SOME POLYNOMIALS ON $\mathbb{P}_2(\mathbb{C}) \# \overline{\mathbb{P}_2(\mathbb{C})}$

by

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INTRODUCTION

Let $\hat{\mathbb{P}}_2 = \mathbb{P}_2(\mathbb{C}) \# \overline{\mathbb{P}_2(\mathbb{C})}$ denote the blow-up of the complex projective plane $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})$ at one point. In this note we explain how one can apply simple arguments to derive on $\hat{\mathbb{P}}_2$ certain polynomials in $\mathrm{Sym}^5(H^2(\hat{\mathbb{P}}_2;\mathbb{Z}))$ defined by the method of constructing polynomial invariants introduced by Donaldson.

It has been discussed in [M] that given any integer $k \geq 2$ we can always make use of Yang–Mills moduli spaces to define assignments

$$\Gamma_X^k : \mathscr{C}_X^k \longrightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$$

for smooth compact simply-connected oriented 4-manifolds X with $b_2^+(X) = 1$. Here we apply this construction to the manifold $\hat{\mathbb{P}}_2$ for a small value $k = 2$. It is well known $H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$ is a free group on two generators h_0 and h_∞ subject to the relations

$$h_0 \cdot h_0 = -h_\infty \cdot h_\infty = 1 \ , \ h_0 \cdot h_\infty = 0 \, .$$

Thus the set $\mathscr{C}^2_{\hat{\mathbb{P}}_2}$ in this framework consists of four "chambers" as elements which are components of the positive cone

$$\Omega_{\hat{\mathbb{P}}_2} = \{a_0 h_0 + a_\infty h_\infty \in H^2(\hat{\mathbb{P}}_2;\mathbb{R}) \mid a_0^2 - a_\infty^2 > 0\}$$

after dividing by a single "wall" $<h_\infty>^\perp$, the span of h_0 over $\mathbb{R}$ in this situation. We let

$$\hat{C} = \{a_0 > -a_\infty > 0\}\ ,\ \sigma^*\hat{C} = \{a_0 > a_\infty > 0\}$$

be two of these components so that $\mathscr{C}^2_{\hat{\mathbb{P}}_2} = \{\hat{C}, \sigma^*\hat{C}, -\hat{C}, -\sigma^*\hat{C}\}$.

<u>THEOREM</u> The assignment

$$\Gamma^2_{\hat{\mathbb{P}}_2} : \mathscr{C}^2_{\hat{\mathbb{P}}_2} \longrightarrow \mathrm{Sym}^5(H^2(\hat{\mathbb{P}}_2;\mathbb{Z}))$$

is given by

$$\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C}) = -\Gamma^2_{\hat{\mathbb{P}}_2}(-\hat{C}) = h_0^5 - 10(h_0^2h_\infty^3) + 15(h_0h_\infty^4) - 6h_\infty^5\ ,$$

$$\Gamma^2_{\hat{\mathbb{P}}_2}(\sigma^*\hat{C}) = -\Gamma^2_{\hat{\mathbb{P}}_2}(-\sigma^*\hat{C}) = h_0^5 + 10(h_0^2h_\infty^3) + 15(h_0h_\infty^4) + 6h_\infty^5\ .$$

The motive of this work is to understand the <u>difference</u> between $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$ and $\Gamma^2_{\hat{\mathbb{P}}_2}(\sigma^*\hat{C})$ rather than to determine the polynomials themselves. In general, the knowledge of comparing $\Gamma^k_X(C_1)$ and $\Gamma^k_X(C_{-1})$ in $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$ for two chambers C_1 , C_{-1} adjacent to a common wall in the system

$$\cup\,\{<e>^\perp \subset H^2(X;\mathbb{R})\,|\,e \in H^2(X;\mathbb{Z})\ ;\ -1 \le e\cdot e \le -k\}$$

will lead to a complete definition of differential invariants for X (c.f. [M]). It appears at present only in the case when $e \cdot e = -k$ does such a comparison being known. The simplest example outreaches this understanding is the assignment $\Gamma^2_{\hat{\mathbb{P}}_2}$ we are considering. It is easy to check in the definition of $\Gamma^2_{\hat{\mathbb{P}}_2}$ there is no lattice solution to $e \cdot e = -2$ but rather one solves $e \cdot e = -1$ for $e = \pm h_\infty \in H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$ which defines the wall $< h_\infty >^\perp$ separating $\hat{C}$ and $\sigma^* \hat{C}$. Here we work out explicitly these two polynomials giving in particular an example for this kind of comparison. It is hoped that the discussion would hint a procedure of finding the differences between polynomials $\Gamma^k_X(C_1)$ and $\Gamma^k_X(C_{-1})$ for the cases $-1 \leq e \cdot e \leq -k+1$ on general ground.

This is a companion article of [M] and some introductory material is unavoidably overlapped in these two papers. To keep such amount low we could be sketchy at some points and, if required, one is referred to [M] for details.

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§ 1. A review of background material.

In this section we recall briefly the construction of the polynomial $\Gamma_X^k(C)$ in $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$ associated to a chamber $C \in \mathscr{C}_X^k$ in general. Also we explain an observation of Donaldson on how to derive for the complex projective plane $\mathbb{P}_2$ a polynomial $\Gamma_{\mathbb{P}_2}^2 \in \mathrm{Sym}^5(H^2(\mathbb{P}_2;\mathbb{Z}))$ which not only serves as an example for this kind of construction but indeed will play a crucial role in the calculation of $\Gamma_{\hat{\mathbb{P}}_2}^2$.

