat{w}\cdot\lambda|_1}),$$

which is canonical up to the choice of a generator of $H^0(\mathbf{0}, \pm 1, \chi_\infty \otimes \epsilon'_{2,\infty} \otimes \hat{w}\cdot\lambda|_2)$. In particular, we have

$$H^i({}^0\mathfrak{m}, \mathrm{O}_{n-1}(\mathbb{R})^{M_P}, \hat{\pi}[\chi]_\infty \otimes \epsilon'_\infty \otimes M_{\hat{w}\cdot\lambda}) \cong \begin{cases} \mathbb{C} & \text{if} \quad i = b_{n-2} \\ 0 & \text{if} \quad i < b_{n-2} \end{cases} \tag{5.9}$$

and using Wigner's lemma we deduce

$$\chi_{\hat{\pi}[\chi]_\infty} = \chi_{-\hat{w}\cdot\lambda+\check{\rho}}|_{\mathfrak{b}_\mathbb{C}}, \tag{5.10}$$

where $\chi_{\hat{\pi}[\chi]_\infty}$ denotes the infinitesimal character of $\hat{\pi}[\chi]_\infty$. The second condition in [B-W], Theorem 3.3 (i) therefore is satisfied and since the inducing parameter is $\Lambda_{\hat{w}} = -\hat{w}(\lambda + \check{\rho})|_{A_P^0}$ the first condition in [B-W], Theorem 3.3 (i) is trivially satisfied. We deduce form [loc. cit.]

$$\begin{aligned} H^{b_{n-2}+\ell(\hat{w})}&(\mathfrak{g}_{n-1}, \mathrm{O}_{n-1}Z^0_{n-1}(\mathbb{R}), \mathrm{I}_\mathbb{R}(\mathbb{C}_{\Lambda_{\hat{w}}} \otimes \epsilon'_\infty \otimes \hat{\pi}[\chi]_\infty) \otimes M_\lambda) \\ &= \bigoplus_{i,j} H^i({}^0\mathfrak{m}, \mathrm{O}^M_{n-1}(\mathbb{R}), \epsilon'_\infty \otimes \hat{\pi}[\chi]_\infty \otimes M_{\hat{w}\cdot\lambda}) \otimes \bigwedge^j \mathfrak{a}^*_{P,\mathbb{C}}, \end{aligned}$$

where i, j run over all non-negative integers satisfying $i + j = b_{n-2}$. Together with (5.9) we deduce $\mathrm{mult}_{\hat{\pi}[\chi],\infty} = 1$. On the other hand, the assertion on the action of π_0 is an immediate consequence of (5.8) and the proof of the Lemma is complete.

We denote by $E_{\hat{\pi}}$ the field of definition of $\hat{\pi}_f$. Using [C], Théorème 3.13 the assumptions on $\hat{\pi}_\infty$ imply that $E_{\hat{\pi}}$ is a finite extension of $\mathbb{Q}$. Obviously, $\hat{\pi}[\chi]_f$ is defined over $E_{\hat{\pi}}(\chi)$.

Lemma. 5.2. *Let λ be an integral and dominant weight. For any $w \in W^P$ we have*

$$H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], w]_\mathbb{C} = H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], w]_{E_{\pi'}} \otimes \mathbb{C},$$

where

$$H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})[\hat\pi[\chi],w]_{E_{\hat\pi[\chi]}}$$
$$= H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})[\hat\pi[\chi],w]_{\mathbb{C}}\cap H^{b_{n-2}+\ell(w)}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,E_{\hat\pi[\chi]}}).$$

Proof. The weight λ is assumed to be integral, which implies that the character $a\mapsto |a|^{\Lambda_w}_{\mathbb{A},f}$, $a\in S_P(\mathbb{A})$ is $\mathbb{Q}$-valued. Thus, since induction is rational, we see that $\mathrm{Ind}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)}_{P(\mathbb{A}_f)}(|\cdot|^{\Lambda_w}_{\mathbb{A},f}|\otimes\epsilon'_f\otimes\hat\pi[\chi]_f)$ is defined over $E_{\hat\pi}(\chi)$. The cohomology carries a structure of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})\times\mathrm{GL}_{n-1}(\mathbb{A}_f)$-module and hence, the assertion of the Lemma follows from the following statement

(bMD) $H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})[\hat\pi[\chi],w]_{\mathbb{C}}$ is a direct summand, which does not weakly intertwine with its complement.

This is the Manin-Drinfeld principle for the boundary (cf. [Ha 2]). We look to the cohomology of $\tilde e\times A^0_P$ as module under the Hecke algebra. In the same way as in the case of the cohomology of $\tilde e(P)$ one establishes an isomorphism

$$H^\bullet(\tilde e(P)\times A^0_P,\mathcal{M}_\lambda)\cong {\mathrm{Ind}^a}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)}_{P(\mathbb{A}_f)}H^\bullet({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}^\bullet(\mathfrak{n}_P,M_\lambda)).$$

The space $H^{b_{n-2}}_{\mathrm{cusp}}({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}_{w\cdot\lambda})((\Lambda_w+\check\rho)_{\mathbb{A}}\otimes\hat\pi[\chi])$ appears as a direct summand in $H^{b_{n-2}}({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}^{\ell(w)}(\mathfrak{n}_P,M_\lambda))$ and therefore

$$H^{b_{n-2}+\ell(w)}_{\mathrm{cusp}}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})[\hat\pi[\chi],w]_{\mathbb{C}}=\mathrm{I}^a_{\mathbb{A}_f}\,H^{b_{n-2}}_{\mathrm{cusp}}({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}_{w\cdot\lambda})((\Lambda_w+\check\rho)_{\mathbb{A}}\otimes\hat\pi[\chi])$$

is a direct summand in $H^{b_{n-2}+\ell(w)}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})$. Thus, we are left with proving that there is no weak intertwining with its complement.

We assume that there is an irreducible subquotient (ξ_f,V_{ξ_f}) in $H^{b_{n-2}+\ell(w)}(\tilde e(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})$, which is isomorphic to a constituent of the Jordan-Hoelder series of $\mathrm{I}^a\,(\Lambda_w+\check\rho)_f\otimes\hat\pi[\chi]_f$. We denote by ξ the automorphic representation, whose component at infinity is isomorphic to ξ_f and V_ξ is the subquotient of the space of automorphic forms corresponding to ξ. Eq. (5.2) implies that there is a $w'\in W_P$ and a subquotient $(\xi_{P,f},V_{\xi_{P,f}})$ in

$$H^{b_{n-2}+\ell(w)-\ell(w')}({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}_{w'\cdot\lambda,\mathbb{C}})$$

such that V_{ξ_f} is a subquotient of the induced representation ${\mathrm{Ind}^a}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)}_{P(\mathbb{A}_f)}V_{\xi_{P,f}}$. The cohomology of ${}^0\tilde S^{M_P}_{n-1}\times A^0_P$ can be entirely described by automorphic forms

$$H^\bullet({}^0\tilde S^{M_P}_{n-1}\times A^0_P,\mathcal{H}_{w'\cdot\lambda,\mathbb{C}})\cong H^\bullet(\mathfrak{m},K^{M_P}_{n-1,\infty},\mathcal{A}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))\otimes H_{w'\cdot\lambda})$$

and [L], Proposition 2 shows that there is a parabolic subgroup $Q \leq M_P$ and a cuspidal representation (ξ_Q, V_{ξ_Q}), $V_{\xi_Q} \subset L^2_0(M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A}))$ of the Levi component M_Q of Q such that $V_{\xi_{P,f}}$ is a subquotient of ${}^a\mathrm{Ind}_{Q(\mathbb{A}_f)}^{M_P(\mathbb{A}_f)} V_{\xi_{Q,f}}$. Using induction in stages we see that the representations ${}^a\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} (\Lambda_w + \check{\rho})_f \otimes \hat{\pi}[\chi]_f$ and ${}^a\mathrm{Ind}_{Q(\mathbb{A}_f)}^{\mathrm{GL}_{n-1}(\mathbb{A}_f)} V_{\xi_{Q,f}}$ are weakly intertwining. Since these representations are admissible we deduce from [J-S] (4.4) Theorem that the pairs $(P, |\cdot|_{\mathbb{A}}^{\Lambda_w+\check{\rho}} \otimes \hat{\pi}[\chi])$ and (Q, ξ_Q) are associate. Since $Q \leq P$ and

$$W^{(M_P)} = N_{\mathrm{GL}_{n-1}}(M_P)/M_P = 1$$

this implies $P = Q$ and $\xi_Q \cong (\Lambda_w + \check{\rho})_{\mathbb{A}} \otimes \hat{\pi}[\chi]$, i.e. $V_{\xi_P} = (\Lambda_w + \check{\rho})_{\mathbb{A}} \otimes V_{\hat{\pi}[\chi]}$. In other words we have

$$V_{\xi_f} \subset H^{b_{n-2}+\ell(w)}({}^0\tilde{S}_{n-1}^{M_P} \times A_P^0, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], w'].$$

Applying the Kunneth rule we see that

$$\begin{aligned} &H^\bullet(\mathfrak{m}, K_{n-1,\infty}^{M_P}, |\cdot|^{\Lambda_{\hat{w}}+\check{\rho}} \otimes \hat{\pi}[\chi] \otimes H_{w'\cdot\lambda}) \\ &\qquad \cong H^\bullet({}^0\mathfrak{m}, K_{n-1,\infty}^{M_P}, \hat{\pi}[\chi] \otimes H_{w'\cdot\lambda}) \otimes H^\bullet(\mathfrak{m}, K_{n-1,\infty}^{M_P}, (\Lambda_{\hat{w}} + \check{\rho})_{\mathbb{A}} \otimes H_{w'\cdot\lambda}). \end{aligned}$$

Since V_{ξ_f} does not vanish Wigner's Lemma on the infinitesimal character immediately yields that

$$w'(\lambda + \check{\rho})_{A_P^0} + \Lambda_{\hat{w}} = 0$$

and similarly the non-vanishing of the subspaces in cohomology V_{ξ_f} and $V_{\hat{\pi}[\chi]}$ implies

$$-\chi_{w(\lambda+\check{\rho})|_{\mathfrak{b}_{\mathbb{C}}}} = \chi_{\hat{\pi}[\chi]} = -\chi_{w'(\lambda+\check{\rho})|_{\mathfrak{b}_{\mathbb{C}}}} = 0$$

(compare Eq. (5.10)). The regularity of $\lambda + \check{\rho}$ then leads to $w = w'$, i.e. V_{ξ_f} is a subquotient of

$$H_{\mathrm{cusp}}^{b_{n-2}+\ell(w)}(\tilde{e}(P) \times A_P^0, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], w]_{\mathbb{C}}$$

This proves (bMD) and also the Lemma.

