Monodromy quasisemisimple D-modules over the arrangements of hyperplanes

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MPI / 93-30

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March 31, 1993

Abstract

Let $\mathcal{A}$ be a collection of hyperplanes in complex affine space and $\mathcal{D}_X$ be a sheaf of differential operators over corresponding stratified space X. We introduce categories of quasisemisimple $\mathcal{D}_X$-modules which are characterized by natural conditions on eigenvalues of monodromie operators for nearby and vanishing cycles functors Ψ and Φ. The main result of this paper is the description of this categories in terms of quivers with quadratic relations. We describe explicitely both functors establishing equivalence of categories. As a consequence we obtain a description of all quasisemisimple $\mathcal{D}_X$-modules in terms of generators and relations. Application of this results to direct images of local systems over the complement to the arrangement of hyperplanes produces a natural complex which coincides with Orlik–Solomon complex in the case of trivial monodromies.

1 Arrangements of hyperplanes and quasisemisimple D-modules

Let us consider complex affine space $X = \mathbf{C}^N$ and a set of complex hyperplanes $X_i = \{f_i = 0\}$ in $\mathbf{C}^N$. Following the tradition of [VS1] we call this set an arrangement $\mathcal{A}$ of hyperplanes. One may attach to this arrangement a natural stratification of X.

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The closed stratum $\overline{X}_\alpha \subset \mathbf{C}^N$ is an intersection of some hyperplanes

$$\overline{X}_\alpha = \bigcap_{i \in I_\alpha} \{f_i = 0\}$$

and its interiour $X_\alpha \subset \overline{X}_\alpha$ consists of points $x \in \overline{X}_\alpha$ that do not belong to other hyperplanes: $x \notin X_j = \{f_j = 0\}$, if $X_j \not\supset \overline{X}_\alpha$. Let us denote this statified space as $X_{\mathcal{A}}$ or $\mathbf{C}^N_{\mathcal{A}}$.

Let $\mathcal{D}_X$ be a sheaf of differential operators over X. Consider holonomic $\mathcal{D}_X$-modules with regular singularities flat along this stratification. These $\mathcal{D}_X$-modules form an abelian category $\mathcal{C}_{\mathcal{A}}$ which is equivalent to the category of constructible perverse sheaves (with respect to a middle perversity) over the stratified space $X_{\mathcal{A}}$ [BBD].

In this paper we study full subcategory $\mathcal{C}^{qs}_{\mathcal{A}}$ of $\mathcal{C}_{\mathcal{A}}$ which we call category of monodromy quasisemisimple $\mathcal{D}$-modules. The definition of $\mathcal{C}^{qs}_{\mathcal{A}}$ looks as follows.

Let $X_\alpha \subset \mathbf{C}^N$ be an arbitrary stratum of $X_{\mathcal{A}}$, $U_\alpha \supset X_\alpha$ be a small neibourhood of X_α in $\mathbf{C}^N$and $\{f_\alpha = 0\}$ be a generic hyperplane going through X_α: $X_\alpha \subset \{f_\alpha = 0\}$ (it means in particular that $\{f_\alpha = 0\}$ does not belong to an arrangement A, if $\mathrm{codim} X_\alpha > 1$). To any $\mathcal{D}_X$-module $M \in \mathcal{C}_{\mathcal{A}}$ we may apply nearby and vanishing cycles functors $\mathbf{\Psi}_{f_\alpha}$ and $\mathbf{\Phi}_{f_\alpha}$ [BBD]. It is clear that $\mathcal{D}_X$-module $\Phi_{f_\alpha}(M)$ has a support on $\overline{X}_\alpha$ only and, as a consequence, the restriction of $\mathcal{D}_{\overline{X}_\alpha}$-module $\Phi_{f_\alpha}(M)$ to an open part $X_\alpha \subset \overline{X}_\alpha$ is equivalent to some local system.

Definition 1.1 *$\mathcal{D}_X$-module M is called (monodromy) quasisemisimple along X_α if it can be decomposed into direct sum of $\mathcal{D}_X$- modules M_i: $M = \bigoplus_i M_i$ with M_i satisfying the following conditions for any stratum X_α of ${\mathbf{C}^N}_{\mathcal{A}}$:*

(i) The action of canonical monodromy operator T on $\mathbf{\Psi}_{f_\alpha}(M_i)$ restricted to U_α is singleeigenvalued:

$$(T - e^{2\pi i\lambda})^n \mathbf{\Psi}_{f_\alpha}(M_i) \mid_{U_\alpha} = 0 \tag{1.1}$$

for some $\lambda \in \mathbf{C}$, $0 \leq \mathrm{Re}\,\lambda < 1$ and and n sufficiently large;

(ii) The local system $\mathbf{\Phi}_{f_\alpha}(M_i) \mid_{X_\alpha}$ can be described as a flat connection θ_i with constant eigenvalued matrix coefficients:

$$\omega(\theta_i) = \sum A^i_\beta \frac{df_\beta}{f_\beta} \tag{1.2}$$

where f_β are some linear functions on $\overline{X}_\alpha$ and A^i_β are some singleeigenvalued finite-dimensional linear operators (the eigenvalues may be different for different β and i).

One can easily prove that the conditions *(i)* and *(ii)* do not depend on the choise of generic linear functions f_α, $f_\alpha \mid_{X_\alpha} = 0$.

For the convenience of notations we call $\mathcal{D}$-module M_i from $C^{qs}_{\mathcal{A}}$ to be singleeigenvalued if it satisfies (1.1) and (1.2) by itself with some fixed values of eigenvalues.

The category $C^{qs}_{\mathcal{A}}$ is rich enough; it contains at least two subcategories which are the most important in applications (and being defined a bit more naturally).

Definition 1.2 *An abelian category $C^0_{\mathcal{A}}$ consists of all $\mathcal{D}_X$-modules M from $C_{\mathcal{A}}$ satisfying the condition (ii) of Definition 1.1 with nilpotent matrix coeffitients:*

The local system $\Phi_{f_\alpha}(M) \mid_{X_\alpha}$ is presented by a flat connection θ:

$$\omega(\theta) = \sum A^i_\beta \frac{df_\beta}{f_\beta} \tag{1.3}$$

where A^i_β are nilpotent linear operators for all strata X_α.

The category $C^0_{\mathcal{A}}$ may be viewed as the smallest abelian subcategory of $C_{\mathcal{A}}$ containing all δ-functions over closed strata $\overline{X_\alpha}$.

Definition 1.3 *(Nonabelian) category $C^{\mathrm{ind}}_{\mathcal{A}}$ of locally indecomposable modules consists of all $\mathcal{D}_X$-modules M from $C_{\mathcal{A}}$ whose restriction to any open subset $U \subset \mathbf{C}^N$ is nonzero indecomposable module.*

Remark 1.1 *In the Definition 1.3, like everywhere throughout this paper we impose the condition of vanishing growth at infinity for $\mathcal{D}_X$-modules M; it can be expressed as a condition for all $\Phi_{f_\alpha}(M) \mid_{X_\alpha}$: they are flat connections θ with constant matrix coefficients, $\omega(\theta) = \sum A_\beta d \log f_\beta$.*

It is clear that for any indecomposable local system Ω over the complement U to the arrangement of hyperplanes their direct images $j_*\Omega$ and $j_!\Omega$ belong to $C^{\mathrm{ind}}_{\mathcal{A}}$. Here $j : U \to \mathbf{C}^N$ is an inclusion.

Proposition 1.1 *Both $C^0_{\mathcal{A}}$ and $C^{\mathrm{ind}}_{\mathcal{A}}$ are subcategories of $C^{qs}_{\mathcal{A}}$.*

We describe here the category $\mathcal{C}_{\mathcal{A}}^{qs}$ of all quasisemisimple $\mathcal{D}_X$-modules in $\mathcal{C}_{\mathcal{A}}$ in terms of a quiver (which means that we establish an equivalence of $\mathcal{C}_{\mathcal{A}}^{qs}$ to a category of representations of some quiver). The corresponding inverse functor from quiver-category to $\mathcal{C}_{\mathcal{A}}^{qs}$ is also described quite explicitely: we describe $\mathcal{D}_X$-modules attached to representations of a quiver in terms of generators and relations. Subcategories $\mathcal{C}_{\mathcal{A}}^{0}$ and $\mathcal{C}_{\mathcal{A}}^{\mathrm{ind}}$ are initialized in a quiver language by some conditions on eigenvalues.

An inductive procedure of describing the category $\mathcal{C}_{\mathcal{A}}^{qs}$ include the description of localizations of modules from $\mathcal{C}_{\mathcal{A}}^{qs}$ to certain open subsets of $\mathbf{C}^N$. Namely, these open subsets $U_n \subset \mathbf{C}^N$ are the complements to the union of fixed generic hyperplanes $\{f_\alpha = 0\}$, containing strata X_{α_n} of codimension n.

This circumstance enable us to describe direct images of one-dimensional local systems on the complement U_1 to an arrangement A (or, more generally, direct images of quasisemisimple local systems). These calculations may be considered as a basic point for computing the cohomology of local systems [Scl]. The answers are presented in the next section.

2 Combinatorial description of qusisemisimple $\mathcal{D}$-modules

2.1 The quiver's descrition of $\mathcal{C}_{\mathcal{A}}^{qs}$

Let us introduce first some notations.

Let X_α and X_β be two strata of $X_{\mathcal{A}}$. We write

$$\alpha < \beta \qquad \text{if } X_\alpha \subset \overline{X}_\beta \tag{2.1}$$

and

$$\alpha \leftarrow \beta \qquad \text{if } X_\alpha \subset \overline{X}_\beta \quad \text{and} \quad \mathrm{codim} X_\alpha - \mathrm{codim} X_\beta = 1 \tag{2.2}$$

For the open stratum of $\mathbf{C}^N$we fix an index $\emptyset$ (so $X_\emptyset \subset \mathbf{C}^N$ is an open stratum). It is convenient to describe the partial order (2.2) on the strata of $X_{\mathcal{A}}$ in terms of the graph $\Gamma_{\mathcal{A}}$ of the stratification. Graph $\Gamma_{\mathcal{A}}$ is an oriented connected graph with vertices identified with the indices of possible strata, and an arrow $\alpha \leftarrow \beta$ exists iff $\alpha \leftarrow \beta$ in a sense of (2.2). Analogously, we may define graph Γ_α , where X_α is an arbitrary stratum of $X_{\mathcal{A}}$. Graph

Γ_α describes the topology of induced stratification on affine space $\overline{X}_\alpha$. The vertices of Γ_α are those indices of strata X_β, for which $X_\beta \subset \overline{X}_\alpha$ and the arrows are the same.

One may equip a graph $\Gamma_{\mathcal{A}}$ with a colouring by complex numbers. Namely, C-coloured graph $\Gamma_{\mathcal{A}}(a_{\beta_i}^{\beta_j})$ is the graph $\Gamma_{\mathcal{A}}$ together with numbers $a_{\beta_i}^{\beta_j}$ attached to all arrows $\beta_i \rightarrow \beta_j$.

Definition 2.1 *$\mathbf{C}$-coloured graph $\Gamma_{\mathcal{A}}(a_{\beta_i}^{\beta_j})$ is called to be (self) compatible (or, in other words, the $\mathbf{C}$-colouring $a_{\beta_i}^{\beta_j}$ of $\Gamma_{\mathcal{A}}$ is compatible) if the following relation takes place for any link of two arrows*

$$\lambda \leftarrow \beta \leftarrow \gamma \tag{2.3}$$

in $\Gamma_{\mathcal{A}}$:

$$a_\beta^\lambda = \sum_\delta a_\gamma^\delta \tag{2.4}$$

where the sum is taken for all δ such that

$$\lambda \leftarrow \delta \leftarrow \gamma, \qquad \delta \neq \beta \tag{2.5}$$

Let now $\zeta_1, \zeta_2, \ldots, \zeta_n$ be arbitrary complex numbers (weights) putting in one-to-one correspondance with all codimension one strata $X_1, X_2, \ldots, X_n$ of affine space $\mathbf{C}^N$.

Proposition 2.1 *There exists unique $\mathbf{C}$-compatible colouring $a_{\beta_i}^{\beta_j} = a_{\beta_i}^{\beta_j}(\zeta_k)$ of $\Gamma_{\mathcal{A}}$ such that $a_\alpha^k = \zeta_k$ for all codimension one strata $X_1, X_2, \ldots, X_n$ of $\mathbf{C}^N$.*

We denote this compatible C-coloured graph by $\Gamma_{\mathcal{A}}(\overline{\zeta}) = \Gamma_{\mathcal{A}}(\zeta_1, \zeta_2, \ldots, \zeta_n)$

Proof of the Proposition 2.1. We prove the existence and uniqueness of compatible coefficients a_β^λ by induction on codimension of X_β. The uniqueness is evident from the defining relations

$$a_\beta^\lambda = \sum_{\substack{\delta:\lambda \leftarrow \delta \leftarrow \gamma \\ \delta \neq \beta}} a_\gamma^\delta \tag{2.6}$$

We may use (2.6) also for definition of a_β^λ by induction on $\mathrm{codim} X_\beta$ provided the defining formula (2.6) for a_β^λ does not depend on a choise of γ. But we may give an alternative expression for a_β^λ, codim$X_\beta = n$, if we know that the relation (2.6) is already valid for all links $\delta : \lambda \leftarrow \delta \leftarrow \gamma$, codim$X_\delta < n$. Indeed, let us fix some flag

$$\lambda_{n+1} \leftarrow \beta_n \leftarrow \gamma_{n-1} \leftarrow \ldots \leftarrow \gamma_1 \leftarrow \gamma_0 = \emptyset$$

where subindices remind of codimensions of strata. Then

$$a_{\beta_n}^{\lambda_{n+1}} = \sum_{\substack{\delta_n : \delta_n \neq \beta_n \\ \lambda_{n+1} \leftarrow \delta_n \leftarrow \gamma_{n-1}}} a_{\gamma_{n-1}}^{\delta_n} = \sum_{\substack{\delta_n : \delta_n \neq \beta_n \\ \lambda_{n+1} \leftarrow \delta_n \leftarrow \gamma_{n-1}}} \sum_{\substack{\delta_{n-1} : \delta_{n-1} \neq \gamma_{n-1} \\ \delta_n \leftarrow \delta_{n-1} \leftarrow \gamma_{n-2}}} a_{\gamma_{n-2}}^{\delta_{n-1}} = \ldots$$

$$\ldots = \sum_{\delta_1, \delta_2, \ldots, \delta_n} a_{\emptyset}^{\delta_1}$$

where the last sum is taken over all the flags

$$\lambda_{n+1} \leftarrow \delta_n \leftarrow \delta_{n-1} \leftarrow \ldots \leftarrow \delta_1 \leftarrow \alpha$$

such that $\delta_n \neq \beta_n$, $\delta_{n-1} \neq \gamma_{n-1}, \ldots \delta_1 \neq \gamma_1$ and $\delta_n \leftarrow \gamma_{n-1}$, $\delta_{n-1} \leftarrow \gamma_{n-2}, \ldots$, $\delta_2 \leftarrow \gamma_1$. It is not difficult to see that if δ_1 is such that for hyperplane $\overline{X}_{\delta_1}$ we have

$$\overline{X}_{\delta_1} \supset X_{\lambda_{n+1}}, \qquad \text{but} \qquad \overline{X}_{\delta_1} \not\supset X_{\beta_n}$$

then there is unique flag satisfying the above conditions: $\overline{X}_{\delta_k} = \overline{X}_{\gamma_k} \cap \overline{X}_{\delta_1}$ and there is no otherwise. So we have

$$a_{\beta_n}^{\lambda_{n+1}} = \sum_{\substack{\delta_1 > \lambda_{n+1}, \delta_1 \not> \beta_n \\ \operatorname{codim} X_{\delta_1} = 1}} a_{\emptyset}^{\delta_1} \tag{2.7}$$

The rhs of (2.7) depends only on λ_{n+1} and β_n, which proves the proposition.

