ORBIFOLDING FROBENIUS ALGEBRAS

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Abstract. We study the general theory of Frobenius algebras with group actions. These structures arise when one is studying the algebraic structures associated to a geometry stemming from a physical theory with a global finite gauge group, i.e. orbifold theories. In this context, we introduce and axiomatize these algebras. Furthermore, we define a geometric cobordism categories whose functors to the category of vector spaces are parameterized by these algebras. The theory is also extended to the graded and super–graded cases. As an application, we consider Frobenius algebras having some additional properties making them more tractable. These properties are present in Frobenius algebras arising as quotients of Jacobian ideal, such as those having their origin in quasi–homogeneous singularities and their symmetries.

Introduction

The subject of this exposition is the general theory of Frobenius algebras with group actions. These structures arise when one is studying the algebraic structures stemming from a geometry associated to a physical theory with a global finite gauge group [DW,DVVV,IV,V]. The most prominent example of this type in mathematics is the Gromov–Witten theory of orbifolds [CR], which are global quotients. The use of orbifold constructions, however, is much older and is the cornerstone of the original mirror construction [GP]. The orbifolds under study in that context are so–called Landau–Ginzburg orbifold theories, which have so far nor been studied mathematically. These correspond to the Frobenius manifolds coming from singularities and are studied in detail in the present paper.

A common aspect of the physical treatment of quotients by group actions is the appearance of so called twisted sectors. This roughly means that if one wishes to take the quantum version of quotient by a group action, one first has to construct an object for each element in the group together with a group action on this object and in a second step take invariants in all of these components. Whereas classically one

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considers only the G–invariants of the original object which constitutes the sector associated to the identity, the untwisted sector.

The present work axiomatizes this procedure and points out that the construction of twisted sectors is not merely an auxiliary artifact, but is essential. This is clearly visible in the case of the Frobenius algebra associated to the singularity of type A_n together with a $\mathbf{Z}/n\mathbf{Z}+1$ action and the singularity of type A_{2n-3} with $\mathbf{Z}/2\mathbf{Z}$ action, which are worked out in detail in the last paragraph. This particular example exhibits a version of Mirror symmetry in which it is self–dual. The twisted sectors,however, shed some light on this duality, since in reality the sum of twisted sectors is dual to the untwisted one.

In the present work, we develop the theory of orbifold Frobenius algebras along the now classical lines of Atiyah, Dubrovin, Dijkgraaf and Segal. I.e. we start by introducing the algebraic structures in an axiomatic fashion. There is a slight difference to the theory with trivial group in that there are two structures with slightly different G-action to be considered which are nevertheless present. These versions differ by a twists with a character. In the singularity version, on the non–twisted level, they correspond to the different G–modules on the cohomology and the Milnor ring [W]. In physical terms these two structures are related by spectral flow. To be even more precise, one structure carries the natural multiplication and the other the natural scalar product. The next step is a cobordism realization of the theory. There are ramifications of the present exposition with [FQ, T], if the above mentioned character is trivial. This is, however, not necessarily the case and in the most interesting examples worked out in the last paragraph, this is not the case. The key structure here is a non–commutative multiplication on the sum of all the twisted sectors, even before taking invariants.

To provide a larger basis of examples we introduce so–called special G–Frobenius algebras, which have as a subgroup Jacobian G–Frobenius algebras which in turn encompass the singularity examples or those of manifolds whose cohomology ring can be described as a quotient by a Jacobian ideal. Here it is important to note that everything can be done in a super (i.e. $\mathbf{Z}/2\mathbf{Z}$–graded) version. This introduces a new degree of freedom into the construction corresponding to the choice of parity for the twisted sectors. Lastly, we explicitely work out several examples including the transition from the singluarity A_{2n+3} to D_n via a quotient by $\mathbf{Z}/2\mathbf{Z}$.

In the case that the Frobenius algebra one starts out with comes from a semi–simple Frobenius manifold and the quotient of the twised sector is not trivial, then there has a unique extension to the level of

Frobenius manifolds. This is the case in the above example of A_{2n+3} and D_n.

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1. Frobenius Algebras and Cobordisms

In this paragraph, we recall the definition of a Frobenius algebra and its relation to the cobordism–category definition of a topological field theory [A,Du,Dij].

1.1. **Frobenius algebras.**

1.1.1. **Definition.** A *Frobenius algebra* (FA) over a field k of characteristic 0 is $\langle A, \circ, \eta, e\rangle$, where

A is a finite dim k–vector space,
$\circ$ is a multiplication on A: $\circ : A \otimes A \to A$,
η is a non-degenerate bilinear form on A, and
1 is a fixed element in A – the unit.

satisfying the following axioms:

a) *Associativity*
$$(a \circ b) \circ c = a \circ (b \circ c)$$
b) *Commutativity*
$$(a \circ b) = b \circ a$$
c) *Unit*:
$$\forall \in a : 1 \circ a = a \circ 1 = a$$
d) *Invariance*:
$$\eta(a, b \circ c) = \eta(a \circ b, c)$$

1.1.2. **Remark.** By using η to identify A and A^* – the dual vector space of A– these objects define a one–form $\epsilon \in A^*$ called the co–unit and a three–tensor $\mu \in A^* \otimes A^* \otimes A^*$

Using dualization and invariance these data are interchangeable with η and $\circ$ via the following formulas. Explicitly, after fixing a basis $(\Delta_i)_{i\in I}$ of A, setting $\eta_{ij} := \eta(\Delta_i, \Delta_j)$ and denoting the inverse metric by η^{ij}

$\epsilon(a) := \eta(a, 1)$,
$\mu(a, b, c) := \eta(a \circ b, c) = \epsilon(a \circ b \circ c)$,
$a \circ b = \sum_{ij} \mu(a, b, \Delta_i)\eta^{ij}\Delta_j$ and
$\eta(a, b) = \epsilon(a \circ b)$.

1.1.3. **Notation.** We call $\rho \in A$ the element dual to ϵ. This is the element which is Poincaré dual to 1.

1.1.4. **Grading.** A graded Frobenius algebra is a Frobenius algebra together with an group grading of the vector space A: $A = \bigoplus_{i\in I} A_i$ together with the following compatibility equations: denote the I–degree of an element by deg.

1) 1 is homogeneous; $1 \in A_d$ for some $d \in I$
2) η is homogeneous of degree $d + D$.
   I.e. for homogeneous elements a, b $\eta(a, b) = 0$ unless $\deg(a) + \deg(b) = d + D$.
   This means that ϵ and ρ are of degree D.
3) $\circ$ is of degree d, this means that μ is of degree $2d+D$, where again this means that
   $\deg(a \circ b) = \deg(a) + \deg(b) + d$

1.1.5. **Definition.** An even derivation $E \in Der(A, A)$ of a G twisted Frobenius algebra A is called an *Euler field*, if it is conformal and is natural w.r.t. the multiplication. i.e. for some $d, D \in k$ it satisfies:

$$\eta(Ea, b) + \eta(a, Eb) = D\eta(a, b) \tag{1.1}$$

and

$$E(ab) = Ea\, b + a\, Eb = d\, ab \tag{1.2}$$

Such derivation will define a grading on A by its set of eigenvalues.

1.1.6. **Remark.** For this type of grading, we will use the group $\mathbf{Q}$. There are two more versions of grading: 1) a $\mathbf{Z}/2\mathbf{Z}$ super-grading, which will be discussed in 1.1.13 and 2) a grading by a finite group G, which is the content section 2.1.

1.1.7. **Definition.** Given an I–graded Frobenius algebra A, we define its *characteristic series* as

$$\chi_A(t) := \sum_{i\in I} \dim(A_i)t^i \tag{1.3}$$

We refer to the set $\{d, D; i : \dim A_i \neq 0\}$ as the (I–)spectrum of A.

1.1.8. **Scaling.** If the group indexing the grading has the structure of a Λ-module, where Λ is a ring, we can scale the grading by an element $\lambda \in \Lambda$. We denote the scaled Frobenius algebra by $\lambda A := \bigoplus_{i\in I} A_{\lambda i}$ where $(\lambda A)_i = A_{\lambda i}$

$$\chi_{\lambda A}(t) := \sum_{i\in I} \dim(\lambda A_i)t^i = t^\lambda \chi_A(t) \tag{1.4}$$

It is sometimes –but not always– convenient to normalize in such a way that $\deg(e) = e$ In the case that the grading is given by $\mathbf{Q}$, this means $\deg_{\mathbf{Q}}(1) = 0$ and in the finite group case $\deg_G(1) = e$.

1.1.9. **Operations.** There are two natural operations on Frobenius algebras, the direct sum and the tensor product. Both of these operation extend to the level of Frobenius manifold, while the generalization of the direct sum is straight forward, the generalization of the tensor product to the level of Frobenius manifolds is quite intricate [K1].

Consider two Frobenius algebras $\mathcal{A}' = \langle A', \circ', \eta', e'\rangle$ and $\mathcal{A}'' = \langle A'', \circ'', \eta'', e''\rangle$.

1.1.10. **Direct sum.** Set $\mathcal{A}' \oplus \mathcal{A}'' := \langle A, \circ, \eta, e\rangle = \langle A' \oplus A'', \circ' \oplus \circ'', \eta' \oplus \eta'', e' \oplus e''\rangle$. E.g. $(a', a'') \circ (b', b'') = (a' \circ' b', a'' \circ'' b'')$. The unit is $1 = 1' \oplus 1''$ and the co-unit is $\epsilon = (\epsilon', \epsilon'')$.

1.1.11. **Lemma.** If both Frobenius algebras are graded by the same I then their direct sum inherits a natural grading if and only if the gradings can be scaled s.t.

$$D' + d' = D'' + d'' := D + d \tag{1.5}$$

where

$$D = D' = D'' \tag{1.6}$$

in *this* scaling.

Furthermore, the unit will have degree $d' = d'' = d$.

Proof

The equation 1.5 insures that the three tensor μ is homogeneous of degree $D + d$. The homogeneity of η yields the second condition: for

η to be homogeneous, we must have that after scaling ρ' and ρ'' are homogeneous of the same degree $D' = D'' = D$. The two equations together imply the homogeneity of $1 = (1', 1'')$ of degree $d = d' = d''$.

1.1.12. **Tensor product.** Set $\mathcal{A}' \otimes \mathcal{A}'' := \langle A, \circ, \eta, e\rangle = \langle A' \otimes A'', \circ' \otimes \circ'', \eta' \otimes \eta'', e' \otimes e''\rangle$.
E.g. $(a', a'') \circ (b', b'') = (a' \circ' b', a'' \circ'' b'')$. The unit is $1 = 1' \otimes 1''$ and the co-unit is $\epsilon = \epsilon' \otimes \epsilon''$.

There are*no conditions* for grading. I.e. if both Frobenius algebras are I–graded there is a natural induced I–grading on their tensor product. The unit is of degree $d = d' + d''$, the co–unit has degree $D' + D''$ and the multiplication is homogeneous of degree $d = d' + d''$.

1.1.13. **Super-grading.** For element a of a super vector space $A = A_0 \oplus A_1$ denote by $\tilde{a}$ its $\mathbf{Z}/2\mathbf{Z}$ degree, i.e. $\tilde{a} = 0$ if it is even ($a \in A_0$) and $\tilde{a} = 1$ if it is odd ($a \in A_1$).

1.1.14. **Definition.** A *super Frobenius algebra* a field k of characteristic 0 is $\langle A \circ, \eta, e\rangle$, where

A is a finite dim k–super vector space,
$\circ$ is a mutiplication on A: $\circ : A \otimes A \to A$,
which preserves the $\mathbf{Z}/2\mathbf{Z}$–grading
η is a non-degenerate even bilinear form on A, and
1 is a fixed even element in A_0 – the unit.

satisfying the following axioms:

a) *Associativity*
$$(a \circ b) \circ c = a \circ (b \circ c)$$
b) *Super–commutativity*
$$(a \circ b) = (-1)^{\tilde{a}\tilde{b}} b \circ a$$
c) *Unit*:
$$\forall \in a : 1 \circ a = a \circ 1 = a$$
d) *Invariance*:
$$\eta(a, b \circ c) = \eta(a \circ b, c)$$

The grading for super Frobenius algebras carries over verbatim.

1.2. **Cobordisms.**

1.2.1. **Definition.** Let $\mathcal{COB}$ be the category whose objects are one–dimensional closed oriented (topological) manifolds considered up to orientation preserving homeomorphism and whose morphisms are cobordisms of these objects.

I.e. $\Sigma \in Hom(S_1, S_2)$ if Σ is an oriented surface with boundary $\partial\Sigma \equiv -S_1 \coprod S_2$.

The composition of morphisms is given by glueing along boundaries with respect to orientation reversing homeomorphisms.

1.2.2. **Remark.** The operation of disjoint union makes this category into monoidal category with unit $\emptyset$.

1.2.3. **Remark.** The objects can be chosen to be reprsented by disjoint unions of the circle with the natural orientation S^1 and the circle with opposite orientation $\bar{S}^1$. Thus a typical object looks like $S = \coprod_{i\in I} S^1 \coprod_{j\in J} \bar{S}^1$. Two standard morpisms are given by the cylinder, and thrice punctured sphere.

1.2.4. **Definition.** Let $\mathcal{VECT}_k$ be the monoidal category of finite dimensional k–vector spaces with linear morphisms with the tensor product providing a monoidal structure with unit k.

