Deninger's conjecture on L-functions of elliptic curves at s=3.

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Abstract

I compute the regulator map on $K_4(X)$ for an arbitrary curve X over a number field. Using this and Beilinson's theorem about regulators for modular curves I prove a formula expressing the value of the L-function $L(E, s)$ of a modular elliptic curve E over $\mathbb{Q}$ at $s = 3$ by the double Eisenstein-Kronecker series.

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1 Introduction

1. In this paper I compute explicitly the regulator map on $K_4(X)$ for an arbitrary curve X over a number field. Using this and Beilinson's theorem about regulators for modular curves ([B2]) I prove a formula expressing the value of the L-function $L(E, s)$ of a modular elliptic curve E over $\mathbb{Q}$ at $s = 3$ by the double Eisenstein-Kronecker series. It was conjectured by C. Deninger [D1].

2. Generalized Eisenstein-Kronecker series. Let E be an elliptic curve and $\Gamma := H_1(E(\mathbb{C}), \mathbb{Z})$. Choose a holomorphic 1-form ω. It defines an embedding $\Gamma \hookrightarrow \mathbb{C}$ together with an isomorphism $E(\mathbb{C}) = \mathbb{C}/\Gamma = \Gamma \otimes \mathbb{R}/\Gamma$. The Poincare duality provides a nondegenerate pairing $\Gamma \times \Gamma \longrightarrow \mathbb{Z}(1)$. Let

$$(\cdot, \cdot) : E(\mathbb{C}) \times \Gamma \longrightarrow \mathbb{R}(1)/\mathbb{Z}(1) = U(1) \subset \mathbb{C}^*$$

be the corresponding pairing. If $\Gamma = \mathbb{Z}u + \mathbb{Z}v \subset \mathbb{C}$ with $Im(u/v) > 0$ then $(z, \gamma) = \exp A(\Gamma)^{-1}(z\bar{\gamma} - \bar{z}\gamma)$ where $A(\Gamma) = \frac{1}{2\pi i}(\bar{u}v - u\bar{v})$.

Let $x, y, z \in E(\mathbb{C})$ and $n \geq 3$. The function

$$K_n(x,y,z) := \sum_{\gamma_1+\ldots+\gamma_n=0}{}' \frac{(x,\gamma_1)(y,\gamma_2+\ldots+\gamma_{n-1})(z,\gamma_n)(\bar{\gamma}_n-\bar{\gamma}_{n-1})}{|\gamma_1|^2|\gamma_2|^2\ldots|\gamma_n|^2} \quad (1)$$

will be called generalized Eisenstein-Kronecker series. It is invariant under the shift $(x, y, z) \to (x+t, y+t, z+t)$ and so lives actually on $E(\mathbb{C}) \times E(\mathbb{C})$.

To formulate the results I have first to recall the definition of

4. The group $B_2(F)$. Let F be a field and $\mathbb{Z}[F^*]$ be the free abelian group generated by symbols $\{x\}$ where $x \in F^*$. Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^*]$ generated by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(x_1,\ldots,\hat{x}_i,\ldots,x_5)\} \quad (2)$$

where $x_1, \ldots, x_5$ run through all 5-tuples of distinct F-points of P^1. By definition $B_2(F) := \mathbb{Z}[F^*]/R_2(F)$. Symbol $\{x\}_2$ denotes the projection of $\{x\}$ to $B_2(F)$.

One can show that the formulas $\delta : \{x\} \longmapsto (1-x) \wedge x$; $\quad \{1\} \longmapsto 0$ provide us with a homomorphism of groups $\delta : B_2(F) \longrightarrow \wedge^2 F^*$. (In other words $\delta(R_2(F)) = 0$). This is one of the most important properties of the group $B_2(F)$.

4. Special values of L-functions. Let $v_x : \mathbb{Q}(E)^* \longrightarrow \mathbb{Z}$ be the valuation defined by a point $x \in E(\bar{\mathbb{Q}})$. Denote by f_E the conductor of E. Let $\omega \in H^0(E, \Omega^1_{E/\mathbb{R}})$, Denote by $\Omega = \int_{E(\mathbb{R})}\omega$ the real period of E.

Theorem 1.1 *Let E be a modular elliptic curve over $\mathbb{Q}$. Then there exist rational functions $f_i, g_i \in \mathbb{Q}(E)^*$ satisfying the conditions:*

$$\sum_i (1-f_i) \wedge f_i \wedge g_i = 0 \quad in \quad \Lambda^3\mathbb{Q}(E)^* \quad (3)$$

$$\sum_i v_x(g_i)\{f_i(x)\}_2 = 0 \quad in \quad B_2(\bar{\mathbb{Q}}) \quad for\ any \quad x \in E(\bar{\mathbb{Q}}) \quad (4)$$

such that

$$L(E,3) = q\Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^2\Omega \cdot \sum_i \sum_{\gamma_1+\gamma_2+\gamma_3=0}{}' \frac{(x_i,\gamma_1)(y_i,\gamma_2)(z_i,\gamma_3)(\bar{\gamma}_3-\bar{\gamma}_2)}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2} \quad (5)$$

where q is a non-zero rational number and x_i, y_i, z_i are the divisors of the functions $g_i, f_i, 1-f_i$ respectively.

It is interesting that the right-hand side of (5) depends only on the divisors of the functions $g_i, f_i, 1-f_i$.

A similar formula expressing $L(E,2)$ for a modular elliptic curve over $\mathbb{Q}$ by the classical Eisenstein-Kronecker series

$$L(E,2) = q\Big(\frac{2\pi A(\Gamma)}{f_E}\Big)\Omega \cdot \sum_i \sum_\gamma{}' \frac{(x_i - y_i, \gamma)\bar{\gamma}}{|\gamma|^4}$$

was known thanks to Bloch and Beilinson [Bl1], [B1].

A formula (5) for an arbitrary elliptic curve E over $\mathbf{Q}$ in a slightly different form was conjectured by C.Deninger ([D1]), who used Massey products in Deligne cohomology to guess a formula for $L(E,3)$. We do not use Massey products in the formulation or proof of the theorem.

One can define ([G2]) for an arbitrary field F an abelian group

$$\mathcal{B}_n(F) := \mathbb{Z}[P^1_F]/\mathcal{R}_n(F)$$

together with a homomorphism

$$\mathcal{B}_n(F) \xrightarrow{\delta} B_{n-1}(F) \otimes F^* \qquad \{x\}_n \longmapsto \{x\}_{n-1} \otimes x$$

I will recall the definition of $\mathcal{R}_n(F)$ in chapter 3 below. Roughly speaking it is the "connected component of zero" of Kerδ. One can show that $\mathcal{R}_n(\mathbb{C})$ is the subgroup of all functional equations for the classical n-logarithm, see [G2].

Conjecture 1.2 *Let E be an elliptic curve over $\mathbf{Q}$. Then there exist rational functions $f_i, g_i \in \mathbb{Q}(E)^*$ satisfying the condition $(n > 3)$*

$$\sum_i \{f_i\}_{n-2} \otimes f_i \wedge g_i \in \mathcal{B}_{n-2}(\mathbb{Q}(E)) \otimes \Lambda^2 \mathbb{Q}(E)^* \qquad (6)$$

$$\sum_i v_x(g_i)\{f_i(x)\}_{n-1} = 0 \quad \textit{in} \quad \mathcal{B}_{n-1}(\bar{\mathbb{Q}}) \quad \textit{for any} \quad x \in E(\bar{\mathbb{Q}}) \qquad (7)$$

such that

$$q \cdot L(E,n) = \Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^{n-1} \Omega \cdot \sum_i K_n(x_i, y_i, z_i) \qquad (8)$$

where q is a non-zero rational number and x_i, y_i, z_i are the divisors of the functions $g_i, f_i, 1-f_i$.

Conjecture 1.3 *For any $f_i, g_i \in \mathbb{Q}(E)^*$ satisfying the conditions of theorem (1.1) (resp. conjecture (1.2)) one has (8) with $q \in \mathbb{Q}$.*

Beilinson's conjecture on L-functions permits to formulate a similar conjecture for an elliptic curve over any number field F in which we have in the right hand side a determinant whose entries are the functions $K_n(x, y, z)$.

Remark. If $n > 2$ then for a regular proper model $E_{\mathbb{Z}}$ of E over $Spec(\mathbb{Z})$ one has $gr_n^{\gamma} K_{2n-2}(E_{\mathbb{Z}}) = gr_n^{\gamma} K_{2n-2}(E)$, so, unlike to the $n = 2$ case, we don't have to worry about the "integrality condition".

5. Explicit formulas for the regulators for curves. Let X be a curve over $\mathbb{R}$ and $n > 1$. Then the real Deligne cohomology $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$ equals $H^1(X/\mathbb{R}, \mathbb{R}(n-1))$. Further, cup product with $\omega \in \Omega^1(\bar{X})$ provides an isomorphism of vector spaces over $\mathbb{R}$:

$$H^1(X/\mathbb{R}, \mathbb{R}(n-1)) \longrightarrow H^0(\bar{X}, \Omega^1)^{\vee}$$

So we will present elements of $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$ as functionals on $H^0(\bar{X}, \Omega^1)^{\vee}$.

In chapter 3 we prove the following explicit formulas for the regulators

$$r_{\mathcal{D}}(3) : K_4(X) \longrightarrow H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(3))$$

which generalize the famous symbol on $K_2(X)$ of Beilinson and Deligne ([B3], [Del]). (For simplicity we formulate results for curves over $\mathbb{Q}$).

I will use the notation

$$\alpha(f, g) := \log|f| d\log|g| - \log|g| d\log|f| \tag{9}$$

Theorem 1.4 *Let X be a regular curve over $\mathbb{Q}$. Then for each element $\gamma_4 \in K_4(X)$ there are rational functions $f_i, g_i \in \mathbb{Q}(X)$ satisfying the conditions (3) and (4) such that for any $\omega \in \Omega^1(\bar{X})$ one has*

$$\int_{X(\mathbb{C})} r_{\mathcal{D}}(3)(\gamma_4) \wedge \omega = \int_{X(\mathbb{C})} \log|g_i| \alpha(1 - f_i, f_i) \wedge \omega$$

The proof of theorem (1.4) is based on the results of [G2], [G3].

The regulator map on a certain subgroup of $K_4^{(3)}$ of curves over number fields was also computed by R. de Jeu [J].

In general the Beilinson regulator is a map

$$r_{\mathcal{D}}(n+1) : K_{2n}(X) \longrightarrow H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n+1))$$

We expect the following to be true.

Conjecture 1.5 *Let X be a nonsingular curve over $\mathbb{Q}$. Then for each element $\gamma_{2n} \in K_{2n}(X)$ there are rational functions $f_i, g_i \in \mathbb{Q}(X)$ satisfying the conditions (6) and (7) such that for any $\omega \in \Omega^1(\bar{X})$*

$$\int_{X(\mathbf{C})} r_{\mathcal{D}}(n+1)(\gamma_{2n}) \wedge \omega = c_{n+1} \cdot \sum_i \int_{X(\mathbf{C})} \log|g_i| \log^{n-2}|f_i| \alpha(1-f_i, f_i) \wedge \omega$$

where $c_{n+1} \in \mathbb{Q}^*$ is a certain explicitly computable constant.

Moreover, one can prove that condition (3), or respectively (6) if $n > 3$, implies that the right-hand side of these formulas depends only on the divisors of the functions $f_i, g_i, 1-f_i$. When X is an elliptic curve this together with Fourier transform and Beilinson's theorem ([B2]) lead to formulas for $L(E,n)$ from theorem 1.1 and conjectures 1.2 - 1.3.

The proof of this conjecture for $K_6(X)$ will be published in [G4].

In chapter 4 we will see that conjecture (1.5) follows from the main conjecture in [G2] which tells us that the complexes $\Gamma(F,n)$ constructed there catch all of the rational algebraic K-theory of an *arbitrary* field F.

The crucial role in the proof of these results is played by the classical n-logarithms $Li_n(z) = \int_0^z Li_{n-1}(t) d\log t$. The single-valued version of the n-logarithm is the following function ([Z1]):

$$\mathcal{L}_n(z) \quad := \quad \begin{matrix} \text{Re} & (n:\ \text{odd}) \\ \text{Im} & (n:\ \text{even}) \end{matrix} \left(\sum_{k=0}^{n} \beta_k \log^k|z| \cdot Li_{n-k}(z) \right), \quad n \geq 2$$

Here $\beta_k = B_k \cdot 2^k/k!$ and B_k are Bernoulli numbers: $\sum_{k=0}^{\infty} \beta_k x^k = \frac{2x}{e^{2x}-1}$.