Remark 2.1 *The relation (2.7) gives us direct geometrical description of the colouring of* $\Gamma_{\mathcal{A}}(\zeta_1, \zeta_2, \ldots, \zeta_n)$.

Let us again consider (uncoloured) graph $\Gamma_{\mathcal{A}}$ describing the stratification of $\mathbf{C}^N$. We attach to this graph a quiver $Q_{\mathcal{A}}$ as follows.

Definition 2.2 *A quiver $Q_{\mathcal{A}}$ consists of a collection of finite-dimensional complex vectorspaces V_β, where β are vertices of $\Gamma_{\mathcal{A}}$, and of linear maps*

$$A_{\lambda\beta}^- : V_\beta \to V_\lambda, \qquad A_{\beta\lambda}^+ : V_\lambda \to V_\beta$$

attached to all the arrows $\beta \to \lambda$ of $\Gamma_{\mathcal{A}}$. These linear maps should satisfy the following relations:

$$\sum_{\beta:\lambda\leftarrow\beta\leftarrow\gamma} A^-_{\lambda\beta}A^-_{\beta\gamma} = 0 \tag{2.8}$$

for any two vertices $\lambda, \gamma,$: $\lambda < \gamma$, $codimX_\lambda = codimX_\gamma + 2$,

$$\sum_{\beta:\gamma\to\beta\to\lambda} A^+_{\gamma\beta}A^+_{\beta\lambda} = 0 \tag{2.9}$$

for any two vertices $\lambda, \gamma,$: $\lambda < \gamma$, $codimX_\lambda = codimX_\gamma + 2$,

$$A^+_{\beta\lambda}A^-_{\lambda\mu} + A^-_{\beta\gamma}A^+_{\gamma\mu} = 0 \tag{2.10}$$

for any quadruple $\begin{array}{ccc} & \gamma & \\ \swarrow & & \searrow \\ \beta & & \mu \\ \searrow & & \swarrow \\ & \lambda & \end{array}$,

$$A^+_{\beta\lambda}A^-_{\lambda\mu} = 0 \tag{2.11}$$

for any triple $\begin{array}{ccc} \beta & & \mu \\ \searrow & & \swarrow \\ & \lambda & \end{array}$, *with no γ such that* $\begin{array}{ccc} & \gamma & \\ \swarrow & & \searrow \\ \beta & & \mu \end{array}$.

In other words, quiver $Q_{\mathcal{A}}$ is finite-dimensional representation of unital associative algebra $\overline{Q_{\mathcal{A}}}$ with a set of idempotents e_β, β being the vertices of $\Gamma_{\mathcal{A}}$, $\sum_\beta e_\beta = 1$, (degree one) generators $A^-_{\lambda\beta}$ and $A^+_{\beta\lambda}$ for any arrow $\beta \to \lambda$ with natural commutation relations with idempotents e_γ:

$$A^\pm_{\lambda\beta}e_\gamma = \delta_{\beta,\gamma}A^\pm_{\lambda\beta} \qquad e_\gamma A^\pm_{\lambda\beta} = \delta_{\gamma,\lambda}A^\pm_{\lambda\beta}$$

and quadratic relations (2.8)–(2.11).

We denote by $B_{\mathcal{A}}$ the category of all quivers $Q_{\mathcal{A}}$. In other words $B_{\mathcal{A}}$ is a category of all finite-dimensional representations of algebra $\overline{Q_{\mathcal{A}}}$.

Let now $\Gamma_{\mathcal{A}}(\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a compatible $\mathbf{C}$-coloured graph with weights $\zeta_1, \ldots, \zeta_n$ and $a^{\beta_j}_{\beta_i}$ be its coloures. We define the full subcategory $B_{\mathcal{A}}(\bar{\zeta}) = B_{\mathcal{A}}(\zeta_1, \ldots, \zeta_n)$ of B_α in the following way.

Definition 2.3 *The category $B_{\mathcal{A}}(\bar{\zeta})$ consists of all quivers $Q_{\mathcal{A}}$ with a condition*

$$\text{the single eigenvalue of } A^+_{\beta\lambda}A^-_{\lambda\beta} \text{ and of } A^-_{\lambda\beta}A^+_{\beta\lambda} \text{ is equal to } a^\lambda_\beta \tag{2.12}$$

for all arrows $\beta \to \lambda$ in $\Gamma_{\mathcal{A}}$.

Let now $Q_{\mathcal{A}}$ be a quiver from Definition 2.2. We may define a *support* of $Q_{\mathcal{A}}$ as a set of all vertices β of $\Gamma_{\mathcal{A}}$ such that $V_\beta \neq \{0\}$:

$$\operatorname{supp} Q_{\mathcal{A}} = \{\beta : V_\beta \neq \{0\}\}.$$

The vertice $\beta \in \operatorname{supp} Q_{\mathcal{A}}$ is called a *source* of $Q_{\mathcal{A}}$ if there is no $\alpha \in \operatorname{supp} Q_{\mathcal{A}}$ such that $\alpha > \beta$.

We say also that a *depth* of a vertice $\beta \in \Gamma_{\mathcal{A}}$ is equal to m, $d(\beta) = m$, if there is a source α of $Q_{\mathcal{A}}$, $\alpha > \beta$ such that $\operatorname{codim}_{\overline{X}_\alpha} X_\beta = m$ and there is no source γ of $Q_{\mathcal{A}}$, $\gamma > \beta$ with $\operatorname{codim}_{\overline{X}_\gamma} X_\beta > m$.

In these notations the category $B_{\mathcal{A}}^{\text{quasi}}$ of quasisemisimple quivers is defined as follows.

Definition 2.4 *Quasisemisimple quiver is a direct sum of quivers Q from $B_{\mathcal{A}}$ satisfying the following conditions:*

(i) *The composition*

$$A^{+}_{\beta\lambda} A^{-}_{\lambda\beta} \tag{2.13}$$

has only one eigenvalue $a^\lambda_\beta = \text{eig.v.}(A^{+}_{\beta\lambda} A^{-}_{\lambda\beta})$ for any arrow $\beta \to \lambda$ in $\Gamma_{\mathcal{A}}$;

(ii) *An inequality*

$$0 < \operatorname{Re} a^\lambda_\beta < 1 \tag{2.14}$$

takes place for any source β of Q and for any arrow $\beta \to \lambda$;

(iii) *If β is a vertice of depth one then*

$$a^\gamma_\beta = a^{\gamma\prime}_\beta \tag{2.15}$$

for any two arrows $\gamma \to \beta$, $\gamma\prime \to \beta$ with γ and $\gamma\prime$ being sources of Q;

(iv) *An operator*

$$\bigoplus_{\substack{\alpha,\alpha\prime: \\ \alpha\to\beta,\alpha\prime\to\beta}} A^{+}_{\alpha\beta} A^{-}_{\beta\alpha\prime} \tag{2.16}$$

is nilpotent in $\bigoplus_{\alpha:\alpha\to\beta} V_\alpha$ for any vertice β of depth more than one in Q.

Now we are able to present a combinatorial description of the category $C^{qs}_{\mathcal{A}}$ of quasisemisimple $\mathcal{D}$-modules over the arrangement of hyperplanes.

Theorem 2.1 *The category $C^{qs}_{\mathcal{A}}$ is equivalent to the category $B^{qs}_{\mathcal{A}}$.*

The functor establishing an equivalence of categories looks as follows. Let M be a singleeigenvalued $\mathcal{D}$-module from $\mathcal{C}_{\mathcal{A}}^{qs}$ and X_α be a stratum. Then the space V_α of a quiver is the space of flat sections of $\Psi_{f_\alpha}(M)\mid_{X_\alpha}$ where the corresponding flat connection has a form $w = \sum_{\beta:\alpha\to\beta} A_\alpha^\beta d\log f_\beta,\ A_\alpha^\beta \in$ End (V_α).

The operators $A^+_{\alpha\beta}$ and $A^-_{\beta\alpha}$ are built from the canonical maps

$$\Psi_{f_\alpha}(M)\mid_{X_\alpha} \overset{u}{\underset{v}{\rightleftarrows}} \Phi_{f_\alpha}\mid_{X_\alpha}(M).$$

Their explicit expressions are given by formulas (3.2), (3.3), (5.21)–(5.24).

The most important in applications are the following two theorems. The first of them describes an extension closure of all δ-functions of strata $\overline{X}_\alpha$. The second describes locally indecomposable modules.

Theorem 2.2 *The category $\mathcal{C}_{\mathcal{A}}^0$ is equivalent to $B_\emptyset(0,\cdots,0)$.*

Let $\mathcal{C}_{\mathcal{A}}^{\text{ind}}(\overline{\zeta}) = \mathcal{C}_{\mathcal{A}}^{\text{ind}}(\zeta_1,\zeta_2,\ldots,\zeta_n)$ consists of those locally indecomposable modules from $\mathcal{C}_{\mathcal{A}}^{\text{ind}}$, whose restrictions to the open stratum $X_\emptyset$ are described by flat connections $\omega = \sum_i A_i d\log f_i$ with eig.v.$(A_i) = \zeta_i, i = 1,\ldots,n$, acting in nonzero space $V_\emptyset$ of flat sections.

Theorem 2.3 *The category $\mathcal{C}_{\mathcal{A}}^{\text{ind}}(\overline{\zeta})$ is equivalent to the category of indecomposable objects of $B_\emptyset(\overline{\zeta})$ with nonzero space $V_\emptyset$.*

In the same way we can describe locally indecomposable modules with a support on some stratum X_α. The only thing to do is to exchange graph $\Gamma_{\mathcal{A}}$ by $\Gamma_{\mathcal{A}}$.

In the next subsection we show how to restore the $\mathcal{D}$-modules from their quiver data.

2.2 Restoring $\mathcal{D}$-modules from quiver's data

Let us remind once more that throughout this paper we are in agreement that for each stratum X_α we fix once forever a generic hyperplane $f_\alpha = 0$ such that $f_\alpha\mid_{X_\alpha}= 0$. Moreover, for codimension n stratum X_α we need sometimes a generic *flag* $\vec{f}_\alpha = f_\alpha^1,\ \ldots,\ f_\alpha^n,\ f_\alpha^1 = f_\alpha$ of functions being equal zero on X_α and generating a basis of $(\mathbf{C}^N/\overline{X}_\alpha)^*$.

We also fix nondegenerated complex skewsymmetric form

$$<,>: \oplus \wedge^k ((\mathbf{C}^N)^*) \to \mathbf{C}$$

which we use inambiguesly for all flag manifolds implicitely appearing in the calculations. One may think of a fixed generic coordinate system $x_1, \ldots, x_N$ in $\mathbf{C}^N$and put

$$< f_{\alpha_1}, \ldots, f_{\alpha_k} > = \det \left(\frac{\partial f_{\alpha_i}}{\partial x_j} \right), \qquad 1 \le i, j \le k$$

We often simplify the notations writing $< \alpha, \beta >$ instead of $< f_\alpha, f_\beta >$, $< \vec{\alpha} >$ instead of $< \vec{f_\alpha} >$ and so on. The vector fields which we use here are always linear, that is , have a form

$$L = \sum_{i=1}^{N} a_i \frac{\partial}{\partial x_i}, \qquad a_i \in \mathbf{C}$$

If we use the notation L_α for a vectorfield with an index of some stratum X_α, it means as a rule that this vector field goes along stratum X_α, in particular $L_\alpha(f_\alpha) = 0$.

Now let Q_α be a quiver with vectorspaces V_β attached to vertices β and linear operators $A^-_{\lambda\beta} : V_\beta \to V_\lambda$, $A^+_{\beta\lambda} : V_\lambda \to V_\beta$ attached to the arrows $\beta \to \lambda$ (see Definition 2.2). We associate to this quiver $\mathcal{D}_X$-module $M(Q_\alpha)$ in the following way: $M(Q_\alpha)$ is a free $\mathcal{D}_X$-module generated by the space $\bigoplus_\beta V_\beta$, β being the vertices of $\Gamma_{\mathcal{A}}$ modulo the following relations:

$$L_\beta(v_\beta) = \sum_{\lambda:\beta\to\lambda} \frac{< \vec{f_\beta} >}{< f_l \vec{f_\beta} >} L_\beta(f_\lambda) A^-_{\lambda\beta}(v_\beta) =$$

$$= \sum_{\lambda:\beta\to\lambda} \frac{< \vec{f_\beta} >}{< \vec{f_\lambda} >} \frac{L_\beta(\vec{f_\lambda})}{\vec{f_\beta}} A^-_{\lambda\beta}(v_\beta), \qquad v_\beta \in V_\beta \qquad (2.17)$$

if L_β is a linear vector field along stratum X_β and

$$f \cdot v_\beta = \sum_{\gamma:\gamma\to\beta} \frac{< f, \vec{f_\gamma} >}{< \vec{f_\gamma} >} A^+_{\gamma\beta}(v_\beta), \qquad v_\beta \in V_\beta \qquad (2.18)$$

if f is a linear function, $f \mid_{X_\beta} = 0$.

Theorem 2.4 *The functor $Q_\alpha \to M(Q_\alpha)$ establishes an equivalence of categories stated in the Theorems 2.1–2.3.*

Note also that the relations (2.18) give possibility to describe $M(Q_\alpha)$ as a sheaf of $\mathcal{O}_X$-modules. For the basic open sets

$$Y_n^\lambda = \left(\mathbf{C}^N \setminus \bigcup_{\gamma:\, \mathrm{codim} X_\gamma = n} \{f_\gamma = 0\} \right) \cup \{f_\lambda = 0\}$$

where $\mathrm{codim}\, X_\lambda = n$, the space of sections $\Gamma(M(Q_\alpha), Y_n^\lambda)$ is free $\mathcal{O}_{Y_n^\lambda}$-module generated by the spaces V_β, $\alpha \to \beta$, $\mathrm{codim}\, X_\beta < n$ and V_λ modulo the relations (2.18).