5.2. Definition of the class $\omega_{\lambda,\hat{\pi}[\chi],\hat{w}}$.

We denote by $W(\tau)$ the space of Whittaker functions on $\mathrm{GL}_{n-1}(\mathbb{A})$ with respect to τ and $W(\pi,\tau) \subset W(\tau)$ is the π-isotypical component. We also denote by

$$\begin{array}{cccccc} \beta: & \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) & \to & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \to & \hat{\mathbb{Z}}^* \\ & \sigma & \mapsto & \sigma|_{\bar{\mathbb{Q}}} & \mapsto & t_\sigma = \prod_{v\neq\infty} t_{\sigma,v} \end{array}$$

the cyclotomic character and we set

$$\mathbf{t}_{\sigma,n-1} = \mathbf{t}_\sigma = \mathrm{diag}(t_\sigma^{-(n-2)}, t_\sigma^{-(n-3)}, \ldots, 1).$$

This defines a map

$$\beta_{n-1}: \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \to T_{n-1}(\hat{\mathbb{Z}}).$$

The automorphism group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on $W(\tau_f)$

$$w^{\tilde\sigma}(g) = w(\mathbf{t}_\sigma g)^\sigma$$

and since $\hat\pi[\chi]_f$ is defined over $E_{\hat\pi[\chi]}$ we obtain

$$W(\hat\pi[\chi]_f, \tau_f) = W(\hat\pi[\chi]_f, \tau_f)_{E_{\hat\pi[\chi]}} \otimes \mathbb{C},$$

where the subscript $E_{\hat\pi[\chi]}$ indicates the space of $\mathrm{Aut}(\mathbb{C}/E_{\hat\pi[\chi]})$ -invariants. As a consequence we see that there is a $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ -operation on induced representations: every $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ defines a σ -linear map

$$\begin{array}{rccl} \tilde\sigma : & \mathrm{I}_{\mathbb{A}_f} |\cdot|_{\mathbb{A},f}^{\Lambda} \otimes W(\hat\pi[\chi]_f, \tau_f) & \to & \mathrm{I}_{\mathbb{A}_f} |\cdot|_{\mathbb{A},f}^{\Lambda} W(\hat\pi[\chi]_f{}^{\sigma}, \tau_f) \\ & f & \mapsto & f^{\tilde\sigma}(g) = (f(g))^{\tilde\sigma} = f(g; \mathbf{t}_\sigma h_1, h_2)^\sigma. \end{array}$$

In particular, $\mathrm{I}_{\mathbb{A}_f}(|\cdot|_{\mathbb{A},f}^{\Lambda} \otimes W(\hat\pi[\chi]_f, \tau_f)_{E_{\hat\pi[\chi]}})$ is a $\mathrm{GL}_{n-1}(\mathbb{A}_f)$ -invariant subspace. We recall that $\mathrm{I}^\star_{P,\Lambda_{\hat w}}\hat\pi[\chi] = \mathrm{I}_{\mathbb{A}}|\cdot|_{\mathbb{A}}^{\Lambda_{\hat w}} \otimes W(\hat\pi[\chi], \tau)$. We want to choose a generator $\xi^\star_{\hat\pi,\infty}$ of the 1 -dimensional vector space $H^{b_{n-2}+\ell(\hat w)}(\mathfrak{g}_{n-1}, K_{n-1,\infty}, \mathrm{I}^\star_{P,\Lambda_{\hat w},\infty}\hat\pi[1] \otimes M_\lambda)$. We consider the diagram

$$\begin{array}{ccc} \mathrm{I}^\star_{P,\Lambda_{\hat w},f}\hat\pi[1] = \mathrm{I}_{\mathbb{A}_f}|\cdot|_{\mathbb{A},f}^{\Lambda_{\hat w}} \otimes W(\hat\pi[1]_f, \tau_f) & & \\ \downarrow \xi^\star_{\hat\pi,\infty} & & \\ H^{b_{n-2}+\ell(\hat w)}(\mathfrak{g}, K^\infty_{n-1}, \mathrm{I}_{P,\Lambda_{\hat w}}\hat\pi[1] \otimes M_\lambda) & \overset{\mathcal{P}^{\star}_{\mathrm{coh}}{}^{-1}}{\to} & H^{b_{n-2}+\ell(\hat w)}(\mathfrak{g}, K^\infty_{n-1}, \mathrm{I}_{\mathbb{A}}|\cdot|^{\Lambda_{\hat w}} \otimes V_{\hat\pi[1]} \otimes M_\lambda) \\ & & \| \\ & & H^{b_{n-2}+\ell(\hat w)}(\tilde e(P) \times A^0_P, \mathcal{M}_\lambda)[\hat w, \hat\pi[1]]. \end{array}$$

Here, we have set $\mathfrak{g} = \mathfrak{g}_{n-1}$; the map $\mathcal{P}^\star_{\mathrm{coh}}$ on cohomology is induced by the map $\mathcal{P}^\star$ defined in section 1, i.e. $\mathcal{P}^\star_{\mathrm{coh}} : \sum_{\mathbf{i}} f_{\mathbf{i}}\,\omega_{\mathbf{i}} \mapsto \sum_{\mathbf{i}} \mathcal{P}^\star(f_{\mathbf{i}})\,\omega_{\mathbf{i}}$ and the first isomorphism is determined by the choice of a generator $\xi^\star_{\hat\pi,\infty}$. In particular, any choice of a system of generators $\xi^\star_{\hat\pi,\infty}$ defines a square of maps with the horizontal arrows being given by the composition of the maps in the above diagram

$$(\star) \qquad \begin{array}{ccc} H^{b_{n-2}+\ell(\hat w)}(\tilde e(P) \times A^0_P, \mathcal{M}_\lambda)[\hat w, \hat\pi[1]] & \to & \mathrm{I}^\star_{P,\Lambda_{\hat w},f}\hat\pi[1] \\ \sigma \downarrow & & \downarrow \tilde\sigma \\ H^{b_{n-2}+\ell(\hat w)}(\tilde e(P) \times A^0_P, \mathcal{M}_\lambda)[\hat w, \hat\pi[1]^\sigma] & \overset{\mathcal{P}^{\star}_{\mathrm{coh}}{}^{-1} \circ \xi^\star_{\hat\pi^\sigma,\infty}}{\to} & \mathrm{I}^\star_{P,\Lambda_{\hat w},f}\hat\pi[1]^\sigma. \end{array}$$

Here, we use the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the infinity component of the representation $\hat\pi[\chi]$ (cf. [C], Définition 3.6); indeed, we have $\pi[\chi]^\sigma_\infty = \pi[\chi]_\infty$.

Lemma 5.3 *The submodule* $\mathbb{C}[\mathrm{GL}_{n-1}(\mathbb{Q}_v)] \cdot f^*_{\hat\pi[\chi],\Lambda,v}$ *of* $\mathrm{I}^*_{P,\Lambda,v}\hat\pi[\chi]$ *is defined over* $E_{\hat\pi}[\chi]$

$$\mathbb{C}[\mathrm{GL}_{n-1}(\mathbb{Q}_v)] \cdot f_{\hat\pi[\chi],\Lambda,v} = E_{\hat\pi}(\chi)[\mathrm{GL}_{n-1}(\mathbb{Q}_v)] \cdot f_{\hat\pi[\chi],\Lambda,v} \otimes \mathbb{C}.$$

Moreover, the $E_{\hat{\pi}}(\chi)$ *-structure on* $\mathbb{C}[\mathrm{GL}_{n-1}(\mathbb{Q}_v)]\cdot f_{\hat{\pi}[\chi],\Lambda,v}$ *is unique up to homotheties.*

Proof. Analyzing the proof of Lemme I.1 in [W] one sees that to prove the Lemma it is sufficient that the following properties hold

$$\left(\mathrm{I}^*_{P,\Lambda,v}\hat{\pi}[\chi]\right)^{K_1(n-1,f_v)} = \langle f^*_{\hat{\pi}[\chi],\Lambda,v}\rangle.$$

and $f^*_{\hat{\pi}[\chi],\Lambda,v}$ generates $\mathbb{C}[\mathrm{GL}_{n-1}(\mathbb{Q}_v)]\cdot f^*_{\hat{\pi}[\chi],\Lambda,v}$. The second property is trivially satisfied, while the first one is immediate by the definition of the $f^*_{\hat{\pi}[\chi],\Lambda,v}$. Thus, the Lemma is proven.

As a consequence of Lemma 5.3 we see that it is possible to choose the generators $\xi^*_{\hat{\pi},\infty}$ in such a way that on the $\mathrm{GL}_{n-1}(\mathbb{A}_f)$ -submodules generated by $f^*_{\hat{\pi}[\chi],\Lambda,f}$ the diagram $(\star)$ is defined over $E_{\hat{\pi}}(\chi)$ and commutes.

