Arithmetic groups and the length spectrum of Riemann surfaces

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MPI / 95-7

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Abstract The trace set $Tr(\Gamma)$ of a Fuchsian group Γ is the set of the non-negative traces of the elements of Γ, without counting multiplicities. I propose the following new characterization of arithmetic Fuchsian groups. A cofinite Fuchsian group is arithmetic iff its trace set has linear growth. An arithmetic group is derived from a quaternion algebra iff $Gap(\Gamma) := \inf\{|a-b| : a, b \in Tr(\Gamma), a \neq b\} > 0$. I prove this characterization in the non-compact case and conjecture it in the compact case. Further, I show that the principal congruence subgroups $\Gamma(N)$ of $PSL(2, \mathsf{Z})$ have the property that every trace in its trace set listed in ascending order is a global maximum compared with the trace sets (listed in ascending order) of all other Fuchsian groups in the same moduli space as $\Gamma(N)$. This generalizes the main result of [10] which shows that the systole, the shortest positive trace bigger than 2, is a global maximum for $\Gamma(N)$ in its moduli space.

1 Introduction

Arithmetic groups in the general context of Lie groups and algebraic groups have been defined in the sixties, see Borel/Harish-Chandra [1]. In the case of arithmetic Fuchsian groups (discrete subgroups of $PSL(2, \mathsf{R})$) Takeuchi [11] found a characterization in terms of the trace set which, for a Fuchsian group Γ, is defined as

$$Tr(\Gamma) = \{|tr(\gamma)| : \gamma \in \Gamma\}.$$

Takeuchi's characterization (see section 3 below) is a number theoretic one, but since it is related to the trace set, it contains a geometric meaning, namely, if $M = \mathsf{H}/\Gamma$ is the Riemann surface corresponding to a Fuchsian group Γ (H is the hyperbolic plane), then $Tr(\Gamma)$ can be defined as the set of the lengths of the closed geodesics of M, more precisely

$$Tr(\Gamma) = Tr(M) := \{2\cosh(L(a)/2) : a \text{ a closed geodesic of } M\} \cup \{2\}$$

where $L(a)$ stands for the length of a. This geometric meaning can be given more explicitely. I shall prove.

Theorem Let Γ be a non-elementary cofinite Fuchsian group which contains at least one parabolic element. Then
(i) Γ is an arithmetic group if and only if there exists a finite constant C such that

$$\#\{a \in Tr(\Gamma) : a \leq n\} \leq 1 + Cn, \forall n \geq 0.$$

(ii) Γ is an arithmetic group derived from a quaternion algebra if and only if

$$Gap(\Gamma) := \inf\{|a-b| : a, b \in Tr(\Gamma), a \neq b\} > 0.$$

Corollary Let Γ and Γ' be two cofinite, non-compact Fuchsian groups. Let the trace sets $Tr(\Gamma) = \{a_1 < a_2 < a_3 < \dots\}$ and $Tr(\Gamma') = \{a'_1 < a'_2 < a'_3 < \dots\}$ both be listed in ascending order. Assume that Γ is arithmetic and Γ' is not arithmetic. Then there exists an integer $N = N(\Gamma, \Gamma')$, depending on the two groups, such that

$$a'_i \leq a_i, \forall i \geq N.$$

I conjecture that these results also hold in the compact case that is if Γ contains no parabolic element.

The above theorem is proved in section 3 where some consequences of this result are also given. Before, in section 2, I prove some other rather surprising facts about the trace set which are in contrast to the properties of the *complete* length spectrum which is the set of the lengths of the closed geodesics of a hyperbolic surface, counted *with* multiplicities. Namely, I show the existence of surfaces with a so-called maximal length spectrum. Let M and M' be surfaces of the same moduli space. Let $Tr(M) = \{a_1, a_2, \dots\}$ and $Tr(M') = \{a'_1, a'_2, \dots\}$ be listed in ascending order. Then M is a *surface with maximal length spectrum* if $a'_i \leq a_i$, $\forall i$, and for every surface M' in the moduli space of M. The surfaces corresponding to the principal congruence subgroups $\Gamma(N)$ of $PSL(2, \mathbb{Z})$ are surfaces with maximal length spectrum. If one considers the complete length spectrum, then such surfaces cannot exist.

In the same section (and with the same methods) it is also shown that there exist families of infinitely many different hyperbolic surfaces with mutually different topology which all have the same trace set, while on the other hand it is well known that the complete length spectrum determines the topology of a surface. Moreover, there exist only finitely many non-isometric surfaces with the same complete length spectrum.

A priori, the existence of surfaces with maximal length spectrum cannot be expected. However, in view of the above mentioned corollary, their existence is less surprising.

In the proofs, the main building blocks of hyperbolic Riemann surfaces, namely the pairs of pants or Y-pieces, will play an important role. Most of the results can be derived from the analysis of the situation of particular Y-pieces.

