On a result of Levin and Fainleib involving multiplicative functions

Pieter Moree

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-46

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Abstract

Let f be a multiplicative function that is constant, on average, in prime arguments. B. V. Levin and A. S. Fainleib derived in 1967 by a bootstrapping method not involving complex analysis an expression for $\sum_{n\le x} f(n)$, under some further conditions on f. In this note we prove a slightly weaker version of their result under rather more transparent conditions on f.

1 Introduction

An arithmetic function f, that is a function $f : \mathbb{N} \to \mathbb{C}$, is said to be multiplicative if $f(ab) = f(a)f(b)$ for all coprime natural numbers a and b. It is said to be completely multiplicative if $f(ab) = f(a)f(b)$ for all natural numbers a and b. Throughout we will assume that $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$. In their classical work [5] Levin and Fainleib gave an elementary method (a method not involving complex analysis) to deal with sums of multiplicative functions constant on average in prime arguments. A. Selberg [13, pp. 183-185] discovered a short elegant method to do so as well, however it is severely limited in both powerfulness and generality [8]. Since 1967 no other elementary method of comparible generality has been found improving on that of Levin and Fainleib.

The main result of [5] reads as follows:

Theorem 1 [5, Theorem 2.1.1]
Let $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be a multiplicative function satisfying the conditions

$$\sum_{n\le x} \frac{\Lambda_f(n)}{n} = \tau \log x + B + h(x), \tag{1}$$

$$h(x) = O(\log^{-\gamma} x), \tag{2}$$

$$\sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \to 0 \text{ as } p \to \infty, \tag{3}$$

$$\prod_{p\le x} (1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots) = O(\log^A x), \tag{4}$$

where $A, B, \gamma > 0$ and $\tau > 0$, are constants. Then, for $\epsilon > 0$,

$$\sum_{n\le x}\frac{f(n)}{n} = \sum_{0\le\nu<\tau+\gamma+1-A} a_\nu \log^{\tau-\nu} x + O_\epsilon(\log^{A-1-\gamma+\epsilon} x) \tag{5}$$

and, for $\lambda > 0$,

$$\sum_{n\le x} f(n)n^{\lambda-1} = x^\lambda \sum_{0\le\nu<\tau+\gamma-A} b_\nu \log^{\tau-1-\nu} x + O_\epsilon(x^\lambda \log^{A-1-\gamma+\epsilon} x), \tag{6}$$

where $b_0 = a_0\tau/\lambda > 0$.

Here Λ_f denotes the Von Mangoldt function associated to the function f (see §2). The letter p is exclusively used to denote primes. The proofs given by Levin and Fainleib are rather succinct. A. G. Postnikov [12, §4.11], however, gave in somewhat more detail an account of their method for a particular multiplicative function, $b(n)$, the characteristic function of the set of integers that can be represented as a sum of two squares. His proof is easily extended to a proof of Theorem 1 in the case that the estimate $h(x) = O(\log^{-\gamma} x)$ holds for every γ.

Unfortunately the conditions of Theorem 1 are, certainly for the non-specialist in multiplicative functions, not so easy to check. The purpose of this note is to prove the following 'consumer friendly' version of Theorem 1. (As usual $\mathrm{Li}(x)$ denotes the logarithmic integral.)

Theorem 2 *Let $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ be a multiplicative function satisfying*

$$0 \le f(p^r) \le c_1 c_2^r,\ c_1 \ge 1,\ 1 \le c_2 < 2, \tag{7}$$

and

$$\sum_{p\le x} f(p) = \tau\mathrm{Li}(x) + O\left(\frac{x}{\log^{2+\gamma} x}\right), \tag{8}$$

where $\tau > 0$ and $\gamma > 0$ are fixed, then, for $\epsilon > 0$,

$$\sum_{n\le x}\frac{f(n)}{n} = \sum_{0\le\nu<\gamma+1} a_\nu \log^{\tau-\nu} x + O_\epsilon(\log^{\tau-1-\gamma+\epsilon} x), \tag{9}$$

