ON THE CLASSIFICATION OF HYPERBOLIC ROOT SYSTEMS OF THE RANK THREE

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Abstract. This is the first monograph devoted to classification of hyperbolic root systems which are appropriate for the theory of Lorentzian (or hyperbolic) Kac–Moody algebras. These hyperbolic root systems should have restricted arithmetic type and a generalized lattice Weyl vector. One can consider them as an appropriate hyperbolic analog of finite and affine root systems. The author obtained finiteness results for the hyperbolic root systems. The classification of these root systems is considered for the first non-trivial and the most reach rank three case. It requires very non-trivial and long calculations. One can consider this work as a starting point for developing of the complete theory of Lorentzian Kac–Moody algebras for the rank three case. The rank three case is the hyperbolic analog of sl_2.

Contents

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Introduction

It was recently understood that from the point of view of the theory of Lorentzian (or hyperbolic) Kac–Moody algebras, some aspects of the theory of Mirror Symmetry, and some other theories, interesting hyperbolic root systems should have some very restricted properties: they have *restricted arithmetic type* and *a generalized lattice Weyl vector* (see Sect. 1.3). One can consider hyperbolic root systems with these properties as an appropriate hyperbolic analog of the classical finite and affine root systems. These properties: restricted arithmetic type and existence of a generalized lattice Weyl vector, are so strong that there is a hope to describe all hyperbolic root systems with these properties. It was shown in [N4], [N5], [N11], [N13] that for the fixed rank ≥ 3 number of such root systems is essentially finite. For example number of such maximal root systems of a fixed rank ≥ 3 is finite. Moreover, rank is also bounded [N2], [N3], [N5], [N9], [N11], [N13], [V3]. It would be very interesting to classify all hyperbolic root systems of restricted arithmetic type and with a generalized lattice Weyl vector. It seems of the same importance as classification of finite and affine root systems having many applications in Lie algebras theory and other mathematical and physical theories.

This work is devoted to classification of hyperbolic root systems of restricted arithmetic type and with a generalized lattice Weyl vector, having the rank 3. For the ranks 1 and 2 the classification is trivial. Thus, the rank 3 case is the first non-trivial one. On the other hand, the rank 3 case is probably the most reach, complicated and important in applications.

For higher rank ≥ 5 some results about classification for elliptic type (see the definition below) had been known: see É.B. Vinberg [V3] for the rank ≥ 30, F. Esselmann [Es] for the rank ≥ 21, C. Walhorn [W] for the rank 5, R. Scharlau and C. Walhorn [SW] about some results for the non-compact case of rank 4. The main reason was that hyperbolic lattices of the rank ≥ 5 represent zero, fundamental domains of their automorphism groups are not compact. The rank 3 case is mainly compact.

Below we present results in more details.

Let S be a hyperbolic lattice (i.e. a non-degenerate integral symmetric bilinear form of signature $(1, n)$). We denote by $(\, . \, , \, . \,)$ the form of the lattice S. An element $\alpha \in S$ is called a *root* if $\alpha^2 = (\alpha, \alpha) < 0$ and $\alpha^2 | 2(\alpha, S)$. A root $\alpha \in S$ defines a *reflection*

$$s_\alpha : x \mapsto x - (2(x,\alpha)/\alpha^2)\alpha$$

of the lattice S. A subset $\Delta \subset S$ of roots of S is called a *root system with the root lattice* S if properties a) and b) below are valid:

a) $s_\alpha(\Delta) = \Delta$ for any $\alpha \in \Delta$, it follows that Δ is invariant with respect to the reflection group $W(\Delta) \subset O(S)$ generated by all reflections s_α in $\alpha \in \Delta$;

b) Δ generates $S \otimes \mathbb{Q}$.