To begin with, let P be an SU(2)–bundle over X with $c_2(P) = k$ and $\mathscr{A}$ be the space of connections on P . The gauge group $\mathscr{G} = \mathrm{Aut}\, P$ acts on $\mathscr{A}$ preserving anti–self–dual (ASD) connections and we denote by

$$M_k(m) = \{A \in \mathscr{A} \mid *_m F(A) = -F(A)\}/\mathscr{G}$$

the moduli space of ASD connections on P relative to a Riemannian metric m on X . In general $M_k(m)$ is a smooth oriented manifold of (real) dimension $8k-6$ assuming $b_2^+(X) = 1$. Associated to any given metric m on X , there is an L_2–normalized self–dual harmonic 2–form ω_m which is unique up to a sign. A choice of ω_m determines a standard orientation of $M_k(m)$ and we write $M_k(\omega_m)$ for $M_k(m)$ with such an assigned orientation understood. In this convention $M_k(-\omega_m)$ has the opposite orientation compared with $M_k(\omega_m)$.

Given any smooth oriented real surface $\Sigma \subset X$ we can define a line bundle over $\mathscr{A}$ by assigning to each connection $A \in \mathscr{A}$ the complex line

$$\mathscr{L}_\Sigma(A) = \Lambda^{\max}(\ker \mathscr{D}_{A|_\Sigma})^* \otimes \Lambda^{\max}(\mathrm{coker}\, \mathscr{D}_{A|_\Sigma})$$

where $\not{\partial}_{A|_\Sigma}$ denotes the Dirac operator coupled with $A|_\Sigma$. If the metric m on X is sufficiently general, such assignments factor through the gauge group action and descend to the manifold $M_k(m)$ defining a line bundle $\mathscr{L}_\Sigma \longrightarrow M_k(m)$ provided the surfaces Σ are suitably chosen. In this situation we consider the zero sets $V_\Sigma \cap M_k(m)$ for certain transversal sections of $\mathscr{L}_\Sigma$ and by working with $4k-3$ such surfaces as a whole we obtain an assignment

$$(\Sigma_1, \dots, \Sigma_{4k-3}) \xrightarrow{q_{k,\omega_m}} \text{the algebraic sum of a transversal intersection } V_{\Sigma_1} \cap \dots\dots \cap V_{\Sigma_{4k-3}} \cap M_k(\omega_m)$$

which is well–defined for $k \geq 2$. Regarding q_{k,ω_m} an element of $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$ this construction defines a map

$$\Gamma_X^k : \mathscr{C}_X^k \longrightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$$

putting $\Gamma_X^k(C) = q_{k,\omega_m}$ for $[\omega_m] \in C$. It is clear that

$$\Gamma_X^k(C) = -\Gamma_X^k(-C) \tag{1.1}$$

for all $C \in \mathscr{C}_X^k$.

Now we explain this construction for $\Gamma_{\mathbb{P}_2}^2$. Let h be the standard generator of $H^2(\mathbb{P}_2;\mathbb{Z}) \simeq \mathbb{Z}$ so that the positive cone $\Omega_{\mathbb{P}_2}$ of $\mathbb{P}_2$ is the part $\mathbb{R}^+ \cdot \langle h \rangle \subset H^2(\mathbb{P}_2;\mathbb{R})$. Note in the first place for all non–negative k we always have

$$\mathscr{C}_{\mathbb{P}_2}^k = \{\Omega_{\mathbb{P}_2}\}.$$

This follows from the fact that the intersection product on $H^2(\mathbb{P}_2;\mathbb{Z})$ is positive definite and so there is no solution to $-1 \le e \cdot e \le -k$ for $e \in H^2(\mathbb{P}_2;\mathbb{Z})$. (Thus any polynomial $\Gamma^k_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2})$ is independent of metrics and in particular defines a differential invariant for $\mathbb{P}_2$.)

(1.2) LEMMA (Donaldson) On the complex projective plane $\mathbb{P}_2$, we have

$$\Gamma^2_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2}) = h^5 \in \mathrm{Sym}^5(H^2(\mathbb{P}_2;\mathbb{Z}))\,.$$

The idea of proving this lemma is contained in [D1] and has further been used in [M]. As the argument is closely related to the calculation of $\Gamma^2_{\hat{\mathbb{P}}_2}$, we include the proof here.