and, for $\lambda > 0$,

$$\sum_{n\le x} f(n)n^{\lambda-1} = x^\lambda \sum_{0\le\nu<\gamma} b_\nu \log^{\tau-1-\nu} x + O_\epsilon(x^\lambda \log^{\tau-1-\gamma+\epsilon} x), \tag{10}$$

where $b_0 = a_0\tau/\lambda > 0$. In case f is completely multiplicative, condition (7) can be weakened to:

$$\sum_{p,\ r\ge 2,\ p^r>x} \frac{f(p)^r \log p}{p^r} = O(\log^{-\gamma} x). \tag{11}$$

The price that has to be paid for this consumer friendliness is a slight loss of generality. For a better understanding some remarks to this result have to be made. I can do no better here than to refer the reader to the remarks (A), (B) and (C) following [11, Theorem III]. A further remark is that the method of Levin and Fainleib has an analytic 'equivalent'. This is the so called Selberg-Delange method (see [15, Ch. 5] for an introductory account). Theorem 1 'corresponds' to Theorems 3 and 5 of [15, Ch. 5] taken together (these results have, like Theorem 1, conditions that are not so easy to check in a given case). The analytic methods have their roots in the proof of Landau [4] of the estimate

$$\sum_{n\le x} b(n) \sim c\frac{x}{\sqrt{\log x}},$$

with $c > 0$ a constant, which he obtained using contour integration of a certain L-series (see [3, pp. 61-63] for a more leisurely account). It was folklore that this result could be extended to what we following I. S. Luthar [6] will call an asymptotic series in the sense of Poincaré:

$$\sum_{n\le x} b(n) = \frac{x}{\sqrt{\log x}}(a_0 + \frac{a_1}{\log x} + \frac{a_2}{\log^2 x} + \ldots + \frac{a_k}{\log^k x} + O_k(\frac{1}{\log^{k+1} x})),$$

where $k \ge 1$ is arbitrary (already the last sentence of Landau's paper seems to hint at this possibility). The proof of this folkore result was written down by J.-P. Serre [14] for the larger class of so called Frobenian multiplicative functions. These functions were considered subsequently by R. W. K. Odoni in a series of papers (see [10] for a survey).

As to the structure of this paper; in §2 the function Λ_f is studied, so as to pave the way for the proof of Theorem 2 in §3. Finally in §4 the paper of Serre mentioned above and a paper of K. Wiertelak [17] are reconsidered in the light of Theorem 2.

To avoid notation like $b_{0,10}, b_{1,10}, b_{2,10}, \cdots$, the notation $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ is used to denote sequences of generic constants. At each occurrence the elements of these sequences may have a different value. Instead of $O(R(x))$, $(x \to \infty)$, always just $O(R(x))$ is being written.

I'd like to thank Prof. Fainleib for informing me about his recent work on multiplicative functions and Prof. Moroz for bringing me into contact with Prof. Fainleib. This paper owes much to an unpublished manuscript kindly sent to me by Prof. Halberstam [2].

2 On the Von Mangoldt function associated to a multiplicative function

A very important role in the method of Levin and Fainleib is played by an arithmetic function associated to the multiplicative function under consideration, which is similar to the Von Mangoldt function. Recall that the Von Mangoldt function is defined by $\Lambda(n) = 0$ if n is not a prime power exceeding one and $\Lambda(p^r) = \log p$ otherwise.

Note that $\log n = \sum_{d|n} \Lambda(d)$. Given a multiplicative function f, we implicitly define a function Λ_f by

$$f(n)\log n = \sum_{d|n} f(\frac{n}{d})\Lambda_f(d). \tag{12}$$

If $\mathbf{1}$ denotes the function that is 1 in every integer, then clearly $\Lambda_{\mathbf{1}} = \Lambda$. We will now argue that Λ_f is a Von Mangoldt function in the sense that it only lives on prime powers. Recall that $L_f(s) = \sum_n f(n)n^{-s}$. Then $-L'_f(s) = \sum_n f(n)(\log n)n^{-s}$ and so $-L'_f(s)/L_f(s) = \sum_n \Lambda_f(n)n^{-s}$. Using $L_f(s) = \prod_p(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots)$, we find

