The mixed Dirichlet-Neumann-Cauchy problem for second order hyperbolic operators

Joseph Bennish

Department of Mathematics
California State University
Long Beach
California 90840
USA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

Max-Planck-Arbeitsgruppe
"Part. Differentialgl. u. kompl. Analysis"
Inst. f. Mathematik, Univ. Potsdam
14469 Potsdam
GERMANY

MPI/95-29

The Mixed Dirichlet–Neumann–Cauchy Problem for Second Order Hyperbolic Operators

Joseph Bennish[1]

Department of Mathematics,
California State University, Long Beach, California 90840

Introduction

Well–posedness, regularity and asymptotic results are obtained for the Dirichlet–Neumann–Cauchy problem on an $(n+1)$–dimensional C^∞ manifold X with boundary Σ. The boundary Σ is time–like and divided into two open regions Σ_1 and Σ_2 by an $(n-1)$–dimensional C^∞ time–like submanifold Σ_0. Dirichlet and Neumann boundary conditions are given on Σ_1 and Σ_2, respectively. Regularity and asymptotic results are also obtained for mixed initial boundary–value problems for second–order hyperbolic operators for which the boundary condition is discontinuous across an interface but satisfies the uniform Lopatiski condition. These problems were considered in [5, Eskin].

The existence and uniqueness theorem (Theorem 1.1) for the Dirichlet–Neumann–Cauchy problem is deduced from a priori estimates obtained by combining energy estimates for second-order hyperbolic initial boundary–value problems, and for elliptic mixed boundary–value problems. Existence is proved by a duality argument. This basic theorem includes already additional regularity in the tangential directions to the interface Σ_0. This regularity is necessary in order to obtain the asymptotics of the solution near the interface, and, unlike elliptic equations, tangential regularity must be included already in the a priori estimates. Technically this is due to the loss of one derivative in the estimate of a hyperbolic problem compared to an elliptic problem. The existence proof for the class of problems considered in [5, Eskin] is reworked so as to include tangential regularity.

The Wiener–Hopf method for obtaining the asymptotics for mixed elliptic problems is also used for obtaining the asymptotics for the hyperbolic problems considered in this paper. If the hyperbolic operator P is elliptic on the conormal bundle of Σ_0, the boundary operators b_1 and b_2 (2.4) are elliptic, and conormal regularity (that is, the regularity given in Theorem 2.1) holds,

[1]This work was supported by the Max-Planck- Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis", Potsdam University.

then the asymptotics near the interface can be derived by the Wiener–Hopf method as in the case of mixed elliptic boundary–value problems studied by M.I. Visik and G.I. Eskin [4, Eskin]. Conormal regularity combines both tangential regularity and regularity in weighted function spaces. The problems treated in this paper satisfy the above sufficient conditions. Note that the asymptotics for the Dirichlet–Neumann–Cauchy problem is particularly simple (Theorem 2.2): given C^∞ data the solution is C^∞ outside of the interface Σ_0, and it has (at worst) a square–root type singularity at Σ_0. The general case gives rise to a more complicated asymptotic expansion (Theorem 1.2) in which the power of the logarithm undergoes a "jump" when the factorization index κ equals a non–negative integer.

In local coordinates $x = (x_0, \dots, x_n)$ the second order hyperbolic operator P is given by

$$P(x, D) = \sum_{j,k=0}^{n} g^{jk}(x) D_j D_k + \sum_{j=0}^{n} b_j(x) D_j + c(x), \tag{0.1}$$

where $D = (D_0, \dots, D_n)$, $D_j = i\partial_j$, the coefficients are C^∞, and $\sum g^{jk}(x)\xi_j\xi_k$ is a real symmetric quadratic form having Lorentz signature, that is, (g^{jk}) has one positive and n negative eigenvalues.

To avoid considering compatability conditions we assume that the data is initially zero. More precisely, let ϕ be a C^∞ function from X to $\mathbf{R}$ with space–like level surfaces, and assume, moreover, that ϕ is a proper map. Then the Dirichlet–Neumann–Cauchy problem is given by

$$\begin{aligned} P(x, D)u &= f \qquad \text{in } X^\circ && (0.2)\\ u &= g_1 \qquad \text{on } \Sigma_1 && (0.3)\\ B(x, D)u &= g_2 \qquad \text{on } \Sigma_2 && (0.4)\\ u &= 0 \qquad \text{when } \phi < 0, && (0.5) \end{aligned}$$

where f, g_1, and g_2 vanish for $\phi < 0$ and X° is the interior of X. The boundary operator B equals $\partial_N + d(x)$, where ∂_N is the Neumann operator with respect to the operator P. In local coordinates $\partial_N = \sum g^{jk}(x)\nu_j(x)\partial_k$, where $\nu(x)$ is a C^∞ section of the conormal bundle of Σ.

The mixed Dirichlet–Neumann–Cauchy problem models the sound radiation field of an enclosed cavity in which sound is radiated into the cavity from one part of the wall of the enclosure, and sound is reflected back from the remainder of the wall. Assuming that the wall is not absorbent, sound radiation and reflection correspond to a Dirichlet and homogeneous Neumann condition, respectively, where the unknown function is the acoustic pressure.

This mixed problem can also be considered an idealized scalar model for the dynamic problem of linear elasticity in which the displacement is known on one part of the boundary and the traction is known on the complement. (The traction is analogous to the Neumann condition.) Existence and uniqueness results for the elasticity problem were given in [3, Section III.4], but regularity of the solution was not treated.

The Dirichlet–Neumann–Cauchy problem is one in a class of mixed initial–boundary value problems treated in [1] by reduction to a pseudo–differential equation on the boundary, however the operators were assumed to be constant coefficient. Moreover, the a priori estimate for the problems treated in [5] were obtained by reduction to a pseudo–differential equation on the boundary, and the existence proof was carried out by the method of elliptic regularization. A duality argument, rather than elliptic regularization, is used in the present treatment of the Dirichlet–Neumann–Cauchy problem, due to the non–ellipticity of the boundary symbols. Nevertheless, it may be possible to adapt the method of elliptic regularization to prove existence for this problem. See Remark 1.3 for further explanation.

In Section 1 energy estimates are derived and existence and uniqueness is proved. Section 2 contains the results on regularity in weighted function spaces (the weights vanish on Σ_0), and the asymptotic expansion of the solution in a neighborhood of Σ_0.

The author is indebted to Professor Gregory Eskin for suggesting this problem as well as for reading a part of the manuscript and offering helpful comments. This paper was completed in the academic year 1994–95 while the author was a guest at the Max Planck Institute, and the author wishes to take this opportunity to express his gratitude to the institute for its generous support.

1 Existence and Uniqueness Theorem

Before defining Sobolev spaces on the manifold X we make the following assumption:

(i) For some c sufficiently large there are coordinates $(x_0, \dots, x_n)$, $x_0 = \phi(x)$, in which $\phi > c$ has the form $(c, \infty) \times G$, where G is a compact C^∞ n–dimensional manifold with boundary, and $\Sigma_0 = \mathbf{R} \times \Gamma$ where Γ is an $(n-2)$–dimensional submanifold of the boundary of G. Moreover, P and B are independent of x_0 for $\phi > c$, $g^{00} = 1$ and $(g^{jk})_{j,k=1,\dots,n}$ is negative definite. Analogous assumptions are made for $\phi < -c$.

Remark 1.1. The above assumption can of course be dropped if the finite time problem is considered, i.e., the analogous problem in which X is replaced by $\phi^{-1}[S, T]$, $S < 0 < T$. This assumption might also be avoided by working in local Sobolev spaces as in [6, Section 24.1].

Choose a finite open covering $\{U_j\}$ of X and charts $\{\kappa_j\}$, $j \in I$. By making use of a Riemannian metric a neighborhood of Σ can be identified with $\Sigma \times [0,3)$. Similarly a neighborhood of Σ_0 can be identified with $\Sigma_0 \times (-3,3) \times [0,3)$. In local coordinates $x^{(j)}$ in U_j the boundary Σ is defined by $x_n^{(j)} = 0$ and the interface Σ_0 is defined by $x_{n-1}^{(j)} = x_n^{(j)} = 0$. Put $x_0 = \phi(x)$. By assumption (i), we may assume that $g^{00} = 1$, and $(g^{jk})_{j,k=1,\dots,n}$ is negative definite. Henceforth, we assume that these conditions are satisfied.

Upon multiplying the equations (0.2)–(0.4) by $e^{-\tau x_0}$ we get in local coordinates that

$$P(x, D + i\tau N)u_\tau = f_\tau \quad \text{in } X^\circ \tag{1.1}$$

$$u_\tau = g_{1\tau} \quad \text{on } \Sigma_1 \tag{1.2}$$

$$B(x, D + i\tau N)u_\tau = g_{2\tau} \quad \text{on } \Sigma_2, \tag{1.3}$$

where $u_\tau(x) = e^{-\tau x_0}u(x)$, and f_τ, etc., are defined similarly, and $N = (1, 0, \dots, 0)$ is conormal to the level surfaces of $\phi(x)$.

The equations (1.1)–(1.3) motivate the introduction of Sobolev spaces which depend on a parameter τ. Let $H_{s,\tau}(\mathbf{R}^{n+1})$ denote the normed space which consists of the distributions $u(\cdot, \tau)$ for which

$$\|u(\cdot,\tau)\|_s^2 = \int \Lambda^{2s} |\tilde{u}(\xi,\tau)|^2 \, d\xi \tag{1.4}$$

is finite, where

$$\Lambda(\xi, \tau) = (\xi_0^2 + \tau^2 + |\xi_1|^2 + \cdots + |\xi_n|^2)^{1/2}.$$

This defines the norm in the aforementioned space. Here $\tilde{v}$ represents the Fourier transform:

$$\tilde{v}(\xi) = \int e^{ix\cdot\xi} v(x) \, dx.$$

The normed quotient space $H_{s,\tau}(\mathbf{R}_+^{n+1})$ consists of all distributions in $\mathbf{R}_+^{n+1} = \{x : x_n > 0\}$ having an extension $lu(\cdot,\tau)$ in $H_{s,\tau}(\mathbf{R}^{n+1})$. The quotient norm is given by

$$\|u\|_s^+ = \inf \|lu\|_s$$

where the infimum is taken over all extensions lu in $H_{s,\tau}(\mathbf{R}^{n+1})$.