Since $\chi_\infty = \mathrm{id}$ we deduce embeddings

$$\mathrm{I}^\star_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi] \to H^{b_{n-2}+\ell(\hat{w})}(\tilde{e}(P)\times A^0_P,\mathcal{M}_\lambda)[\hat{\pi}[\chi],\hat{w}]$$

for all $\hat{\pi}[\chi] = \hat{\pi}\otimes\chi$. Since the spaces $W(\hat{\pi}[\chi]_f,\tau_f) = W(\hat{\pi}_f,\tau_f)\otimes\mathbb{C}_\chi$ and $W(\hat{\pi}[1]_f,\tau_f) = W(\hat{\pi}_f,\tau_f)\otimes\mathbb{C}_1$ can be canonically identified we obtain a natural identification

$$\mathrm{I}^\star_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi] = \mathrm{I}^\star_{P,\Lambda_{\hat{w}},f}\hat{\pi}[1],$$

which is equivariant for the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. The diagram

$$\begin{array}{ccc}
\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[1] & \xi^\star_{\hat{\pi},\infty} & \\
\| & \searrow & \\
\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi] & \xrightarrow{\xi^\star_{\hat{\pi},\infty}} & H^{b_{n-2}+\ell(\hat{w})}(\tilde{e}(P)\times A^0_P,\mathcal{M}_\lambda)[\hat{\pi}[\chi],\hat{w}]
\end{array}$$

obviously commutes and we deduce that the map, which is induced by $\xi^\star_{\hat{\pi},\infty}$ on $\mathrm{I}^\star_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi]$ too is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ -equivariant.

In section 1 we defined vectors $f^\star_{\hat{\pi}[\chi]}\in\mathrm{I}^\star_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi]$ for all $\chi\in\mathbf{X}_p$. The mulitiplicity-1 property, which is satisfied for all finite places v and the equality $w^0_{\hat{\pi},v}(\mathbf{t}_{\sigma,v}) = w^0_{\hat{\pi},v}(1)^\sigma = 1$, which holds because $\mathbf{t}_{\sigma,v}\in K_1(n-1,f_v)$ then show

$$(f^\star_{\hat{\pi}[\chi],v})^\sigma = f^\star_{\hat{\pi}[\chi]^\sigma,v}$$

for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E_\lambda)$ and all finite places v .

We now define

$$\omega_{\lambda,\hat{\pi}[\chi],\hat{w}} = {\mathcal{P}^\star_{\mathrm{coh}}}^{-1}\circ\xi^\star_{\hat{\pi},\infty}(f^\star_{\hat{\pi}[\chi]}) \in H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P)\times A^0_P,\mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi],w]_{E_{\hat{\pi}}(\chi)}.$$

Obviously we have $\omega^\sigma_{\lambda,\hat{\pi}[\chi],\hat{w}} = \omega_{\lambda,\hat{\pi}[\chi]^\sigma,\hat{w}}$.

Since the elements of the relative Weyl group have length

$$\ell(w_i) = n - 1 - i$$

we see that $\omega_{\lambda,\hat{\pi}[\chi],\hat{w}}$ is a cohomology class in degree

$$b_{n-2} + \ell(\hat{w}) = \frac{n^2 - 2n}{4}.$$

We choose basis $\mathcal{B}_{n-1} = \{\omega_1, \ldots, \omega_{\dim X^1_{n-1}}\}$ of the dual $(\mathfrak{g}_{n-1}/\mathfrak{k}_{n-1})^*$ of the tangent space of the symmetric space $X^1_{n-1} = \mathrm{GL}^+_{n-1}(\mathbb{R})/K_{n-1,\infty}$ as well as a basis $\{b\}$ of M_λ and we write

$$\xi^\star_{\hat{\pi},\infty} = \sum_{\mathbf{i}=(i_1,\ldots,i_{b_{n-2}+\ell(\hat{w})})} \sum_b f^\star_{\mathbf{i},b,\infty}\, \omega_{\mathbf{i}} \otimes b.$$

Here, $\omega_{\mathbf{i}} = \omega_{i_1} \wedge \ldots \wedge \omega_{b_{n-2}+\ell(\hat{w})}$. Following the above construction we then we find

$$\omega_{\lambda,\hat{\pi}[\chi],\hat{w}} = \sum_{\mathbf{i},b} f_{\hat{\pi}[\chi],\mathbf{i},b}\, \omega_{\mathbf{i}} \otimes b,$$

where we have set $f^\star_{\hat{\pi}[\chi],\mathbf{i},b} = f^\star_{\hat{\pi}[\chi],f} f^\star_{\mathbf{i},b,\infty}$ and $f_{\hat{\pi}[\chi],\mathbf{i},b} = \mathcal{P}^{\star\,-1}(f^\star_{\hat{\pi}[\chi],\mathbf{i},b})$.

5.3. Eisenstein cohomology.

Using the theory of Eisenstein series we may extend classes from the boundary to the whole space $S_{n-1}(K)$. For the moment we let $P \leq \mathrm{GL}_{n-1}$ be any parabolic subgroup and π' be a cuspidal representation in $L^2_0(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))$. The cylinder $e(P,K) \times A^0_P$ is then isomorphic to $e(P,K) \times A^0_P \cong P(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})/KK_{n-1,\infty}$ and we therefore may define the map

$$\mathrm{Eis}:\ H^\bullet(\tilde{e}(P) \times A^0_P, \mathcal{M}_\lambda) \to H^\bullet(\tilde{S}_{n-1}, \mathcal{M}_\lambda)$$

by assigning to $\omega \in H^\bullet(\tilde{e}(P) \times A^0_P, \mathcal{M}_\lambda)$ the class

$$\mathrm{Eis}(\omega) = \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{Q})} \gamma \circ \omega.$$

This operator is defined by analytic continuation except for possible poles which we want to discuss a little closer. Since we may realize the induced representation $\mathrm{I}_{\mathbb{A}}(|\cdot|^\Lambda_{\mathbb{A}} \otimes \pi')$, $\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ as subspace of $\mathcal{C}^\infty(P(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})/K)$ we obtain a map

$$\begin{aligned}\mathrm{Eis}_{P,\pi',\Lambda}:\ & H^\bullet(\mathfrak{g}_{n-1}, K_{n-1,\infty}Z^0_{n-1}(\mathbb{R}), \mathrm{I}_{\mathbb{A}}(|\cdot|^\Lambda_{\mathbb{A}} \otimes \pi')) \\ & \to H^\bullet(\mathfrak{g}_{n-1}, K_{n-1,\infty}, \mathcal{A}(\mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})).\end{aligned}$$

It is a consequence of the general theory that $\mathrm{Eis}_{P,\pi',\Lambda}$ only has a finite number of simple poles in the range

$$\{\Lambda \in \mathfrak{a}^*_{P,\mathbb{C}}:\ \mathrm{Im}(\Lambda) = 0,\ \langle \check{\rho}_P, \alpha\check{\ }\rangle \geq \langle \Lambda, \alpha\check{\ }\rangle > 0,\ \alpha \in \Delta(P, A_P)\},$$

where $\alpha\check{}$ denotes the dual root. We go back to the situation of the previous subsection, i.e. from now on $P \leq \mathrm{GL}_{n-1}$ denotes again the parabolic subgroup of type $(n-2,1)$. A small calculation proves that

$$\hat{w}(\lambda+\check{\rho}) = \left(\lambda_1 + \frac{n-2}{2}, \ldots, \lambda_{\frac{n}{2}-1}+1, -\lambda_{\frac{n}{2}-1}-1, \ldots, -\lambda_1 - \frac{n-2}{2}, \lambda_{\frac{n}{2}}\right)$$

if π is of type I and

$$\hat{w}(\lambda+\check{\rho}) = \left(\lambda_1 + \frac{n-2}{2}, \ldots, \lambda_{\frac{n}{2}-1}+1, -\lambda_{\frac{n}{2}-1}-3, \ldots, -\lambda_1 - \frac{n+2}{2}, \lambda_{\frac{n}{2}}\right)$$

if π is of type II. In particular using $(\mathbf{W}.\mathbf{2})$ and $(\mathbf{W}.\mathbf{4})$ we deduce

$$\Lambda_{\hat{w}} = \begin{cases} (0, l-\frac{1}{2}) & \text{if } \pi \text{ is of type I} \\ (n-2, l-\frac{1}{2}) & \pi \qquad\qquad \text{II.} \end{cases} \tag{5.11}$$

The element $\Lambda_{\hat{w}}$ obviously is real. On the other hand since $l \leq 1/2$ we have $\langle \Lambda_{\hat{w}}, \alpha\check{}\rangle > 0$ for $\alpha \in \Delta(P, A_P)$. Our choice of the parabolic subgroup P together with Proposition 5.3 in [F-Sch] then implies that $\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda}$ is regular at $\Lambda_{\hat{w}}$. As a result we obtain the well defined map

$$\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}} : H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P)\times A_P^0, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi],\hat{w}]_{\mathbb{C}} \to H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{Eis}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}). \tag{5.12}$$

Using this map we finally define a special class in the cohomology of $\tilde{S}_{n-1}$: we denote by $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ the image of $\omega_{\lambda,\hat{\pi}[\chi],\hat{w}}$ under the above map. Using Lie algebra cocycles we have

$$\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}} = \sum_{\mathbf{i}} E_{\hat{\pi}[\chi],\mathbf{i},b}\, \omega_{\mathbf{i}} \otimes b,$$

where we have set $E_{\hat{\pi}[\chi],\mathbf{i},b} = \mathrm{Eis}(f_{\hat{\pi}[\chi],\mathbf{i},b})$.