1.2.5. **Theorem.** (Atiyah, Dijkgraaf, Dubrovin) [A,Dij, Du] There is a 1–1 correspondence between Frobenius algebras over k and isomorphism classes of covariant functors of monoidal categories from $\mathcal{COB}$ to $\mathcal{VECT}_k$, natural with respect to orientation preserving homeomorphisms of cobordisms and whose value on cylinders $S \times I \in Hom(S,S)$ is the identity.

Under this identification, the Frobenius algebra A is the image of S^1, the multiplication or rather μ is the image of a thrice punctured sphere and the metric is the image of an annulus.

2. Orbifold Frobenius algebras

2.1. G–Frobenius algebras.

We fix a finite group G and denote its neutral element by e.

2.1.1. **Definition.** A *G–Frobenius algebra* (FA) over a field k of characteristic 0 is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where

G	finite group
A	finite dim G-graded k–vector space $A = \oplus_{g\in G} A_g$ A_e is called the untwisted sector and the A_g for $g \neq e$ are called the twisted sectors.
$\circ$	a multiplication on A which respects the grading: $\circ : A_g \otimes A_h \to A_{gh}$
1	a fixed element in A_e–the unit
η	non-degenerate bilinear form which respects grading i.e. $g|_{A_g\otimes A_h} = 0$ unless $gh = e$.
φ	an action by algebra automorphisms of G on A, $\varphi \in \mathrm{Hom}_{k-alg}(G, A)$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$
χ	a character $\chi \in \mathrm{Hom}(G, k^*)$

Satisfying the following axioms:
NOTATION: We use a subscript on an element of A to signify that it has homogeneous group degree –e.g. a_g means $a_g \in A_g$– and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.

a) *Associativity*
$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$

b) *Twisted commutativity*
$a_g \circ a_h = \varphi_g(a_h) \circ a_g$

c) *G Invariant Unit*:
$1 \circ a_g = a_g \circ 1 = a_g$
and
$\varphi_g(1) = 1$

d) *Invariance of the metric*:
$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$

i) *Projective self–invariance of the twisted sectors*
$\varphi_g|A_g = \chi_g^{-1} id$

ii) *G–Invariance of the multiplication*
$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$

iii) *Projective G–invariance of the metric*
$\varphi_g^*(\eta) = \chi_g^{-2}\eta$

iv) *Projective trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

An alternate choice of data is given by a one–form ϵ the co–unit with $\epsilon \in A_e^*$ and a three–tensor $\mu \in A^* \otimes A^* \otimes A^*$ which is of group degree e, i.e. $\mu|_{A_g\otimes A_h\otimes A_k} = 0$ unless $ghk = e$.

The relations between $\eta, \circ$ and ϵ, μ are analogous to those of 1.1.2.

Again, we denote by $\rho \in A_e$ the element dual to $\epsilon \in A_e^*$ and Poincaré dual to $1 \in A_e$.

2.1.2. **Remarks.**

1) A_e is central by twisted commutativity and $\langle A_e, \circ, \eta|_{A_e \otimes A_e}, 1\rangle$ is a Frobenius algebra.
2) All A_g are A_e-modules.
3) Notice that χ satisfies the following equation which completely determine it in terms of φ. Setting $h = e, c = 1$ in axiom iv)
$$\dim A_g = \chi_{g^{-1}} \mathrm{Tr}(\varphi_g|_{A_e}) \tag{2.1}$$
by axiom iii) the action of φ on ρ determines χ up to a sign
$$\chi_g^{-2} = \chi_g^{-2}\eta(\rho, 1) = \eta(\varphi_g(\rho), \varphi_g(1)) = \eta(\varphi_g(\rho), 1) \tag{2.2}$$
4) Axiom iv) forces the χ to be group homomorphisms, so it would be enough to assume in the data that they are just maps.

2.1.3. **Proposition.** The G invariants A^G of a G–Frobenius algebra A form an associative and commutative algebra with unit. This algebra with the induced bilinear form is a Frobenius algebra if and only if $\sum_g \chi_g^{-2} = |G|$. If $k = \mathbf{C}$ and $\chi \in \mathrm{Hom}(G, U(1))$ this implies $\forall g : \chi_g = \pm 1$

Proof. Due to axiom ii) the algebra is associative and commutative. And since 1 is G invariant, the algebra has a unit.

Now suppose $\sum_g \chi_g^{-2} = |G|$. Then $\eta|_{A^G \otimes A^G}$ is non–degenerate: Let $a \in A^G$ and choose $b \in A$ s.t. $\eta(a, b) \neq 0$. Set $\tilde{b} = \frac{1}{|G|}\sum_{g \in G} \varphi_g(b) \in A^G$. Then

$$\begin{aligned}
\eta(a, \tilde{b}) &= \sum_{g \in G} \eta(a, \varphi_g(b)) = \sum_{g \in G} \eta(\varphi_g(a)\varphi_g(b)) = \frac{1}{|G|}\sum_{g \in G} \chi_g^{-2}\eta(a, b) \\
&= \eta(a, b) \neq 0
\end{aligned}$$

On the other hand if $\eta|_{A^G \otimes A^G}$ is non–degenerate then let $a, b \in A^G$ be such that $\eta(a, b) = 1$. It follows:

$$1 = \eta(a, b) = \frac{1}{|G|}\sum_{g \in G} \eta(\varphi_g(a), \varphi_g(b)) = \frac{1}{|G|}\sum_{g \in G} \chi_g^{-2}\eta(a, b) = \frac{1}{|G|}\sum_{g \in G} \chi_g^{-2}$$

so that $\sum_g \chi_g^{-2} = |G|$.

The last statement follows from the simple fact that if $|\chi_g| = 1$ and $\sum_g \chi_g^{-2} = |G|$ then $\chi_g^{-2} = 1$ and hence $\chi_g \in \{-1, 1\}$.

2.1.4. **Definition.** A G–Frobenius algebra is called an *orbifold model* if the data $\langle A^G, \circ, 1\rangle$ can be augmented by a compatible metric to yield a Frobenius algebra. In this case, we call the Frobenius algebra A^G a *G–orbifold Frobenius algebra.*

2.2. **Super-grading.** We can enlarge the framework by considering super-algebras rather than algebras. This will introduce the standard signs.

2.2.1. **Definition.** A G-twisted Frobenius super-algebra is over a field k of characteristic 0 is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where

G finite group
A finite dimensional $\mathbf{Z}/2\mathbf{Z} \times G$-graded k–vector space
$A = A_0 \oplus A_1 = \oplus_{g\in G}(A_{g,0} \oplus A_{g,1}) = \oplus_{g\in G} A_g$
A_e is called the untwisted sector and is even.
The A_g for $g \neq e$ are called the twisted sectors.
$\circ$ a multiplication on A which respects both gradings:
$\circ : A_{g,i} \otimes A_{h,j} \to A_{gh,ij}$
1 a fixed element in A_e–the unit
η non-degenerate even bilinear form
which respects grading i.e. $g|_{A_g\otimes A_h} = 0$ unless $gh = e$.
φ an action by even algebra automorphisms of G on A,
$\varphi \in \mathrm{Hom}_{k-alg}(G, A)$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$
χ a character $\chi \in \mathrm{Hom}(G, k^*)$
or if $k = \mathbf{C}$, $\chi \in \mathrm{Hom}(G, U(1))$

Satisfying the axioms a)–d) and i)–iii) of a G–Frobenius algebra with the following alteration:

b$^\sigma$) *Twisted super–commutativity*
$a_g \circ a_h = (-1)^{\tilde{a}_g\tilde{a}_h}\varphi_g(a_h) \circ a_g$
iv$^\sigma$) *Projective super–trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{STr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h})$
where STr is the super–trace.

2.2.2. **Operations.** Restriction
If $H \subset G$ and $A = \oplus_{g\in G} A_g$ then $\tilde{A} := \bigoplus_{h\in H} A_h$ is naturally a H–Frobenius algebra.

Direct Sum Given a G–Frobenius algebra A and an H–Frobenius algebra B then $A \oplus B$ is naturally a $G \times H$–Frobenius algebra with the graded pieces $(A \oplus B)_{(g,h)} = A_g \oplus B_h$.

We define the direct sum of two G–Frobenius algebras to be the G–Frobenius subalgebra corresponding to the diagonal $\Delta : G \to G \times G$ in $A \oplus A$.

TENSOR PRODUCT Given a G–Frobenius algebra A and an H–Frobenius algebra B then $\bigoplus_{(g,h)}(A_g \otimes B_h)$ is naturally a $G \times H$–Frobenius algebra $(A \otimes B)_{(g,h)} = A_g \otimes B_h$.

We define the tensor product of two G–Frobenius algebras to be the G–Frobenius subalgebra corresponding to the diagonal $G \to G \times G$ in $A \oplus A$.

BRAIDED TENSOR PRODUCT. If A and B are two G–Frobenius algebras with the same character χ, we can define a tensor on $A \otimes B$ by setting $(A \otimes B)_g := \oplus_{k \in G} A_k \otimes B_{k^{-1}g}$. For the multiplication we use the sequence

$$A_k \otimes B_{k^{-1}g} \otimes A_l \otimes B_{l^{-1}h} \xrightarrow{(id \otimes id \otimes \varphi_{k^{-1}g} \otimes id) \circ \tau_{2,3}} A_k \otimes A_{k^{-1}glg^{-1}k} \otimes B_{k^{-1}g} \otimes B_{l^{-1}h} \overset{\circ \otimes \circ}{\to} A_{glg^{-1}k} \otimes B_{k^{-1}gl^{-1}h} \quad (2.3)$$

and $\bigoplus_k (\varphi_k \otimes \varphi_{kh^{-1}})$ for the action of h on $\bigoplus_k A_k \otimes B_{k^{-1}g}$

Remark. If one thinks in terms of cohomology of spaces the the direct sum corresponds to the disjoint union and the tensor product corresponds to the Cartesian product. The origin of the braided tensor product, however, is not clear yet.

2.3. Geometric model – spectral flow.

The axioms of the G–Frobenius algebra are well suited for taking the quotient, since the multiplication is G–invariant. However, this is not the right framework for a geometric interpretation. In order to accommodate a more natural coboundary description, we need the following definition:

2.3.1. **Definition.** A *Ramond G–algebra* over a field k of characteristic 0 is $< G, V, \bar{\circ}, v, \bar{\eta}, \bar{\varphi}, \chi >$

- G finite group
- V finite dim G-graded k–vector space
  $V = \oplus_{g \in G} V_g$
  V_e is called the untwisted sector and
  the V_g for $g \neq e$ are called the twisted sectors.
- $\bar{\circ}$ a multiplication on V which respects the grading:
  $\bar{\circ} : V_g \otimes V_h \to V_{gh}$
- 1 a fixed element in V_e–the unit

$\bar{\eta}$ non-degenerate bilinear form
which respects grading i.e. $\bar{\eta}|_{V_g \otimes V_h} = 0$ unless $gh = e$.
$\bar{\varphi}$ an action by algebra automorphisms of G on V,
$\bar{\varphi} \in \mathrm{Hom}_{k-alg}(G, V)$, s.t. $\bar{\varphi}_g(V_h) \subset V_{ghg^{-1}}$
χ a character $\chi \in \mathrm{Hom}(G, k^*)$

Satisfying the following axioms:
NOTATION: We use a subscript on an element of V to signify that it has homogeneous group degree –e.g. v_g means $v_g \in V_g$– and we write $\bar{\varphi}_g := \bar{\varphi}(g)$ and $\chi_g := \chi(g)$.

a) *Associativity*
$(v_g \bar{\circ} v_h) \bar{\circ} v_k = v_g \bar{\circ} (v_h \bar{\circ} v_k)$

b') *Projective twisted commutativity*
$v_g \bar{\circ} v_h = \chi_g \bar{\varphi}_g(v_h) \bar{\circ} v_g = \bar{\varphi}_g(v_h \bar{\circ} v_g)$

c') *Projectively invariant unit*:
$v \bar{\circ} v_g = v_g \bar{\circ} v = v_g$
and
$\bar{\varphi}_g(v_0) = 1\chi_g v_0$

d) *Invariance of the mtric*:
$\eta(v_g, v_h \bar{\circ} v_k) = \eta(v_g \bar{\circ} v_h, v_k)$

1') *Self–invariance of the twisted sectors*
$\bar{\varphi}_g | V_g = id$

2') *Projective G–invariance of multiplication*
$\bar{\varphi}_k(v_g \bar{\circ} v_h) = \chi_k \bar{\varphi}_k(v_g) \bar{\circ} \bar{\varphi}_k(v_h)$

3') *G–Invariance of metric*
$\bar{\varphi}_g^*(\bar{g}) = g$

4') *Trace axiom*
$\forall c \in V_{[g,h]}$ and l_c left multiplication by c:
$\mathrm{Tr}(l_c \bar{\circ} \bar{\varphi}_g|_{V_h}) = \mathrm{Tr}(\bar{\varphi}_{h^{-1}} \bar{\circ} l_c|_{V_g})$

2.3.2. **Definition.** A *state space* for a G–Frobenius algebra A is a triple $\langle V, v, \bar{\eta}, \bar{\varphi} \rangle$, where

V is a G-graded free rank one A–module: $V = \bigoplus_{g \in G} V_g$,
v is a fixed generator of V– called the vacuum,
$\bar{\eta}$ is non–degenreate bilinear form on V
$\bar{\varphi}$ is a linear G-action on V fixing v projectively
i.e. $\bar{\varphi}(g)(span(v)) \subset span(v)$

such that these stuctures are compatible with those of A (we denote $\bar{\varphi}_g := \bar{\varphi}(g)$

a) The action of A respects the grading: $A_g V_h \subset V_{gh}$.
b) V_h is a rank one free A_h–module and $V_h = A_h v$.
c) $\bar{\varphi}_g(av) = \varphi_g(a) \bar{\varphi}(v) : \ \forall a \in A, v \in V$

d) For $a, b \in A : \bar\eta(av, bv) = \eta(a, b)$
e) $\forall g\beta \in \pi, c \in A_{[g,h]}$ $\forall c \in V_{[g,h]}$ and l_c left multiplication by c:
$\mathrm{Tr}(l_c \bar\circ \bar\varphi_g|_{V_h}) = \mathrm{Tr}(\bar\varphi_{h^{-1}} \bar\circ l_c|_{V_g})$

2.3.3. **Remark.** A state space inherits an associative multiplication $\bar\circ$ with unit from A via

$$\forall a, b \in A : (av)\bar\circ(bv) := (a \circ b)v \tag{2.4}$$

2.3.4. **Definition.** We call two state spaces isomorphic, if there is a A–module isomorphism between the two.