Put $\hat{x} = (x_{n-1}, x_n)$. Choose a finite partition of unity $\{\phi_j\}$ of X, $j \in I$, subordinate to the coordinate neighborhoods U_j such that ϕ_j is independent of x_0 for $|x_0|$ large. The index set I is the disjoint union of I_1 and I_2, where $j \in I_1$ if and only if $\overline{U_j}$ intersects the interface Σ_0. Moreover, the coordinate neighborhoods U_j, $j \in I_1$, are contained in the "collar" $|\hat{x}| < 2$ of

Σ_0. The Sobolev space $H_{s,\sigma,t,\tau}(X^\circ)$, for real numbers s, σ, and t, consists of distributions $u(\cdot,\tau)$, $\tau > 0$, in X° with finite norm

$$\|u\|_{s,\sigma,t} = \sum_{j\in I_1} \|\Lambda_0^t(\Lambda_j'')^\sigma \kappa_{j*}\phi_j u\|_s^+ + \sum_{j\in I_2} \|\Lambda_0^t \kappa_{j*}\phi_j u\|_{s+\sigma}, \tag{1.5}$$

where $\Lambda_0^t = (D_0 + i\tau)^t$, and the symbol of the pseudo–differential operator $(\Lambda_j'')^\sigma$ is given by

$$\Lambda_j''(\xi_0,\xi_j'',\tau) = (\xi_0^2 + \tau^2 + |\xi_j''|^2)^{1/2}; \qquad \xi_j'' = (\xi_1^{(j)},\cdots,\xi_{n-2}^{(j)}). \tag{1.6}$$

Here κ_{j*} is the push–forward under the diffeomorphism κ_j from U_j to $\overline{\mathbf{R}_+^{n+1}}$ where $\mathbf{R}^{n+1} = \{x \in \mathbf{R}^{n+1} : x_n \geq 0\}$, $x = (x_0,\ldots,x_n)$. The main theorem below uses the spaces

$$H_{s,\sigma,\tau}(X^\circ) := H_{s,\sigma,0,\tau}(X^\circ).$$

The norm in this space is denoted by $\|\ \|_{s,\sigma}$.

N.B. Another atlas of charts and a subordinated partition of unity in the definition (1.6) leads typically to a norm which is *not* equivalent to the original one. We will employ in the following theorem and lemmas such norms which are defined with different partitions of unity. Indeed, one could use instead of (U_j,ϕ_j) the system (U_j',ψ_j) where $\psi_j \geq 0$ is in $C_0^\infty(U_j')$ and $\sum\psi_j$ never vanishes. In particular, the Sobolev spaces occuring on the right–hand side of estimates will typically employ functions ψ_j satisfying $\psi_j\phi_j = \phi_j$, where ϕ_j are the functions used in the definition of the norm on the left–side of the inequality. No distinction in notation will be used. For a specific norm which appears either in an estimate or the statement of a proof the coordinate systems U_j' and functions ψ_j are fixed, in particular, they are independent of the parameter τ.

The spaces $H_{s,t,\tau}(\Sigma)$, $H_{s,\tau}(\Sigma)$, and $H_{s,t,\tau}(\Sigma_i)$, $H_{s,\tau}(\Sigma_i)$, $i = 1,2$, are defined analogously. The norm of v in $H_{s,\tau}(\mathbf{R}^n)$ is given by

$$[v(\cdot,\tau)]_s^2 = \int \Lambda'^{2s}|\tilde{v}|^2\, d\xi_0 d\xi',$$

where the symbol of Λ' is

$$\Lambda'(\xi_0,\xi',\tau) = (\xi_0^2 + \tau^2 + |\xi'|^2)^{1/2}; \qquad \xi' = (\xi_1,\ldots,\xi_{n-1}).$$

The norm of v in $H_{s,t,\tau}(\Sigma)$ is given by

$$[v]_{s,t} = \sum[\Lambda_0^t\kappa_{j*}\phi_j' v]_s,$$

where ϕ_j' is the restriction of ϕ to Σ, and the sum is taken over those j for which the support of ϕ_j intersects Σ. The norms in $H_{s,t,\tau}(\Sigma)$ and in the quotient spaces $H_{s,t,\tau}(\Sigma_i)$, $i = 1, 2$, are denoted by $[\ ]_{s,t}$, $[\ ]'_{s,t}$ and $[\ ]''_{s,t}$, respectively. We set $H_{s,\tau}(\Sigma) := H_{s,0,\tau}(\Sigma)$, etc., with corresponding norms $[\ ]_s$, etc. The space $H_{s,\tau}(\mathbf{R}^n)$ is defined to be $H_{s,0,\tau}(\mathbf{R}^n)$, etc.

We give next the well-posedness result for the Dirichlet–Neumann–Cauchy problem and present its proof after some preliminary lemmas.

Theorem 1.1 *Assume that P is hyperbolic with respect to the level surfaces of ϕ, ϕ is proper, Σ and Σ_0 are time-like, and assumption (i) holds. Let f, g_1 and g_2 be data which vanish for $\phi < 0$ and which satisfy $f_\tau \in H_{0,\sigma,\tau}(X^\circ)$, $g_{1\tau} \in H_{3/2+\sigma,\tau}(\Sigma_1)$ and $g_{2\tau} \in H_{1/2+\sigma,\tau}(\Sigma_2)$ for $\sigma \geq 0$ and τ large. Then there is a unique solution u of the mixed problem (0.2)–(0.5) for which $u_\tau \in H_{1,\sigma,\tau}(X^\circ)$ if τ is sufficiently large.*

Remark 1.2. The above result could be somewhat stregthened by using more involved Sobolev spaces. The result then is roughly that compared to the analogous mixed elliptic problem there is a loss of one derivative, and near Σ_0 this loss occurs just along the interface Σ_0. Note that there is an additional loss of one–half derivative in the Dirichlet data $g_{1\tau}$ compared to the result for the Dirichlet–Cauchy problem given in [6, Section 24.1].

We now define auxillary Sobolev spaces which are used in the proof of the main result. Decompose I_2 into a disjoint union of I_3 and I_4 where $j \in I_3$ if and only if $\overline{U_j}$ has non-trivial intersection with Σ. Moreover, the partition of unity is sufficiently refined so that U_j is contained in $x_n < 2$ if $j \in I_3$. The norm in $\mathcal{H}_{s,p,\sigma,t,\tau}(X^\circ)$, $\tau > 0$, is given by

$$\begin{aligned} |u|_{s,p,\sigma,t} &= \sum_{j\in I_1} \|\Lambda_0^t \Lambda'^p (\Lambda_j'')^\sigma \kappa_{j*}\phi_j u\|_s^+ + \sum_{j\in I_3} \|\Lambda_0^t (\Lambda')^{p+\sigma} \kappa_{j*}\phi_j u\|_s^+ \\ &\quad + \sum_{j\in I_4} \|\Lambda_0^t \kappa_{j*}\phi_j u\|_{s+p+\sigma}. \end{aligned} \tag{1.7}$$

The other type of auxillary Sobolev space is defined by

$$\mathcal{H}_{p,\sigma,\tau}(X^\circ) := \mathcal{H}_{0,p,\sigma,0,\tau}(X^\circ)$$

with norm $|\ |_{p,\sigma}$.

In the following four lemmas basic energy estimates for the Dirichlet–Neumann–Cauchy problem are derived. In *all* instances in Section 1 the constant C denotes a constant independent of τ. This is repeated in the statement of the theorems and lemmas for emphasis.

Lemma 1.1 *Assume $u(\cdot,\tau)$ is a solution of the boundary-value problem (1.1)–(1.3) belonging to $H_{1,0,t+1,\tau}(X^\circ)$, and that $f_\tau \in H_{0,0,t,\tau}(X^\circ)$, $g_{1\tau} \in$*

$H_{1/2,t+1,\tau}(\Sigma_1)$, and $g_{2\tau} \in H_{-1/2,t+1,\tau}(\Sigma_2)$, $t \in \mathbf{R}$. Then u satisfies the a priori estimate

$$\sqrt{\tau}\, \|u\|_{1,0,t} \le C\|f_\tau\|_{-1,0,t+1} + C[g_{1\tau}]'_{1/2,t+1} + C[g_{2\tau}]''_{-1/2,t+1}, \tag{1.8}$$

for τ sufficiently large, and C independent of τ.

N.B. Lemma 1.1 with $t = -1$ implies the uniqueness assertion of Theorem 1.1.

Proof. We start with the half-space problem $x_n > 0$ and assume for now that $t = 0$ and $P(D)$ is given by $\sum g^{jk} D_j D_k$, g^{jk} are constants, $B = \partial_N$ and that Σ_1 (Σ_2) is given by $x_n = 0$ and $x_{n-1} > 0$ ($x_{n-1} < 0$). Multiplying the equation $P(D + i\tau N)u(\cdot,\tau) = f_\tau$ by $2(\partial_0 + \tau)\overline{u}$, taking the real part, and integrating by parts we get

$$\tau \int_{x_n>0} \left(|(\partial_0+\tau)u|^2 - \sum_{j,k\ge 1} g^{jk}\partial_j u \overline{\partial_k u} \right) dx = \tag{1.9}$$
$$\Re \int_{x_n>0} f_\tau (\partial_0+\tau)\overline{u}\, dx - \Re \int_{x_n=0} Bu\, (\partial_0+\tau)\overline{u}\, dS;$$

where ν is the exterior normal $(0,\ldots,0,-1)$, B is the Neumann operator (0.4), and where we have used the identities:

$$2\Re((\partial_0+\tau)\overline{u}(\partial_0+\tau)^2 u) = \partial_0\left(|(\partial_0+\tau)u|^2\right) + 2\tau|(\partial_0+\tau)u|^2;$$

$$2\Re \sum_{j,k=1}^{n} (\partial_0+\tau)\overline{u} g^{jk}\partial_j\partial_k u = 2\Re \sum_{j,k=1}^{n} \partial_k\left(g^{jk}\partial_j u(\partial_0+\tau)\overline{u}\right)$$
$$- \partial_0 \sum_{j,k=1}^{n} \left(g^{jk}\partial_j u\overline{\partial_k u}\right) - 2\tau \sum_{j,k=1}^{n} g^{jk}\partial_j u\overline{\partial_k u},$$

$$2\Re\left((\partial_0+\tau)\overline{u} g^{0k}(\partial_0+\tau)\partial_k u\right) = \partial_k(g^{0k}|(\partial_0+\tau)u|^2), \qquad \text{for } k > 0.$$

We obtain a bound for the surface integral that appears in (1.9) assuming Dirichlet–Neumann boundary conditions: $u = g_1$ on Σ_1 and $Bu = g_2$ on Σ_2. Let

$$(v,w) = \int_{x_n=0} v\overline{w}\, dS.$$

Choose extensions lg_i of g_i, $i = 1,2$, such that

$$[\Lambda_0 lg_{1\tau}]_{1/2} < 2[\Lambda_0 g_{1\tau}]'_{1/2}, \qquad [\Lambda_0 lg_{2\tau}]_{-1/2} < 2[\Lambda_0 g_{2\tau}]''_{-1/2}.$$

