AN EVIDENCE FOR A PHASE TRANSITION IN THREE DIMENSIONAL LATTICE MODELS

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ABSTRACT. As it was recently discovered, an eigenvector structure of commutative families of layer-to-layer matrices in three dimensional lattice models is described by a two dimensional spin lattices generalizing the notion of one dimensional spin chains. In this paper the relations between the two dimensional spin lattice in the thermodynamic limit and the phase structure of three dimensional lattice models are conjectured. Two simplest cases are considered: the homogeneous spin lattice related to the Zamolodchikov - Bazhanov - Baxter model and a "chess spin lattice" related to the Stroganov - Mangazeev elliptic solution of the modified tetrahedron equation. The evidences for the phase transition in the second case are obtained.

1. INTRODUCTION

The most known model of the statistical mechanics on the cubic lattice is the two-state Zamolodchikov model [1] with the partition function calculated by Baxter [2]. Later Bazhanov and Baxter found the generalization of Zamolodchikov's model to N states [3], and then Mangazeev, Stroganov and Sergeev found its vertex formulation [4]. The Zamolodchikov - Bazhanov - Baxter model can not be considered as a good model of the statistical mechanics since the Boltzmann weights of the model are not defined positively (moreover, some of them are imaginary). Nevertheless the leading term of the partition function in the thermodynamic limit is supposed to be positive, so one may talk about the statistical mechanics. But the statistical mechanics is trivial: the Zamolodchikov-Bazhanov-Baxter model appeared to be critical one, see [5]. To observe the phase transition phenomena in three dimensional models is the problem of the significant interest.

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The almost resent, a few years old result is the derivation of the auxiliary linear problem for three dimensional lattice models providing firstly the method of obtaining a more wide class of integrable models. For instance, the next to the Zamolodchikov - Bazhanov - Baxter model in this class is the "chess model" parameterized in the terms of elliptic functions [7] [1]. Secondly, the auxiliary linear problem provides a way to generate the complete set of integrals of motion directly as the decomposition of some operator-valued polynomial $\mathbf{J}(\lambda, \mu)$ with respect to two free complex parameters λ and μ, see [6].

Similarly to the two dimensional case, when a generating function for integrals of motion for an evolution in $1+1$ dimensional space-time is an auxiliary transfer matrix associated with one dimensional spin system (i.e. a spin chain), the auxiliary transfer matrix $\mathbf{J}(\lambda, \mu)$ for the evolution in $2+1$ dimensional space-time is associated naturally with a two dimensional spin system (i.e. to a spin lattice).

The most interesting feature of $\mathbf{J}(\lambda, \mu)$ is that it does not form the commutative family. The reason is that the whole set of integrals of motion is over-defined – this resembles the invariance of a mass center in usual quantum mechanics, but the commutative subset is complete. Practically it means that all the family of $\mathbf{J}(\lambda, \mu)$ may be presented simultaneously in an $N \times N$ matrix form for N-state models [2]. This matrix structure allows one to derive a functional equation for the spectrum of the commutative subset [6].

The most resent result [10] is that layer-to-layer transfer matrices of the three dimensional lattice models are related to the algebraic supplements of $\mathbf{J}$ as $N \times N$ matrices. This observation gives us a key for the investigation of the phase structure of three dimensional lattice models. Since the models are indeed too complicated for the calculations (e.g. the partition function is known only for Zamolodchikov - Bazhanov - Baxter model), one may investigate $\mathbf{J}$ in the thermodynamic limit in order to predict the main features of initial three dimensional lattice models.

Besides, since $\mathbf{J}(\lambda, \mu)$ contains the complete set of commutative integrals of motion and so the two dimensional spin lattice forms the integrable model without any mentioning of three dimensional lattice of statistical mechanics, the investigation of $\mathbf{J}(\lambda, \mu)$ is the separate

[1] In the general situation one may talk about "M-chess models" parameterized in the terms of theta-functions on a genus $g = (M-1)^2$ algebraic curve, see [6, 8, 9]

[2] in comparison, a usual family of commutative transfer matrices may be diagonalized simultaneously.

important problem. The importance follows from the observation that for two state case all the commutative integrals are hermitian and so the system is physical.

This paper is organized as follows. At first, we give briefly the notion of two dimensional square spin lattice, the definition of $\mathbf{J}(\lambda,\mu)$ and other important objects, give the functional equation for the spectrum of $\mathbf{J}$, and then formulate several conjectures concerning the structure of $\mathbf{J}$ in the thermodynamic limit. Secondly, we test the conjectures on the homogeneous spin lattice related to the Zamolodchikov - Bazhanov - Baxter model and reproduce the trivial phase structure of it [5]. Finally, we investigate the simplest chess lattice and observe a nontrivial structure of it in the thermodynamic limit.

2. Formulation of the square spin lattice

2.1. **Set of commutative Hamiltonians.** Consider the square lattice of the sizes $A\times B$ on the torus. The vertices of the lattice may be enumerated by the pairs of the indices $v=(\alpha,\beta)$, $\alpha\in\mathbb{Z}_A$, $\beta\in\mathbb{Z}_B$.

Let the "algebra of observables" $\mathcal{A}$ be the tensor product of AB copies of the Pauli σ-matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

so that $\sigma_{j:\alpha,\beta}\in\mathcal{A}$ are

$$\sigma_{j:\alpha,\beta} = 1\otimes 1\otimes \ldots \otimes \underbrace{\sigma_j}_{\substack{\alpha,\beta-th\\ place}} \otimes \ldots, \quad \alpha\in\mathbb{Z}_A,\quad \beta\in\mathbb{Z}_B\,. \tag{2}$$

Pauli matrices are 2×2 ones, so that according to the conventional terminology we deal with two-state case.