We denote by $W(\Delta) \subset O(S)$ the group generated by all reflections in roots of Δ and by

$$A(\Delta) = \{\phi \in O^+(S) \mid \phi(\Delta) = \Delta\}$$

the group of symmetries of Δ. Let $\mathcal{M}$ be a fundamental polyhedron of $W(\Delta)$ in the hyperbolic space defined by S (see Sect. 1) and by

$$A(\Delta,\, \mathcal{M}) = \{\phi \in A(\Delta) \mid \phi(\mathcal{M}) = \mathcal{M}\}$$

the group of symmetries of $\mathcal{M}$. Then $A(\Delta) = W(\Delta) \rtimes A(\Delta,\, \mathcal{M})$. The root system Δ has *restricted arithmetic type* if $A(\Delta)$ has finite index in $O(S)$. The root system Δ has a *generalized lattice Weyl vector* if there exists a non-zero $\rho \in S$ and a subgroup $A \subset A(\Delta,\, \mathcal{M})$ of finite index such that $A(\rho) = \rho$. The ρ is called a *generalized lattice Weyl vector* of Δ for $\mathcal{M}$.

It is sufficient to describe *primitive root systems* when all elements of Δ are primitive (non-divisible) elements of S. Any other root system *is twisted* to a primitive one: one can get it by an appropriate multiplication on restricted natural numbers of elements of a primitive root system. Further we suppose that Δ is primitive. Then Δ is equivalent to a reflection subgroup $W = W(\Delta) \subset O(S)$. Really, Δ is the set of all primitive roots α of S such that the reflection $s_\alpha \in W$. Therefor, by definition, W has *restricted arithmetic type and a generalized lattice Weyl vector* if Δ does. It is easy to see that if $W \subset W' \subset O(S)$ are two reflection subgroups, both groups W and W' have restricted arithmetic type and the group W has a generalized lattice Weyl vector, then W' also has a generalized lattice Weyl vector. It follows that the full reflection group $W(S)$ of the lattice S (generated by reflections in all roots of S) has restricted arithmetic type and a generalized lattice Weyl vector if there exists a reflection subgroup $W \subset W(S)$ having restricted arithmetic type and a generalized lattice Weyl vector. A hyperbolic lattice S having this property is called *reflective*. Depending on square of the generalized lattice Weyl vector ρ of $W(S)$: $(\rho,\rho) > 0$, $(\rho,\rho) = 0$ or $(\rho,\rho) < 0$, the reflective lattice S is called reflective of *elliptic, parabolic or hyperbolic type respectively.* Therefor, any primitive root system Δ of restricted arithmetic type and with a generalized lattice Weyl vector is a subsystem of the primitive root system $\Delta(S)$ of all primitive roots of the reflective lattice S. Root systems $\Delta(S)$ of reflective hyperbolic lattices S are especially important: the root system $\Delta(S)$ is the maximal hyperbolic root system of restricted arithmetic type and with a generalized lattice Weyl vector having the root lattice S.

Obviously, S has elliptic type if and only if $[O(S) : W(S)] < \infty$, and this type of reflective lattices was considered long time ago in papers of É.B. Vinberg and the author (see references). Reflective lattices of parabolic and hyperbolic type were recently studied in [N11] and [N13]. We have the following general result where a lattice is called *primitive* if one cannot divide its form by any natural number greater than 1:

Theorem 1 ([N4], [N5], [N11], [N13]). *For the fixed rank rk $S \ge 3$ the set of primitive (see Sect. 1.4) reflective (of any type) hyperbolic lattices S is finite. In particular, for the fixed rank rk $S \ge 3$, the set of primitive root lattices S of hyperbolic root systems of restricted arithmetic type and with a generalized lattice Weyl vector is finite.*

To make a further reduction, we say that an embedding of lattices $S_1 \subset S$ of the same rank is called *equivariant* if it defines the embedding of automorphism groups $O(S_1) \subset O(S)$. For an equivariant embedding $S_1 \subset S$, the lattice S is reflective if S_1 does. It is easy to prove (and is well-known) that any lattice S_1 has an equivariant (and actually canonical) embedding into an elementary lattice S. (See Proposition 2.2.2.) Here a lattice L is called *elementary* if its discriminant group $A_L = L^*/L$ is elementary: any its p-component $A_{L_p} \cong (\mathbb{Z}/p\mathbb{Z})^{r_p}$ is a p-elementary Abelian group. To describe all reflective hyperbolic lattices, it is sufficient to describe all reflective primitive elementary hyperbolic lattices S and to describe their primitive reflective equivariant sublattices $S_1 \subset S$ (of finite index).