In the identity

$$(Bu, \Lambda_0 u) = -(lg_{2\tau}, \Lambda_0 lg_{1\tau}) + (Bu - lg_{2\tau}, \Lambda_0(u - lg_{1\tau})) + (lg_{2\tau}, \Lambda_0 u) + (Bu, \Lambda_0 lg_{1\tau})$$

the second integral on the right–hand side vanishes. Thus, we get

$$|(Bu, \Lambda_0 u)| \leq C\left\{[\Lambda_0 lg_{1\tau}]_{1/2} + [\Lambda_0 lg_{2\tau}]_{-1/2} + \|u\|_1\right\}. \tag{1.10}$$

By virtue of the inequality

$$|\int_{x_n>0} f_\tau(\partial_0 + \tau)\overline{u}\, dx| = |\int_{x_n>0} (\tau - \partial_0) f_\tau \overline{u}\, dx| \leq \|(\partial_0 - \tau) f_\tau\|_{-1}\|u\|_1$$

and the negative definiteness of $(g^{jk})_{j,k=1,\dots,n}$, we get by (1.9) and (1.10) that for τ sufficiently large

$$\sqrt{\tau}\|u\|_1 \leq C\|\Lambda_0 f_\tau\|_{-1} + C[\Lambda_0 g_{1\tau}]'_{1/2} + C[\Lambda_0 g_{2\tau}]''_{-1/2} \tag{1.11}$$

If the original operators $P(x, D + i\tau N)$ (see (0.1)) and B are used we get an identity which is the same as (1.9) aside from an additional term which is a sesquilinear form in (u, u'). The absolute value of this term is bounded by $C\|u\|_1$. Hence, for τ large, the estimate (1.11) still holds.

Now assume that $\Lambda_0^{t+1} u$ is in $H_{1,\tau}(\mathbf{R}_+^{n+1})$, $t \in \mathbf{R}$. Putting $v = \Lambda_0^t u$, we get

$$\begin{aligned} Pv &= \Lambda_0^t f_\tau - \Lambda_0^t [P, \Lambda_0^{-t}]v, && \text{in } x_n > 0, \\ v &= \Lambda_0^t g_{1\tau}, && \text{on } x_n = 0,\ x_{n-1} > 0 \\ Bv &= \Lambda_0^t g_{2\tau} - \Lambda_0^t [B, \Lambda_0^{-t}]v, && \text{on } x_n = 0,\ x_{n-1} < 0. \end{aligned}$$

Applying estimate (1.11) to v we get the estimate

$$\sqrt{\tau}\|\Lambda_0^t u\|_1 \leq C\|\Lambda_0^{t+1} f_\tau\|_{-1} + C[\Lambda_0^{t+1} g_{1\tau}]'_{1/2} + C[\Lambda_0^{t+1} g_{2\tau}]''_{-1/2}, \tag{1.12}$$

for τ large, since the term involving the commutator can then be absorbed by the left–hand side.

Estimates analogous to (1.12) hold for the Dirichlet problem, Neumann problem and the problem without boundary (interior estimate). Thus we get for any j

$$\sqrt{\tau}\|\Lambda_0^t \kappa_{j*}\phi_j u\|_1 \leq C\|\Lambda_0^{t+1} P\kappa_{j*}\phi_j u\|_{-1} + C[\Lambda_0^{t+1}\kappa_{j*}\phi_j u]'_{1/2} + C[\Lambda_0^{t+1} B\kappa_{j*}\phi_j u]''_{-1/2}$$

$$\leq C\|\Lambda_0^{t+1}\kappa_{j*}\phi_j f_\tau\|_{-1} + C[\Lambda_0^{t+1}\kappa_{j*}\phi_j g_{1\tau}]'_{1/2} + C[\Lambda_0^{t+1}\kappa_{j*}\phi_j g_{2\tau}]''_{-1/2} + C\|\Lambda_0^t \psi_j u\|_1.$$

Here $\psi_j \in C_0^\infty(U_j)$ with $\psi_j\phi_j = \phi_j$. Summing over j and taking τ large, we get the global estimate (1.8).

Lemma 1.2 *If $u(\cdot,\tau) \in C_0^\infty(X^\circ)$, then for $\sigma \geq 0$ the a priori estimate*

$$|u|_{1,\sigma} \leq C|Pu|_{-1,\sigma+1} \tag{1.13}$$

holds for all τ sufficiently large. The constant C is independent of the parameter τ.

Proof. If C_0 is sufficiently large then

$$E(x, D + i\tau N) = P(x, D + i\tau N) - C_0(D_0 + i\tau)^2 \tag{1.14}$$

is an elliptic operator. The elliptic estimate for the Dirichlet problem gives

$$|u|_{2,-1,0,0} \leq C|Eu|_{0,-1,0,0} + C|u|_{1,-1,0,0}. \tag{1.15}$$

The definition of the norms which appear on the right–hand side is modified from the one given in (1.7) in that the functions ψ_j are used instead of ϕ_j, with $\psi_j\phi_j = \phi_j$. Consequently the last norm on the right side in the above inequality cannot be absorbed by the left–hand side even for τ large.

Put

$$\Lambda_{1j}^\sigma = \kappa_{j*}^{-1}\psi_j(\kappa_j^{-1}(x))\Lambda_{2j}^\sigma\kappa_{j*}\phi_j,$$

where Λ_{2j} equals Λ_j'' if $j \in I_1$, Λ_j' if $j \in I_3$, and Λ if $j \in I_4$. Here $\psi_j \in C_0^\infty(U_j)$ with $\psi_j\phi_j = \phi_j$. Applying this pseudodifferential operator to the elliptic equation gives

$$E(\Lambda_{1j}^\sigma u) = \Lambda_{1j}^\sigma Eu + [E, \Lambda_{1j}^\sigma]u.$$

The above elliptic estimate gives (after summing over all $j \in I$)

$$\begin{aligned} |u|_{2,-1,\sigma,0} &\leq C|Eu|_{0,-1,\sigma,0} + C|u|_{2,-1,\sigma-1,0} \\ &\leq C|Pu|_{0,-1,\sigma,0} + C|u|_{0,-1,\sigma,2} + C|u|_{2,-1,\sigma-1,0} \\ &\leq C|Pu|_{0,-1,\sigma,0} + C|u|_{2,-1,\sigma-\delta,\delta}, \end{aligned} \tag{1.16}$$

for $0 < \delta \leq 1$. (This estimate holds for all real σ.) For $\sigma \geq 0$ we get by induction

$$|u|_{2,-1,\sigma,0} \leq C|Pu|_{0,-1,\sigma,0} + C|u|_{2,-1,0,\sigma}.$$

By a straightforward analogue of [6, Theorem B.2.9] (partial hypoellipticity) we get

$$|u|_{2,-1,0,\sigma} \leq C|Pu|_{0,-1,\sigma,0} + C\|u\|_{1,0,\sigma}.$$

The last two inequalities and estimate (1.8) now give

$$|u|_{2,-1,\sigma,0} \leq C|Pu|_{0,-1,\sigma+1,0}.$$

This implies estimate (1.13).

Lemma 1.3 *If $u \in \mathcal{H}_{1,\sigma,\tau}$, $\sigma \in \boldsymbol{R}$, $Pu = 0$ in X°, $u = 0$ on Σ_1, $Bu = 0$ on Σ_2, $\tau > \tau_0$ for τ_0 sufficiently large (independent of u), then $u = 0$ on X°.*

Proof. Without loss of generality we assume that $\sigma < 0$. It suffices, by estimate (1.8), to show that $u \in H_{1,0,\sigma,\tau}(X^\circ)$. A regularity argument is used to show that u belongs to this smoother space.

By partial hypoellipticity, $u \in \mathcal{H}_{2,-1,\sigma,0,\tau}$. We obtain the a priori estimate

$$|u|_{2,-1,\sigma,t} \leq C|Eu|_{0,-1,\sigma,t} + C|u|_{2,-1,\sigma-1,t}, \quad \sigma,\ t \in \mathbf{R},$$

for the solution u to the elliptic mixed–boundary value problem $Eu = -C_0\Lambda_0^2 u$ in X°, $u = 0$ on Σ_1, $Bu = 0$ on Σ_2, where E is defined by (1.14). Indeed, the solution u satisfies the elliptic estimate (1.15) essentially by [4]Example 13.1 and [4]Thm. 22.1. The case $t = 0$ then follows as in the proof of (1.16). For general t this can be proved as in the proof of Lemma 1.1. That is, the operator Λ_0^t is applied locally, and the commutator term can then (in the global estimate) be absorbed by the left side.

Put

$$\Lambda_{1j\epsilon}^r = \kappa_{j*}^{-1}\psi_j(\kappa_j^{-1}(x))\Lambda_{2j\epsilon}^r\kappa_{j*}\phi_j,$$

where the symbol $\sigma(\Lambda_{2j\epsilon}^r)$, $\epsilon > 0$, of the operator $\Lambda_{2j\epsilon}^r$ is given by

$$\sigma(\Lambda_{2j}^r)/\left(1 + \epsilon\sigma(\Lambda_{2j}^r)\right).$$

The operator Λ_{2j}^r is defined as in the previous lemma. The functions $\phi_j \in C_0^\infty(U_j)$ form a partition of unity, and $\psi_j \in C_0^\infty(U_j)$ satisfies $\psi_j\phi_j = \phi_j$. Applying $\Lambda_{1j\epsilon}^r$, for $0 < r < 1$, to the mixed boundary–value problem $Eu = -C_0\Lambda_0^2 u$, etc., gives

$$E(\Lambda_{1j\epsilon}^r u) = \Lambda_{1j\epsilon}^r\Lambda_0^2 u + [E, \Lambda_{1j\epsilon}^r]u,$$

$\Lambda_{1j\epsilon}^r u = 0$ on Σ_1 and $B(\Lambda_{1j\epsilon}^r u) = 0$ on Σ_2, since, without loss of generality, the coefficient of D_n in the operator B is constant. By the previous a priori estimate applied to this mixed problem we get

$$\begin{aligned}|\Lambda_{1j\epsilon}^r u|_{2,-1,\sigma,-r} &\leq C|\Lambda_{1j\epsilon}^r\Lambda_0^2 u|_{0,-1,\sigma,-r} + C|[E,\Lambda_{1j\epsilon}^r]u|_{0,-1,\sigma,-r} \\ &\quad + C|\Lambda_{1j\epsilon}^r u|_{2,-1,\sigma-1,-r} \leq C|u|_{2,-1,\sigma,0}.\end{aligned}$$

This shows that $|\Lambda_{1j\epsilon}^r u|_{2,-1,\sigma,-r}$ is bounded independently of $\epsilon > 0$, and implies that there is a sequence $\Lambda_{1j\epsilon_n}^r u$ converging weakly in $\mathcal{H}_{2,-1,\sigma,-r,\tau}$ with $\epsilon_n \to 0$. Hence $u \in \mathcal{H}_{2,-1,\sigma+r,-r,\tau}$. By induction $u \in \mathcal{H}_{2,-1,0,\sigma,\tau}$, which implies that $u \in H_{1,0,\sigma,\tau}$. Lemma 1.3 is proved.