2.3 An example: Direct images of local systems

Let $\Omega(A_1, A_2, \ldots, A_n)$ be a local system over the complement $U_\emptyset$ to the arrangement of hyperplanes $\{f_i = 0\}$ defined by a flat connection

$$\omega = \sum A_i d \log f_i$$

Let $eig.v.(A_i) = \zeta_i$. Then we can find direct images $j_* \Omega$ and $j_! \Omega$, where $j : U_\emptyset \hookrightarrow \mathbf{C}^N$ is an inclusion, as universal object in $\mathcal{C}_{\mathcal{A}}^{\mathrm{ind}}(\zeta_1, \zeta_2, \ldots, \zeta_n)$ representing the functors $F_\Omega^* : F_\Omega^*(M) = \mathrm{Hom}_{\mathcal{D}(U)}(j^* M, \Omega)$ and $F_\Omega^! : F_\Omega^!(M) = \mathrm{Hom}_{\mathcal{D}(U)}(\Omega, j^! M)$. Due to the equivalence of categories one can make these calculations inside $B_\emptyset(\vec{\zeta})$ by means of usual linear algebra.

Let us describe an answer for $j_* \Omega$, where Ω is one-dimensional local system

$$\omega = \sum a_i d \log f_i \qquad a_i \in \mathbf{C} \tag{2.19}$$

Denote by W_m a vector space over Cwith a basis $< e_\beta >, \mathrm{codim}\, e_\beta = m$. Let X_α be a stratum of codimension n. Denote by $\overline{V_\alpha}$ the following subspace of $W_0 \otimes W_1 \otimes \cdots \otimes W_n$:

$$\overline{V_\alpha} = \bigoplus_{\substack{\text{all flags } \alpha_0 \to \alpha_1 \to \cdots \to \alpha_n: \\ \alpha_0 = \emptyset,\, \alpha_n = \alpha}} \mathbf{C}\, e_{\alpha_0} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \tag{2.20}$$

and let $V_\alpha \subset \overline{V_\alpha}$ consists of all the elements

$$v_\alpha = \sum x_{\alpha_0, \ldots, \alpha_n} e_{\alpha_0} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}, \qquad v_a \in \overline{V_\alpha} \tag{2.21}$$

satisfying the equations

$$\sum_{\beta:\alpha_{i-1}\to\beta\to\alpha_{i+1}} x_{\alpha_0,\dots\alpha_{i-1}\beta\alpha_{i+1}\dots\alpha_n} e_{\alpha_0}\cdots\otimes e_{\alpha_{i-1}}\otimes e_\beta\otimes e_{\alpha_{i+1}}\cdots\otimes e_{\alpha_n}=0 \quad (2.22)$$

for any fixed degenerated flag

$$\emptyset=\alpha_0\to\alpha_1\to\cdots\to\alpha_{i-1}>\alpha_{i+1}\to\cdots\to\alpha_n=\alpha$$

Let us fix some arrow $\alpha\to\beta$ in graph $\Gamma_{\mathcal{A}}$,codim $X_\alpha=n$. Then we can define operators $\overline{A^+_{\alpha\beta}}:\overline{V_\beta}\to\overline{V_\alpha}$ and $\overline{A^-_{\beta\alpha}}:\overline{V_\alpha}\to\overline{V_\beta}$ as follows:

$$\overline{A^+_{\alpha\beta}}(e_{\alpha_0}\otimes\cdots\otimes e_{\alpha_n}\otimes e_{\alpha_{n+1}=\beta})=\delta_{\alpha,\alpha_n}\cdot e_{\alpha_0}\otimes\cdots\otimes e_{\alpha_n} \quad (2.23)$$

and

$$\overline{A^-_{\beta\alpha}}(e_{\alpha_0}\otimes\cdots\otimes e_{\alpha_n=\alpha})=\sum_{\substack{j:\,\mathrm{codim}\,X_j=1,\\ j>\beta,j\not>\alpha}} a_j\, e_{\alpha_0}\otimes(e_{\alpha_1}-e_{j\cap\alpha_0})\otimes$$

$$\otimes(e_{\alpha_2}-e_{j\cap\alpha_1})\otimes\cdots\otimes(e_{\alpha_n}-e_{j\cap\alpha_{n-1}})\otimes e_\beta \quad (2.24)$$

where $j\cap\alpha_i$ is an index of a stratum $X_j\cap X_{\alpha_i}$: $X_{j\cap\alpha_i}=X_j\cap X_{\alpha_i}$.

One can check that operators $\overline{A^+_{\alpha\beta}}$ and $\overline{A^-_{\beta\alpha}}$ correctly define by restriction the operators

$$A^+_{\alpha\beta}:V_\beta\to V_\alpha \qquad \text{and} \qquad A^-_{\beta\alpha}:V_\alpha\to V_\beta \quad (2.25)$$

Proposition 2.2 *The quiver $Q(a_1,\dots,a_n)$ defined in (2.20)-(2.25) describes direct image of local system (2.19) via the equivalence of categories of Theorem 2.3.*

Remark 2.2 *$\mathcal{D}_X$-module $j_*\Omega$ can be realized in the space*

$$\mathbf{C}[X][f_1^{-1},\dots,f_n^{-1}]\cdot f^{\vec{a}}$$

where $f^{\vec{a}}=f_1^{a_1}\cdots f_n^{a_n}$ should be treated as a formal symbol defining an action of first order differential operator by Leibnitz rule. Then the space V_α, codim $X_\alpha=m$ (see (2.20)-(2.22)) is isomorphic to linear envelop of $f_{i_1}^{-1}\cdots f_{i_m}^{-1}\cdot f^{\vec{a}}$ with $\{f_{i_1}=0\}\cap\cdots\cap\{f_{i_m}=0\}=X_\alpha$. The equations (2.22) are equivalent to well known Orlik-Solomon relations for the products of $d\log f_i$.

We can attach to a quiver $Q(a_1, \ldots, a_n)$ a natural complex $C^\cdot(a_1, \ldots, a_n)$:

$$C^i(a_1, \ldots, a_n) = \bigoplus_{\alpha : \operatorname{codim} X_\alpha = i} V_\alpha$$

and differential $d : C^i \to C^{i+1}$ being equal to $\oplus A^-_{\beta\alpha}$, $\operatorname{codim} X_\alpha = i$ ($d^2 = 0$ due to (2.8)).

In the case of zero weights complex $C^\cdot(0, \ldots, 0)$ coincides with Orlik-Solomon algebra [OS], [Br] and its homology are equal to $H^*(U)$. Moreover, as it was proved in [VS1], the homologies of $C^\cdot(a_1, \ldots, a_n)$ are isomorphic to $H^*(U, \Omega(a_1, \ldots, a_n)$ for $(a_1, \ldots, a_n)$ being close enough to zero.

3 Beilinson's glueing theorem

3.1 Glueing of perversed sheaves

Let X be a smooth algebraic variety over $\mathbf{C}$, $\mathcal{M}(X)$ be a category of perversed constructible sheaves with respect to a middle perversity [BBD]. Let $f \to \mathbf{C}$ be an algebraic function, $Y = f^{-1}(0)$, $U = f^{-1}(\mathbf{C} \setminus \{0\})$, $j : U \to X$, $i : \to X$ be corresponding imbeddings. Let

$$\mathbf{\Psi}_f^{\mathrm{geom}} : \mathcal{M}(U) \to \mathcal{M}(Y)$$

and

$$\mathbf{\Phi}_f^{\mathrm{geom}} : \mathcal{M}(X) \to \mathcal{M}(Y)$$

denote functors of neaby and vanishing cycles (in the notations of [D] $\mathbf{\Psi}_f^{\mathrm{geom}} = \mathbf{\Psi}_{\eta,f}\, j^*$). The functors $\mathbf{\Psi}_f^{\mathrm{geom}}$ and $\mathbf{\Phi}_f^{\mathrm{geom}}$ come up with a canonical automorphism $T : \mathbf{\Psi}_f^{\mathrm{geom}} \to \mathbf{\Psi}_f^{\mathrm{geom}}$ (monodromy) and with natural transformations

$$u : \mathbf{\Psi}_f^{\mathrm{geom}} \to \mathbf{\Phi}_f^{\mathrm{geom}}, \qquad v : \mathbf{\Phi}_f^{\mathrm{geom}} \to \mathbf{\Psi}_f^{\mathrm{geom}}$$

such that $vu = T - 1$.

Let us denote by $\mathcal{M}_f(U, Y)$ the category whose objects are quadruples $(M_U, M_Y; u, v)$ where $M_Y \in \mathcal{M}(U)$, $M_Y \in \mathcal{M}(Y)$ and $u : \mathbf{\Psi}_{\eta,f}(M_U) \to M_Y$, $v : M_Y \to \mathbf{\Psi}_{\eta,f}(M_U)$ are such that $vu = T - 1$.

The assignment

$$M \to (j^* M, \mathbf{\Phi}_f^{\mathrm{geom}}(M); u, v)$$

defines a functor

$$G : \mathcal{M}(X) \to \mathcal{M}_f(U, Y).$$

Theorem 3.1 *[B] (see also [Ver]). G is an equivalence of categories.*

Let λ be a complex number, $0 \leq \mathrm{Re}\lambda < 1$. We may define a full subcategory $\mathcal{M}_{f,\lambda}(U,Y)$ of $\mathcal{M}_f(U,Y)$ consisting of the quadruples $(M_U, M_Y; u, v)$ with a condition that endomorphisms $1 + uv$ and $1 + vu$ have $e^{2\pi i\lambda}$ as a single eigenvalue. We put also

$$\mathcal{M}_{f,\lambda}(X) = G^{-1}(\mathcal{M}_{f,\lambda}(U,Y)), \qquad \mathcal{M}_{f,\lambda}(U) = j^*(\mathcal{M}_{f,\lambda}(X)).$$

We call the perverse sheaves from $\mathcal{M}_{f,\lambda}(X)$ and $\mathcal{M}_{f,\lambda}(U)$ λ-monodromic with respect to f. Standard arguments from linear algebra show that for $\lambda \notin \mathbf{Z}$ any perverse sheave from $\mathcal{M}_{f,\lambda}(X)$ is uniquely determined by its restriction to U, in other words,

$$j^* : \mathcal{M}_{f,\lambda}(X) \rightarrow \mathcal{M}_{f,\lambda}(U)$$

is an equivalence of categories for $\lambda \notin \mathbf{Z}$ and there is a decomposition

$$\mathcal{M}(X) \simeq \bigoplus_{\lambda:\, 0 \leq \mathrm{Re}\lambda < 1} \mathcal{M}_{f,\lambda}(X).$$

3.2 Glueing in terms of $\mathcal{D}$-modules

Let us keep the previous notation. Let $\mathcal{D}(X)$ be a category of holonomic $\mathcal{D}_X$-modules with regular singularities. Due to the comparison theorem [BBD] de Rham functor $DR : \mathcal{D}(X) \rightarrow \mathcal{M}(X)$ establishes an equivalence of categories. In particular we are able to introduce via this equivalence the categories $\mathcal{D}_{\lambda,f}(X)$ and $\mathcal{D}_{\lambda,f}(U)$ of λ-monodromic with respect to f $\mathcal{D}$-modules. Following [B] we may describe functor Ψ and glueing theorem 3.1 more explicitely.

Let N be a holonomic $\mathcal{D}(U)$-module. We define $\Psi^\lambda(N)$ to be maximal $\mathcal{D}_X[[s+\lambda]]$ factormodule of $j_*(N \cdot f^s)[[s+\lambda]]$ with a support on Y. Here s is a formal variable, λ is a complex number, $0 \leq \mathrm{Re}\lambda < 1$. $\mathcal{D}_X[[s+\lambda]]$ module $j_*(N \cdot f^s)[[s+\lambda]]$ consists of expressions $\sum_i g_i n_i f^{k_i} P_i(s) \cdot f^s$ where $g_i \in \mathcal{O}(X)$, n_i are the elements of N, $k_i \in \mathbf{Z}$, $P_i(s)$ are Taylor series on $s+\lambda$, f^s is a formal symbol, defining an action of vector fields L on X:

$$L(n \cdot f^s) = (L(n) + snL(f)f^{-1}) \cdot f^s$$

Then, if $N \in \mathcal{D}_{\lambda,f}(U)$, $\Phi_f^{geom}(DR(M)) \simeq DR(\Psi^\lambda(N))$ and the multiplication by $-s$ defines the action of logariphm of monodromy $S = \log T$ on $\Psi^\lambda(N)$ with $eig.value(S) = \lambda$.

Moreover, it is not difficult to see that if $\lambda \notin \mathbf{Z}$ then $\Psi^\lambda(N) \simeq \Psi^0(N \otimes f^{-\lambda})$, where $f^{-\lambda}$ is irreducible $\mathcal{D}_U$-module which corresponds to one-dimensional local system on U with a flat connection

$$\omega = -\lambda d \log f$$

The Beilinson's glueing theorem can be read in $\mathcal{D}$-modules language as follows.

Theorem 3.2 *The category $\mathcal{D}_{\lambda,f}(X)$ of λ-monodromic with respect to f $\mathcal{D}$-modules is equivalent to the category $\mathcal{D}_{\lambda,f}(U,Y)$ of quadruples $(M,N;\alpha,\beta)$ where $M \in \mathcal{D}_\lambda(U)$, $N \in \mathcal{D}(Y)$ and $\alpha : \Psi^\lambda(M) \to N$, $\beta : N \to \Psi^\lambda(M)$ are such that $S = \alpha\beta$ with a condition that $eig.v.(\beta\alpha) = eig.v.(\alpha\beta) = \lambda$.*

We need also an explicit form of an inverse functor

$$F_{\lambda,f} : \mathcal{D}_{\lambda,f}(U,Y) \to \mathcal{D}_{\lambda,f}(X) \tag{3.1}$$

Consider first the case $\lambda = 0$. For any $M \in \mathcal{D}(U)$ let $\Xi^0(M)$ be the maximal factormodule of $j_*(N \cdot f^s)[[s]]$ coinciding with M on U. Let now $(M,N;\alpha,\beta) \in \mathcal{D}_{0,f}(U,Y)$. We put $F_{0,f}(M,N;\alpha,\beta)$ to be the homology of a complex

$$\begin{array}{ccccc} & & \Xi^0(M) & & \\ & \overset{s}{\nearrow} & & \overset{pr}{\searrow} & \\ \Psi^0(M) & & \oplus & & \Psi^0(M) \\ & \underset{\alpha}{\searrow} & & \underset{\beta}{\nearrow} & \\ & & N & & \end{array} \tag{3.2}$$

The functor $F_{0,f}$ establishes an equivalence of categories $\mathcal{D}_{0,f}(U,Y)$ and $\mathcal{D}_{0,f}(X)$. Note also that the canonical $\mathcal{D}_Y$-module $\Phi_f^0(M)$, $M \in \mathcal{D}_0(X)$ is defined in the case $\lambda = 0$ also as homology of natural complex

$$j_! j^* M \to \Xi^0(j^* M) \oplus M \to j_* j^* M$$

For $\lambda \neq 0$, $0 < \mathrm{Re}\lambda < 1$ we know that thr categories $\mathcal{D}_{\lambda,f}(X)$ and $\mathcal{D}_{\lambda,f}(U,Y)$ are equivalent to the category $\mathcal{D}_{\lambda,f}(U)$. To make the construction to be consistent with the case $\lambda = 0$ we define an equivalence $F_{\lambda,f}$: $\mathcal{D}_{\lambda,f}(U,Y) \to \mathcal{D}_{\lambda,f}(X)$ as (a bit nonnatural) the following composition:

$$F_{\lambda,f} = (j_* f^\lambda \otimes) \cdot F_{0,f} \cdot (f^{-\lambda}\otimes, \Psi^\lambda) : \quad (M, N; \alpha, \beta) \stackrel{(f^{-\lambda}\otimes, \Psi^\lambda)}{\longrightarrow}$$

$$\to (M \otimes f^{-\lambda}, \Psi^\lambda(M); \alpha\beta - \lambda, 1) \stackrel{F_{0,f}}{\longrightarrow} F_{0,f}(M \otimes f^{-\lambda}, \Psi^\lambda(M); \alpha\beta - \lambda, 1) \to$$

$$\stackrel{j_* f^\lambda \otimes}{\longrightarrow} j_* f^\lambda \otimes F_{0,f}(M \otimes f^{-\lambda}, \Psi^\lambda(M); \alpha\beta - \lambda, 1) \qquad (3.3)$$

4 Inductive description of the category $\mathcal{C}_{\mathcal{A}}^{qs}$

4.1 Plan of the construction

We come back to the notation of the sections 1 and 2. We describe the category $\mathcal{C}_{\mathcal{A}}^{qs}$ by induction on codimension of strata.