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s} = -\frac{L'_f(s)}{L_f(s)} &= \sum_p (-\log(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots))' \\
&= \sum_p \frac{\log p(f(p)p^{-s} + 2f(p^2)p^{-2s} + 3f(p^3)p^{-3s} + \cdots)}{(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots)} \\
&= \sum_p \sum_j \frac{\Lambda_f(p^j)}{p^{js}}.
\end{aligned}$$

Hence the assertion follows. It now follows from (12) that

$$f(n)\log n = \sum_{p^j|n} f(\frac{n}{p^j})\Lambda_f(p^j). \tag{13}$$

Using (13) and induction we find:

Proposition 1 *We have*

$$\Lambda_f(p^r) = c_f(p^r)\log p, \tag{14}$$

where

$$c_f(p) = f(p), \tag{15}$$

and $c_f(p^r)$ is defined recursively for $r > 1$ by

$$c_f(p^r) = rf(p^r) - \sum_{j=1}^{r-1} c_f(p^j)f(p^{r-j}). \tag{16}$$

For a function f that is completely multiplicative the behaviour of $c_f(p^r)$ is particularly simple; we have $c_f(p^r) = f(p)^r$. Again this is easily proved by induction. This suggests that for arbitrary multiplicative functions one can factor out a completely multiplicative function.

Proposition 2 *Suppose $f(n) = g(n)h(n)$, where g is multiplicative and h is completely multiplicative. Then $\Lambda_f(n) = \Lambda_g(n)h(n)$.*

Proof: Note that f is multiplicative. We are done if we show that $c_f(p^r) = c_g(p^r)h(p)^r$, for all prime powers. Once again we proceed by induction. For $r = 1$ the assertion is true by (15). Suppose it is true for $r = 1, \cdots, s-1$ $(s \geq 2)$. Then, using (16),

$$\begin{aligned} c_f(p^s) &= sg(p^s)h(p)^s - \sum_{j=1}^{s-1} c_f(p^j)g(p^{s-j})h(p)^{s-j} \\ &= sg(p^s)h(p)^s - \sum_{j=1}^{s-1} c_g(p^j)g(p^{s-j})h(p)^s \\ &= c_g(p^s)h(p)^s. \ \blacksquare \end{aligned}$$

Proposition 3 *Suppose that f is multiplicative and satisfies $|f(n)| \leq 1$. Then*

$$|\Lambda_f(p^r)| \leq (2^r - 1)\log p.$$

Proof: We prove by induction that $|c_f(p^r)| \leq 2^r - 1$. The result then follows from (14). For $r = 0$ and $r = 1$ the inequality clearly holds. Also, for $r \geq 2$,

$$|c_f(p^r)| \leq r + \sum_{j=1}^{r-1}(2^j - 1) = 2^r - 1. \ \blacksquare$$

Lemma 1 *Suppose that f is multiplicative and satisfies* (7). *Then*

$$|\Lambda_f(p^r)| \leq (2c_1c_2)^r \log p.$$

Proof: Consider the multiplicative function $g(n) = f(n)(c_1c_2)^{-\Omega(n)}$, where as usual $\Omega(n) = \sum_{p^\alpha \| n} 1$. Notice that $|g(n)| \leq 1$. Since $(c_1c_2)^{\Omega(n)}$ is completely multiplicative Proposition 2 can be invoked to deduce that $\Lambda_f(p^r) = \Lambda_g(p^r)(c_1c_2)^r$. Proposition 3 yields $|\Lambda_g(p^r)| \leq (2^r - 1)\log p$. Thus the lemma follows. $\blacksquare$

To a multiplicative function f satisfying (7) we associate a function f_0 as follows. Let S be the set consisting of 1 and the natural numbers having no prime factor less than p_0. Now put $f_0(n) = f(n)$ if n is in S and $f_0(n) = 0$ otherwise. Notice that f_0 is a multiplicative function. We say f_0 is the *companion* of f. Some of the reasons for introducing this notion will become apparent from the proof of the next lemma.