For the later computation of the values of the L-function using cohomology we shall introduce the Whittaker function corresponding to the cohomology class $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$. We denote by $v_{\hat{\pi}[\chi],\mathbf{i},b}$ the Whittaker function of $E_{\hat{\pi}[\chi],\mathbf{i},b}$ and may then visualize our situation in the picture

$$\begin{array}{ccccccc}
f_{\hat{\pi}[\chi],\mathbf{i},b} & \in & \mathrm{I}_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi] & \overset{\mathcal{P}^\star_{\hat{\pi}[\chi]}}{\rightarrow} & \mathrm{I}^\star_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi] & \ni & f^\star_{\hat{\pi}[\chi],\mathbf{i},b} \\
 & & \mathrm{Eis}\downarrow & & \downarrow \mathcal{P}^P_{\mathrm{GL}_{n-1}} & & \\
\mathrm{Eis}(f_{\hat{\pi}[\chi],\mathbf{i},b}) & \in & V_{I_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi]} & \overset{E\mapsto E^\tau}{\rightarrow} & W(\mathrm{I}_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi],\tau) & \ni & v_{\hat{\pi}[\chi],\mathbf{i},b}.
\end{array}$$

This is the same diagram as (1.3) and we want to transfer it to cohomology. To this end we choose a generator $\xi_{\hat{\pi},\infty}$ of $H^{b_{n-2}+\ell(\hat{w})}(\mathfrak{g}_{n-1}, K_{n-1,\infty}, \mathrm{I}^\star_{P,\Lambda_{\hat{w}},\infty}\hat{\pi}[\chi] \otimes M_\lambda)$. We obtain a chain of maps

$$\begin{array}{ccc}
W(\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi]) & & \\
\downarrow \xi_{\hat{\pi},\infty} & & \\
H^{b_{n-2}+\ell(\hat{w})}(\mathfrak{g},K_{n-1}^{\infty},W(\mathrm{I}_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi],\tau)\otimes M_{\lambda}) & \stackrel{\{E\mapsto E^{\tau}\}^{-1}}{\to} & H^{b_{n-2}+\ell(\hat{w})}(\mathfrak{g},K_{n-1}^{\infty},V_{\mathrm{I}_{P,\Lambda_{\hat{w}}}\hat{\pi}[\chi]}\otimes M_{\lambda}) \\
& & \cap \\
& & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1},\mathcal{M}_{\lambda,\mathbb{C}}).
\end{array}$$

Again we choose the $\xi_{\hat{\pi},\infty}$'s such that the diagram

$$\begin{array}{ccc}
W(\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi],\tau_f)_{E_{\hat{\pi}[\chi]}} & \to & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1},\mathcal{M}_{\lambda,\mathbb{C}})_{E_{\hat{\pi}[\chi]}} \\
\sigma\downarrow & & \downarrow\sigma \\
W(\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi]^{\sigma},\tau_f)_{E_{\hat{\pi}[\chi]}} & \to & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1},\mathcal{M}_{\lambda,\mathbb{C}})_{E_{\hat{\pi}[\chi]}}
\end{array}$$

is defined over E_{λ} . We write

$$\xi_{\hat{\pi},\infty}=\sum_{\mathbf{i},b} v_{\mathbf{i},b,\infty}\,\omega_{\mathbf{i}}\otimes b$$

We then obtain a corresponding diagram in cohomology

$$\begin{array}{ccccc}
f^{\star}_{\hat{\pi}[\chi],f} & \in & \mathrm{I}^{\star}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi] & \stackrel{{\mathcal{P}^{\star}_{\mathrm{coh}}}^{-1}\circ\xi^{\star}_{\hat{\pi},\infty}}{\to} & H^{b_{n-2}+\ell(\hat{w})}(\tilde{e}(P)\times A_P^0,\mathcal{M}_{\lambda})[\hat{w},\hat{\pi}[\chi]] \\
& & \mathcal{P}^{P}_{\mathrm{GL}_{n-1},f}\downarrow & & \downarrow\mathrm{Eis} \\
v_{\hat{\pi}[\chi],f} & \in & W(\mathrm{I}_{P,\Lambda_{\hat{w}},f}\hat{\pi}[\chi],\tau_f)_{E_{\hat{\pi}[\chi]}} & \stackrel{\{E\mapsto E^{\tau}\}^{-1}\circ\xi_{\hat{\pi},\infty}}{\to} & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1},\mathcal{M}_{\lambda,\mathbb{C}})_{E_{\hat{\pi}[\chi]}},
\end{array}$$

which commutes at up to some scalar. More precisely, we define $v_{\hat{\pi}[\chi],f}$ to be the finite part in the decomposition $v_{\hat{\pi}[\chi],\mathbf{i},b}=v_{\hat{\pi}[\chi],f}v_{\mathbf{i},b,\infty}$; the section $f^{\star}_{\hat{\pi}[\chi],f}$ is then being mapped to $v_{\hat{\pi}[\chi],f}$ under the map given by the composition of the three arrows in the above diagram.

5.4. Non-triviality of $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$.

We proceed by computing the restriction of $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ to the boundary of $\tilde{S}_{n-1}$. As an application we shall derive the non-triviality of the Eisenstein cohomology class as well as obtain the rationality of $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$. Since the boundary is a disjoint union

$$\partial S_{n-1}(K)=\bigcup_{P\in\mathcal{P},P\neq\mathrm{GL}_{n-1}} e(P,K)$$

for any $P\neq\mathrm{GL}_{n-1}$ there is a restriction map

$$r_P:\,H^{\bullet}(\tilde{S}_{n-1},\mathcal{M}_{\lambda})\to H^{\bullet}(\tilde{e}(P),\mathcal{M}_{\lambda}).$$

Let $\omega\in H^{\bullet}(\tilde{S}_{n-1},\mathcal{M}_{\lambda})$. The space $e(P,K)\times\bar{A}_P^0$ is a neighbourhood of $E(P,K)$ in $S_{n-1}(K)$ and the restriction $\omega|_{a\cdot e(P,K)}$ does not depend on $a\in A_P^0$, hence we may set $\omega|_{e(P,K)} = \omega|_{a\cdot e(P,K)}$ for some $a \in A_P^0$. For any parabolic subgroup P' in GL_{n-1} we define the constant term

$$\omega|_{P'} = \int_{N_{P'}(\mathbb{Q})\backslash N_{P'}(\mathbb{A})} \omega(ng)\, dn.$$

In coordinates we have

$$\omega|_{P'} = \sum_{\mathbf{i}} f_{\mathbf{i}P'}\, \omega_{i_1} \wedge \ldots \wedge \omega_{b_{n-2+\ell(\hat{w})}},$$

where $\omega = \sum_{\mathbf{i}} f_{\mathbf{i}}\, \omega_{\mathbf{i}}$. The restriction is then given by

$$r_{P'}([\omega]) = [\omega_{P'}]|_{e(P,K)}$$

(cf. [Sch], 1.10 Satz).

We recall the general formula for the constant term. Let $P \leq \mathrm{GL}_{n-1}$ a parabolic subgroup, π' a cuspidal representation of the Levi component M_P and $\Lambda \in \mathfrak{a}_{P,\mathbb{C}}$. Let furthermore f_Λ be a section in $\mathrm{I}_{P,\Lambda}\pi'$ and let $P' \leq \mathrm{GL}_{n-1}$ a parabolic subgroup. We denote by $\{P\}_{\text{ass}}$ resp. $\{P\}_{\text{conj}}$ the equivalence class of all parabolic subgroups $P' \leq \mathrm{GL}_{n-1}$, which are associate resp. conjugate to P and $W(A_P, A_{P'})$ is the set of all inner automorphisms $s : A_P \to A_{P'}$ which induce an isomorphism $M_P \to M_{P'}$. We then obtain

$$\mathrm{Eis}_{P,\pi',\Lambda}(f_\Lambda)_{P'} = \sum_{s \in W(A_P, A_{P'})} c(s, \Lambda_{\hat{w}}) f_\Lambda,$$

where $c(s,\Lambda)$ is the standard intertwining operator. Representing cohomology classes by $(\mathfrak{g}_{n-1}, K_{n-1\infty})$ -cocycles we may write

$$\mathrm{Eis}_{P,\pi',\Lambda}(\omega)|_P = \sum_{s \in W(A_P, A_{P'})} \mathbf{c}(s, \Lambda)\omega_\Lambda,$$

where $\omega_\Lambda \in H^\bullet(\mathfrak{g}_{n-1}, K_{n-1,\infty}Z_{n-1}^0(\mathbb{R}), \mathrm{I}_{\mathbb{A}}(\Lambda_{\mathbb{A}} \otimes \pi'))$ is as above and

$$\mathbf{c}(s, \Lambda)\omega_\Lambda = \sum c(s, \lambda) f_{\mathbf{i}}\, \omega_{\mathbf{i}}.$$

We want to apply this to compute the restriction of $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ to the boundary. We have $\{P_{(n-2,1)}\}_{\text{ass}} = \{P_{(n-2,1)}\}_{\text{conj}} \cup \{P_{(1,n-2)}\}_{\text{conj}}$ and $W(P_{(n-2,1)}, P_{(n-2,1)}) = \{1\}$ and $W(P_{(n,1)}, P_{(1,n)}) = \{w_{l,n-1}\}$. Using the general theory of the constant term we see that $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{P'}$ vanishes unless $P' = P_{(1,n-2)}$ or $P' = P_{(n-2,1)}$ and in these cases we obtain

$$\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{P_{(n-2,1)}} = \omega_{\hat{\pi}[\chi],\Lambda_{\hat{w}}}$$

and

$$\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{P_{(1,n-2)}} = \mathbf{c}(w_{n-1,l}, \Lambda)\, \omega_{\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{P'}.$$