We introduce a positive density on X (which is independent of x_0 for $|x_0|$ large) and a corresponding scalar product $(\cdot,\cdot)_0$ on $L^2(X)$. This scalar product for $u,\ v \in C_0^\infty(X)$ extends uniquely to a non-degenerate sesquilinear pairing

$$(\cdot,\cdot)_0 : \ (\mathcal{H}_{p,\sigma,\tau}(X^\circ))^* \times \mathcal{H}_{p,\sigma,\tau}(X^\circ).$$

Let $|u|_{(p,\sigma)}$, for $u \in C_0^\infty(X)$, represent its norm as an element in the dual space $(\mathcal{H}_{-p,-\sigma,\tau}(X^\circ))^*$ with respect to this sesquilinear form, that is,

$$|u|_{(p,\sigma)} = \sup\{|(u,v)_0|; \quad |v|_{-p,-\sigma} = 1\}.$$

Lemma 1.4 *The dual space of $\mathcal{H}_{-p,-\sigma,\tau}(X^\circ)$ has a continuous imbedding into $\mathcal{H}_{p,\sigma,\tau}(X^\circ)$ for appropriately chosen partitions of unity used in the definition (1.7). In particular, if $p,\ \sigma \geq 0$, then the partitions of unity $\{U_j, \phi_j\}$, $j \in I$, and $\{U_k^{(1)}, \phi_k^{(1)}\}$, $k \in I^{(1)}$, are appropriate choices for the first and second spaces, respectively, provided that the coverings satisfy $U_k^{(1)} \cap U_j$ is empty whenever (i) $k \in I_4^{(1)}$ and $j \in I_1 \cup I_3$, or (ii) $k \in I_3^{(1)}$ and $j \in I_1$. Here I_m, $I_m^{(1)}$, $m = 1,3,4$, are the subsets of the index sets I and $I^{(1)}$, respectively, used in the definition (1.7).*

The converse also holds, that is, $\mathcal{H}_{p,\sigma,\tau}(X^\circ)$ has a continuous imbedding into the dual space of $\mathcal{H}_{-p,-\sigma,\tau}(X^\circ)$ for (other) appropriately chosen partitions of unity.

Proof. We prove the first assertion for the case $p,\ \sigma \geq 0$. The converse and the other cases are proved similarly.

Let $k \in I_4^{(1)}$. For simplicity of notation we do not write the push-forwards and pullbacks of the diffeomorphisms in the local coordinate systems. We have for $u \in C_0^\infty(X)$:

$$|\phi_k^{(1)}u|_{p+\sigma} = \sup\{\,|(\phi_k^{(1)}u, v)_0| : \ |v|_{-p-\sigma} = 1\}.$$

But $(\phi_k^{(1)}u, v)_0 = (\phi_k^{(1)}u, \psi_k^{(1)}v)_0$ and $|\psi_k^{(1)}v|_{-p-\sigma} \leq C|v|_{-p-\sigma}$, where $\psi_k^{(1)} \in C_0^\infty(U_k^{(1)})$, $\psi_k^{(1)}\phi_k^{(1)} = \phi_k^{(1)}$, and the constant C depends only on the function $\psi_k^{(1)}$. Hence

$$|\phi_k^{(1)}u|_{p+\sigma} \leq C\sup\{\,|(\phi_k^{(1)}u, \psi_k^{(1)}v)_0| : \ |\psi_k^{(1)}v|_{-p-\sigma} = 1\}.$$

Furthermore, by the hypothesis on the partitions of unity, we get that

$$|\psi_k^{(1)}v|_{-p,-\sigma} = |\psi_k^{(1)}v|_{-p-\sigma}.$$

Thus,

$$\begin{aligned} |\phi_k^{(1)}u|_{p+\sigma} &\leq C\sup\{\,|(\phi_k^{(1)}u, v)_0| : \ |v|_{-p,-\sigma} = 1\} \\ &\leq C|\phi_k^{(1)}u|_{(p,\sigma)}. \end{aligned}$$

However, it can easily be shown that multiplication by a compact C^∞ function is a continuous operator on the dual space. Thus we get

$$|\phi_k^{(1)}u|_{p+\sigma} \leq C|u|_{(p,\sigma)}, \quad k \in I_4^{(1)}.$$

The proof of analogous estimates for the other values of k is proved similarly. By summing over $k \in I^{(1)}$ we obtain the assertion.

Lemma 1.5 *If $u \in C_0^\infty(X)$, $u = 0$ on Σ_1, $Bu = 0$ on Σ_2, then for $\sigma \leq -1$ and τ sufficiently large the following a priori estimate holds:*

$$|u|_{(1,\sigma)} \leq C|Pu|_{-1,\sigma+1}. \tag{1.17}$$

Proof. Let P^* be the formal adjoint of P with respect to the above scalar product, that is,

$$(u, P^*w)_0 = (Pu, w)_0, \qquad u,\ w \in C_0^\infty(X^\circ).$$

By Lemmas 1.2 and 1.4 we get the inequality

$$|(u, P^*w)_0| \leq |Pu|_{-1,\sigma+1}|w|_{(1,-\sigma-1)} \leq C|Pu|_{-1,\sigma+1}|P^*w|_{-1,-\sigma}.$$

We consider the Hilbert space given by the direct sum

$$(\mathcal{H}_{-1,-\sigma,\tau}(X^\circ))^* \oplus \mathcal{H}_{-1,\sigma+1,\tau}(X^\circ),$$

and a closed subspace S given by all 2-tuples of the form (u, Pu) with $u = 0$ on Σ_1 and $Bu = 0$ on Σ_2. (Note that by the condition on Pu the trace of Bu exists.) The set of all linear functionals on S of the form $(u, P^*w)_0$, $w \in C_0^\infty(X^\circ)$, is dense in the dual space of S. Indeed, if the 2-tuple (u, Pu) annihilates all linear functionals of this form then $Pu = 0$ on X°, $u = 0$ on Σ_1 and $Bu = 0$ on Σ_2, hence, by Lemmas 1.3 and 1.4 u vanishes on X°. The estimate (1.17) follows from the above inequality.

Proof of Theorem 1.1. Uniqueness is an immediate consequence of Lemma 1.1 (set $t = -1$). We prove existence using first a duality argument which resembles the proof of [6, Lemma 24.1.6].

Let $(u, v)_0$ represent the sesquilinear form introduced prior to the proof of Lemma 1.4. Near the boundary Σ we use coordinates in which Σ is defined by $x_n = 0$ and x_n is invariant. Then the restriction of the density to the boundary is also a positive density, and we let $\langle g, h\rangle$ be the sesquilinear form on the boundary with respect to this density.

If P^* is the formal adjoint with respect to the given density, we get for $u \in H_1(X^\circ)$ and $v \in C_0^\infty(X)$ the identity

$$(P^*v, u)_0 = (v, Pu)_0 - i\langle v, Bu\rangle - i\langle B_1 v, u\rangle,$$

where $B_1 = B + d(x)$ for a function $d \in C^\infty(X)$ which is independent of x_0 for $|x_0|$ large. By assumption $f_\tau \in H_{0,\sigma,\tau}(X^\circ)$, hence f belongs to $\mathcal{H}_{-1,\sigma+1,\tau}$. If $v \in C_0^\infty(X)$, $v = 0$ on Σ_1, and $B_1 v = 0$ on Σ_2, we assert that

$$|(v, f_\tau)_0 - i\langle v, \ell g_{2\tau}\rangle - i\langle B_1 v, \ell g_{1\tau}\rangle| \leq C|P^* v|_{-1,-\sigma-1},$$

where $\ell g_{i\tau}$ are extensions of $g_{i\tau}$, $i = 1, 2$, in the appropriate function spaces. Indeed, by Lemma 1.5,

$$\begin{aligned} |(v, f_\tau)_0| &\leq |v|_{(1,-\sigma-1)}\, |f_\tau|_{-1,\sigma+1} \\ &\leq C|P^* u|_{-1,-\sigma}. \end{aligned}$$

The other terms are estimated as follows:

$$\begin{aligned} |\langle v, \ell g_{2\tau}\rangle| + |\langle B_1 v, \ell g_{1\tau}\rangle| &\leq C[v]_{-1/2-\sigma} \\ &\leq C\|v\|_{1,-\sigma-1} \leq C|v|_{1,-\sigma-1} + C|P^* v|_{-1,-\sigma-1} \\ &\leq C|v|_{(1,-\sigma-1)} + C|P^* v|_{-1,-\sigma-1} \leq C|P^* v|_{-1,-\sigma} \end{aligned}$$

In this estimate partial hypoellipticity (an analogue of [6, Theorem B.2.9]), Lemma 1.4 and Lemma 1.5 were used.

By the Hahn–Banach theorem there is a linear form L on $C_0^\infty(X)$ such that

$$\begin{aligned} |L(w)| &\leq |w|_{-1,-\sigma}, \qquad w \in C_0^\infty(X), \\ L(P^* v) &= (v, f_\tau)_0 - i\langle v, \ell g_{2\tau}\rangle - i\langle B_1 v, \ell g_{1\tau}\rangle, \end{aligned}$$

for $v \in C_0^\infty(X^\circ)$, $v = 0$ on Σ_1, and $B_1 v = 0$. Thus, there is a function u in the dual of $\mathcal{H}_{-1,-\sigma,\tau}(X^\circ)$ such that $L(w) = (w, u)_0$. By Lemma 1.5 the function u is in $\mathcal{H}_{1,\sigma,\tau}(X^\circ)$. We have $Pu = f_\tau$ in X°, $u = g_{1\tau}$ in Σ_1 and $Bu = g_{2\tau}$ in Σ_2. By partial hypoellipticity (an analogue of [6, Theorem B.2.9]) $u \in H_{1,\sigma,\tau}(X^\circ)$.