Let further $\mathbf{L}=||\mathbf{L}_{v,v'}||$ be the $AB\times AB$ matrix with the operator-valued entries:

$$\begin{aligned} \mathbf{L}_{\alpha,\beta|\alpha,\beta} &= 1\,, & \mathbf{L}_{\alpha+1,\beta+1|\alpha,\beta} &= -xy\, c_{\alpha,\beta}\,\sigma_{3:\alpha,\beta}\,,\\ \mathbf{L}_{\alpha+1,\beta|\alpha,\beta} &= -x\, a_{\alpha,\beta}\,\sigma_{1:\alpha,\beta}\,, & \mathbf{L}_{\alpha,\beta+1|\alpha,\beta} &= -y\, b_{\alpha,\beta}\,\sigma_{2:\alpha,\beta}\,. \end{aligned} \tag{3}$$

Here $a_{\alpha,\beta},b_{\alpha,\beta},c_{\alpha,\beta},x,y\in\mathbb{C}$ in general. All the other matrix elements $\mathbf{L}_{v,v'}$ are identically zeros. We will call

$$\lambda = x^A\,,\quad \mu = y^B \tag{4}$$

the spectral parameters. Due to the ultra-locality (2) the formal determinant of $\mathbf{L}$ is well defined.

$$\det ||\mathbf{L}|| \stackrel{\text{def}}{=} \mathbf{J}(\lambda,\mu) = \sum_{\alpha=0}^{A}\sum_{\beta=0}^{B} \lambda^\beta \mu^\alpha \mathbf{J}_{\alpha,\beta} \tag{5}$$

Note, $\mathbf{J}$ depends on x, y via λ, μ, see (4), and besides $\mathbf{J}_{0,0} = 1$. Let

$$\mathbf{u} = \prod_{\alpha\in\mathbb{Z}_A} \sigma_{1:\alpha,0} , \quad \mathbf{v} = \prod_{\beta\in\mathbb{Z}_B} \sigma_{2:0,\beta} . \tag{6}$$

It is convenient to change a bit the normalization of the operator-valued coefficients $\mathbf{J}_{\alpha,\beta}$ as follows:

$$\mathbf{J}_{\alpha,\beta} = i^{\alpha\beta}\mathbf{u}^\beta\mathbf{v}^\alpha\mathbf{t}_{\alpha,\beta} \tag{7}$$

Proposition 1. *The set of $\mathbf{t}_{\alpha,\beta}$ is the commutative one, also $[\mathbf{u},\mathbf{t}_{\alpha,\beta}] = [\mathbf{v},\mathbf{t}_{\alpha,\beta}] = 0$. Moreover, the independent subset of $\mathbf{t}_{\alpha,\beta}$ and any combination of $\mathbf{u},\mathbf{v}$ is the complete set of commutative matrices in $\mathcal{A}$.*

Proof: see [6]. The case considered is the case of the local Weyl algebra with $N = 2$ and $q = -1$.

Note a couple of the simple relations:

$$\mathbf{J}(\lambda,\mu) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{J}(\lambda,-\mu) , \quad \mathbf{J}(\lambda,\mu) \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{J}(-\lambda,\mu) . \tag{8}$$

Remark 1. *If the parameters $a_{\alpha,\beta}$, $b_{\alpha,\beta}$, $c_{\alpha,\beta}$ are real, then all $\mathbf{t}_{\alpha,\beta}$ are hermitian and so are the Hamiltonians. Thus it is formulated is the real physical model.*

The Hamiltonians $\mathbf{t}_{\alpha,\beta}$ have a bit cumbersome combinatorial structure for $0 < \alpha < A$, $0 < \beta < B$, so that one can not extract any smart combinations of local $\sigma_{j:\alpha,\beta}$ resembling e.g. simple Hamiltonians of one dimensional spin chains.

2.2. **Functional equation.** Together with the matrix $\mathbf{L}$ with the operator-valued entries (3) consider the $\mathbf{C}$-valued matrix L:

$$\begin{aligned} &L_{\alpha,\beta|\alpha,\beta} = 1 , && L_{\alpha+1,\beta+1|\alpha,\beta} = -x^2y^2\, c^2_{\alpha,\beta} , \\ &L_{\alpha+1,\beta|\alpha,\beta} = -x^2\, a^2_{\alpha,\beta} , && L_{\alpha,\beta+1|\alpha,\beta} = -y^2\, b^2_{\alpha,\beta} . \end{aligned} \tag{9}$$

As previously, the determinant of L is in use:

$$\det ||L|| = J_f(\lambda^2,\mu^2) \tag{10}$$

Proposition 2. *Abelian algebra of* $\mathbf{t}_{\alpha,\beta}$ *is generated by the decomposition of*

$$\det_{\mathbf{u},\mathbf{v}} ||\mathbf{J}(\lambda,\mu)|| \;=\; J_f(\lambda^2,\mu^2)\;, \tag{11}$$

with respect to λ^2,μ^2. *In (11)* $\det\limits_{\mathbf{u},\mathbf{v}}$ *is the determinant of* $\mathbf{J}$ *as* 2×2 *matrix in the block of* $\mathbf{u},\mathbf{v}$. *From the other hand side, this is the functional equation for the spectrum of* $\mathbf{t}_{\alpha,\beta}$.

Proof: see [6].