Since state spaces are free rank one A–modules It is clear that all automorphisms are rescalings of v.

2.4. **Proposition.** Given a G–twisted Frobenius algebra A then there is at most a unique state space up to isomorphism and the form $\bar\eta$ is G–invariant (i.e. $\bar\varphi_g^*(\bar\eta) = \bar\eta$).

Proof. We start with a free rank one A–module V and reconstruct all other data. The G–grading on V is uniquely determined from that of A by axiom b) and this grading satisfies axiom a). Up to isomorphism, we may assume a generator $v \in V$ is fixed, then axiom d) determines $\bar\eta$ from η. We denote the Eigenvalue of $\bar\varphi_g$ on v by λ_g: $\bar\varphi_g v = \lambda_g v$ Notice that due to c) $\bar\varphi$ is determined by λ_g. Using axioms b), c) and e), we find that $\lambda_g = \chi_g$, thus fixing the G–action $\bar\varphi$.

Namely with $c = 1$ and $h = e$ in e):

$$\mathrm{Tr}(\bar\varphi_g|_{V_e}) = \lambda_g \mathrm{Tr}(\varphi_g|_{A_e}) = \lambda_g \chi_g^{-1} \mathrm{Tr}(\varphi_e|_{A_g}) = \lambda_g \chi_g^{-1} \mathrm{Tr}(\varphi_e|_{V_g})$$

The equality $\bar\varphi_g = \varphi_g \lambda_g$ implies that $\bar\eta$ is $\bar\varphi$ invariant:

$$\bar\eta(\bar\varphi_g(av), \bar\varphi_g(bv)) = \lambda_g^2 \eta(\varphi_g(a), \varphi_g(b)) = \lambda_g^2 \chi_g^{-2} \eta(a, b) = \bar\eta(av, bv)$$

In general:

$$\begin{aligned}
\mathrm{Tr}(l_c \bar\varphi_g|_{V_h}) &= \lambda_g \mathrm{Tr}(l_c \varphi_g|_{A_h}) \\
&= \lambda_g \chi_g^{-1} \chi_{h^{-1}} \mathrm{Tr}(\varphi_{h^{-1}} \circ l_c|_{A_g}) \\
&= \lambda_g \chi_g^{-1} \chi_{h^{-1}} \lambda_{h^{-1}}^{-1} \mathrm{Tr}(\bar\varphi_{h^{-1}} \bar\circ l_c|_{V_g}) \\
&= \mathrm{Tr}(\bar\varphi_{h^{-1}} \bar\circ l_c|_{V_g})
\end{aligned}$$

so that with this choice of $\bar\varphi$ axiom e) is satisfied.

This fact leads us to the following definitions:

2.4.1. **Definition.** The *Ramond space* of a G–Frobenius algebra A is the G–graded vector–space

$$V := \oplus_g V_g := \bigoplus_g A_g \otimes k$$

together with the G-action $\bar{\varphi} := \varphi \otimes \chi$, the induced metric $\bar{g}$ and the induce multiplication $\bar{\circ}$, Replacing A_g by V_a this changes the axioms as follows:

2.4.2. **Theorem.** There is a one–to–one correspondence between isomorphism classes of G–Ramond algebras and G-Frobenius algebras

Proof. The association of a Ramond space to a G–Frobenius algebra provides the correspondence. The inverse being the obvious reverse twist by χ.

2.4.3. **Remark.** In the theory of singularities, the untwisted Ramond sector corresponds to the forms $H^{n-1}(V_\epsilon, \mathbf{C})$ while the Frobenius algebra corresponds to the Milnor ring [W]. These are naturally isomorphic, but have different G–module structures. In that situation, one takes the invariants of the Ramond sector, while we will be interested in invariants of the G–Frobenius algebra and not only the untwisted sector.

3. Bundle cobordisms, finite gauge groups, orbifolding and G–Ramond algebras

In this section, we introduce two cobordism categories which correspond to orbifold Frobenius algebras and G–Ramond respectively. G–Frobenius algebras. Again G is a fixed finite group.

3.1. **Bundle cobordisms.** Gluing along boundaries induces the composition and the disjoint union provides a monoidal structure.

3.1.1. **Definition.** Let $\mathcal{GBCOB}$ be the category whose objects are principal G–bundles over one–dimensional closed oriented (topological) manifolds whose morphisms cobordisms of these objects (i.e. principal G–bundles over oriented surfaces with boundary)

More precisely, $B_\Sigma \in Hom(B_1, B_2)$ if Σ is an oriented surface with boundary $\partial\Sigma = -S_1 \coprod S_2$ and B_Σ is a bundle on Σ which restricts to B_1 and B_2 on the boundary.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms on the base and covering bundle isomorphisms.

3.1.2. **Remark.** The operation of disjoint union makes this category into monoidal category with unit $\emptyset$ formally regarded as a principal G bundle over $\emptyset$.

3.1.3. **Remark.** Typical objects are bundles B over $S = \coprod_{i\in I} S^1 \coprod_{j\in J} \bar{S}^1$

3.1.4. **Structure Lemma.** The space $Bun(S^1, G)$ of G bundles on (S^1, ν) can be described as follows:

$$Bun(S^1, G) = (G \times F)/G$$

where F is a generic fiber regarded as a principal G-space and G acts on itself by conjugation. and the quotient is taken by the diagonal action. Furthermore the space $Bun'(S^1, G)$ of pointed G bundles on S^1 can be described as follows:

$$Bun(S^1, G) = G \times F \sim G \times G.$$

3.1.5. **Remark.** Usually one uses the identification $Bun(M, G) = \mathrm{Hom}(\pi_1(M), G)/G$. Which we also use in the proof. However, for certain aspects of the theory – more precisely to include non–trivial characters– it is vital to include a point in the bundle respectively a trivialization rather then just a point in the base.

Proof of the structure Lemma. Given a pointed principal G bundle (B, S^1, π, G, G), $b \in B$ we set $\nu = \pi(b) \in S^1$. The choice of $b \in F = B_\nu$ gives and identification of F with G, i.e. we let $\beta : G \mapsto F$ be the admissible map that satisfies $\beta(e) = b$. We set $g \in G$ to be the element corresponding to the monodromy around the generator of $\pi_1(S^1)$. Notice, that since we fixed an admissible map everything is rigid – there are no automorphisms– and the monodromy is given by an element, not a conjugacy class. Thus we associate to (B, b) the tuple (g, b).

Vice versa, given (g, b) we start with the pointed (S^1, ν) and construct the bundle with fiber G, monodromy g and marked point $b = \beta(e) \in B_\nu$.

The bijectiveness of this construction follows by the classical results quoted above [St].

The choice in this construction corresponds to are a choice of a point $b \in B$. Changing b amounts to changing β and the monodromy g. Moreover, Moving the point $\nu = \phi(b)$ and moving b inside the fiber by parallel transport keeps everything fixed. Moving b inside the fixed fiber by translation (once ν is fixed) corresponds to translation by the group action in the fiber i.e. the translation action of G on G and simultaneous conjugation of the monodromy. Hence, the first claim follows.

This observation leads us to the following definition:

3.1.6. **Definition.** We call a bundle over a closed one–dimensional manifold rigidified if its compomnents are labelled and the bundle is pointed in each component. or equivalently if a point and a trivialization around that point are fixed in each component. We denote such a bundle (B, b) where $b \subset B : b = (b_0, \dots, b_n)$ is the set of base–points for each component. Furthemore, if the surface has genus zero we realize it in the plane as a pointed disc with all boundaries being S^1. We label the outside circle by 0. If π is the bundle projection, we set $x_i := \pi(b_i)$ we call b_0 the base–point, the B_{x_i} the special fibers and B_{x_0} the initial fiber.

3.2. **Rigidified bundle cobordisms and G–Frobenius algebras.**

3.2.1. **Definition.** Let $\mathcal{GBCOB}^*$ be the category whose objects are rigidified bundles principal G–bundles over one–dimensional closed oriented (topological) manifolds considered up to pull–back under orientation preserving homeomorphism of the base respecting the markings and whose morphisms are cobordisms of these objects (i.e. principal G–bundles over oriented surfaces with boundary together with rigidification on the boundary, a choice of base–point $x_0 \in \partial\Sigma$ compatible with the rigidification and a choice of curves Γ_i from x_0 to x_i.

I.e. objects are bundles B over $S = \coprod_{i\in I} S^1 \coprod_{j\in J} \bar{S}^1$ with base–points on the various components $\{(b_i \in B|_{S^1_i} : i \in I\}\{b_j \in B|_{\bar{S}^1_j} : j \in J\})$. and $B_\Sigma \in Hom((B_1, b_1), (B_2, b_2))$ if Σ is an oriented surface with boundary considered up to orientation preserving homeomorphism with boundary $\partial\Sigma = -S_1 \coprod S_2$ – again up to homeomorphism – and B_Σ is a bundle on Σ which restricts to B_1 and B_2 on the boundary together with rigidification data for the boundary i.e.(b_1, b_2) with $b_1 \subset B_1$ and $b_2 \subset B_2)$.

The extra structure of curves and base–point allows us to identify the special fibers with the initial fiber via parallel transport. Thus we can describe the rigidification data in terms of b_0, and the group elements g_i defined via $\Gamma_i(b_0) = g_i b_i$.

The composition of morphisms is given by gluing along boundaries with respect to orientation reversing homeomorphisms of the base and covering bundle isomorphisms identifying the base–points.

3.2.2. **Remark.** The operation of disjoint union makes this category into monoidal category with unit G regarded as a bundle over $\emptyset$ with base point e.

3.2.3. **Remark.** By the structure Lemma 3.1.4 we can identify the monoidal generators of objects of $\mathcal{R}GBCOB$ with $G \times G$.

A generating object can thus be depicted by an oriented circle with labels (g, h) where $(g, h) \in G \times G$. We use the notation as (g, h) for positively oriented circles and $\overline{(g, h)}$ for negatively oriented circles. Due to the involutive property we have that $\overline{(g, h)} \simeq (g^{-1}, h)^*$. General objects are then just disjoint unions of these, i.e. tuples (g_i, h_i). homomorphisms on the generators are given by the trivial bundle cylinder with different trivializations on both ends represented the natural diagonal action of G by conjugation and translation described in 3.1.4, so that this diagonal G–action is realized in terms of cobordisms.

The natural $G \times G$ action, however, cannot be realized by these cobordisms and we would like to enrich our situation to this case by adding morphisms corresponding to the G action on the trivializations.

3.2.4. **Definition.** Let $\mathcal{R}GBCOB$ be the category obtained $\mathcal{G}BCOB*$ by adding the following morphisms. For any n–tuple $(k_1, \ldots, k_n) : k_i \in G$ and any object $(g_i.h_i) : i = 1, \ldots n$ we set

$$\tau(k_1, \ldots, k_n)(g_i.h_i) : (g_i, k_i h_i)_{i \in \{1, \ldots n\}}$$

we call these morphisms of type II and the morphisms coming from $\mathcal{G}BCOB*$ of type I.

3.2.5. **Remark.** There is a natural forgetful functor from $\mathcal{RGBCOB}$ to $\mathcal{GBCOB}$.

Given a character $\chi \in Hom(G, k^*)$ we can form the fiber–product $G \times_\chi k$ this gives a functor $k[G]_\chi$ from the category $\mathcal{VECT}_k$ to $k[G] - \mathcal{MOD}$, the category of $k[G]$–modules.

3.2.6. **Definition.** A G_χ–orbifold theory is a monoidal functor V from $\mathcal{RGBCOB}$ to $k[G] - \mathcal{MOD}$ satisfying the following conditions:

i) The image of V lies in the image of $k[G]_\chi$
ii) Objects of $\mathcal{R}GBCOB$ which differ by morphisms of type II are mapped to the same object in $k[G] - \mathcal{MOD}$.
iii) The value on morphisms of type I does not depend on the choice of connecting curves or base–point.
iv) The morphisms of type II are mapped to the G–action by χ

v) The functor is natural with respect to morphisms of the type $\tau(k,\dots,k)$. I.e. $V(\tau(k,\dots,k)\circ\Sigma) = V(\tau_{out}(k^{-1},\dots,k^{-1}))\circ V(\Sigma)\circ V_{in}(\tau(k,\dots,k))$

vi) V associates id to cylinders $B\times I, b\in B\times 0, b\in B\times 1$ considered as cobordisms from (B,b) to itself.

vii) V is involutory: $V(\bar{S}) = V^*(S)$ where $*$ denotes the dual vector space with induced k[G]–module structure. In accordance, the morphism of type II commute with involution, i.e. they are mapped to the G–action by χ^{-1}.

vii) The functor is natural with respect to homeomorphisms of the underlying surface of a cobordism and pull–back of the bundle.