One can show (see proposition (4.6)) that for any functions f_i, g_i satisfying (7) one can write the regulator integrals

$$\int_{X(\mathbf{C})} \log|g_i| \log^{n-2}|f_i| \alpha(1-f_i, f_i) \wedge \omega = b_{n+1} \cdot \sum_i \int_{X(\mathbf{C})} \mathcal{L}_n(f_i) d\log|g_i| \wedge \omega$$

where b_{n+1} are certain explicitly computable non zero rational constant.

6. The structure of the paper. Let $\mathcal{O}$ be a local ring with infinite residue field. In chapter 2 we will construct homomorphisms

$$K_{6-i}^{[i]}(\mathcal{O})_{\mathbf{Q}} \longrightarrow H^i(\Gamma(\mathcal{O},3)) \tag{10}$$

where $K_n^{[i]}(\mathcal{O})$ are the graded quotients of the rank filtration on Quillen's K-groups of the ring $\mathcal{O}$. Hypothetically modulo torsion it is opposit to the Adams filtration.

Now let X be a curve over a number field F. Then we define a complex $\Gamma(X,3)$ and homomorphisms

$$K_{6-i}^{[i]}(X)_{\mathbb{Q}} \longrightarrow H^i(\Gamma(X,3)) \tag{11}$$

We impose the condition that F is a number field only because of one argument "ad hoc" in proof which is based on the Borel theorem. Hopefully after some modification the proof should work for an arbitrary field F.

In chapter 3 we will prove that the composition of this map with the natural map from $\Gamma(X,3)$ to Deligne cohomology coincides with Beilinson's regulator.

In the end of chapter 3 and in chapter 4 we compute the regulator integrals for curves. In particular we show that conjecture 1.5 essentially follows from the main conjecture of [G1] on the structure of motivic complexes. Then we apply these results to elliptic curves and get the generalized Eisenstein-Kronecker series. Therefore we finish the proof of theorem 1.1 and deduce conjecture 1.2 from conjecture 1.5.

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2 Motivic complexes for a curve and algebraic K-theory

1 The weight 3 motivic complex . We will call by this name the complex $\Gamma(X,3)$ introduced in [G1-G2] for an arbitrary regular scheme X. If $X = Spec(F)$ where F is an *arbitrary* field, it looks as follows:

$$B_3(F) \longrightarrow B_2(F) \otimes F^* \longrightarrow \Lambda^3 F^* \tag{12}$$

Here $B_3(F) := \mathbb{Z}[F^*]/R_3(F)$ where the subgroup $R_3(F)$ will be defined below, after theorem (2.3). The group $B_3(F)$ is placed in degree 1. The differential has degree $+1$ and is defined as follows:

$$\{x\}_3 \longmapsto \{x\}_2 \otimes x; \quad \{x\}_2 \otimes y \longmapsto (1-x) \wedge x \wedge y$$

In [G1] we have constructed homomorphisms of groups

$$c_{2,3} : K_4(F) \otimes \mathbb{Q} \longrightarrow H^2\Gamma(Spec(F);3) \otimes \mathbb{Q}$$

We will recall the definition of this homomorphism below.

The goal of this chapter is to get a similar homomorphism for curves over number fields.

2. The complex $\Gamma(X;3)$ for a curve X over a field F. Let K be an arbitrary field with discrete valuation v and residue class k_v. The group of units U has a natural homomorphism $U \longrightarrow k_v^*$, $u \mapsto \bar{u}$. An element $\pi \in K^*$ is prime if $\mathrm{ord}_v \pi = 1$.

Let us define the residue homomorphism

$$\partial_v : \Gamma(K,3) \longrightarrow \Gamma(k_v,2)[-1] \tag{13}$$

There is a homomorphism $\theta_n : \Lambda^n K^* \longrightarrow \Lambda^{n-1} k_v^*$ uniquely defined by the properties ($u_i \in U$):

$\theta_n(\pi \wedge u_1 \wedge \cdots \wedge u_{n-1}) = \bar{u}_1 \wedge \cdots \wedge \bar{u}_{n-1}$ and $\theta_n(u_1 \wedge \cdots \wedge u_n) = 0$.

It clearly does not depend on the choice of π. Let us define a homomorphism $s_v : \mathbb{Z}[K^*] \longrightarrow \mathbb{Z}[k_v^*]$ as follows

$$s_v\{x\} = \left\{ \begin{array}{ll} \{\bar{x}\} & \text{if } x \text{ is a unit} \\ 0 & \text{otherwise} \end{array} \right\}$$

Then it induces a homomorphism $s_v : B_2(K) \longrightarrow B_2(k_v)$ (see s. 9 §1 of [G1]). We get a homomorphism

$$s_v \otimes \theta_1 : B_2(K) \otimes K^* \longrightarrow B_2(k_v)$$

Let us consider the following map ∂_v of complexes:

$$\begin{array}{ccccc} B_3(K) & \xrightarrow{\delta} & B_2(K) \otimes K^* & \xrightarrow{\delta} & \Lambda^3 K^* \\ & & \downarrow s_v \otimes \theta_1 & & \downarrow \theta_3 \\ & & B_2(k_v) & \xrightarrow{\delta} & \Lambda^2 k_v^* \end{array} \tag{14}$$

The maps ∂_v define a homomorphism of complexes, see s. 14 of §1 in [G1].

Let $\mathcal{O}_x$ be the local ring at the point $x \in X$ and k_x the residue field.

By definition $\Gamma(X;3)$ is the total complex associated with the bicomplex

$$\begin{array}{ccccc} B_3(F(X)) & \xrightarrow{\delta} & B_2(F(X)) \otimes F(X)^* & \xrightarrow{\delta} & \Lambda^3 F(X)^* \\ & & \downarrow \partial & & \downarrow \partial \\ & & \coprod_{x \in X_1} B_2(k_x) & \xrightarrow{\delta} & \coprod_{x \in X_1} \Lambda^2 k_x^* \end{array}$$

Here $\partial = \coprod_{x \in X_1} \partial_x$, where ∂_x is the residue homomorphism related to the valuation on $F(X)$ corresponding to the point x; the very left group placed in degree 1 and the differentials have degree $+1$.

3. The key result. The main point is to show that homomorphism $c_{i,3}$ carries the residue map in Quillen K-theory to the one on Γ-complexes.

One has the exact localization sequence

$$\longrightarrow K_n(X) \longrightarrow K_n(F(X)) \xrightarrow{\tilde{\delta}} \coprod_{x \in X_1} K_{n-1}(k_x) \longrightarrow \tag{15}$$

where X_1 is the set of all codimension one points of a scheme X and $\tilde{\delta}$ is the residue homomorphism in the Quillen K-theory.

So keeping in mind the localization sequence we see that in order to construct a homomorphism of groups

$$K_4(X) \otimes \mathbb{Q} \longrightarrow H^2(\Gamma(X;3) \otimes \mathbb{Q})$$

the only thing we have to prove is the following

Theorem 2.1 *The diagram*

$$\begin{array}{ccc} K_4(F(X)) & \xrightarrow{c_{2,3}} & H^2\Gamma(Spec(F(X));3)_{\mathbb{Q}} \\ \downarrow \tilde{\delta} & & \downarrow \delta \\ K_3(k_x) & \xrightarrow{c_{1,2}} & H^1\Gamma(Spec(k_x);2)_{\mathbb{Q}} \end{array}$$

is commutative.

Proof. $K_4(F(X))$ is generated by $K_4(\mathcal{O}_x)$ and $K_3(k_x) \cdot F(X)^*$ where $\cdot$ is the product in Quillen's K-theory. So we have to show that

a) *The composition*

$$K_4(\mathcal{O}_x) \longrightarrow K_4(F(X)) \xrightarrow{c_{2,3}} H^2\Gamma(Spec(F(X));3)_{\mathbb{Q}} \xrightarrow{\delta} H^1\Gamma(Spec(k_x);2)_{\mathbb{Q}}$$

is zero.

b) The statement of the theorem is true for the subgroup $K_3(k_x) \cdot F(X)^*$.

The rest of this chapter is devoted to the proof of statement a). The statement b) is proved in s. 4.7 below.

4. Proof of a): the beginning. In this section we suppose that $\mathcal{O}$ is a local ring with an infinite residue field k. Let $\mathcal{O}^\infty$ be the free $\mathcal{O}$-module with the basis $e_1, ..., e_n, ...$ and $\mathcal{O}^n$ be the submodule with the basis $e_1, ..., e_n$. A vector $v \in \mathcal{O}^n$ will be identified with the corresponding column of height n. By definition a set of vectors $v_1, ..., v_m \in \mathcal{O}^n$ is jointly unimodular if the matrix $(v_1, ..., v_m)$ is left invertible in $M_{nm}(\mathcal{O})$. Any projective module over $\mathcal{O}$ is free, so one can show that any jointly unimodular set of vectors can be completed to a basis of $\mathcal{O}^n$.

Let V be a free $\mathcal{O}$-module of rank n and $v_1, ..., v_m \in V$. We will say that the vectors v_i are in general position if any $min(n, m)$ of them are jointly unimodular. This notion is independent of choice of a basis in V.

Let $\tilde{C}_k(\mathcal{O}^n)$ be the free abelian group generated by $k+1$-tuples of vectors in generic position in $\mathcal{O}^n$. They form a complex $\tilde{C}_*(\mathcal{O}^n)$ with the differential d given by the usual formula

$$d : \tilde{C}_m(\mathcal{O}^n) \to \tilde{C}_{m-1}(\mathcal{O}^n); \quad d : (v_1, \dots, v_{m+1}) \mapsto \sum_{i=1}^{m+1} (-1)^{i-1} (v_1, \dots, \hat{v}_i, \dots, v_{m+1}) \; .$$

This complex is acyclic in degrees bigger then 0 and so is a resolution of the trivial $GL_n(\mathcal{O})$-module $\mathbb{Z}$. The group $GL_n(\mathcal{O})$ acts on $\tilde{C}_k(\mathcal{O}^n)$.

Configurations of m vectors in $\mathcal{O}^n$ are m-tuples of vectors considered modulo $GL(\mathcal{O}^n)$-equivalence.

We get a complex $C_*(\mathcal{O}^n)$. One has canonical homomorphism

$$H_i(GL_n(\mathcal{O}) \longrightarrow H_i(C_*(\mathcal{O}^n))$$

Set

$$p_2 : \mathbb{Z}[F^*] \longrightarrow B_2(F) \qquad p_3 : \mathbb{Z}[F^*] \longrightarrow B_3(F)$$
$$B_2(\mathcal{O}) := p_2(\mathbb{Z}[\mathcal{O}^*]) \qquad B_2(\mathcal{O}) := p_3(\mathbb{Z}[\mathcal{O}^*])$$

Then one has complexes

$$B_2(\mathcal{O}) \longrightarrow \Lambda^2 \mathcal{O}^*$$
$$B_3(\mathcal{O}) \longrightarrow B_2(\mathcal{O}) \otimes \mathcal{O}^* \longrightarrow \Lambda^3 \mathcal{O}^*$$

which are subcomplexes of $\Gamma(Spec(K), 2)$ and $\Gamma(Spec(K), 3)$. Let us construct a homomorphism of complexes

$$\begin{tikzcd}
{} \arrow[r] & C_5(\mathcal{O}^3) \arrow[r] \arrow[d, "f_6(3)"] & C_4(\mathcal{O}^3) \arrow[r] \arrow[d, "f_5(3)"] & C_3(\mathcal{O}^3) \arrow[d, "f_4(3)"] \\
& B_3(\mathcal{O}) \arrow[r] & B_2(\mathcal{O}) \otimes \mathcal{O}^* \arrow[r] & \Lambda^3 \mathcal{O}^*
\end{tikzcd}$$

We will use the following notation. For any n vectors $v_1, ..., v_n$ in $\mathcal{O}^n$ set $\Delta(v_1, ..., v_n) := det(v_1, ..., v_n)$ where $det(v_1, ..., v_n)$ is the $n \times n$ matrix formed by the columns of coordinates of vectors v_i in the canonical basis $e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1)$. Notice that vectors $v_1, ..., v_n$ are in generic position (= jointly unimodular) if and only if $\Delta(v_1, ..., v_n) \in \mathcal{O}^*$

Let $\mathrm{Alt}_n\ f(v_1, \ldots, v_n) := \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(v_{\sigma(1)}, \ldots, f_{\sigma(n)})$. Set

$$f_4(3) : (v_1, \ldots, v_4) \mapsto \mathrm{Alt}_4\ \Delta(v_1, v_2, v_3) \wedge \Delta(v_1, v_2, v_4) \wedge \Delta(v_1, v_3, v_4)$$

$$f_5(3)(v_1, \ldots, v_5) := \frac{1}{2}\mathrm{Alt}_5(\{r(v_1|v_2, \ldots, v_5)\}_2 \otimes \Delta(v_1, v_2, v_3)) \ .$$

Here $(v_1|v_2, \ldots, v_5)$ is the configuration of four vectors in $V/ < v_1 >$ obtained by the projection of vectors $v_2, ..., v_5$. We take then the cross-ratio of the corresponding points on the projective line. Now put

$$f_6(3) : (v_1, \ldots, v_6) \mapsto \frac{1}{15}\mathrm{Alt}_6 \left\{ \frac{\Delta(v_1, v_2, v_4)\Delta(v_2, v_3, v_5)\Delta(v_3, v_1, v_6)}{\Delta(v_1, v_2, v_5)\Delta(v_2, v_3, v_6)\Delta(v_3, v_1, v_4)} \right\} \quad (16)$$

Definition 2.2 *The subgroup $R_3(F) \subset \mathbb{Z}[F^*]$ is generated by the elements* $\sum_{i=1}^{7} (-1)^i f_6(3)(v_1, ..., \hat{v}_i, ..., v_7)$.