Remark 2. $\mathbf{J}(\lambda,\mu)$ *is the generating function for the integrals of motion for a three dimensional lattice integrable model. With the help of three dimensional vertex weights one may construct the layer-to-layer transfer degenerative matrix* $\mathbf{Q}_0(\lambda,\mu)$, *such that*

$$\begin{aligned}
&\mathbf{Q}_0(\lambda,\mu)\,\cdot\,\mathbf{t}_{\alpha,\beta} \;=\; \mathbf{t}_{\alpha,\beta}\,\cdot\,\mathbf{Q}_0(\lambda,\mu)\quad \forall\,\alpha,\beta\;,\\
&\mathbf{Q}_0(\lambda,\mu)\,\cdot\,\mathbf{u} \;=\; \mathbf{u}\,\cdot\,\mathbf{Q}_0(\lambda,-\mu)\;,\quad \mathbf{Q}_0(\lambda,\mu)\,\cdot\,\mathbf{v} \;=\; \mathbf{v}\,\cdot\,\mathbf{Q}_0(-\lambda,\mu)
\end{aligned} \tag{12}$$

and the usual layer-to-layer transfer matrix is

$$\mathbf{Q}(\lambda^2,\mu^2) = \mathbf{Q}_0(\lambda,\mu)+\mathbf{Q}_0(\lambda,-\mu)+\mathbf{Q}_0(-\lambda,-\mu)+\mathbf{Q}_0(-\lambda,-\mu)\;. \tag{13}$$

The most important feature of $\mathbf{Q}_0$ *is that it obeys the Baxter equation:*

$$\mathbf{J}(\lambda,\mu)\,\mathbf{Q}_0(\lambda,\mu) \;=\; 0\;, \tag{14}$$

so that λ,μ *are not arbitrary complex numbers but belong to the "quantum spectral curve"* $J_f(\lambda^2,\mu^2) \;=\; 0$. *The matrix* $\mathbf{Q}_0$ *is proportional to the matrix of algebraic supplement of the degenerative* $\mathbf{J}$. *Note, in three dimensional model* x *and* y *(and so* λ *and* μ*) are complex in general, and so the physical interpretation is in question. The reader may find the details in* [10].

Since $\mathbf{u}^2=\mathbf{v}^2=1$, we may decompose

$$\mathbf{J}(\lambda,\mu) \;=\; T_{0,0}(\lambda^2,\mu^2)+\lambda\mathbf{u}T_{1,0}(\lambda^2,\mu^2)+\mu\mathbf{v}T_{0,1}(\lambda^2,\mu^2)+i\lambda\mu\mathbf{u}\mathbf{v}T_{1,1}(\lambda^2,\mu^2), \tag{15}$$

so that

$$T(\lambda,\mu) \;=\; T_{0,0}(\lambda^2,\mu^2)+\lambda T_{1,0}(\lambda^2,\mu^2)+\mu T_{0,1}(\lambda^2,\mu^2)+i\lambda\mu T_{1,1}(\lambda^2,\mu^2) \tag{16}$$

generates the commutative set of integrals of motion. In particular, $\mathbf{J}(\lambda,\mu) = T(\lambda\mathbf{u},\mu\mathbf{v})$ up to the ordering, and

$$\begin{aligned}\det_{\mathbf{u},\mathbf{v}} ||\mathbf{J}|| &= T_{0,0}^2 - \lambda^2 T_{1,0} - \mu^2 T_{0,1} - \lambda^2\mu T_{1,1} \\ &= \frac{1}{2}\left(T(\lambda,\mu)T(-\lambda,-\mu) + T(-\lambda,\mu)T(\lambda,-\mu)\right) .\end{aligned} \tag{17}$$

2.3. **Thermodynamic limit.** As usual, one may understand the thermodynamic limit as the asymptotic when $A, B \mapsto \infty$.

Conjecture 1. *If in the thermodynamic limit asymptotically*

$$J_f \mapsto e^{AB\,\mathfrak{f}} , \quad \mathfrak{f} \in \mathbb{R} , \mathfrak{f} > 0 , \tag{18}$$

then the maximal eigenvalues of all

$$|T(\pm\lambda,\pm\mu)|, \quad |T_{0,0}|, \quad |T_{1,0}|, \quad |T_{0,1}|, \quad |T_{1,1}| \;\mapsto\; e^{AB\,\mathfrak{f}/2} . \tag{19}$$

Of course, the states where $T_{k,l}$ take their maximal eigenvalues, have to be different.

(19) seems to be valid even if $J_f(\lambda^2,\mu^2) = 0$ for finite A, B. In this case $\mathfrak{f}$ may still be defined. As the result, the matrix elements of $\mathbf{Q}_0(\lambda,\mu)$ in remark 2 on page 5 should be also proportional to $\mathrm{e}^{AB\,\mathfrak{f}/2}$.

Conjecture 2. *The asymptotic $\mathfrak{f}$ of the case $J_f(\lambda^2,\mu^2) = 0$ contains the principal transcendental expression of a free energy of three dimensional lattice integrable models.*

Below we will argue this conjecture on the example of the Zamolodchikov - Bazhanov - Baxter model.

Conjecture 3. *The analytical structure of the $\mathfrak{f}$ reflects the phase structure of the set of $\mathbf{t}_{\alpha,\beta}$ in the thermodynamic limit.*

It is interesting to note that for the well known two dimensional lattice-statistical model know as the free fermion model the partition function may also be presented as a determinant (see e.g. [11]). The structure of our determinant J_f is a bit different. The principle difference is that free fermion determinants can not be zero for finite volume AB, while in our case $J_f(\lambda^2,\mu^2) = 0$ is the most interesting class.