3.2.7. **Corollary.** G_χ–orbifold theories are homotopy invariant.

Proof. By standard arguments using naturality and vi) a homotopy of an object S induces an identity on its image. More precisely given a homotopy of objects $f_t: S\mapsto S$, it induces a map $F: S\times I\mapsto S\times I$ and we have the commutative diagram:

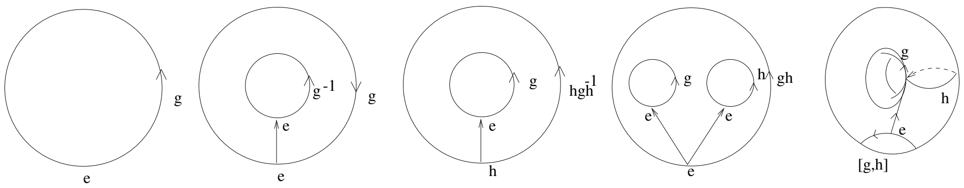
$$\begin{array}{ccc} V(S) & \overset{V(S\times I)=id}{\rightarrow} & V(S) \\ V(f_0)\downarrow & & \downarrow V(f_1) \\ V(S) & \overset{V(S\times I)=id}{\rightarrow} & V(S) \end{array}$$

In particular, if $V(f_0)=id$ then $V(f_1)=id$

3.2.8. **Remark.** Given a choice of connecting curves, we can identify all fibers over special points. Therefore after fixing one identification of a fiber with G, we can identify the other marked points as translations of points of parallel transport and label them by group elements, which we will do.

The action of $\tau(k,\dots,k)$ corresponds to a change of identification for one point and simultaneous re-gauging of all other points via this translation, i.e. a diagonal gauging. Therefore given a cobordism, we can fix an identification of all fibers with G.

3.2.9. **Definition and Notation.** We will fix some standard bundle cobordisms pictured below.



I II III IV V

I: The *standard disc bundle* is the disc with a trivial bundle and positively oriented boundary considered as a cobordism between $\emptyset$ and (e,e) it will be denoted by D.

II: The *standard g–cylinder bundle* is the cylinder $S^1 \times I$ with the bundle having monodromy g around $((S^1,0))$ considered as a cobordism between $(g,e)\coprod\overline{(g,e)}$ and $\emptyset$ it will be denoted by C_g.

III: The *$(g,k)^h$–cylinder bundle* is the cylinder with a bundle having monodromy g around $(S^1,0)$ considered as a cobordism between (g,e) and (hgh^{-1},h) it will be denoted by $C_{g,h}$.

IV: The*standard (g,h)–trinion bundle* is the trinion with the bundle having monodromies g around S^1_1 and h around S^2 and translations e for τ_{01},τ_{02} considered as a cobordism between $(g,e),(h,e)$ and (gh,e) it will be denoted by $T_{g,h}$.

V: The *(g,h)–torus bundle* is the once–punctured torus with the principal G–bundle having monodromies g and h around the two standard cycles considered as a cobordism between $([g,h],e)$ and $\emptyset$ it will be denoted by $E^k_{g,h}$.

3.2.10. **Lemma.** The bundles over a cylinder that form cobordisms between (g,e) and (h,k) are parameterized by G; it is necessary that $g=khk^{-1}$, these cobordisms are given by the $C_{(g,h)}$.

Proof. Given such a bundle over the cylinder $\Sigma_0 := S^1 \times I$ the translation from $B_{\nu,0}$ to $B_{\nu,1}$ along $\gamma(t) := (\nu,t)$ is a complete invariant. We fix this element $k\in G$ Since in $\pi_1(\Sigma_0,(\nu,0)$ $C_1=\gamma C_2\gamma^{-1}$ we must have $g=khk^{-1}$ we call these bundles $\Phi_{k,g}$

3.2.11. **Proposition.** To fix a G_χ–orbifold theory on the objects of the type (g,e) and to fix a G_χ–orbifold theory on the morphisms it suffices fix it on bundles over the standard cylinder bundle C, the (g,h)–cylinder bundles $C_{g,h}$, and on the standard trinion T.

Proof. The first claim follows from the condition ii) and the monoidal structure. For the second claim first notice that due to the homotopy Lemma 3.2.7 V is fixed on D. Also, any bundle over the cylinder is trivial, furthermore by v) we may regard the cylinder as a cobordism from $B|_{S^1}$ to itself and after applying a morphism of type II we can assume that the boundary objects ore of the type $(g,e),(h,k)$. Therefore we know the functor V on all bundles over cylinders. Dualizing and gluing on cylinders, we find that once the functor is defined on the standard trinion, it is defined on all bundles over all trinions. Lastly, given any surface, we can choose a decomposition by a marking into discs, cylinders and trinions, then gluing determines the value of V on this surface.

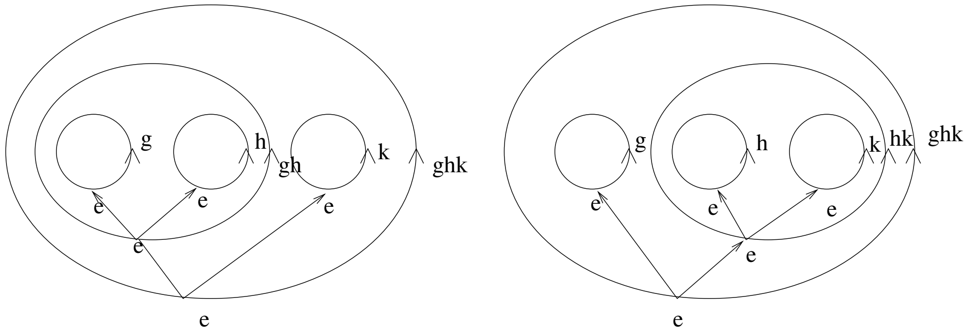
3.2.12. **Proposition and Definition.** For any G_χ–orbifold theory V set

$$\begin{array}{lcl} V((g,e)) & = & V_g \\ V(T_{g,h}) & = & \bar{\circ}: V_g \otimes V_h \to V_{gh} \\ V(D)(1_k) & = & v_0 \\ C_g & = & \bar{\eta}|_{V_g \otimes V_{g^{-1}}} \\ V(C_{g,h}) & = & \bar{\varphi}_h : V_g \mapsto V_{hgh^{-1}} \end{array}$$

then this data together with g and χ form a Ramond–G algebra $\langle G, V, \bar{\circ}, v, \bar{\eta}, \bar{\varphi}, \chi \rangle$ which we call the *associated G–Ramond algebra to V*.

Proof. It is clear by 3.2.11 that given a G_χ orbifold theory it is completely fixed by its associated G–Ramond algebra. The converse is also true:

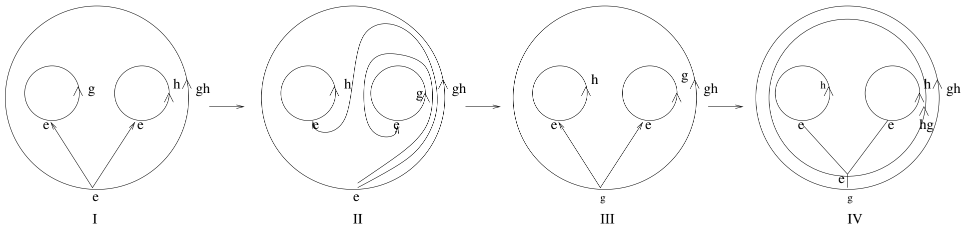
The axiom a) follows from the standard gluing procedures of TFT. I.e. the usual gluing of a disc with 3 holes from two discs with two holes in two different ways.



For axiom b') we regard the following commutative diagrams

$$\begin{array}{ccccccc} (g,e)\coprod(h,e) & \xrightarrow{\tau_{12}} & (h,e)\coprod(g,e) & & (h,e)\coprod(g,e) & & (g,e)\coprod(h,e) \\ \downarrow T^e_{g,h} & \Rightarrow & \downarrow II & \Rightarrow & \downarrow III & \Rightarrow & \downarrow IV \\ (gh,e) & \xrightarrow{id} & (gh,e) & & (gh,g) & & (gh,g) \\ & & & \Downarrow V & & & \\ V_g \otimes V_h & \xrightarrow{\tau_{12}} & V_h \otimes V_g & \xrightarrow{id} & V_h \otimes V_g & \xrightarrow{id} & V_h \otimes V_g \\ \downarrow \bar{\circ} & & & & & & \downarrow \bar{\varphi}_g \bar{\circ} \\ V_{gh} & \xrightarrow{id} & V_{gh} & \xrightarrow{id} & V_{gh} & \xrightarrow{id} & V_{gh} \end{array}$$

where we have used axiom vii) for the first move and axiom iii) for the second and gluing for the last one.

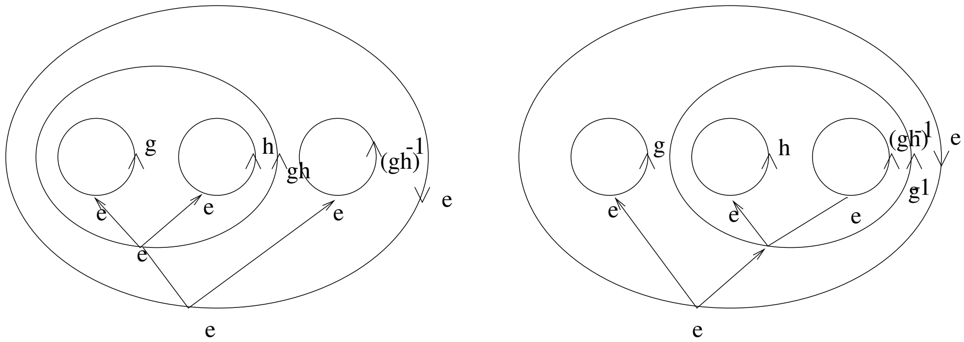


The unit of axiom c) is given by $\bar{D}^e$. The projective invariance follows from

$$\begin{array}{ccccccc}
\emptyset & & & \xrightarrow{D} & & & (e,e) \\
\downarrow \| & & & & & & \downarrow \| \\
\emptyset & \xrightarrow{D} & (e,e) & \xrightarrow{C^k_{(e,e)}} & (e,k) & \xrightarrow{II_{k-1}} & (e,e) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k & \xrightarrow{1_e} & V_e & \xrightarrow{\bar{\varphi}_k} & V_e & \xrightarrow{\bar{\chi}_{k-1}} & V_e \\
\downarrow \| & & & & & & \downarrow \| \\
k & & & \xrightarrow{1^e} & & & V_e
\end{array}$$

where in the third line $1_k \mapsto v_0 \mapsto \chi_k v_0 \mapsto v_0$

The axiom d) of the invariance of the metric is again a standard gluing argument.



I.e. using $\bar{\epsilon}$ and associativity:

$$\bar{\eta}(v_g \bar{\circ} v_h, v_{(gh)^{-1}}) = \bar{\epsilon}((v_g \bar{\circ} v_h) \bar{\circ} v_{(gh)^{-1}}) = \bar{\epsilon}(v_g \bar{\circ} (v_h \bar{\circ} v_{(gh)^{-1}})) = \bar{\eta}(v_g, v_h \bar{\circ} v_{(gh)^{-1}})$$

For axiom 1')

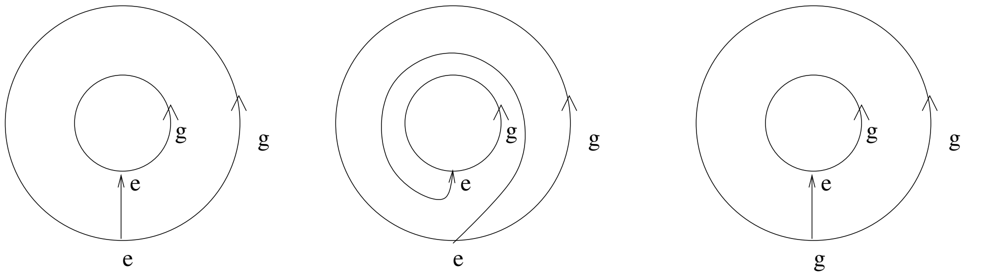
We use the following diagram:

$$\begin{array}{lll}
(g,e)\stackrel{id}{\to}(g,e) & & (g,e)\\
\downarrow id \Rightarrow \downarrow \tilde{C} & \Rightarrow & \downarrow C\\
(g,e)\stackrel{id}{\to}(g,e) & & (g,g)
\end{array}$$

$$\Downarrow V$$

$$\begin{array}{ccccc}
V_g & \stackrel{id}{\to} & V_g & \stackrel{id}{\to} & V_g\\
\downarrow id & & & & \downarrow \bar{\varphi}_g\\
V_g & \stackrel{id}{\to} & V_g & \stackrel{id}{\to} & V_g
\end{array}$$

where we used axiom vii) for the first move and axiom iii) for the second.