Let us first consider an open set $U_1 = X \setminus \bigcup_i \{f_i = 0\}$, $X = \mathbf{C}^N$ of complement to the arranged hyperplanes; $j_1 : U_1 \hookrightarrow X$ being the inclusion. Then, due to the Definition 1.2, the category $\mathcal{C}_1 = j^* \mathcal{C}_{\mathcal{A}}^{qs}$ is defined as a category of local systems described by flat connections

$$\omega = \sum_i A_i d \log f_i \qquad (4.1)$$

with a condition (1.2), meaning that all A_i admit simultenious Jordan block decomposition. Category $\mathcal{C}_1$ is equivalent to a full subcategory of finite-dimensional representations of quadratic algebra with generators A_i and the relations which one can recover rewriting the flatness condition of (4.1).

Next we look to the complement U_2 to the union of fixed generic hyperplanes going through codimension 2 strata:

$$U_2 = X \setminus \bigcup_{\alpha:\, \mathrm{codim}\, X_\alpha = 2} \{f_\alpha = 0\},$$

$j_2 : U_2 \hookrightarrow X$ being corresponding inclusion and describe the category $\mathcal{C}_2 = j_2^* \mathcal{C}_{\mathcal{A}}^{qs}$. This description comes into steps. First we choose some codimension one stratum $X_i = \{f_i = 0\}$ and apply the glueing construction in order to glue X_i with U_1. It means that we consider a triple $Y^i \hookrightarrow X_2^i \hookleftarrow U_{1,2}$, where

$$U_{1,2} = X \setminus \bigcup_{\alpha:\, \mathrm{codim}\, X_\alpha = 1,2} \{f_\alpha = 0\}, \tag{4.2}$$

$$X_2^i = \{f_i = 0\} \bigcup U_{1,2}, \qquad Y^i = \{f_i = 0\} \bigcap X_2^i \tag{4.3}$$

and apply Beilinson's construction to this triple. Now the spaces X_2^i are open sets in U_2, $j_2^i : X_2^i \hookrightarrow U_2$ being the inclusions and $U_2 = \bigcup_i X_2^i$. Using the axioms of a sheaf for any $M \in \mathcal{C}_2$ we recover M by its restrictions j_2^{i*} to all the X_2^i.

In quite a similar manner we perform a general induction step. Assuming the knowledge of the category $\mathcal{C}_n = j_n^*(\mathcal{C}_{\mathcal{A}}^{qs})$ on an open set U_n we obtain the description of $\mathcal{C}_{n+1} = j_{n+1}^*(\mathcal{C}_{\mathcal{A}}^{qs})$ on an open set U_{n+1}. Here $U_k = X \setminus \bigcup_{\alpha:\mathrm{codim}\, X_\alpha = k}\{f_\alpha = 0\}$, $j_k : U_k \hookrightarrow X$ being the corresponding inclusion. This is done by application of glueing construction to a triple $Y^\beta \hookrightarrow X_n^\beta \hookleftarrow U_{n,n+1}$, $\mathrm{codim}\, X_\beta = n$, where

$$U_{n,n+1} = X \setminus \bigcup_{\alpha:\, \mathrm{codim}\, X_\alpha = n,n+1} \{f_\alpha = 0\},$$

$$X_n^\beta = \{f_\beta = 0\} \bigcup U_{n,n+1}, \qquad Y^\beta = \{f_\beta = 0\} \bigcap X_n^\beta \tag{4.4}$$

and then by recovering the sheaf of $\mathcal{D}_{U_n}$-modules by restrictions to X_n^β, $\bigcup_\beta X_n^\beta = U_{n+1}$.

It is important to emphazise that we have to use the glueing procedure for (4.4) twice, in two different ways. First we are to obtain the combinatorial data and calculate all new relations appearing in the glueing. Here we use directly Beilinson's Theorem 3.2. Then we need to realize explicitely $\mathcal{D}$-module given by these combinatorial data using functor $F_{\lambda,f}$ (see (3.1)–(3.3)). Following these calculations we discover also that for each inductive step we have a splitting of the corresponding category $\mathcal{C}_n$ described by the consistency conditions on the eigenvalues of monodromies (2.4). The new splitted terms should be supported on subglued stratum and do not appear in the description of locally indecomposable modules (Theorem 2.3).

The rest of this section is devoted to explicit inductive description of $\mathcal{D}$-modules from $\mathcal{C}_{\mathcal{A}}^{qs}$. In order to make exposition readible we first demonstrate the technique on more simple examples of codimension one and two strata and then pass to general induction step.

4.2 The flatness condition

Let us first make simple exercise and compute the relations on matrices A_i coming from the flatness of the connection (4.1). These relations should be well known (see, for instance [Ko] for Knizhnik–Zamolodchikov configuration) but we prefer to repeat them in our terms.

It is more convenient for us to admit poles along hyperplanes $f_\alpha = 0$, $\operatorname{codim} X_\alpha = 2$, in other words, to work in the space

$$U_2 = \mathbf{C}^N \setminus \bigcup_{\alpha:\operatorname{codim} X_\alpha = 2} \{f_\alpha = 0\}.$$

Let us fix some stratum X_α of codimension two. We can choose a pair of commuting linear vector fields L_α and M_α as follows: L_α be a generic vectorfield along X_α: $L_\alpha(f_\alpha) = 0$, $L_\alpha(f_i) \neq 0$ for all $i : i \to \alpha$ and M_α be transversal to f_α (like a gradient): $M_\alpha(f_\alpha) \neq 0$.

Let $X_i \cup X_j = \bar{X}_\alpha$. Then the functions f_i, f_j and f_α are linear dependent:

$$f_j \cdot < \alpha, i > = f_\alpha \cdot < j, i > + f_i \cdot < \alpha, j > \tag{4.5}$$

or

$$\frac{< \alpha, i >}{f_\alpha f_i} = \frac{< j, i >}{f_i f_j} + \frac{< \alpha, j >}{f_j f_\alpha} \tag{4.6}$$

We know that for any linear vector field L

$$L(w) = \sum_j \frac{L(f_j)}{f_j} A_j(w), \qquad w \in W \tag{4.7}$$

where W is a basic space of sections of a vector bundle over U_1.

Substitutung L_α and M_α into (4.7) we see that

$$[L_\alpha, M_\alpha](w) = \sum_{\substack{i,j: i \neq j \\ i \to \alpha,\, j \to \alpha}} \frac{L_\alpha(f_i) M_\alpha(f_j) - L_\alpha(f_j) M_\alpha(f_i)}{f_i f_j} A_i A_j(w) +$$

$$+ \text{ other terms} \tag{4.8}$$

where "other terms" have in denominator functions f_k and f_l such that $\overline{X}_\alpha \not\subset \{f_k = 0\} \cap \{f_l = 0\}$. Using linear dependance conditions (4.5) and (4.6) we rewrite the first sum in rhs of (4.8) as

$$\sum_{\alpha,i:\,i\to\alpha} \frac{L_\alpha(f_i)M_\alpha(f_\alpha)}{f_\alpha f_i}\left(\sum_{\substack{j:\,j\to\alpha\\ j\neq i}}[A_i,A_j](w)\right)$$

The functions $\frac{1}{f_\alpha f_i}$ are linear independent now so we have the following relation which takes place for any flag $\emptyset \to i \to \alpha$:

$$\sum_{\substack{j:\,j\to\alpha\\ j\neq i}}[A_i,A_j]=0 \tag{4.9}$$

We conclude that the category $\mathcal{C}_0$ of flat connections (4.1) is equivalent to the category of finite dimensional representations of quadratic algebra with generators A_i and relations (4.9). This algebra may be viewed as infinitesimal version of fundamental group $\pi_1(\mathbb{C}^N \setminus \bigcup_i\{f_i=0\})$.

4.3 Glueing of codimension 1 strata

For the simplisity of notations we reserve symbol X in this subsection for an open subset U_2

$$U_2=\mathbb{C}^N\setminus \bigcup_{\alpha:\,\operatorname{codim}X_\alpha=2}\{f_\alpha=0\}.$$

All the games of this subsection will be inside $X=\tilde{U}_2$.

Just as before we start from $\mathcal{D}_{X_\emptyset}$-module M on an open stratum $X_\emptyset$ which is generated by finite-dimensional vectorspace W, free over the ring of functions $\mathcal{O}(X_\emptyset)$ with the following action of linear vector fields L on $X_\emptyset$:

$$L(w)=\sum_{j:\,j\leftarrow\emptyset}\frac{L(f_j)}{f_j}A_j(w), \qquad w\in W \tag{4.10}$$

where A_j are some linear operators $A_j: W\to W$ with fixed eigenvalues a_j, $0\le \operatorname{Re}a_j<1$ subjected to relations (4.9).

Let us fix some codimension one stratum X_i. We may assume that $\mathcal{D}_X$-module N from $\mathcal{C}_{\mathcal{A}}^{q'}$ with a support incide X_i is generated by some finite-dimensional vectorspace V_i and is described by the relations

$$f_iv_i=0, \qquad L_i(v_i)=\sum_{\alpha:\,\alpha\leftarrow i}\frac{L_i(f_\alpha)}{f_\alpha}A_i^\alpha(v_i), \qquad v_i\in V_i \tag{4.11}$$

where $L_i(f_i) = 0$, $A_i^\alpha : V_i \to V_i$, with eig.v.$(A_i^\alpha) = a_i^\alpha$. Let us compute $\Psi_i(M)$, where $\Psi_i = \Psi_{f_i}^{a_i}$. Applying some vectorfield L to $wf_i^{s+k} \stackrel{dfn}{=} f_i^k w \cdot f_i^s$:

$$L(wf_i^{s+k}) = L(f_i)(s+k+A_i)wf_i^{s+k-1} + \sum_{j:j\neq i} \frac{L(f_j)}{f_j} A_j(w) f_i^{s+k} \qquad (4.12)$$

we see that one can invert this operator inside $\mathcal{D}_X[[s+a_i]]$ every time except $k=0$. So $\Psi_i(M)$ is generated by the elements wf_i^{s-1}, $w \in W$ and all the expressions wf_i^s should be treated as zero. The relation (4.12) gives us also the action of monodromie:

$$S(wf_i^{s-1}) = -swf_i^{s-1} = A_i w f_i^{s-1} \qquad (4.13)$$

From (4.12) we deduce also the action of vector fields L_i, $L_i(f_i) = 0$ on wf_i^{s-1}:

$$L_i(wf_i^{s-1}) = \sum_{j:j\neq i} \frac{L_i(f_j)}{f_j} A_j(w) f_i^{s-1} \qquad (4.14)$$

In order to find morphisms between $\Psi(M)$ and N, we have to rewrite (4.14) in a form (4.11), in other words, to replace all the f_i, codim$X_i = 1$, in denominator of rhs of (4.14) by f_α, codim$X_\alpha = 2$. This may be done by substituting (4.5) and (4.6) into (4.14). Finally we have

$$L_i(wf_i^{s-1}) = \sum_{\alpha:\alpha\leftarrow i} \frac{L_i(f_\alpha)}{f_\alpha} \sum_{\substack{j:j\to\alpha \\ j\neq i}} A_j(w) f_i^{s-1} \qquad (4.15)$$

We can interpret morphisms $\Psi_i(M) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} N$ from Theorem 3.2 in our case as commutative diagramm

$$\begin{array}{ccc} W & \xrightarrow{\sum_{j:j\to\alpha,\, j\neq i} A_j} & W \\ A_{i\theta}^- \Big\downarrow \Big\uparrow A_{\theta i}^+ & & A_{i\theta}^- \Big\downarrow \Big\uparrow A_{\theta i}^+ \\ V_i & \xrightarrow[A_i^\alpha]{} & V_i \end{array} \qquad (4.16)$$

with the relations

$$A^+_{\emptyset i} A^-_{i\emptyset} = A_i, \tag{4.17}$$

$$A^-_{i\emptyset} \left(\sum_{\substack{j:\, j\to\alpha \\ j\neq i}} A_j \right) = A^\alpha_i A^-_{i\emptyset}, \tag{4.18}$$

$$\left(\sum_{\substack{j:\, j\to\alpha \\ j\neq i}} A_j \right) A^+_{\emptyset i} = A^+_{\emptyset i} A^\alpha_i \tag{4.19}$$

Let us look now to the eigenvalues of operators from (4.16)

Lemma 4.1 *Linear operator*

$$\sum_{\substack{j;\, j\to\alpha \\ j\neq i}} A_j$$

has a single eigenvalue equal to $\sum_{\substack{j;\, j\to\alpha \\ j\neq i}} a_j$.

Proof of the Lemma. We use standard linear algebra arguments and the basic relation (4.9). We see from (4.9) that an element $C_\alpha = \sum_{j:\,\alpha\leftarrow j} A_j$ commutes with all the A_i, $\alpha \leftarrow i$. The Jordan block decomposition of W to generalized eigenspaces of C_α is thus consistent with the action of all the A_i, $\alpha \leftarrow i$, because this decomposition can be performed by the action of some analytical functions of C_α. So we can restrict ourselves to a single eigenvalue of C_α and, applying trace arguments, observe that this eigenvalue is equal to $\sum_{j:\,\alpha\leftarrow j} a_j$. The statement of the Lemma now follows from the equality

$$\sum_{\substack{j;\, j\to\alpha \\ j\neq i}} A_j = C_\alpha - A_i, \qquad [C_\alpha, A_i] = 0$$

Let us remind that we can freely change the matrices A_j in (4.10) or A^α_i in (4.11) by adding identiy matrices: it is equivalent to choosing another basis of sections in a vector bundle:$A_j \to A_j + k \Leftrightarrow w \to w f^k_j$, $k \in \mathbf{Z}$ and this is the only gauge freedom we have for the connections with constant coeffitients and singleeigenvalued matrices.