2.4. **The change of notations.** Below we will investigate the asymptotic value of J_f. For the sake of the shortness, we will omit the square signs, choosing the following final notations:

$$\begin{array}{ll} L_{\alpha,\beta|\alpha,\beta} \;=\; 1\;, & L_{\alpha+1,\beta+1|\alpha,\beta} \;=\; -xy\,c_{\alpha,\beta}\;, \\ L_{\alpha+1,\beta|\alpha,\beta} \;=\; -x\,a_{\alpha,\beta}\;, & L_{\alpha,\beta+1|\alpha,\beta} \;=\; -y\,b_{\alpha,\beta}\;, \end{array} \tag{20}$$

$\lambda = x^A$, $\mu = y^B$, $J_f(\lambda,\mu) = \det||L||$, and

$$\mathfrak{f} \;=\; \lim_{A,B\mapsto\infty} \frac{1}{AB}\log\det||L||\;. \tag{21}$$

3. Homogeneous case

Let us start the investigation of J_f and $\mathfrak{f}$ from the simplest homogeneous case:

$$a_{\alpha,\beta} \;=\; 1\;,\;\; b_{\alpha,\beta} \;=\; 1\;,\;\; c_{\alpha,\beta} \;=\; c\;. \tag{22}$$

Then with the help of Fourier transformation one calculates

$$J_f(\lambda,\mu) \;=\; \prod_{\alpha=0}^{A-1}\prod_{\beta=0}^{B-1}\left(1 \;-\; x\mathrm{e}^{2\pi i\alpha/A} \;-\; y\mathrm{e}^{2\pi i\beta/B} \;-\; xyc\mathrm{e}^{2\pi i(\alpha/A+\beta/B)}\right) \tag{23}$$

where $x^A = \lambda$ and $y^B = \mu$. Let

$$\chi_1(x,y) \;=\; 1 \;-\; x \;-\; y \;-\; xyc\;. \tag{24}$$

Then in the thermodynamic limit

$$\mathfrak{f} \;=\; \frac{1}{(2\pi)^2}\int\!\!\int_0^{2\pi} d\phi\, d\phi'\, \log|\chi_1(x\mathrm{e}^{i\phi}, y\mathrm{e}^{i\phi'})|\;. \tag{25}$$

Note at the first, $\mathfrak{f}$ depends on $|x|$ and $|y|$. c for the physical regime should be real positive [3] Condition $J_f(\lambda,\mu) = 0$ corresponds to the case when one factor in the product (23) is zero, i.e. when $\chi_1(x,y) = 0$ in one or two points on the unit circles $x = |x|\mathrm{e}^{i\phi}$, $y = |y|\mathrm{e}^{i\phi'}$. Since the logarithmic singularity is integrable, (25) is nevertheless valid.

In the regime of two zeros of $\chi_1(x,y)$ one may parameterize without lost of generality

$$|x| \;=\; \frac{\sin\beta_2}{\sin\beta_1}\;,\quad |y| \;=\; \frac{\sin\beta_3}{\sin\beta_1}\;,\quad c \;=\; \frac{\sin\beta_0\sin\beta_1}{\sin\beta_2\sin\beta_3} \tag{26}$$

[3] In general, $c\in\mathbb{R}$ is the necessary condition of $\mathfrak{f}\in\mathbb{R}$. $c>0$ comes from remark 1 on page 4 up to the change of notation, eq. (20).

where by the definition

$$\beta_0 \;=\; \pi-\beta_1-\beta_2-\beta_3\;. \tag{27}$$

Zeros of χ_1 are

$$\chi_1\left(\frac{1-\mathrm{e}^{-2i\beta_2}}{1-\mathrm{e}^{2i\beta_1}},\frac{1-\mathrm{e}^{2i\beta_3}}{1-\mathrm{e}^{-2i\beta_1}}\right) \;=\; \chi_1\left(\frac{1-\mathrm{e}^{2i\beta_2}}{1-\mathrm{e}^{-2i\beta_1}},\frac{1-\mathrm{e}^{-2i\beta_3}}{1-\mathrm{e}^{2i\beta_1}}\right) \;=\; 0\;. \tag{28}$$

Due to the evident symmetries

$$\mathfrak{f}(\beta_1,\beta_2,\beta_3)=\mathfrak{f}(\pi+\beta_1,\beta_2,\beta_3)=\mathfrak{f}(\beta_1,\pi+\beta_2,\beta_3)=\mathfrak{f}(\beta_1,\beta_2,\pi+\beta_3)=\mathfrak{f}(-\beta_1,-\beta_2,-\beta_3) \tag{29}$$

any set of β_1,β_2,β_3 may be reduced to the canonical one:

$$0\le\beta_1+\beta_2<\pi\;,\quad 0\le\beta_1+\beta_3<\pi\;,\quad 0\le\beta_2+\beta_3<\pi \tag{30}$$

In the regime (30)

$$\mathfrak{f} \;=\; \mathfrak{I}(\beta_0)+\mathfrak{I}(\beta_1)+\mathfrak{I}(\beta_2)+\mathfrak{I}(\beta_3)-\log|2\sin\beta_1|\;. \tag{31}$$

where $\mathfrak{I}$ is the kind of the dilogarithm:

$$\mathfrak{I}(\beta)\;\stackrel{\mathrm{def}}{=}\;\frac{1}{\pi}\int_0^\beta \alpha\cot\alpha\,d\alpha\;,\quad -\pi\;<\;\beta\;<\;\pi\;, \tag{32}$$

or equivalently

$$\mathfrak{I}(\beta)\;=\;\frac{\beta}{\pi}\log|2\sin\beta|\;+\;\sum_{m=1}^{\infty}\frac{\sin 2m\beta}{2\pi m^2}\;, \tag{33}$$

with the main useful properties

$$\mathfrak{I}(\beta)+\mathfrak{I}(-\beta)\;=\;0\;,\quad \mathfrak{I}(\beta)+\mathfrak{I}(\pi-\beta)\;=\;\log|2\sin\beta|\;. \tag{34}$$

The answer includes the physical as well as non-physical regimes for c.