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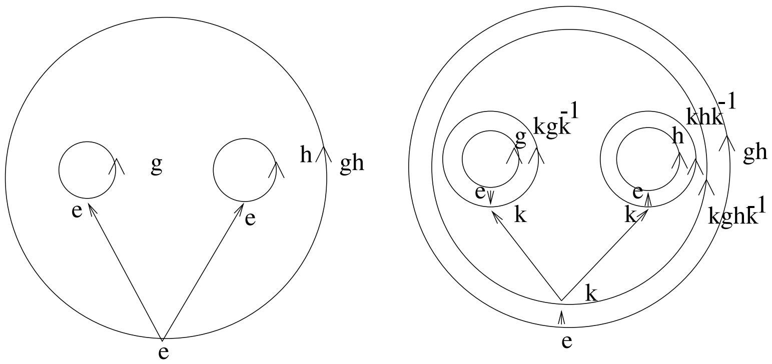
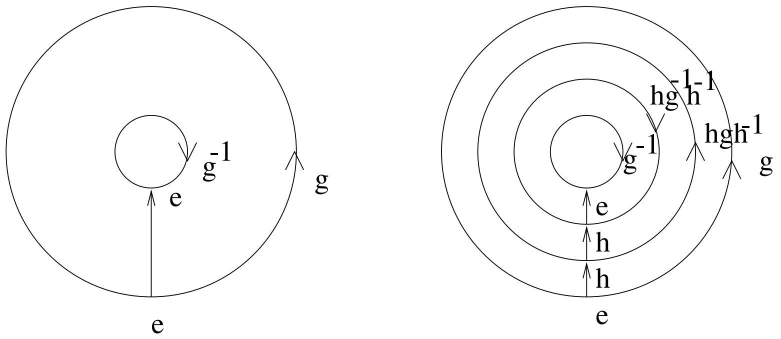
The axiom 2') follows from the diagrams

$$\begin{array}{ccccccc}
(g,e)\coprod(h,e) & \stackrel{(C^k_{(g,e)},C^k_{(h,e)})}{\longrightarrow} & (kgk^{-1},k)\coprod(khk^{-1},k) & \stackrel{T^k_{kgk^{-1},khk^{-1}}}{\longrightarrow} & (kghk^{-1},k) & \stackrel{C^{k^{-1}}_{(kghk^{-1},k)}}{\longrightarrow} & (gh,e)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
V_g\otimes V_h & \stackrel{\bar{\varphi}_k\otimes\bar{\varphi}_k}{\longrightarrow} & V_{kgk^{-1}}\otimes V_{khk^{-1}} & \stackrel{\circ^k}{\longrightarrow} & V_{kghk^{-1}} & \stackrel{\bar{\varphi}_{k^{-1}}}{\longrightarrow} & V_{gh}
\end{array}$$



$$\begin{array}{ccccccc}
(g,e)\coprod(h,e) & & & \xrightarrow{T^e_{gh}} & & & (gh,e) \\
\downarrow \| & & & & & & \downarrow \| \\
(g,e)\coprod(h,e) & \xrightarrow{(II_k,II_k)} & (g,k)\coprod(h,k) & \xrightarrow{T^k_{g,h}} & (gh,k) & \xrightarrow{II_{k^{-1}}} & (gh,e) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V_g\otimes V_h & \xrightarrow{\bar\chi_k\otimes\bar\chi_k} & V_g\otimes V_h & \xrightarrow{\circ^k} & V_{gh} & \xrightarrow{\bar\chi_{k^{-1}}} & V_{gh} \\
\downarrow \| & & & & & & \downarrow \| \\
V_g\otimes V_h & & & \xrightarrow{\bar\circ} & & & V_{gh}
\end{array}$$

The axiom 3') – G–invariance of the metric – follows from the following diagrams



$$\begin{array}{ccccc}
(g,e)\coprod\overline{(g,e)} & & \xrightarrow{A^e} & & K \\
\downarrow \| & & & & \downarrow \| \\
(g,e)\coprod\overline{(g,e)} & \overset{(C,\bar C)}{\to} & (kgk^{-1},k)\coprod\overline{(kgk^{-1},k)} & \overset{A^k}{\to} & K \\
\downarrow (id,\bar{}) & & \downarrow (id,\bar{}) & & \downarrow (id,\bar{}) \\
(g,e)\coprod(g^{-1},e) & \overset{(C,C)}{\to} & (kgk^{-1},k)\coprod(kg^{-1}k^{-1},k) & \overset{A^k}{\to} & K \\
\downarrow & & \downarrow & & \downarrow \\
V_g\otimes V_{g^{-1}} & \overset{\bar\varphi_k\otimes\bar\varphi_k}{\to} & V_{kgk^{-1}}\otimes V_{kg^{-1}k^{-1}} & \overset{\eta^k}{\to} & K
\end{array}$$

and

$$\begin{array}{ccccc}
(g,e)\coprod\overline{(g,e)} & & \xrightarrow{A^e} & & K \\
\downarrow \| & & & & \downarrow \| \\
(g,e)\coprod\overline{(g,e)} & \overset{(II_k,II_k)}{\to} & (g,,k)\coprod\overline{(g,k)} & \overset{A^k}{\to} & K \\
\downarrow (id,\bar{}) & & \downarrow (id,\bar{}) & & \downarrow (id,\bar{}) \\
(g,e)\coprod(g^{-1},e) & \overset{(II_k,II_k}{\to} & (g,,k)\coprod(g^{1},k) & \overset{A^k}{\to} & K \\
\downarrow & & \downarrow & & \downarrow \\
V_g\otimes V_{g^{-1}} & \overset{\bar\chi_k\otimes\bar\chi_{k^{-1}}}{\to} & V_g\otimes V_{g^{-1}k} & \overset{\eta^k}{\to} & K \\
\downarrow \| & & & & \downarrow \| \\
V_g\otimes V_{g^{-1}} & & \xrightarrow{\eta} & & K
\end{array}$$

Lastly axiom 4') comes from gluing a once punctured torus in two different ways.

3.2.13. **Proposition.** Given a G–Ramond algebra V there is a unique G_χ orbifold theory $\mathcal{V}$ s.t. V is its associated G–Ramond algebra.

Proof.

We need to show that the data is sufficient to reconstruct the functor. For the objects this is clear, due to the monoidal structure. For discs, annuli and trinions the functor is defined by its basic ingredients, glueing annuli $C_{g,h}$ to the basic trinion, and gauging with morphisms of type II. For other morphisms of type I we notice that we can always decompose a surface into trinions, annuli and discs. For each decomposition there are three choices. The choice of a marking for the decomposition, a choice of orientation and a choice of pairs of points on over the marked curves. The second and the third choice can be seen to be irrelevant by insering two annuli $C_{e,k}$ and $A_{k,e}$ with suitable orientation in a normal neighborhood of the curve in question. The first choice is unique up to two operations [HT] which are correspond to associativity and the trace axiom, and is thus also irrelevant.

Combining the Propositions 3.2.11 and 3.2.13 with 2.4.2 we obtain:

3.2.14. **Theorem.** There is a 1–1 correspondence between G-twisted FA and isomorphism classes of G_χ–orbifold theories as χ runs though the characters of G.

Proof. In the standard way, we make the G–twisted Frobenius algebras and the G_χ–orbifold theories into categories by introducing the following morphisms. For G–twisted Frobenius algebras we use algebra homomorphism respecting all the additional structures and for G_χ–orbifold theories we use natrural transformations among functors. The map of associating a G-twisted FA to a G_χ–orbifold theory then turns into a full and faithful funcor which is by reconstruction surjective on the objects.

An immediate consequence of this is:

3.2.15. **Corollary.** There is a 1–1 correspondence between G-orbifold FA and isomorphism classes of monoidal functors which are identity on cylinders and have the involutary property from $\mathcal{GBCOB}$ to $\mathcal{V}ECT$ which lift to $\mathcal{RGBCOB}$.

3.3. **Spectral flow.** In the previous paragraph, we chose the geometric version to correspond to the Ramond picture, as is suggested by physics, since we are considering the vacuum states in their Hilbert–spaces at the punctures. By spectral flow, however, the vacuum states should bijective correspond to the chiral algebra. In the current setting the difference only manifests itself in a change of G–action given by a twist resulting from the character χ. We can directly produce this G–action and thereby the G–Frobenius algebra by considering the G–action not by $C^e_{e,k}$, but by $II_{k^{-1}} \circ C^e_{e,k}$. This statement is proven by re-inspection the commutative diagrams in the proof of 2.4.2

4. Special G–Frobenius algebras

4.1. **Definition.** A *special G–Frobenius algebra* is a G–twisted Frobenius algebra whose components A_g are cyclic A_e–modules, together with two collections of maps $(r_g), (i_g)$ indexed by G where

$r_g : A_e \to A_g$ is the map of A_e–modules induced by multiplication, we write $a_g := r_g(a)$ in particular if one sets $1_g := r_g(1)$ we obtain $r_g(a) = a1_g$. Notice that it is equivalent to specify the map r_g or generators 1_g of A_g as cyclic A_e–modules.
$i_g : A_g \mapsto A_e$ are collections of maps s.t. each i_g is an injection which splits the exact sequence of A_e-modules.

$$0 \longrightarrow I_g \longrightarrow A \underset{i_g}{\overset{r_g}{\rightleftarrows}} A_g \longrightarrow 0 \tag{4.1}$$

Here $I_g \subset A_e$ is the annihilator of 1_g and thus of A_g.

Furthermore the following should hold:

$$\forall g \in G : \varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}$$

with $\varphi_{g,h} \in K$ and $\varphi_{g,1} = 1$.

We denote the concatenation of i_g and r_g by π_g

$$\pi_g = r_g \circ i_g : A_e \to A_e$$

Notice that multiplication by 1_g acts as projection: $a1_g = \pi_g(a)1_g$.

4.1.1. **Special super G–Frobenius algebra.** The super version of special G–Frobenius algebras is straightforward. Notice that since each A_g is a cyclic A_e–algebra its parity is fixed to be $(-1)^{\tilde{g}} := \widetilde{1_g}$ times that of A_e. I.e. $a_g = i_g(a_g)1_g$ and thus $\widetilde{a_g} = \widetilde{i_g(a_g)}\widetilde{1_g}$. In particular if A_e is purely even A_g is purely of degree $\tilde{g}$.

4.2. **Proposition.** Let $\eta_g : i_g(A_g) \otimes i_g(A_g) \to K$: be given by the following formula.

$$\eta_g(a, b) = \eta(r_{g^{-1}}(a), r_g(b))$$

Then $i_g(A_g) \subset A_e$ together with η_g and the induced multiplication $a \circ_g b = i_g(a \circ r_g(b)) = \pi_g(a \circ b)$ is a Frobenius sub–algebra. Furthermore η_g establishes an isomorphism of A_e–sub–modules between $i_g(A_g)$ and $i_{g^{-1}}(A_{g^{-1}})$.

Proof. That the multiplication is commutative and associative can be seen as follows

$$a \circ_g b = i_g(ab1_g)) = i_g(ba1_g) = b \circ_g c$$

likewise
$(a\circ_g b)\circ_g c = i_g((ab)c1_g)) = i_g(a(bc)1_g) = a\circ_g(b\circ_g c)$ η_g is an invariant non–degenerate form.

The non–degeneracy follows directly from the non–degeneracy of η: fix $a \in i_g(A_g)$ and let $a_g \in A_g$ and $b_{g^{-1}} \in A_{g^{-1}}$ s.t. $i_g(a_g) = a$ and $\eta(a_g, b_{g^{-1}}) \neq 0$. Set $b := i_{g^{-1}}(b_{g^{-1}})$ and $\tilde{b} = \pi_g(b)$

Then

$$\begin{aligned} \eta_g(a,\tilde{b}) &= \eta(a1_g, \tilde{b}1_{g^{-1}}) = \eta(\tilde{b}a1_g, 1_{g^{-1}}) = \eta(ba1_g, 1_{g^{-1}}) \\ &= \eta(a1_g, b1_{g^{-1}}) = \eta(a_g, b_{g^{-1}}) \neq 0 \end{aligned}$$

The invariance follows from

$$\eta_g(ab,c) = \eta(r_{g^{-1}}(ab), r_g(c)) = \eta(ab1_{g^{-1}}, c1_g) = \eta(a1_{g^{-1}}, bc1_g) = \eta_g(a, bc)$$

With these two properties, we get a map $i_g(A_g) \to i_g(A_g)^*$ and η gives an isomorphism $i_g(A_g)^* \to i_{g^{-1}}(A_{g^{-1}})$ via $(a,b) \in A_g \otimes A_{g^{-1}} \mapsto \eta(a1_g, b1_{g^{-1}})$.

4.3. **Proposition.** The following equality holds $\pi_{g^{-1}} \circ \pi_g = \pi_g$ showing that $\pi_{g^{-1}}|_{i_g(A_g)} = id$ identifying $i_g(A_g)$ and $i_{g^{-1}}(A_{g^{-1}})$.

Proof. We have the following:

$$\begin{aligned} \eta(\pi_{g^{-1}} \circ \pi_g(a)1_g, b_{g^{-1}}) &= \eta(1_g, \pi_{g^{-1}}(\pi_g(a) i_{g^{-1}}(b_{g^{-1}}))1_{g^{-1}}) \\ &= \eta(\pi_g(a)\eta(1_g, i_{g^{-1}}(b_{g^{-1}})1_{g^{-1}}) \\ &= \eta(\pi_g(a)1_g, b_{g^{-1}}) \end{aligned}$$

which proves the statement due to the non–degeneracy of η.