In general Yang–Mills moduli spaces $M_k(m)$ are difficult to determine but if X is moreover an algebraic surface, contained in $\mathbb{P}_N$ say, then we can take ω_m to be the restricted Fubini–Study form on X and whereby identify $M_k(\omega_m)$ with the moduli space $M^s_k(\omega_m)$ of ω_m–stable 2–bundles $V \longrightarrow X$ with $(c_1(V),c_2(V)) = (0,k)$. Recall that a 2–bundle V over an algebraic surface X is ω_m–stable if for any line bundle $\mathcal{L} \longrightarrow X$ admitting a non–zero holomorphic bundle map $\mathcal{L} \longrightarrow V$ we have

$$\mathcal{L} \cdot \omega_m < \tfrac{1}{2}\,(\Lambda^2 V) \cdot \omega_m\,.$$

Here we write for a line bundle $L \longrightarrow X$ that

$$L \cdot \omega_m = \int_X c_1(L) \wedge \omega_m\,.$$

In this complex setting, one finds if Σ is an algebraic curve then the bundle $\mathscr{L}_\Sigma \longrightarrow M_k^s(\omega_m)$ has a canonical section with zero set $V_\Sigma \cap M_k^s(\omega_m)$ given by

$$\{[V] \in M_k^s(\omega_m) \mid H^0(V|_\Sigma \otimes K_\Sigma^{\frac{1}{2}}) \neq 0\,;\ \ H^1(V|_\Sigma \otimes K_\Sigma^{\frac{1}{2}}) \neq 0\}$$

where $K_\Sigma^{\frac{1}{2}}$ denotes a square root of the canonical bundle K_Σ. For a projective line $\mathbb{P}_1 \subset X$ in particular, it is easy to deduce

$$V_{\mathbb{P}_1} \cap M_k^s(\omega_m) = \{[V] \in M_k^s(\omega_m) \mid V|_{\mathbb{P}_1} \text{ is non–trivial}\} \tag{1.3}$$

and we denote V_Σ^J for this kind of zero sets in the case when $\Sigma \simeq \mathbb{P}_1$. The non–triviality of $V|_{\mathbb{P}_1}$ in (1.3) means the following. It is well–known 2–bundles on $\mathbb{P}_1$ always split and so

$$V|_{\mathbb{P}_1} \simeq \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(-a)$$

for some integer $a \geq 0$. We say $V|_{\mathbb{P}_1}$ is trivial if $a = 0$ and otherwise non–trivial. Now, as $H_2(\mathbb{P}_2;\mathbb{Z})$ is generated by a projective line $H \simeq \mathbb{P}_1$, we can apply this framework to determine $\Gamma^2_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2})$ in terms of algebraic geometry as follows.

Working over $\mathbb{P}_2$ it is no loss to write $M_2(\mathbb{P}_2)$ for $M_2^s(\omega_m)$ as the choice of ω_m is immaterial. By a theorem of Barth, $M_2(\mathbb{P}_2)$ identifies with the set of all non–singular conics α in the dual plane $\mathbb{P}_2^*$ and so is Zariski open in $\mathbb{P}_5 = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_2^*}(2))\setminus\{0\})$. We denote $\alpha_V \in \mathbb{P}_5$ the conic on $\mathbb{P}_2^*$ corresponding to the element $[V] \in M_2(\mathbb{P}_2)$.

Also write $\ell_H \in \mathbb{P}_2^*$ for the point determined by a line $H \subset \mathbb{P}_2$. As a consequence of his theorem, we can interpret particularly the zero set $V_H^J \cap M_2(\mathbb{P}_2)$ as (a jumping divisor)

$$\text{(1.4)} \qquad \{[V] \in M_2(\mathbb{P}_2) \mid \alpha_v \text{ passes through } \ell_H \in \mathbb{P}_2^*\},$$

cutting transversally by a hyperplane in $\mathbb{P}_5$ (cf. [OSS] for instance). Now it is easy to deduce from this $\Gamma^2_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2}) = h^5$. Indeed, given five general lines $H_1, H_2, \ldots, H_5$ on $\mathbb{P}_5$, we know from (1.4) that

$$V^J_{H_1} \cap \ldots \cap V^J_{H_5} \cap M_2(\mathbb{P}_2)$$

can be represented by the intersection of five hyperplanes of $\mathbb{P}_5$ lying in general positions. The associated intersection number is therefore equal to one counted positively. The lemma follows.

Put it differently, this calculation shows the coefficient of h^5 for the polynomial $\Gamma^2_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2})$ is one. As we shall see, this coefficient in essence determines all the non–zero coefficients for $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$ and $\Gamma^2_{\hat{\mathbb{P}}_2}(\sigma^* \hat{C})$.

§ 2. The method of the calculation

The aim of this section is to explain the idea of finding the assignment $\Gamma^2_{\hat{\mathbb{P}}_2}$. Via the isomorphism $H^1(\hat{\mathbb{P}}_2;\mathcal{O}^*) \simeq H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$, we shall not distinguish in following discussions an element of $H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$ from the divisor or the line bundle it defines on $\hat{\mathbb{P}}_2$.

Fix a point $x \in \mathbb{P}_2$ and let $E_0 \subset \mathbb{P}_2$ be a line not containing x . Assume $\hat{\mathbb{P}}_2$ is the blow–up of $\mathbb{P}_2$ at x and let $\pi : \hat{\mathbb{P}}_2 \longrightarrow \mathbb{P}_2$ be the natural projection. Denote by $E_\infty = \pi^{-1}(x)$ the exceptional curve on $\hat{\mathbb{P}}_2$ and identify E_0 with its natural image in $\hat{\mathbb{P}}_2$. Clearly then E_0 and E_∞ freely generate $H_2(\hat{\mathbb{P}}_2;\mathbb{Z})$. It is well–known $\hat{\mathbb{P}}_2$ admits the structure of a (non–trivial) $\mathbb{P}_1$–fibration over E_0 . A fibre $F \subset \hat{\mathbb{P}}_2$ over a point $y \in E_0$ is the proper transformation of the line in $\mathbb{P}_2$ joining the point x and y . Thus we have

$$F = E_0 - E_\infty \tag{2.1}$$

as an element in $H_2(\hat{\mathbb{P}}_2;\mathbb{Z})$ or a divisor on $\hat{\mathbb{P}}_2$. The following table of intersection products is easy to obtain.