Theorem 2.3 *a) $f_4(3)$ and $f_5(3)$ do not depend on the choice of ω.*

b) The homomorphisms $f_*(3)$ provide a morphism of complexes.

Proof. See the appendix.

5. What remains to be done. Just by the construction we have a commutative diagram where the vertical arrows are the natural inclusions:

$$\begin{array}{ccccc} C_*(\mathcal{O}^3) & \xrightarrow{f_*(3)} & \Gamma(\mathcal{O}; 3) & \xrightarrow{\delta} & 0 \\ \downarrow & & \downarrow & & \downarrow \\ C_*(K^3) & \xrightarrow{f_*(3)} & \Gamma(K; 3) & \xrightarrow{\delta} & \Gamma(k; 2) \end{array}$$

Thus we have constructed a homomorphism

$$H_4(GL_3(\mathcal{O})) \longrightarrow H^2\Gamma(\mathcal{O}; 3) \quad (17)$$

such that the composition

$$H_4(GL_3(\mathcal{O})) \longrightarrow H_4(GL_3(K)) \xrightarrow{c_{2,3}} H^2\Gamma(K; 3)_{\mathbf{Q}} \xrightarrow{\delta} H^1\Gamma(k; 2)_{\mathbf{Q}}$$

is equal to zero.

To complete the part a) of our program we have to do the stabilisation, i.e. for any $n > 3$ to extend the homomorphism (17) to a homomorphism

$$H_4(GL_n(\mathcal{O})) \longrightarrow H^2\Gamma(\mathcal{O};3)$$

which fits into a commutative diagram

$$\begin{array}{ccc} H_4(GL_n(\mathcal{O})) & \longrightarrow & H^2\Gamma(\mathcal{O};3) \\ \downarrow & & \downarrow \\ H_4(GL_n(K)) & \longrightarrow & H^2\Gamma(K;3) \end{array}$$

This will be done in the next three sections.

6. The bi-Grassmannian complex over a field ([G2]) The bicomplex

$$\begin{array}{ccccccccc} & & & & & & \downarrow & & \downarrow \\ & & & & & \xrightarrow{d} & C_7(5) & \xrightarrow{d} & C_6(5) \\ & & & & \downarrow d' & & \downarrow d' & & \downarrow d' \\ & & & \xrightarrow{d} & C_7(4) & \xrightarrow{d} & C_6(4) & \xrightarrow{d} & C_5(4) \\ & & \downarrow d' & & \downarrow d' & & \downarrow d' & & \downarrow d' \\ \xrightarrow{d} & & C_7(3) & \xrightarrow{d} & C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \end{array} \qquad (18)$$

where

$$d' : (l_1, \dots, l_m) \mapsto \sum_{i=1}^{m} (-1)^{i-1} (l_i | l_1, \dots, \hat{l}_i, \dots, l_m)$$

is the weight three Grassmannian bicomplex. Denote by $(BC_*(3), \partial)$ the corresponding total complex. We place $C_4(3)$ in degree 3 and ∂ has degree -1. We define a map of complexes $\psi_*(3)$:

$$\begin{array}{ccccc} BC_5(3) & \longrightarrow & BC_4(3) & \longrightarrow & BC_3(3) \\ \Big\downarrow \psi_5(3) & & \Big\downarrow \psi_4(3) & & \Big\downarrow \psi_3(3) \\ B_3(F) & \longrightarrow & B_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^* \end{array} \qquad (3.19)$$

by setting it to be zero on the groups $C_*(k)$ for $k > 3$ and using the formulas above for the map on the subcomplex $C_*(3)$.

Theorem 2.4 *The map ψ is a morphism of complexes.*

Proof. See chapter 3 in [G2].

7. Complex of affine flags over a field ([G3], §3). A p-flag in a vector space V is a sequence of subspaces

$$0 = L^0 \subset L^1 \subset L^2 \subset ... \subset L^p \qquad dimL^i = i$$

An affine p-flag is a p-flag $L^\bullet$ together with a choice of vectors $l^i \in L^i/L^{i-1}$ for all $1 \leq i \leq p$. We will denote affine p-flags as $(l_1, ..., l_p)$.

Several affine p-flags are in general position if all the corresponding subspaces L^i are in generic position.

Let $A^p(m)$ be the manifold of all affine p-flags in an m-dimensional vector space V^m over a field F. The group $GL(V^m)$ acts on it.

Let X be a G-scheme. Then there is a simplicial scheme $BX_\bullet$ where $BX_{(k)} := G\backslash X^{k+1}$. Let $\tau_{\geq n}BX_\bullet$ be the n-truncated simplicial scheme, where $\tau_{\geq n}BX_{(k)} = 0$ for $k < n$ and $BX_{(k)}$ otherwise.

Let $\hat{B}A^p(m)_\bullet \subset BA^p(m)_\bullet$ be the simplicial scheme where $\hat{B}A^p(m)_{(k)}$ consists of *configurations* of $(n+1)$-tuples of affine p-flags in generic position in V^m (i.e. $(n+1)$-tuples considered modulo the action of $GL(V^m)$).

Further, let me recall the definition of the bi-Grassmannian $\hat{G}(n)$ ([G3]). Let $(e_0, ..., e_{k+l})$ be a basis in a vector space V. Denote by $\hat{G}_l^k$ the open part of the Grassmannian consisting of l-dimensional subspaces in V transversal to the coordinate hyperplanes. It is canonically isomorphic to the set of all l-planes in $k+l$-dimensional *affine* space A^{k+l} transversal to a given $k+l$-simplex. Indeed, consider the affine hyperplane in V passing through the ends of basis vectors $e_0, ..., e_{k+l}$. There is canonical isomorphism $m : \hat{G}_l^k \to$ $\{$ configurations of $k+l+1$ vectors in general position in a k-dimensional vector space $\}$. The configuration $m(\xi)$ consists of the images of e_i in V/ξ.

The bi-Grassmannian $\hat{G}(n)$ is the following diagram of manifolds

$$\hat{G}(n) := \begin{array}{ccccccc} & & & & \hat{G}_1^{n+2} & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_0^{n+2} \\ & & & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\ & & \hat{G}_2^{n+1} & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_1^{n+1} & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_0^{n+1} \\ & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\ \dots & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_2^n & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_1^n & \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & \hat{G}_0^n \end{array}$$

Here the horizontal arrows are provided by intersection with coordinate hyperplanes and the vertical ones by factorisation along coordinate axes. The bi-Grassmannian $\hat{G}(n)$ is a truncated simplicial scheme: $\hat{G}(n)_{(k)} := \coprod_{p+q=k} \hat{G}_p^q$.

Remark. The bi-Grassmannian $\hat{G}(n)$ is not a bisimplicial scheme. It is a *hypersimplicial* scheme. To explain what it means let me recall that the hypersimplex $\Delta^{k,l}$ is the convex hull of centers of k-faces of the standard simplex Δ^{k+l+1} ([GGL]). Its boundary is a union of hypersimplices of type $\Delta^{k-1,l}$ and $\Delta^{k,l-1}$. More precisely, if A and S are 2 disjoint finite sets, $\Delta^{A;S}$ is defined as convex hull of centers of all those $k+|S|$-dimensional faces of $\Delta^{A\cup S}$ which contain all vertices of S. Then

$$\partial \Delta^{k,l}(e_0, ..., e_{k+l}) = \sum (-1)^i \Delta^{k-1,l}(e_0, ..., \hat{e}_i, ..., e_{k+l}; e_i) +$$
$$\sum (-1)^i \Delta^{k,l-1}(e_0, ..., \hat{e}_i, ..., e_{k+l})$$

Exercise. Define hypersimplicial sets, schemes, ..., and check that bi-Grassmannian is a $(0,n)$-truncated hypersimplicial scheme.

A correspondence between simplicial schemes $X_\bullet$ and $Y_\bullet$ is a simplicial subscheme $Z_\bullet \subset X_\bullet \times Y_\bullet$ finite over $X_\bullet$.

There is the following correspondence T between the truncated simplicial schemes $\tau_{\geq n} B\hat{A}^p(m)_\bullet$ and $\hat{G}(n)_\bullet$. For a point

$$a = (v_0^1, ..., v_0^{p+1}; ...; v_k^1, ..., v_k^{p+1}) \in \tau_{\geq n} B\hat{A}^{p+1}(n+p)_{(k)}$$

set

$$T(a) := \cup_{q=0}^{k-n} \cup_{i_0+...+i_k=p-q} m^{-1}(L_0^{i_0} \oplus ... \oplus L_k^{i_k} | v_0^{i_0+1}, ..., v_k^{i_k+1})$$

Here $(L_0^{i_0} \oplus ... \oplus L_k^{i_k} | v_0^{i_0+1}, ..., v_k^{i_k+1})$ is the configuration of vectors in the space $V^m / \oplus_{s=0}^k L_s^{i_s}$ obtained by projection of the vectors $v_0^{i_0+1}, ..., v_k^{i_k+1}$ and $m^{-1}(...)$ is the corresponding point of the appropriate Grassmannian.

Theorem 2.5 *T is a correspondence between the truncated simplicial schemes $\tau_{\geq n} B\hat{A}^p(m)_\bullet$ and $\hat{G}(n)_\bullet$*

Proof. Follows essentially from the proof of the Key lemma in s.2.1 of [G3].

Let X be a set. Denote by $\mathbb{Z}[X]$ the free abelian group generated by the points of X. Applying the functor $X \to \mathbb{Z}[X(F)]$ to our simplicial schemes we get simplicial free abelian groups $C_\bullet(A^p(m))$ and $BC_\bullet(n)$. After normalisation we get the complex $C_*(A^p(m))$ of affine flags in generic position :

$$... \xrightarrow{d} C_{n+1}(A^p(m)) \xrightarrow{d} C_n(A^p(m)) \xrightarrow{d} C_{n-1}(A^p(m)) \xrightarrow{d} ... \qquad (20)$$

and the bi-Grassmannian complex $BC_*(n)$. Theorem (2.5) transforms to

Theorem 2.6 *There is a homomorphism of complexes*

$$T : C_*(A^{p+1}(p+n)) \longrightarrow BC_*(n)$$

One has a canonical homomorphism

$$H_*(GL_m(F), \mathbb{Z}) \longrightarrow H_*(C_*(A^p(m))) \qquad (21)$$

So we get for any $p \geq 0$ canonical homomorphisms

$$H_*(GL_{n+p}(F), \mathbb{Z}) \longrightarrow H_*(BC_*(n)) \qquad (22)$$

It is sufficient for our purposes to consider homomorphism (22) for sufficiently big p.