2 Some concrete length spectra and surfaces with maximal length spectrum

Definition (i) A *surface* is a Riemann surface of constant negative curvature and of finite area. Its *signature* (g, n) indicates the *genus* g and the *number of boundary components* n which, if not otherwise stated, are simple closed geodesics or cusps. The *moduli space* of a surface M is denoted by $T(M)$. If M has boundary geodesics, then, by convention, $T(M)$ contains only the surfaces M' of the same signature as M when the boundary geodesics of M' have the same lengths as the boundary geodesics of M.
(ii) Let M be a surface. Let a be a closed geodesic of M. Then, by abuse of notation, the *length* of a is also denoted by a. Define

$$tr(a) = 2\cosh(a/2),$$

the *trace* of a. Define

$$Tr(M) = \{tr(a) : a \text{ a closed geodesic of } M\} \cup \{2\},$$

the *trace set of* M.
(iii) A Fuchsian group Γ is called *cofinite* if the Riemann surface $M = \mathsf{H}/\Gamma$ has finite area where H denotes the hyperbolic plane. The *trace set of* Γ is the set

$$Tr(\Gamma) = \{|tr(a)| : a \in \Gamma\}.$$

(iv) The trace set of a surface M or a Fuchsian group Γ is always *listed in ascending order.* Let M and M' be two surfaces and let

$$Tr(M) = \{a_1, a_2, a_3, \ldots\},$$

$$Tr(M') = \{b_1, b_2, b_3, \ldots\}.$$

Then

$$Tr(M) \leq Tr(M') \text{ means } a_i \leq b_i, \quad \forall i.$$

If $a_1 = b_1 = 2$, then

$$Tr(M) < Tr(M') \text{ means } a_i < b_i, \quad \forall i > 1.$$

(v) The *systole* of a surface M is its shortest closed geodesic which is not a boundary geodesic.

(vi) A *maximal surface* M is a surface with the property that the length of its systole is a local maximum with respect to an open neighborhood of M in $T(M)$.

A *global maximal surface* M is a surface with the property that the length of its systole is a global maximum in $T(M)$.

(vii) A surface M has *maximal length spectrum* if for all surfaces M' in $T(M)$ the inequality

$$Tr(M') \leq Tr(M)$$

holds.

Remark (i) I underline the fact that some aspects of the above definition do not correspond to some usual definitions. The signature of a surface is not the same as the usual definition of the signature of its fundamental group. Further, in the definition of the trace set, the multiplicities of the traces (or of the lengths of closed geodesics) play no role. This is in contrast to the usual definition of the length spectrum of a surface which is called *complete length spectrum* in the sequel.

(ii) If a surface M has maximal length spectrum, then it is necessarely a global maximal surface. But while every moduli space contains a global maximal surface (see [9]), the existence of a surface with maximal length spectrum is a priori not at all clear and it is even rather surprising that such surfaces exist.

(iii) If, in contrast, we consider the complete length spectrum, then the existence of a surface M with a "maximal complete length spectrum" is excluded. This can be seen for example like follows. If M has "maximal complete length spectrum", then it must be a global maximal surface. Let M' be in a small neighborhood of M in $T(M)$ such that the closed geodesics which were systoles in M all remain shorter than all other closed geodesics. It then follows by a result of [9] that at least one of the closed geodesics which is a systole in M is longer in M' than in M which contradicts the assumption that M has "maximal complete length spectrum".

Definition (i) A *Y-piece* is a surface of signature $(0,3)$. For non-negative reals a, b, c the symbol $Y(a,b,c)$ stands for a Y-piece with boundary geodesics of lengths a, b, c.

(ii) If a Y-piece M has two cusps, then instead of $M = Y(0,0,c)$, I also write Y_c.

Lemma 1 *(i) $Tr(Y(a,b,0))$ contains the set*

$$\{n(tr(a) + tr(b)) - tr(b) : n = 1, 2, 3, \ldots\}.$$

(ii) For all positive integers n, $Tr(Y(a,b,0))$ contains $Tr(Y_n(z_n,b,0))$ with

$$tr(z_n) = n(tr(a) + tr(b)) - tr(b).$$

Proof. (i) We replace $Y(a,b,0)$ by $Y(a,b,2\epsilon)$ and we work in a covering surface of $Y(a,b,2\epsilon)$. For every positive integer n, $Y(a,b,2\epsilon)$ has a closed geodesic z_n

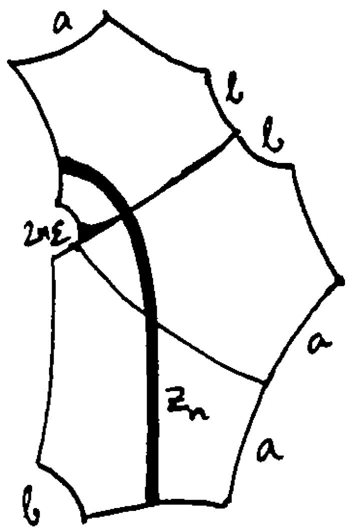

Figure 1: The geodesic z_n for $n = 3$. The figure shows the halfs of the different closed geodesics.

(compare Figure 1) with

$$\cosh(z_n/2) = \frac{\cosh(a/2) + \cosh(b/2)\cosh\epsilon}{\sinh(b/2)\sinh\epsilon}\sinh(b/2)\sinh n\epsilon - \cosh(b/2)\cosh n\epsilon.$$

This implies for $\epsilon \to 0$

$$\cosh(z_n/2) = n(\cosh(a/2) + \cosh(b/2)) - \cosh(b/2).$$

(ii) In the $n-$folded covering of (i), z_n is a simple closed geodesic and, together with b and a cusp, the boundary geodesic of a Y-piece, compare Figure 1. □

Definition For positive integers A and N let

$$\Gamma_A(N) = \left\{ \begin{bmatrix} 1 + aAN & bAN \\ cN & 1 + dAN \end{bmatrix} \in SL(2, \mathsf{Z}) \middle| a, b, c, d \in \mathsf{Z} \right\}.$$

If $A = 1$, then I also write $\Gamma(N)$ instead of $\Gamma_1(N)$.