The three dimensional lattice integrable model, corresponding to the homogeneous coefficients (22), is Zamolodchikov - Bazhanov - Baxter's one [1, 2, 3]. Parameters β_j are the linear excesses of the spherical triangle in Baxter's parameterization [2], and the expression (31) is very close to Baxter's formula for the partition function per site

$$\log z\;=\;\sum_{j=0}^{3}\left(\mathfrak{I}(\beta_j)-\frac{\beta_j}{2\pi}\log(2\sin\beta_j)\right) \tag{35}$$

in the "geometrical" regime $0\le\beta_j<\pi$, $j=0,1,2,3$, see [2]. This observation argues the conjecture 2 on page 6.

The regime when $\chi_1(x,y)=0$ has no solution on unit circles (or only one, e.g. $\chi_1(1,1)=0$), may be parameterized by

$$x\;=\;\frac{\sinh\beta_2}{\sinh\beta_1}\;,\quad y\;=\;\frac{\sinh\beta_3}{\sinh\beta_1}\;,\quad xyc\;=\;\frac{\sinh(\beta_1+\beta_2+\beta_3)}{\sinh\beta_1}\tag{36}$$

with $\beta_j\in\mathbb{R}\mod i\pi$. This parameterization is not useful, because now

$$\mathfrak{f}\;=\;\log\max(1,|x|,|y|,|xyc|)\;.\tag{37}$$

Remark 3. *$\mathfrak{f}$, given by (37), is not smooth, but the reason is trivial. For finite A,B and $\lambda=x^A$, $\mu=y^B$ the set of integer pairs (α,β) of the polynomial $J_f=\sum_{\alpha,\beta}\lambda^\beta\mu^\alpha J_{\alpha,\beta}$ is called the Newton polygon. For instance, the corners of it are given by*

$$J_f(\lambda,\mu)\;=\;1\,+\,\ldots\,+\,(-)^Bx^{AB}\,+\,\ldots\,(-)^Ay^{AB}\,+\,(-)^{A+B}(xyc)^{AB}\tag{38}$$

Then (37) just describes the regimes when one corner of Newton's polygon dominates.

Since the homogeneous model is related to the Zamolodchikov model, which is critical [5], one has no critical features of $\mathfrak{f}$ (31,37).

4. Chess model

The next to the homogeneous is the case of "2-chess model". The lattice of the parameters $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ and $c_{\alpha,\beta}$ has the period 2 in both directions. Of course, A and B should be even. Corresponding three dimensional spin model was discovered in [7], its integrability is based on the modified tetrahedron equation.

Let

$$\chi_4(x^2,y^2)\;=\;\det\begin{vmatrix}1&-xa_{1,0}&-yb_{0,1}&-xyc_{1,1}\\-xa_{0,0}&1&-xyc_{0,1}&-yb_{1,1}\\-yb_{0,0}&-xyc_{1,0}&1&-xa_{1,1}\\-xyc_{0,0}&-yb_{1,0}&-xa_{0,1}&1\end{vmatrix}\;.\tag{39}$$

In comparison to homogeneous case [4], if $a_{\alpha,\beta}=a$, $b_{\alpha,\beta}=b$, $c_{\alpha,\beta}=c$ then $\chi_4(x^2,y^2)=(1-ax-by+cxy)(1+ax-by-cxy)(1-ax+by-cxy)(1+ax+by+cxy)$. The Fourier

[4] $\chi_4(x^2,y^2)$ may be decomposed completely for a bit more general set of $a_{\alpha,\beta}$, $b_{\alpha,\beta}$, $c_{\alpha,\beta}$: this corresponds to "inhomogeneous Zamolodchikov model" and in any case is related to the previous consideration.

transformation now gives

$$J_f(\lambda,\mu) \;=\; \prod_{\alpha=0}^{A/2-1}\prod_{\beta=0}^{B/2-1} \chi_4(x^2 \mathrm{e}^{4\pi i\alpha/A}, y^2\mathrm{e}^{4\pi i\beta/B})\;, \tag{40}$$

where $\lambda = x^A$, $\mu = y^B$, and the leading term of J_f in the thermodynamic limit is

$$\mathfrak{f} \;=\; \frac{1}{4(2\pi)^2}\int\!\!\int_0^{2\pi} d\phi\, d\phi' \,\log|\chi_4(x^2\mathrm{e}^{i\phi}, y^2\mathrm{e}^{i\phi'})|\;. \tag{41}$$

For the simplicity we will consider the so-called "two-color chess model", i.e. the case when

$$\begin{aligned} &a_{1,1} = a_{0,0} = a_0\;,\quad b_{1,1} = b_{0,0} = b_0\;,\quad c_{1,1} = c_{0,0} = c_0\;,\\ &a_{1,0} = a_{0,1} = a_1\;,\quad b_{0,1} = b_{1,0} = b_1\;,\quad c_{1,0} = c_{0,1} = c_1\;. \end{aligned} \tag{42}$$

Now χ_4 may be decomposed to two irreducible multipliers:

$$\chi_4(x^2,y^2) \;=\; \chi_2'(x,y)\;\chi_2'(-x,y)\;, \tag{43}$$

where

$$\chi_2'(x,y) \;=\; 1 \;-\; a_0a_1x^2 \;-\; b_0b_1y^2 \;+\; c_0c_1x^2y^2 \;-\; (c_0+c_1+a_0b_1+a_1b_0)xy\;. \tag{44}$$