Lemma 2 *Suppose f is multiplicative and satisfies* (7) *and* (8). *Let f_0 be the companion of f. Then*

$$\sum_{n \leq x} \frac{\Lambda_{f_0}(n)}{n} = \tau \log x + B_{f_0} + O(\log^{-\gamma} x),$$

where B_{f_0} is a constant.

Proof: Clearly if f satisfies (7) and (8), so does its companion. We have, using Lemma 1,

$$\sum_{\substack{p, r \geq 2 \\ p^r \leq x}} \frac{|\Lambda_{f_0}(p^r)|}{p^r} \leq \sum_{\substack{p, r \geq 2,\ p^r \leq x \\ p \geq p_0 > 2c_1c_2}} \frac{(2c_1c_2)^r \log p}{p^r} \ll \sum_p \frac{\log p}{p^2} \ll 1.$$

Hence

$$\begin{aligned}
\sum_{\substack{p,r\geq 2\\ p^r\leq x}} \frac{\Lambda_{f_0}(p^r)}{p^r} &= B + O\Big(\sum_{\substack{p,r\geq 2\\ p^r> x}} \frac{|\Lambda_{f_0}(p^r)|}{p^r}\Big)\\
&= B + O\Big(\sum_{p>\sqrt{x}} \frac{\log p}{p^2}\Big)\\
&= B + O(x^{-1/2}),
\end{aligned}$$

for some constant B. Using (14) and (15) it now follows that

$$\begin{aligned}
\sum_{n\leq x} \frac{\Lambda_{f_0}(n)}{n} &= \sum_{p\leq x} \frac{f_0(p)\log p}{p} + \sum_{\substack{p,r\geq 2\\ p^r\leq x}} \frac{\Lambda_{f_0}(p^r)}{p^r}\\
&= \sum_{p\leq x} \frac{f_0(p)\log p}{p} + B + O(x^{-1/2}),\\
&= \tau \log x + B_{f_0} + O(\log^{-\gamma} x),
\end{aligned}$$

where the last step follows from (8) on applying Abel summation. ■

Remark. For future reference we note that from the positivity of f and the above proof it follows that

$$\sum_{n\leq x} \frac{|\Lambda_{f_0}(n)|}{n} = \sum_{n\leq x} \frac{\Lambda_{f_0}(n)}{n} + O(1). \tag{17}$$

For completeness we give two further results on Λ_f. They are not needed for the sequel.

Proposition 4 *Let f and g be arithmetic functions. Let $(f*g)(n) = \sum_{d|n} f(d)g(n/d)$. Then $\Lambda_{f*g} = \Lambda_f + \Lambda_g$.*

Proof: As is well-known, $L_{f*g} = L_f L_g$. So

$$-\frac{L'_{f*g}(s)}{L_{f*g}(s)} = -\frac{L'_f(s)}{L_f(s)} - \frac{L'_g(s)}{L_g(s)},$$

thus

$$\sum_n \frac{\Lambda_{f*g}(n)}{n^s} = \sum_n \frac{\Lambda_f(n)}{n^s} + \sum_n \frac{\Lambda_g(n)}{n^s},$$

hence the result follows on comparing the coefficients of n^{-s} on both sides. ■

Proposition 5 *Let f be multiplicative. Then*

$$\Lambda_f(n) = r\log p \sum_{m=1}^{r} \frac{(-1)^{m-1}}{m} \sum_{k_1+k_2+\cdots+k_m=r} f(p^{k_1})\cdots f(p^{k_m}),$$

if $n = p^r$ and 0 otherwise, or alternatively,

$$\Lambda_f(n) = r\log p \sum_{l_1+2l_2+\cdots+rl_r=r} \frac{(-1)^{l_1+\cdots+l_r-1}}{l_1+\cdots+l_r}\binom{l_1+\cdots+l_r}{l_1!l_2!\cdots l_r!} f(p)^{l_1}f(p^2)^{l_2}\cdots f(p^r)^{l_r},$$

if $n = p^r$ and 0 otherwise, where the k_i run through the natural numbers and the l_i through the non-negative integers.