4.4. **Definition.** Let A be special G–twisted Frobenius algebra A. We define a *G graded 2–cocycle with values in A_e* to be a map $\gamma : G \times G \to A_e$ which satisfies

$$\gamma_{g,h} := \gamma(g,h) \in i_{gh}(A_{gh}) \tag{4.2}$$

and

$$\pi_{ghk}(\gamma_{g,h}\gamma_{gh,k}) = \pi_{ghk}(\gamma_{h,k}\gamma_{g,hk}) \tag{4.3}$$

A *non–abelian G 2–cocycle with values in K^** is a map $\varphi : G \times G \to K$ which satisfies:

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k} \tag{4.4}$$

where $\varphi_{g,h} := \varphi(g,h)$ and

$$\varphi_{e,g} = \varphi_{g,e} = 1$$

Notice that in the case of a commutative group G this says that the $\varphi_{g,h}$ form a two cocycle with values in K^*.

Furthermore setting $g = h^{-1}$, we find:

$$\varphi_{g^{-1},ghg^{-1}} = \varphi_{g,h}^{-1}$$

4.5. **Proposition.** A special G–twisted Frobenius algebra gives rise A to a g 2–cocycle γ with values in A and to an non–abelian G 2–cocycle φ with values in K^*, which satisfy the compatibility equations

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \tag{4.5}$$

and

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{4.6}$$

Proof.

Given a special G twisted Frobenius structure on A, we define $\gamma_{g,h} \in i_{gh}(A_{gh})$ by

$$1_g 1_h = \gamma_{g,h} 1_{gh}$$

and $\varphi_{g,h} \in K^*$ by:

$$\varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}.$$

By associativity of the multiplication, we find that the $\gamma_{g,h}$ define a graded 2–cocycle with values in A.

$$\gamma_{g,\beta}\gamma_{gh,k}1_{ghk} = \gamma_{g,h}1_{gh}1_k = (1_g 1_h)1_k = 1_g(1_h 1_k) = 1_g\gamma_{h,k}1_{hk} = \gamma_{h,k}\gamma_{g,hk}1_{ghk} \tag{4.7}$$

So that

$$\pi_{ghk}(\gamma_{g,h}\gamma_{gh,k}) = \pi_{ghk}(\gamma_{h,k}\gamma_{g,hk}).$$

φ is a group homomorphism so that

$$\varphi_{gh,k}1_{ghkh^{-1}g^{-1}} = \varphi_{gh}(1_k) = \varphi_g(\varphi_h(1_k)) =$$
$$\varphi_g(\varphi_{h,d}1_{hkh^{-1}}) = \varphi_{g,hkh^{-1}}\varphi_{h,k}1_{ghkh^{-1}g^{-1}}$$

which yields

$$\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k} \tag{4.8}$$

By G–twisted commutativity

$$\gamma_{g,h}1_{gh} = 1_g 1_h = \varphi_g(1_h)1_g = \varphi_{g,h}1_{ghg^{-1}}1_g = \varphi_{g,h}\gamma_{ghg^{-1},g}1_{gh}$$

So $\gamma_{g,h}$ and the $\varphi_{g,h}$ satisfy

$$\varphi_{g,h}\pi_{gh}(\gamma_{ghg^{-1},g}) = \pi_{gh}\gamma_{g,h} \tag{4.9}$$

φ is also an algebra automorphism:

$$\varphi_k(1_g)\varphi_k(1_h) = \varphi_k(1_g 1_h)$$

Expressed in the φ's and γ's:

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{4.10}$$

which gives a formula for the action of φ on the γ's

4.5.1. **Corollary.** A special G-Frobenius algebra gives rise to collection of Frobenius–algebras $(A_g, \circ_g, 1_g, \eta_g)_{g\in G}$ together with a G–action on A_e. a A graded G 2–cocycle with values in A_e and a compatible non–abelian G 2–cocycle with values in k^*. Furthermore:

i) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2}\eta_e(a,b)$
ii) $\eta_g(a_g, b_g) = \eta(i_g(a_g)i_g(b_g))\gamma_{gg^{-1}}, 1)$
iii) The projective trace axiom $\forall c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\mathrm{Tr}(\varphi_{g^{-1}}l_c|_{A_h}) \tag{4.11}$$

4.6. **Proposition.** (Reconstruction) Given a Frobenius algebra (A_e, η_e) and a G-action on A:$\varphi : G \times A \to A$ together with the following data

- Frobenius algebras $(A_g, \eta_g | g \in G \setminus \{1\})$.
- Injective algebra homomorphisms $i_g : A_g \to A_e$ s.t. $i_g(A_g) = i_{g^{-1}}(A_{g^{-1}})$.
- A graded G 2–cocycle γ with values in A_e and a compatible non–abelian G 2–cocycle φ with values in K^*
- A group homomorphism $\chi : G \to K^*$

such that i)-iii) of 4.5.1 hold, then there is a unique extension of this set of data to a special G–twisted Frobenius algebra, i.e. there is a unique special G–twisted Frobenius algebra with these underlying data.

Proof.
Denote the unit of A_e by 1 and the unit of A_g by 1_g.
Define the algebra epimorphism $r_g : A_g \mapsto A$ by $r_g(1) := 1_g$
Denote by $\pi_g : A_e \to i_g(A_g)$ the projection to A_g.
Let the A_e module structure on A_g be given by

$$ab_g := i_g^{-1}(\pi_g(a i_g(b_g))$$

it is clear that A_g is a cyclic A_e–module generated by 1_g.
The algebra structure is then determined by:

$$a_g b_h := i_g(a_g)i_h(b_h)\gamma_{g,h}1_{gh}$$

This multiplication is associative as can be seen by reading 4.7 backwards.

The φ's determine the G-action by: $\varphi_g(b_h) := \varphi_g(i_h(b_h))\varphi_{g,h}1_{ghg^{-1}}$.

The compatibility and 4.6 guarantee that this is a representation and the G action is indeed an action by algebra automorphisms.

The G–twisted commutativity $a_g b_h = \varphi_g(b_h)a_g$ also follows from the compatibility.

The form η is defined the following way:

$$\eta(a_g, h_{g^{-1}}) := \eta_e(i_g(a_g)i_{g^{-1}}(b_{a^{-1}})\gamma_{gg^{-1}}, 1) = \eta_g(i_g(a_g), i_{g^{-1}}(b_{a^{-1}}))$$

and

$$\eta(a_g, h_h) := 0 \text{ if } gh \neq 1$$

This form is non–degenerate by assumption and it is invariant:

$$\begin{aligned}
\eta(a_g b_h, d_{(gh)^{-1}}) &= \eta_e(i_g(a_g)i_h(b_h)i_{h^{-1}g^{-1}}(d_{h^{-1}g^{-1}})\gamma_{gh}\gamma_{gh,h^{-1}g^{-1}}, 1) \\
&= \eta_e(i_g(a_g)i_h(b_h)i_{h^{-1}g^{-1}}(d_{h^{-1}g^{-1}})\gamma_{h,h^{-1}g^{-1}}\gamma_{g,g^{-1}}, 1) \\
&= \eta(g_g, b_h d_{h^{-1}g^{-1}})
\end{aligned}$$

4.6.1. **Remark.** By straightforward calculation we have that the projective trace axiom is equivalent to
$\forall g, h \in G; c1_{ghg^{-1}h^{-1}} \in A_{[g,h]}, c \in i_{hgg^{-1}h^{-1}}(A_{[gh]})$

$$\begin{aligned}
&\chi_h\varphi_{h,g}\mathrm{Tr}(l_{\gamma_{ghg^{-1}h^{-1},hgh^{-1}}c}\varphi_h|_{i_g(A_g)}) = \\
&\qquad \chi_{g^{-1}}\varphi_{g^{-1},ghg^{-1}}\mathrm{Tr}(\varphi_{g^{-1}}l_{\gamma_{hg^{-1}h^{-1}g,g^{-1}hg})}c|_{i_h(A_h)}) \quad (4.12)
\end{aligned}$$

Notice that in the graded case (see below) this condition only needs to be checked for $\deg(\gamma_{ghg^{-1}h^{-1},hgh^{-1}}c) \neq 0$ and $\deg(\gamma_{hg^{-1}h^{-1}g,g^{-1}hg})c) \neq 0$.

Furthermore if $[g,h] = e$ then $\gamma_{ghg^{-1}h^{-1},hgh^{-1}} = \gamma_{hg^{-1}h^{-1}g,g^{-1}hg}) = 1_e$ and $\varphi_{g^{-1},ghg^{-1}} = \varphi_{g,h}^{-1} = \varphi_{g^{-1},h}$.

4.7. **Remarks.**

1) If $A_g A_h \neq 0$ the compatibility condition (4.5) already determines the $\varphi_{g,h} \in K^*$.
2) In particular: $\gamma_{g,g} = 0$ unless $\chi_g = 1$ and if $[g,h] = e$ we have $\varphi_{g,h}\varphi_{hg} = 1$ or $\gamma_{g,h} = \gamma_{h,g} = 0$.
3) If also $A_g A_h A_k \neq 0$ the elements defined by (4.5) automatically satisfy the conditions of non–abelian 2–cocycles and the condition (4.6) is automatically satisfied.

Proof Without loss of generality, we may assume that $1_g 1_h 1_k \neq 0$ then due to the condition 4.5

$1_g 1_h 1_k = (1_g 1_h) 1_k = \varphi_{gh}(1_k)(1_g 1_h) = (\varphi_{gh}(1_k) 1_g) 1_h = \varphi_{gh,k} 1_{ghkh^{-1}g^{-1}} 1_g 1_h$

and using associativity, we similarly obtain

$1_g 1_h 1_k = 1_g(1_h 1_k) = 1_g(\varphi_h(1_k) 1_h = \varphi_g(\varphi_{b,k} 1_{hkh^{-1}}) 1_g 1_h = \varphi_{g,hkh^{-1}} \varphi_{h,k} 1_{ghkh^{-1}g^{-1}} 1_g 1_h$

For the first statement in 2) one just needs to plug in $g = h$ into (4.5) and for the second one has to apply the formula twice. For 3) notice that $1_k 1_g 1_h = 1_k(1_g 1_h) = \varphi_k(1_g 1_h) 1_k$ and on the other hand $1_k 1_g 1_h = \varphi_k(1_g) 1_k 1_h) = \varphi_k(1_g) \varphi_k(1_h) 1_k$.

4.7.1. **Graded special G–Frobenius algebras.** Consider a set of graded Frobenius algebras satisfying the reconstruction data: $\{(A_g, \eta_g) : g \in G\}$ with degrees $d_g := \deg(\eta_g)$
s.t. $A_g \simeq A_{g^{-1}}$. E.g. in the cohomology of fixed point sets d_g is given by the dimension for the Jacobian Frobenius manifolds d_g fixed by the degree of $Hess(f_g) = \rho_g$. Furthermore, the reconstructed $\{\eta|_{(A_g \otimes A_{g^{-1}}}, g \in G\}$ have degree $d_g = d_{g^{-1}}$.

For a G–twisted FA the degrees all need to be equal to $d := d_e$. To achieve this, one can shift the grading in each A_g by s_g this amounts to assigning degree s_g to 1_g. This is the only freedom, since the multiplication should be degree–preserving and all A_g are cyclic..

Set
$s_g^+ := s_g + s_{g^{-1}}$
$s_g^- := s_g - s_{g^{-1}}$

Then $s_g^+ := d - d_g$ for grading reasons, but the shift s^- is more elusive.

4.7.2. **Definition.** The standard shift for a Jacobian Frobenius algebra is given by

$$s_g^+ := d - d_g$$

and

$$s_g^- := \frac{1}{2\pi i}\mathrm{tr}(\log(g)) - \mathrm{tr}(\log(g^{-1})) := \frac{1}{2\pi i}(\sum_i \lambda_i(g) - \sum_i \lambda_i(g^{-1}))$$
$$= \sum_{i:\lambda_i \neq 0} 2(\frac{1}{2\pi i}\lambda_i(g) - 1)$$

where the $\lambda_i(g)$ are the logarithms of the eigenvalues of g using the branch with values in $[0, 2\pi)$ i.e. cut along the positive real axis.

In this case we obtain:

$$s_g = \frac{1}{2}(s_g^+ + s_g^-) = \frac{1}{2}(d - d_g) + \sum_i (\frac{1}{2\pi i}\lambda_i(g) - \frac{1}{2})$$

4.7.3. **Remark.** The shift s_g^- is canonical in the case of quasi–homogeneous singularities upon replacing the classical monodromy operator J by Jg. This will be discussed in the elsewhere [K2].

The degree of γ_{g,g^1} is s^+ from comparing degrees in the equation $1_g 1_{g^{-1}} = \gamma_{g,g^1} 1_e$

4.7.4. **Reconstruction for graded special G–Frobenius algebras.** In the Reconstruction program-m the presence of a non–trivial grading can greatly simplify the check of the trace axiom. E.g. if $A_{[g,h]}$ has no element of degree 0 then both sides of this requirement are 0 and if $[g,h] = e$ in venereal one needs only to look at the special choice c with $\deg(c) = 0$ which most often is just $c = 1$ the identity.