Now we need to make a statement comparing homology of the complex $BC_*(3)$ and cohomology of the complex $\Gamma(F, 3)$. For this reason we will introduce the cohomological version $BC^*(3)$ of the complex $BC_*(3)$ setting $BC^i(3) := BC_{6-i}(3)$ and keeping the same differential, now considered as a cohomological one. One can do the same trick with the complex $C_*(A^p(m))$, getting its cohomological version $C^*(A^p(m))$.

Combining the map (22) with

$$\psi^* : H^*(BC_*(3)) \longrightarrow H^*(\Gamma(F, 3))$$

we get the desired homomorphisms

$$H_4(GL(F),\mathbb{Q}) \longrightarrow H^2(\Gamma(F,3)\otimes\mathbb{Q})$$

8. The affine flag complexes over $\mathcal{O}$. We will construct in the affine flag complex $C_*A^{p+1}(3+p)$ a natural subcomplex $C_*A^{p+1}(\mathcal{O},3+p)$ corresponding to the ring $\mathcal{O}$ such that

1) One has canonical homomorphism

$$H_*(GL_m(\mathcal{O}),\mathbb{Z}) \longrightarrow H_*(C_*(A^p(\mathcal{O},m))) \tag{23}$$

together with commutative diagram

$$\begin{array}{ccc} H_*(GL_m(\mathcal{O}),\mathbb{Z}) & \longrightarrow & H_*(C_*(A^p(\mathcal{O},m))) \\ \downarrow & & \downarrow \\ H_*(GL_m(K),\mathbb{Z}) & \longrightarrow & H_*(C_*(A^p(m))) \end{array}$$

(the down arrows are provided by the natural embedding $\mathcal{O} \hookrightarrow K$).

2) The restriction of the composition $\psi \circ T$ to the subcomplex $C_*A^{p+1}(\mathcal{O},3+p)$ lands in $\Gamma(\mathcal{O},3)$:

$$\psi\circ T : C_*A^{p+1}(\mathcal{O},3+p) \longrightarrow \Gamma(\mathcal{O},3)$$

In particulary this implies that the composition

$$C_*A^{p+1}(\mathcal{O},3+p) \longrightarrow \Gamma(K,3) \longrightarrow \Gamma(k,2)[-1]$$

is zero.

We will represent a $p+1$-flag in $p+3$-dimensional vector space by vectors $(l_1,...,l_{p+1})$; the subspaces of the flag are given by $< l_1,...,l_k >$. Consider m affine flags $a_1,...,a_m$. To define them $(p+1)\cdot m$ vectors is needed. We would like to define a class of *admissible* set of vectors among them. Namely take *first* k_1 vectors from the flag a_1, then first k_2 vectors from a_2 and so on. The set of vectors we get this way is called an admissible set of vectors related to the affine flags $a_1,...,a_m$.

Choose a basis $e_1,...,e_{p+3}$ in $\mathcal{O}^{p+3}$. Let us say that the affine flags $a_1,...,a_m$ are in $\mathcal{O}$-generic position if any admissible $p+3$-tuple of vectors are in generic position. This just means that $\Delta(v_1,...,v_{p+3}) \in \mathcal{O}^*$ for every admissible $p+3$-tuple of vectors $v_1,...,v_{p+3}$ related to the affine flags $a_1,...,a_m$.

The affine flags in $\mathcal{O}$-generic position provide a complex $C_*A^{p+1}(\mathcal{O},3+p)$ with the all described above properties.

The condition 2) holds for the following reason. To compute $\psi \circ T$ we take a set of admissible vectors $(v_1, ..., v_p, ...)$, construct from them a configuration $(v_1, ..., v_p | v_{p+1}, ...)$ in a 3-dimensional space and then apply one of homomorphisms $f_*(3)$. The homomorphisms $f_*(3)$ were defined explicitely using only products and ratios of determinants $\Delta^{\omega}(x, y, z)$. To compute such a determinant we need to choose a volume form ω in the three dimensional vector space, and the result (homomorphisms $f_*(3)$) does not depend on that choise. So for each individual configuration coming as described above from an admissible configuration of vectors $(v_1, ..., v_p, ...)$, one can choose a specific volume form setting $\Delta^{\omega(v)}(x, y, z) := \Delta(v_1, ..., v_p, x, y, z)$. Then for affine flags in $\mathcal{O}$-generic position all the determinants we need will be in $\mathcal{O}^*$.

3 Proof of Deninger's conjecture

1. A regulator from $\Gamma(X, 3)$ to $\mathbb{R}(3)_{\mathcal{D}}$([G2-G3]). We have defined complexes $\Gamma(X, 3)$ so far only when $X = Spec(F)$ or X is a curve. In general $\Gamma(X, 3)$ is the total complex associated with the bicomplex

$$\Gamma(F(X), 3) \xrightarrow{\partial_x} \coprod_{x \in X_1} \Gamma(F(x), 2)[-1] \xrightarrow{\partial_x} \coprod_{x \in X_2} F(x)^*[-2] \xrightarrow{\partial_x} \coprod_{x \in X_3} \mathbb{Q}[-3] \tag{24}$$

Let $S^i(X)$ be the space of smooth i-forms at the generic point of X. (This means that each is defined on a Zariski open domain of X).

For any variety X over $\mathbb{C}$ one has *canonical* homomorphism of complexes

$$\begin{array}{ccccc}
B_3(\mathbb{C}(X)) & \xrightarrow{\delta} & B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \wedge^3 \mathbb{C}(X)^* \\
\downarrow r_3(1) & & \downarrow r_3(2) & & \downarrow r_3(3) \\
S^0(X) & \xrightarrow{d} & S^1(X) & \xrightarrow{d} & S^2(X)
\end{array}$$

given by the following formulas ($\alpha(f, g)$ was defined in (9)).

$$r_3(1) : \{f\}_3 \mapsto \mathcal{L}_3(f)$$

$$r_3(2) : \{f\}_2 \otimes g \longmapsto -\mathcal{L}_2(f) d \arg g + \quad + \frac{1}{3} \log |g| \cdot \alpha(1 - f, f)$$

$$r_3(3) : f_1 \wedge f_2 \wedge f_3 \mapsto \text{Alt} \left(\frac{1}{2} \cdot \log |f_1| d \arg f_2 \wedge d \arg f_3 - \right.$$

$$\left. - \frac{1}{6} \log |f_1| d \log |f_2| d \log |f_3| \right)$$

It enjoys the properties

a) $dr_3(3)(f_1 \wedge ... \wedge f_3) + \pi_3 d\log f_1 \wedge ... \wedge d\log f_3 = 0$ where π_3 means real part.

b) Let Y be an irreducible divisor in X and v_Y be the corresponding valuation on the field $\mathbb{C}(X)$. Then $r_3(\cdot)$ carries the residue homomorphism ∂_{v_Y} to the usual residue homomorphism on the DeRham complex $S^*(X) \longrightarrow S^{*-1}(Y)[-1]$.

A similar homomorphisms exists for the complexes $\Gamma(X,2)$ and $\Gamma(X,1)$ (see [G2-G3] or do it as an easy exercise)

This just means that these formulas provide a homomorphism from the complex $\Gamma(X,3)$ to the weight 3 Deligne complex $\mathbb{R}(3)_{\mathcal{D}}$ on X.

2. Relation with Beilinson's regulator. Recall that we have constructed in s.2-4 canonical homomorphisms

$$c_{i,3} : K_{6-i}(\mathbb{C}(X))_{\mathbb{Q}} \longrightarrow H^i(\Gamma(\mathbb{C}(X)), 3)_{\mathbb{Q}}$$

and in this section

$$r_{\mathcal{D}} : H^i(\Gamma(\mathbb{C}(X)), 3)_{\mathbb{Q}} \longrightarrow H^i_{\mathcal{D}}(Spec\mathbb{C}(X)), \mathbb{R}(3))$$

Theorem 3.1 *The composition*

$$r_{\mathcal{D}} \circ c_{i,3} : K_{6-i}(\mathbb{C}(X))_{\mathbb{Q}} \longrightarrow H^i_{\mathcal{D}}(Spec\mathbb{C}(X)), \mathbb{R}(3))$$

coincides with Beilinson's regulator.

To prove this theorem we will remind an explicit construction of the universal Chern class $c_3^{\mathcal{D}} \in H^6_{\mathcal{D}}(BGL_\bullet, \mathbb{R}(3))$ given in [G3]. We will first construct the corresponding "motivic" class $c_3 \in H^6_{\mathcal{M}}(BGL_\bullet, \Gamma(3))$ and then apply canonical homomorphism from $\Gamma(3)$ to Deligne cohomology.

2. Explicit construction of the class $c_3 \in H^6_{\mathcal{M}}(BGL_{3\bullet}, \Gamma(3))$. Recall that

$$BG_\bullet := pt \;\overset{s_0}{\underset{s_1}{\leftleftarrows}}\; G \;\overset{s_0}{\underset{s_2}{\Lleftarrow}}\; G^2 \;\overset{s_0}{\underset{s_3}{\Lleftarrow}}\; G^3 \;\overset{s_0}{\underset{s_3}{\cdots}}$$

Choose an affine flag $a \in A^{p+1}(n+p)$. Consider simplicial subscheme $B\hat{G}L(n+p)_\bullet \subset BGL(n+p)_\bullet$ consisting of simplices $(g_0, ..., g_k)$ such that

$(g_0a, ..., g_ka)$ is in generic position. So there is a morphism of simplicial schemes

$$A : BGL(n+p)_\bullet \longrightarrow B\hat{A}^{p+1}(n+p)_\bullet$$

defined by formula $(g_0, ..., g_k) \longrightarrow (g_0a, ..., g_ka)$. Further, in s.2.? we have constructed a morphism of truncated simplicial schemes

$$\tau_{\geq n}\hat{B}A^{p+1}(n+p)_\bullet \longrightarrow \hat{G}(n)_\bullet$$

So we get a morphism of truncated simplicial schemes

$$A : \tau_{\geq n}BGL(n+p)_\bullet \longrightarrow \hat{G}(n)_\bullet \qquad (25)$$

Our formulas for the homomorphism of complexes (see s.2-4)

$$BC^*(3)(F) \xrightarrow{\psi} \Gamma(F, 3)$$

give us a cocycle representing a cohomology class in $H^6_{\mathcal{M}}(\hat{G}(n)_\bullet, \Gamma(3))$. So pulling it back by (25) we get a cocycle $\hat{c}_3$ representing $H^6_{\mathcal{M}}(\hat{B}GL(n+p)_\bullet, \Gamma(3))$. It is not a cocycle on the whole $BGL(n+p)_\bullet$ because it has nontrivial residues on some divisors in the complement of $\hat{B}GL(n+p)_\bullet$ in $BGL(n+p)_\bullet$. Fortunately it is easy to check that all residues of the components of $\hat{c}_3$ in $\Gamma(G^i, 3)$ are zero for $i > 3$. For $i = 3$ there are nontrivial residues, but the corresponding problem was already solved in s.4.2 of [G3]. (Recall that by construction components of $\hat{c}_3$ on G^i for $i < 3$ are zero.) Namely, in s.4 of [G3] there was constructed a cocycle c_n^M representing the Chern class in $H^{2n}(BGL(n+p)_\bullet, \mathcal{K}_n^M[-n])$. Here $\mathcal{K}_n^M$ is the sheaf of Milnor's K-groups. In our case $n = 3$ and the component of $\hat{c}_3$ on G^3 coincides with the one of c_3^M. Therefore we can simply add all components of c_3^M on G^i for $i < 3$ and the new cochain we get will be a cocycle. Moreover, the canonical morphism

$$H^6_{\mathcal{M}}(BGL(3+p)_\bullet, \Gamma(3)) \longrightarrow H^6(BGL(3+p)_\bullet, \mathcal{K}_3^M[-3])$$

provided by the obvious morphism $\Gamma(F, 3) \longrightarrow K_3^M(F)[-3]$ carries c_3 to c_3^M just by the construction. In particular the cohomology class of c_3 is nonzero. Now we apply the constructed map to Deligne cohomology $\Gamma(X, 3) \longrightarrow \mathbb{R}(3)_{\mathcal{D}}$ and get a cocycle $c_3^{\mathcal{D}}$ representing a class in $H^6_{\mathcal{D}}(BGL(3+p)_\bullet, \mathbb{R}(3))$. It was proved in [G3] (see s.5.7 in [G3]) that the image of class $[c_3^M]$ in $H^3(BGL(3+p)_\bullet, \Omega^3_{cl})$ coincides with the Chern class of universal bundle over $BG_\bullet$. So the commutative diagram ($\Omega^3_{cl} \hookrightarrow \Omega^{\geq 3}$)

$$\begin{array}{ccc} H^6_{\mathcal{M}}(BGL(3+p)_\bullet, \Gamma(3)) & \longrightarrow & H^3(BGL(3+p)_\bullet, \mathcal{K}^M_3) \\ \downarrow r_{\mathcal{D}} & & \downarrow d\log^{\wedge 3} \\ H^6_{\mathcal{D}}(BGL(3+p)_\bullet, \mathbf{Q}(3)) & \longrightarrow & H^3(BGL(3+p)_\bullet, \Omega^{\geq 3}) \end{array}$$

implies

Theorem 3.2 *The cohomology class $[c_3^{\mathcal{D}}] \in H^6_{\mathcal{D}}(BGL(3+p)_\bullet, \mathbb{R}(3))$ coincides with the third Chern class of the universal bundle.*

