EMMY NOETHER AND TOPOLOGY

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My knowledge about the role of Emmy Noether in Algebraic Topology stems from my encounters with Paul Alexandroff and Heinz Hopf. I studied in Zürich at the Eidgenössische Technische Hochschule for three terms, from the summer term 1949 to the summer term 1950. Heinz Hopf and Beno Eckmann were my teachers. For the first time I attended courses on Algebraic Topology and its applications in other fields. I also learnt some facts about the famous book Topologie I by Alexandroff and Hopf [A-H] and how the cooperation of Alexandroff and Hopf started.

In May 1925 Hopf received his doctor's degree at the University of Berlin under Erhard Schmidt. He spent the Academic Year 1925/26 at the University of Göttingen where Alexandroff and Hopf met for the first time in the summer term 1926. I quote from Alexandroff [A]:

"Die Bekanntschaft zwischen Hopf und mir wurde im selben Sommer zu einer engen Freundschaft. Wir gehörten beide zum Mathematiker-Kreis um Courant und Emmy Noether, zu dieser unvergeßlichen menschlichen Gemeinschaft mit ihren Musikabenden und ihren Bootsfahrten bei und mit Courant, mit ihren "algebraisch-topologischen" Spaziergängen unter der Führung von Emmy Noether und nicht zuletzt mit ihren verschiedenen Badepartien und Badeunterhaltungen, die sich in der Universitäts-Badeanstalt an der Leine abspielten.
...
Die Kliesche Schwimmanstalt war nicht nur ein Studentenbad, sie wurde auch von vielen Universitätsdozenten besucht, darunter von Hilbert, Courant, Emmy Noether, Prandtl, Friedrichs, Deuring, Hans Lewy, Neugebauer und vielen anderen. Von auswärtigen Mathematikern seien etwa Jakob Nielsen, Harald Bohr, van der Waerden, von Neumann, André Weil als Klies ständige Badegäste erwähnt."

Here one sees Göttingen as a mathematical world center. As Alexandroff points out in [A] two Göttingen schools of this time were especially prominent and active, one headed by Courant (Applied Mathematics) and one by Emmy Noether (Modern abstract algebra) both closely related to Hilbert. Alexandroff was in Göttingen during all three summers 1926-28, Hopf during the summers of 1926 and 1928 and part of 1927. Emmy Noether attended their lectures. As Hopf writes in [H4] they learnt from her "die Begründung der Homologietheorie in simplizialen Komplexen - nämlich: (continued in English)

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Let X^r be the group of r-dimensional chains. Let ∂ denote the homomorphism $X^{r+1} \to X^r$ given by taking the boundary of a simplex, then $\partial\partial = 0$ as can be verified for a single simplex. This means: the image ∂X^{r+1} is contained in the kernel Z^r of the map $\partial : X^r \to X^{r-1}$. The quotient group $H^r = Z^r/\partial X^{r+1}$ is the r-th homology group."

Nowadays we speak of a chain complex $\{X^r\}$ and consider the exact sequences

$$\begin{aligned} &0 \to Z^r \to X^r \to \partial X^r \to 0 \\ &0 \to \partial X^{r+1} \to Z^r \to H^r \to 0 \end{aligned} \tag{1}$$

Assuming that for each r there are only finitely many simplices, namely α_r, then all ranks of the abelian groups occuring in (1) are finite. The rank of H^r is the r-th Betti number b_r. The rank of Z^r is denoted by z_r. The rank ∂X^r is denoted by $\bar{\alpha}_r$. Then

$$\begin{aligned} \alpha_r &= z_r + \bar{\alpha}_r \\ z_r &= \bar{\alpha}_{r+1} + b_r \\ b_r &= \alpha_r - \bar{\alpha}_r - \bar{\alpha}_{r+1} \end{aligned} \tag{2}$$

If we assume that there are no simplices for r sufficiently large, then we have for the Euler-Poincaré characteristic the formula

$$\Sigma(-1)^r b_r = \Sigma(-1)^r \alpha_r, \tag{3}$$

as follows from the last formula of (2). Hopf points out [H4] that this basis free definition (avoiding the Poincaré incidence matrices) was completely new at that time; "(und ich weiß nicht einmal, ob der Begriff der "Homologiegruppe" schon irgendwo schwarz auf weiß in der Literatur vorgekommen war)". We come back to this later. In the preface of the book [A-H] the authors write in September 1935

"Die allgemeine mathematische Einsicht von Emmy Noether beschränkte sich nicht auf ihr spezielles Wirkungsgebiet, die Algebra, sondern übte einen lebhaften Einfluß auf jeden aus, der zu ihr in mathematische Beziehung kam. Für uns war dieser Einfluß von der größten Bedeutung, und er spiegelt sich auch in diesem Buch wieder. Die Tendenz der starken Algebraisierung der Topologie auf gruppentheoretischer Grundlage, der wir in unserer Darstellung folgen, geht durchaus auf Emmy Noether zurück. Diese Tendenz scheint heute selbstverständlich ; sie war es vor acht Jahren nicht; es bedurfte der Energie und des Temperamentes von Emmy Noether, um sie zum Allgemeingut der Topologen zu machen und sie in der Topologie, ihren Fragestellungen und ihren Methoden, diejenige Rolle spielen zu lassen, die sie heute spielt."