In this work, we give the complete list of all reflective elementary hyperbolic lattices of rank 3. See Sect. 2.3 and Sect. 7. We use the same method: existence of "narrow parts" of polyhedra in hyperbolic spaces, which have been used to prove the finiteness Theorem 1. Moreover, we use the additional arithmetic invariant: number of classes of central symmetries of the fundamental polyhedron $\mathcal{M}$. Moreover, we give the global estimate of invariants of reflective hyperbolic lattices of the rank three, see Theorems 4.3.6.1 and 8.5.10.1. In particular, it gives the estimate of the index $[S : S_1]$ of reflective sublattices S_1 in elementary reflective hyperbolic lattices S. Actually, using methods of this work, these estimates can be significantly improved for each concrete elementary hyperbolic lattice S, and even give classification of all equivariant reflective sublattices S_1 of S. We hope to demonstrate this in further papers where we shall consider applications of the classification of this work.

In Part I we consider classification for elliptic and parabolic cases. In Part II we consider hyperbolic case. This case is much more complicated. This work is the first where classification is considered for this case.

We hope that our classification will be important in some theories. We mention that from the point of view of the theory of Lorentzian Kac–Moody algebras, the rank 3 case is the hyperbolic analog of sl_2. Moreover the rank 3 case is more visual than the rank ≥ 4 case. For the rank 3 case one has to work with polygons (i.e. fundamental polygons of reflection groups) which are much simpler and much more visual than polyhedra of dimension ≥ 3.

A preliminary variant of this work was published in preprints [N15], [N16] and [N17].

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Part I. Classification of maximal hyperbolic root systems of the rank three and of elliptic or parabolic type

1. Hyperbolic reflection groups and hyperbolic root systems

1.1. Klein model of hyperbolic space.

Let Φ be a *hyperbolic linear space* (i.e. Φ is a $\mathbb{R}$-linear space equipped with a non-degenerate symmetric $\mathbb{R}$-bilinear form of signature $(1,n)$ where $n+1=\dim\Phi$). For $x,y\in\Phi$ we denote by (x,y) the value at the pair (x,y) of the bilinear form corresponding to Φ, we also denote $x^2=(x,x)$. One can associate to Φ an open cone

$$V=V(\Phi)=\{x\in\Phi\mid x^2>0\}. \tag{1.1.1}$$

The V is the disjoint union of two open convex half-cones. We denote by $V^+=V^+(\Phi)$ one of them. The *hyperbolic space* corresponding to Φ is the set of rays

$$\mathcal{L}=\mathcal{L}(\Phi)=V^+/\mathbb{R}_{++}=\{\mathbb{R}_{++}x\mid x\in V^+\} \tag{1.1.2}$$

where $\mathbb{R}_{++}$ denote the set of positive real numbers. The distance ρ in $\mathcal{L}$ is defined by the formula

$$\cosh\rho(\mathbb{R}_{++}x,\mathbb{R}_{++}y)=(x,y)/\sqrt{x^2y^2}. \tag{1.1.3}$$

Then the curvature of $\mathcal{L}$ is equal to -1. The dimension $\dim\mathcal{L}=\dim\Phi-1$.

Let $O(\Phi)$ be the group of automorphisms of the hyperbolic linear space Φ. Then the group of motions of $\mathcal{L}$ is naturally identified with the subgroup $O^+(\Phi)\subset O(\Phi)$ of index 2 of automorphisms which fix the half-cone V^+.

A *half-space* of $\mathcal{L}$ is the set

$$\mathcal{H}_\delta^+=\{\mathbb{R}_{++}x\in\mathcal{L}\mid (x,\delta)\ge 0\}, \tag{1.1.4}$$

where $\delta\in\Phi$ and $\delta^2<0$. The half-space $\mathcal{H}_\delta^+$ is bounded by the *hyperplane*

$$\mathcal{H}_\delta=\{\mathbb{R}_{++}x\in\mathcal{L}\mid (x,\delta)=0\}. \tag{1.1.5}$$

The element $\delta\in\Phi$ is defined by the half-space $\mathcal{H}_\delta^+$ (respectively, by the hyperplane $\mathcal{H}_\delta$) up to multiplication by elements of $\mathbb{R}_{++}$ (respectively, by elements of the set $\mathbb{R}^*$ of non-zero real numbers). The δ is called the *orthogonal vector* to the half-space $\mathcal{H}_\delta^+$ (respectively to the hyperplane $\mathcal{H}_\delta$).