Comparing (4.11) and (4.17) we conclude via Lemma 4.1 that nontrivial morphisms between $\Psi_i(M)$ and N could exist only if the following relation on eigenvalues is valid:

$$a_i^\alpha = \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} a_j \pmod{\mathbf{Z}} \tag{4.20}$$

Supposing the validness of (4.20) we can normalize the realization of N in such a way that (4.20) takes place on the level of complex numbers:

$$a_i^\alpha = \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} a_j. \tag{4.21}$$

If the relation (4.20) is not satisfied then both $A^-_{i\emptyset}$ and $A^+_{\emptyset i}$ are equal to zero and the glued $\mathcal{D}$-module is a direct sum of $\mathcal{D}$-module without singularities along X_i and of $\mathcal{D}$-module, concentrated on X_ia

Let us now realize corresponding $\mathcal{D}$-module in terms of generators and relations. Assume first that $a_i = \text{eig.v.}(A_i) = 0$. Then on the level of generators monada (3.2) looks as follows:

$$\begin{array}{ccccccc} & & & wf_i^s & & \longrightarrow & 0 \\ & & & swf_i^{s-1} \quad wf_i^{s-1} & & & \\ & \overset{s}{\nearrow} & & \oplus & & \overset{id}{\searrow} & \\ wf_i^{s-1} & \xrightarrow{A^-_{i\emptyset}} & & v_i & & \xrightarrow{A^+_{\emptyset i}} & wf_i^{s-1} \end{array} \tag{4.22}$$

so the homology of (4.22) are generated by the elements $\overline{w} = wf_i^s$, $w \in W$ and $\overline{v}_i = -v_i + (A^+_{\emptyset i} v_i) f_i^{s-1}$.

Let us compute the action of vector fields L on $\overline{w}$, L_i on $\overline{v}_i$ and the result of multiplication of $\overline{v}_i$ by f_i. We have

$$L(\overline{w}) = L(f_i)(A^+_{\emptyset i} A^-_{i\emptyset} + s) w f_i^{s-1} + \sum_{j:\, j\neq i} \frac{L(f_j)}{f_j} A_j(w) f_i^s \tag{4.23}$$

because of (4.10). From (4.22) we see that elements $swf_i^{s-1} \oplus A^-_{i\emptyset}(w)$ are boundaries and (4.23) may be rewritten as

$$L(\overline{w}) = L(f_i)(A^+_{\emptyset i} A^-_{i\emptyset} w f_i^{s-1} - A^-_{i\emptyset} w) + \sum_{j:\, j\neq i} \frac{L(f_j)}{f_j} A_j(w) f_i^s$$

or

$$L(\overline{w}) = L(f_i)\overline{A_{i\emptyset}^{-}w} + \sum_{j:\, j\neq i} \frac{L(f_j)}{f_j}\overline{A_j w} \tag{4.24}$$

Next,

$$f_i\overline{v}_i = f_i(-v_i + (A_{\emptyset i}^{+}v_i)f_i^{s-1} = A_{\emptyset i}^{+}(v_i)f_i^{s} = \overline{A_{\emptyset i}^{+}v_i}, \tag{4.25}$$

and, finally, the most difficult calculation:

$$L(\overline{v_i}) = -\sum_{\alpha:\,\alpha\leftarrow i} \frac{L_i(f_\alpha)}{f_\alpha} A_i^\alpha v_i + \sum_{\alpha:\,\alpha\leftarrow i}\ \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{L(f_j)}{f_j} A_j A_{\emptyset i}^{+} v_i f_i^{s-1} \tag{4.26}$$

Using linear dependance condition (4.6) we rewrite (4.26) as

$$L(\overline{v_i}) = \sum_{\alpha:\,\alpha\leftarrow i} \frac{L_i(f_\alpha)}{f_\alpha}\left(-A_i^\alpha v_i + \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} A_j A_{\emptyset i}^{+} v_i f_i^{s-1}\right) +$$

$$+ \sum_{\alpha:\,\alpha\leftarrow i} \frac{L(f_j)}{f_j} \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{<\alpha, j>}{<i,\alpha>} \frac{A_j A_{\emptyset i}^{+}(v_i)}{f_j} f_i^{s} \tag{4.27}$$

Then we substitute the relation (4.19) into (4.27):

$$L(\overline{v_i}) = \sum_{\alpha:\,\alpha\leftarrow i} \frac{L_i(f_\alpha)}{f_\alpha}\left(-A_i^\alpha v_i + A_{\emptyset i}^{+} A_i^\alpha v_i f_i^{s-1}\right) +$$

$$+ \sum_{\alpha:\,\alpha\leftarrow i} \frac{L(f_j)}{f_j} \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{<\alpha, j>}{<i,\alpha>} \frac{A_j A_{\emptyset i}^{+} v_i}{f_j}$$

which means that

$$L(\overline{v_i}) = \sum_{\alpha:\,\alpha\leftarrow i} \frac{L_i(f_\alpha)}{f_\alpha}\left(\overline{A_i^\alpha v_i} + \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{<\alpha, j>}{<i,\alpha>} \frac{\overline{A_j A_{\emptyset i}^{+} v_i}}{f_j}\right) \tag{4.28}$$

We conclude that the glued $\mathcal{D}$-module corresponding to diagram (4.16) is generated by elements $\overline{w}$, $w \in W$ and $\overline{v_i}$, $v_i \in V_i$, and is defined by the relations (4.24), (4.25) and (4.28).

In the case of nonintegral eigenvalue of monodromie S (more precisely, $0 < \mathrm{Re}\, a_i < 1$, $a_i = \text{eig.v.}\, A_i$) we should, due to prescription (3.3), put first in the diagram (4.16) W instead of V_i, identity operator instead of $A^+_{\emptyset i}$, and $A_i - a_i \cdot \mathrm{Id}$ instead of $A^-_{i\emptyset}$ and compute the $\mathcal{D}$-module coming from corresponding monada (4.22). Then we should tensor multiply the result by $j_i^*(f_i^{a_i})$ where $j_i : X \setminus X_i \hookrightarrow X$ being the inclusion.

The generators of the resulting module are $\tilde{w} = \overline{w} \otimes f_i^{a_i}$ and $\tilde{v}_i = \overline{v_i} \otimes f_i^{a_i}$ where $\overline{w}$ and $\overline{v_i}$ are generators of homology of monada; moreover from (4.25) we can treate $\tilde{v}_i$ also as $\overline{w} \otimes f_i^{a_i - 1}$. Then, for instance,

$$L(\tilde{w}) = L(\overline{w}) \otimes f_i^{a_i} + a_i \overline{w} \otimes f_i^{a_i - 1} =$$

$$L(f_i)\overline{(A_i - a_i)w} \otimes f_i^{a_i - 1} + \sum_{j:\, j \neq i} \frac{L(f_j)}{f_j} \overline{A_j w} \otimes f_i^{a_i} + a_i \overline{w} \otimes f_i^{a_i - 1} =$$

$$L(f_i)\overline{A_i w} \otimes f_i^{a_i - 1} + \sum_{j:\, j \neq i} \frac{L(f_j)}{f_j} \overline{A_j w} \otimes f_i^{a_i}$$

and if we denote $A^+_{\emptyset i} = id \otimes f : \overline{w} \otimes f_i^{a_i - 1} \to \overline{w} \otimes f_i^{a_i}$, $A^-_{i\emptyset} = A_i \otimes f_i^{-1} : \overline{w} \otimes f_i^{a_i} \to \overline{w} \otimes f_i^{a_i - 1}$, $A_i^{\alpha} = \sum_{j:\, j \to \alpha,\, j \neq i} A_j \otimes id$ we observe that the relations (4.17)–(4.19) remain unchanged just as defining relations (4.24), (4.25) and (4.28) for the glued $\mathcal{D}$-module.

Moreover, we can decompose invertible operator A_i into some other product $A_i = A^+_{\emptyset i} A^-_{i\emptyset}$ of invertible operators and make a change of variables in the space $\widetilde{V_i}$, identifying $\tilde{v}_i$ with $\overline{A^+_{\emptyset i} v_i} \otimes f_i^{a_i - 1}$. Then $A_i^{\alpha} = (A^+_{\emptyset i})^{-1} \left(\sum_{j:\, j \to \alpha,\, j \neq i} A_j \right) A^+_{\emptyset i}$ and the condition (4.18) follows from (4.9). The defining relations (4.24), (4.25) and (4.28) have the same form due to the rules of changes of variables.

Let M now be some $\mathcal{D}_X$-module (remind once more that X is still $X = U_2 = \mathbf{C}^N \setminus \bigcup_{\alpha:\, \mathrm{codim}\, X_\alpha = 2} \{f_\alpha = 0\}$). The formulas (4.24), (4.25) and (4.28) define restriction of M to open sets $X_2^i = U_2 \bigcap_{j:\, \mathrm{codim}\, X_j = 1,\, j \neq i} \{f_j \neq 0\}$. We can restore M as a sheaf and define it by its global sections. These sections are, due to (4.25)

$$w, \qquad w \in W$$

and

$$v_i = \begin{cases} v_i \ \text{ over } X_2^i \\ f_i^{-1} A^+_{\emptyset i} v_i \ \text{ over } X_2^j, \quad j \neq i \end{cases} \qquad v_i \in V_i$$

In terms of these global sections $\mathcal{D}_X$-module M is described after the renormalization $A^-_{i\emptyset} \to (< f_i >)^{-1} A^-_{i\emptyset}$, $A^+_{\emptyset i} \to < f_i > A^+_{\emptyset i}$ by the following relations on its generators $w \in W$ and $v_i \in V_i$.

$$L(w) = \sum_{i;\,\emptyset \to i} \frac{L(f_i)}{< f_i >} A^-_{i\emptyset} w \tag{4.29}$$

$$L_i(v_i) = \sum_{\alpha:\, i \to \alpha} \frac{L_i(f_\alpha)}{f_\alpha} (A^\alpha_i v_i + \sum_{\substack{j:\, j \to \alpha \\ j \neq i}} \frac{< \alpha j >}{< j >} \cdot \frac{< i >}{< i\alpha >} A^-_{j\emptyset} A^+_{\emptyset i} v_i) \tag{4.30}$$

if $L_i(f_i) = 0$ and

$$f_i v_i = < f_i > A^+_{\emptyset i} v_i \tag{4.31}$$

with operators $A^+_{\emptyset i} : W \to V_i$, $A^-_{i\emptyset} : V_i \to W$ and $A^\alpha_i : V_i \to V_i$ subjected to the relations

$$A^-_{i\emptyset} \left(\sum_{\substack{j:\, j \to \alpha \\ j \neq i}} A^+_{\emptyset j} A^-_{j\emptyset} \right) = A^\alpha_i A^-_{i\emptyset} \tag{4.32}$$

$$\left(\sum_{\substack{j:\, j \to \alpha \\ j \neq i}} A^+_{\emptyset j} A^-_{j\emptyset} \right) A^+_{\emptyset i} = A^+_{\emptyset i} A^\alpha_i \tag{4.33}$$

The results of this subsection may be resumed in the following proposition.

Proposition 4.1 *Let $j_2 : U_2 \hookrightarrow \mathbf{C}^N$ be an inclusion. Then any $\mathcal{D}_{U_2}$-module M from $j_2^*(C^{q_*}_{\mathcal{A}})$ can be defined by the formulas (4.29)-(4.31). Corresponding linear algebra data W, V_i, $A^+_{\emptyset i} : W \to V_i$, $A^-_{i\emptyset} : V_i \to W$ and $A^\alpha_i : V_i \to V_i$ are subjected to the relations (4.32), (4.33).*

Moreover, we have the following restriction on eigenvalues

$$a_i = \text{eig.v.}\,(A^+_{\emptyset i} A^-_{i\emptyset}), \;\; a'_i = \text{eig.v.}\,(A^-_{i\emptyset} A^+_{\emptyset i}), \;\; a^\alpha_i = \text{eig.v.}\, A^\alpha_i :$$

$$a'_i = a_i \qquad a^\alpha_i = \sum_{\substack{j:\, j \to \alpha \\ j \neq i}} a_j$$

which are not valid only if M is a direct sum of a module without singularities on some strata $X_{i_1}, \ldots, X_{i_k}$ and of modules supported on these strata.

4.4 Glueing of codimension 2 strata

Let now

$$X = U_3 = \mathbf{C}^N \setminus \bigcup_{\alpha:\,\mathrm{codim}\, X_\alpha = 3} \{f_\alpha = 0\}.$$

We start from $\mathcal{D}_X$-module M, whose restriction to

$$U_{2,3} = \mathbf{C}^N \setminus \bigcup_{\alpha:\,\mathrm{codim}\, X_\alpha = 2,3} \{f_\alpha = 0\}$$

is given by the relations (4.29) and (4.30). Let X_α be a codimension two stratum. Following the Definition 1.1 we may assume that $\Phi_\alpha(M) \overset{dfn}{=} \Phi_{f_\alpha}(M)$ is generated by vectorspace V_α with the relations

$$L_\alpha(v_\alpha) = \sum_{\lambda:\,\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} A_\alpha^\lambda v_\alpha \tag{4.34}$$

for any vectorfield L_α along X_α and

$$f v_\alpha = 0 \qquad \text{if } f\,|_{X_\alpha} = 0. \tag{4.35}$$

Let us compute $\Psi_\alpha(M) \overset{dfn}{=} \Psi_{f_\alpha}(M\,|_{U_{2,3}})$. Applying arbitrary vector field L, $L(f_\alpha) \neq 0$ to wf_α^s:

$$L(wf_\alpha^s) = sL(f_\alpha)wf_\alpha^{s-1} + \sum_i \frac{L(f_i)}{<i>} A_{i\emptyset}^-(w) f_\alpha^s$$

we see that the only possibility we have is to put wf_α^s to be equal zero in $\Psi_\alpha(M)$ and monodromie operator $S(wf_\alpha^s) = 0$. Analogously, applying $L_i, L_i(f_i) = 0$ to $v_i f_\alpha^s$ we see that

$$v_i f_\alpha^s = 0 \tag{4.36}$$

in $\Psi_\alpha(M)$ and

$$S(v_i f_\alpha^{s-1}) = -s v_i f_\alpha^{s-1} =$$
$$A_i^\alpha v_i f_\alpha^{s-1} + \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{<\alpha j>}{<j>} \cdot \frac{<i>}{<i\alpha>} A_{j\emptyset}^- A_{\emptyset i}^+ v_i f_\alpha^{s-1}. \tag{4.37}$$

The last two statements are based on the following Lemma, which is proved analogously to Lemma 4.1.

Lemma 4.2 *An operator*

$$\bigoplus_{i:\, i\to\alpha}\left(A_i^\alpha + \sum_{\substack{j:\, j\to\alpha\\ j\neq i}} \frac{<\alpha j>}{<j>}\cdot\frac{<i>}{<i\alpha>}A^{-}_{j\emptyset}A^{+}_{\emptyset i}\right)$$

is nilpotent in $\bigoplus_{i:\, i\to\alpha} V_i$ *provided the relations (4.32), (4.33) (4.21) are satisfied.*

Note that for $\mathcal{D}$-modules supported on codimension one strata we also have a basic statement (4.36), because this case reduces to the previous subsection and if such a module is a direct summand of singleeigenvalued module $M\mid_{U_{2,3}}$ then by Definition 1.1 zero monodromie eigenvalue and again the statements (4.36) and (4.37).