To put it into one line we have

$$\bigoplus_{\substack{P' \in \mathcal{P} \\ P' \neq \mathrm{GL}_{n-1}}} r_{P'} \omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}} = \omega_{\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{e(P_{(n-2,1)},K)} + \mathbf{c}(w_{n-1,l},\Lambda)\, \omega_{\hat{\pi}[\chi],\Lambda_{\hat{w}}}|_{e(P_{(1,n-2)},K)}.$$

In particular, $r_{\partial \tilde{S}_{n-1}} \omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ and hence $\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ do not vanish.
As another application we want to prove that the map

$$\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}} : H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{\mathbb{C}} \to H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}})$$

already is defined over $E_{\hat{\pi}[\chi]}$. Let $\sigma \in \mathrm{Aut}(\mathbb{C}, E_\lambda)$, i.e. $\mathcal{M}_\lambda^\sigma = \mathcal{M}_\lambda$. Since

$$(\mathrm{I}^a(\Lambda_{\hat{w},\mathbb{A}} \otimes \hat{\pi}[\chi]))^\sigma \cong \mathrm{I}^a(\Lambda_{\hat{w},\mathbb{A}} \otimes \hat{\pi}[\chi]^\sigma),$$

the multiplicity-1 result on the isotypical component (cf. Lemma 5.1) implies

$$(H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{\mathbb{C}})^\sigma = H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi]^\sigma, \hat{w}]_{\mathbb{C}}.$$

We consider the diagram

$$\begin{array}{ccc} H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{\mathbb{C}} & \overset{\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}}}{\to} & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}) \\ \sigma \downarrow & & \sigma \downarrow \\ H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}^\sigma]_{\mathbb{C}} & \overset{\mathrm{Eis}_{P,\hat{\pi}[\chi]^\sigma,\Lambda_{\hat{w}}}}{\to} & H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}). \end{array}$$

The family of operators $\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}}$ is defined over E_λ if and only if the above diagram commutes for all $\sigma \in \mathrm{Aut}(\mathbb{C}, E_\lambda)$. To prove this, following [Ha 3], ch. 4.3 we look at the differnce $\Delta_\sigma = \sigma \circ \mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}} - \mathrm{Eis}_{P,\hat{\pi}[\chi]^\sigma,\Lambda_{\hat{w}}} \circ \sigma$. If Δ_σ does not vanish it defines a $\mathrm{GL}_{n-1}(\mathbb{A}_f)$-equivariant map

$$\Delta_\sigma : H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{\mathbb{C}} \to H^{b_{n-2}+\ell(\hat{w})}_{\partial S_{n-1}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}),$$

where $H^{b_{n-2}+\ell(\hat{w})}_{\partial S_{n-1}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}) = \bigcap_{\substack{P' \in \mathcal{P} \\ P' \neq \mathrm{GL}_{n-1}}} \ker r_{P'}$. The space of classes, which vanish on the boundary decomposes

$$H^\bullet_{\partial S_{n-1}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}) = H^\bullet_{\mathrm{res}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}) \oplus H^\bullet_{\mathrm{cusp}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}).$$

On the other hand we assume that λ is regular and hence $H^\bullet_{\mathrm{res}}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,\mathbb{C}}) = 0$ (cf. [F], Theorem 19.II). Thus, Δ_σ intertwines $\mathrm{I}_{\mathbb{A}_f}(\Lambda_{\hat{w},\mathbb{A}} \otimes \hat{\pi}[\chi]_f)$ with the space of cusp forms, which is impossible. We record our result:

Theorem 5.4. *The operators*

$$\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}} : H^{b_{n-2}+\ell(\hat{w})}_{\mathrm{cusp}}(\tilde{e}(P) \times A^0_P, \mathcal{M}_{\lambda,\mathbb{C}})[\hat{\pi}[\chi], \hat{w}]_{E_{\hat{\pi}[\chi]}} \to H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,E_{\hat{\pi}[\chi]}})$$

are defined over $E_{\hat{\pi}[\chi]}$ *and for all* $\sigma \in \mathrm{Aut}(\mathbb{C}/E_\lambda)$ *we have* $(\mathrm{Eis}_{P,\hat{\pi}[\chi],\Lambda_{\hat{w}}})^\sigma = \mathrm{Eis}_{P,\hat{\pi}[\chi]^\sigma,\Lambda_{\hat{w}}}$.

Corollary 5.5. *For all* $\sigma \in \mathrm{Aut}(\mathbb{C}/E_\lambda)$ *we have* $(\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}})^\sigma = \omega_{\mathrm{Eis},\hat{\pi}[\chi]^\sigma,\Lambda_{\hat{w}}}$. *In particular,*

$$\omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}} \in H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda,E_{\hat{\pi}[\chi]}}).$$

Proof. Immediate from $(\omega_{\hat{w},\hat{\pi}[\chi]})^\sigma = \omega_{\hat{w},\hat{\pi}[\chi]^\sigma}$.

6 The critical values of automorphic L-functions

In this last section we finally turn to the computation of the values $L(\pi \otimes \chi\eta, l)$ for critical pairs (η, l) . To this end we assume that n is even and π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ satisfying (**Coh**) and (**Reg**) . Thus, π has non-vanishing cohomology with regular coefficients. Our method is as follows. We select a weight λ such that the p-adic Mellin transform of $d\mathbf{P}^{\Lambda_{\hat{w}}}_{\eta,l}$ computes the values of the L-function at critical pairs (η, l) . For these weights λ we then compute the values of $d\mathbf{P}^{\pi}_{\eta,l}$.

6.1. The topological calculation of the distribution $d\mathbf{P}^{\pi}_{\Lambda_{\hat{w}},\eta}$.

Let $\mathbf{m}$ and w be the quantities attached to π as in section 4. We let (η, l) be any critical pair for π , where we may assume that $l \leq 1/2$. Depending on $\mathbf{m}$ and l we fix a weight $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ which satisfies the conditions $(\mathbf{W.1})$ - $(\mathbf{W.5})$.

In this subsection we shall state the consequences of properties $(\mathbf{W.3})$ and $(\mathbf{W.4})$ of such a choice of λ .

- $(\mathbf{W.4})$ *yields the computation of the critical values of the* L*-function.* We denote by $v_{\hat{\pi}[\chi],\mathbf{i},b} = \otimes_v v_{\hat{\pi}[\chi],\mathbf{i},b,v}$ the Whittaker function of $\mathrm{Eis}(f_{\hat{\pi}[\chi],\mathbf{i},b})$. We define the distribution

$$d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta,\pi,\hat{\pi}[\chi]} = \sum_{\mathbf{i},\mathbf{j}} \sum_{a,b} \mathrm{sgn}[\mathbf{i},\mathbf{j}] \mathrm{tr}(a \otimes b)\; d\mathbf{P}^{\Lambda_{\hat{w}},\eta}_{w_{\infty,\mathbf{j},a}, v_{\infty,\mathbf{i},b}}$$

as well as the polynomial

$$P_{l,\flat} = \sum_{\mathbf{i},\mathbf{j}} \sum_{a,b} \mathrm{sgn}[\mathbf{i},\mathbf{j}]\ \mathrm{tr}(a \otimes b)\ P^{\eta}_{w_{\mathbf{j},a,\infty}, v_{\mathbf{i},b,\infty}}.$$

Here, $\mathrm{sgn}[\mathbf{i},\mathbf{j}]$ denotes the signum of the permutation $[\mathbf{i},\mathbf{j}]$. Note that $v_{\mathbf{i},b,\infty}$ depends on λ , hence on $l = \lambda_{n/2}$. We will mostly drop the indices $\pi, \hat{\pi}[\chi]$ to facilitate the notation. Eq. (5.11) together with Theorem 3.5 then yields

$$(\gamma\kappa q^{n-1})^e \clubsuit \int_{\mathbb{Z}_p^*} \chi_p \eta_p^{n-1}\, d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta} = P_{l,\flat}(1/2) \frac{L(\pi \otimes \chi\eta, l)}{L(\hat{\pi} \otimes \chi^{-1}, -l + 3/2 + \Delta)}, \tag{6.1}$$

where $\Delta = 0$ in case I, $\Delta = n - 2$ in case II. Thus, the p-adic Mellin transform of $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$ computes the critical value $L(\pi \otimes \chi\eta, l)$.

• $(\mathbf{W}.\mathbf{3})$ *yields a cohomological interpretation of* $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$. Using [F-H], 6.12 we immediately obtain (cf. [M])

Lemma 6.1. *Let* $\rho|_{\mathrm{GL}_{n-1}}$ *be the restriction of* ρ *to* GL_{n-1}. *Then the restriction*

$$\rho|_{\mathrm{GL}_{n-1}} = \rho_\lambda \oplus V$$

contains the representation ρ_λ *with highest weight* λ *as a direct summand.*

As a consequence we see that there is a non-trivial pairing $\mathrm{tr}: \tilde{M}_{\mathbf{m}}|_{\mathrm{GL}_{n-1}} \times M_\lambda \to \mathbb{C}$. This pairing enables us to construct the distribution $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$ in a topological way as follows. We introduce for any compact open subgroup $K \leq \mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$ the manifold

$$F_n(K) = \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})/KK_{n,\infty} \cong \mathbb{R}_+^* \times S_n(K).$$

The embedding $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$, $g \mapsto \mathrm{diag}(g,1)$ induces mappings

$$i:\ F_{n-1}(K(n-1,N)) \to S_n(K(n,N))$$

where $K(n,N) \leq \mathrm{GL}_n(\mathbb{A}_f)$ denotes the principal congruence subgroup of level N as well as a map

$$j^*:\ (\mathfrak{g}_n/\mathfrak{k}_n)^* \to (\mathfrak{g}_{n-1}/\mathfrak{so}_{n-1})^*$$

In particular, since i is proper we obtain a map in cohomology with compact supports

$$i^*:\ H_c^\bullet(\tilde{S}_n, \tilde{\mathcal{M}}_{\mathbf{m}}) \to H_c^\bullet(\tilde{F}_{n-1}, i^*\tilde{\mathcal{M}}_{\mathbf{m}}).$$

To simplify the exposition we assume that the basis vectors ω_i and ω'_j are chosen so that

$$j^*(\omega'_j) = \begin{cases} \omega_j & \text{if} \quad j \leq \dim \mathcal{X}_{n-1}+1 \\ 0 & \text{else.} \end{cases}$$

We denote by $p:\ F_{n-1}(K) \to S_{n-1}(K)$ the canonical projection and by

$$p^*:\ H^\bullet(\tilde{S}_{n-1}, \mathcal{M}_\lambda) \to H^\bullet(\tilde{F}_{n-1}, p^*\mathcal{M}_\lambda)$$

the induced map on cohomology.