(2.2) TABLE

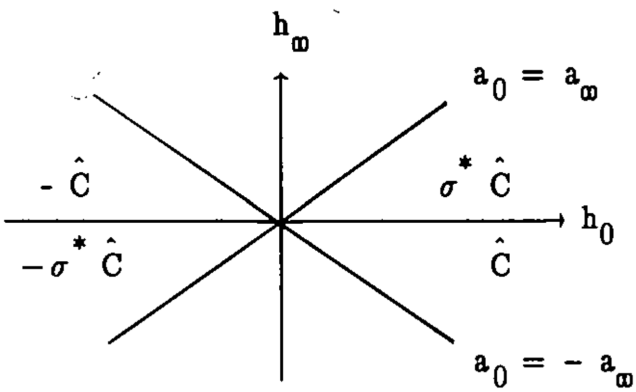
	E_0	E_∞	F
E_0	1	0	1
E_∞	0	-1	1
F	1	1	0

Suppose h_0, $h_\infty \in H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$ are the Poincaré dual classes of E_0, E_∞ respectively so that

$$h_0(E_0) = -h_\infty(E_\infty) = 1 , \ h_0(E_\infty) = h_\infty(E_0) = 0 .$$

By previous discussions, $\mathscr{C}^2_{\hat{\mathbb{P}}_2}$ is a set consisting of as elements four connected regions indicated below.

DIAGRAM



Notice that $\sigma^* \hat{C}$ is simply the image of the chamber

$$\hat{C} = \{a_0h_0 + a_\infty h_\infty \in H^2(\hat{\mathbb{P}}_2;\mathbb{R}) \mid a_0 > -a_\infty > o\} \tag{2.3}$$

via the reflection σ^* on $H^2(\hat{\mathbb{P}}_2;\mathbb{R})$ sending h_∞ to $-h_\infty$. Note also precisely lattice points in $\hat{C}$ define ample line bundles on $\hat{\mathbb{P}}_2$.

To determine $\Gamma^2_{\hat{\mathbb{P}}_2}$ one is in principle to find two polynomials $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$ and $\Gamma^2_{\hat{\mathbb{P}}_2}(\sigma^* \hat{C})$. It turns out that only a single calculation of

(2.4) $$\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C}) = a_0h_0^5 + a_1(h_0^4h_\infty) + a_2(h_0^3h_\infty^2) + a_3(h_0^2h_\infty^3) + a_4(h_0h_\infty^4) + a_5h_\infty^5$$

will suffice. We benefit from the fact that the reflection σ^* on $H^2(\hat{\mathbb{P}}_2;\mathbb{R})$ can be realized by an orientation preserving diffeomorphism σ on $\hat{\mathbb{P}}_2$ (cf. [FM] for instance). By naturality we derive

(2.5) $$\begin{aligned}\Gamma^2_{\hat{\mathbb{P}}_2}(\sigma^*\hat{C}) &= \sigma^*\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C}) \\ &= a_0h_0^5 - a_1(h_0^4h_\infty) + a_2(h_0^3h_\infty^2) - a_3(h_0^2h_\infty^3) + a_4(h_0h_\infty^4) - a_5h_\infty^5 .\end{aligned}$$

This polynomial is therefore determined by $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$. In following discussions we shall be concentrating on the calculation of $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$ only. For brevity sometimes we write $\hat{q}$ in place of $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$.

<u>REMARK</u> As differentiable manifold $\hat{\mathbb{P}}_2$ is the connected sum $\mathbb{P}_2 \# \overline{\mathbb{P}}_2$ where $\overline{\mathbb{P}}_2$ denotes the projective plane $\mathbb{P}_2$ with the opposite orientation. It is known that the complex conjugation on $\overline{\mathbb{P}}_2$ extends and defines an orientation preserving diffeomorphism σ on $\hat{\mathbb{P}}_2$ realizing the reflection σ^*. The same argument applies to the reflection $h_0 \longmapsto -h_0$ on $H^2(\hat{\mathbb{P}}_2;\mathbb{R})$ taking $\hat{C}$ to $-\sigma^*\hat{C}$ and $\sigma^*\hat{C}$ to $-\hat{C}$. This time however the naturality argument yields no more information as the polynomials $\Gamma^2_{\hat{\mathbb{P}}_2}(-\hat{C})$ and $\Gamma^2_{\hat{\mathbb{P}}_2}(-\sigma^*\hat{C})$ are just $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$ and $\Gamma^2_{\hat{\mathbb{P}}_2}(-\sigma^*\hat{C})$ with the opposite sign by (1.1).