We are going now to compute $\mathcal{D}_{\{f_\alpha=0\}}$-module structure of $\Psi_\alpha(M) = \Psi^0_\alpha(M)$ in terms of (4.29)-(4.30). Let $i\to\alpha$ and g_α be some linear function, $g_\alpha\mid_{X_\alpha}=0$, g_α not proportional to f_α. Then f_α, g_α and f_i are linear dependant:

$$g_\alpha <f_\alpha f_i> = f_i <f_\alpha g_\alpha> + f_\alpha <g_\alpha f_i> \tag{4.38}$$

and from (4.31) and (4.36) we have

$$g_\alpha\cdot v_i f_\alpha^{s-1} = \frac{<f_\alpha g_\alpha><f_i>}{<f_\alpha f_i>}A^{+}_{\emptyset i} v_i f_\alpha^{s-1} \tag{4.39}$$

We have also

$$M_\alpha(w f_\alpha^{s-1}) = \sum_i \frac{M_\alpha(f_i)}{<f_i>}A^{-}_{i\emptyset} w f_\alpha^{s-1} \tag{4.40}$$

for any vectorfield M_α along $\{f_\alpha\}=0$: $M_\alpha(f_\alpha)=0$ and

$$L_\alpha(v_i f_\alpha^{s-1}) = \sum_{\substack{\beta:\, i\to\beta\\ \beta\neq\alpha}} \frac{L_\alpha(f_\beta)}{f_\beta}\Big(A_i^\beta v_i + \sum_{\substack{j:\, j\to\beta\\ j\neq i}} \frac{<\beta j>}{<j>}\cdot\frac{<i\beta>}{<i>}A^{-}_{j\emptyset}A^{+}_{\emptyset i} v_i\Big) f_\alpha^{s-1} \tag{4.41}$$

In order to compare the expressions (4.34), (4.40) and (4.41) we have to put together all the terms with $i\to\alpha$ in rhs of (4.40), and for any given stratum X_λ of codimension 3 all the terms with $\beta\to\lambda$ in rhs of (4.41). Using (4.38) again, we obtain first

$$M_\alpha(w f_\alpha^{s-1}) = \sum_{i:\, i\to\alpha} \frac{M_\alpha(g_\alpha)<f_i f_\alpha>}{<g_\alpha f_\alpha><f_i>}A^{-}_{i\emptyset} w f_\alpha^{s-1} + \sum_{j:\, j\not\to\alpha} \frac{M_\alpha(f_j)}{<f_j>}A^{-}_{j\emptyset} w f_\alpha^{s-1} \tag{4.42}$$

We now turn to the relation (4.41). Fix some $i : i \to \alpha$ and $\lambda : \alpha \to \alpha$. let X_β be codimension two stratum with $i \to \beta \to \lambda$ and X_j be codimension one stratum with $j \to \beta$. Then f_i, f_β, f_λ and f_α are linear dependant and

$$f_i < \beta\lambda\alpha > -f_\beta < i\lambda\alpha > +f_\lambda < i\beta\alpha > -f_\alpha < i\beta\lambda >= 0 \tag{4.43}$$

As a consequence we have

$$L_\alpha(f_\beta) = L_\alpha(f_\lambda)\frac{< i\beta\alpha >}{< i\lambda\alpha >} \tag{4.44}$$

and

$$\frac{1}{f_\alpha f_\beta} = \frac{< i\lambda\alpha >}{< i\beta\alpha >}\cdot\frac{1}{f_\alpha f_\lambda} + \frac{< i\beta\lambda >}{< i\beta\alpha >}\cdot\frac{1}{f_\beta f_\lambda} - \frac{< \beta\lambda\alpha >}{< i\beta\alpha >}\cdot\frac{f_i}{f_\alpha f_\beta f_\lambda} \tag{4.45}$$

We have also linear dependance of f_j, f_β, f_λ and f_α and thus the conditions (4.43)–(4.45) with index i replaced by j. Substituting (4.43)–(4.45) and their analogs for f_j, f_β, f_λ and f_α into (4.41)and using linear dependance conditions for f_i, f_j and f_β we obtain after some calculations the following expression:

$$L_\alpha(v_i f_\alpha^{s-1}) =$$

$$= \sum_{\lambda;\alpha\to\lambda}\sum_{\substack{\beta:i\to\beta\to\lambda\\ \beta\neq\alpha}}\frac{L_\alpha(f_\lambda)}{f_\lambda}\left(A_i^\beta - \sum_{\substack{j:j\to\beta\\ j\neq i}}\frac{< j\lambda\alpha >< i >}{< i\lambda\alpha >< j >}A_{j\theta}^- A_{\theta i}^+\right)v_i f_\alpha^{s-1} -$$

$$- \sum_{\lambda;\alpha\to\lambda}\sum_{\substack{\beta:i\to\beta\to\lambda\\ \beta\neq\alpha}}\frac{L_\alpha(f_\lambda) < i >}{f_\lambda f_\beta}\left(A_{\theta i}^+ A_i^\beta - \sum_{\substack{j:j\to\beta\\ j\neq i}}A_{\theta j}^+ A_{j\theta}^- A_{\theta i}^+\right)v_i f_\alpha^{s-1}$$

The last summand is equal to zero in accordance with (4.33). Finally we have the following identity in $\Psi_\alpha(M)$:

$$L_\alpha(v_i f_\alpha^{s-1}) =$$

$$= \sum_{\lambda;\alpha\to\lambda}\sum_{\substack{\beta:i\to\beta\to\lambda\\ \beta\neq\alpha}}\frac{L_\alpha(f_\lambda)}{f_\lambda}\left(A_i^\beta + \sum_{\substack{j:j\to\beta\\ j\neq i}}\frac{< i >< j\lambda\alpha >}{< j >< \lambda i\alpha >}A_{j\theta}^- A_{\theta i}^+\right)v_i f_\alpha^{s-1} \tag{4.46}$$

Now, using (4.39), (4.42) and (4.46) we can represent canonical morphisms $\Psi_\alpha(M) \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} \Phi_\alpha(M)$ in terms of the following commutative diagram

$$\begin{array}{ccccc} W & \underset{\widehat{A^+_{\theta i}}}{\overset{\widehat{A^-_{i\theta}}}{\rightleftharpoons}} & \bigoplus_{i:\, i\to\alpha} V_\alpha & \xrightarrow{\oplus_{i:\, i\to\alpha} \sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} A_i^\beta} & \bigoplus_{i:\, i\to\alpha} V_\alpha \\ & & \widehat{A^-_{\alpha i}} \Big\downarrow \Big\uparrow \widehat{A^+_{i\alpha}} & & \widehat{A^-_{\alpha i}} \Big\downarrow \Big\uparrow \widehat{A^+_{i\alpha}} \\ & & V_\alpha & \xrightarrow{A_\alpha^\lambda} & V_\alpha \end{array} \tag{4.47}$$

where $\widehat{A^-_{i\theta}} = \frac{<\alpha i>}{<i><g_\alpha f_\alpha>} A^-_{i\theta}$, $\widehat{A^+_{\theta i}} = \frac{<i><g_\alpha f_\alpha>}{<\alpha i>}$ and $\widehat{A^-_{\alpha i}} = \frac{<i>}{<\alpha i>} A^-_{\alpha i}$, $\widehat{A^+_{i\alpha}} = \frac{<\alpha i>}{<i>} A^+_{i\alpha}$ with monodromie operator S:

$$S(w) = 0, \qquad S(v_i) = A_i^\alpha v_i + \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{<\alpha j><i>}{<j><i\alpha>} A^-_{j\theta} A^+_{\theta i} v_i$$

(just as in the case of codimension one we renormalize operators $A^+_{i\alpha}$ and $A^-_{\alpha i}$ to avoid coefficients in quiver relations).

Thus we have the relations

$$A^+_{j\alpha} A^-_{\alpha i} + A^-_{j\theta} A^+_{\theta i} = 0 \qquad i \neq j, \quad \overline{X_i} \cap \overline{X_j} = \overline{X_\alpha} \tag{4.48}$$

$$\sum_{i:\, i\to\alpha} A^-_{\alpha i} A^-_{i\theta} = 0 \tag{4.49}$$

$$\sum_{i:\, i\to\alpha} A^+_{\theta i} A^+_{i\alpha} = 0 \tag{4.50}$$

$$A^-_{\alpha i} \left(\sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} A_i^\beta \right) = A_\alpha^\lambda A^-_{\alpha i} \tag{4.51}$$

for a fixed flag $i \to \alpha \to \lambda$,

$$\left(\sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} A_i^\beta \right) A^+_{i\alpha} = A^+_{i\alpha} A_\alpha^\lambda \tag{4.52}$$

for the same flag $i \to \alpha \to \lambda$,

$$A^{+}_{i\alpha}A^{-}_{\alpha i} = A^{\alpha}_{i} \tag{4.53}$$

if $i \to \alpha$.

Let us describe the glued $\mathcal{D}$-module in terms of generators and relations.The differential in the monada (3.2) is given by the formulas

$$d_0(wf_\alpha^{s-1}) = swf_\alpha^{s-1} \qquad d_0(v_i f_\alpha^{s-1}) = sv_i f_\alpha^{s-1} \oplus \frac{< i >}{< \alpha i >} A^{-}_{\alpha i}(v_\alpha),$$

$$d_1(wf_\alpha^{s-1}) = wf_\alpha^{s-1}, \qquad d_1(v_i f_\alpha^{s-1}) = v_i f_\alpha^{s-1}$$

and

$$d_1(v_\alpha) = \frac{< \alpha i >}{< i >} A^{+}_{i\alpha}(v_\alpha) f_\alpha^{s-1}$$

The homologies of monada (3.2) are generated by the elements

$$\overline{w} = wf_\alpha^s, \qquad \overline{v_i} = v_i f_\alpha^s \qquad \text{and } \overline{v_\alpha} = -v_\alpha + \sum_{i:\, i\to\alpha} \frac{< \alpha i >}{< i >} A^{+}_{i\alpha} v_\alpha f_\alpha^{s-1}$$

We can easily see that just as in (4.29) and (4.31),

$$L(\overline{w}) = \sum_{i;\,\emptyset\to i} \frac{L(f_i)}{< f_i >} \overline{A^{-}_{i\emptyset} w}, \qquad w \in W \tag{4.54}$$

and

$$f_i \overline{v_i} = < f_i > \overline{A^{+}_{\emptyset i} v_i}, \qquad v_i \in V_i \tag{4.55}$$

Further,

$$L_i(\overline{v_i}) = L_i(f_\alpha) \left(A^{\alpha}_{i} + s + \sum_{\substack{j:\, j\to\alpha \\ j\neq i}} \frac{< \alpha j >}{< j >} \cdot \frac{< i >}{< i\alpha >} A^{-}_{j\emptyset} A^{+}_{\emptyset i} \right) v_i f_\alpha^{s-1} +$$

$$+ \sum_{\substack{\beta:\, i\to\beta \\ \beta\neq\alpha}} \frac{L_i(f_\beta)}{f_\beta} \left(A^{\beta}_{i} + \sum_{\substack{j:\, j\to\beta \\ j\neq i}} \frac{< \beta j >}{< j >} \cdot \frac{< i >}{< i\beta >} A^{-}_{j\emptyset} A^{+}_{\emptyset i} \right) v_i f_\alpha^{s} \tag{4.56}$$

We replace $sv_i f_\alpha^{s-1}$ in rhs of (4.56) by $-\frac{<i>}{<\alpha i>} A_{\alpha i}^{-}(v_\alpha)$ and using (4.48) and (4.52) we reduce (4.56) to

$$L_i(\overline{v_i}) = L_i(f_\alpha)\frac{<i>}{<\alpha i>}\overline{A_{\alpha i}^{-} v_i}+$$

$$+\sum_{\substack{\beta: i\to\beta \\ \beta\neq\alpha}} \frac{L_i(f_\beta)}{f_\beta}\left(\overline{A_i^\beta v_i} + \sum_{\substack{j: j\to\beta \\ j\neq i}} \frac{<\beta j>}{<j>}\cdot\frac{<i>}{<i\beta>}\overline{A_{j\beta}^{-} A_{\beta i}^{+} v_i}\right) \tag{4.57}$$

Let now linear function $g(x)$ is such that $g\ |_{X_\alpha} = 0$. Then, using linear dependance condition for g, f_α and f_i, if $i\to\alpha$, we have

$$g(\overline{v_\alpha}) = g(-v_\alpha + \sum_{i: i\to\alpha} \frac{<\alpha i>}{<i>} A_{i\alpha}^{+} v_\alpha f_\alpha^{s-1}) = \sum_{i: i\to\alpha} \frac{<\alpha i>}{<i>} g(A_{i\alpha}^{+} v_\alpha) f_\alpha^{s-1} =$$

$$= \sum_{i: i\to\alpha} \frac{<gf_i>}{<i>} A_{i\alpha}^{+} v_\alpha f_\alpha^{s} + <gf_\alpha> \sum_{i: i\to\alpha} A_{\emptyset i}^{+} A_{i\alpha}^{+} v_\alpha f_\alpha^{s-1}$$

The last sum is equal to zero, due to (4.50) so finally

$$g(\overline{v_\alpha}) = \sum_{i: i\to\alpha} \frac{<gf_i>}{<i>}\overline{A_{i\alpha}^{+} v_\alpha} \tag{4.58}$$

The calculations of $L_\alpha(\overline{v_\alpha})$ are the longest ones. The computations use linear dependance conditions (4.43)–(4.45), the defining relations (4.48)–(4.52) and simple identities for determinants. The answer is

$$L_\alpha(\overline{v_\alpha}) = \sum_{\lambda: \alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda}\left(\overline{A_\alpha^\lambda v_\alpha} + \sum_{i: i\to\alpha}\sum_{\substack{\beta: i\to\beta\to\lambda \\ \beta\neq\alpha}} \frac{<i\beta\lambda>}{<i\lambda\alpha>}\left(\frac{<\alpha i>}{<i>}\frac{\overline{A_i^\beta A_{i\alpha}^{+} v_\alpha}}{f_\beta}\right.\right.$$

$$\left.\left.+\sum_{\substack{j: j\to\beta \\ j\neq i}} \frac{<\beta j><\alpha i>}{<i\beta><j>}\frac{\overline{A_{j\beta}^{-} A_{\beta i}^{+} A_{i\alpha}^{+} v_\alpha}}{f_\beta}\right)\right) \tag{4.59}$$

The relations (4.54), (4.57), (4.58) and (4.59) define $\mathcal{D}$-module in the open set

$$U_3^\alpha = \left(\mathbb{C}^N \setminus \bigcup_{\beta: \operatorname{codim} X_\beta = 2,3} \{f_\beta = 0\}\right) \cup \{f_\alpha = 0\}$$

We can restore $\mathcal{D}_X$-module M as a sheaf by its restrictions to U_3^α and define it by the global sections (remind that now $X = U_3 = \mathbf{C}^N \setminus \bigcup_{\alpha:\,\mathrm{codim}\, X_\alpha = 3}\{f_\alpha = 0\}$). There are three types of sections: w, v_i, codim$X_i = 1$ and v_α, codim$X_\alpha = 2$. the first two are identified with $\overline{w}$ and $\overline{v_i}$ for all U_3^α and

$$v_\alpha = \begin{cases} \overline{v_\alpha} \text{ over } U_3^\alpha \\ \sum_{i:\, i\to\alpha} \frac{<\alpha i>}{<i>} \frac{\overline{A_{i\alpha}^+ v_\alpha}}{f_\alpha} \text{ over } U_3^\beta,\ \beta \neq \alpha \end{cases}$$

The relation (4.59) can be tranformed in a form

$$L_\alpha(\overline{v_\alpha}) = \sum_{\lambda:\,\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} \left(\overline{A_\alpha^\lambda v_\alpha} + \sum_{i:\,i\to\alpha} \sum_{\substack{\beta:\,i\to\beta\to\lambda \\ \beta\neq\alpha}} \frac{< i\beta\lambda >< \alpha i >}{< i\lambda\alpha >< \beta i >} \cdot \right.$$

$$\left. \left(\frac{< \beta i > A_{i\beta}^+}{< i > f_\beta} \overline{A_{\beta i}^- A_{i\alpha}^+ v_\alpha} + \sum_{\substack{j:\,j\to\beta \\ j\neq i}} \frac{< \beta j > A_{i\beta}^+}{< j > f_\beta} \overline{A_{\beta i}^- A_{i\alpha}^+ v_\alpha} \right) \right.$$

which means that

$$L_\alpha(v_\alpha) = \sum_{\lambda:\,\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} \left(A_\alpha^\lambda v_\alpha + \sum_{\substack{\beta:\,\beta\to\lambda \\ \beta\neq\alpha}} \frac{< i\beta\lambda >< \alpha i >}{< i\lambda\alpha >< \beta i >} A_{\beta i}^- A_{i\alpha}^+ v_\alpha \right) \tag{4.60}$$

where i is an index of codimension one stratum X_i whose closure contains both X_α and X_β. There are two possibilities: there exists unique such X_i for given X_α and X_β and there are no. In the second case corresponding summand in (4.60) is treated as zero.