Proposition 1 *For positive integers A and N we have*

$$Tr(\Gamma_A(N)) = \{nAN^2 \pm 2 : n \in \mathsf{N}\} \setminus \{-1, -2\}.$$

Proof. Since the determinant of every group element is 1, it is easy to see that

$$Tr(\Gamma_A(N)) \subset \{nAN^2 \pm 2 : n \in \mathsf{N}\}.$$

On the other hand for every integer n

$$\begin{bmatrix} 1+nAN^2 & AN \\ nN & 1 \end{bmatrix}$$

is in $\Gamma_A(N)$. $\square$

Proposition 2 *Choose Y_x such that $tr(x)$ is an integer. Then*

$$Tr(Y_x) = \{n(tr(x)+2) \pm 2 : n \in \mathbb{N}\} \setminus \{-2\}.$$

Proof. It follows by Lemma 1 that

$$\{n(tr(x)+2) - 2 : n = 1, 2, 3, \ldots\} \subset Tr(Y_x).$$

When we apply Lemma 1(i) to Y_{z_n} with $tr(z_n) = n(tr(x)+2) - 2$ (compare Lemma 1(ii)), we see that also

$$\{n(tr(x)+2) + 2 : n \in \mathbb{N}\} \subset Tr(Y_x)$$

holds since $(n+1)(tr(x)+2) - tr(x) = n(tr(x)+2) + 2$. By Proposition 1,

$$Tr(\Gamma_{tr(x)+2}(1)) = \{n(tr(x)+2) \pm 2 : n \in \mathbb{N}\} \setminus \{-2\}.$$

The surface $\mathsf{H}/\Gamma_{tr(x)+2}(1)$ contains a Y-piece Y_x which is generated by the two group elements

$$\begin{bmatrix} 1 & tr(x)+2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore $Tr(Y_x) \subset Tr(\Gamma_{tr(x)+2}(1))$ and the proposition follows. $\square$

Corollary 1 *Choose Y_x such that $tr(x)$ is an integer. Then*

$$Tr(Y_x) = Tr(\Gamma_{tr(x)+2}(1)).$$

Proof. Clear by the proof of Proposition 2. $\square$

Lemma 2 *Let Y_x and Y_y be such that $tr(x)$ is an integer and such that $2 < tr(y) < tr(x)$. Then $Tr(Y_y) < Tr(Y_x)$.*

Proof. By Proposition 2 we have

$$Tr(Y_x) = \{n(tr(x)+2) \pm 2 : n \in \mathbb{N}\} \setminus \{-2\}.$$

By Lemma 1, $Tr(Y_y)$ contains the set

$$\{n(tr(y)+2) \pm 2 : n \in \mathbb{N}\} \setminus \{-2\}.$$

Since $2 < tr(y) < tr(x)$, the lemma follows. $\square$

Theorem 1 *For every integer $N \geq 2$, $\mathsf{H}/\Gamma(N)$ has maximal length spectrum.*

Proof. The case $N = 2$ is trivial, so assume that $N > 2$. Let $C(N) = \mathsf{H}/\Gamma(N)$ and let $M \in T(C(N))$. Let $Y_y \subset M$ be the Y-piece in M such that y is minimal. It was shown in [10] that

$$tr(y) \leq N^2 - 2.$$

By Proposition 1 we have

$$Tr(\Gamma(N)) = \{nN^2 \pm 2 : n \in \mathsf{N}\} \setminus \{-2\}$$

and by the proof of Proposition 2, $C(N)$ contains a Y-piece Y_x with $tr(x) = N^2 - 2$. Moreover,

$$Tr(\Gamma(N)) = Tr(Y_x)$$

by Corollary 1. The theorem now follows by Lemma 2. □

Corollary 2 *Let $C(N) = \mathsf{H}/\Gamma(N)$ be defined as in the proof of Theorem 1 and let $M \in T(C(N))$. If M is not a global maximal surface, then $Tr(M) < Tr(C(N))$.*

Proof. Clear by the proof of Theorem 1. □

Remark (i) Let Γ be a normal subgroup of $SL(2, \mathsf{Z})$ of level N. If for the systole x of M we have $tr(x) = N^2 - 2$, then M is a global maximal surface, see [10]. If moreover $Tr(M) = Tr(\Gamma(N))$, then M has maximal length spectrum. In particular, this is the case if M is a cover of $\mathsf{H}/\Gamma(N)$.

(ii) I conjecture that there exist other types of surfaces with maximal length spectrum, see the following two examples.

Conjecture (i) Let M be the (unique, see [9]) maximal surface of signature $(1, 1)$ and let the boundary be a cusp. Then M has maximal length spectrum.

(ii) Let M be the (unique, see [9]) maximal surface of signature $(2, 0)$. Then M has maximal length spectrum.

Remark Let us consider for a moment the case genus 1, the tori with Euclidean metric of normalized area. Then, as it was already known more than 2000 years ago, the socalled hexagon lattice has the largest possible systole and hence is the unique global maximal surface. The question if the hexagonal lattice also has maximal length spectrum however has not been considered until now as far as I know.