The assignment of finite dimensional representations $M \mapsto \eta \otimes M$ induces an assignment of sheafs on $S_{n-1}(K)$

$$\psi:\ \mathcal{M} \mapsto \mathcal{M}^\psi.$$

and hence a map

$$H^\bullet(\tilde{S}_{n-1}, \mathcal{M}) \to H^\bullet(\tilde{S}_{n-1}, \mathcal{M}^\psi)$$

which we denote as $\omega \mapsto \psi \otimes \omega$. We note that if $\mathbb{Q}(\eta) \leq E$ we have $\mathcal{M}_E^\eta = \mathcal{M}_E$. In particular we obtain a class

$$\eta \otimes \omega_{\mathrm{Eis},\hat{\pi}[\chi],\Lambda_{\hat{w}}} \in H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1}, \mathcal{M}_{\lambda, E_{\hat{\pi}[\chi]}(\eta)}).$$

The pairing tr induces a pairing of the associated sheafs

$$\mathrm{tr}:\ i^*\tilde{\mathcal{M}}_{\mathbf{m},\mathbb{C}}\otimes p^*\mathcal{M}_{\lambda,\mathbb{C}}\to\underline{\mathbb{C}}.$$

and the formulas for b_n as well as $b_{n-2}^{\hat w}=b_{n-2}+\ell(\hat w)$ show that

$$b_n+b_{n-2}+\ell(\hat w)=\frac{n^2-n}{2}=\dim(\mathcal{X}_{n-1}\times\mathbb{R}).$$

Together with the cup product we obtain the same diagram as in [M]

(6.2)

$$\begin{array}{ccccc} H_c^{b_n}(\tilde F_{n-1},i^*\tilde{\mathcal{M}}_{m,\mathbb{C}}) & \times & H^{b_{n-2}^{\hat w}}(\tilde F_{n-1},p^*\mathcal{M}_{\lambda,\mathbb{C}}) & \overset{\mathrm{tr}\circ\cup}{\to} & H_c^{\dim\tilde F_{n-1}}(\tilde F_{n-1},\underline{\mathbb{C}})\ =\mathbb{C}\\ \uparrow i^* & & \uparrow p^* & & \\ H_c^{b_n}(\tilde S_n,\tilde{\mathcal{M}}_{m,\mathbb{C}}) & & H^{b_{n-2}+\ell(\hat w)}(\tilde S_{n-1},\mathcal{M}_{\lambda,\mathbb{C}}). & & \end{array}$$

From this we deduce the existence of a pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle:\ H_c^{b_n}(\tilde S_{n-1},\mathcal{M}_{m,\mathbb{C}})\times H^{b_{n-2}+\ell(\hat w)}(\tilde{S_{n-2}},\mathcal{M}_{\lambda,\mathbb{C}}) & \to & \mathbb{C}\\ (\omega,\omega') & \mapsto & \mathrm{tr}\ i^*\omega\cup p^*\omega'. \end{array}$$

As in [M] we denote by

$$r_u^*:\ H_c^{b_n}(\tilde S_n,\mathcal{M}_{\mathbf{m}})\to H_c^{b_n}(\tilde S_n,\mathcal{M}_{\mathbf{m}}),$$

the right translation with $u\in\mathrm{GL}_n(\mathbb{A}_f)$. We set

$$u_{p^e}=\begin{pmatrix}1 & & & 0\\ & \ddots & & \vdots\\ & & 1 & 1/p^e\\ & & & 1\end{pmatrix}.$$

The construction of the classes $\omega_{\mathrm{Eis},\hat\pi[\chi],\Lambda_{\hat w}}$ applies to any representation $\pi[\mu]=\hat\pi\otimes\mu$, where $\mu\in\mathbf{X}_e$ and μ has conductor $f_\mu=p^e$. As in section 3 we extend this definition to all characters $\mu\in\mathbf{X}_e$ and set

$$\omega_{\mathrm{Eis},\pi(\mu),\Lambda_{\hat w},p^e}=\kappa^{-(e-e_\mu)}\,r^*_{D^{-(e-e_\mu)}}(\omega_{\mathrm{Eis},\pi(\mu),\Lambda_{\hat w}}).$$

In the same way as in section 3 we define a Fourier transform

$$\mathcal{F}:\ \mathbf{X}_e^{H^\bullet(\tilde S_{n-1},\mathcal{M}_\lambda)}\to\mathbf{O}_e^{H^\bullet(\tilde S_{n-1},\mathcal{M}_\lambda)}$$

and we set

$$\omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat w}}=\mathcal{F}_{\varepsilon+p^e\mathbb{Z}_p}(\omega_{\mathrm{Eis},\hat\pi\otimes\mu,\Lambda_{\hat w},p^e}).$$

All this is analogous to the constructions in [M] and with the same proof as in [M] we obtain the following description of the values of $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{hatw},\eta}$.

Proposition 6.2. *For any* $\varepsilon \in \mathbb{Z}_p^*$ *and* $e \geq 2$ *we have*

$$d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}(\varepsilon + p^e\mathbb{Z}_p) = (\gamma\kappa q^{n-1})^{-e}\, \langle \eta \otimes \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}}, r^*_{u_{p^e}}\omega \rangle.$$

Remark 6.3. Since the p-adic Mellin transform of $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$ computes the values of our L-function (cf. Eq. (6.1)) this yields a topological calculation of special values of $L(\pi, s)$.

The group of connected components.– We consider the action of the group of connected components $\mathbb{Z}/2\mathbb{Z} = \mathrm{O}_{n-1}(\mathbb{R})/\mathrm{SO}_{n-1}(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ on $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$. We denote by -1 the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. Using differential forms we find $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}(\varepsilon + p^e\mathbb{Z}_p) = \mathbf{J}(\omega, \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}})$, where

$$\mathbf{J}(\omega, \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}}) = \int_{F_{n-1}(K)} \omega\left(\begin{pmatrix} g & \\ & 1\end{pmatrix} u_{p^e}\right) \wedge \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}}(g).$$

Changing integration variables we see that

$$\mathbf{J}(\omega, \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p}, \Lambda_{\hat{w}}) = \mathbf{J}(-1\cdot\omega, -1\cdot\omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p}, \Lambda_{\hat{w}}).$$

On the other hand recalling Eq. (4.7) and Lemma 5.1 we see that

$$-1\cdot r^*_{u_{p^e}}\omega \wedge -1\cdot\omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}}, = (-1)^k\, r^*_{u_{p^e}}\omega \wedge \omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}},$$

where

$$k \equiv \flat + l + (n-1)/2 \pmod 2.$$

Hence, $d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$ and therefore $P_{l,\flat}$ vanish unless

$$\flat \equiv l + (n-2)/2 \pmod 2,$$

which is satisfied.

6.2. Definition of the complex period $\Omega(\pi)$ *.*

We denote by $E_{\mathbf{m}}$ the finite extension of $\mathbb{Q}$ over which the finite dimensional representation $\rho : \mathrm{GL}_n \to M_{\mathbf{m}}$ is defined and we denote by $M_{\mathbf{m},E_{\mathbf{m}}} \leq M_{\mathbf{m}}$ a GL_n-stable $E_{\mathbf{m}}$-subspace. We let $\mathcal{M}_{\mathbf{m}}$ be the sheaf on $S_n(K)$ corresponding to $M_{\mathbf{m},E_{\mathbf{m}}}$. Using Theorem 3.13 and 3.19 in [C] we see that π_f is defined over a finite extension E_π of $\mathbb{Q}$ (containing $E_{\mathbf{m}}$). Moreover π_f can be realized on a E_π-subspace in the Betti cohomology:

$$H^\bullet(\tilde{S}_n, \mathcal{M}_{\mathbf{m}})(\pi_f) = H^\bullet(\tilde{S}_n, \mathcal{M}_m)(\pi_f)_{E_\pi},$$

where

$$H^\bullet(\tilde{S}_n, \mathcal{M}_{\mathbf{m}})(\pi_f)_{E_\pi} = H^\bullet(\tilde{S}_n, \mathcal{M}_{\mathbf{m}})(\pi_f) \cap H^\bullet(\tilde{S}_n, \mathcal{M}_{m,E_\pi}).$$

In the same way as in section 5 we define a twisted action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the space of Whittaker functions on $\mathrm{GL}_n(\mathbb{A}_f)$ and obtain

$$W(\pi_f,\tau_f)=W(\pi_f,\tau_f)_{E_\pi}\otimes\mathbb{C},$$

where $W(\pi_f,\tau_f)_{\pi_E}$ is the space of $\mathrm{Aut}(\mathbb{C}/E_\pi)$ -invariants.