Consider two half-spaces $\mathcal{H}_{\delta_1}^+$, $\mathcal{H}_{\delta_2}^+$ orthogonal to $\delta_1,\ \delta_2\in\Phi$. Assume that $\mathcal{H}_{\delta_1}^+\not\subset\mathcal{H}_{\delta_2}^+$, $\mathcal{H}_{\delta_2}^+\not\subset\mathcal{H}_{\delta_1}^+$, and intersection $\mathcal{H}_{\delta_1}^+\cap\mathcal{H}_{\delta_2}^+$ contains a non-empty open subset of $\mathcal{L}(\Phi)$. Then we have two cases:

a) $\mathcal{H}_{\delta_1}^+\cap\mathcal{H}_{\delta_2}^+$ is an angle of a value ϕ where

$$\cos\phi=(\delta_1,\delta_2)/\sqrt{\delta_1^2\delta_2^2} \tag{1.1.6}$$

if $-1\le(\delta_1,\delta_2)/\sqrt{\delta_1^2\delta_2^2}\le 1$.

b) Intersection of the hyperplanes $\mathcal{H}_{\delta_1}$, $\mathcal{H}_{\delta_2}$ is empty (they are hyperparallel) and the distance ρ between them is equal to

$$\cosh\rho=(\delta_1,\delta_2)/\sqrt{\delta_1^2\delta_2^2} \tag{1.1.7}$$

if $1 \le (\delta_1, \delta_2)/\sqrt{\delta_1^2\delta_2^2}$.

As usual, we complete $\mathcal{L}$ by the set of *infinite points* $\mathbb{R}_{++}c$ where $c \in \Phi$, $c^2 = 0$ and $(c, V^+) > 0$. Then $\mathcal{L}$ with its infinite points is the closed ball

$$\overline{\mathcal{L}} = (\overline{V^+} - \{0\})/\mathbb{R}_{++}, \tag{1.1.8}$$

where $\overline{V^+}$ is the closure of V^+ in the linear space Φ. The boundary $\mathcal{L}_\infty = \overline{\mathcal{L}} - \mathcal{L}$ is the sphere of dimension $n-1$. It is called ∞*-sphere* or *infinity* of $\mathcal{L}$. We also add infinite points to half-spaces, hyperplanes and other subsets of $\mathcal{L}$ considering their closure in $\overline{\mathcal{L}}$.

A non-degenerate convex locally finite polyhedron $\mathcal{M}$ in a hyperbolic space $\mathcal{L}$ is the intersection of a set of half-spaces:

$$\mathcal{M} = \bigcap_{\delta \in P(\mathcal{M})} \mathcal{H}_\delta^+. \tag{1.1.9}$$

The $\mathcal{M}$ is called *non-degenerate* if $\mathcal{M}$ contains a non-empty open subset of $\mathcal{L}$. The $\mathcal{M}$ is called *locally finite* if for any $X \in \mathcal{L}$ there exists its open neighbourhood $U \subset \mathcal{L}$ such that $\mathcal{M} \cap U$ is the intersection with U of a finite set of half-spaces in $\mathcal{L}$. Here $P(\mathcal{M}) \subset \Phi$ is a finite or countable subset of elements with negative square. We assume that no two of these elements are proportional and every half-space $\mathcal{H}_\delta^+$, $\delta \in P(\mathcal{M})$, defines a face of $\mathcal{M}$ of the highest dimension $\dim \mathcal{L} - 1$ (it is clear what this means, since $\mathcal{M}$ is locally finite). Then, the polyhedron $\mathcal{M}$ defines the subset $P(\mathcal{M}) \subset \Phi$ uniquely up to multiplication by $\mathbb{R}_{++}$ elements of $P(\mathcal{M})$. The set $P(\mathcal{M})$ is called *the set of vectors orthogonal to* $\mathcal{M}$ *(or to faces of highest dimension of* $\mathcal{M}$*).* Below, we consider only locally finite and convex polyhedra.