Theorem 2 *Let M be a surface of genus g with n cusps, $n > 0$. Let $Tr(M) = \{2 = a_1, a_2, a_3, \ldots\}$ be its trace set. Then*

$$tr(a_{2i}) \leq -2 + \frac{36i(2g + n - 2)}{n^2}, i = 1, 2, \ldots$$

$$tr(a_{2i+1}) \leq 2 + \frac{36i(2g + n - 2)}{n^2}, i = 0, 1, 2, \ldots$$

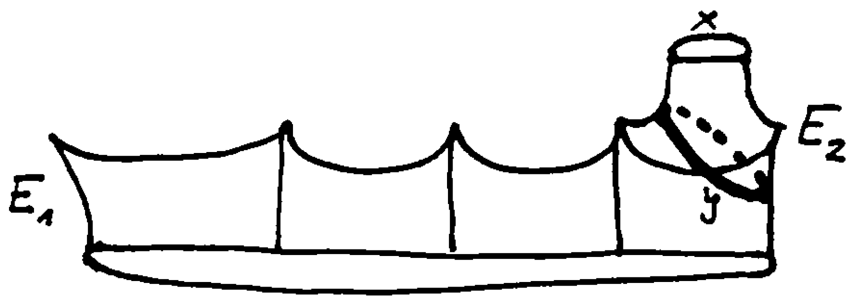

Figure 2: The surface M with its two ends E_1 and E_2 and the closed geodesic y.

and we have equality for all i if M corresponds to a principal congruence subgroup $\Gamma(N)$. Equality for all i is also possible if M corresponds to another normal subgroup of $SL(2,\mathbb{Z})$.

Proof. It follows by [10], that M contains a Y-piece Y_x with

$$tr(x) \leq -2 + \frac{36(2g+n-2)}{n^2}.$$

The theorem follows. $\square$

Remark The complete length spectrum of a surface determines the topology of a surface and, in the case where all boundary components are cusps, also the boundary. Moreover, there exist only finitely many non-isometric surfaces with the same complete length spectrum, see Buser [2] for the compact and Müller [7] for the non-compact case. If however we only take the trace set, then there exist infinitely many surfaces of a very different topology which have the same trace sets.

Theorem 3 *There exist infinitely many different surfaces of a mutually different topology which all have the same trace set as the groups $\Gamma_{AN^2}(1)$, $\Gamma_A(N)$ and the surface Y_x with $tr(x) = AN^2 - 2$ for all $AN > 1$.*

Proof. That the groups $\Gamma_{AN^2}(1)$ and $\Gamma_A(N)$ and the surface Y_x with $tr(x) = AN^2 - 2$ have the same trace set is a consequence of Propostion 1 and Corollary 1.

Take a finite covering surface M of Y_x as in Figure 2. One end E_1 of M consists of a unique copy of Y_x, the other end E_2 of two copies of Y_x.

(i) Let us firstly look at E_2. The subsurface consisting of two copies of Y_x contains a Y-piece $N = Y(0,x,y)$ where y is the closed geodesic of Figure 2. We have $tr(y) = tr(x) + 4$. By Lemma 1, $Tr(N)$ contains the sets

$\{n(2tr(x)+4) - tr(x) : n = 1,2,\dots\}$ and $\{n(2tr(x)+4) - (tr(x)+4) : n = 1,2,\dots\}$

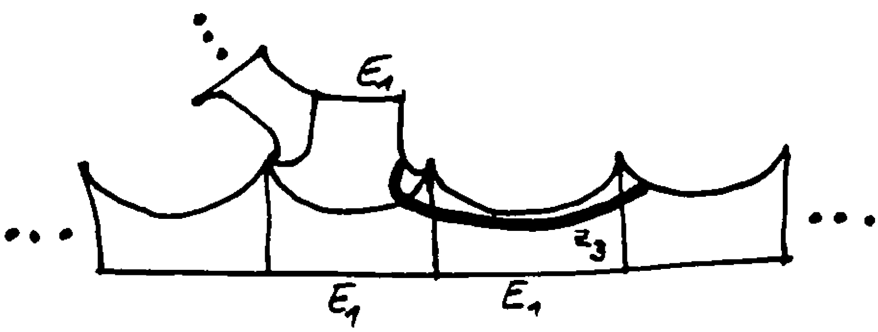

Figure 3: The geodesic z_n for $n = 3$. The figure shows the halfs of the different geodesics.

which implies that $Tr(M)$ contains the set

$$\{n(tr(x) + 2) \pm 2 : n \in \mathsf{N}, n \text{ odd}\}.$$

(ii) Look now at E_1. We take a finite covering surface of M and apply the argument of the proof of Lemma 1, see Figure 3. It follows that $Tr(M)$ contains the sets $Tr(Y_{z_k})$ for $k \in \mathsf{N} \setminus \{0\}$ and

$$tr(z_k) = 2k(tr(x) + 2) - 2.$$

This implies, again by Lemma 1, that $Tr(M)$ contains the set

$$\{n(tr(x) + 2) \pm 2 : n \in \mathsf{N}, n \text{ even}\} \setminus \{-2\}.$$

(iii) It follows by (i), (ii) and Corollary 1 that $Tr(M) \supset Tr(Y_x)$. Since M is a finite covering surface of Y_x we also have $Tr(M) \subset Tr(Y_x)$. Finally, we can choose in M the part between the two ends arbitrarely long and the theorem follows. □

3 The growth of the length spectrum

Definition I denote by $Gap(M)$ or $Gap(\Gamma)$ the minimal gap of the trace set of a surface M or a Fuchsian group Γ which is defined as

$$Gap(M) = \inf\{|a - b| : a, b \in Tr(M), a \neq b\}.$$

Definition (i) A cofinite Fuchsian group Γ is called *arithmetic* if

(a) $K = \mathsf{Q}(Tr(\Gamma))$ is a number field of finite degree and $Tr(\Gamma) \subset \mathcal{O}_K$, the ring of integers of K.