To define the complex periods $\Omega(\pi)$ in a sensful way we first have to normalize the choice of the generators $\xi_{\pi,\infty}$. To this end we have to make the

Assumption. The special value $P_{l,\flat}(1/2)$ does not vanish.

Once this assumption is made we normalize our choose a scalar multiple of the generator $\xi_{\pi,\infty}$ such that $P_{l,\flat}(1/2)=1$. We consider the diagram:

$$\begin{array}{ccc} W(\pi_f,\tau_f) & \overset{\xi_{\pi,\infty}}{\to} & H^{b_{n-1}}(\tilde{S}_n,\mathcal{M}_{m,\mathbb{C}})(\pi)_{E_\pi} \\ \tilde{\sigma}\downarrow & & \downarrow\sigma \\ W(\pi_f^\sigma,\tau_f) & \overset{\xi_{\pi^\sigma,\infty}}{\to} & H^{b_{n-1}}(\tilde{S}_n,\mathcal{M}_{m,\mathbb{C}})(\pi^\sigma)_{E_\pi}. \end{array}$$

We choose $\tilde{\Omega}(\pi)\in\mathbb{C}^*$ such that the arrows $\tilde{\Omega}(\pi)\xi_{\pi,\infty}$ are defined over E_π and the above diagram commutes for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E_{\mathbf{m}})$. The existence of such a choice follows form [W], Lemme I.1 together with Schur's lemma.

The weight λ and hence $\mathrm{I}_{P,\Lambda_{\hat{w}},\infty}\hat{\pi}[\chi]$ and $v_{\hat{\pi}[\chi],\mathbf{i},b}$ still depend on l . Therefore the normalization of the $\xi_{\pi,\infty}$'s and also $\tilde{\Omega}(\pi)$ depend on l . We write $\tilde{\Omega}_l(\pi)$ to indicate this.

An Application of the functional equation.– We want to reformulate Eq. (6.1) using the functional equation for the L -function attached to the representation $\hat{\pi}\otimes\chi^{-1}$. In its general form the functional equation reads

$$L(\hat{\pi}\otimes\chi^{-1},s)=\varepsilon(\hat{\pi}\otimes\chi^{-1},s)L(\tilde{\pi}_1'\otimes\chi,1-s).$$

The ε -factor decomposes

$$\varepsilon(\hat{\pi}\otimes\chi^{-1},s)=(D_F^{n-2}N_F(f_{\hat{\pi}\otimes\chi})^{1/2-s}W(\hat{\pi}\otimes\chi^{-1}).$$

where D_F is the discriminant and N_F the absolute Norm of the number field F . Since in our case $F=\mathbb{Q}$ and the conductors of $\hat{\pi}$ and χ are coprime we have $D_F=1$ and $N_F(f_{\hat{\pi}\otimes\chi})=f_{\hat{\pi}}f_\chi$. Moreover, since in addition χ is unramified at infinity we deduce from [Ba-Ra] Proposition 4.1

$$W(\hat{\pi}\otimes\chi^{-1})=\omega_{\hat{\pi}}(f_\chi)\chi(f_{\hat{\pi}})W(\hat{\pi})W(\chi)^{n-2}.$$

Taking into account that $W(\chi)=f_\chi^{-1/2}G(\chi_p)$ we find

$$L(\hat{\pi}\otimes\chi^{-1},l)=f_{\hat{\pi}}^{2l-3/2}f_\chi^{2l-n/2-1/2}\omega_{\hat{\pi}}(f_\chi)\chi(f_{\hat{\pi}})W(\hat{\pi})G(\chi_p)^{n-2}L(\tilde{\pi}_1'\otimes\chi,1-l).$$

We note that the root number $W(\hat{\pi})$ is contained in $E_{\hat{\pi}}$ and satisfies $W(\hat{\pi})^\sigma = W(\hat{\pi}^\sigma)$. This is an easy consequence of [C], Lemme 4.6 and the local functional equation. Thus, except for the Gaussian sum, all terms appearing in the functional equation are contained in $E_{\hat{\pi}[\chi]}$ and behave equivariantly under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Before replacing $L(\hat{\pi} \otimes \chi^{-1}, -l + \Delta)$ in Eq. () by the above expression we calculate the behaviour of the factor $\clubsuit = \clubsuit_{\hat{\pi},\chi}$ under automorphisms. We recall the definition

$$\clubsuit = c\, G(\chi_p \eta_p^{n-1})^{-1} G(\chi_p)^{2-n} |f_\chi|_p^{-1} v_{\hat{\pi}[\chi],S}(g_S^0)^{-1} w^0_{\hat{\pi},p}(\varpi^{\mathbf{v}'})^{-1}.$$

The Gaussian sum $G(\chi)$ is contained in $\mathbb{Q}(\chi)$. Moreover, $w^0_{\hat{\pi},p}$ was defined the spherical Whittaker function satisfying $w_{\hat{\pi},p}(e) = 1$. This uniqueness property immediately implies $w^{\tilde{\sigma}}_{\hat{\pi},p} = w_{\hat{\pi}^\sigma,p}$ for $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Analogously, the considerations in section 4.3 show that $v_{\hat{\pi}[\chi],v} \in W(\hat{\pi}[\chi]_v, \tau_v)_{E_{\hat{\pi}[\chi]}}$ and $v^{\tilde{\sigma}}_{\hat{\pi}[\chi],v} = v_{\hat{\pi}[\chi]^\sigma,v}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and $v \in S$. We choose a finite abelian extension $K = \mathbb{Q}(\zeta)$ large enough so that

$$\mathrm{Aut}(\mathbb{C}/K) \xrightarrow{\beta_{n-1}} \prod_{v\in S} (K^*_{n-1,v})^{g_v^0} \cap T_{n-1}(\hat{\mathbb{Z}}) \leq T_{n-1}(\hat{\mathbb{Z}}).$$

and calculate for $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and $v \in S$

$$v_{\hat{\pi}[\chi],v}(g_v^0)^\sigma = v^{\tilde{\sigma}}_{\hat{\pi}[\chi],v}(\mathbf{t}^{-1}_{\sigma,n-1} g_v^0) = v_{\hat{\pi}[\chi]^\sigma,v}(g_v^0),$$

because $v_{\hat{\pi}[\chi],v}$ is invariant on the right under $K^*_{n-1,v}$. In particular, we obtain $v_{\hat{\pi}[\chi],S}(g_S^0) \in KE_{\hat{\pi}[\chi]}$. Since $w_{\hat{\pi},p}$ is $K_{n-2,v}$-invariant we deduce in the same way that $w_{\hat{\pi},p}(t)^\sigma = w_{\hat{\pi}^\sigma,p}(t)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and $t \in T_{n-2}(F_p)$; in particular $w_{\hat{\pi},p}(\varpi^{\mathbf{v}'}) \in E_{\hat{\pi}}$.

Enlargening K we may assume that $\mathbb{Q}(\eta) \leq K$ and altogether we have seen that $c^{-1}\clubsuit_{\hat{\pi},\chi} \in KE_{\hat{\pi}[\chi]}$ and

$$(c^{-1}G(\chi_p)^{n-2}G(\chi_p\eta_p^{n-1})\clubsuit_{\hat{\pi},\chi})^\sigma = c^{-1}G(\chi_p^\sigma)^{n-2}G(\chi_p^\sigma\eta_p^{n-1})\clubsuit_{\hat{\pi}^\sigma,\chi^\sigma}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. To complete the discussion of the factors appearing in the computation of the values of the L-function, we note that we may choose $\kappa \in \mathbb{Q}$ and that below we shall see that γ_π is contained in E_π, satisfying $\gamma_\pi^\sigma = \gamma_{\pi^\sigma}$ for $\sigma \in E_m$. Thus combining (6.1) with the functional equation we find

$$(6.3)\quad \int_{\mathbb{Z}_p^*} \chi_p \, d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}$$
$$= P_{l,\flat}(1/2)\hat{\clubsuit}\, G(\chi_p) L(\pi \times \eta \otimes \hat{\pi}[\chi], 1/2 + \Delta) \frac{L(\pi \otimes \chi\eta, l)}{L(|\cdot|^{-1/2} \otimes \tilde{\pi}'_1 \otimes \chi, l - \Delta)},$$

where $\hat{\clubsuit} = \hat{\clubsuit}_{\pi,\hat{\pi}[\chi]} \in E_\pi E_{\hat{\pi}[\chi]}$ is equivariant with respect to the action of $\sigma \in \mathrm{Aut}(\mathbb{C}/E_{\mathbf{m}}E_\lambda)$.

We want to use Eq. (6.3) to deduce the properties of the critical values of the L-function $L(\pi\otimes\eta\chi,l)$. To this end we have to make sure that all the special values of L-functions appearing in (6.3) are regular. First we note that all the L-functions are holomorphic in $\mathbb{C}$; in case of the convolution this is proven in [M-W], Appendice. Also we may assume that $L(\pi\otimes\eta\chi,l)$ does not vanish. Eq (6.3) implies that the product of all three values is regular, hence we know that all three special values are regular as soon as we know that

$$L(\pi\times\eta\otimes\hat{\pi},1/2+\Delta)\neq 0.$$

- π is of type I. The value in question equals $L(\pi\times|\cdot|^{1/2}\otimes\hat{\pi},0)$, where $|\cdot|^{1/2}\otimes\pi$ is unitary. (**C.2**) implies that the local factors outside the places where π or $\hat{\pi}$ ramify are equal to one. Hence, the functional equation together with non-vanishing of the convolution at $s=1$ (cf. [Sha 3], 2.2, e) and h)) shows that $L_{\text{fin}}(\pi\times\eta\otimes\hat{\pi},1/2)$ does not vanish. Since the Γ-factors at infinity can not contribute a zero we see that $L(\pi\times\otimes\hat{\pi},1/2)$ does not vanish.