Proof: For a proof of the first part of the assertion, see [5, Lemma 1.1.2]. The deduction of the latter part from the first is an easy combinatorial exercise left to the interested reader. ■

3 Proof of Theorem 2

In this section we will prove Theorem 2. We will need a rough upper bound for $\sum_{n\le x} f(n)/n$ that is given in the next proposition. (Actually it is quite sharp as comparison with (9) shows that it is off by a constant only for those f satisfying the conditions of Theorem 2.)

Proposition 6 *Suppose that f is a multiplicative function satisfying* (7) *and*

$$\sum_{p\le x} f(p) = \tau\frac{x}{\log x} + O\left(\frac{x}{\log x(\log\log x)^{1+\epsilon}}\right), \tag{18}$$

for some $\epsilon > 0$. Then

$$\sum_{n\le x}\frac{f(n)}{n} \le \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right) = O(\log^\tau x). \tag{19}$$

Proof: We have

$$\sum_{r=2}^{\infty}\frac{f(p^r)}{p^r} \le c_1\sum_{r=2}^{\infty}\left(\frac{c_2}{p}\right)^r \le \frac{c_1c_2^2}{1-c_2/2}\frac{1}{p^2} = O\left(\frac{1}{p^2}\right), \tag{20}$$

where the implied constant does not depend on p. So

$$\begin{aligned}\sum_{n\le x}\frac{f(n)}{n} &\le \prod_{p\le x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right) \ll \prod_{p\le x}\left(1+\frac{f(p)}{p}+O\left(\frac{1}{p^2}\right)\right)\\ &\ll \exp\left\{\sum_{p\le x}\frac{f(p)}{p}+O(1)\right\} \ll \exp\{\tau\log\log x+O(1)\} \ll \log^\tau x,\end{aligned}$$

where we use that $\sum_{p\le x} f(p)/p = \tau\log\log x + O(1)$, which follows from (7) on using Abel summation, and the fact that

$$\int_2^\infty \frac{dt}{t\log t(\log\log t)^{1+\epsilon}}$$

converges. ■

Suppose f is a multiplicative function satisfying (7). Let f_0 denote its companion. The outline of the proof is as follows: we show that f_0 satisfies (1), (2), (3) and (4). Thus by Theorem 1 we find an estimate for $\sum_{n\le x} f_0(n)/n$. Then we use an idea of Odoni [11] to derive from this the required estimate (9) for $\sum_{n\le x} f(n)/n$. By Abel summation the estimate (10) follows from (9). I should like to add that all elementary approaches known to me leading to results similar to Theorem 1, proceed by estimating $\sum_{n\le x} f(n)/n$ before $\sum_{n\le x} f(n)n^{\lambda-1}$, $\lambda>0$.

Proof of Theorem 2. By Lemma 2 it follows that (1) and (2) hold for f_0. That (3) is satisfied follows from inequality (20). By Proposition 6 inequality (4) is satisfied with $A=\tau$. (As we will see later A cannot be chosen to be smaller than τ, which would yield a sharper result.) Thus by Theorem 1 we find that

$$\sum_{n\le x}\frac{f_0(n)}{n}=\sum_{0\le\nu<1+\gamma} a_\nu \log^{\tau-\nu} x+O(\log^{\tau-1-\gamma+\epsilon} x), \tag{21}$$

where $a_0>0$. Next we establish (21) with f_0 replaced by f. Every integer $n>1$ can be expressed uniquely as $n=bk$ with k the largest divisor of n that is in S. Since $(b,k)=1$, $f(n)=f(b)f_0(k)$, and therefore, if we reserve the letter b for natural numbers not having prime numbers $\ge p_0$,

$$\begin{aligned}\sum_{n\le x}\frac{f(n)}{n} &= \sum_{bk\le x}\frac{f(b)f_0(k)}{bk}=\sum_{b\le x}\frac{f(b)}{b}+\sum_{b\le x/p_0}\frac{f(b)}{b}\sum_{1<k\le x/b}\frac{f_0(k)}{k}\\ &= \sum_{x/p_0<b\le x}\frac{f(b)}{b}+\sum_{b\le x/p_0}\frac{f(b)}{b}\sum_{k\le x/b}\frac{f_0(k)}{k}.\end{aligned}$$