A polyhedron $\mathcal{M}$ is called *elliptic* (or it has *geometrically finite volume*) if $\mathcal{M}$ is a convex envelope of a finite set of points in $\mathcal{L}$, some of them at infinity of $\mathcal{L}$.

1.2. Reflection groups of hyperbolic lattices, and hyperbolic root systems.

Let S be a hyperbolic (that is, of signature $(1, n)$) integral symmetric bilinear form over $\mathbb{Z}$. This means that S is a free $\mathbb{Z}$-module of finite rank equipped with an integral valued symmetric $\mathbb{Z}$-bilinear form and the corresponding real form $\Phi = S \otimes \mathbb{R}$ is hyperbolic (it has signature $(1, n)$ like in Sect. 1.1). For brevity, S will be referred to as the *hyperbolic lattice*. Like in Sect. 1.1, we correspond to a hyperbolic lattice S the hyperbolic space $\mathcal{L} = \mathcal{L}(S) = \mathcal{L}(S \otimes \mathbb{R})$. From the theory of integral quadratic forms (more generally, from the theory of arithmetic groups), it is known that the automorphism group $O^+(S)$ of a hyperbolic lattice S is discrete in $\mathcal{L}$; if $\mathrm{rk}\, S \ge 3$, the group $O^+(S)$ has a fundamental domain of finite volume which is an elliptic convex polyhedron (e. g. see [C], [R1], [VSh]).

An element $\alpha \in S$ is called *root* if $\alpha^2 < 0$ and $\alpha^2 | 2(S, \alpha)$. If α is a root, then $-\alpha$ is also a root. A non-zero element $x \in S$ is called *primitive* if $x/m \notin S$ for any natural $m > 1$. Obviously, any root α is multiple to a primitive root which is called *the primitive root of* α. If α_0 is a primitive root, then $\lambda\alpha_0$ is a root only for a finite number of $\lambda \in \mathbb{N}$. Really, $\frac{\alpha_0}{(\lambda/2)} \in S^* = Hom(S, \mathbb{Z})$, and $\lambda \le 2a(S)$ where $a(S)$ is the exponent (the maximal order of elements) of the discriminant group $A_S = S^*/S$ of S. In particular, $\lambda \le 2|\det(S)|$ where the *determinant of* S is equal to $\det(S) = \det((e_i, e_j))$ for a bases $e_1, \dots, e_n$ of S. Thus, to find roots, it is sufficient to find primitive roots and then multiply them on restricted multiplicities.

Any root $\alpha \in S$ defines the *reflection* $s_\alpha \in O^+(S)$ which is given by the formula

$$x \mapsto x - \frac{2(x,\alpha)}{\alpha^2}\alpha, \quad x \in S. \tag{1.2.1}$$

The reflection depends only on $\pm\alpha_0$ where α_0 is the primitive root of α. The reflection s_α acts in $\mathcal{L} = \mathcal{L}(S)$ as the reflection in the hyperplane $\mathcal{H}_\alpha$ orthogonal to the root α (i.e. s_α is identical on $\mathcal{H}_\alpha$ and changes two half-spaces bounded by $\mathcal{H}_\alpha$). It is easy to see that an automorphism $\phi \in O^+(S)$ gives a reflection in a hyperplane of $\mathcal{L}$ if and only if ϕ is a reflection in some root of the lattice S.

All reflections s_α in all roots of S generate the *reflection group* $W(S) \subset O^+(S)$ of the lattice S. We shall consider subgroups $W \subset W(S)$ generated by some set of reflections of S. We shall call W by *reflection subgroup of* S.

