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CHOW RING OF GENERIC FLAG VARIETIES

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Abstract. Let G be a split semisimple algebraic group over a field k and let X be the flag variety (i.e., the variety of Borel subgroups) of G twisted by a generic G-torsor. We start a systematic study of the conjecture, raised in [8] in form of a question, that the canonical epimorphism of the Chow ring of X onto the associated graded ring of the topological filtration on the Grothendieck ring of X is an isomorphism. Since the topological filtration in this case is known to coincide with the computable gamma filtration, this conjecture indicates a way to compute the Chow ring. We reduce its proof to the case of $k = \mathbb{Q}$. For simply-connected or adjoint G, we reduce the proof to the case of simple G. Finally, we provide a list of types of simple groups for which the conjecture holds. Besides of some classical types considered previously (namely, A, C, and the special orthogonal groups of types B and D), the list contains the exceptional types G_2, F_4, and simply-connected E_6.

Contents

1. The conjecture

Let k be a field and let G be a split semisimple algebraic group over k. A *(standard) generic G-torsor* E is the generic fiber of a *(standard) versal* G-torsor $U \to S = U/G$, whose total space U is a non-empty open G-equivariant subvariety in a finite-dimensional linear representation V of G. (For short, we omit the word "standard" in the sequel.)

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Therefore, the base S of the versal torsor is an absolutely integral variety over k whose function field $F := k(S)$ is the base of the generic torsor E; in particular, E is a principle homogeneous G-space over the field F. Versal and therefore generic G-torsors exist for any k and any G, see, e.g., [10, Example 2.6].

The *generic flag variety* X of G, given by E, is defined as the flag variety (i.e., the variety of Borel subgroups) of G_F, twisted by E. A choice of a Borel subgroup $B \subset G_F$ identifies X with the quotient variety E/B. We write $\operatorname{CH} X$ for the Chow ring of X (graded by codimension of cycles), $K(X)$ for the Grothendieck group of X, and $GK(X)$ for the graded ring associated with the topological filtration (i.e., the filtration by codimension of support) on $K(X)$. We consider the epimorphism of graded rings $\operatorname{CH} X \twoheadrightarrow GK(X)$, associating to the class of a closed subvariety $Z \subset X$ of codimension j the class in the j-th graded piece of $GK(X)$ of the structure bundle of Z.

Conjecture 1.1. *For any k, G, and E as above, the epimorphism $\operatorname{CH} X \twoheadrightarrow GK(X)$ is an isomorphism.*

The ring $K(X)$ is known due to [12]. Moreover, the topological filtration on $K(X)$ coincides with the gamma filtration (see [7, Example 2.4]), which is computable. Therefore Conjecture 1.1 is a way to compute $\operatorname{CH} X$. Let us mention a recent [18] where the problem of computation of $\operatorname{CH} X$ is also investigated.

As shown in Section 2, Conjecture 1.1 does not depend on E. In Section 3, we show that Conjecture 1.1 only needs to be proven for $k = \mathbb{Q}$. In Section 4, we show that Conjecture 1.1 holds for $G = G_1 \times G_2$ provided it holds for G_1 and G_2; in particular, for simply-connected or adjoint G, we reduce the proof of Conjecture 1.1 to the case of simple G.

In the final Section 5, we provide a list of simple G for which Conjecture 1.1 holds. Besides of some classical types considered previously (namely, A, C, and the special orthogonal groups of types B and D), the list contains the exceptional types G_2, F_4, and simply-connected E_6. As by now, all remaining types (besides the spinor group Spin_n with $n \leq 10$) seem to be open.

Summarizing, we prove:

Theorem 1.2. *Conjecture 1.1 holds for G (with arbitrary k and E) provided that G is a product of simple groups none of which is (isomorphic to): a spinor group Spin_n with $n \geq 11$, a semispinor group $\operatorname{Spin}^{\pm}_{4n}$ with $n \geq 2$, an adjoint group of type D_n with $n \geq 4$, an adjoint group of type E_6, any of type E_7 or of type E_8.*

2. Variation of E

For arbitrarily fixed k and G, the ring $\operatorname{CH} X$ does not depend on the choice of a generic G-torsor E:

Lemma 2.1. *For k and G as in Conjecture 1.1 and for $i = 1, 2$, let E_i be a generic G-torsor, and let X_i the the generic flag variety of G given by E_i. The rings $\operatorname{CH} X_1$ and $\operatorname{CH} X_2$ are canonically isomorphic.*