The proof that $u(\cdot, \tau) = \exp(-\tau x_0) u(x)$ and u vanishes for $x_0 < 0$ is carried out as in [5, p. 544]. By estimate (1.8) we get that

$$\|(\Delta\tau)^{-1}\left(u(\cdot, \tau + \Delta\tau) - u(\cdot, \tau)\right)\|_{1,0,\sigma-1} \quad \text{and} \quad \|(\Delta\tau)^{-1}\left(e^{ix_0\Delta\tau} - 1\right) u(\cdot, \tau)\|_{1,0,\sigma-1}$$

are bounded independent of $\Delta\tau$. Taking a subsequence which converges weakly in $H_{1,\sigma-1,\tau}(X^\circ)$, we conclude that $du/d\tau$ and $x_0 u$ are both in $H_{1,\sigma-1,\tau}(X^\circ)$ for $\tau \geq \tau_0$. Since $D_0 + i\tau$ and $\partial_\tau + x_0$ commute and the latter operator annihilates the data, we have that $(\partial_\tau + x_0)u$ satisfies the boundary–value problem (1.1)–(1.3) with zero right-hand side. By estimate (1.8) we obtain a distribution-valued ordinary differential equation:

$$\frac{du}{d\tau} + x_0 u = 0,$$

so that $u(x,\tau) = e^{-\tau x_0}u(x)$ for some distribution $u(x)$. Since $\|e^{-\tau x_0}u(x)\|_{1,\sigma-1} \leq C$ for $\tau \geq \tau_0$ where C is independent of τ we conclude that $u(x) = 0$ for $x_0 \leq 0$. Q.E.D.

Tangential regularity for a general class of hyperbolic mixed problems. Tangential regularity also holds for the class of problems investigated in [5]: namely, mixed initial–boundary value problems for second order hyperbolic equations in which the uniform Lopatinski condition is satisfied by both boundary operators B_k on Σ_k, $k = 1,2$, where the defining equations (0.2)′–(0.5)′ are obtained from equations (0.2)–(0.5) by replacing the left-hand sides of (0.3) and (0.4) by $B_1(x,D)u$ and $B_2(x,D)u$, respectively. Regularity in weighted function spaces and asymptotic behavior of the solution near the interface Σ_0 will be given in Section 2.

The next theorem is a refinement of Theorem 1.1 in [5]. Coordinates are chosen as before: Σ is given by $x_n = 0$ and Σ_0 is given by $x_{n-1} = x_n = 0$. In local coordinates let $\lambda_i(x,\xi_0+i\tau,\xi')$, $i = 1,2$, be the roots of $p(x,\xi+i\tau N)$, the principal homogeneous symbol of $P(x,\xi+i\tau N)$, with respect to ξ_n; $\Im\lambda_2 < 0$ for $\tau > 0$. Let $B_k^{(0)}$ be the principal part of the boundary operator B_k. The uniform Lopatinski condition means that $B_k^{(0)}(x,\xi_0+i\tau,\xi',\lambda_2)$ does not vanish for $x \in \overline{\Sigma}_k$, $\tau \geq 0$, and $(\xi_0+i\tau,\xi') \neq 0$, $k = 1,2$. Let $\kappa(x_0,x'')$ be the factorization index of the homogeneous elliptic symbol $B_1B_2^{-1}(x,\xi_0+i\tau,\xi',\lambda_2)$, $x \in \Sigma_0$, with respect to ξ_{n-1}. Let m_k be the degree of B_k, $k = 1,2$. The space $H_{s,r,\sigma,\tau}(X^\circ)$ for real numbers s, p, and σ, is the quotient space of distributions $u(\cdot,\tau)$, $\tau > 0$, in X° with finite norm

$$\|u\|_{s,r,\sigma,t} = \sum_{j\in I_1} \|(\Lambda')^r(\Lambda_j'')^\sigma \kappa_{j*}\phi_j u\|_s^+ + \sum_{j\in I_2} \|\kappa_{j*}\phi_j u\|_{s+r+\sigma}. \tag{1.18}$$

The notation used in the last definition is defined as in (1.5). The spaces $H_{s,\sigma,\tau}(\Sigma_i)$, $i = 1,2$, are defined analogously.

Theorem 1.2 *Assume that P is hyperbolic with respect to the level surfaces of ϕ, ϕ is proper, Σ and Σ_0 are time-like, and assumption (i) holds. Let f, g_1 and g_2 be data which vanish for $\phi < 0$, and which satisfy, for τ large, $f_\tau \in H_{s,r,\sigma,\tau}(X^\circ)$, $h_{1\tau} \in H_{s+r+1-m_1,\sigma,\tau}(\Sigma_1)$ and $h_{2\tau} \in H_{s+r+1-m_2,\sigma,\tau}(\Sigma_2)$, where $\sigma \geq 0$ is an integer and $s \geq 0$. Assume s and r satisfy*

$$\begin{aligned} |s+r+1-m_2-\Re\kappa(x_0,x'')| &< 1/2, \\ s-m_k+1 &> 0, \qquad k = 1,2. \end{aligned}$$

Then there is a unique solution u of the mixed problem (0.2)′–(0.5)′ for which u_τ belongs to $H_{3/2+s,r,\sigma-1/2,\tau}(X^\circ)$ for τ sufficiently large, and such that the trace of u_τ on Σ belongs to $H_{s+r+1,\sigma,\tau}(\Sigma)$.

N.B. By assumption the real part of the factorization index has oscillation less than 1. This restriction on the oscillation can be dropped, but function spaces of piecewise-constant (or variable) order of smoothness must then be used. Cf. [5]Theorem 1.2.

Sketch of proof. We will obtain the a priori estimate

$$\tau\|u\|^2_{s+1,r,\sigma} + [u]^2_{s+r+1,\sigma} \leq C\|f_\tau\|^2_{s,r,\sigma} + C\sum_{k=1}^{2}[h_{k\tau}]^2_{s+r+1-m_k,\sigma} \tag{1.19}$$

for τ sufficiently large, under the assumption that $u(\cdot,\tau)$ belongs to the space $H_{s+3/2,r,\sigma,\tau}(X^\circ)$. The constant C represents a constant independent of τ. The existence of a solution u with u_τ in $H_{s+1,r,\sigma,\tau}$ can be carried out as in [5, Sect. 3]; the hyperbolic operator was regularized into an elliptic operator depending on a parameter $\epsilon > 0$. Arguing as in [5, pp. 542–544] it can be shown that the solution actually belongs to the space $H_{s+3/2,r,\sigma-1/2,\tau}(X^\circ)$.

Near Σ the operator $P(x,D)$ can be expressed in local coordinates by $a(x_{(j)}, D_0^{(j)}, D'_{(j)}) - D_n^2$, although x_0 is not necessarily given by $\phi(x)$.

To obtain tangential regularity near Σ_0 we follow the same argument given in [5, Section 3] but modify the symmetrizers by including additional differentiation in the direction of a forward-directed time-like vector field v tangent to the boundary in a "collar" of Σ, and constant for $|\phi(x)|$ large. A first-order differential operator $h(x,D)$ can be associated with v, whose symbol $h(x,\xi)$ is given locally by $v\cdot\xi$, $\xi \in T^*(X)$. Local coordinates are chosen so that $a^{(0)}_{(j)}(x_{(j)}, \xi_0^{(j)}, \xi'_{(j)}) < 0$ when $h = 0$ and $(\xi_0, \xi') \neq 0$. The symbol $a^{(0)}$ is the principal part of a.

Local estimates are first done in a collar of Σ. We take a sufficiently refined finite covering of

$$\{(x,\xi_0,\xi',\tau) : \ (\xi_0,\xi',\tau) \neq 0, \tau \geq 0, \ x_n < 1\}$$

contained in a collar of Σ, and a partition of unity $\{\phi_j(x,\xi_0,\xi',\tau)\}$ subordinate to it in which the functions ϕ_j are C^∞ and homogeneous of degree 0 in the variables (ξ_0,ξ',τ). In the jth coordinate neighborhood let $\chi_\delta^{(j)}(D_0^{(j)}, D'_{(j)}, \tau)$ represent the pseudodifferential operator with symbol

$$\chi_0\left(\delta^{-1}h(x,\xi_0,\xi')/\Lambda'_j(\xi_0,\xi',\tau)\right),$$

where χ_0 is in $C_0^\infty(-2,2)$ and equals one in $[-1,1]$, $\delta > 0$ is small. For simplicity of notation the dependence of the variables on the local coordinate system is not always indicated.

For the case in which $a^{(0)}(x,\xi_0+i\tau,\xi') \neq 0$ in the jth coordinate neighborhood (this corresponds to containment of the coordinate neighborhood in the elliptic and hyperbolic regions), the symmetrizer in local coordinates is given by (cf. [5](3.6))

$$S_0^{(j)} = \phi_j^* L_j^* \Lambda_j'^p \left(-(D_n - \lambda_2^{(j)}) + \delta(D_n - \lambda_1^{(j)})\right) \Lambda_j'^p L_j \phi_j,$$

where $L_j = (1 - \chi_\delta^{(j)}(D_0^{(j)}, D'_{(j)}, \tau))(\Lambda_0^{(j)})^t$, $t \geq 0$ an integer. The symbol of the differential operator Λ_0 is given by $h(x,\xi) + i\tau$, whilst the symbol of the operator $\lambda_i^{(j)}$ is $(-1)^{i+1}\sqrt{a_j^{(0)}}$, $i = 1, 2$, where, as before, $\Im \lambda_2^{(j)} < 0$ for $\tau > 0$. The constant $\delta \geq 0$ is small.

If the support of ϕ_j contains zeros of $a^{(0)}$ (the glancing region case) we use for the symmetrizer (cf. [5](3.25))

$$S_0^{(j)} = \phi_j^* L_j^* (\Lambda_j')^p (-\delta s_1^{(j)} - i\tau\delta^{-2} s_2^{(j)} - 2D_n)(\Lambda_j')^p L_j \phi_j,$$

where $s_1^{(j)} = \partial a_j^{(0)}/\partial \xi_0^{(j)}$, and $s_2^{(j)} = \partial^2 a_j^{(0)}/\partial \xi_0^2$.