Without loss of generality we may re-scale the parameters

$$a_0a_1x^2 \;=\; \frac{u^2}{w^2}\;,\quad b_0b_1y^2 \;=\; \frac{v^2}{w^2}\;,\quad c_0c_1x^2y^2 \;=\; u^2v^2\;, \tag{45}$$

and define

$$\chi_2(u,v) \;=\; 1 \;-\; \frac{u^2}{w^2} \;-\; \frac{v^2}{w^2} \;+\; u^2\,v^2 \;-\; 4\,h\,\frac{u\,v}{w}\;, \tag{46}$$

where we suppose $w\in\mathbb{R}$, $w>0$, and put

$$k^{-2} \;=\; h^2 - \left(\frac{w-w^{-1}}{2}\right)^2\;,\quad h \;=\; \sqrt{k^{-2} + \left(\frac{w-w^{-1}}{2}\right)^2}\;. \tag{47}$$

We will use the short notation $\chi_2(u,v)$ as well as $\chi_2(u,v;w,k)$ or $\chi_2(u,v;w,h)$. The homogeneous case corresponds to $k=1$, correspondingly $h=\dfrac{w+w^{-1}}{2}$ and

$$\chi_2(u,v;w,k=1) = \frac{(w-u-v-uvw)(w+u+v-uvw)}{w^2}\;. \tag{48}$$

Now

$$\mathfrak{f} \;=\; \frac{1}{2(2\pi)^2}\int\!\!\int_0^{2\pi} d\phi\, d\phi'\;\log|\chi_2(u\mathrm{e}^{i\phi}, v\mathrm{e}^{i\phi'})|\;. \tag{49}$$

Since the value of $\mathfrak{f}$ depends on $|u|$ and $|v|$, we may always regard all u, v, w, h as real positive numbers. Several symmetry properties of $\mathfrak{f}$ may be easily established. Using e.g. $\chi_2(u,v,w,k) = \chi_2(v,u,w,k)$ or $\chi_2(u,v,w,k) = -\dfrac{u^2}{w^2}\,\chi_2(u^{-1},v,w^{-1},k)$, one may obtain

$$\mathfrak{f}(u,v,w,k) \;=\; \frac{1}{2}\log\frac{u\,v}{w} \;+\; \mathfrak{f}_0(u,v,w,k)\;, \tag{50}$$

where

$$\mathfrak{f}_0(u,v,w,k) = \mathfrak{f}_0(v,u,w,k) = \mathfrak{f}(u^{-1},v,w^{-1},k) = \mathfrak{f}(u^{-1},v^{-1},w,k)\;. \tag{51}$$

To integrate (49) with respect to ϕ', the decomposition is useful:

$$\chi_2(u,v) \;=\; \left(1 \;-\; \frac{u^2}{w^2}\right)\,(1\;-\;\lambda_+(u,v))\,(1\;-\;\lambda_-(u,v))\;, \tag{52}$$

where

$$\lambda_\pm(u,v) \;=\; \frac{2\dfrac{uv}{w}}{1\;-\;\dfrac{u^2}{w^2}}\;\left(\sqrt{k^{-2}+\left(\frac{w-w^{-1}}{2}\right)^2}\;\pm\;\sqrt{k^{-2}+\left(\frac{u-u^{-1}}{2}\right)^2}\right)\;. \tag{53}$$

Let $\phi\in\Sigma_\pm \;\;\Leftrightarrow\;\; |\lambda_\pm(u\mathrm{e}^{i\phi},v)|>1$, then

$$\begin{aligned}\mathfrak{f} \;&=\; \frac{1}{4\pi}\int_0^{2\pi} d\phi\;\log\left|1-\frac{u^2}{w^2}\mathrm{e}^{2i\phi}\right|\\ &+\frac{1}{4\pi}\int_{\phi\in\Sigma_+} d\phi\;\log|\lambda_+(u\mathrm{e}^{i\phi},v)| \;+\; \frac{1}{4\pi}\int_{\phi\in\Sigma_-} d\phi\;\log|\lambda_-(u\mathrm{e}^{i\phi},v)|\;.\end{aligned} \tag{54}$$

4.1. **Integration for $u=v=1$.** For the test of the phase transition we choose at the first $u=v=1$. In advance, in this case the opposite corners of the Newton polygon are dominant (see the remark 3 on page 9), and any change of $\mathfrak{f}$ should be related to the change of the ground state of the integrals of motion from the middle of the Newton polygon of J_f.

Three regimes appear when $u=v=1$: The first one, $0<k\le 1$, i.e. $h\ge\dfrac{w+w^{-1}}{2}$. Then

$$\mathfrak{f}_0(w,k) = \mathfrak{T}(w,k) \tag{55}$$

where

$$\mathfrak{T}(w,k) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_0^{\pi/2} d\phi\, \log\left(2\sqrt{k^{-2}+\left(\frac{w-w^{-1}}{2}\right)^2} \;+\; 2\sqrt{k^{-2}-\sin^2\phi}\right)\;. \tag{56}$$