(b) If $\phi : K \to \mathbb{C}$ is any embedding which is not the identity if restricted to $Tr^2(\Gamma)$, then $\phi(Tr(\Gamma))$ is bounded where $Tr^2(\Gamma) = \{t^2 : t \in Tr(\Gamma)\}$.

(ii) An arithmetic group Γ is derived from a quaternion algebra if

(c) If $\phi : K \to \mathbb{C}$ is any embedding which is not the identity, then $\phi(Tr(\Gamma))$ is bounded.

This characterization of arithmetic Fuchsian groups is due to Takeuchi [11]. I will also need the following result of Takeuchi [12] which, as the author writes, has been cited without proof in Magnus [6] (the first formulation is from Fricke).

Theorem 4 *Let $\{a_1, a_2, \ldots, a_n\}$ be a set of generators of a cofinite Fuchsian group Γ. For any subset $\{i_1, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$ let $t_{i_1 \cdots i_m} = tr(a_{i_1} \cdots a_{i_m})$. Then $Tr(\Gamma)$ is contained in the ring*

$$\mathbb{Z}[t_{i_1 \cdots i_m} | \{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, n\}].$$

□

Definition A Fuchsian group Γ has the *B-C property* if there exists a constant $B(\Gamma)$ such that for all integers n the set $Tr(\Gamma) \cap [n, n+1]$ has less than $B(\Gamma)$ elements.

"B-C property" stands for bounded clustering property. The above definition has been introduced in Luo/Sarnak [5] where the following result is proved.

Theorem 5 *(i) An arithmetic group has the B-C property.*

(ii) For an arithmetic group Γ derived from a quaternion algebra we have $Gap(\Gamma) > 0$. □

Definition The trace set of a Fuchsian group Γ has *linear growth* if there exist finite positive constants C and D so that

$$\#\{a \in Tr(\Gamma) : a \leq n\} \leq D + nC, \forall n.$$

Analogously we define that the trace set of a *surface* has linear growth.

Remark The property "linear growth" is slightly weaker than the "B-C property".

Corollary 3 *he trace set of an arithmetic group has linear growth.*

Proof. This is clear by Theorem 5. □

In this section a partial converse of Theorem 5 and Corollary 3 is proved (I shall firstly prove part (ii) and then part (i) of the following theorem).

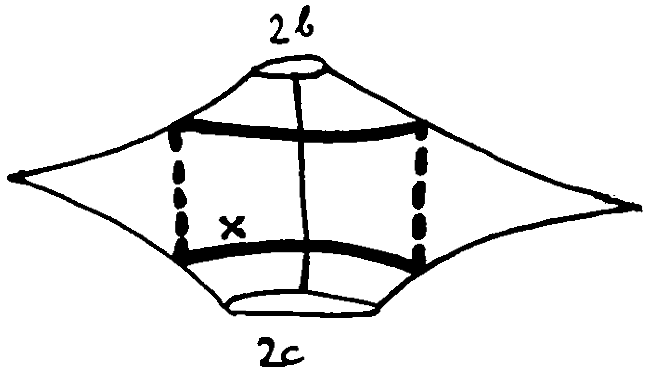

Figure 4: The double cover M' and the closed geodesic x.

Theorem 6 *Let Γ be a cofinite non-elementary Fuchsian group which contains at least one parabolic element.*

(i) If $Tr(\Gamma)$ has linear growth, then Γ is an arithmetic group.

(ii) If $Gap(\Gamma) > 0$, then Γ is an arithmetic group derived from a quaternion algebra.

Conjecture Theorem 6 also holds if Γ has no parabolic element, that is in the compact case.

Remark (i) More precisely, I shall prove the following. Let Γ be a cofinite non-elementary Fuchsian group. Then $Gap(\Gamma) > 0$ implies that $Tr(\Gamma) \subset \mathbb{Z}$ (and therefore in this case, Γ is an arithmetic group derived from a quaternion algebra).

Moreover, I shall prove that if $Tr(\Gamma)$ has linear growth then $Tr(\Gamma)$ contains only square roots of rational integers and hence is arithmetic by the following result of Takeuchi [11]: A cofinite Fuchsian group Γ is an arithmetic group if and only if $\Gamma^{(2)}$ is an arithmetic group derived from a quaternion algebra where $\Gamma^{(2)}$ is the subgroup of Γ generated by $\{a^2 : a \in \Gamma\}$. Therefore, in the non-compact case, we conclude that a cofinite Fuchsian group Γ is an arithmetic group if and only if $Tr(\Gamma^{(2)}) \subset \mathbb{Z}$.

Lemma 3 *Let $M = Y(0, b, c)$. For $n \in \mathbb{N} \setminus \{0\}$ let $N_n = Y_{x_n}$ with*

$$tr(x_n) = n(tr(b) + tr(c))^2 - 2.$$

Then for all n, $Tr(N_n) \subset Tr(M)$.