By using these symmetrizers and applying the same arguments as in [5, Section 3] we get for all integral $t \geq 0$ (cf. [5](3.41))

$$C_{\delta_1}\tau \|(1 - \chi_\delta^{(j)})(\Lambda_0^{(j)})^t \phi_j u\|_{s+1,r,0}^2 + [(1 - \chi_\delta^{(j)})(\Lambda_0)^t (D_n - \lambda_0)\phi_j u]_{s+r,0}^2$$

$$\leq C\delta_1 [(\Lambda_0^{(j)})^t \psi_j u]_{s+r+1,0}^2 + C\|(\Lambda_0^{(j)})^t \psi_j u\|_{s+1,r,0}^2 + C\|(\Lambda_0^{(j)})^t \psi_j f_\tau\|_{s,r,0}^2, \quad (1.20)$$

$\psi_j(x)$ is a C^∞ function supported in a coordinate neighborhood and satisfying $\psi_j(x)\phi_j = \phi_j$. The constant δ_1 is arbitrarily small. The symbol of λ_0 is $\sum \lambda_2 \phi_j|_{x_n=0}$, the sum is taken only over those j for which the support of ϕ_j does not intersect the glancing region. (The push–forward κ_{j*} of the local chart is not written.)

For x near Σ_0 put

$$E(x,\xi,\tau) = (1 - \chi_{3\delta})P(x, \xi + i(\tau + \Lambda')N) + \chi_{3\delta} P(x, \xi + i\tau N).$$

(The dependence on the local coordinates is not indicated.) Then $E(x, D, \tau)$ is an elliptic operator whose symbol agrees with the symbol of $P(x, D + i\tau N)$ on the support of χ_δ.

Since E satisfies the transmission property a boundary–value problem for the operator E makes sense. Applying $\chi_\delta(\Lambda_0^{(j)})^t \phi_j$ to the equation $Pu = f_\tau$ we get the elliptic equation

$$E v_+^{(j)} = g_\tau^{(j)}, \qquad \text{in } x_n > 0$$

where $v_+^{(j)} = \chi_\delta(\Lambda_0^{(j)})^t \phi_j u$ if $x_n > 0$ and vanishes if $x_n < 0$;

$$g_\tau^{(j)} = \chi_\delta(\Lambda_0^{(j)})^t \phi_j f_\tau + [P, \chi_\delta(\Lambda_0^{(j)})^t]\phi_j u + \chi_\delta(\Lambda_0^{(j)})^t R\psi_j u,$$

R has order ≤ 1, and $\psi_j(x)\phi_j = \phi_j$. The theory of elliptic boundary value problems for smooth pseudodifferential operators (see [4, Sect. 23]) gives the following a priori estimate for the Dirichlet problem:

$$\|v_+\|^2_{s+1,r,0} \leq C\|g_\tau\|^2_{s-1,r,0} + C[v_+]^2_{s+r+1/2,0} + C\|v_+\|^2_{s,r,0}.$$

Furthermore

$$\|[P, \chi_\delta(\Lambda_0^{(j)})^t]\phi_j u\|_{s-1,r,0} \leq C\tau^{-1}\|\phi_j u\|_{s+1,r,0},$$

and $[(D_n - \lambda_0)v_+]^2_{s+r,0}$ is bounded by the right-hand side of (1.20).

These estimates imply the analogue of (1.20) in which $1 - \chi_\delta^{(j)}$ is replaced by $\chi_\delta^{(j)}$. Summing (1.20) and its analogue gives an estimate which is identical to the estimate (1.20) except that $1 - \chi_\delta^{(j)}$ has been deleted. Finally, the analogous interior estimate is obtained; then by summing these estimates over all j we get

$$C_{\delta_1}\tau\|(\Lambda_0)^t u\|^2_{s+1,r,0} + [(D_n - \lambda_0)(\Lambda_0)^t u]^2_{s+r,0} \leq C\delta_1[(\Lambda_0)^t u]^2_{s+r+1,0} + C\|(\Lambda_0)^t f_\tau\|^2_{s,r,0}.$$

A proof very similar to the one of Lemma 1.2 shows that the above estimate implies the following estimate involving tangential derivatives:

$$C_{\delta_1}\tau\|u\|^2_{s+1,r,\sigma} + [(D_n - \lambda_0)u]^2_{s+r,\sigma} \leq C\delta_1[u]^2_{s+r+1,\sigma} + C\|f_\tau\|^2_{s,r,\sigma}. \quad (1.21)$$

In the proof the elliptic differential operator with symbol

$$P(x, \xi + i(\tau + h(x, \xi))N) \quad (1.22)$$

is used instead of the elliptic operator (1.14).

By regularity of elliptic pseudodifferential equations in a domain (see [4], [2]) applied to the solution of the equation [5](3.52) we get (cf. [5](3.54))

$$[u]^2_{s+r+1,\sigma} \leq C\sum_{k=1}^{2}[h_{k\tau}]^2_{s+r+1-m_k,\sigma} + C[(D_n - \lambda_0)u]^2_{s+r,\sigma} + C[u]^2_{s+r,\sigma}. \quad (1.23)$$

By (1.21) and (1.23) we get the a priori estimate (1.19).

Remark 1.3. The reader may ask why the existence proof for the Dirichlet–Neumann–Cauchy problem used a rather complicated duality argument, rather than the simpler method for proving existence used, for instance, in the case that the uniform Lopatinski condition is satisfied. The latter approach involves perturbing the hyperbolic operator P (which depends on the parameter τ) into an operator P_ϵ which is parameter–dependent elliptic for $\epsilon > 0$ small. By constructing a parametrix similarly to [4]Theorem 22.1 but using Sobolev spaces dependent on the paramter τ one obtains a remainder with small norm (less than 1) *uniformly* for τ large and ϵ small, which implies the invertibility of the operator corresponding to the boundary value problem. The uniform bound on the norm of the remainder is not satisfied in the absence of the uniform Lopatinski condition.

2 Regularity and Asymptotics

1. Conormal regularity. In this section regularity in weighted function spaces for the solution to the Dirichlet–Neumann–Cauchy problem is given, and the behavior of the solution near the interface Σ_0 is described by an asymptotic expansion. We first define the weighted function spaces $W_{s,N,\sigma}(X^\circ)$.

The space $W_{s,N,\sigma}(X^\circ)$, $\sigma \geq 0$, consists of all functions u in X° with finite norm

$$|[u]|_{s,N,\sigma} = \sum_{j\in I_1} \sum_{k+l=0}^{N} \|\Lambda_j''^{\sigma} x_{n-1}^k x_n^l \phi_j u\|_{s+k+l}^{+} + \sum_{j\in I_2} \|\phi_j u\|_{s+\sigma+N}$$

with notation as in (1.5). The space $W_{s,r,N,\sigma}(X^\circ)$ is defined analogously, where r refers to the number of derivatives in Λ' near Σ_0 and the additional number of derivatives away from Σ_0 (cf. (1.18)). Here Σ_0 is given locally by $x_{n-1} = x_n = 0$, and Σ is given by $x_n = 0$. Note that for functions in these spaces multiplying the function by $x_{n-1}^j x_n^k$, $j + k \leq N$, increases the smoothness *in all directions* by order $j + k$. The spaces $H_{s,N}(\Sigma_1)$ and $H_{s,N}(\Sigma_2)$ are defined similarly with weights which are powers of x_{n-1}.

Regularity results in the spaces $W_{s,N,\sigma}$, which combines regularity both in the tangential direction and with weights in the normal direction, will be referred to as conormal regularity.

Theorem 2.1 *If, in addition to the hypothesis of Theorem 1.1, $f_\tau \in W_{0,N,\sigma}(X^\circ)$, $g_{1\tau} \in H_{3/2+\sigma,N}(\Sigma_1)$, $g_{2\tau} \in H_{1/2,N,\sigma}(\Sigma_2)$, $\sigma \geq N$, then the unique solution u_τ of (1.1)–(1.3) with $u_\tau \in H_{1,\sigma,\tau}(X^\circ)$ belongs to the weighted function space $W_{1-\epsilon,N,\sigma-N}(X^\circ)$, $\epsilon > 0$ arbitrarily small.*

Proof. We apply the conormal regularity result [2]Theorem 2.5 for the general mixed elliptic boundary value problem investigated in [4, Sect 24] to the mixed elliptic boundary value problem given by $Eu = h$ in X°, $u = g_{1\tau}$ in Σ_1 and $Bu = g_{2\tau}$ in Σ_2 where the elliptic operator E is defined by (1.14). By this result if $u \in W_{1,0,\sigma}$ is a solution of this mixed elliptic problem with $h \in W_{-1-\epsilon,N,\sigma}$, $g_{1\tau} \in H_{1/2+\sigma,N}$ and $g_{2\tau} \in H_{-1/2+\sigma,N}$, then $u \in W_{1-\epsilon,N,\sigma}$, for $\epsilon > 0$ arbitrarily small.

The proof is inductive. Assume u_τ is in $W_{1-\epsilon,k,\sigma-k}$, $0 \leq k < N$. Then $\Lambda_0^2 u_\tau$ is in $W_{-1-\epsilon,k+1,\sigma-k-1}$. Applying the elliptic regularity result to the aforementioned mixed boundary value problem we get that u_τ belongs to $W_{1-\epsilon,k+1,\sigma-k-1}$. Q.E.D.

Remark 1.3. For the sake of brevity these remarks are informal. The analogue of Theorem 2.1 holds for the class of hyperbolic boundary-value problems considered in Theorem 1.2. The following modifications need to be made in the proof of Lemma 2.1. Instead of the elliptic operator E defined by (1.14) the operator with symbol

$$P(x, \xi + i(\tau + \delta h(x,\xi))N)$$

is used, where the positive number δ is chosen sufficiently small so that the Lopatinski condition is satisfied (cf. (1.22)). In general, an elliptic regularity result in spaces with piecewise order of smoothness is needed. Such a result involves spaces of the type $H_{(s_i,r_i),N}$ with norm

$$\sum_i \|\phi_i u\|_{s_i,r_i,N},$$

where $\{\phi_i\}$ is a partition of unity, the first index refers to differentiation in all variables, the second index refers to differentiation only along the boundary Σ, and the last index represents the weight. If the indices vary only slightly in overlapping coordinate neighborhoods, then the parametrix construction as in [4, Section 25] leads to a compact remainder which has a gain of $1-\epsilon$, $\epsilon > 0$ small.

2. Asymptotics of the solution near Σ_0. We give a general result on asymptotics which includes: (a) the Dirichlet–Neumann–Cauchy problem; (b) mixed initial boundary value problems for second–order hyperbolic operators satisfying the uniform Lopatinski condition; (c) mixed elliptic boundary value problems for second–order strongly elliptic operators on a compact C^∞ manifold with boundary. The method we discuss here is applicable so long as conormal regularity has been established, the conormal bundle of Σ_0 lies in the elliptic region of the differential operator, the boundary operators b_1 and b_2 (see (2.4)) are elliptic, and, with respect to the first index s, there is no loss of differentiation compared to the elliptic case, that is, (2.8) holds. Conormal regularity means regularity of the solution u in the space $W_{s,N,\sigma}(X^\circ)$.