Proof. The following proof is due to A. S. Merkurjev. For $i = 1, 2$, let E_i be the generic fiber of a versal G-torsor $U_i \to S_i$, where U_i is a non-empty open G-equivariant subvariety of a linear representation V_i of G. In particular, the base field of E_i is the function field $F_i := k(S_i)$. Then $U := U_1 \times U_2$ is a non-empty open G-equivariant subvariety of the G-representation $V := V_1 \oplus V_2$ and we have a versal G-torsor $U \to S := S_1 \times S_2$. There is a commutative diagram:

$$\begin{array}{ccccc} U_1 & \longleftarrow & U & \longrightarrow & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \longleftarrow & S & \longrightarrow & S_2 \end{array}$$

Passing to the generic fibers of the vertical morphisms, we get a commutative diagram

$$\begin{array}{ccccc} E_1 & \longleftarrow & E & \longrightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} F_1 & \longleftarrow & \operatorname{Spec} F & \longrightarrow & \operatorname{Spec} F_2 \end{array}$$

where E is the generic fiber of $U \to S$ and F is the function field of the k-variety S. Since the vertical morphisms are G-torsors, both the left and the right squares of the diagram are cartesian, giving identifications $E = (E_i)_F$ and, therefore, $X = (X_i)_F$, where X is the generic flag variety corresponding to E. By no-name lemma [15, Lemma 2.1], the field extensions F/F_i are purely transcendental. It follows that the change of field homomorphisms $\operatorname{CH} X_i \to \operatorname{CH} X$ are isomorphisms. □

3. Variation of k

The following lemma reduces Conjecture 1.1 to prime fields:

Lemma 3.1. *If Conjecture 1.1 holds for some field k and some k-group G, then it holds for any field extension k'/k and the k'-group $G' := G_{k'}$.*

Proof. Let $U \to S$ be a versal G-torsor and let E be the generic G-torsor given by its generic fiber. Then $U_{k'} =: U' \to S' := S_{k'}$ is a versal G'-torsor whose generic fiber E' is a generic G'-torsor. The base of E' is the function field $F' := k'(S)$. The corresponding to E' flag variety X' is then the base change F'/F of the F-variety X. We are using the characteristic maps described in Appendix. In the commutative square

$$\begin{array}{ccc} S(\hat{T}') & \longrightarrow & \operatorname{CH} X' \\ \uparrow & & \uparrow \\ S(\hat{T}) & \longrightarrow & \operatorname{CH} X \end{array}$$

with $T' := T_{k'}$ and surjective horizontal maps, the left vertical map is an isomorphism. It follows that the change of field homomorphism $\operatorname{CH} X \to \operatorname{CH} X'$ is surjective.

Similarly, from the commutative square

$$\begin{array}{ccc} \mathbb{Z}[\hat{T}'] & \longrightarrow & K(X') \\ \uparrow & & \uparrow \\ \mathbb{Z}[\hat{T}] & \longrightarrow & K(X) \end{array}$$

with surjective horizontal maps and an isomorphism on the left, we deduce that the change of field homomorphism $K(X) \to K(X')$ is surjective. However, by [12], it is as well injective (for an arbitrary projective homogeneous variety X and an arbitrary change of field homomorphism out of $K(X)$). It follows that the right map of the square is an isomorphism. Since the topological filtrations on both $K(X)$ and $K(X')$ coincide with gamma filtrations ([7, Example 2.4]), this is an isomorphism of rings with filtrations. Consequently, the change of field homomorphism of the associated graded rings $GK(X) \to GK(X')$ is also an isomorphism.

We have shown that the left map in the commutative square

$$\begin{array}{ccc} \mathrm{CH}\, X' & \longrightarrow & GK(X') \\ \uparrow & & \uparrow \\ \mathrm{CH}\, X & \longrightarrow & GK(X) \end{array}$$

is surjective whereas the right one is an isomorphism. Therefore the top epimorphism has to be an isomorphism provided that the bottom epimorphism is so. □

And the next proposition reduces Conjecture 1.1 to the field $\mathbb{Q}$. We recall that any split semisimple group over any field k is the base change $\mathbb{Z} \to k$ of certain Chevalley group over the integers.

Proposition 3.2. *Let G be a Chevalley group over $\mathbb{Z}$. If Conjecture 1.1 holds for the field $\mathbb{Q}$ and the $\mathbb{Q}$-group $G_{\mathbb{Q}}$, then it holds for any field k and the k-group G_k.*

Proof. We assume that Conjecture 1.1 holds for $\mathbb{Q}$ and $G_{\mathbb{Q}}$. By Lemma 3.1, it then holds for any field k of characteristic 0 and the group G_k. Therefore we may assume that $\operatorname{char} k$ is a prime p.