To deal with the sums involving $\sum_b f(b)/b$ we use:

Proposition 7 *Suppose f is multiplicative and satisfies* (7). *Put $\alpha=\log c_2/\log 2$. Then*

$$\sum_{b\le y} f(b)=O(y^\alpha \log^{\pi(p_0)} y).$$

Proof: Notice that $f(n)\le c_1^{\omega(n)}c_2^{\Omega(n)}$, where $\omega(n)$ and $\Omega(n)$ denote the number of different respectively the total number of prime factors of n. It follows that $f(b)\le c_1^{\pi(p_0)}c_2^{\Omega(b)}$. Using that $\Omega(b)\le \log b/\log 2$ we find $f(b)=O(b^\alpha)$. In combination with the trivial estimate $\sum_{b\le y}1=O(\log^{\pi(p_0)} y)$, the estimate follows. ■

Corollary 1 *Let $j\ge 0$. By Abel summation applied to the inequality of Proposition 7 we find that for arbitrary $j\ge 0$, $\sum_b f(b)\log^j b/b$ is convergent, to B_j say, and*

$$\sum_{b>y}\frac{f(b)\log^j b}{b}=O(y^{\alpha-1}\log^{\pi(p_0)+j} y).$$

Put $S_4=x^{\alpha-1}\log^{\pi(p_0)} x$. By (22), (21) and Corollary 1

$$\begin{aligned}\sum_{n\le x}\frac{f(n)}{n} &= \sum_{b\le x/p_0}\frac{f(b)}{b}\sum_{k\le x/b}\frac{f_0(k)}{k}+O(S_4)\\ &= \sum_{b\le z}\frac{f(b)}{b}\sum_{k\le x/b}\frac{f_0(k)}{k}+\sum_{z<b\le x/p_0}\frac{f(b)}{b}\sum_{k\le x/b}\frac{f_0(k)}{k}+O(S_4).\end{aligned}$$

$$
\begin{aligned}
&= \sum_{b\le z}\frac{f(b)}{b}\sum_{0\le\nu<\gamma+1} a_\nu \log^{\tau-\nu}(\frac{x}{b}) + O(\sum_{b\le z}\frac{f(b)}{b}\log^{\tau-1-\gamma+\epsilon}(\frac{x}{b})) \\
&\quad + \sum_{z<b\le x/p_0}\frac{f(b)}{b}\sum_{k\le x/b}\frac{f_0(k)}{k} + O(S_4). \\
&= S_1 + O(S_2) + O(S_3) + O(S_4),
\end{aligned}
\tag{22}
$$

say, where the parameter z satisfying $\log z/\log x = o(1)$, will be chosen later. Now, using Corollary 1,

$$
\begin{aligned}
S_1 &= \sum_{0\le\nu<\gamma+1} a_\nu \sum_{0\le j<\gamma-\nu+1}(-1)^j \log^{\tau-\nu-j} x \binom{\tau-\nu}{j}\sum_{b\le z}\frac{f(b)\log^j b}{b} + O(\log^{\tau-1-\gamma} x). \\
&= x\sum_{0\le\nu<\gamma} a_\nu \log^{\tau-\nu} x + +O(z^{\alpha-1}\sum_{0\le\nu<1+\gamma}\log^{\tau-1-\nu} x \sum_{0\le j<\gamma-\nu+1}\frac{\log^{\pi(p_0)+j} z}{\log^j x}) \\
&= x\sum_{0\le\nu<\gamma+1} b_\nu \log^{\tau-1-\nu} x + O(z^{\alpha-1}\log^{\pi(p_0)} z \log^{\tau-1} x) + O(\log^{\tau-1-\gamma} x).
\end{aligned}
$$