4. Proof of Theorem (3.1). Let me first recall the definition of Beilinson's regulator for affine schemes. Let X be an affine scheme over k and $BGL_\bullet$ be the simplicial scheme representing the classifying space for the group GL. Then $Hom_{Sch}(X, BGL_\bullet) = BGL(X)_\bullet$ is a simplicial set. We will treat it as a 0-dimensional simplicial scheme. So one has canonical morphism of simplicial schemes:

$$X \times BGL(X)_\bullet \longrightarrow BGL_\bullet$$

In particular we have canonical morphism

$$e : X(\mathbb{C}) \times BGL(X)_\bullet \longrightarrow BGL_\bullet(\mathbb{C})$$

If $c_n \in H^{2n}_{\mathcal{D}}(BGL_\bullet(\mathbb{C}), \mathbb{R}(n))$ is the universal Chern class in Deligne cohomology, then

$$e^* c_n \in H^{2n}_{\mathcal{D}}(X(\mathbb{C}) \times BGL_\bullet(X), \mathbb{R}(n))$$

Therefore we get a homomorphism

$$< e^* c_n, \cdot >: H_i(GL(X), \mathbb{Z}) \longrightarrow H^{2n-i}_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(n))$$

In particular composed with the Hurevitc map $K_i(X) \longrightarrow H_i(GL(X), \mathbb{Z})$ it leads to the Beilinson' regulator

$$r_{Be} : K_i(X) \longrightarrow H^{2n-i}_{\mathcal{D}}(X, \mathbb{R}(n))$$

Now suppose we have an i-cycle γ in the complex obtained by normalisation of the simplicial set $BGL(\mathbb{C}[X])_\bullet$. Then to compute $< e^* c_n, [\gamma] > \in H^{2n-i}_{\mathcal{D}}(X(\mathbb{C}), \mathbb{R}(n))$ one can proceed as follows. Let $\gamma = \sum n_j (g_0^{(j)}, ..., g_i^{(j)})$.

Each $(g_0^{(j)}, ..., g_i^{(j)})$ defines a map $\gamma_j : X(\mathbb{C}) \longrightarrow G^i$. Let $c_{\mathcal{D}}^i$ be the component in $\mathbb{R}\Gamma(G^i, \mathbb{R}(n)_{\mathcal{D}})$ of the cocycle representing the Chern class in the bicomplex $\mathbb{R}\Gamma(BG_\bullet, \mathbb{R}(n)_{\mathcal{D}})$. Then $\sum_j \gamma_j^* c_{\mathcal{D}}^i \in \mathbb{R}\Gamma(X(\mathbb{C}), \mathbb{R}(n)_{\mathcal{D}})$ is a cocycle representing the class $< e^* c_n, [\gamma] >$.

The last problem is that the cocycle $c_n^{\mathcal{D}}$ is represented by currents on $BG_\bullet$, so there might be a trouble with pulling it back by γ_j. However on a certain generic part of $U \subset BG_\bullet$ the cocycle $c_n^{\mathcal{D}}$ is represented by smooth forms. We will show that

one can always find such a representative $\tilde{\gamma}_j$ for the homology class class $[\gamma_j]$ that $\tilde{\gamma}_j(X(\mathbb{C})) \subset U \subset G^i$.

Currents can always be restricted to an open part of a manifold thanks to the map $C_0^\infty(U) \to C_0^\infty(X)$. So presenting $\tilde{\gamma}_j$ as a composition $X(\mathbb{C}) \hookrightarrow U \hookrightarrow G^i$and using pull back of currents for open embeddings we see that $\gamma_j^*[c_n^{\mathcal{D}}]$ is represented by $\tilde{\gamma}_j^* c_n^{\mathcal{D}}|_U$.

Now let us prove the formulated above statement. Let $a \in V^n$, $G := GL(V^n)$. Say that $(m+1)$-tuple of elements $(g_0, ..., g_{m+1})$ of $GL_n(F)$ is a-generic if the $(m+1)$-tuple of vectors $(g_0 a, ..., g_{m+1} a)$ in V^n is in generic position, i.e. any $k \leq n$ of these vectors generate a k-dimensional subspace.

Let $G^{m+1}(a) \in G^{m+1}$ be the subset of a-generic $(m+1)$-tuples of elements. Then $\mathbb{Z}[G^{m+1}(a)]$ is a simplicial abelian group and the corresponding complex is a free resolution of the trivial G-module $\mathbb{Z}$. (Standard proof: if $\sum n_i (g_0^{(i)}, ..., g_m^{(i)})$ is a cycle, choose an g such that ga is in generic position with all $g_0^{(i)} a$. Then the boundary of $\sum n_i (g, g_0^{(i)}, ..., g_m^{(i)})$ is $\sum n_i (g_0^{(i)}, ..., g_m^{(i)})$. Therefore $H_*(G, \mathbb{Z}) = H_{*-1}(\mathbb{Z}[G^\bullet(a)]_G, \mathbb{Z})$, i.e. *all homology classes of G can be represented by a-generic cycles.* (In fact the above argument shows that this statement is true for any reasonable notion of generic cycles.) Theorem (3.1) is proved.

Remark. Similary one can define a version of continuos cohomolgy of the Lie group G as follows:

$$H_{a-c}^*(G, \mathbb{R}) := H^{*-1}(C(G^{m+1}(a)^G)$$

where $C(G^{m+1}(a))$ is the space of continuos functions on $G^{m+1}(a)$. The restriction map

$$H_c^*(G, \mathbb{R}) \longrightarrow H_{a-c}^*(G, \mathbb{R})$$

is an isomorphism. Indeed, $H_c^*(G, \mathbb{R}) = H_c^*(G, C(G^\bullet(a))$. The spectral sequence for computation of the last group degenerates to the complex $C(G^\bullet(a)^G$ because $H_c^i(G, C(G^m(a)) = 0$ for positive i by Shapiro lemma.

Moreover the obvious pairing

$$H_{*-1}(\mathbb{Z}[G^\bullet(a)]_G) \times H^{*-1}(C(G^{m+1}(a)^G) \longrightarrow \mathbb{R}$$

coincides with the natural pairing $H_*(G,\mathbb{Z}) \times H_c^*(G,\mathbb{R}) \longrightarrow \mathbb{R}$ after identification of the left sides.

5. Computations for curves over $\mathbb{C}$.

Theorem 3.3 *Let X be a compact curve over $\mathbb{C}$ and ω is a holomorphic 1-form on X. and $f_i, g_i \in \mathbb{C}(X)$ are rational functions. Then*

$$\int_X r_3(2)(\sum_i \{f_i\}_2 \otimes g_i) \wedge \bar{\omega} = c_3 \cdot \int_X \log|g_i|\alpha(1-f_i, f_i) \wedge \bar{\omega} \tag{26}$$

where $c_3 \in \mathbb{Q}^*$ is a constant and x_i, y_i, z_i are divisors of functions $g_i, f_i, 1-f_i$.

Proof. One has $F(x) d\log g \wedge \omega = 0$ for a function $F(x)$ on X, and so

$$F(x) d\arg g \wedge \omega = iF(x) \cdot d\log g \wedge \omega \tag{27}$$

Therefore

$$\int_{X(\mathbb{C})} \mathcal{L}_2(f) d\arg g \wedge \omega = i \cdot \int_X \mathcal{L}_2(f) d\log|g| \wedge \omega = -i \cdot \int_{X(\mathbb{C})} d\mathcal{L}_2(f) \log|g| \wedge \omega$$

Here we can integrate by parts because $\mathcal{L}_2(f)$ has only integrable singularities. Applying the formula

$$d\mathcal{L}_2(f) = -\log|1-f| d\arg|f| + \log|f| d\arg|1-f| \tag{28}$$

and (27) we get the proof for $n = 3$.

We did not use the crucial condition $\sum_i (1-f_i) \wedge f_i \wedge g_i = 0$ in $\Lambda^3 \mathbb{Q}(E)^*$ in this computation.

6. The case of elliptic curves over $\mathbb{C}$.

Theorem 3.4 *Let E be an elliptic curve over $\mathbb{C}$ and $\omega \in \Omega^1(\bar{E})$ is normalized by $\int_{E(\mathbb{C})} \omega \wedge \bar{\omega} = 1$. Suppose $f_i, g_i \in \mathbb{C}(E)^*$ satisfy the condition*

$$\sum_i (1-f_i) \wedge f_i \wedge g_i = 0 \quad in \quad \Lambda^3 \mathbb{Q}(E)^* \tag{29}$$

Then

$$\int_{E(\mathbb{C})} \log|g_i|\alpha(1-f_i, f_i) \wedge \omega = \sum_{\gamma_1 + \ldots + \gamma_3 = 0}' \frac{(x_i, \gamma_1)(y_i, \gamma_2)(z_i, \gamma_3)(\bar{\gamma}_3 - \bar{\gamma}_2)}{|\gamma_1|^2 |\gamma_2|^2 |\gamma_3|^2} \tag{30}$$

where x_i, y_i, z_i are divisors of functions $g_i, f_i, 1-f_i$.

Proof. For a rational function $f(z) \in \mathbb{C}(E)^*$ with $div f(z) = \sum \alpha_i x_i$ one has the Fourier expansion

$$\log|f(z)| = \sum_i \sum_{\gamma \in \Gamma} \frac{\alpha_i(x_i, \gamma)}{|\gamma|^2} + C_f, \qquad C_f \in \mathbb{R} \tag{31}$$

in the sence of distributions. Indeed, $\partial\bar{\partial}\log|f(z)| = \sum \alpha_i \delta_{x_i} = \sum_{\gamma \in \Gamma} \alpha_i(x_i, \gamma)$, so $\partial\bar{\partial} C_f = 0$, and hence C_f is a constant.

The Fourier transform carries product to the convolution and $\int_{E(\mathbb{C})}$ to the functional "value at zero". So if we suppose all the constants C_f are zero, then we immediately get formula (30) from these properties of the Fourier transform. In general $C_f \neq 0$. However it turns out condition (29), guarantee that (30) is independent of C_{f_i}, C_{g_i} and C_{1-f_i}. More precisely, $f \longmapsto C_f$ is a homomorphism $\mathbb{C}(X)^* \longrightarrow \mathbb{R}$,. We will show that (30) will not change if we replace this homomorphism by a different one. Let us prove this statement. In fact we will prove that (30) written for any complex curve X depends only on divisors of $f_i, g_i, 1 - f_i$.

It was shown above that

$$\sum_i \int_{E(\mathbb{C})} \log|g_i|\alpha(1 - f_i, f_i) \wedge \omega = i \sum_i \int_{E(\mathbb{C})} \log|g_i| d\mathcal{L}_2(f_i) \wedge \omega =$$

$$-i \sum_i \int_{E(\mathbb{C})} \mathcal{L}_2(f) d\log|g| \wedge \omega$$

So the left hand side of (30) does not depend on C_{g_i}.

Choose a basis in $V_E := \mathbb{C}(E)^* \otimes \mathbb{Q}$. Decompose $\sum_i (1 - f_i) \wedge f_i \otimes g_i$ in this basis and collect all the terms where a given basis element h appears. We get $\sum_i (a_i \wedge h) \otimes b_i + \sum_j (c_j \wedge d_j) \otimes h$. Let us show that (30) is independent of constant C_h.