- π is of type II. Since we assume $n\geq 3$ the point $1/2+\Delta$ lies in the range of convergence of the Euler product, whence the claim.

The ratio of L-function on GL_n and GL_{n-2}.– We continue to cite from [M].

Lemma 6.4. *There is a finite abelian extension $K=\mathbb{Q}(\zeta)$ and a complex number $\Omega'\in\mathbb{C}^*$ such that $\Omega' w_f\in W(\pi_f,\tau_f)_{E_\pi(\zeta)}$ and $(\Omega' w_f)^{\tilde{\sigma}}=\Omega' w_f$ for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E_{\mathbf{m}}(\zeta))$. Here, w_f denotes the finite part of the Whittaker function defined in section 1. This in particular implies $\gamma_\pi\in E_\pi$ and $\gamma_\pi^\sigma=\gamma_{\pi^\sigma}$ for $\sigma\in\mathrm{Aut}(\mathbb{C}/E_{\mathbf{m}})$.*

Proof. This is the same proof as in [M].

We set $\Omega_l'(\pi)=\tilde{\Omega}_l(\pi)\Omega'$ and as an immediate consequence we obtain

$$\frac{1}{\Omega_l'(\pi)}\omega\in H_c^{b_n}(\tilde{S}_n,\mathcal{M}_{m,E_\pi(\zeta)}).$$

On the other hand the averaging process over all characters $\mu\in\mathbf{X}_e$ immediately yields

$$\omega_{\mathrm{Eis},\varepsilon+p^e\mathbb{Z}_p,\Lambda_{\hat{w}}}\in H^{b_{n-2}+\ell(\hat{w})}(\tilde{S}_{n-1},\mathcal{M}_{\lambda,\mathbb{C}})_{E_{\hat{\pi}}}.$$

We obtain

Proposition 6.5. *There is a finite abelian extension $K=\mathbb{Q}(\zeta)$ such that for any $U\leq\mathbb{Z}_p^*$ the following holds:*

$$d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta}(U)\in\Omega_l'(\pi)\cdot E_\pi E_{\hat{\pi}}(\zeta)$$

and

$$\{d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta,\pi,\hat{\pi}[\chi]}(U)/\Omega_l'(\pi)\}^\sigma=d\mathbf{P}^{\mathrm{coh}}_{\Lambda_{\hat{w}},\eta,\pi^\sigma,\hat{\pi}[\chi]^\sigma}(U)/\Omega_l'(\pi^\sigma)$$

for all $\sigma\in\mathrm{Aut}(\mathbb{C}/E_{\mathbf{m}}E_\lambda(\zeta))$.

Proof. This follows immediately from Proposition 5.3 and the fact that the diagram (6.2) defining $\langle\ ,\ \rangle$ and hence the pairing $\langle\ ,\ \rangle$ itself are defined over $\mathbb{Q}$.

We define
$$\Omega_l(\pi,\hat\pi)=\Omega_l'(\pi)L(\pi\times\eta\otimes\hat\pi,1/2+\Delta)$$
and we abbreviate
$$\mathbb{L}(\pi\otimes\chi\eta,l)=\frac{G(\chi_p)^{[n/2]}}{\Omega_l(\pi,\hat\pi)}\,\frac{L(\pi\otimes\chi\eta,l)}{L(|\cdot|^{-1/2}\otimes\tilde\pi_1'\otimes\chi,l-\Delta)}.$$

Together with Eq. (6.3) we arrive at the following result on the special values of L-functions on GL_n.

Theorem 6.6. *Let n be even and let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ satisfying* (**Coh**) *and* (**Reg**). *Let (l,η), $l\le 1/2$ be a critical pair for π. Let furthermore $\lambda\in\mathbb{Z}^{n-1}$ be any weight satisfying* (**W.1**) - (**W.5**). *Then there is a finite abelian extension K of $E_{\mathbf{m}}E_\lambda$ such that for all characters χ with infinity component $\chi_\infty=\mathrm{id}$ and conductor $f_\chi=p^e$, $e\ge 2$ the following holds:*
$$\mathbb{L}(\pi\otimes\chi\eta,l)\in KE_\pi E_{\hat\pi}(\chi)$$
and for all $\sigma\in\mathrm{Aut}(\mathbb{C}/K)$ we have
$$\mathbb{L}(\pi\otimes\chi\eta,l)^\sigma=\mathbb{L}(\pi^\sigma\otimes\chi^\sigma\eta,l).$$

6.3. The induction step.

We want to apply Theorem 6.6 to inductively establish our final result on the values at critical points of the L-function.

Theorem 6.7 *Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$, n even, which is unramified at $\mathfrak{p}$ and satisfies* (**Coh**) *and* (**Reg**). *Let $\eta:\mathbb{Q}^*\backslash\mathbb{A}^*\to\mathbb{C}^*$ be an idele class character of finite order and conductor satisfying* (**C.1**). *Let $l\le 1/2$ be any critical point for $\eta\otimes\pi$. Then there is a complex period $\Omega_l(\pi)\in\mathbb{C}^*$ and a finite extension $K/\mathbb{Q}$ such that for all $\chi\in\mathbf{X}_{\mathfrak{p}}$*
$$\frac{L(\pi\otimes\eta\chi,l)}{\Omega_l(\pi)}\in KE_{\mathbf{m}}$$
and for all $\sigma\in\mathrm{Aut}(\mathbb{C}/KE_{\mathbf{m}})$ we have
$$\left(G(\chi)^{[n/2]}\frac{L(\pi\otimes\chi\eta,l)}{\Omega_l(\pi)}\right)^\sigma=G(\chi^\sigma)^{[n/2]}\frac{L(\pi^\sigma\otimes\chi^\sigma\eta,l)}{\Omega_l(\pi^\sigma)}.$$

In detail, to *prove* the Theorem, we proceed as follows. We write $\mathbf{m}=\mathbf{m}_\pi$ for the weight attached to π. We choose a weight $\lambda=(\lambda_1,\dots,\lambda_{n-1})$ satisfying (**W.1**) -

$(\mathbf{W.5})$. Hence, λ depends on l and π . We then choose a cuspidal representation $\hat{\pi}$ of $\mathrm{GL}_{n-2}(\mathbb{A})$ satisfying (5.6) with respect to λ . If now $|\cdot|^{-1/2} \otimes \hat{\pi}[\chi]$ again satisfies $(\mathbf{W.1})$ - $(\mathbf{W.5})$ resp. $(\mathbf{Reg})$ and l turns out to be a critical point for $|\cdot|^{-1/2} \otimes \hat{\pi}[\chi]$ we may apply Theorem 6.6 to inductively prove the assertion of the Theorem on the critical value $L(\pi \otimes \eta\chi, l)$. Since n is even the induction is over all even integers $n, n-2, \dots, 2$ and we are reduced to the computation of $L(\hat{\pi}[\chi] \otimes \chi, l)$, where $\hat{\pi}[\chi]$ is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with non-vanishing cohomology and l is critical for $\hat{\pi}[\chi]$. In this case the statement about the values at critical points is classical.

Thus it remains to verify that $|\cdot|^{-1/2} \otimes \hat{\pi}[\chi]$ satisfies $(\mathbf{Coh})$ and $(\mathbf{Reg})$ and that l is a critical point for $|\cdot|^{-1/2} \otimes \hat{\pi}[\chi]$. We shall seperately treat the cases introduced in Definition 4.3, into which the representation π falls.

- π is of type I. By definition $|\cdot|^{-1/2} \otimes \hat{\pi}_\infty = \pi(-1/2, \mathbf{l}')$ with $l' \equiv 0 \pmod 2$, hence, $|\cdot|^{-1/2} \otimes \hat{\pi}$ is of type I for GL_{n-2} . $(\mathbf{W.5})$ immediately implies that $(\mathbf{Reg})$ holds for $\hat{\pi}$. Moreover,

$$\mathbf{m}_{|\cdot|^{-1/2}\otimes\hat{\pi}} = w \cdot \lambda|_1 = (\lambda_1, \dots, \lambda_{\frac{n-2}{2}}, -\lambda_{\frac{n-2}{2}} - 1, \dots, -\lambda_1 - 1).$$

Remark 4.2 then shows that all points $l \in 1/2+\mathbb{Z}$ in the range $-\lambda_{n/2-1}+1/2 \leq l \leq 1/2$ are critical for $|\cdot|^{-1/2} \otimes \hat{\pi}$. Since $\lambda_{n/2-1} \geq m_{n/2}$ and $-m_{n/2} + 1/2 \leq l \leq 1/2$ we deduce that l is critical for $|\cdot|^{-1/2} \otimes \hat{\pi}$.

- π is of type II. Again, $|\cdot|^{-1/2}\otimes\hat{\pi}$ is of type I and $(\mathbf{W.5})$ implies $(\mathbf{Reg})$. Moreover, $l \geq m_{n/2} + 1/2$ implies that $l - (n-2) \geq -m_{n/2} + 1/2 - (n-2) = \lambda_{n/2-1} + 1/2$ (compare the special choice of λ) and hence $l-(n-2)$ is a critical point for $|\cdot|^{-1/2}\otimes\hat{\pi}$ by Remark 4.2.

$$\begin{array}{cccc}
\text{type II} & & \text{type I} & n \\
 & \searrow & \downarrow & \\
 & & \text{type I} & n-2 \\
 & & \downarrow & \\
 & & \vdots & \vdots \\
 & & \downarrow & \\
 & & \text{type I} & 4 \\
 & & \downarrow & \\
 & & \text{type I} & 2 \\
 & & & \\
 & & (n \text{ even}) &
\end{array}$$

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