Notice that

$$
S_2 = \log^{\tau-1-\gamma+\epsilon} x \sum_{b\le z}\frac{f(b)}{b}(1-\frac{\log b}{\log x})^{\tau-1-\gamma+\epsilon} \ll \log^{\tau-1-\gamma+\epsilon} x.
$$

Finally we deal with S_3. We have

$$
S_3 \ll \sum_{z<b\le x/p_0}\frac{f(b)}{b}\log^\tau(\frac{x}{b}) \ll \log^\tau x \sum_{b>z}\frac{f(b)}{b} \ll \log^\tau x z^{\alpha-1}\log^{\pi(p_0)} z.
$$

Taking $z = \log^{\frac{\gamma+1}{1-\alpha}} x$ in (22) leads to (9) with $a_0 > 0$. (As $\log z/\log x = o(1)$, we are allowed to make this choice of z.) We have

$$
\sum_{n\le x} f(n)n^{\lambda-1} = x^\lambda \sum_{n\le x}\frac{f(n)}{n} - \lambda\int_2^x\{\sum_{n\le t}\frac{f(n)}{n}\}t^{\lambda-1}dt + O(1).
$$

Using that for arbitrary $s\ge 1$, $s\in\mathbf{N}$ and $\lambda, a\in\mathbf{R}$, $\lambda>0$,

$$
\int_2^x t^{\lambda-1}\log^a t dt = \frac{x^\lambda}{\lambda}\log^a x\{1-\frac{a}{\lambda\log x}+\frac{b_2}{\log^2 x}+\cdots+\frac{b_{s-1}}{\log^{s-1} x}+O(\frac{1}{\log^s x})\},
$$

the inequality (10) is deduced with $b_0 = a_0\tau/\lambda > 0$.

To prove Theorem 2 in the case f is completely multiplicative it is not necessary to make use of the companion of f. All we need to do is to check that (1), (2), (3) and (4) are satisfied for f and invoke Theorem 1. For completely multiplicative functions f life is easier in that we know $\Lambda_f(p^r)$ explicitly;

$$
\Lambda_f(p^r) = f(p)^r \log p. \tag{23}
$$

Going through the proof of Lemma 2, we find, on making use of (23), (8) and (11), that

$$\sum_{n\le x}\frac{\Lambda_f(n)}{n}=\tau\log x+B_f+O(\log^{-\gamma}x).$$

Thus (1) and (2) are satisfied. For $p>x$ the inequality

$$\sum_{p,r\ge 2}^{\infty}\frac{f(p^r)}{p^r}\le\frac{1}{\log 2}\sum_{\substack{p,r\ge 2\\ p^r>x}}\frac{f(p)^r\log p}{p^r}$$

holds. Thus (11) implies that $\sum_{r=2}^{\infty}f(p)^r/p^r\to 0$ as $p\to\infty$. But then we also have $\sum_{r=1}^{\infty}f(p)^r/p^r\to 0$ as $p\to\infty$ and hence (3) is satisfied. We leave it to the reader to check that if f is a completely multiplicative function satisfying (11) and (18), then (19) holds (cf. the proof of Proposition 6). Thus (4) is satisfied with $A=\tau$. ∎

Remark. By (9), (19) and $a_0>0$ the choice $A=\tau$ in (4) is best possible for multiplicative and completely multiplicative functions satisfying the conditions of Theorem 2.

4 Some applications of Theorem 2

In the introduction it was remarked that using complex analysis it can be shown that $\sum_{n\le x}b(n)$ has an asymptotic series in the sense of Poincaré. We leave it to the reader to check that this result can be also obtained using Theorem 2. The function $b(n)$ is one of many interesting multiplicative functions arising in arithmetic that satisfy the conditions of Theorem 2. In the remainder of this section a paper of Serre and a paper of Wiertelak will be reconsidered in the light of Theorem 2. This might convince the sceptic reader of the usefulness of Theorem 2.