Conjecture 1.1 holds, in particular, for the p-adic field $\mathbb{Q}_p$ and the group $G_{\mathbb{Q}_p}$. Proceeding like in the previous proof, using specialization homomorphisms of Chow and Grothendieck rings, given by the discrete valuation ring $\mathbb{Z}_p$ (as in [2, Example 20.3.1]), in place of change of field homomorphisms, we show that it also holds for the prime subfield of k (and the corresponding base change of G). Finally, again by Lemma 3.1, it holds for k itself (and G_k). □

4. Variation of G

Proposition 4.1. *Let $G := G_1 \times G_2$ for some split semisimple algebraic groups G_1 and G_2 over the field $k = \mathbb{Q}$. Conjecture 1.1 holds for G provided it does for both G_1 and G_2.*

The proof is given in the end if this section. We start with some preparations.

For $i = 1, 2$, let E_i/F_i be a generic G_i-torsor, obtained as the generic fiber of a versal G_i-torsor $U_i \to S_i$. The product $U_1 \times_k U_2 := U \to S := S_1 \times_k S_2$ is a versal G-torsor and its generic fiber E is a generic G-torsor whose base is the field $F := k(S)$. Note that $(E_i)_F$ is a generic $(G_i)_{F_{3-i}}$-torsor with the same base F. And E coincides with the product of the torsors $(E_1)_F$ and $(E_2)_F$ over F.

We write X_i for the generic flag variety variety of $(G_i)_{F_{3-i}}$, given by $(E_i)_F$. And we write X for the generic flag variety variety of G, given by E. Then X and X_i are F-varieties satisfying $X = X_1 \times X_2$.

Lemma 4.2. *The exterior product homomorphism* $\operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to \operatorname{CH} X$ *is surjective.*

Proof. Let T_i be a maximal split torus in G_i. Then $T := T_1 \times_k T_2$ is a maximal split torus in G. The composition

$$S(\hat{T}) = S(\hat{T}_1) \otimes S(\hat{T}_2) \twoheadrightarrow \operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to \operatorname{CH} X,$$

where the middle epimorphism is tensor product of the usual epimorphisms

$$S(\hat{T}_i) \twoheadrightarrow \operatorname{CH} X_i,$$

is the usual epimorphism $S(\hat{T}) \twoheadrightarrow \operatorname{CH} X$. □

Lemma 4.3. *The exterior product homomorphism* $K(X_1) \otimes K(X_2) \to K(X)$ *is an isomorphism.*

Proof. Replacing in the proof of the previous lemma the Chow ring by the Grothendieck ring and the symmetric algebra by the group algebra, we get a proof of surjectivity for the homomorphism in question. Injectivity (for arbitrary projective homogeneous varieties X_1, X_2 and their product X) follows by [11, Theorem 16]. □

Corollary 4.4. *The exterior product homomorphism* $GK(X_1) \otimes GK(X_2) \to GK(X)$ *is an isomorphism.* □

Proof of Proposition 4.1. Tensor product of the isomorphisms $\operatorname{CH} X_i \to GK(X_i)$ gives rise to an isomorphism

$$\operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to GK(X_1) \otimes GK(X_2).$$

Composing it with the isomorphism of Corollary 4.4, we get an isomophism

$$\operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to GK(X),$$

which also decomposes as

$$\operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to \operatorname{CH} X \to GK(X),$$

where the first map is surjective by Lemma 4.2. It follows that the second (as well as the first) map of the composition is an isomorphism. □

Remark 4.5. As a byproduct of the proof of Proposition 4.1, we see that the exterior product homomorphism $\operatorname{CH} X_1 \otimes \operatorname{CH} X_2 \to \operatorname{CH} X$ of Lemma 4.2 is an isomorphism provided that Conjecture 1.1 holds for G_1 and G_2.

5. Simple groups

5a. **Types** A **and** C. For any $n \geq 1$, Conjecture 1.1 has been proved for all simple (split) groups of type A_n and of type C_n in [8, Theorem 1.1]. Note that unlike the positive cases of Conjecture 1.1 discussed in the next subsection, the Chow group of the generic flag variety here usually contains a non-trivial, even a large torsion subgroup (see [8, Examples 3.17 – 3.21]).