For the calculation of $\hat{q}$ we work with the moduli space $M_2(\hat{\mathbb{P}}_2)$ of stable 2–bundles $\hat{V} \longrightarrow \hat{\mathbb{P}}_2$ with $(c_1(\hat{V}),c_2(\hat{V})) = (0,2)$. By [M] lemma (4.4), stability condition is uniformly defined on $\hat{C}$ and so there is no ambiguity of writing $M_2(\hat{\mathbb{P}}_2)$. To determine a coefficient a_i of $\hat{q}$, the most direct way is to obtain the algebraic sum associated to a transversal intersection

$$\underbrace{V_{E_0} \cap \ldots \cap V_{E_0}}_{(5-i)\ \text{times}} \cap \underbrace{V_{E_\infty} \cap \ldots \cap V_{E_\infty}}_{i\ \text{times}} \cap M_2(\hat{\mathbb{P}}_2)$$

giving the coefficient a_i rightaway. This could be done but requires efforts to show the transversality for such intersections. We find it easier to work out intersection numbers associated to some other combinations of homology classes in $H_2(\hat{\mathbb{P}}_2;\mathbb{Z})$. This is summarized as below.

(2.6) <u>TABLE</u>

Number of			Evaluations of $\Gamma^2_{\hat{\mathbb{P}}_2}(\hat{C})$
E_0	E_∞	F	
0	2	3	0
1	1	3	0
2	0	3	0
3	0	2	1
4	0	1	1
5	0	0	1

By the fact $F = E_0 - E_\infty$ one finds (2.6) imposes six (independent) conditions on as many unknowns a_i, the coefficients of $\hat{q}$. Now it is elementary to solve a_i using linear algebra and the calculation gives

$$\hat{q} = h_0^5 - 10(h_0^2 h_\infty^3) + 15(h_0 h_\infty^4) - 6h_\infty^5$$

as asserted. Here $(h_0^{\ell_0} h_\infty^{\ell_\infty})$ denotes the symmetrization of the polynomial $h_0^{\ell_0} h_\infty^{\ell_\infty}$ in $(H^2(\hat{\mathbb{P}}_2;\mathbb{Z}))^{\otimes 5}$.

Our task therefore is to establish (2.6). The calculation for the first three rows in the table follows from

(2.7) $$V_F^J \cap V_F^J \cap V_F^J \cap M_2(\hat{\mathbb{P}}_2) = \phi$$

for three (distinct) fibres $F \subset \hat{\mathbb{P}}_2$. This will be shown in § 3. To find the non–zero intersection numbers in the last three row of (2.6), we need the fact that $\pi^* M_2(\mathbb{P}_2)$ is naturally contained in $M_2(\hat{\mathbb{P}}_2)$. Our argument for this is in the spirit of [SC].

(2.8) <u>LEMMA</u> If $[V] \in M_2(\mathbb{P}_2)$, then the pullback bundle $\pi^* V \longrightarrow \hat{\mathbb{P}}_2$ is stable relative to ample line bundles $N_0 E_0 + N_\infty E_\infty$ on $\hat{\mathbb{P}}_2$ where N_0 and N_∞ are integers satisfying $N_0 > -N_\infty > 0$.

<u>PROOF</u> Note first for elements $[V] \in M_2(\mathbb{P}_2)$ the pullback bundle $\pi^* V \longrightarrow \hat{\mathbb{P}}_2$ restricts trivially on projective lines E_0 and F of $\hat{\mathbb{P}}_2$ in general. To show $\pi^* V$ is stable we are to check for all line bundles $\mathscr{L} \longrightarrow \hat{\mathbb{P}}_2$, fitting into an exact sequence

(2.9) $$0 \longrightarrow \mathscr{L} \xrightarrow{\varphi} \pi^* V \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0 ,$$

the inequality

(2.10) $$\mathscr{L} \cdot (N_0 E_0 + N_\infty E_\infty) < 0$$

holds provided $N_0 > -N_\infty > 0$. In (2.9) the bundle map φ is assumed to be non–trivial defining an ideal sheaf I of isolated zeros on $\hat{\mathbb{P}}_2$.

To obtain (2.10) we write $\mathscr{L} = a_0 E_0 + a_\infty E_\infty$ for some integers a_0 , a_∞ . By restricting the exact sequence (2.9) to a general $E_0 \subset \hat{\mathbb{P}}_2$ with the properties that

(i) $\varphi|_{E_0}$ is non–vanishing , and

(ii) $\pi^* V|_{E_0}$ is trivial,

then one infers readily

(2.11) $$(a_0 E_0 + a_\infty E_\infty) \cdot E_0 = a_0 \leq 0 .$$

Similar argument when applied to a general fibre $F \subset \hat{\mathbb{P}}_2$ gives

(2.12) $$(a_0 E_0 + a_\infty E_\infty) \cdot F = a_0 + a_\infty \leq 0$$

using $F = E_0 - E_\infty$. Now suppose in (2.10) the contrary holds, or that

(2.13) $$0 \leq (a_0 E_0 + a_\infty E_\infty) \cdot (N_0 E_0 + N_\infty E_\infty) = a_0 N_0 - a_\infty N_\infty .$$

It is elementary to check assuming $N_0 > -N_\infty > 0$ the solution to (2.11), (2.12) and (2.13) is only $a_0 = a_\infty = 0$. In this situation $\mathcal{L}$ is the trivial bundle $\mathcal{O}$. However, putting $L = \mathcal{O}$ in (2.9) one finds $h^0(\pi^* V) \neq 0$ which contradicts the fact that $h^0(\pi^* V) = h^0(V) = 0$ by the stability of $V \longrightarrow \mathbb{P}_2$. This shows the lemma.