In the beautiful paper [14] Serre proves, using contour integration and properties of certain L-functions, a folklore result similar to Theorem 2. He then proceeds by giving numerous examples, for the greater part involving Fourier coefficients of cusp forms. I will formulate the main result of [14] and then show that it immediately follows from Theorem 2 and a sufficiently strong form of Chebotarev's density theorem.

A set of primes $\mathcal{P}$ is called Frobenius of density δ, if there exists a finite Galois extension $K/\mathbb{Q}$ and a subset H of $G:=\mathrm{Gal}(K/\mathbb{Q})$ such that

1. H is stable under conjugation;
2. $|H|/|G|=\delta$;
3. for every prime p, with at most finitely many exceptions, one has $p\in\mathcal{P}$ if and only if $\sigma_p(K/\mathbb{Q})\in H$, where $\sigma_p(K/\mathbb{Q})$ denotes the Frobenius map of p in G (defined modulo conjugation in case p does not divide the discriminant of K).

A set of integers S is said to be multiplicative if for all coprime integers m and n, $mn\in S$ if and only if $m,n\in S$. (Notice that the characteristic function of a multiplicative set is a multiplicative function.)

Theorem 3 [14, Théorème 2.8]
Suppose that V is a multiplicative set and that the set of primes appearing in V is Frobenius of density δ, where $0 < \delta < 1$. Then $V(x)$, the counting function of the elements in V not exceeding x has an asymptotic expansion in the sense of Poincaré with main term $cx \log^{\delta-1} x$, $c > 0$.

On taking f to be the characteristic function of V in Theorem 2 and using the Chebotarev density theorem with error $O(e^{-c\sqrt{\log x}})$ (see e.g. [1, Satz 4]), Theorem 3 results.

Next we reconsider a paper of Wiertelak [16]. Let a, $a \neq 0, \pm 1$, and m be integers. Denote by $\mathrm{ord}_m(a)$ the multiplicative order of $a (\mathrm{mod}\ m)$. Put $N(x, n, a)$ for the number of integers m, $1 \leq m \leq x$, $(m, a) = 1$, for which $(\mathrm{ord}_m(a), n) = 1$. By [16, Theorem 1] the estimate

$$\sum_{\substack{p \leq x,\ p \nmid a \\ (\mathrm{ord}_p(a), n) = 1}} 1 = \tau \mathrm{Li}(x) + O\Big(\frac{(\log \log x)^4}{\log^3 x}\Big), \tag{24}$$

where $0 < \tau < 1$ and the implied constant depends at most on a and n, holds. In [17] Wiertelak proves that, for $\epsilon > 0$,

$$N(x, n, a) = c\frac{x}{\log^{1-\tau} x} + O(\frac{x}{\log^{2-\alpha-\epsilon} x}), \tag{25}$$

for some $c > 0$, thus improving substantially on the result of Odoni [9], who proved that $N(x, n, a) \sim cx \log^{\tau-1} x$. Wiertelak proves (25) by checking, making use of (24), that the conditions of Theorem 1 are satisfied for the function $\epsilon(m)$, with $\epsilon(m) = 1$ for those m satisfying both $(m, a) = 1$ and $(\mathrm{ord}_m(a), n) = 1$ and $\epsilon(m) = 0$ for the other m. This requires several pages. Noticing that $\epsilon(m)$ is multiplicative, (25) immediately follows from (24) and Theorem 2, however.

Remark. It would be interesting to extend Theorem 2 to functions $f : G \to \mathbb{R}_{\geq 0}$, where G is a free arithmetical semigroup (see e.g. [12, p. 92] or [7, p. 85] for a definition). A partial result in this direction was proved in [7] (Theorem 4.2).

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Max-Planck-Institut für Mathematik
Gottfried-Claren Str. 26
53225 Bonn
Germany
e-mail: moree@antigone.mpim-bonn.mpg.de