Of course, the book [A-H] presents proofs that the homology groups H^r depend only on the topological space and not on its simplicial decomposition, just as the book by H. Seifert and W. Threlfall which appeared one year earlier.

Let us look at two very important papers of Hermann Künneth [K1], [K2]. Only three to four years before Alexandroff and Hopf met in Göttingen, Künneth still uses the incidence matrices, refers to the original papers of Poincaré, and is very close to Poincaré's notation. For example he speaks of the

"Poincarésche Berandungsrelationen (Kongruenzen)" and writes:

$$
(4) \qquad \begin{aligned}
a^n_{r_n} &\Rrightarrow \sum_{r_{n-1}=1}^{\alpha_{n-1}} \varepsilon^n_{r_n r_{n-1}} a^{n-1}_{r_{n-1}} \\
\sum_{r_{n-1}=1}^{\alpha_{n-1}} \varepsilon^n_{r_n r_{n-1}} a^{n-1}_{r_{n-1}} &\Rrightarrow 0
\end{aligned}
$$

Here $a^n_{r_n}$ is an n-dimensional cell. The index r_n runs from 1 to α_n (number of n-dimensional cells). The matrix $E_n = \left(\varepsilon^n_{r_n r_{n-1}}\right)$ consisting of ± 1 or 0 is the incidence matrix giving the boundary (denoted by Poincaré by $\equiv$ and here by $\Rrightarrow$) from X^n to X^{n-1} (Hopf's notation, see above). The rank of the matrix E_n is $\bar{\alpha}_n$. The two formulas (4) of [K1] state $\partial\partial = 0$. For the n-th Betti number Künneth quotes Poincaré [P] p. 299

$$P_n - 1 = \alpha_n - \bar{\alpha}_n - \bar{\alpha}_{n+1} \quad \text{(see (2))}$$

Indeed, in the classical terminology P_n is the n-th Betti number. For us $P_n - 1 = b_n$ is the n-th Betti number as rank of the homology group. Already Riemann writes [R] "Wenn im Innern einer stetig ausgedehnten Mannigfaltigkeit mit Hülfe von m festen, für sich nicht begrenzenden, n-Streckstücken jedes unbegrenzte n-Streck begrenzt ist, so hat diese Mannigfaltigkeit einen $(m + 1)$ -fachen Zusammenhang $n$ter Dimension". The concept of $(m + 1)$ -facher Zusammenhang corresponds to introducing P_n instead of $b_n = m$. The group H^n (for finite simplicial complexes) has rank b_n and has a torsion part which is given by the elementary divisors of the matrix E_{n+1} (torsion numbers). The famous Künneth formulas of [K1] and [K2] give the Betti numbers and the torsion numbers of a cartesian product in terms of the Betti numbers and the torsion numbers of the factors. Everything is formulated and proved without the concept of homology group. Künneth works entirely with matrices and writes in the introduction of [K1]:

"Da die Bestimmung der Bettischen Zahlen zurückgeht auf die Bestimmung der Rangzahlen gewisser Matrizen, werden hier in einem eingeschobenen, algebraischen Teil einige Sätze über in besonderer Weise zusammengesetzte Matrizen und daraus gebildete Linearformen abgeleitet."

The papers [K1], [K2] can serve as an excellent example how algebraic topology was written in the time when one only used numerical invariants and not the homology groups.

During the academic year 1927/28 Alexandroff and Hopf were in Princeton. Solomon Lefschetz, Professor at Princeton University, had many discussions with Hopf. A result was the paper [H2] (received April 18, 1928) on the Lefschetz fixed point theorem which appeared later than the paper [H1]. But actually [H1] (received December 12, 1928) was written later in connection with Hopf's course in

Göttingen during the summer term 1928 where Emmy Noether proposed essential simplifications and improvements. Hopf writes in [H1]:

"Meinen ursprünglichen Beweis dieser Verallgemeinerung der Euler-Poincaréschen Formel konnte ich im Verlauf einer im Sommer 1928 in Göttingen von mir gehaltenen Vorlesung durch Heranziehung gruppentheoretischer Begriffe unter dem Einfluß von Fräulein E. Noether wesentlich durchsichtiger und einfacher gestalten."