Proof. Let x be the closed geodesic of M', a double cover of M as in Figure 4. Then

$$\cosh(x/2) = \frac{1 + \cosh(b/2)\cosh(c/2)}{\sinh(b/2)\sinh(c/2)} \sinh b \sinh c - \cosh b \cosh c$$

which implies

$$tr(x) = (tr(b) + tr(c))^2 - 2.$$

In M', x is a boundary geodesic of a subsurface Y_x. We now apply Lemma 1 to Y_x which proves the lemma. $\square$

Definition For two integers a and b, (a,b) denotes the greatest positive common factor.

Proposition 3 *$Gap(Y_x) > 0$ if and only if $tr(x)$ is an integer. More precisely, if $tr(x) \neq 2$ is an integer, then $Gap(Y_x) = \min\{4, tr(x) - 2\}$. If $tr(x) = 2$, then $Gap(Y_x) = 4$.*

Proof. (i) Assume that $tr(x) = -2 + a/b$, with positive integers a and $b > 1$ and $(a,b) = 1$. By Lemma 3, $Tr(Y_x)$ contains $Tr(Y_y)$ with $tr(y) = -2 + a^2/b^2$. Since we can take y as the new x, we obtain by induction that $Tr(Y_x)$ contains $Tr(Y_{y_k})$ with

$$tr(y_k) = -2 + a^{(2^k)}/b^{(2^k)}, k \in \mathsf{N}.$$

By Lemma 1, $Tr(Y_x)$ contains the set $\{-2+na/b : n = 1,2,3,\ldots\}$. Since $Tr(Y_{y_k})$ contains the analogous set, it follows that $Tr(Y_x)$.contains the set

$$\{-2 + na^{(2^k)}/b^{(2^k)} : k \in \mathsf{N}, n \in \mathsf{N} \setminus \{0\}\}.$$

This implies

$$Gap(Y_x) \leq \min\{|(-2 + \xi a^{(2^k)}/b^{(2^k)}) - (-2 + \eta a/b)| \neq 0 : \xi, \eta \in \mathsf{N} \setminus \{0\}\}.$$

Since $(a,b) = 1$, this minimum is

$$a/b^{(2^k)}.$$

Since $b > 1$ by hypothesis, we obtain for $k \to \infty$ that $Gap(Y_x) = 0$.

(ii) Assume now that $z - 2 := tr(x) \notin \mathsf{Q}$. Then, by Lemma 3, $Tr(Y_x)$ contains $nz^2 - 2$ for every positive integer n and, by Lemma 1, $Tr(Y_x)$ contains $kz - 2$ for every positive integer k. Then, since z can be approximated by a continued fraction (see for example Rockett/Szüsz [8]), for every $\epsilon > 0$ there exist integers ξ and η such that

$$0 < |z^2\xi - z\eta| = z|z\xi - \eta| < \epsilon$$

which proves that $Gap(Y_x) = 0$.

(iii) Assume finally that $tr(x)$ is an integer. Then, by Proposition 2, we have $Gap(Y_x) = \min\{4, tr(x) - 2\}$ if $tr(x) > 2$ and $Gap(Y_x) = 4$ if $tr(x) = 2$. $\square$

Lemma 4 *Let $M = Y(0,b,c)$ and $N = Y(0,2b,z)$ with $tr(z) = 2 + tr(b)tr(c)$. Then $Tr(N) \subset Tr(M)$.*

Proof. Look at a double cover of M, see Figure 5. The lemma follows if one calculates $tr(z)$ where z is the closed geodesic in Figure 5. $\square$

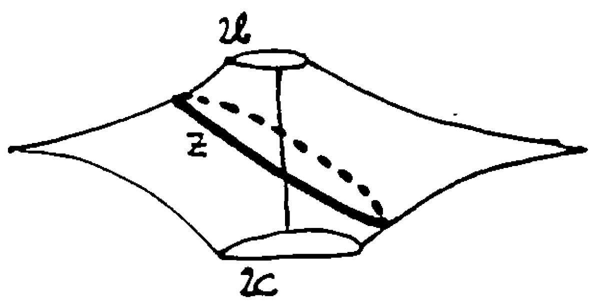

Figure 5: The double cover of M with the closed geodesic z.

Proposition 4 *Let $M = Y(0, b, c)$. Then $Gap(M) = 0$ if at least one of the three numbers $tr^2 b$, $tr^2 c$, and $tr(b)tr(c)$ is not an integer.*

Proof. Assume that $Gap(M) > 0$. By Lemma 3, $Tr(M)$ contains $Tr(Y_x)$ with $tr(x) = (tr(b) + tr(c))^2 - 2$ and by Proposition 3, $tr(x)$ must be an integer.

(i) Assume that $tr(b) = a/d$, $tr(c) = e/f$, with integers a, d, e, f and $(a, d) = (e, f) = 1$. Since $tr(x)$ is an integer,

$$\frac{(af + de)^2}{d^2 f^2}$$

is an integer and since $(a, d) = (e, f) = 1$, it follows that $d = f$ and that $a + e \equiv 0 \bmod (d)$.