Using an identities on polyvectors in $\mathbf{C}^N/\overline{X_\alpha}$ and in $\mathbf{C}^N/\overline{X_\beta}$:

$$< \vec{f_\alpha} > f_\alpha \wedge f_i =< \alpha i > \vec{f_a}, \qquad < \vec{f_\beta} > f_\alpha \wedge f_i =< \beta i > \vec{f_b}$$

(for the notations see section 2.2) we can rewrite (4.60) as

$$L_\alpha(v_\alpha) = \sum_{\lambda:\,\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} \left(A_\alpha^\lambda v_\alpha - \sum_{\substack{\beta:\,\beta\to\lambda \\ \beta\neq\alpha}} \frac{< \lambda\vec{\beta} >}{< \lambda\vec{\alpha} >} \cdot \frac{< \vec{\alpha} >}{< \vec{\beta} >} A_{\beta i}^- A_{i\alpha}^+ v_\alpha \right) \tag{4.61}$$

The transformation of the formula (4.57) is evident (see relation (4.63)).

Summarizing the calculations we conclude that $\mathcal{D}_X$-module M over $X = U_2$ can be defined by generators w, $w \in W$, v_i, $v_i \in V_i$ ($\mathrm{codim} X_i = 1$) and $v_\alpha\, v_\alpha \in V_\alpha$ ($\mathrm{codim} X_\alpha = 2$) with the relations

$$L(w) = \sum_{i;\,\emptyset \to i} \frac{L(f_i)}{< f_i >} A^-_{i\emptyset} w, \qquad w \in W \tag{4.62}$$

$$L_i(v_i) = \sum_{\alpha:\, i \to \alpha} \frac{< i >}{< \alpha i >} L_i(f_\alpha) A^-_{\alpha i} v_i, \qquad v_i \in V_i \tag{4.63}$$

if L_i is a vector field along X_i,

$$f_i v_i = A^+_{\emptyset i} v_i, \qquad v_i \in V_i, \tag{4.64}$$

$$g(v_\alpha) = \sum_{i:\, i \to \alpha} \frac{< g f_i >}{< i >} A^+_{i\alpha} v_\alpha, \qquad v_\alpha \in V_\alpha \tag{4.65}$$

for any linear function $g(x)$ such that $g\,|_{X_\alpha} = 0$,

$$L_\alpha(v_\alpha) = \sum_{\lambda:\, \alpha \to \lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} \left(A^\lambda_\alpha v_\alpha - \sum_{\substack{\beta:\, \beta \to \lambda \\ \beta \neq \alpha}} \frac{< \lambda \vec{\beta} >}{< \lambda \vec{\alpha} >} \cdot \frac{< \vec{\alpha} >}{< \vec{\beta} >} A^-_{\beta i} A^+_{i\alpha} v_\alpha \right), \tag{4.66}$$

($v_\alpha \in V_\alpha$) for any vector field L_α along X_α with i being an index of codimension one stratum X_i whose closure contains both X_α and X_β.

The linear operators $A^+_{\emptyset i} : W \to V_i$, $A^-_{i\emptyset} : V_i \to W$, $A^-_{\alpha i} : V_i \to V_\alpha$, $A^+_{i\alpha} : V_\alpha \to V_i$ and $\mathcal{A}^l_\alpha : V_\alpha \to V_\alpha$ satisfy the following conditions:

$$A^+_{j\alpha} A^-_{\alpha i} + A^-_{j\emptyset} A^+_{\emptyset i} = 0 \qquad i \neq j, \quad \overline{X_i} \cap \overline{X_j} = \overline{X_\alpha} \tag{4.67}$$

$$\sum_{i:\, i \to \alpha} A^-_{\alpha i} A^-_{i\emptyset} = 0 \tag{4.68}$$

$$\sum_{i:\, i \to \alpha} A^+_{\emptyset i} A^+_{i\alpha} = 0 \tag{4.69}$$

$$A^-_{\alpha i} \left(\sum_{\substack{\beta:\, i \to \beta \to \lambda \\ \beta \neq \alpha}} A^+_{i\beta} A^-_{\beta i} \right) = A^\lambda_\alpha A^-_{\alpha i} \tag{4.70}$$

for a fixed flag $i \to \alpha \to \lambda$,

$$\left(\sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} A^+_{i\beta} A^-_{\beta i} \right) A^+_{i\alpha} = A^+_{i\alpha} A^\lambda_\alpha \tag{4.71}$$

for the same flag $i \to \alpha \to \lambda$.

Remark 4.1 *The relations (4.32), (4.33) and also (4.9) follow from (4.67)–(4.68) if we remind that* $A^\alpha_i = A^+_{i\alpha} A^-_{\alpha i}$, $A_i = A^+_{\beta i} A^-_{i\beta}$.

Let us look now to the eigenvalues of operators $A^+_{i\alpha} A^-_{\alpha i}$, $A^-_{\alpha i} A^+_{i\alpha}$ and A^λ_α. Let $a^\alpha_i = \text{eig.v.}(A^+_{i\alpha} A^-_{\alpha i})$, $a^{\alpha\prime}_i = \text{eig.v.}(A^-_{\alpha i} A^+_{i\alpha})$, $a^\lambda_\alpha = \text{eig.v.}(A^\lambda_\alpha)$. We can apply Lemma 4.1 to a local system over hyperplane $\{f_\alpha = 0\}$. As a result we have the following

Lemma 4.3 *Linear operator*

$$\sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} A^+_{i\beta} A^-_{\beta i}$$

has a single eigenvalue equal to $\sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} a^\beta_i$.

Now we see that the morphisms in the diagramm (4.54) are nontrivial only if for any flag $i \to \alpha \to \lambda$ we have

$$a^\lambda_\alpha = \sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} a^\beta_i \pmod{\mathbf{Z}} \tag{4.72}$$

again we can renormilize the local system on codimension two strata in such a way that instead of (4.72) we have

$$a^\lambda_\alpha = \sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} a^\beta_i \tag{4.73}$$

on the level of complex numbers.

If the relation (4.73) is not satisfied then both $A^+_{i\alpha}$ and $A^+_{\beta i}$ are equal to zero and the glued $\mathcal{D}$-module is a direct sum of $\mathcal{D}$-module without singularities along X_α and of $\mathcal{D}$-module, concentrated on X_α. We resume the results of this subsection in the following proposition.

Proposition 4.2 *Let $j_3 : U_3 \hookrightarrow \mathbf{C}^N$ be an inclusion. Then any $\mathcal{D}_{U_3}$-module M from $j_3^*(C_{\mathcal{A}}^{q^s})$ can be defined by the formulas (4.62)-(4.66). Corresponding linear algebra data W, V_i, V_α, $A^+_{\emptyset i} : W \to V_i$, $A^-_{i\emptyset} : V_i \to W$, $A^-_{\alpha i} : V_\alpha \to V_i$, $A^+_{i\alpha} : V_i \to V_\alpha$ and $A^\lambda_\alpha : V_\alpha \to V_\alpha$ are subjected to the relations (4.67)-(4.71).*

Moreover, we have the following restriction on eigenvalues

$$a_i^\alpha = \text{eig.v.}(A^+_{i\alpha}A^-_{\alpha i}), \quad a_i^{\alpha\prime} = \text{eig.v.}(A^-_{\alpha i}A^+_{i\alpha}), \quad a_\alpha^\lambda = \text{eig.v.}(A^\lambda_\alpha)$$

for any flag $i \to \alpha \to \lambda$:

$$a_i^{\alpha\prime} = a_i^\alpha \qquad\qquad a_\alpha^\lambda = \sum_{\substack{\beta:\, i\to\beta\to\lambda \\ \beta\neq\alpha}} a_i^\beta$$

which are not valid only if M is a direct sum of a module without singularities on some codimension two strata $X_{\alpha_{i_1}}, \ldots, X_{\alpha_{i_k}}$ and of modules supported on these strata.

5 General induction step

Here we give a precise formulation of general induction statement. We omit the calculations supposing the reader can find enough of them in the previous section. The only difference is that in general case one should more often use identities with polyvectors $\vec{f_\alpha}$ instead of identities with linear functions. The main induction statement looks as follows.

Let

$$X = U_n = \mathbf{C}^N \setminus \bigcup_{\alpha:\, \operatorname{codim} X_\alpha = n} \{f_\alpha = 0\}$$

and let the restriction of singleeigenvalued $\mathcal{D}_X$-module M to an open subset

$$U_{n-1,n} = \mathbf{C}^N \setminus \bigcup_{\alpha:\, \operatorname{codim} X_\alpha = n-1,n} \{f_\alpha = 0\}$$

can be described as a $\mathcal{D}$-module generated by vector space $\oplus_\beta V_\beta$, $\operatorname{codim} X_\beta < n$ with the relations

$$L_\beta(v_\beta) = \sum_{\lambda:\, \beta\to\lambda} \frac{< \vec{f_\beta} >}{< f_l \vec{f_\beta} >} L_\beta(f_\lambda) A^-_{\lambda\beta}(v_\beta) =$$

$$= \sum_{\lambda:\beta\to\lambda} \frac{<\vec{f_\beta}>}{<\vec{f_\lambda}>} \frac{L_\beta(\vec{f_\lambda})}{\vec{f_\beta}} A^-_{\lambda\beta}(v_\beta), \qquad v_\beta \in V_\beta \tag{5.1}$$

if L_β is a linear vector field along stratum X_β, $\operatorname{codim} X_\beta < n-1$,

$$f \cdot v_\beta = \sum_{\gamma:\gamma\to\beta} \frac{<f, \vec{f_\gamma}>}{<\vec{f_\gamma}>} A^+_{\gamma\beta}(v_\beta), \qquad v_\beta \in V_\beta \tag{5.2}$$

if f is a linear function, $f\,|_{X_\beta} = 0$, $\operatorname{codim} X_\beta < n$ and

$$L_\alpha(v_\alpha) = \sum_{\lambda:\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} \left(A^\lambda_\alpha v_\alpha - \sum_{\substack{\beta:\beta\to\lambda \\ \beta\neq\alpha}} \frac{<\lambda\vec{\beta}>}{<\lambda\vec{\alpha}>} \cdot \frac{<\vec{\alpha}>}{<\vec{\beta}>} A^-_{\beta\gamma} A^+_{\gamma\alpha} v_\alpha \right), \tag{5.3}$$

($v_a \in V_\alpha$), codim $X_\alpha = n-1$ for any vector field L_α along X_α with γ being an index of codimension $n-2$ stratum X_γ whose closure contains both X_α and X_β.

The operators $A^-_{\lambda\alpha} : V_\alpha \to V_\lambda$, $A^+_{\alpha\lambda} : V_\lambda \to V_\alpha$, $\operatorname{codim} X_\alpha < n-1$ and $A^\lambda_\alpha : V_\alpha \to V_\alpha$, $\operatorname{codim} X_\alpha = n-1$ satisfy the following relations:

$$\sum_{\beta:\lambda\leftarrow\beta\leftarrow\gamma} A^-_{\lambda\beta} A^-_{\beta\gamma} = 0 \tag{5.4}$$

for any two vertices $\lambda, \gamma,$: $\lambda < \gamma, n > \operatorname{codim} X_\lambda = \operatorname{codim} X_\gamma + 2$,

$$\sum_{\beta:\gamma\to\beta\to\lambda} A^+_{\gamma\beta} A^+_{\beta\lambda} = 0 \tag{5.5}$$

for any two vertices $\lambda, \gamma,$: $\lambda < \gamma, n > \operatorname{codim} X_\lambda = \operatorname{codim} X_\gamma + 2$,

$$A^+_{\beta\lambda} A^-_{\lambda\mu} + A^-_{\beta\gamma} A^+_{\gamma\mu} = 0 \tag{5.6}$$

for any quadruple $\beta \swarrow\!\!\!\!\!\nwarrow\; \gamma \searrow\; \mu \;\swarrow \lambda$ (with $\beta \leftarrow \gamma \to \mu$, $\beta \to \lambda \leftarrow \mu$), $n > \operatorname{codim} X_\lambda$;

$$A^+_{\beta\lambda} A^-_{\lambda\mu} = 0 \tag{5.7}$$

for any triple $\beta\searrow_{\lambda}\swarrow\mu$, $n > \mathrm{codim}X_\lambda$ with no γ such that $\beta\swarrow^{\gamma}\searrow\mu$;

$$A^-_{\alpha\gamma}\left(\sum_{\substack{\beta:\gamma\to\beta\to\lambda\\ \beta\neq\alpha}} A^+_{\gamma\beta}A^-_{\beta\gamma}\right) = A^\lambda_\alpha A^-_{\alpha\gamma} \tag{5.8}$$

for any two vertices λ,γ,: $\lambda<\gamma, n=\mathrm{codim}X_\lambda=\mathrm{codim}X_\gamma+2$;

$$\left(\sum_{\substack{\beta:\gamma\to\beta\to\lambda\\ \beta\neq\alpha}} A^+_{\gamma\beta}A^-_{\beta\gamma}\right)A^+_{\gamma\alpha} = A^+_{\gamma\alpha}A^\lambda_\alpha \tag{5.9}$$

for any two vertices λ,γ,: $\lambda<\gamma, n=\mathrm{codim}X_\lambda=\mathrm{codim}X_\gamma+2$.