Perhaps Emmy Noether told Hopf that (1) should be considered equivariantly for a simplicial map f. Then the traces of all endomorphisms of the groups occuring in (1) are well-defined. The trace replaces the rank. The formulas (2) hold with traces instead of ranks and the resulting Euler-Poincaré formula generalising (3) is

$$\Sigma(-1)^r tr(f \mid H^r) = \Sigma(-1)^r tr(f \mid C^r). \tag{5}$$

Then $tr(f \mid C^r)$ counts with proper multiplicity the r-dimensional simplices which are mapped to themselves. Formula (5) is a first approximation of relating the global Lefschetz number $\Sigma(-1)^r tr(f \mid H^r)$ to the fixed points.

In 1964 the Selecta of Heinz Hopf appeared [H3] on the occasion of his 70th birthday. Hopf added to [H1], the second paper of the Selecta, a new footnote

"Die obige Note ist wohl die erste Publikation gewesen, in der die heutige geläufige, von EMMY NOETHER stammende, gruppentheoretische Auffassung der Topologie zur Geltung kommt."

It is Hopf's first paper where he fully uses Emmy Noether's ideas. However the concept of homology group appeared earlier and independently of Emmy Noether in the literature.

When I prepared this lecture, I wanted to have some other early topological paper as example besides the Künneth papers. I chose the important and often used Mayer-Vietoris theorem relating the homology groups of $A \cup B$, A, B and $A \cap B$ with each other (under suitable assumptions). For the Euler-Poincaré characteristic we have

$$e(A \cup B) = e(A) + e(B) - e(A \cap B)$$

which generalises a trivial fact on the cardinality of finite sets. I looked at the original literature. I began with the paper of Walter Mayer [M] which was received on November 16, 1927. Here Mayer essentially introduces the abstract concept of chain complex and its homology groups and obtains as an application the Mayer-Vietoris formulas for the Betti numbers (See also [V4]). Mayer mentions that he learnt topology from Leopold Vietoris in his Vienna course of 1926/27. I wrote to Vietoris on September 9, 1996. I had seen him last at the celebration of his 100. birthday in Innsbruck (June 4, 1991) and was of course a little hesitant to write to somebody 105 years old. He answered on September 25, 1996. He wrote a letter five pages long and answered all my questions. In his Vienna course he introduced homology groups. He was in no way under the influence of Emmy Noether. The paper [V1] is the first paper where the concepts Zusammenhangsgruppe (Homologiegruppe mit Koeffizienten in $\mathbb{Z}_2$) and Homologiegruppe (Koeffizienten in $\mathbb{Z}$) are defined. Vietoris says that Hopf when writing [H1] did not know this paper and also not the later paper [V2] where the results of [V1] are treated in more detail. Vietoris' definition of homology is much more general than the usual one. "Ich brauchte nämlich diese Allgemeinheit, weil ich die kombinatorische Topologie der Polyeder auf kompakte metrische Räume übertragen wollte." Here the homology groups may have infinite rank "z.B. die *Cantor*sche Dreiteilungsmenge und deren mehrdimensionale Analoga." In his letter Vietoris also speaks about the article of J. Dieudonné [D1] where Dieudonné says about Mayer's paper [M]: "He was at that time in Vienna and did not mention Emmy Noether at all in his paper. However, by that time the spirit of "modern algebra" had spread to many German universities under the efforts of E. Artin, R. Baer, R. Brauer, H. Hasse, W. Krull and J. von Neumann, it is not unlikely that it could also have reached Vienna." But it seems that the Vienna results had difficulty to spread to Germany in spite of Vietoris' lecture in the annual meeting 1926 of the German Mathematical Society [V3] where Alexandroff was in the chair. In a letter to Dieter Puppe of November 11, 1990 Vietoris writes:

"*J. Dieudonné* hat in seinem Aufsatz "Emmy Noether and Algebraic Topology" Walter Mayer getadelt, weil er in der genannten Arbeit Emmy Noether nicht erwähnt hat. Er tut ihm damit Unrecht. Denn Mayer konnte ebenso wenig wie ich von Verdiensten E. Noethers um die algebraische Topologie etwas wissen." Vietoris continues with a reference to [ML]: "Saunders MacLane hat in seinem Aufsatz "Topology becomes algebraic with Vietoris and Noether" die Aussagen Dieudonn*é*s widerlegt und eine Lanze für die österreichische Wissenschaft gebrochen." But we should also read [D2], p. 38-39, where Dieudonné writes "Independently, in 1926, Vietoris also needed to get rid of matrices in order to define homology for more general spaces than simplicial complexes ...". In [V1], [V2], [V3] the main theme of Vietoris was to prove his well-known theorem for a continuous map $f : Y \to X$ of compact metric spaces that the triviality of the homology for all inverse images $f^{-1}(x)$ in dimensions $\leq n$ implies that f induces an isomorphism of the homology groups of Y and X in dimensions $\leq n$ ([D2], p. 123-124). I learnt about this theorem of Vietoris for the first time in Princeton 1953 because it is used in Armand Borel's thesis (Ann. of Math. 57, 115-207 (1953)).