By Lemma 4, $Tr(M)$ contains $Tr(N)$ with $N = Y(0, 2b, z)$ and $tr(z) = 2 + ae/d^2$ and $tr(2b) = -2 + a^2/d^2$. It follows by the same argument as above that $a^2 + ae = a(a+e) \equiv 0 \bmod (d^2)$. Since $(a, d) = 1$ it follows $a+e \equiv 0 \bmod (d^2)$.

Repeating this argument we conclude that $a + e \equiv 0 \bmod (d^k)$ for every positive integer k which implies that $d = 1$ and that $tr(b)$ and $tr(c)$ are integers.

(ii) We now treat the general case. By Lemma 4, $Tr(M)$ contains $Tr(N)$ with $N = Y(0, 2b, z)$ and $tr(z) = 2 + tr(b)tr(c)$. We apply Lemma 3 and Proposition 3 to N and conclude that

$$tr^2(b)(tr(b) + tr(c))^2$$

is an integer. Since we already saw that $(tr(b) + tr(c))^2$ is an integer, it follows that $tr^2(b) \in \mathbb{Q}$ and hence $tr(2b) \in \mathbb{Q}$. By an analogous argument $tr^2(c) \in \mathbb{Q}$, and since $(tr(b) + tr(c))^2$ is an integer, we conclude that $2tr(b)tr(c) \in \mathbb{Q}$ and thus $tr(z) \in \mathbb{Q}$. Therefore, we can apply part (i) to $N = Y(0, 2b, z)$ which proves that $tr(2b)$ and $tr(z)$ are integers. By the analogous argument $tr(2c)$ is also an integer and the proposition follows. □

Theorem 7 *Let $M = Y(0, c, d)$. Then $Gap(M) > 0$ if and only if $tr(c)$ and $tr(d)$ are integers.*

Proof. If $tr(c)$ and $tr(d)$ are integers, then it follows by Theorem 4 that $Tr(M) \subset \mathsf{N}$ and therefore $Gap(M) \geq 1$.

Assume now that $Gap(M) > 0$. Then, by Proposition 4, $tr^2(c)$, $tr^2(d)$ and $tr(c)tr(d)$ are integers. Therefore, we can write $tr(c) = a\sqrt{n}$, $tr(d) = b\sqrt{n}$ where a, b and n are integers and n is square free or 1. We have to prove that $n = 1$. Assume that this is not the case.

By Lemma 1, $Tr(M)$ contains the set

$$\{(a+b)y\sqrt{n} - a\sqrt{n} : y = 1, 2, 3, \ldots\}.$$

By Lemma 3, $Tr(M)$ contains the set

$$\{n(a+b)^2 x - 2 : x = 1, 2, 3, \ldots\}.$$

We will show that that there exist positive integers x and y such that

$$[n(a+b)^2 x - 2] - [(a+b)y\sqrt{n} - a\sqrt{n}]$$

is arbitrarely small. We multiply the above difference by its "conjugate"

$$[n(a+b)^2 x - 2] + [(a+b)y\sqrt{n} - a\sqrt{n}]$$

and obtain

$$[n(a+b)^2 x - 2]^2 - n[(a+b)y - a]^2.$$

Let $z = n(a+b)^2 - 2]^2 - n[(a+b) - a]^2$, $u_x = n(a+b)^2 x - 2$, and $v_y = (a+b)y - a$. It then follows by the theory of binary quadratic forms (see for example Zagier [14]) that the Diophantic equation

$$u^2 - nv^2 = z$$

has infinetely many different solutions (u_i, v_j), i, j integers, since it has one solution (u_1, v_1). Therefore, there are solutions (u_i, v_j) with i and j arbitrarely big so that

$$[n(a+b)^2 x - 2] + [(a+b)y\sqrt{n} - a\sqrt{n}]$$

is arbitrarely big and hence, since z is a constant,

$$[n(a+b)^2 x - 2] - [(a+b)y\sqrt{n} - a\sqrt{n}]$$

is arbitrarely small which has to been proved. □

Proof of Theorem 6 (ii).

Assume that $Gap(\Gamma) > 0$ for a non-elementary cofinite Fuchsian group Γ with at least one parabolic element.

(i) We firstly assume that Γ has no elliptic element. Let $M = \mathsf{H}/\Gamma$. Let a be a simple closed geodesic of M. Then M contains a second simple closed geodesic

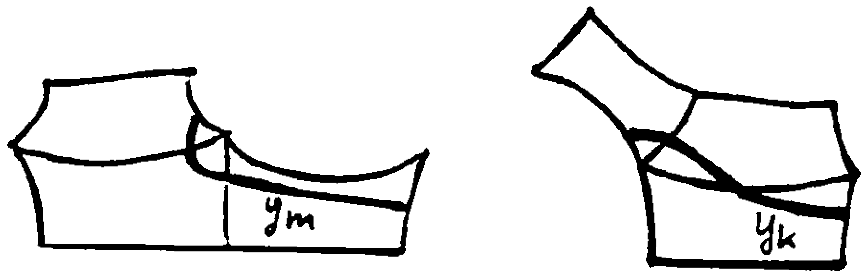

Figure 6: The geodesics y_k and y_m in a $k+m$-fold cover of Y_x

b such that M contains the Y-piece $Y(a,b,0)$. It follows be Theorem 7 that $tr(a)$ and $tr(b)$ are integers. In Theorem 3 we can choose a generating set consisting only of simple closed geodesics such that the products also correspond to simple closed geodesics. It thus follows by Theorem 3 that $Tr(M)$ is a subset of the integers and hence Γ is an arithmetic group derived from a quaternion algebra.