In particular, when $k=1$ and $w=\tan\beta_1$, $0\leq\beta_1\leq\pi/2$,

$$\begin{aligned}\mathfrak{T}(w=\tan\beta_1,k=1) &= \frac{2}{\pi}\,\text{Catalan} + \frac{1}{\pi}\int_0^{\pi/2-2\beta_1}\frac{\alpha\,d\alpha}{\cos\alpha}\\ &= 2\mathfrak{I}(\beta_1)\;+\;2\mathfrak{I}(\pi/2-\beta_1)\;-\;\frac{1}{2}\log(2\sin 2\beta_1)\;,\end{aligned} \tag{57}$$

where $\mathfrak{I}$ is defined by (32), and Catalan $\stackrel{\text{def}}{=} \displaystyle\sum_{j=0}^{\infty}\frac{(-)^j}{(2j+1)^2} \sim 0.9159655942$. In this case $\mathfrak{f}$ coincides with (31) with $\beta_0=\beta_1$ and $\beta_2=\beta_2=\pi/2-\beta_1$. So in some sense (56) is an elliptic continuation of the dilogarithm function. The derivative of (55) (see (47) for the relations between h,k,w)

$$\frac{\partial\mathfrak{f}_0(w,k)}{\partial k^{-1}} = \frac{h^{-1}}{\pi}\,K(k)\;,\quad\text{where}\quad K(k)\stackrel{\text{def}}{=}\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{58}$$

has the logarithmic singularity when $k\mapsto 1$.

The second regime is $0\leq k^{-1}\leq 1$, i.e. $\dfrac{|w-w^{-1}|}{2}\leq h\leq\dfrac{w+w^{-1}}{2}$. Then

$$\mathfrak{f}_0(w,k) = \frac{1}{2}\,|\log w|\;+\;\frac{1}{\pi}\int_0^{\arcsin(k^{-1})} d\phi\,\log\left(\frac{h\;+\;\sqrt{k^{-2}-\sin^2\phi}}{\sqrt{\left(\frac{w-w^{-1}}{2}\right)^2+\sin^2\phi}}\right)\;, \tag{59}$$

where $0\leq\arcsin(k^{-1})\leq\pi/2$, and the derivative

$$\frac{\partial\mathfrak{f}_0(w,k)}{\partial k^{-1}} = \frac{k^{-1}h^{-1}}{\pi}\,K(k^{-1})\;. \tag{60}$$

The third regime is $-\left(\dfrac{w-w^{-1}}{2}\right)^2\leq -k^{-2}\leq 0$, i.e. $0\leq h\leq\dfrac{|w-w^{-1}|}{2}$. Then

$$\mathfrak{f}(w,k) = \frac{1}{2}\,|\log w|\;. \tag{61}$$

4.2. **Integration for** $u, v \neq 1$**.** The situation with $u, v \neq 1$ is a bit different and more complicated. Let $u, v, w > 0$ and

$$h_+ \;=\; \frac{w^2+u^2+v^2+u^2v^2w^2}{4uvw}\;,\quad h_- \;=\; \frac{w^2-u^2-v^2+u^2v^2w^2}{4uvw}\;. \tag{62}$$

Due to the symmetries (51) we may consider only the cases when

$$v \;\le\; u \;\le\; 1\;. \tag{63}$$

If $h_+ \le h$, then $\mathfrak{f}_0(u,v,w,k) = \mathfrak{T}(w,k)$ and so coincides with (55). The phase transition point is $h = h_+ > (w+w^{-1})/2$, i.e. $k_{cr}^{-2} \ge 1$.

The region $|h_-| \le h \le h_+$ with the restriction (63) provides $\Sigma_+ = (-\alpha,\alpha)\cup(\pi-\alpha,\pi+\alpha)$ and $\Sigma_- = \emptyset$ in the formal expression (54). α here is the solution of

$$|\lambda_+(u\mathrm{e}^{i\alpha},v)| \;=\; 1\;,\quad 0\le\alpha\le\frac{\pi}{2}\;. \tag{64}$$

Formally, the expression for $\mathfrak{f}$ is the following:

$$\mathfrak{f} \;=\; \frac{1}{4\pi}\int_0^{2\pi} d\phi\,\log\left|1 - \frac{u^2}{w^2}\mathrm{e}^{2i\phi}\right| \;+\; \frac{1}{2\pi}\int_{-\alpha}^{\alpha} d\phi\,\log|\lambda_+(u\mathrm{e}^{i\phi},v)|\;. \tag{65}$$

(64) may be rewritten as a quadratic equation for $\cos 2\alpha$, so that only one solution obeys $|\cos 2\alpha| \le 1$. When $k=1$, this equation is reduced to the cosine theorem for the spherical triangle, and the parameterization like (26) arises naturally. When $k\neq 1$, this quadratic equation is irreducible, and we do not succeed in its appropriate parameterization in the terms of u, v, w, h But one may show, $\partial^2\mathfrak{f}_0(u,v,w,k)/\partial k^{-2}$ is singular when $h\mapsto h_+ - 0$.

To write out $\mathfrak{f}_0$ in this region, it is necessary to introduce the elliptic parameterization. Let $k' = \sqrt{1-k^2}$ and $K(k)$ is given by (58). For the notations of Jacobi functions and periods see [12]. For $k^2>0$ introduce

$$\tau(x) \;\stackrel{\mathrm{def}}{=}\; \frac{k\,\mathrm{sn}(x,k')}{\mathrm{dn}(x,k')\,+\,\mathrm{cn}(x,k')}\;,\quad \frac{\tau'(x)}{\tau(x)} \;=\; \sqrt{1\,+\,k^2\left(\frac{\tau(x)-\tau(x)^{-1}}{2}\right)^2} \;=\; \frac{1}{\mathrm{sn}(x,k')}\;. \tag{66}$$

The general parameterization of the elliptic curve

$$\chi_2(u\mathrm{e}^{i\alpha}, v\mathrm{e}^{-i\alpha'}, w, k) \;=\; 0 \tag{67}$$