There is also the following eigenvalues restriction. $\mathcal{D}_{U_{n-1,n}}$-module $M\mid_{U_{n-1,n}}$ can be decomposed into direct sum

$$M\mid_{U_{n-1,n}} = M^{(1)}\oplus M^{(2)}$$

For the first module $M^{(1)}$ we have an equality

$$a^\lambda_\alpha = \sum_{\substack{\beta:\gamma\to\beta\to\lambda\\ \beta\neq\alpha}} a^\beta_\gamma \tag{5.10}$$

for any flag $\gamma\to\alpha\to\lambda$: $\lambda<\gamma$, $\mathrm{codim}X_\alpha = n-1$, where

$$a^\alpha_\gamma = \text{eig.v.}\,(A^+_{\gamma\alpha}A^-_{\alpha\gamma}) = \text{eig.v.}\,(A^-_{\alpha i}A^+_{i\alpha}),\quad a^\lambda_\alpha = \text{eig.v.}\,(A^\lambda_\alpha)$$

The second module $M^{(2)}$ has a support on codimension $n-1$ strata (and thus is realized by (5.1)–(5.9) with all V_β equal zero for β: $\mathrm{codim}\,X_\beta < n-1$) and

$$0 \leq \mathrm{Re}\, a^\lambda_\alpha < 1 \tag{5.11}$$

for any two α,λ, $\alpha\to\lambda$, $\mathrm{codim}\,X_\alpha = n-1$.

Proposition 5.1 *In assumption of (5.1)–(5.11) $\mathcal{D}_X$-module M is generated by vector space $\oplus_\beta V_\beta$, codim $X_\beta \leq n$ with the relations (5.1)–(5.3) and operators $A^-_{\lambda\alpha}: V_\alpha\to V_\lambda$, $A^+_{\alpha\lambda}: V_\lambda\to V_\alpha$, codim $X_\alpha < n$ and $A^\lambda_\alpha : V_\alpha\to V_\alpha$, codim $X_\alpha = n$ subjected to (5.4)–(5.9).*

The spaces V_α and operators $A^-_{\lambda\alpha} : V_\alpha \to V_\lambda$, $A^+_{\alpha\lambda} : V_\lambda \to V_\alpha$, $\operatorname{codim} X_\alpha < n-1$ are the same as for $M\,|_{U_{n-1,n}}$ and $A^+_{\alpha\lambda}A^-_{\lambda\alpha} = A^l_\alpha$ for all α, $\operatorname{codim} X_\alpha = n-1$.

Moreover, $\mathcal{D}_X$-module M can be decomposed into direct sum $M = M^{(1)} \oplus M^{(2)}$ For the first module $M^{(1)}$ we have an equality

$$a^\lambda_\alpha = \sum_{\substack{\beta:\, \gamma\to\beta\to\lambda \\ \beta\neq\alpha}} a^\beta_\gamma \tag{5.12}$$

for any flag $\gamma \to \alpha \to \lambda$: $\lambda < \gamma$, $\operatorname{codim} X_\alpha = n$, where

$$a^\alpha_\gamma = \text{eig.v.}\,(A^+_{\gamma\alpha}A^-_{\alpha\gamma}) = \text{eig.v.}\,(A^-_{\alpha i}A^+_{i\alpha}), \quad a^\lambda_\alpha = \text{eig.v.}\,(A^\lambda_\alpha)$$

The second module $M^{(2)}$ has a support on codimension n strata.

The proof of the proposition is based on the calculation of the functor $\Psi_{f_\alpha}(M\,|_{U_{n-1,n}})$ for a stratum X_α of codimension n.

Let us fix some codimension n stratum X_α. For any $\mathcal{D}_X$-module M we denote for simplicity of notations $\Psi_\alpha(M) \overset{dfn}{=} \Psi_{f_\alpha}(M\,|_{U_{n-1,n}})$ for simplicity of notations. The computation of $\Psi_\alpha(M)$ for $M\,|_{U_{n-1,n}}$ being supported on codimension $n-1$ strata reduces to the codimension one case and was completely described in subsection 4.3. Let us consider the case when the condition (5.10) is satisfied. Then we state that

(i) $\Psi_\alpha(M)$ is generated by elements $v_\beta f^{s-1}_\alpha$, $v_\beta \in V_\beta$, $\operatorname{codim} X_\beta < n$ with $v_\beta f^s_\alpha$ being treated as zero elements;

(ii) Canonical monodromie operator $S = (-s)\cdot$ is nilpotent on $\Psi_\alpha(M)$;

(iii) An action of S on $\Psi_\alpha(M)$ is described by the relations

$$S v_\beta f^{s-1}_\alpha = 0 \qquad \text{if } \operatorname{codim} X_\beta < n-1 \tag{5.13}$$

$$S v_\beta f^{s-1}_\alpha = -s v_\beta f^{s-1}_\alpha =$$

$$= A^\alpha_\beta v_\beta f^{s-1}_\alpha - \sum_{\substack{\delta:\, \delta\to\alpha \\ \delta\neq\beta}} \frac{<\alpha\vec{\delta}>}{<\vec{\delta}>} \cdot \frac{<\vec{\beta}>}{<\alpha\vec{\beta}>} A^-_{\delta\gamma}A^+_{\gamma\beta} v_\beta f^{s-1}_\alpha. \tag{5.14}$$

if $\operatorname{codim} X_\beta = n-1$.

Note that the statement *(ii)* follows from the following counterpart of Lemma 4.2:

Lemma 5.1 *An operator*

$$\bigoplus_{\beta:\beta\to\alpha}\left(A^{\alpha}_{\beta}-\sum_{\substack{\delta:\delta\to\alpha\\ \delta\neq\beta}}\frac{<\alpha\vec{\delta}>}{<\vec{\delta}>}\cdot\frac{<\vec{\beta}>}{<\alpha\vec{\beta}>}A^{-}_{\delta\gamma}A^{+}_{\gamma\beta}\right) \tag{5.15}$$

is nilpotent in $\bigoplus_{\beta:\beta\to\alpha}V_\beta$ *provided the relations (5.4)–(5.10) are satisfied.*

Now, considering $\Psi(M)$ as $\mathcal{D}_{\{f_\alpha=0\}}$-module we see that it can be described in terms of relations (5.1)–(5.3) if we take $v_\beta f_\alpha^{s-1}$ as generators. The graph structure of strata $X_\beta\cap\{f_\alpha=0\}$, $\operatorname{codim}X_\beta<n$ remains unchanged except the strata X_β, $\beta\to\alpha$. Instead of their intersection with hyperplane $\{f_\alpha=0\}$ we should consider the only stratum X_α and attach to this stratum vectorspace $\bigoplus_{\beta:\beta\to\alpha}V_\beta$. The formulas which we need for drawing a commutative diagramm representing canonical fourtuple $\Psi_\alpha(M)\underset{v}{\overset{u}{\rightleftarrows}}\Phi_\alpha(M)$ are as follows.

Let us fix some new deneric linear function $g_\alpha(x)$ which cut stratum X_α in hyperplane $\{f_\alpha=0\}$: $g_\alpha\mid_{X_\alpha}=0$, $g\not\sim f_\alpha$. Then we have

$$M_{\alpha,\gamma}(v_\gamma f_\alpha^{s-1})=\sum_{\beta:\gamma\to\beta\to\alpha}M_{\alpha,\gamma}(g_\alpha)\frac{<\vec{\gamma}><\alpha\beta\gamma>}{<\beta\vec{\gamma}><f_\alpha g_\alpha\vec{f_\gamma}>}A^{-}_{\beta\gamma}v_\gamma f_\alpha^{s-1}+$$

$$+\sum_{\substack{\beta':\gamma\to\beta'\\ \beta'\not\to\alpha}}M_{\alpha,\gamma}(f_{\beta'})\frac{<\vec{\gamma}>}{<\beta'\vec{\gamma}>}A^{-}_{\beta'\gamma}v_\gamma f_\alpha^{s-1} \tag{5.16}$$

for any stratum X_γ, $\operatorname{codim}X_\gamma=n-2$, $\gamma>\alpha$ and for any linear vector field $M_{\alpha,\gamma}$ along $\{f_\alpha=0\}\cap X_\gamma$: $M_{\alpha,\gamma}(f_\alpha)=0$, $M_{\alpha,\gamma}(f)=0$ if $f\mid_{\{f_\alpha=0\}\cap X_\gamma}=0$,

$$g_\alpha\cdot v_\beta f_\alpha^{s-1}=\sum_{\gamma:\gamma\to\beta}\frac{<f_\alpha g_\alpha\vec{f_\gamma}>}{<\alpha\beta\vec{\gamma}>}\frac{<\vec{\gamma}>}{<\beta\vec{\gamma}>}A^{+}_{\gamma\beta}v_\beta f_\alpha^{s-1} \tag{5.17}$$

for any stratum X_β, $\operatorname{codim}X_\gamma=n-1$, $\beta\to\alpha$ and

$$L_\alpha(v_\beta f_\alpha^{s-1})=$$

$$\sum_{\lambda:\alpha\to\lambda}\sum_{\substack{\delta:\beta\to\delta\to\lambda\\ \delta\neq\alpha}}\frac{L_\alpha(f_\lambda)}{f_\lambda}\left(A^{\delta}_{\beta}-\sum_{\substack{\gamma:\gamma\to\delta\\ \gamma\neq\beta}}\frac{<\vec{\beta}><\lambda\alpha\vec{\gamma}>}{<\vec{\gamma}><\lambda\alpha\vec{\beta}>}A^{-}_{\gamma\mu}A^{+}_{\mu\beta}\right)v_i f_\alpha^{s-1} \tag{5.18}$$

also for a stratum X_β, $\operatorname{codim} X_\gamma = n-1$, $\beta \to \alpha$ with μ being an index of codimension $(n-2)$ stratum X_μ whose closure contains both X_β and X_γ and L_α being a vector field along X_α.

Suppose now that $\Phi_\alpha(M)$ is given as singleeigenvalued local system generated by vectorspace V_α with the relations

$$L_\alpha(v_\alpha) = \sum_{\lambda:\,\alpha\to\lambda} \frac{L_\alpha(f_\lambda)}{f_\lambda} A_\alpha^\lambda v_\alpha \tag{5.19}$$

for any linear vector field L_α along X_α and

$$f v_\alpha = 0 \qquad \text{if} \quad f\,|_{X_\alpha} = 0. \tag{5.20}$$

Comparing the relations (5.16)–(5.18) with (5.19)–(5.20) we observe that the diagram $\Psi_\alpha(M) \underset{v}{\overset{u}{\rightleftarrows}} \Phi_\alpha(M)$ is completely defined by linear maps (which we renormalize to avoid factors in commutation relations):

$$\bigoplus_{\beta;\,\beta\to\alpha} {A_{\alpha\beta}^{-}}' : \bigoplus_{\beta;\,\beta\to\alpha} V_\beta \longrightarrow V_\alpha, \tag{5.21}$$

$$\bigoplus_{\beta;\,\beta\to\alpha} {A_{\beta\alpha}^{+}}' : V_\alpha \longrightarrow \bigoplus_{\beta;\,\beta\to\alpha} V_\beta \tag{5.22}$$

with

$$A_{\alpha\beta}^{-} = \frac{< \alpha\vec{\beta} >}{< \vec{\beta} >} {A_{\alpha\beta}^{-}}' \tag{5.23}$$

and

$$A_{\beta\alpha}^{+} = \frac{< \vec{\beta} >}{< \alpha\vec{\beta} >} {A_{\beta\alpha}^{+}}' \tag{5.24}$$

satisfying the relations (5.4), (5.5), (5.8) and (5.9). An equality $S = vu$ provides the relations $A_\beta^\alpha = A_{\beta\alpha}^{+} A_{\alpha\beta}^{-}$ for all β; $\beta \to \alpha$ and (5.6), (5.7). Just as in sections 4.3 and 4.4 we observe also that the relation (5.10) on eigenvalues is not satisfied only if the glued module is a direct sum of a module without singularities along X_α and of a module supported on X_α for which one may freely assume the conditions (5.11). Next, applying monada (3.2) technique we describe the glued module in terms of generators and relations and finally restore M as a sheaf by its restrictions to open sets

$$U_{n-1,n} \cup \{f_\alpha = 0\}$$

for all α: $\operatorname{codim} X_\alpha = n$. The calculations analogous to that of sections 4.3 and 4.4 show that M is defined by the formulas (5.1)-(5.3).

We have described a scetch of proof of Proposition 5.1. Theorems 2.1 and 2.4 are direct consequences of Proposition 5.1. The only thing which we want to explain is a simple remark that an operator

$$\bigoplus_{\beta:\beta\to\alpha} \left(A^\alpha_\beta - \sum_{\substack{\delta:\delta\to\alpha \\ \delta\neq\beta}} \frac{<\alpha\vec{\delta}>}{<\vec{\delta}>} \cdot \frac{<\vec{\beta}>}{<\alpha\vec{\beta}>} A^-_{\delta\gamma} A^+_{\gamma\beta} \right)$$

from lemma 5.1 is conjugated to

$$\bigoplus_{\substack{\beta,\delta: \\ \delta\to\alpha,\beta\to\alpha}} A^+_{\delta\alpha} A^-_{\alpha\beta} \tag{5.25}$$

in $\bigoplus_{\beta:\beta\to\alpha} V_\alpha$ provided the relations (5.6) and (5.7) are satisfied. The nilpotence of operator (5.25) for the vertices of depth more then one follows from the nilpotence of monodromie operator S in the second step of glueing. The decomposition of a restriction of $\mathcal{D}$-module to some open set U_n to a direct sum of $\mathcal{D}_{U_n}$-modules if a condition (5.10) is not satisfied contradicts to the definition of local indecomposability. But if these conditions are satisfied we conclude from Lemma 5.1 that $\Psi_{f_\alpha}(M\ |_{U_{n-1,n}})$ coincides with its unipotent part $\Psi^0_{f_\alpha}(M\ |_{U_{n-1,n}})$ for all α: $\operatorname{codim} X_\alpha > 1$ and we have no need in singleeigenvalued restriction (1.1) on $\Psi_{f_\alpha}(M)$. So we have the proof of the second part of Proposition 1.1 and of the Theorem 2.3. We can also easily see by induction that for category $\mathcal{C}^0_{\mathcal{A}}$ the conditions of Lemma 5.1 are automatically satisfied (all the eigenvalues remain being equal zero). So we have no need in the condition (1.1) for $\mathcal{C}^0_{\mathcal{A}}$, from where we deduce the rest of Proposition 1.1 and can also prove an equivalence of categories in Theorem 2.2 .

Acknowledgements

This text was written when the author was a visitor in Max-Planck Institut für Mathematik and he uses an opportunity to express his best regards to the Institute. The author is sincerely grateful to Vadim Schechtman for numerous fruitful and stimulating discussions.

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