Let W be a reflection subgroup of S. From the theory of groups generated by reflections (e.g. see [V1], [V4], [VSh]), it follows that W has a fundamental polyhedron $\mathcal{M}$ in $\mathcal{L}$ which is the closure of a connected component of $\mathcal{L}$ without hyperplanes of all reflections from W. The polyhedron $\mathcal{M}$ is a convex locally finite polyhedron. We always assume that the set $P(\mathcal{M})$ of orthogonal vectors to $\mathcal{M}$ is a set of roots of S. It is also called the *set of simple roots*. The group W is generated by reflections s_α, $\alpha \in P(\mathcal{M})$. The set $\{s_\alpha \mid \alpha \in W(P(\mathcal{M}))\}$ gives then all reflections of the reflection subgroup W. The set $\Delta = W(P(\mathcal{M}))$ is called a *(hyperbolic) root system* of the lattice S. One can then look at the lattice S as a *root lattice*. Hyperbolic root systems are very important for some theories, for example, for the theory of Lorentzian (or hyperbolic) Kac–Moody algebras (see [Kac]). It is clear that to find all possible root systems, it is sufficient to restrict by *primitive root systems* containing only primitive roots of S. All other root systems are *twisted* to primitive root systems: there roots are multiple (by bounded multiplicities) to roots of a primitive root system. To find a root system Δ it is sufficient to find a system of its simple roots $P(\mathcal{M})$ since $\Delta = W(P(\mathcal{M}))$ where W is generated by reflections in $P(\mathcal{M})$. To find a system of simple roots $P(\mathcal{M})$, it is sufficient to find a system of simple primitive roots $P(\mathcal{M})_{\mathrm{pr}}$ which is equivalent to the reflection subgroup $W \subset W(S)$. The system of simple roots $P(\mathcal{M})$ is twisted to $P(\mathcal{M})_{\mathrm{pr}}$: one can get it by multiplying of elements of $P(\mathcal{M})_{\mathrm{pr}}$ by some bounded natural numbers (*twisting coefficients*).

1.3. Restricted arithmetic type and a generalized lattice Weyl vector.

We denote by

$$A(P(\mathcal{M})) = \{g \in O^+(S) \mid g(P(\mathcal{M})) = P(\mathcal{M})\} \tag{1.3.1}$$

the *group of symmetries of the fundamental polyhedron (together with the set* $P(\mathcal{M})$ *of its orthogonal roots).* The group W with the set $P(\mathcal{M})$ of simple roots (equivalently, the root system $\Delta = W(P(\mathcal{M}))$) have *restricted arithmetic type* (see [N11], [N10]), if the semi-direct product $\mathrm{Aut}(\Delta) = W \rtimes A(P(\mathcal{M}))$ has finite index in $O^+(S)$. Obviously, $A(P(\mathcal{M})) \subset A(P(\mathcal{M})_{\mathrm{pr}})$, and $P(\mathcal{M})_{\mathrm{pr}}$ has restricted arithmetic type if $P(\mathcal{M})$ does. If W and $P(\mathcal{M})_{\mathrm{pr}}$ have restricted arithmetic type, we say that the reflection subgroup W *has restricted arithmetic type*. A non-zero element $\rho \in S$ is called a *generalized lattice Weyl vector* if $A(\rho) = \rho$ for a subgroup $A \subset A(P(\mathcal{M}))$ of finite index (equivalently, the orbit $A(P(\mathcal{M}))(\rho)$ is finite). Reflection subgroups W with $P(\mathcal{M})$ (and corresponding root systems $\Delta = W(P(\mathcal{M}))$) of restricted arithmetic type and with a generalized lattice Weyl vector are important for the theory of K3 and other surfaces (see [P-ŠŠ], [N1]—[N9], [N12], [N13], [AN1], [AN2]), the theory of Borcherds type automorphic products and theory of automorphic Lorentzian Kac–Moody algebras (see [B2]—[B6], [GN1]—[GN7], [N10], [N11]) and other theories (e. g. see [HM1], [HM2], [M], [Kaw1], [Kaw2] on applications in Physics). It seems, hyperbolic root systems satisfying these conditions are of the same importance as well-known finite and affine root systems.

It is an interesting but difficult problem to find all hyperbolic root systems Δ (or corresponding systems of simple roots $P(\mathcal{M})$) of restricted arithmetic type and with a generalized lattice Weyl vector.

Below we shall make some general remarks about this problem.