5b. **Special orthogonal groups.** Let G be the adjoint split simple group of type B_n for some $n \geq 1$. (Since $\mathrm{B}_1 = \mathrm{C}_1$ and $\mathrm{B}_2 = \mathrm{C}_2$, we may assume that $n \geq 3$.) This means that G is isomorphic to the split special orthogonal group O^+_{2n+1}. The corresponding generic flag variety X is then the variety of complete flags of totally isotropic subspaces of the generic $2n+1$-dimensional non-degenerate quadratic form q (given by a generic G-torsor). The variety X projects onto the highest orthogonal Grassmannian Y of q – the variety of n-dimensional totally isotropic subspaces in q. This way X is identified with the flag variety of the tautological vector bundle on Y. In particular, the Chow motive of X is a direct sum of several shifted copies of the motive of Y.

It has been shown in [14] (see also [16]) that the additive group of $\mathrm{CH}\, Y$ is torsion-free. This implies the same for $\mathrm{CH}\, X$. Since in general every element of the kernel of the epimorphism $\mathrm{CH}\, X \to GK(X)$ is of finite order, it follows that the kernel is trivial for our X meaning that Conjecture 1.1 holds for G.

The remaining split simple group of type B_n – the simply-connected one – is the spinor group Spin_{2n+1} for which Conjecture 1.1 is wide open.[1] Even the question if the Chow group of zero cycles $\mathrm{CH}_0\, X$ is torsion-free (equivalent to the same question on $\mathrm{CH}_0\, Y$) is open. If Conjecture 1.1 holds, then the homomorphism $\mathrm{CH}_0\, X \to K(X)$ is injective so that $\mathrm{CH}_0\, X$ is torsion-free by the reason that the $K(X)$ is so.

Now let G be the split special orthogonal group O^+_{2n} for some $n \geq 3$. Therefore G is a split simple group of type D_n. Since $\mathrm{D}_3 = \mathrm{A}_3$, we may assume that $n \geq 4$. We explain below that Conjecture 1.1 holds for this G. However, it is open for every of the remaining groups of type D_n, namely: the spinor group Spin_{2n} (simply-connected) – besides of $n = 4, 5$;[1] the projective orthogonal group PGO^+_{2n} (adjoint); and – in the case of even n – the semispinor group $\mathrm{Spin}^{\pm}_{2n}$.

Generic flag variety X of $G = O^+_{2n}$ is the variety of flags of totally isotropic subspaces of dimensions $1, 2, \dots, n-1$ of the generic $2n$-dimensional non-degenerate quadratic form q (of trivial discriminant) given by a generic G-torsor. The variety X projects onto a component Y of the highest orthogonal Grassmannian of q, i.e., a component of the variety of n-dimensional totally isotropic subspaces in q. (Note that Y is isomorphic to the highest orthogonal Grassmannian of a $2n-1$-dimensional subform $q' \subset q$, providing a link with the case of adjoint B_{n-1}, considered above.) This way X is identified with the flag variety of the tautological vector bundle on Y. In particular, the Chow motive of X is a direct sum of several shifted copies of the motive of Y.

It has been shown in [14] as well (see also [16]) that $\mathrm{CH}\, Y$ is torsion-free. This implies the same for $\mathrm{CH}\, X$. So, Conjecture 1.1 holds for G by the same reason as in the case of adjoint B_n.

5g. **Type** G_2**.** Let G be a split simple group of type G_2 over a field k. Conjecture 1.1 holds for such G because of the following stronger result:

Proposition 5.1. *For G as above and* **any** *G-torsor E over k, the epimorphism $\mathrm{CH}\, X \to GK(X)$ is an isomorphism, where X is the flag variety of G, twisted by E.*

[1] The case of $G = \mathrm{Spin}_n$ for $n = 7, 8, 9, 10$ is known and easy: because of relationship between G-torsors over k and 3-Pfister forms, one has the statement of Proposition 5.1 for such G as well.

Proof. By [8, Lemma 4.2] and since any parabolic subgroup of G is special, we may replace X by any variety of parabolic subgroups in G, twisted by E. One of these varieties is isomorphic to the projective quadric Y given by a 7-dimensional non-degenerate subform of a 3-fold Pfister form π (which is anisotropic if and only if E is not split). We may assume that π is anisotropic (otherwise the statement we want is trivial). The Chow motive (and therefore also the GK-motive) of Y decomposes into a direct sum, where each summand is a shift of the Rost motive R associated with π. Thus we only need to check that $\mathrm{CH}\, R \to GK(R)$ is an isomorphism. The motive R is a direct summand of the motive of a 3-dimensional smooth projective quadric. We are done because for any projective dimensional quadric Q of dimension ≤ 3 the epimorphism $\mathrm{CH}\, Q \to GK(Q)$ has trivial kernel. □

5f. F_4 **and simply-connected** E_6**.** We have a statement similar to Proposition 5.1, but we need the characteristic-0 assumption here (mainly, to have a computation of Chow groups of Rost motives related to prime 3). But by Proposition 3.2 this is fine to ensure that Conjecture 1.1 holds for F_4 and simply-connected E_6 in general.