It is clear bundles $\pi^* V \longrightarrow \hat{\mathbb{P}}_2$ restricts trivially on E_∞ . Conversely we can prove for $[\hat{V}] \in M_2(\hat{\mathbb{P}}_2)$ satisfying $\hat{V}|_{E_\infty} \simeq \mathcal{O}_{E_\infty}^{\oplus 2}$ the direct image sheaf $\pi_* \hat{V}$ defines a stable 2–bundle on $\mathbb{P}_2$ with $(c_1(\hat{V}), c_2(\hat{V})) = (0,2)$. It follows

(2.14) $$M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2) = \{ [\hat{V}] \in M_2(\hat{\mathbb{P}}) \,|\, \hat{V}|_{E_\infty} \text{ is non–trivial}\}$$

using $\pi^* \pi_* \hat{V} \simeq \hat{V}$ for $[\hat{V}] \in \pi^* M_2(\mathbb{P}_2)$. In § 3 we identify elements in (2.14) only to find that

(2.15) $$V^J_{E_0} \cap V^J_{E_0} \cap V^J_{E_0} \cap \{M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\hat{\mathbb{P}}_2)\} = \phi$$

if the three lines E_0 lie in general positions. Thus for the calculation of the last three rows in (2.6) we can do the counting of intersection numbers entirely in $\pi^* M_2(\hat{\mathbb{P}}_2)$, the complement of $M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2)$ in $M_2(\hat{\mathbb{P}}_2)$. By passing to the moduli space $M_2(\mathbb{P}_2)$, which is naturally identified with $\pi^* M_2(\mathbb{P}_2)$, we find such calculations has essentially been done in lemma (1.2) should one observe E_0 and F project to lines on $\mathbb{P}_2$ under the map π . (Note $\pi(F)$ passes through the point $x \in \mathbb{P}_2$). Now we adapt the fact $\Gamma^2_{\mathbb{P}_2}(\Omega_{\mathbb{P}_2}) = 1$ to deduce these intersection numbers are one, completing the calculation of the table (2.6).

Before closing this section, it might worth pointing out (2.14) discloses the relation

$$M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2) = V^J_{E_\infty}$$

as both sets contain precisely elements $[\hat{V}] \in M_2(\hat{\mathbb{P}}_2)$ which restrict non–trivially on E_∞. This suggests $M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2)$, as $V^J_{E_\infty}$, is the zero set of a canonical section of $\mathscr{L}_{E_\infty} \longrightarrow M_2(\hat{\mathbb{P}}_2)$ and thus if in the non–trivial case it should be of codimension one. We shall see in the appendix it is indeed the case as we check

$$M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2) \simeq \mathbb{C}^4$$

while it is known that $\dim_{\mathbb{C}} M_2(\hat{\mathbb{P}}_2) = 5$. However the description of this component is not important in the calculation of $\hat{q}$.

§ 3. The calculation of the polynomial

As explained in § 2 the calculation of $\Gamma^2_{\hat{\mathbb{P}}_2}$ reduces to the determination of a single polynomial $\hat{q}$. A key point of finding $\hat{q}$ is to understand an element $[\hat{V}]$ in $M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2) = V^J_{E_\infty}$ via an alternative characterization $h^0(\hat{V} \otimes E_\infty) = 1$. To obtain this, observe first the cohomology for any $[\hat{V}] \in M_2(\hat{\mathbb{P}}_2)$ is identically zero, i.e.

$$(3.1) \qquad h^0(\hat{V}) = h^1(\hat{V}) = h^2(\hat{V}) = 0 \, .$$

Thus if $\mathcal{L}$ is a line bundle over $\hat{\mathbb{P}}_2$ we find by the Riemann–Roch formula

$$(3.2) \qquad \chi(\hat{V} \otimes \mathcal{L}) = \mathcal{L} \cdot (\mathcal{L} - K_{\hat{\mathbb{P}}_2}) \quad \text{where}$$

$$(3.3) \qquad K_{\hat{\mathbb{P}}_2} = -3E_0 + E_\infty$$

denotes the canonical bundle for $\hat{\mathbb{P}}_2$.

(3.4) LEMMA Let $L \in \{E_0, E_\infty, F\}$ be a projective line on $\hat{\mathbb{P}}_2$. Then for $[\hat{V}] \in M_2(\hat{\mathbb{P}}_2)$ we have

$$\hat{V}|_L \simeq \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L & \text{or} \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \, . \end{cases}$$

The argument for (3.4) is in parallel with [M] lemma (3.4) using (3.1), (3.2). For this reason we omit the proof here.

(3.5) LEMMA. As subsets of $M_2(\hat{\mathbb{P}}_2)$,

$$V^J_{E_\infty} = \{[\hat{V}] \in M_2(\hat{\mathbb{P}}_2) \mid h^0(\hat{V} \otimes E_\infty) = 1\} .$$