4.7.5. **Ramond–grading.** The grading in the Ramond–sector is by the following definition

$$\deg(v_0) := -\frac{d}{2}$$

This yields

$$\deg(\bar{\eta}) = 0 \text{ and } \deg(\bar{\circ}) = \frac{d}{2}$$

5. Jacobian Frobenius algebras

5.1. **Definition.** A Frobenius algebra A is called *Jacobian*

if it can be represented as the Milnor ring of a function f. I.e. If there is a function $f \in \mathcal{O}_{\mathbf{A}^n_K}$ s.t. $A = \mathcal{O}_{\mathbf{A}^n_K}/J_f$ where J_f is the Jacobian ideal of f. And the bilinear form is given by the residue pairing. This is the form given by the the Hessian of $\rho = \mathrm{Hess}_f$.

If we write $\mathcal{O}_{\mathbf{A}^n_K} = K[x_1 \dots x_n]$, J_f is the ideal spanned by the $\frac{\partial f}{\partial x_i}$.

A *realization of a Jacobian Frobenius algebra* is a pair (A, f) of a Jacobian Frobenius algebra and a function f on some affine K space $\mathbf{A}^n_K$, i.e. $f \in \mathcal{O}_{\mathbf{A}^n_K} = K[x_1 \dots x_n]$ s.t. $A = K[x_1 \dots x_n]$ and $\rho := \det(\frac{kel^2 f}{kel x_i \partial x_j})$.

A *small realization of a Jacobian Frobenius algebra* is a realization of minimal dimension, i.e. of minimal n.

5.2. **Definition.** A *natural G action on a realization of a Jacobian Frobenius algebra* (A_e, f) is a linear G action on $\mathbf{A}^n_K$ which leaves f invariant.

Given a natural G action on a realization of a Jacobian Frobenius algebra (A, f) set for each $g \in G$, $\mathcal{O}_g := \mathcal{O}_{\mathrm{Fix}_g(\mathbf{A}^n_K)}$ this is the ring of functions of the fixed point set of g for the G action on $\mathbf{A}^n_K$. These are the functions fixed by g: $\mathcal{O}_g = K[x_1, \dots, x_n]^g$.

Denote by $J_g := J_{f|_{\mathrm{Fix}_g(\mathbf{A}^n_K)}}$ the Jacobian ideal of f restricted to the fixed point set of g.

Define

$$A_g := \mathcal{O}_g / J_g \tag{5.1}$$

The A_g will be called twisted sectors for $g \neq 1$. Notice that each A_g is a Jacobian Frobenius algebra with the natural realization given by $(A_g, f|_{\mathrm{Fix}_g})$. In particular, it comes equipped with an invariant bilinear form η_g defined by the element $\mathrm{Hess}(f|_{\mathrm{Fix}_g})$

For $g = 1$ the definition of A_e is just the realization of the original Frobenius algebra, which we also call the untwisted sector.

Notice there is a restriction morphism $r_g : A_e \to A_g$ given by $a \mapsto a \bmod J_g$.

Denote $r_g(1)$ by 1_g. This is a non–zero element of A_g since the action was linear. Furthermore it generates A_g as a cyclic A_e module.

The set $\mathrm{Fix}_g \mathbf{A}^n_K$ is a linear subspace. Let I_g be the vanishing ideal of this space.

We obtain a sequence

$$0 \to I_g \to A_e \stackrel{r_g}{\to} A_g \to 0$$

Let i_a be any splitting of this sequence induced by the inclusion: $\hat{i}_g : \mathcal{O}_g \to \mathcal{O}_e$ which descends due to the invariance of f.

In coordinates, we have the following description. Let $\mathrm{Fix}_g \mathbf{A}^n_K$ given by equations $x_i = 0 : i \in N_g$ for some index set N_g.

Choosing complementary generators $x_j : j \in T_g$ we have $\mathcal{O}_g = K[x_j : j \in T_g]$ and $\mathcal{O}_e = K[x_j, x_i : j \in T_g, i \in N_g]$. Then $I_g = (x_i : i \in N_g)_{\prime_e}$ the ideal in $\mathcal{O}_e$ generated by the x_i then $\mathcal{O}_e = I_g \oplus i_g(A_g)$ using the splitting i_g coming from the natural inclusion $\hat{i}_g : K[x_j : j \in T_g] \to K[x_j, x_i : j \in T_g, i \in N_g]$ We also define the projections

$$\pi_g : A_1 \to A_g; \pi_g = i_g \circ r_g$$

which in coordinates is given by $f \mapsto f|_{x_j = 0 : j \in N_g}$ Let

$$A := \bigoplus_{g \in G} A_g$$

where the sum is a sum of A_e modules.

5.3. **Theorem.** Given a natural G action on a realization of a Jacobian Frobenius algebra (A_e, f) with a quasi–homogeneous function f of degree d and type $\mathbf{q} = (q_1, \dots, q_n)$ together with the natural choice of splittings i_g the possible structures of special G twisted Frobenius algebra on the A_e module $A := \bigoplus_{g \in G} A_g$ are in 1–1 correspondence with elements of $\bar{Z}^2(G, A^G)$. the $\bar{Z}^2$ are G graded cocycles and a choice of parity which corresponds to a character $\sigma \in \mathrm{Hom}(G, \mathbf{Z}/2\mathbf{Z})$, s.t. for all elements of order 2 the character is trivial.

Proof By the reconstruction Theorem the structure of special G twisted Frobenius algebra on the given data A_g, η_g, i_g is determined by:

- A graded G 2–cocycle with values in A_e and a compatible non–abelian G 2–cocycle with values in K^*
- A group homomorphism $\chi : G \to K^*$

subject to the conditions:

i) $\eta_e(\varphi_g(a), \varphi_g(b)) = \chi_g^{-2} \eta_e(a, b)$
ii) $\eta_g(a_g, b_g) = \eta(i_g(a_g b_g))\gamma_{gg^{-1}}, 1)$
iii) For any $c \in A_{[gh]}$:

$$\chi_h \mathrm{STr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h})$$

where we can restrict to the c with $\deg(c) = 0$.

The χ_g are fixed by the equation

$$(-1)^{\tilde{g}} \dim(A_g) = \mathrm{STr}(id|_{A_{g^{-1}}}) = \chi_g \mathrm{STr}(\varphi_g|_{A_e}) = \chi_g \mathrm{Tr}(\varphi_g|_{A_e})$$

To calculate the trace on the RHS, we use the character function for a morphism g of degree 0 on graded module $V = \oplus_n V_n$:

$$\chi_{V_n}(g, z) := \sum_{n,\mu} \mu \dim(V_{\mu,n}) z^n$$

where $V_{\mu,n}$ is the Eigenspace of Eigenvalue μ on the space V_n. We will use the grading induced by the quasi–homogeneity. I.e. let N be such that $q_i = Q_i/N$ with $Q_i \in \mathbf{N}$ and N s.t. $|G|\big|N$. Then a monomial has degree n if its quasi homogeneous degree is n/N. This is the natural grading for the quasi–homogeneous map $\mathrm{grad}(f)$. Notice that since g commutes with f it preserves the grading. It is clear that this character behaves multiplicatively under concatenations of quasi–homogeneous functions. Therefore by applying Arnold's method, we can pass to a cover of K^n with the projection map $T : T(x_1, \dots, x_n) = (x^{q_1}, \dots, x^{q_n})$ and calculate the character for T and for $\mathrm{grad}(f) \circ T$. Then repeating the argument in a simultaneous Eigenbasis of g and the grading of [A], we obtain:

$$\chi_{A_e}(g,z) = \prod_{i=1}^{n} \frac{(\tilde{\mu}_i) z^{N-Q_i} - 1}{(\tilde{\mu}_i z)^{Q_i} - 1}$$

where the $\tilde{\mu}_i$ are the Eigenvalues of some lift of the action of g i.e. $\tilde{\mu}_i^{Q_i} = \mu_i$. Notice that since $|G|\big|N$ $\mu^N = 1$, so that in the limit of $z \to 1$, we obtain:

$$\mathrm{Tr}(\varphi_g|_{A_e}) = \prod_{i:\mu_i \neq 1} -\mu_i^{-1} \prod_{i:\mu_i = 1} \frac{1}{q_i} - 1 = (-1)^{|N_g|} \mathrm{det}(g)^{-1} \dim(A_g)$$

so that

$$\chi_g = (-1)^{\tilde{g}} (-1)^{|N_g|} \mathrm{det}(g)$$

We set

$$\sigma(g) := \tilde{g} + |N_g| \mod 2$$

and call it the sign of g. Notice that $\sigma \in \mathrm{Hom}(G, \mathbf{Z}/2\mathbf{Z})$ since both det and χ are characters. Also notice that a choice of sign corresponds to a choice of parity and vice–versa. Thus we obtain

$$\chi_g = (-1)^{\sigma(g)} \det(g)$$

This ensures condition i). Since for $\rho = \mathrm{Hess}(f)$:

$$\varphi_g(\rho) = \mathrm{det}(g)^{-2} \rho$$

denote the element defining η_g by $\tilde{\rho}_g$ and by $\rho_g := i_g(\tilde{\rho}_g)$ condition ii) implies

$$\gamma_{g,g^{-1}} \rho_g = \rho$$

also we need

$$\gamma_{g,g^{-1}} = \chi_g \gamma_{g^{-1},g} (-1)^{\tilde{g}\widetilde{g^{-1}}}$$

Here we have a freedom of scaling, which we choose to distribute equally. More precisely, let $Jac(f)$ the Jacobian matrix of f in the above basis be given by the block form

$$Jac(f) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then $\rho_g = \det(D)$ and this equation is solved by

$$\gamma(g, g^{-1}) = ((-1)^{\tilde{g}} \chi_g)^{1/2} \mathrm{det}(A - BD^{-1}C)$$

where we choose to cut the plane along the negative real axis. This uniquely defines a square root unless $(-1)^{\tilde{g}} \chi_g = -1$ in the case that $g^2 \neq e$ we can choose roots i and $-i$ for g respectively g^{-1}, the only case that has no solution is the case of $g^2 = e$ and $(-1)^{\tilde{g}} \chi_g = -1$, since there is no non–degenerate invariant form on an odd cyclic algebra. This means that if $g^2 = e$ either $\chi_g = -1$ and $\tilde{g} = 1$ or $\chi_g = -1$ and $\tilde{g} = 1$ or in other words we need $(-1)^| N_g| = \det(g)$.

Finally, we need to check the validity of iii). Notice that since the multiplication is graded the traces are 0 unless $\deg(\gamma c) = 0$ so that we can assume that $\gamma c = 1$. In this case, we have to show:

$$\chi_h \mathrm{STr}(\varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{STr}(\varphi_{g^{-1}}|_{A_h})$$

Let x_i be a basis of A_e in which g is diagonal. Then we have to compute the trace of the action of h on the sub-algebra generated by the x_i with eigenvalue 1 under the action of g. This is just the truncated version of the calculation above, so diagonalizing h on $K[x_i : i \in T_g]$ we find using the same characteristic functions and rationale as before:

$$\begin{aligned}
& \chi_h \mathrm{STr}(\varphi_h|_{A_g}) = \chi_h \varphi_{h,g} (-1)^{\tilde{g}} \mathrm{Tr}(\varphi_h|_{i_g(A_g)}) \\
= \;& \chi_h \varphi_{h,g} (-1)^{\tilde{g}} \prod_{j:\nu_j \neq 1} -\nu_j^{-1} \prod_{j:\nu_j = 1} \frac{1}{q_j} - 1 \\
= \;& \chi_h \varphi_{h,g} (-1)^{\tilde{g}} (-1)^{|T_g|} (-1)^{|T_g \cap T_h|} \det(h|_{T_g})^{-1} \dim(i_g(A_g) \cap i_h(A_h)) \\
= \;& \varphi_{h,g} (-1)^{\sigma(g)} (-1)^{\sigma(h)} \det(h) \det(h|_{T_g})^{-1} (-1)^N (-1)^{|T_g \cap T_h|} \dim(i_g(A_g) \cap i_h(A_h)) \\
= \;& \varphi_{h,g} (-1)^{\sigma(hg)} \det(h|_{N_g}) (-1)^{|T_g \cap T_h| + N} \dim(i_g(A_g) \cap i_h(A_h)) \\
= \;& \epsilon(h,g) T(h,g)
\end{aligned}$$

where ν_j are the Eigenvalues of h on $i_g(A_g)$ and

$$\epsilon(g,h) = \varphi_{g,h} (-1)^{\sigma(g)\sigma(h)} \det(g|_{N_h})$$

and

$$T(h,g) = (-1)^{\sigma(g)\sigma(h)} (-1)^{\sigma(g)+\sigma(h)} (-1)^{|T_g \cap T_h| + N} \dim(i_g(A_g) \cap i_h(A_h))$$

we have that

$$T(h,g) = T(g,h) = T(g^{-1},h)$$

Notice also that if $[g,h] = e$

$$\epsilon(gh,k) = \epsilon(g,k)\epsilon(h,k)$$

and

$$T(gh,h) = T(hg,h) = T(g,h)$$

On the other hand

$$\begin{aligned} & \mathrm{STr}(\varphi_{g^{-1}}|_{A_h}) = \epsilon(g^{-1},h)(-1)^{\sigma(g^{-1}h)}(-1)^{|T_g\cap T_h|+N}\dim(i_g(A_g)\cap i_h(A_h)) \\ = \ & \varphi_{h,g}(-1)^{\sigma(g^{-1}h}\det(g^{-1}|_{N_h})(-1)^{\tilde{g}\tilde{h}}(-1)^{|T_g\cap T_h|}\dim(i_g(A_g)\cap i_h(A_h)) \\ = \ & \epsilon(g^{-1},h)T(g^{-1},h) \end{aligned}$$

where μ_j are the Eigenvalues of g^{-1} on $i_h(A_h)$.