Indeed, $\sum_j c_j \wedge d_j = \sum_k (1 - s_k) \wedge s_k$ where $\sum_k \{s_k\}_2$ is a factor with which h appears in $\sum_i \{f_i\}_2 \otimes g_i$. Therefore

$$\sum_i \int_{E(\mathbb{C})} \alpha(c_i, d_i) \wedge \omega = \sum_i \int_{E(\mathbb{C})} \alpha(1 - s_i, s_i) \wedge \omega = \sum_i \int_{E(\mathbb{C})} d\mathcal{L}_2(f_i) \wedge \omega = 0$$

Further, thanks to condition (41) one has $(\sum_i a_i \wedge b_i + \sum_j c_j \wedge d_j) \wedge h = 0$ in $\Lambda^3 V_E$, so $\sum_i a_i \wedge b_i = -\sum_j c_j \wedge d_j$ and hence

$$\sum_i \int_{E(\mathbb{C})} (\log|a_i| d\log|b_i| - \log|b_i| d\log|a_i|) \wedge \omega = \sum_i \int_{E(\mathbb{C})} \alpha(c_i, d_i) \wedge \omega = 0$$

On the other hand

$$\sum_i \int_{E(\mathbf{C})} (\log|a_i| d\log|b_i| + \log|b_i| d\log|a_i|) \wedge \omega = \sum_i \int_{E(\mathbf{C})} d(\log|a_i| \log|b_i|) \wedge \omega = 0$$

So

$$\sum_i \int_{E(\mathbf{C})} \log|a_i| d\log|b_i| \wedge \omega = \sum_i \int_{E(\mathbf{C})} \log|b_i| d\log|a_i| \wedge \omega = 0$$

Therefore the contribution of C_h is $C_h \cdot \int_{E(\mathbf{C})} \log|b_i| d\log|a_i| \wedge \omega$ and so is zero. Theorem (3.4) is proved.

7. Proof of statement b) from s. 2.3. Let F be a number field. One has the following commutative diagram

$$\begin{array}{ccccc} K_4(F(X)) & \xrightarrow{c_{2,3}} & H^2\Gamma(Spec(F(X));3)_{\mathbf{Q}} & \xrightarrow{r_3(\cdot)} & H^2_{\mathcal{D}}(Spec(F(X)\otimes\mathbb{R});3) \\ \downarrow \tilde{\delta} & & \downarrow \delta & & \downarrow res \\ K_3(k_x) & \xrightarrow{c_{1,2}} & H^1\Gamma(Spec(k_x);2)_{\mathbf{Q}} & \xrightarrow{r_2(\cdot)} & H^1_{\mathcal{D}}\Gamma(Spec(k_x\otimes\mathbb{R});2) \end{array}$$

where k_x is also a number field (a finite extension of F). Here $r_3(\cdot)$ and $r_2(\cdot)$ are the regulator constructed explicitely by means of the polylogarithms We proved that $r_3(\cdot)\circ c_{2,3} = r_{\mathcal{D}}$ where $r_{\mathcal{D}}$ is the Beilinson regulator to the Deligne cohomology. Further, it is known that $r_2(\cdot)\circ c_{1,3}$ coincides with the Borel regulator r_{Bo} ([G2]). So we come to the commutative diagram

$$\begin{array}{ccc} K_4(F(X)) & \xrightarrow{r_{\mathcal{D}}} & H^2_{\mathcal{D}}(Spec(F(X)\otimes\mathbb{R});3) \\ \downarrow \tilde{\delta} & & \downarrow res \\ K_3(k_x) & \xrightarrow{r_{Bo}} & H^1_{\mathcal{D}}\Gamma(Spec(k_x\otimes\mathbb{R});2) \end{array}$$

The map $r_2(\cdot)$ is injectiv. This follows from the injectivity of the Borel regulator and the fact that $c_{1,3}$ is an isomorphism. So the statement b) and hence the theorem (2.1) is proved.

It follows from Beilinson's theorem on regulators of modular curves that for a modular elliptic curve E over there exists an element $\gamma_4 \in K_4(E)$ whose regulator (up to standard factors) is $L(E,3)$. More precisely, there exists a covering $X \longrightarrow E$ of E by a cetain modular curve X and an element

$\gamma_4' \in K_4(X)$ such that acting on γ_4' by the transfer map $K_4(X) \longrightarrow K_4(E)$ we get an element γ_4 with the desired property.

The definition of the element γ_4' and moreover the transfer map are very implicit. So we do not get any particular information about the element γ_4. However, applying our results stated in theorems (3.4), (3.3), (3.1) to this element we get theorem (1.1).

4 Generalizations

1. The groups $\mathcal{R}_n(F)$ (see s.1.4 in [G2]). Let us define by induction subgroups $\mathcal{R}_n(F) \subset \mathbb{Z}[P_F^1]$, $n \geq 1$. Set

$$\mathcal{B}_n(F) := \mathbb{Z}[P_F^1]/\mathcal{R}_n(F)$$

Put $\mathcal{R}_1(F) := (\{x\} + \{y\} - \{xy\}, (x, y \in F^*)$. Then $\mathcal{B}_1(F) = F^*$. Consider homomorphisms

$$\mathbb{Z}[F^*] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : \quad n \geq 3 \\ \wedge^2 F^* & : \quad n = 2 \end{cases}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x & : \quad n \geq 3 \\ (1-x) \wedge x & : \quad n = 2 \end{cases}$$

$$\delta_n : \{1\} \mapsto 0 \tag{32}$$

Here $\{x\}_n$ is the projection of $\{x\}$ in $\mathcal{B}_n(F)$. Set $\mathcal{A}_n(F) :=$ Ker δ_n . Any element $\alpha(t) = \Sigma n_i\{f_i(t)\} \in \mathbb{Z}[P^1_{F(t)}]$ has a specialization $\alpha(t_0) := \Sigma n_i\{f_i(t_0)\} \in \mathbb{Z}[F^*]$, $t_0 \in F^*$. If t_0 is a zero or pole of $f_i(t)$, then we put $\{f_i(t_0)\} := 0$.

Definition 4.1 *$\mathcal{R}_n(F)$ is generated by elements $\alpha(0) - \alpha(1)$ where $\alpha(t)$ runs through all elements of $\mathcal{A}_n(F(t))$.*

Lemma 4.2 $\delta_n(\mathcal{R}_n(F)) = 0$.

See proof of lemma 1.16 in [G2]. □

Therefore we get the homomorphisms

$$\delta : \mathcal{B}_n(F) \to \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : \quad n \geq 3 \\ \wedge^2 F^* & : \quad n = 2 \end{cases}$$

and finally the following complex $\Gamma(F,n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \wedge^2 F^* \ldots \xrightarrow{\delta} \mathcal{B}_2 \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

where $\mathcal{B}_n \equiv \mathcal{B}_n(F)$ placed in degree 1 and $\delta : \{x\}_p \otimes \wedge_{i=1}^{n-p} y_i \longmapsto \delta(\{x\}_p) \wedge \wedge_{i=1}^{n-p} y_i$ has degree $+1$.

2. The regulator to Deligne cohomology. Let K be a field with a discrete valuation v and the residue class $\overline{k}_v$. Recall that there is the residue homomorphism (see ... or)

$$\partial_v : \Gamma(K,n) \longrightarrow \Gamma(k_v, n-1)[-1] \tag{33}$$

Recall that $S^i(X)$ be the space of smooth i-forms at the generic point of X. Set

$$\widehat{\mathcal{L}_n}(z) = \begin{cases} \mathcal{L}_n(z) & n: \text{ odd} \\ i\mathcal{L}_n(z) & n: \text{ even} \end{cases}$$

One can show that for $n \geq 3$

$$d\widehat{\mathcal{L}_n}(z) = \widehat{\mathcal{L}_{n-1}}(z) d(i \arg z) \tag{34}$$

$$-\sum_{k=2}^{n-2} \beta_k \log^{k-1}|z| \cdot \widehat{\mathcal{L}_{n-k}}(z) \cdot d\log|z| + \beta_{n-1} \log^{n-2}|z| \alpha(1-z,z) \ .$$

In this formula the same coefficients appear as in the definition of the function $\mathcal{L}_n$.

Theorem 4.3 *There exist* **canonical** *homomorphism of complexes*

$$\begin{array}{ccccccc}
\mathcal{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \ldots & \xrightarrow{\delta} & \wedge^n \mathbb{C}(X)^* \\
\downarrow r_n(1) & & \downarrow r_n(2) & & & & \downarrow r_n(n) \\
S^0(X) & \xrightarrow{d} & S^1(X) & \xrightarrow{d} & \ldots & \xrightarrow{d} & S^{n-1}(X)
\end{array}$$

with the following properties:

a) $dr_n(n)(f_1 \wedge \ldots \wedge f_n) + \pi_n d\log f_1 \wedge \ldots \wedge d\log f_n = 0$ *where* π_n *means real part for* n *odd and imaginary for* n *even.*

b) $r_n(1)\{f(x)\}_n = \mathcal{L}_n(f(x))$ *and*

$$r_{n+1}(n)(\{f\}_{n-1} \otimes g) := \widehat{\mathcal{L}_{n-1}}(f) di \arg g - \tag{35}$$

$$-\sum_{k=2}^{n-2}\beta_k \log^{k-2}|f|\log|g|\cdot\widehat{\mathcal{L}}_{n-k}(f)d\log|f|+\beta_{n-1}\log|g|\cdot\log^{n-3}|f|\cdot\alpha(1-f,f)$$

d) Let Y be an irreducible divisor in X and v_Y be the corresponding valuation on the field $\mathbb{C}(X)$. Then $r_n(\cdot)$ carries the residue homomorphism ∂_{v_Y} (see (33)) to the residue homomorphism on the DeRham complex $S^*(X) \longrightarrow S^{*-1}(Y)[-1]$.

An explicit construction of this homomorphism will be given elsewhere.

Remark. It was conjectured in [G2] that the complex $\mathcal{B}_*(\mathbb{C}(X))$ computes the weight n pieces of the K-theory of the field $\mathbb{C}(X)$. The homomorphism $r_n(\cdot)$ should provide the regulator map to Deligne cohomology.

3. Computations for curves over $\mathbb{C}$.

Theorem 4.4 *Let X be a compact curve over $\mathbb{C}$ and ω is a holomorphic 1-form on X. Suppose that $n > 3$ and $f_i, g_i \in \mathbb{C}(X)$ are rational functions satisfying the following condition:*

$$\sum_i \{f_i\}_{n-2} \otimes f_i \wedge g_i = 0 \quad in \quad \mathcal{B}_{n-2}(\mathbb{C}(X)) \otimes \Lambda^2\mathbb{C}(X)^* \tag{36}$$

Then

$$\int_X r_n(2)(\sum_i\{f_i\}_{n-1}\otimes g_i)\wedge\bar{\omega} = c_n\cdot\int_X \log|g_i|\log|f_i|^{n-3}\alpha(1-f_i,f_i)\wedge\bar{\omega} \tag{37}$$

where $c_n \in \mathbb{Q}^*$ is a constant and x_i, y_i, z_i are divisors of functions $g_i, f_i, 1-f_i$.