Another passage of the letter of Vietoris to Puppe is of special interest. He speaks about the time before his papers [V1] and [V2] and says "Selbstverständlich wußten die Topologen schon vor diesen Arbeiten, daß sie es bei der Addition von Zykelklassen mit Abelschen Gruppen zu tun hatten. Weil sie aber wußten, daß diese Gruppen durch Rang- und Elementarteiler (Torsionszahlen) charakterisiert sind, hielten sie die Beschäftigung mit den Gruppen für überflüssig. Diesen Standpunkt vertritt noch S. Lefschetz in seinem ausgezeichneten Buch Topology (1930). Er schreibt dort auf S. 29: "Indeed everything that follows in this section can be, and frequently is, translated into the theory of groups. It is of course a mere question of a different terminology"."

Emmy Noether did not publish a single paper about her ideas concerning the algebraisation of topology. One can only mention [N1] which is not even printed in her Collected Papers [N2]. She speaks in [N1] about the structure theorem for finitely generated abelian groups and her only reference to topology is "...; in den Anwendungen des Gruppensatzes - z.B. Bettische und Torsionszahlen in der Topologie - ist somit ein Zurückgehen auf die Elementarteilertheorie nicht erforderlich."

The influence of Emmy Noether on the mathematicians around her - also in fields where she did not work herself - could not be shown better than by the case of Alexandroff and Hopf and Algebraic Topology. She published half a sentence and has an everlasting effect through "algebraisch-topologische Spaziergänge", attending courses of her young colleagues, discussions and unselfish help.

Emmy Noether left Germany for the United States in 1933 after the beginning of the Nazi regime and the terror against the German citizens who where Jewish disguised in the law "Zur Wiederherstellung des Berufsbeamtentums". She went to Bryn Mawr College, lectured there and in the nearby Princeton University. She died already on April 14, 1935. The book by Auguste Dick [Di] contains Albert Einstein's commentary in the New York Times, the Obituary by B.L. van der Waerden, the Memorial Adress delivered by Hermann Weyl and the adress "In Memory of Emmy Noether" given by Paul Alexandroff at a meeting of the Moscow Mathematical Society on September 5, 1935, published in Proc. Moscow Math. Soc. 1936. This adress is also printed in the Collected Papers of E. Noether [N2]. On p. 9 of [N2] we can read again about the influence of Emmy Noether on Alexandroff and Hopf and on the development of Algebraic Topology. It is very moving to read the last section of Alexandroff's adress ending with the sentence: "She loved people, science, life with all the warmth, all the joy, all the selflessness and all the tenderness of which a deeply feeling heart - and a woman's heart - was capable."

The meeting of the Moscow Mathematical Society honoring Emmy Noether was part of the first International Conference on Topology. The sensational new concepts and results would have been impossible even to formulate without algebraic objects like groups and rings associated to spaces via functors from topology to algebra as we say nowadays. I mention only two highlights: the cohomology theory including the ring structure (J.W. Alexander, I. Gordon, A.N. Kolmogoroff) and the theory of characteristic classes according to E. Stiefel and H. Whitney (Hopf lectured on Stiefel's dissertation which concerns the characteristic classes of the tangent bundle of a manifold and applications. Whitney spoke about his theory of characteristic classes of a sphere bundle). See Hopf's report on this conference in [H4] or [H5]. The characteristic classes live in the cohomology ring of the manifold or more generally the base space of the bundle. How to distinguish an element in a cohomology group if on e only knows the numerical invariants rank and torsion numbers? The characteristic classes have had numerous applications later (for example, Pontryagin classes and Thom's cobordism theory). Frank Adams solved the vector field problem for spheres using cohomology operations. They are natural operations between cohomology groups of different dimensions. How to define them if only rank and torsion numbers are treated? As reported in [Di] p. 153, S. Lefschetz emphasized at the conclusion of the congress "the great value that Emmy Noether's ideas had for the development of modern topology", quite a step from his point of view in his topology book of 1930 that the algebraisation is a mere question of a different terminology.

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