PROOF. The long exact sequence associated to

$$0 \longrightarrow \hat{V} \longrightarrow \hat{V} \otimes E_\infty \longrightarrow \hat{V} \otimes E_\infty|_{E_\infty} \longrightarrow 0$$

gives an isomorphism

$$H^0(\hat{V} \otimes E_\infty) \xrightarrow{\sim} H^0(\hat{V} \otimes E_\infty|_{E_\infty}) \tag{3.6}$$

using (3.1). From this we find $[\hat{V}] \in \pi^* M_2(\hat{\mathbb{P}}_2)$ precisely when $h^0(\hat{V} \otimes E_\infty) = 0$ by (3.6) since then

$$\hat{V} \otimes E_\infty|_{E_\infty} \simeq \mathcal{O}_{E_\infty}(-1) \oplus \mathcal{O}_{E_\infty}(-1)$$

as $E_\infty|_{E_\infty} \simeq \mathcal{O}_{E_\infty}(-1)$. On the other hand, $[\hat{V}] \in V^J_{E_\infty}$ is characterized by

$$\hat{V}|_{E_\infty} \simeq \mathcal{O}_{E_\infty}(1) \oplus \mathcal{O}_{E_\infty}(-1)$$

as a consequence of lemma (3.4). Thus one finds

$$h^0(\hat{V} \otimes E_\infty) = h^0(\hat{V} \otimes E_\infty|_{E_\infty}) = h^0(\mathcal{O}_{E_\infty} \oplus \mathcal{O}_{E_\infty}(-2)) = 1$$

and the lemma follows.

The crucial point of this characterization for $[\hat{V}] \in V^J_{E_\infty}$ is that the bundle $\hat{V}$ over $\hat{\mathbb{P}}_2$ is obtained from an extension

$$(3.7) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \hat{V} \otimes E_\infty \otimes D^{-1} \longrightarrow E_\infty^{\otimes 2} \otimes D^{-2} \otimes I \longrightarrow 0 \text{, or}$$

$$(3.8) \qquad 0 \longrightarrow E_\infty^{-1} \otimes D \longrightarrow \hat{V} \longrightarrow E_\infty \otimes D^{-1} \otimes I \longrightarrow 0$$

for some effective divisor $D \geq 0$ and ideal sheaf I of isolated zeros on $\hat{\mathbb{P}}_2$.

(3.9) LEMMA If $\hat{V}$ is stable, then $D \equiv 0$.

PROOF Replacing E_∞ by $F = E_0 - E_\infty$ it is no loss to write

$$D = m_1 E_0 + m_2 F$$

for some integers m_1 and m_2 where $m_1 = D \cdot F \geq 0$. Consider first $m_1 = 0$. In this case $D = m_2 F$ for some $m_2 \geq 0$ and we are to exclude $m_2 > 0$. This follows readily from (3.7) by that

$$c_2(\hat{V} \otimes E_\infty \otimes F^{-m_2}) = 1 - 2m_2$$

which is positive only when $m_2 = 0$ assuming $m_2 \geq 0$. Assume now $m_1 \geq 1$. We find then $m_2 < 0$ if $\hat{V}$ is stable; otherwise one checks $m_2 \geq 0$ would imply that

$$(N_0E_0 + N_\infty E_\infty) \cdot (-E_\infty + D) = N_0(m_1 + m_2) + N_\infty + N_\infty m_2$$

is (strictly) positive by $N_0 > -N_\infty > 0$. This however contradicts the stability of $\hat{V}$ in (3.8). Consider therefore $m_2 < 0$. In this situation

$$h^0(D|_{E_\infty}) = h^0(\mathcal{O}_{E_\infty}(m_2)) = 0$$

but then the exact sequence

$$0 \longrightarrow E_\infty^{-1} \otimes D \longrightarrow D \longrightarrow D|_{E_\infty} \longrightarrow 0$$

shows

$$h^0(E_\infty^{-1} \otimes D) = h^0(D) > 0$$

which again contradicts the stability of $\hat{V}$ by (3.8). This shows the lemma.

REMARK For suitably chosen ample line bundles on $\hat{\mathbb{P}}_2$, the proof of $D = 0$ in this lemma becomes simplier. However, as the stability condition is uniformly defined on the chamber $\hat{C}$, the vanishing of D should not depend on such choices. For this reason we purpue here an argument not making any assumption on the ample line bundles of $\hat{\mathbb{P}}_2$ at all.

Now using this lemma we infer readily the intersection

(3.10) $$V^J_{E_0} \cap V^J_{E_0} \cap V^J_{E_0} \cap (M_2(\hat{\mathbb{P}}_2) \setminus \pi^* M_2(\mathbb{P}_2))$$

is generically empty. For this purpose, we choose three E_0–lines on $\hat{\mathbb{P}}_2$ not sharing a common point. Then if on the contrary there is some $[\hat{V}]$ in the intersection (3.10), by lemma (3.9) we find an exact sequence

$$0 \longrightarrow E_\infty^{-1} \longrightarrow \hat{V} \longrightarrow E_\infty \otimes I \longrightarrow 0 \tag{3.11}$$

where I is an ideal sheaf of a <u>simple</u> zero $z \in \hat{\mathbb{P}}_2$ since $c_2(\hat{V} \otimes E_\infty) = 1$. As the point z can lie on at most two of the three E_0–lines under the present assumption, it is always possible to choose one E_0 amongst the three not containing z. Restricting the exact sequence (3.11) to such an E_0 gives a locally free extension

$$0 \longrightarrow E_\infty^{-1}|_{E_0} \longrightarrow \hat{V}|_{E_\infty} \longrightarrow E_\infty|_{E_0} \longrightarrow 0$$

of <u>trivial</u> bundles as $E_\infty \cdot E_0 = 0$. Since $H^1(\mathcal{O}_{E_0}) = 0$ the above exact sequence splits and so

$$\hat{V}|_{E_0} \simeq \mathcal{O}_{E_0} \oplus \mathcal{O}_{E_0}$$

is trivial. This however contradicts to the assumption that $[\hat{V}] \in V^J_{E_0} \cap M_2(\hat{\mathbb{P}}_2)$. It follows intersection (3.10) is generically empty.