Proof. Let us first give detailed proof in the cases $n = 4$ emphasizing certain differences between this case and $n = 3$ case considered in theorem (3.3).

i) $n = 4$.

$$\int_{X(\mathbb{C})}\mathcal{L}_3(f)d\arg g\wedge\omega = i\cdot\int_{X(\mathbb{C})}\mathcal{L}_3(f)d\log|g|\wedge\omega = -i\cdot\int_{X(\mathbb{C})}d\mathcal{L}_3(f)\log|g|\wedge\omega$$

Applying the formula for $d\mathcal{L}_3(f)$ we get

$$-i\cdot\int_{X(\mathbb{C})}\Big(-\mathcal{L}_2(f)d\arg f+\frac{1}{3}\log|f|\cdot\alpha(1-f,f)\Big)\cdot\log|g|\wedge\omega$$

It seems that in general it is impossible to rewrite the individual integral $\int_{X(\mathbb{C})} \mathcal{L}_2(f) log|g| dlog|f| \wedge \omega$ as

$$c \cdot \sum_i \int_{X(\mathbb{C})} \log|t_i| \log|s_i| \alpha(1-s_i, s_i) \wedge \omega$$

for some rational functions s_i and t_i. However assuming condition (36) one can do this for

$$\sum_i \int_{X(\mathbb{C})} \mathcal{L}_2(f_i) \log|g_i| d\log|f_i| \wedge \omega \tag{38}$$

Indeed, (36) just means that one has

$$\sum_i \{f_i\}_2 \otimes f_i \otimes g_i \in B_2(\mathbb{C}(X)) \otimes S^2\mathbb{C}(X)^*$$

The 2 homomorphisms from $B_2(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* \otimes \mathbb{C}(X)^*$ to real C^∞ 1-forms on open part of X given by the formulas

$$\{f\}_2 \otimes f \otimes g \longmapsto \mathcal{L}_2(f) log|f| dlog|g|$$

and

$$\{f\}_2 \otimes f \otimes g \longmapsto \frac{1}{2}\mathcal{L}_2(f)\Big(\log|f| d\log|g| \quad + \quad \log|g| d\, log|f|\Big)$$

coincide on the subgroup $B_2(\mathbb{C}(X)) \otimes S^2\mathbb{C}(X)^*$. Therefore (38) is equal to

$$\frac{1}{2}\sum_i \int_{X(\mathbb{C})} \mathcal{L}_2(f_i) d\Big(log|g_i| \cdot log|f_i|\Big) \wedge \omega = -\frac{1}{2}\sum_i \int_{X(\mathbb{C})} d\mathcal{L}_2(f_i) log|g_i| \cdot log|f_i| \wedge \omega$$

It remaines to use the formula (28) together with (27). The theorem for $n = 4$ is proved.

ii) The proof of the general statement is based on the following

Lemma 4.5 *Let us suppose (36). Then*

$$\sum_i \{f_i\}_{n-k} \otimes \underbrace{f_i \otimes \ldots \otimes f_i}_{k \quad times} \otimes g_i = 0 \quad \in B_{n-k}(\mathbb{C}(X)) \otimes S^{k+1}\mathbb{C}(X)^* \tag{39}$$

Proof. Indeed, according to (36)

$$\sum_i \{f_i\}_{n-2} \otimes f_i \otimes g_i \in B_{n-2} \otimes S^2\mathbb{C}(X)^*$$

and from the other hand

$$\sum_i \{f_i\}_{n-k} \otimes \underbrace{f_i \otimes ... \otimes f_i}_{k \quad \text{times}} \otimes g_i \in \mathcal{B}_{n-k-1} \otimes S^k\mathbb{C}(X)^* \otimes \mathbb{C}(X)^*$$

It remains to use the fact that for a vector space V

$$S^kV \otimes V \cap S^{k-1}V \otimes S^2V = S^{k+1}V$$

For elements f_i, g_i satisfying (36) one has

$$\sum_i \int_{X(\mathbb{C})} \mathcal{L}_{n-k}(f_i) \log^{k-1}|f_i| \log|g_i| d\log|f_i| \wedge \omega = \tag{40}$$

$$\frac{1}{k+1}\int_{X(\mathbb{C})} \mathcal{L}_{n-k}(f_i) d(\log^{k-1}|f_i| \log|g_i|) \wedge \omega$$

Integrating by parts and using the formula for $d\mathcal{L}_{n-k}(f)$ and (27) we get the theorem by induction.

Similar arguments prove the following

Proposition 4.6 *If f_i, g_i satisfy (36) then for any $n-2 \geq k \geq 1$ one has $(q_k \in \mathbb{C}^*$*

$$\sum_i \int_{X(\mathbb{C})} \alpha(1-f_i, f_i) \log^{n-3}|f_i| \log|g_i| \wedge \omega =$$

$$q_k \cdot \sum_i \int_{X(\mathbb{C})} d\mathcal{L}_{n-k}(f_i) \log^{k-1}|f_i| \log|g_i| \wedge \omega$$

4. **The case of elliptic curves over $\mathbb{C}$.**

Theorem 4.7 *Let E be an elliptic curve over $\mathbb{C}$ and $\omega \in \Omega^1(\bar{E})$ is normalized by $\int_{E(\mathbb{C})} \omega \wedge \bar{\omega} = 1$. Suppose $f_i, g_i \in \mathbb{C}(E)^*$ satisfies the condition*

$$\sum_i \{f_i\}_{n-2} \otimes f_i \wedge g_i = 0 \quad \text{in} \quad \mathcal{B}_{n-2}(\mathbb{C}(X)) \otimes \Lambda^2\mathbb{C}(X)^* \tag{41}$$

Then

$$\int_{E(\mathbb{C})} \log|g_i| \log|f_i|^{n-3} \alpha(1-f_i, f_i) \wedge \bar{\omega} = \tag{42}$$

$$\sum_{\gamma_1+...+\gamma_n=0}^{\prime} \frac{(x_i, \gamma_1)(y_i, \gamma_2+...+\gamma_{n-1})(z_i, \gamma_n)(\bar{\gamma}_n - \bar{\gamma}_{n-1})}{|\gamma_1|^2|\gamma_2|^2...|\gamma_n|^2}$$

where x_i, y_i, z_i are divisors of functions $g_i, f_i, 1-f_i$.

Proof. It is similar to the case $n = 3$ considered in theorem (3.4). Recall the Fourier expansion

$$\log|f(z)| = \sum_i \sum_{\gamma \in \Gamma} \frac{\alpha_i(x_i,\gamma)}{|\gamma|^2} + C_f, \qquad C_f \in \mathbb{R} \tag{43}$$

As before, assuming all the constants C_f are zero we immediately get formula (42) from the properties of the Fourier transform. In general $C_f \neq 0$. However the condition (41) guarantee that (42) is independent of C_{f_i}, C_{g_i} and C_{1-f_i}. Let us prove this statement.

We will consider separately cases $n = 4$ and $n > 4$ to emphasize the main points of the calculation. In fact we will prove that (42) written for any complex curve X depends only on divisors of $f_i, g_i, 1 - f_i$.

i) $n = 4$. Consider the expression

$$\sum_i (1 - f_i) \wedge f_i \otimes f_i \otimes g_i \in \otimes^4 V_E \tag{44}$$

Recall that we suppose

$$\sum_i (1 - f_i) \wedge f_i \otimes f_i \wedge g_i = 0 \quad \text{in} \quad \wedge^2 V_E \otimes \wedge^2 V_E \tag{45}$$

Let $s = \prod_i h_i^{<s,h_i>}$ be a notation for decomposition of a function s in chosen basis h_i. Then the component of (44) in $\otimes^3 V_E \otimes h$ is

$$\sum_i < g_i, h > \cdot (1 - f_i) \wedge f_i \otimes f_i \otimes h$$

So the contribution of C_h is

$$C_h \cdot \sum_i < g_i, h > \cdot \int_{E(\mathbf{C})} \alpha(1 - f_i, f_i) \log|f_i| \wedge \omega$$

This is zero because using proposition(4.6) and Stokes formula

$$\int_{E(\mathbf{C})} \alpha(1 - f_i, f_i) \log|f_i| \wedge \omega = 3/2 \cdot \int_{E(\mathbf{C})} d\mathcal{L}_3(f_i) \wedge \omega = 0$$

Now consider the component of (44) in $\wedge^2 V_E \otimes h \otimes V_E$. Let us write it as $\sum_i a_i \wedge b_i \otimes h \wedge c_i$. Then condition (45) implies that

$$\sum_i a_i \wedge b_i \otimes c_i = \sum_i < g_i, h > \cdot (1 - f_i) \wedge f_i \otimes f_i$$

Therefore

$$\int_{E(\mathbf{C})} \alpha(a_i, b_i) \log|c_i| \wedge \omega = \sum_i < g_i, h > \cdot \int_{E(\mathbf{C})} \alpha(1 - f_i, f_i) \log|f_i| \wedge \omega = 0$$

Finally, look at the component of (44) in $h \wedge V_E \otimes \otimes^2 V_E$. It actually belongs to $h \otimes V_E \otimes S^2 V_E$ because of the condition (45). Let us decompose it on 2 components: the first in $h \otimes S^3 V_E$ and the second in $h \otimes \Big(\Lambda^2 V_E \otimes V_E \cap V_E \otimes S^2 V_E\Big)$.

If we write the first component as $\sum_i h \otimes x_i \cdot y_i \cdot z_i$, the corresponding contribution of C_h will be

$$C_h 1/3 \sum_i \int_{E(\mathbf{C})} d(\log|x_i| \log|y_i| \log|z_i|) = 0$$

It remains the second component.The reason the contribution of C_h to be zero in this case is the most funny. Namely, this component can be written as $\sum_i h \wedge (1 - s_i) \otimes s_i \otimes g_i$. So the corresponding integral is

$$-C_h \cdot \int_{E(\mathbf{C})} d\log|1-s_i| \log|s_i| \log|g_i| \wedge \omega = -1/2 C_h \int_{E(\mathbf{C})} \alpha(1-s_i, s_i) \log|s_i| \wedge \omega$$

(We used the fact that $\int_{E(\mathbf{C})} d\log|a_i| \log|b_i| \log|c_i| \wedge \omega = 1/2 \int_{E(\mathbf{C})} \alpha(a_i, b_i) \log|c_i| \wedge \omega$ if $\sum_i a_i \otimes b_i \otimes c_i \Lambda^2 V_E \otimes V_E$). But this integral coincides with the one for $\sum_i (1 - s_i) \wedge s_i \otimes h \otimes s_i$ which already was proved to be zero!

ii) $n > 4$. The reasons are similar to those of the case $n = 4$. Proposition (4.6) for $k = 2$ implies the statement about C_{g_i}. Consider element

$$\sum_i (1 - f_i) \wedge f_i \underbrace{\otimes f_i \otimes \ldots \otimes f_i}_{n-3 \text{ times}} \otimes g_i \in \Lambda^2 V_E \otimes S^{n-3} V_E \otimes V_E \qquad (46)$$

The condition that its projection to $\Lambda^2 V_E \otimes \otimes^{n-4} V_E \otimes \Lambda^2 V_E$ is zero implies that the contribution of C_h related to the term

$$\sum_i (1 - f_i) \wedge f_i \underbrace{\otimes f_i \otimes \ldots \otimes f_i}_{n-4 \text{ times}} \otimes h \otimes g_i$$

is zero (the arguments are in complete analogy with the $n = 4$ case).

Finally, the component of (46) in $h \wedge V_E \otimes \otimes^{n-2} V_E$belongs to $h \otimes V_E \otimes S^{n-2} V_E$ thanks to condition (45). Decomposing it on 2 components: in $h \otimes S^{n-1} V_E$ and the in $h \otimes \Big(\Lambda^2 V_E \otimes^{n-3} V_E \cap V_E \otimes S^{n-2} V_E\Big)$. we get the statement similarly to the case $n = 4$. Theorem is proved.

5 Appendix

1. Proof of theorem 2.3b). Let me remind the formulation of this theorem

Theorem 2.3 a) $f_4(3)$ and $f_5(3)$ *do not depend on the choice of* ω.

b) The homomorphisms $f_*(3)$ *provide a morphism of complexes.*

Proof. a) See the proof of similar results in chapter 3 of [G2].

b) We have to prove that $f_4(3) \circ d = \delta \circ f_5(3)$ and $f_5(3) \circ d = \delta \circ f_6(3)$. For the first result see chapter 3 in [G2].

The second one is much more subtle. As pointed out H.Gangl, the geometric proof given in [G2] (see theorem 3.10 there) has some errors. Namely, in lemma 3.8 $r = -r_3$ but not $r = r_3$ as clamed, and as a result the proof of theorem 3.10 become more involved; further, the correct statement in theorem 3.10 is $f_5(3) \circ d = \delta \circ 1/15 \cdot f_6(3)$ (the coefficient 1/15 in the definition of $f_6(3)$ was missed).

Another proof was given in [G1]. It was actually the first proof of the statement b). However in this proof we used a different definition for homomorphism $f_6(3)$ (the map M_3 in [G1]). Moreover the proof was rather complicated and the relation between the homomorphisms $f_6(3)$ and M_3 not easy to see. Therefore I will present in detail a completely different proof togerther with some corrections to chapter 3 in [G2].

Let us suppose that in a three dimensional vector space V_3 we choose a volume form ω. Then for any two vectors a, b one can define the cross product $a \times b \in V_3^*$ as follows: $< a \times b, c > := \Delta(a, b, c)$. The volume form ω defines the dual volume form in V_3^*, so we can define $\Delta(x, y, z)$ for any three vectors in V_3^*.