All these three types of problems have the form

$$P(x,D)u = f, \quad \text{in } X^\circ; \tag{2.1}$$

$$B_k(x,D)u = g_k, \quad \text{on } \Sigma_k, \quad k = 1,2; \tag{2.2}$$

where P is a second-order differential operator with C^∞ coefficients, B_k are differential operators of order m_k, $k = 1, 2$. As before, X is an $(n+1)$-dimensional manifold with boundary Σ, X° is the interior of X, and Σ is divided into two parts, Σ_1 and Σ_2, by a smooth submanifold Σ_0 of Σ of codimension 1. The non-compactness of X in cases (a) and (b) causes no problems since by assumption (i) a finite partition of unity can be taken.

We take local coordinates near Σ_0 in which this submanifold is given by $x_{n-1} = x_n = 0$, and X is given by $x_n \geq 0$. Let P_0 be the principal symbol of P. Put

$$e(\tilde{x}, \hat{\xi}) = P_0(x, \xi)\Big|_{\hat{x}=0, \tilde{\xi}=0}$$

where $\tilde{x} = (x_0, x'')$, $\hat{x} = (x_{n-1}, x_n)$, and $x'' = (x_1, \ldots, x_{n-2})$. We are assuming that the conormal bundle of Σ_0 is contained in the elliptic region of P, that is,

$$e(\tilde{x}, \hat{\xi}) \neq 0, \qquad \text{for } \tilde{x} \in \Sigma_0, \ \hat{\xi} \neq 0. \tag{2.3}$$

The roots $\lambda_k(\tilde{x}, \xi_{n-1})$, $k = 1, 2$, of $e(\tilde{x}, \hat{\xi}) = 0$ with respect to ξ_n are not real for $\xi_{n-1} \neq 0$, and we can take $\Im\lambda_1 > 0$ and $\Im\lambda_2 < 0$ for $\xi_{n-1} \neq 0$ (this holds for problem (c) since the operator is strongly elliptic). We normalize e by putting $e(\tilde{x}, 0, 1) = 1$.

Let $B_k^{(0)}$ be the principal part of B_k, $k = 1, 2$. The boundary operators b_k given by

$$b_k(\tilde{x}, \xi_{n-1}) = C_k(\tilde{x}, \xi_{n-1}, \lambda_2(\tilde{x}, \xi_{n-1})); \qquad C_k(\tilde{x}, \hat{\xi}) = B_k^{(0)}(x, \xi)\Big|_{\hat{x}=0, \tilde{\xi}=0} \tag{2.4}$$

are assumed to be elliptic, that is,

$$b_k(\tilde{x}, \pm 1) \neq 0, \quad \forall \tilde{x} \in \Sigma_0, \ \ k = 1, 2. \tag{2.5}$$

In problems (a) and (b) and in problem (c) for $n \geq 3$ the symbol $b_1 b_2^{-1}$ has a factorization (see [4]Section 6). This factorization can be written

$$(b_1 b_2^{-1})(\tilde{x}, \xi_{n-1}) = a(\tilde{x})(\xi_{n-1} + i0)^{\kappa(\tilde{x})}(\xi_{n-1} - i0)^{m_1 - m_2 - \kappa(\tilde{x})}. \tag{2.6}$$

where the factorization index $\kappa(\tilde{x})$ is a C^∞ function on the submanifold Σ_0.

Let $\mu_1(\tilde{x}) = \lambda_2(\tilde{x}, 1)$ and $\mu_2(\tilde{x}) = -\lambda_2(\tilde{x}, -1)$. Put

$$z_k = x_{n-1} + \mu_k(\tilde{x}) x_n, \qquad k = 1, 2.$$

Let $\kappa_0(\tilde{x}) = \kappa(\tilde{x}) + m_2$. Put

$$H(z, s) = e^{-i\pi s/2}\Gamma(-s) z^s + \sum_0^\infty \frac{i^k}{k!} \frac{z^k}{s - k}, \tag{2.7}$$

where Γ is the Gamma function. The function H can be extended to an entire function of s with values in the space of distributions in the domain $\Im z > 0$. The space Q_M consists of functions near Σ_0 which have the form (in local coordinates)

$$\sum_{p=0}^{M}\sum_{r=0}^{p}\sum_{m=0}^{2p-r}\Big\{c^{(1)}_{prm}(\tilde{x})z_2^m\frac{d^r}{ds^r}H(-z_1,\kappa_0(\tilde{x})-m+p)$$
$$+\,c^{(2)}_{prm}(\tilde{x})z_1^m\frac{d^r}{ds^r}H(z_2,\kappa_0(\tilde{x})-m+p)\Big\}$$

where $c^{(k)}_{prm}\in C^\infty(\Sigma_0)$. We note that if $s\neq 0,1,2,\dots$ then $d^rH(z,s)/ds^r$ has the expansion $\sum_{k=0}^{r}c_kz^s\log^k z_1$ where c_k depends analytically on s. When $s=0,1,2,\dots$ there is a similar expansion but the upper limit of the sum is then $r+1$.

We now give the theorem on asymptotics. For simplicity we assume that the oscillation of $\Re\kappa$ is less than 1, but this assumption can be dropped by making use of Sobolev spaces of piecewise constant order of smoothness (see [4]Section 25).

Theorem 2.2 *Suppose that (2.3) and (2.5) hold, that is, the conormal bundle of* Σ_0 *is contained in the elliptic region of* P*, and that the boundary operators* $b_k(\tilde{x},D_{n-1})$ *are elliptic. For data* $f\in H_\infty(X^\circ)$, $g_k\in H_\infty(\overline{\Sigma}_k)$, $k=1,2$, *suppose* u *is a solution of problem (2.1)–(2.2) in the function space* $W_{s,\infty,\infty}$, *where* s *satisfies*

$$|s-1/2-\Re\kappa_0(\tilde{x})|<1/2,\quad \forall\tilde{x}\in\Sigma_0. \qquad (2.8)$$

Then the solution has the asymptotic expansion $u(x)=s_M(x)+u_M(x)$, *for all* $M\geq 0$, *in a neighborhood of* Σ_0, *where* $s_M\in Q_M$, *and*

$$D^\alpha_{\tilde{x}}u_M=O(|\hat{x}|^{\Re\kappa_0(\tilde{x})+M+1-\epsilon}),\qquad \forall\epsilon>0,\ |\alpha|\geq 0. \qquad (2.9)$$

This theorem applies to the three problems (a), (b) and (c). The conormal regularity result for (a) was given in Theorem 2.1, and for problem (b) it was discussed in Remark 1.3. Conormal regularity for mixed elliptic boundary–value problems was given in [2]Theorem 2.5.

We first give a lemma.

Lemma 2.1 *Let* $\phi(t)\in C_0^\infty(\boldsymbol{R}^1)$ *equal 1 for* $t<1$. *Then the distribution in the domain* $\Im z>0$ *given by*

$$\int_0^\infty e^{izt}(1-\phi(t))t^a\,dt$$

differs from $H(z,-a-1)$ *by a function which is entire in both* a *and* z.

Proof. The distribution $\chi_+^a = x_+^a/\Gamma(a+1)$ is an entire function of a [6, Sect. 3.2]. Its Fourier–Laplace transform, defined in $\Im z > 0$, is computed in [6, Sect. 7.1]:

$$\int_0^\infty e^{izt}\chi_+^a(t)\,dt = e^{i\pi(a+1)/2}z^{-a-1}.$$

Hence

$$\int_0^\infty e^{izt}(1-\phi(t))t^a\,dt = \Gamma(a+1)\left(e^{i\pi(a+1)/2}z^{-a-1} + f(a,z)\right),$$

where $f(a,z)$ is entire in both a and z. Since the left–hand side is an entire distribution–valued function of s, the singularities on the right–hand side must cancel. The gamma function is a meromorphic function having only simple poles which are located at the integers $k \le 0$. The residue there is $(-1)^k/k!$.

Proof of Theorem 2.2. Let χ be a cut–off function which equals one in a small neighborhood of Σ_0, and let $v = \chi u$. By (2.1)–(2.2) we get (mod C^∞)

$$\begin{aligned} e(\tilde{x},\hat{D})v &= -(P-e)v, \quad \text{in } \Sigma_0 \times \mathbf{R}^2; && (2.10)\\ C_k(\tilde{x},\hat{D})v &= -(B_k - C_k)v, \quad \text{on } \Sigma_0 \times \mathbf{R}^1_\pm,\ k=1,2; && (2.11)\end{aligned}$$

The asymptotics for this mixed boundary–value problem were obtained in [4]Section 13 in the case in which the right–hand side is C^∞. Letting $\mathcal{W}_M$ represent the space $W_{s+M+1,\infty,\infty}$, we shall prove inductively that $v = s_M + r_M$ with $s_M \in \mathcal{Q}_M$ and $r_M \in \mathcal{W}_M$, for all $M \ge -1$. By the Sobolev imbedding theorem u_M will then satisfy (2.9), and the theorem will be proved.

Assume $v \in \mathcal{Q}_M + \mathcal{W}_M$. First we reduce to the case in which the right–hand side of (2.10) vanishes. By conormal regularity $(P-e)r_M$ is in $\mathcal{W}_{M-1}$. The distribution E which is the inverse Fourier transform with respect to $\hat{\xi}$ of $e(\tilde{x},\hat{\xi})^{-1}$ is a fundamental solution of e [6, Sect. 7.1]; convolution by E is an operator of order -2 on the weighted function space $C^\infty(\Sigma_0, W_{s,N}(\mathbf{R}^2))$ (see [4, Lemma 24.2]). Thus there is a solution of $ev = -(P-e)r_M$ in $\mathcal{W}_{M+1}$.

Next, we show that there is a solution of $ev = -(P-e)s_M$ in $\mathcal{Q}_{M+1} + \mathcal{W}_{M+1}$. Note that

$$e(\tilde{x},\hat{D}) = (\mu_1-\mu_2)^2\frac{\partial^2}{\partial z_1\partial z_2},$$

where the partial derivatives on the right–side are given by

$$\partial/\partial z_1 = (\mu_1-\mu_2)^{-1}(\partial_n - \mu_2\partial_{n-1}), \qquad \partial/\partial z_2 = (\mu_2-\mu_1)^{-1}(\partial_n - \mu_1\partial_{n-1}).$$

It follows from this decompostion of e and Lemma 2.1 that the equation

$$e(\tilde{x},\hat{D})v = z_2^m\frac{d^r}{ds^r}H(-z_1, r(\tilde{x})),$$

for r in C^∞ and integral $m \geq 0$, has a solution which differs from

$$i(m+1)^{-1}(\mu_1-\mu_2)^{-2}z_2^{m+1}\frac{d^r}{ds^r}H(-z_1,r(\tilde{x})+1)$$

by a smooth function in $x \in \mathbf{R}^{n+1}$. The analogous statement holds if the roles of z_1 and z_2 are reversed. A tangential derivative of the right-hand side of the last equation is a sum of terms of the same form. Among these terms the power of z_2 increases by at most 1, and for the term in which this increase occurs the power of the logarithm (that is, the number of derivatives in s) will remain unchanged. This indicates the effect of applying $P-e$ to s_M, and substantiates the above claim.