First, root systems, simple root systems and reflection subgroups W of restricted arithmetic type with a generalized lattice Weyl vector are divided on three types:

Elliptic type: There exists a generalized lattice Weyl vector ρ with positive square $\rho^2 > 0$.

Parabolic type: There exists a generalized lattice Weyl vector ρ with zero square $\rho^2 = 0$ and there does not exist a generalized lattice Weyl vector with positive square.

Hyperbolic type: There exists a generalized lattice Weyl vector ρ with negative square $\rho^2 < 0$ and there does not exist a generalized lattice Weyl vector with positive or zero square.

Obviously, W and $P(\mathcal{M})$ of restricted arithmetic type have a generalized lattice Weyl vector ρ if and only if W and $P(\mathcal{M})_{\text{pr}}$ have restricted arithmetic type and the same generalized lattice Weyl vector ρ. Thus, to find all W and $P(\mathcal{M})$ of restricted arithmetic type and with a generalized lattice Weyl vector, we can restrict by primitive sets $P(\mathcal{M})_{\text{pr}}$ which are equivalent to reflection subgroups $W \subset W(S)$. One should find all reflection subgroups $W \subset W(S)$ of restricted arithmetic type and with a generalized lattice Weyl vector for $P(\mathcal{M})_{\text{pr}}$ where $\mathcal{M}$ is a fundamental polyhedron for W. Here the set $P(\mathcal{M})_{\text{pr}}$ is defined by the reflection subgroup $W \subset W(S)$ (up to choosing the fundamental polyhedron $\mathcal{M}$).

1.4. Reflective hyperbolic lattices.

Assume that a reflection subgroup $W \subset W(S)$ has restricted arithmetic type and a generalized lattice Weyl vector ρ (for $P(\mathcal{M})_{\text{pr}}$ where $\mathcal{M}$ is a fundamental polyhedron of W). Suppose that $W \subset W' \subset W(S)$ where W' is a reflection subgroup of restricted arithmetic type. We can choose a fundamental polyhedron $\mathcal{M}'$ for W' such that $\mathcal{M}' \subset \mathcal{M}$. If ρ is a generalized lattice Weyl vector for $P(\mathcal{M})_{\text{pr}}$, then ρ is a generalized lattice Weyl vector for $P(\mathcal{M}')_{\text{pr}}$ (one should use that $A(P(\mathcal{M}')_{\text{pr}}) \cap A(P(\mathcal{M})_{\text{pr}})$ has finite index in $A(P(\mathcal{M}')_{\text{pr}})$ since W has restricted arithmetic type). Thus, W' also has a generalized lattice Weyl vector. If W has elliptic type, then W' also has elliptic type. If W has parabolic type, then W' has parabolic or elliptic type. If W has hyperbolic type, then W' has either hyperbolic, or parabolic, or elliptic type. The full reflection group $W(S)$ of the lattice S obviously has restricted arithmetic type: it is normal in $O^+(S)$. Thus, $W(S)$ itself has restricted arithmetic type and a generalized lattice Weyl vector if $W(S)$ has a reflection subgroup $W \subset W(S)$ of restricted arithmetic type and with a generalized lattice Weyl vector. Hyperbolic lattices S with this property are called *reflective*. Thus, (equivalently) a hyperbolic lattice S is *reflective* if it has a generalized lattice Weyl vector for its reflection group $W(S)$. Depending on the type (elliptic, parabolic or hyperbolic) of $W(S)$, the reflective lattice S is called *reflective of elliptic, parabolic or hyperbolic type* or *elliptically, parabolically or hyperbolically reflective*. It seems that these maximal reflection groups $W(S)$ having a generalized lattice Weyl vector are especially interesting. Moreover, for reflective lattices S, one can try to find out all reflection subgroups $W \subset W(S)$ of restricted arithmetic type and with a generalized lattice Weyl vector. If the lattice S is not reflective, any reflection subgroup $W \subset W(S)$ of restricted arithmetic type does not have a generalizes lattice Weyl vector.