Proposition 5.2. *Let k be a field of characteristic* 0. *Let G be a split simple group of type* F_4 *or a split simply-connected group of type* E_6 *over k. Let E be a G-torsor over k, and let X be the flag variety of G, twisted by E. Then the epimorphism* $\mathrm{CH}\, X \to GK(X)$ *is an isomorphism.*

Proof. For every prime p, let k_p be a maximal (possibly infinite) algebraic field extension of k of degree prime to p. It suffices to check the statement in the case $k = k_p$. We may assume that E is not split (over $k = k_p$) because otherwise the statement we want is trivial. The assumption implies that $p = 2, 3$.

The p-portion of the Rost invariant for G produces a symbol in the Galois cohomology group $H^3(k, \mu_p^{\otimes 2})$, see [4] for references. Since the Rost invariant has trivial kernel (see [3]), the symbol is non-zero and the upper motive of the variety X is a Rost motive R corresponding to the symbol (in the sense of [9]). It follows by [13] (as well as by [6]) that the Chow motive of the variety X decomposes in a finite direct sum of shifts of R. The Chow groups of R, computed in [9] (in characteristic 0), are as follows: $\mathrm{CH}^j R$ is $\mathbb{Z}$ for $j = 0$; $p\mathbb{Z}$ for $j = (p+1)k$ with $k = 1, \ldots, p-1$; $\mathbb{Z}/p\mathbb{Z}$ for $j = (p+1)k - 2$ with $k = 1, \ldots, p-1$; and 0 for the remaining values of j.

Let n be the number of summands in the decomposition of the motive of X into a direct sum of shifted copies of R. The change of field homomorphism $K(X) \to K(\bar{X})$, where $\bar{X}$ is X over an algebraic closure of k, is an isomorphism. The order of the cokernel of $GK(X) \to GK(\bar{X})$ is $(p-1)n$. By the formula of [5, Proposition 2], the order of torsion in $GK(X)$ is also $(p-1)n$. Since the order of torsion in $\mathrm{CH}\, X$ is $(p-1)n$ as well, the statement we want follows. □

Appendix. Characteristic maps

Let G be a split semisimple algebraic group over a field k and let X be a generic flag variety of G. Let $T \subset G$ be a maximal split torus and let $B \supset T$ be a Borel subgroup of G. Let $\hat{T}$ be the group of characters of T. We consider the group ring $\mathbb{Z}[\hat{T}]$ and the ring homomorphism $\mathbb{Z}[\hat{T}] \to K(X)$, mapping each character of T to the class in $K(X)$ of the corresponding linear bundle on X. It is surjective: the ring $\mathbb{Z}[\hat{T}]$ can be interpreted as the B-equivariant Grothendieck ring $K_B(\operatorname{Spec} k)$, and the homomorpshism decomposes as

$$\mathbb{Z}[\hat{T}] = K_B(\operatorname{Spec} k) = K_B(V) \twoheadrightarrow K_B(U) \twoheadrightarrow K_B(E) = K(E/B) = K(X),$$

where U is the open subvariety of the G-representation V for which E is the generic fiber of the G-torsor $U \to U/G$. The onto maps here are surjective by the localization property of equivariant K-groups ([17, Theorem 2.7]), the second map is an isomorphism by homotopy invariance ([17, Theorem 4.1]).

Similarly, we consider the symmetric algebra $S(\hat{T})$ and the ring homomorphism $S(\hat{T}) \to \operatorname{CH} X$, mapping each character of T to the Euler class in $\operatorname{CH}^1 X$ of the corresponding linear bundle on X. It is surjective (by the "same" reason as the above homomorphism $\mathbb{Z}[\hat{T}] \to K(X)$): the ring $S(\hat{T})$ can be interpreted as the B-equivariant Chow ring $\operatorname{CH}_B \operatorname{Spec} k$, and the homomorpshism decomposes as

$$S(\hat{T}) = \operatorname{CH}_B \operatorname{Spec} k = \operatorname{CH}_B V \twoheadrightarrow \operatorname{CH}_B U \twoheadrightarrow \operatorname{CH}_B E = \operatorname{CH} E/B = \operatorname{CH} X.$$

Here we use localization and homotopy invariance properties of equivariant Chow groups ([1]).

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