To finish the calculation for $\hat{q}$ it remains to check

$$V^J_F \cap V^J_F \cap V^J_F \cap M_2(\hat{\mathbb{P}}_2) = \phi$$

for three (distinct) fibres F or $\hat{\mathbb{P}}_2$. This is a consequence of the following lemma the proof of which is comparable with [M] lemma (3.6) and so we shall be brief.

(3.12) LEMMA For $[\hat{V}] \in M_2(\hat{\mathbb{P}}_2)$, the restriction $\hat{V}|_F$ is generically trivial. Furthermore, the number of fibres F on where $\hat{V}$ restricts non–trivially is at most two.

To see this, we consider $\hat{\mathbb{P}}_2 \longrightarrow E_0$ a $\mathbb{P}_1$–fibration with projection map pr. Supposing $\hat{V}|_F$ is not generically trivial, by lemma (3.4) we conclude $\hat{V}|_F$ is uniformly $\mathcal{O}_F(1) \oplus \mathcal{O}_F(-1)$. Then $\hat{V}$ fits into an extension of bundles

$$0 \longrightarrow L \longrightarrow \hat{V} \longrightarrow L^{-1} \longrightarrow 0$$

where $L = E_0 \otimes pr^* pr_*(\hat{V} \otimes E_0^{-1})$ is a line bundle over $\hat{\mathbb{P}}_2$ satisfying $L \cdot L = -2$. This is however impossible since there is no lattice solution to $e \cdot e = -2$ for $e = c_1(L) \in H^2(\hat{\mathbb{P}}_2;\mathbb{Z})$. So $\hat{V}|_F$ is always trivial except on at most

$$< c_2(\hat{V})/[F] \, , \, [E_0] > = 2$$

fibres F on $\hat{\mathbb{P}}_2$ (cf. [M] lemma (3.6)). This completes the calculation of $\hat{q}$.

Despite not required in this discussion, the description of $M_2(\hat{\mathbb{P}}_2)$ has been found in [B]. See also [K].

APPENDIX

We include an appendix to explain the subset $M_2(\hat{\mathbb{P}}_2)\backslash\pi^* M_2(\mathbb{P}_2) = V^J_{E_\infty}$ is a copy of

(a.1) $$(\mathbb{P}_2\backslash E_\infty) \times \left[\frac{\mathbb{C}^3\backslash\mathbb{C}^2}{\mathbb{C}^*}\right] \simeq \mathbb{C}^2 \times \mathbb{C}^2 \simeq \mathbb{C}^4 .$$

The following table facilitates our discussion.

(a.2) TABLE

	E^2_∞	E_∞	E^{-1}_∞	E^{-2}_∞
h^0	1	1	0	0
h^1	1	0	0	2
h^2	0	0	0	0
χ	0	1	0	– 2

Here in (a.2) we denote χ the Euler characteristic and h^i , $i = 0,1,2$, the (complex) dimension of cohomology groups associated complex line bundles E^ℓ_∞ , $\ell = -2,-1, 1, 2$, over $\hat{\mathbb{P}}_2$.

We observe first from (a.2) a bundle $\hat{V}$ obtained from an extension

$$0 \longrightarrow E^{-1}_\infty \xrightarrow{\varphi} \hat{V} \longrightarrow E_\infty \otimes I \longrightarrow 0$$

is stable only if the zero z of the section $\varphi \in H^0(\hat{V} \otimes E_\infty)$ stays away from E_∞. In such situations $h^0(\hat{V} \otimes E_\infty) = 1$. However the zero $z \in \hat{\mathbb{P}}_2 \backslash E_\infty \simeq \mathbb{C}^2$ of φ does not uniquely determine $\hat{V}$ as we deduce from the spectral sequence

$$0 \longrightarrow H^1(E_\infty^{-2}) \longrightarrow \mathrm{Ext}^1(I, E_\infty^{-2}) \longrightarrow \underline{\mathrm{Ext}}\,^1_z(I, E_\infty^{-2}) \longrightarrow H^2(E_\infty^{-2}) \longrightarrow \dots$$

that associated to each $z \in \hat{\mathbb{P}}_2 \backslash E_\infty$ there is a family of inequivalent locally free extensions parametrized by a copy of

$$\frac{\mathrm{Ext}^1(I, E_\infty^{-2}) \backslash H^1(E_\infty^{-2})}{\mathbb{C}^*} \simeq \frac{\mathbb{C}^3 \backslash \mathbb{C}^2}{\mathbb{C}^*} \simeq \mathbb{C}^2$$

as $H^1(E_\infty^{-2}) \simeq \mathbb{C}^2$ while $H^2(E_\infty^{-2}) = 0$ by (a.2). Now the description of $V^J_{E_\infty} \simeq \mathbb{C}^4$ stated in (a.1) follows should one check all bundles $\hat{V}$ so obtained are indeed stable. This can be settled by essentially the same arguments in showing (2.8), (3.12) and for this reason we omit the proof here.

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