(equivalently, $|\lambda_+(u\mathrm{e}^{i\alpha},v)| = 1$ for $v<u<1$ and $k^2>1$) is given by

$$u\mathrm{e}^{i\alpha} \;=\; \tau(a+i\gamma)\;,\quad v\mathrm{e}^{-i\alpha'} \;=\; \tau(b-i\gamma)\;,\quad w \;=\; \tau(a+b)\;, \tag{68}$$

so that $u = |\tau(a+i\gamma)|$, $v = |\tau(b-i\gamma)|$. The conditions $v < u < 1$, $w > 0$ correspond to

$$0 < a+b < 2K(k')\;,\quad a > |b|\;,\quad 0 < a < K(k')\;,\quad 0 < \gamma \leq \Re(K(k))\;. \tag{69}$$

Now the integration may be done in the terms of a, b, γ as follows (recall, $\mathfrak{f}_0 = \mathfrak{f} - \frac{1}{2}\log\frac{uv}{w}$):

$$\begin{aligned} \mathfrak{f}_0 \;&=\; \frac{1}{2\pi}\int_{\alpha}^{\pi-\alpha} d\phi\;\log\left|1 \;-\; \frac{u^2}{w^2}\mathrm{e}^{2i\phi}\right| \;+\; \left(\frac{\alpha}{\pi}-\frac{1}{2}\right)\log\frac{uv}{w} \\ &+\; \frac{1}{2\pi i}\int_{a-i\gamma}^{a+i\gamma}\frac{dz}{\mathrm{sn}(z,k')}\;\log\left(2k^{-1}\mathrm{ns}(a+b,k') \;+\; 2k^{-1}\mathrm{ns}(z,k')\right)\;. \end{aligned} \tag{70}$$

This expression becomes $\mathfrak{T}(w,k)$ when $\gamma = K(k)$ and $0 < k < 1 \Leftrightarrow \alpha = \pi/2$ and $h = h_+$.

Of course, (70) becomes (31) when $k = 1$. In details, at $k = 1$ the integration in (70) may be done in the terms of dilogarithm, and in order to get (31), the Roger pentagonal identity is to be applied twice.

When $k^2 < 1$, $k/i > 0$, another parameterization is useful: let

$$\sigma(x) \;=\; \tau(-ix) = \frac{k}{i}\,\frac{\mathrm{sn}(x,k)}{1\;+\;\mathrm{dn}(x,k)}\;. \tag{71}$$

$$u\mathrm{e}^{i\alpha} = \sigma(a+i\gamma)\;,\quad v\mathrm{e}^{-i\alpha'} = \sigma(b-i\gamma)\;,\quad w = \sigma(a+b)\;, \tag{72}$$

where $0 \leq a < K(k)$, $a > |b|$, $0 \leq a+b < 2K(k)$ and $0 \leq \gamma \leq \gamma_{cr}(a,b,k)$. Expression for $\mathfrak{f}_0$ is similar to (70):

$$\begin{aligned} \mathfrak{f}_0 \;&=\; \frac{1}{2\pi}\int_{\alpha}^{\pi-\alpha} d\phi\;\log\left|1 \;-\; \frac{u^2}{w^2}\mathrm{e}^{2i\phi}\right| \;+\; \left(\frac{\alpha}{\pi}-\frac{1}{2}\right)\log\frac{uv}{w} \\ &+\; \frac{1}{2\pi i}\int_{a-i\gamma}^{a+i\gamma}\frac{dz}{\mathrm{sc}(z,k)}\;\log\left(2ik^{-1}\mathrm{cs}(a+b,k) \;+\; 2ik^{-1}\mathrm{cs}(z,k)\right)\;. \end{aligned} \tag{73}$$

For all k the limit $\gamma \mapsto 0$ (i.e.$\alpha \mapsto 0$) corresponds $h \mapsto h_- > 0$. Below h_- the dependence on h disappears, $\mathfrak{f}_0(u,v,w,h) = \mathfrak{f}_0(u,v,w,h_-)$. Namely (recall, $v < u < 1$),

$$0 \;<\; h \;<\; h_- \qquad \begin{aligned} u \leq w \quad&\Rightarrow\quad \mathfrak{f}_0(u,v,w,h) = \frac{1}{2}\,\log\frac{w}{uv}\;, \\ w \leq u \quad&\Rightarrow\quad \mathfrak{f}_0(u,v,w,h) = \frac{1}{2}\,\log\frac{u}{vw}\;. \end{aligned} \tag{74}$$

The special case is $h_- < 0$ and $h < |h_-|$. The limit $h \mapsto |h_0|$ corresponds to $k^2 < 0$ and $\gamma = \gamma_{cr}(a,b,k)$. We do not succeed in expression of γ_{cr}. When $h_- < 0$ and $h < |h_-|$, both solutions of the quadratic equation for $\cos(2\alpha)$, mentioned right after eq. (64), are valid.

Then Σ_- is not empty, and we do not succeed in any reasonable parameterization of this regime.

5. Discussion

We have observed the evidences for a nontrivial phase structure of the two-colors chess spin lattice parameterized in the terms of elliptic integrals. Expressions (70) and (73) show the importance of the parameterization of the genus 1 algebraic curve $\chi_2(u,v)=0$.

The chess model is the simplest representative of the class of M-chess models. For M-chess model the structure of $M^2\times M^2$ determinant like (39) implies an uniformization in the terms of theta-functions on the Jacobian of $g=(M-1)^2$ compact Riemann surface. g moduli play the roles of a temperature-like parameters, see [8]. So we expect a rather rich phase structure of the higher spin lattices.

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