Thus we can consider the problem (2.10)–(2.11) in which the right-hand side of (2.10) vanishes. The solution of (2.10) has the form

$$v(x) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp[-ix_n\lambda_2(\tilde{x},\xi_{n-1})-ix_{n-1}\xi_{n-1}]\widetilde{v_0}(\tilde{x},\xi_{n-1})d\xi_{n-1}, \quad (2.12)$$

where v_0 is the restriction of v to Σ, and $\widetilde{v_0}$ is the Fourier transform of v_0 with respect to x_{n-1}. Substituting this into (2.11) we get (mod C^∞)

$$b_k(\tilde{x},D_{n-1})v_0 = -(B_k-C_k)v, \quad \text{on } (-1)^k x_{n-1}<0, \ \ x_n=0, \quad k=1,2,$$

where b_k and C_k were defined in (2.4). Let

$$v_k = -b_k(\tilde{x},D_{n-1})v_0-(B_k-C_k)v, \quad \text{on } x_n=0, \quad k=1,2. \qquad (2.13)$$

Then v_1 (v_2) is supported in $x_{n-1}\leq 0$ ($x_{n-1}\geq 0$). We get the pseudodifferential equation in the domain $x_{n-1}>0$

$$(b_1b_2^{-1})(\tilde{x},D_{n-1})v_2(x_{n-1}) = (B_1-C_1)v-b_1b_2^{-1}(B_2-C_2)v \qquad (2.14)$$

modulo a function in H_∞. Here b_2^{-1} is redefined near 0 so as to remove the singularity. We will tacitly use this convention for other symbols as well.

The asymptotics for (2.14) have been worked out in [4, Sect. 26] by the factorization method. Since $v = s_M + r_M$ we see that the right-hand side of (2.14) is a sum of a function of the form

$$F^{-1}_{\xi_{n-1}}\left\{\sum_{p=0}^{M}\sum_{r=0}^{p}c_{pr}(\tilde{x},\operatorname{sgn}\xi_{n-1})|\xi_{n-1}|^{-\kappa_0(\tilde{x})-p+m_1-2}\log^r|\xi_{n-1}|\right\},$$

and a function in $H_{s+M-m_1+3/2,\infty,\infty}$, where c_{pr} are C^∞ functions. Note that this last function space is equivalent to $C^\infty(\Sigma_0, H_{s+M-m_1+3/2,\infty})$. Here $F^{-1}_{\xi_{n-1}}$ represents the inverse Fourier transform.

Applying the factorization method we get (see (2.6)) that the solution v_2 of (2.14) is given by the sum of a function of the form

$$F^{-1}_{\xi_{n-1}}\left\{\sum_{p=0}^{M}\sum_{r=0}^{p} c_{1pr}(\widetilde{x})(\xi_{n-1}+i0)^{-\kappa(\widetilde{x})-p-2}\log^{r+1}(\xi_{n-1}+i0)\right\},$$

and a function in $C^\infty(\Sigma_0, H^+_{s+M-m_2+3/2,\infty})$. The space $H^+_{s,N}$ consists of all functions in $H_{s,N}(\mathbf{R}^1)$ vanishing in $\mathbf{R}^1_-$.

By (2.13) $\widetilde{v}_0$ is given as a sum of a function of the form

$$\sum_{p=0}^{M}\sum_{r=0}^{p} d_{pr}(\widetilde{x}, \operatorname{sgn}\xi_{n-1})|\xi_{n-1}|^{-\kappa_0(\widetilde{x})-p-2}\log^{r+1}|\xi_{n-1}|, \tag{2.15}$$

where $d_{pr}(\widetilde{x}, \pm 1)$ is a C^∞ function, and a function in $C^\infty(\Sigma_0, \widetilde{H}_{s+M+3/2,\infty})$. By (2.12) we have

$$v(x) = (2\pi)^{-1}\left(\int_0^\infty e^{-iz_1\xi_{n-1}}\widetilde{v_0}d\xi_{n-1} + \int_{-\infty}^0 e^{-iz_2\xi_{n-1}}\widetilde{v_0}d\xi_{n-1}\right), \tag{2.16}$$

Substituting (2.15) into (2.16), we get by Lemma 2.1 that $v \in \mathcal{Q}_{M+1} + \mathcal{W}_{M+1}$.

In the case of the Dirichlet–Neumann–Cauchy problem the asymptotics for the solution are much simpler.

Theorem 2.3 *Suppose the conditions of Theorem 1.1 hold, and that the data is smooth, that is, $f_\tau \in H_\infty(X^\circ)$, and $g_{k\tau} \in H_\infty(\Sigma_k)$, $k = 1, 2$. If u_τ is a solution of (1.1)–(1.3) in $H_{1,0,\tau}(X^\circ)$, then there are local coordinates near Σ_0 in which*

$$u_\tau(x) = c_1(x)(x_{n-1}+ix_n)^{1/2} + c_2(x)(x_{n-1}-ix_n)^{1/2} + r(x), \tag{2.17}$$

where c_1, c_2 and r are C^∞ functions.

Proof. Some modifications to the proof of Theorem 2.2 involving the theory of smooth pseudodifferential operators [4, Sect. 10] (smooth means satisfying the transmission property) suffice. We take B_1 and B_2 in (2.2) to be the Neumann and Dirichlet operators, respectively. First, since $P_0(\widetilde{x}, 0, 0, \hat{\xi})$ is a negative definite quadratic form with respect to $\hat{\xi}$ there are local coordinates near Σ_0 in which $e(\widetilde{x}, \hat{\xi})$ is given by $-(\xi^2_{n-1} + \xi^2_n)$. Therefore $z_1 = x_{n-1} - ix_n$, $z_2 = x_{n-1} + ix_n$, $b_1 = |\xi_{n-1}|$, $b_2 = 1$, and $\kappa_0 = 1/2$.

The symbol of $b_1 b_2^{-1} = |\xi_{n-1}|$ equals the product of the "plus" symbol $(\xi_{n-1} + i0)^{1/2}$ (i.e., it extends analytically to the upper half-plane) and the symbol $(\xi_{n-1} - i0)^{-1/2}$ which satisfies the transmission property. We can rewrite (2.14) in the form

$$(D_{n-1} - i0)^{1/2} w = g, \qquad \text{on } \mathbf{R}^n_+, \tag{2.18}$$

where $w = (D_{n-1} + i0)^{1/2} v_2$ and g represents the right-hand side of (2.14).

Let us show inductively that $v = \chi u_\tau$ is given by the sum $s_M + r_M$, where s_M has the form of the right-hand side of (2.17), and r_M is in $\mathcal{W}_M = W_{M+2,\infty,\infty}$. It is clear that the theorem follows from this decomposition of v. Assuming that v can be so expressed for a given value of M, then g is a sum of the function $d_1(\tilde{x}, x_{n-1}) x^{1/2}_{n-1,+}$, $d_1 \in C^\infty$, and a function in $C^\infty(\Sigma_0, H_{M+3/2,\infty}(\mathbf{R}^1))$. By (2.18) w is the restriction of $(D_{n-1} - i0)^{-1/2} g$ to $x_{n-1} > 0$, hence w is in $C^\infty(\Sigma_0, H_{M+2,\infty}(\mathbf{R}^1_+))$ where $H_{s,N}(\mathbf{R}^1_+)$ consists of all functions in $\mathbf{R}^1_+$ which have extensions to $H_{s,N}(\mathbf{R}^1)$. This implies that

$$w = \sum_{k=0}^{M+1} a_k(\tilde{x}) x^k_{n-1,+}, \qquad \text{mod } C^\infty(\Sigma_0, H^+_{M+2,\infty}),$$

where $a_k \in C^\infty(\Sigma_0)$. Therefore,

$$v_2 = d_2(\tilde{x}, x_{n-1}) x^{1/2}_{n-1,+}, \qquad \text{mod } C^\infty(\Sigma_0, H^+_{M+5/2,\infty}),$$

where $d_2 \in C^\infty$. Substituting v_0 (which equals v_2) into (2.16) we obtain the desired result.

This result in which the asymptotics do not contain logarithms holds more generally for all second-order mixed boundary value problems for which the conditions of Theorem 2.2 are satisfied, the symbol $b_1 b_2^{-1}$ is the product of a "plus" symbol and a smooth symbol, and the factorization index κ is constant. Indeed, in this case the solution of the analogue of (2.18) involves no logarithms since the factors in the factorization of a smooth symbol are themselves smooth. If, in addition, the coefficients of $e(\tilde{x}, \hat{\xi})$ are real, then the asymptotic expansion is given in (2.17) with the exponent 1/2 replaced by κ_0.

References

[1] J. Bennish, Mixed Initial–Boundary Value Problems for Hyperbolic Equations with Constant Coefficients, *Journal of Mathematical Analysis and Applications* **153**, No. 2 (1990), 506–532.

[2] J. Bennish, Asymptotics for Elliptic Boundary Value Problems for Systems of Pseudo–differential Equations, to appear in *Journal of Mathematical Analysis and Applications.*

[3] G. Duvaut and J. L. Lions, "Inequalities in Mechanics and Physics," Springer–Verlag, 1976.

[4] G. I. Eskin, "Boundary Value Problems for Elliptic Pseudodifferential Equations," Translations of Math. Monographs v. 52, Amer. Math. Soc. 1981.

[5] G. I. Eskin, Mixed initial–boundary value problems for second order hyperbolic equations, *Comm. Partial Differential Equations* **12**, No. 5 (1987), 503–587.

[6] L. Hörmander, "The Analysis of Linear Partial Differential Equations," Springer Verlag, Berlin/Heidelberg/New York/Tokyo, Vol. 1 and 2 (1983), Vol. 3 and 4 (1985).

[7] J.L. Lions and E. Magenes, "Non–Homogeneous Boundary Value Problems and Applications I, II," Springer Verlag, New York Heidelberg Berlin, 1972.

[8] M.I. Vishik and G.I. Eskin, Convolution equations in bounded domains in spaces with weighted norms. Mat. Sb. **69**, 1 (1966), 65–110.