(ii) Assume now that Γ has an elliptic element a and that Γ is not a triangle group. Then Γ contains a hyperbolic element b and $Tr(\Gamma)$ contains the trace set of a degenerated Y-piece $Y(a,b,0)$ where the "boundary geodesic" a is an elliptic point. By an analogous calculation as in the proof of Lemma 1, $Y(a,b,0)$ contains a closed geodesic d with

$$tr(d) = tr(b) + 2tr(a).$$

Moreover, $Tr(Y(a,b,0))$ contains $Tr(Y(b,d,0))$. Therefore, by Theorem 7, $tr(b)$ and $tr(d)$ are integers which, by the above equation, implies that $2tr(a)$ is also an integer. An analogous argument as in the proof of Lemma 3 shows that $(tr(a)+tr(b))^2$ is an integer hence $tr(a)$ is an integer.

(iii) Assume finally that Γ is a triangle group (a,b,∞). Then it contains a hyperbolic element d and it follows as in Lemma 1 or part (ii) above that $Tr(\Gamma)$ contains the trace set of a degenerated Y-piece $Y(a,d,0)$. Therefore, $tr(a)$ is an integer by part (ii) and hence also $tr(b)$. Theorem 3 finishes the proof. $\square$

Lemma 5 *$Tr(Y_x)$ contains $Tr(Y(0,y_k,y_m))$ with $tr(y_k) = k(tr(x)+2)+2$ and $tr(y_m) = m(tr(x)+2)-2$ for all pairs (k,m) of positive integers.*

Proof. Compare Figure 6 and Lemma 1. $\square$

Proposition 5 *$Tr(Y_x)$ has linear growth if and only if $tr(x)$ is an integer.*

Proof. Assume that $z = tr(x)+2$ is not rational. By Lemma 5 and Lemma 4 $Tr(Y_x)$ contains $tr(y_m)tr(y_k)+2$ and hence the set

$$\{mkz^2 - 2(k-m)z - 2 : m,k \in \mathsf{N} \setminus \{0\}\}.$$

Since $z \notin \mathbb{Q}$ all these numbers are different for different choices of pairs (m, k). We assume that $k \geq m$ and it follows that

$$mkz^2 \geq mkz^2 - 2(k-m)z - 2.$$

This implies

$$\#\{a \in Tr(Y_x) : a \leq Nz^2\} \geq 1/2 \sum_{i=1}^{i=N} \sigma_0(i)$$

where $\sigma_0(i)$ is the sum of (positive) divisors of i. It is well known that the above sum grows like $Nlog(N)$ and therefore, $Tr(Y_x)$ cannot have linear growth.

Assume now that $z = a/b$ for integers a and $b > 1$ with $(a, b) = 1$. It follows by Lemma 3 that $Tr(Y_x)$ contains the trace set of $Y_{x(k)}$ for $tr(x(k)) + 2 = z^{(2^k)}$ for every positive integer k. We have

$$\#\{a \in Tr(Y) : a \leq bz^2\} \geq 1/2 \sum_{i=1}^{i=b} \sigma_0(i)$$

since all numbers $mkz^2 - 2(k-m)z - 2$ are different for different choices of pairs (m, k) with $mk \leq b$. But by the above remark b can be replaced by $b_{(2_k)}$ which implies that $Tr(Y_x)$ cannot have linear growth. □

Theorem 8 *If $Tr(Y(c, d, 0))$ has linear growth, then there exist integers a, b, n such that $tr(c) = a\sqrt{n}$ and $tr(d) = b\sqrt{n}$ and n is squarefree or 1.*

Proof. This follows by the proof of Proposition 4 and by Proposition 5. □

Proof of Theorem 6 (i). Assume that $Tr(M)$ has linear growth. Then, by Theorem 8, $tr^2(a)$ is an integer for every simple closed geodesic a of M and for every closed geodesic a of M which is a simple closed geodesic in a finite covering surface of M. It follows that $Tr^2(M) \subset \mathbb{N}$, which implies Theorem 6 (i). □

Remark I shall give some consequences of the previous results.

Theorem 9 *Let $M = \mathbf{H}/\Gamma$ be a non-compact surface of finite volume. Then Γ is arithmetic if and only if for every Y-piece Y_x which is virtually contained in M (this means that Y_x is contained in a finite cover of M) $tr(x)$ is an integer.*

Proof. This follows by Propostion 5, Theorem 6 and Theorem 8. □

Theorem 10 *Let Γ and Γ' be two cofinite, non-compact Fuchsian groups. Assume that Γ is arithmetic and that Γ' is not arithmetic. Then*

$$Tr(\Gamma') \leq Tr(\Gamma) \qquad \textit{almost everywhere,}$$

this means that there exists a number n, depending on Γ and Γ', such that $a'_i \leq a_i$, for all $i \geq n$, where $Tr(\Gamma') = \{a'_1, a'_2, \ldots\}$ and $Tr(\Gamma) = \{a_1, a_2, \ldots\}$.

Proof. This is a consequence of Theorem 6. □

Corollary 4 *Assume that $M = \mathbf{H}/\Gamma$ is non-compact and of finite volume and has maximal length spectrum and assume that in the moduli space of Γ there is at least one arithmetic group. Then Γ is arithmetic.*

Proof. Clear by Theorem 10. □

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