Lemma 5.1 *For any* 6 *vectors in generic position* $a_1, a_2, a_3, b_1, b_2, b_3$ *in* V_3

$$\Delta(a_1, a_2, b_1) \cdot \Delta(a_2, a_3, b_2) \cdot \Delta(a_3, a_1, b_3) - \Delta(a_1, a_2, b_2) \cdot \Delta(a_2, a_3, b_3) \cdot \Delta(a_3, a_1, b_1) =$$

$$\Delta(a_1, a_2, a_3) \cdot \Delta(a_1 \times b_1, a_2 \times b_2, a_3 \times b_3)$$

Proof. The left hand side is zero if the vectors a_1, a_2, a_3 are linearly dependent. So $\Delta(a_1, a_2, a_3)$ divides it. Similarly the left hand side is zero if a_i is collinear to b_i or $\alpha_1 a_1 + \beta_1 b_1 = \alpha_2 a_2 + \beta_2 b_2 = \alpha_3 a_3 + \beta_3 b_3$ for some numbers α_k, β_k. This implies that $\Delta(a_1 \times b_1, a_2 \times b_2, a_3 \times b_3)$ also divides the left hand side. It is easy to deduce the formula from this.

However it perhaps easier to check the formula directly. Consider the following special configuration of vectors:

a_1	a_2	a_3	b_1	b_2	b_3
1	0	0	x_1	y_1	z_1
0	1	0	x_2	y_2	z_2
0	0	1	x_3	y_3	z_3

Then the left hand side is equal to $x_3y_1z_2 - y_3z_1x_2$, and the computation of the right hand side gives the same result. The lemma is proved.

Remark. Let $a_1, ..., a_n, b_1, ..., b_n$ be a configuration of $2n$ vectors in an n-dimensional vector space V_n. Set $\Delta(\hat{a}_n, b_1) := \Delta(a_1, ..., a_{n-1}, b_1)$ and so on. Then

$$\Delta(\hat{a}_1, b_1) \cdot ... \cdot \Delta(\hat{a}_n, b_n) - \Delta(\hat{a}_1, b_2) \cdot ... \cdot \Delta(\hat{a}_n, b_1) =$$

$$\Delta(a_1, ..., a_n) \cdot \Delta(a_1 \times ... \times a_{n-2} \times b_n, ..., a_n \times ... \times a_{n-3} \times b_{n-1})$$

Notice that $f_5(3) \circ d - \delta \circ f_6(3) \in B_2(F) \otimes F^*$. There is a homomorphism

$$\delta \otimes id : B_2(F) \otimes F^* \longrightarrow \wedge^2 F^* \otimes F^*, \qquad \{x\}_2 \otimes y \longmapsto (1-x) \wedge x \otimes y$$

The crucial step of the proof is the following

Proposition 5.2

$$(\delta \otimes id) \circ \Big(f_5(3) \circ d - \delta \circ f_6(3)\Big)(v_1, ..., v_6) = 0 \quad in \quad \wedge^2 F^* \otimes F^*$$

Proof. We will use notation $\Delta(i, j, k)$ for $\Delta(v_i, v_j, v_k)$. According to lemma (5.1)

$$1 - \frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)} = \frac{\Delta(1,2,3)\Delta(v_1 \times v_4, v_2 \times v_5, v_3 \times v_6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}$$

Using the cyclic permutation $1- > 2- > 3- > 1, 4- > 5- > 6- > 4$ we see that one has to calculate the element

$$3 \cdot \mathrm{Alt}_6 \left\{ \frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)} \wedge \frac{\Delta(1,2,3)\Delta(v_1 \times v_4, v_2 \times v_5, v_3 \times v_6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)} \otimes \frac{\Delta(1,2,4)}{\Delta(1,2,5)} \right\}$$

in $\wedge^2 F^* \otimes F^*$.

Let us do this. We will compute first the contribution of the factor $\otimes\Delta(1,2,4)$. What we need to find is

$$\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\frac{\Delta(1,2,3)\Delta(v_1\times v_4,v_2\times v_5,v_3\times v_6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\right\}$$

in $\wedge^2 F^*$. Here $\mathrm{Alt}_{(1,2,4);(3,5,6)}$ is the skewsymmetrization with respect to the group $S_3\times S_3$ which permutes the indices $(1,2,4)$ and $(3,5,6)$.

i) Consider

$$\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\Delta(v_1\times v_4,v_2\times v_5,v_3\times v_6)\right\}$$

Using the skewsymmetry with respect to the permutation exchanging 1 with 3 (notation: $1<->3$) and $4<->6$ we see that this expression is zero.

ii)Look at

$$-\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\Delta(2,3,6)\otimes\Delta(1,2,4)\right\}$$

The skewsymmetry with respect to $1<->4$ or with respect to $3<->6$ imply that it is also zero.

iii) Consider

$$-\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\Delta(1,2,5)\otimes\Delta(1,2,4)\right\}$$

The skewsymmetry with respect to the permutations $1<->2$ as well as $3<->6$ leads to

$$-\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(2,3,5)}{\Delta(1,3,4)}\wedge\Delta(1,2,5)\otimes\Delta(1,2,4)\right\}$$

iv) Look at the term with $\Delta(3,1,4)$:

$$-\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\Delta(3,1,4)\otimes\Delta(1,2,4)\right\}$$

Using the permutation $1<->4$ we get

$$-\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(3,1,6)}{\Delta(1,2,5)}\wedge\Delta(1,3,4)\otimes\Delta(1,2,4)\right\}$$

v) Finally, using $1 <-> 2$ and $5 <-> 6$ we see that

$$\mathrm{Alt}_{(1,2,4);(3,5,6)}\left\{\frac{\Delta(1,2,4)\Delta(2,3,5)\Delta(3,1,6)}{\Delta(1,2,5)\Delta(2,3,6)\Delta(3,1,4)}\wedge\Delta(1,2,3)\otimes\Delta(1,2,4)\right\}=$$

$$-3\cdot\mathrm{Alt}_{(1,2,4);(3,5,6)}\{\Delta(1,3,5)\wedge\Delta(1,2,3)\otimes\Delta(1,2,4)\}$$

Therefore we get

$$\mathrm{Alt}_{(1,2,4);(3,5,6)}\Big(\Delta(1,2,5)\wedge\frac{\Delta(2,3,5)}{\Delta(1,3,4)}+\Delta(1,3,4)\wedge\frac{\Delta(1,3,6)}{\Delta(1,2,5)}+$$

$$3\cdot\Delta(1,2,3)\wedge\Delta(1,3,5)\Big)\otimes\Delta(1,2,4)=$$

$$\mathrm{Alt}_{(1,2,4);(3,5,6)}\Big(\Delta(1,2,5)\wedge\Delta(2,3,5)+\Delta(1,3,4)\wedge\Delta(1,3,6)+$$

$$3\cdot\Delta(1,2,3)\wedge\Delta(1,3,5)\Big)\otimes\Delta(1,2,4)=$$

$$5\cdot\mathrm{Alt}_{(1,2,4);(3,5,6)}\{\Delta(1,2,3)\wedge\Delta(1,3,5)\otimes\Delta(1,2,4)\}$$

The computation of the contribution of $\Delta(1,2,5)$ goes similarly and gives the same answer. *So the total result of our computation is*

$$-30\cdot\mathrm{Alt}_6\{\Delta(1,2,4)\wedge\Delta(1,4,5)\otimes\Delta(1,2,3)\}\tag{47}$$

Here we get the coefficient -30 taking into account the action of the cyclic group of order 3 generated by $1->2->3->1, 4->5->6->3$.

Now let us compute $f_5(3)\circ d(v_1,...,v_6)$. We will use the formula

$$\delta\{r(v_1,v_2,v_3,v_4)\}_2=1/2\cdot\mathrm{Alt}_4\{\Delta(v_1,v_2)\wedge\Delta(v_1,v_3)\}\tag{48}$$

Neglecting for a moment the constant c, c' we get

$$(\delta\otimes id)\Big(f_5(3)\circ d(v_1,...,v_6)\Big)=c\cdot\mathrm{Alt}_6\{r(v_1|v_2,v_3,v_4,v_5\}_2\otimes\Delta(1,2,3)=$$

$$c'\cdot\mathrm{Alt}_6\Delta(1,2,4)\wedge\Delta(1,4,5)\otimes\Delta(1,2,3)$$

To justify this we used here formula (48) and the symmetry considerations for transpositions $i<->j$ where $1\le i<j\le 3$. More careful consideration shows $c'=-2$. It remains to compare it with (47). That's why we need in the definition of $f_6(3)$ the coefficient $1/15$.

We have proved that

$$\Big(f_5(3)\circ d-\delta\circ f_6(3)\Big)(v_1,...,v_6)=\sum_{1\le i<j<k\le 6}\gamma_{i,j,k}\otimes\Delta(i,j,k)$$

where $\gamma_{i,j,k} \in B_2(F)$ and moreover $\delta(\gamma_{i,j,k}) = 0$ in $\wedge^2 F^*$. According to [S2]

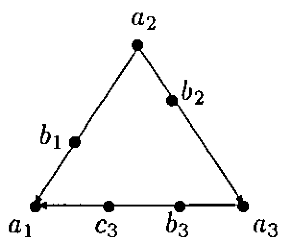
$$Ker\Big(B_2(F) \xrightarrow{\delta} \wedge^2 F^*\Big) \otimes \mathbb{Q} = K_3^{ind}(F) \otimes \mathbb{Q} \tag{49}$$

One knows that $K_3^{ind}(F(t)) \otimes \mathbb{Q} = K_3^{ind}(F) \otimes \mathbb{Q}$. Therefore the left hand side of (49) is rationaly invariant. On the other hand one can connect by a rational curve the configurations $(v_1, v_2, ..., v_6)$ and $(v_2, v_1, ..., v_6)$ (interchanging v_1 with v_2) in the space of all generic configurations. This implies that $\gamma(1,2,3) = \gamma(2,1,3)$ modulo torsion. But $\gamma(1,2,3) = -\gamma(2,1,3)$ modulo torsion by the skewsymmetry. So $\gamma(1,2,3) = 0$ modulo torsion, and the same conclusion is valid for $\gamma(i,j,k)$. With more work one can show that $f_5(3) \circ d - \delta \circ f_6(3) = 0$ at least modulo 6-torsion, but we do not need this. Theorem is proved.

2. The geometrical definition of the homomorphism $f_6(3)$ Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a configuration of 6 distinct points in P^2 as on fig. 1. Let $P^2 = P(V_3)$. Choose vectors in V_3 such that they are projected to points a_i, b_i. We denote them by the same letters. Choose $f_i \in V_3^*$ such that $f_i(a_i) = f_i(a_{i+1}) = 0$. Put

$$r_3'(a_1, a_2, a_3, b_1, b_2, b_3) = \frac{f_1(b_2) \cdot f_2(b_3) \cdot f_3(b_1)}{f_1(b_3) \cdot f_2(b_1) \cdot f_3(b_2)} \,. \tag{50}$$

The right-hand side of (3.10) does not depend on the choice of vectors f_i, b_j.



(fig. 1)

Lemma 3.8 $-r(b_1|a_2, a_3, b_2, b_3) = r_3'(a_1, a_2, a_3, b_1, b_2, b_3)$.

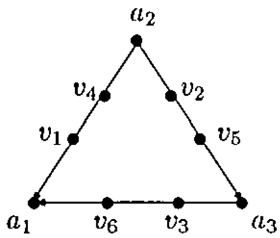
Proof. The same as the one of lemma 3.8 in [G2]

Now let $\hat{b}_3$ be the of the line b_1b_2 with the line a_1a_3. Further, let x be the intersection point of the lines a_1b_2 and a_3b_1. Let us denote by c_3 the intersection point of the line a_2x with the line a_1a_3. Then

$$r_3'(a_1, a_2, a_3, b_1, b_2, b_3) = r(a_1, a_3, c_3, b_3) \tag{51}$$

Indeed, by the well known theorem $r(a_1, a_3, \hat{b}_3, b_3) = -1$.

Now returning to a configuration $(v_1, ..., v_6)$ (see fig 2)



(fig. 2)

we see that one has proceed as follows: Put $b_1 := v_1, b_2 := v_2, b_3 := v_3$ and apply the given above definition to the configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ and then alternate. Notice that the configuration $(a_1, a_2, a_3, b_1, b_2, b_3)$ is defined by three flags $(v_1, v_1v_4), (v_2, v_2v_5), (v_3, v_3v_6)$.

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