Finally we see that the φ are determined by the $\epsilon(g,h)$ which have to satisfy

$$\epsilon(g,h)=\epsilon(h^{-1},g) \quad \epsilon(g,g)=1 \quad \epsilon(g_1g_2,h)=\epsilon(g_1,h)\epsilon(g_2,h)$$

which means that the choices of the projective factors for G–action correspond in a 1–1 fashion to a choice of discrete torsion.

5.3.1. **Remarks.**

1) There is always one choice of cocycle γ given by
$$\gamma_{g,h}=\begin{cases} \chi_g^{-\frac{1}{2}}\pi_g(\rho) & h=g^{-1} \\ 0 & \text{else} \end{cases}$$
2) For any other solution γ by 4.7, we must have $\gamma_{g,h}=0$ if $[g,h]=e$ unless $\phi_{g,h}=\phi_{h,g}$ or in other words
$$\det(g|_{N_h})\det(h)|_{N_g})=1$$
since $\epsilon(g,h)=\epsilon(h^{-1},g)=eps(h,g)^{-1}$

6. Mirror construction for special G–Frobenius algebras

6.1. **Double grading.** We consider Frobenius algebras with grading in some abelian group I.
$$A=\bigoplus_{i\in I}A_i$$

This grading can be trivially extended to a double grading with values in $I\times I$ in two ways

$$A^{cc}=\bigoplus_{i\in I}A_{i,i}$$

and
$$A^{ac}=\bigoplus_{i\in I}A_{i,-i}$$

corresponding to the diagonal $\Delta: I\to I\times I$ and $(id,-)\circ\Delta: I\to I\times I$. We call bi–graded Frobenius algebras of this form of (c,c)–type and of (a,c)–type respectively. In the language of Euler fields we consider the field $(E,\bar{E})=(E,E)$ or $(E,\bar{E})=(E,-E)$

These gradings become interesting for special G–Frobenius algebras, since in that case the shifts will produce a possible non–diagonal grading.

6.2. **Definition.** Given a graded special G–Frobenius algebra we assign the following bi–degrees to 1_g

$$(E,\bar{E})(1_g) := (s_g, s_g^{-1})$$

It is clear that A_e is of (c,c)–type. A is however only of (c,c) type if $s_g = s_g^{-1}$.

Furthermore for the Ramond–space of A we assign the following bi–degree to v_0

$$(E,\bar{E})(v_0) := (-\frac{d}{2}, -\frac{d}{2})$$

6.3. **Euler–twist (spectral flow).** In this section, we consider a graded special G–Frobenius algebra and construct a new vector–space from it. We denote the grading operator by E:

6.3.1. **Definition.** The twist–operator j for an Euler–field E is

$$j := \; exp(2\pi i E)$$

We denote the group generated by j by J.

we call a special G–Frobenius algebra Euler if there is a special $J\times G$–Frobenius algebra $\tilde{A}$ of which A is a subalgebra.

6.3.2. **Definition.** The dual $\check{A}$ to an Euler special G–Frobenius algebra of (c,c)–type is the vector space

$$\check{A} := \bigoplus_{g\in G} \check{A}_g := \bigoplus_{g\in G} V_{gj^{-1}}$$

with the A_e and G–module structure determined by $\tau_j^g : \check{A}_g \simeq V_{gj^{-1}}$. together with the bi–grading

$$(E,\bar{E})(1'_g) := (s_{gj^{-1}} - d, s_{gj^1})$$

where $\check{1}_g$ denote the generator of $\check{A}_g$ as A_e–module and the bi–linear from

$$\check{\eta} := \tau_j^* = (\bar{\eta})$$

where $\tau_j := \oplus_{g\in G}\tau_j^g$.

7. Explicit Examples

7.1. **Self duality of A_n.** We consider the example of the Jacobian Frobenius Algebra associated to the function z^{n+1}

$$A_n := \mathbf{C}[z]/(z^n)$$

together with the induced multiplication and the Grothendieck residue. Explicitly:

$$z^i z^j = \begin{cases} z^{i+j} & \text{if } i+j \leq n \\ 0 & \text{else} \end{cases}$$

the form

$$\eta(z^i, z^j) = \delta_{i,n-1-j}$$

and the grading:

$$E(z^i) := \frac{i}{n+1}$$

which means $\rho = z^{n-1}$ and $d = \frac{n-1}{n+1}$.

We consider just the group $J \simeq \mathbf{Z}/n\mathbf{Z}+1$ with the generator j acting on z by multiplication with $\zeta_{n+1} := \exp 2\pi i n + 1$. We have

$$Fix_{j^i} \begin{cases} \mathbf{C} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

and thus

$$A_{j^i} \begin{cases} A_n & \text{if } i = 0 \\ 1_{j^i} & \text{else} \end{cases}$$

Furthermore we have the following grading;

$$(E, \bar{E})(1_{j^i}) = \begin{cases} (0, 0) & \text{for } i = 0 \\ (\frac{k-1}{n+1}, \frac{n-k}{n+1} & \text{else} \end{cases}$$

which means $\rho_{j^i} = 1_{j^i}$ and $d_{j^i} = 0$.

Using the reconstruction Theorem we have to find a cocycle γ and a compatible action φ. There is no problem for the metric since always $|N_g| = 1$ and if $n+1$ is even $\det(\zeta^{\frac{n+1}{2}}) = -1$. Since the group J is cyclic there is no freedom of choice for ϵ and just two choices of parity are possible corresponding to $j \mapsto \pm 1$.

From the general considerations we know $\gamma_{j^i, j^{n-1-i}} \in A_e$ and $\deg(\gamma_{j^i, j^{n-1-i}}) = d - d_{j^i} = \frac{n-1}{n+1}$ which yields

$$\gamma_{j^i, j^{n-1-i}} = ((-1)^{\tilde{j}i} \zeta^i)^{1/2} \rho = ((-1)^{\tilde{j}} \zeta)^{i/2} z^{n-1}$$

for the other γ notice that $\deg(1_{j^i}) + \deg(1_{j^k}) = \frac{i+k-2}{n+1}$ while $\deg(1_{j^{i+j}}) = \frac{i+k-1}{n+1}$ if $i + k \neq n+1$, but there is no element of degree $\frac{1}{n+1}$ in $A_{j^{i+k}}$ for $i + k \neq n+1$.

Hence

$$\gamma_{j^i,j^k} = \begin{cases} (-1)^{\tilde{j}}\zeta)^{i/2} z^{n-1} & \text{for } i+j=n+1 \\ 0 & \text{else} \end{cases}$$

this means that

$$\varphi_{j^i,j^k} = (-1)^{\tilde{j}^i\tilde{j}^k}\zeta^{-i}$$

Therefore the G–invariants $A^G = A_1$ generated by the identity 1
The Ramond grading of this algebra is

$$(E,\bar{E})(1_{j^i}v_0) = \begin{cases} (0,0) & \text{for } i=0 \\ (\frac{k}{n+1}-\frac{1}{2}, -\frac{k}{n+1}+\frac{1}{2}) & \text{else} \end{cases}$$

Since $j \in G$ the special G–Frobenius algebra is Euler and the dual is defined, moreover $G = J$ so that the vector–space structures of A and $\check{A}$ coincide. The grading is given by:

$$(E,\bar{E})(1_{j^i}v_0) = \begin{cases} (0,0) & \text{for } i=0 \\ -\frac{k-(n+1)}{n+1}-\frac{1}{2}, \frac{n+1-k}{n+1}+\frac{1}{2}) & \text{else} \end{cases}$$

The G–action is given by

$$\check{\varphi}_{j^i,j^k} = \begin{cases} (-1)^{\tilde{j}^i\tilde{j}^k}\zeta^i & \text{for } k=1 \\ (-1)^{\tilde{j}^i\tilde{j}^k} & \text{for } k=1 \end{cases}$$

This G–action leaves all even sectors $\check{A}_{j^i}$ invariant except for $i = 1$. Thus with the choice of all even sectors we have as G–modules

$\check{A} \simeq A_n$

where more explicitly

$\check{1}_{j^i} \mapsto z^{n+1-k} : k = 2, \dots n$ and $\check{1}_0 \mapsto 1$.

Notice that this A_n is of (a,c)–type however. Also since $J = G$ the form $\bar{\eta}$ pulls back and gives a non–degenerate form on $\check{A}^G$, which is the usual form on A_n Furthermore the usual multiplication on A_n is compatible with everything so that we can say that A_n is self–dual under this operation.

7.2. **D_n from a special $\mathbf{Z}/2\mathbf{Z}$–Frobenius algebra based on A_{2n-3}.** In this section, we show how to get D_n from a special $\mathbf{Z}/2\mathbf{Z}$–Frobenius algebra based on A_{2n-3}. The function for the Frobenius algebra A_{2n-3} is z^{2n-2}. Since this is an even function $\mathbf{Z}/2\mathbf{Z}$ acting via $z \mapsto -z$ is a symmetry. There are two sectors the untwisted and the twisted sector containing the element 1_{-1} with degree 0 The multiplication is fixed by $\deg(\gamma_{-1,-1}) = \frac{2n-4}{2n-2}$ thus

$$\gamma_{-1,-1} = z^{2(n-2)}$$

again the group is cyclic so the G–action only depends on the choice of parity of the -1–sector, we have to be careful since $(-1)^2 = 1$, but since $|N_{-1}| = 1$ we have that $(-1)^{\tilde{-}1}\chi_{-1} = 1$ to either $\sigma(-1) = 1$ or $\sigma(-1) = 0$.

In the untwisted sector we have $A_e^{\mathbf{Z}/2\mathbf{Z}} = \langle 1, z^2, \dots, z^{2(n-1)} \rangle \simeq A_{n-1}$ and the action of -1 on 1_{-1} is given by

$$\varphi_{-1,-1} = (-1)^{\widetilde{-1}\widetilde{-1}} + 1)$$

so that if $\sigma(-1) = 1$

$$A^{\mathbf{Z}/2\mathbf{Z}}_{2n-3,\mathbf{Z}/2\mathbf{Z}} \simeq D_n$$

and if $\sigma(-1) = 0$ we just obtain the invariants of the untwisted sector which are isomorphic to A_{n-1}.

The untwisted sector is given by the singularity A_{n-1} as expected upon the transformation $u = z^2$, Notice, that the invariants of the Ramond sector yield the singularity A_{n-2} as expected from [W]. These are of the form $u^i du$or $z^{2i+1}dz$ with $i = 0, \dots, n-3$.

7.3. **Point mod G.** In the theory of Jacobian Frobenius algebras there is the notion of a point is played by a Morse singularity $z_1^2 + \cdots + z_n^2$. Any finite subgroup group $G \subset O(n,k)$ leaves this point invariant.

The G–twisted algebra after possibly stabilizing is the following.

$$A = \bigoplus_{g \in G} k1_g$$

And the grading is $\deg(1_g) = (\frac{1}{2}s_g^-, \frac{1}{2}s_g^-)$, since $d = d_g = 1$

We have several different choices for the cocycle by solving the equation of the previous paragraph with choices for ϵ and the super–sign and the Ansatz:

$\gamma_{g,h} = \psi_{g,h} 1_e$

an in the graded case with the degree restriction: $\frac{1}{2}s_g^- + \frac{1}{2}s_h^- = \frac{1}{2}s_{gh}^-$.

Special case.

If we assume that $G \subset SO(n,\mathbf{C})$ and that $s_g^- = 0$ (i.e. $\sum_{i:\lambda_i \neq 0} \frac{1}{2\pi i}\lambda_i = \frac{|N_g|}{2} \in \mathbf{N}$, then we can choose the cocycle

$$\gamma_{g,h} = 1_e$$

which makes A into the group algebra.

7.3.1. **Point mod $\mathbf{Z}/n\mathbf{Z}$.** By the above analysis we realize $\mathbf{Z}/n\mathbf{Z}$ as the sub–group of rotations of order n in $\mathbf{C}$. We have that $s_g^- = 0$ and thus we can choose the full cocycle making A into A_n, multiplicatively, with trivial grading and trivial G–action one chooses all even sectors.

The metric, however will not be consistent. The identity pairs with itself for instance. Dualizing A,we obtain the following space

$$\check{A} = \oplus_{j^i} A_{j^{i-1}}$$

with again a trivial $(E, \bar{E})$ grading. Choosing the generator $\tilde{j} := j^{-1}$ for J the metric reads

$$\tilde{\eta}(\tilde{1}_{\tilde{j}^i}, \tilde{1}_{\tilde{j}^k}) = \begin{cases} 1 & \text{if } i+j = n-1 \\ 1 & i = j = n \\ 0 & \text{else} \end{cases}$$

This metric is compatible with the following multiplication:

$$\tilde{1}_{\tilde{j}^i} \circ \tilde{1}_{\tilde{j}^k} = \begin{cases} \tilde{1}_{\tilde{j}^{i+k}} & \text{if } i+j \leq n-1 \\ \tilde{1}_{\tilde{j}^{n-1}} & i = j = n \\ 0 & \text{else} \end{cases}$$

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