Multiplication of the form of a lattice S does not change anything. Thus, we can always suppose that the lattice S is *primitive*, i.e. $S(1/k)$ is not a lattice for any natural $k > 1$. Here $K(r)$ denote a lattice which is obtained by multiplication of the form of a lattice K on $r \in \mathbb{Q}$.

We have the following general result:

Theorem 1.4.1 ([N4], [N5], [N11], [N13])**.** *For the fixed rank rk $S \geq 3$ the set of primitive reflective hyperbolic lattices S is finite.*

Cases rk $S = 1$ or rk $S = 2$ are trivial. Any hyperbolic lattice S of rk $S = 1$ is reflective of elliptic type since $O^+(S)$ is trivial. If rk $S = 2$, the lattice S is reflective if and only if either S has a non-zero element with square 0 (then $O^+(S)$ is finite and S is reflective of elliptic type) or S has at least one root (then S is also reflective of elliptic type). We remark that the proof of Theorem 1.4.1 for the hyperbolic type is only outlined in [N13]. For rk $S = 3$ all necessary details will be given in this paper (see Theorem 8.4.1.1 in Part II).

The main subject of this work is to describe hyperbolic reflective lattices of rank 3. We apply the same method (of narrow parts of polyhedra in hyperbolic spaces) which was applied to prove Theorem 1.4.1. But to find all possibilities, we need to improve this method and add some additional arithmetic arguments. Certainly, classification is more delicate problem than proving finiteness.

2. Classification of reflective hyperbolic lattices of the rank 3 and of elliptic or parabolic type: formulations

2.1. The principle of classification.

Let S_1 be a hyperbolic lattice and $S_1 \subset S_2$ its overlattice. In this Section we consider embeddings of lattices of the same rank only. If the embedding $S_1 \subset S_2$ induces an embedding of the automorphism groups $O(S_1) \subset O(S_2)$ (in general, these groups are commensurable), we say that the embedding $S_1 \subset S_2$ is an *equivariant embedding.* We remark that index $[O(S_2) : O(S_1)]$ is always finite because both groups $O^+(S_2)$ and $O^+(S_1)$ have a fundamental domain of finite volume in naturally identified hyperbolic spaces $\mathcal{L}(S_1 \otimes \mathbb{R}) = \mathcal{L}(S_2 \otimes \mathbb{R})$. It follows that an equivariant embedding gives an embedding of the reflection groups $W(S_1) \subset W(S_2)$ where both groups $W(S_1)$ and $W(S_2)$ have restricted arithmetic type. It follows that the lattice S_2 is reflective if the lattice S_1 does. Obviously, S_2 is primitive if S_1 does. A lattice S is called *maximal* if it does not have equivariant embeddings to other lattices of the same rank except the lattice S itself.

Our principle of classification of reflective hyperbolic lattices of the rank 3 is as follows: We define *elementary lattices* such that any lattice has an equivariant embedding into an elementary one. We give the list of *reflective elementary hyperbolic lattices of the rank* 3. Thus, any reflective hyperbolic lattice of the rank 3 has an equivariant embedding into a lattice of this list. For any reflective elementary hyperbolic lattice S of rank 3 we calculate the set of simple roots $P(\mathcal{M})_{\rm pr}$ of $W(S)$ and the Gram matrix $G(P(\mathcal{M})_{\rm pr}) = ((\alpha,\beta))$, $\alpha,\beta \in P(\mathcal{M})_{\rm pr}$. Moreover, we estimate indexes $[S:S_1]$ of its primitive reflective sublattices $S_1 \subset S$, and one can find all of them if one needs (their number is obviously finite). It seems, reflective elementary hyperbolic lattices are especially interesting (e. g. they contain maximal reflective hyperbolic lattices), and finding all of them is especially important. On the other hand, the mere fact that a lattice S_1 has an equivariant embedding $S_1 \subset S$ into a known lattice S contains a lot of information about the lattice S_1 itself. For example, $\det(S_1) = det(S)[S:S_1]^2$, any primitive root α_1 of S_1 is equal to $t\alpha$ where α is a primitive root of S and $t \le [S:S_1]$, in particular, $\alpha_1^2 = t^2\alpha^2$.

2.2. Elementary hyperbolic lattices of the rank 3.

We recall some elements of the *discriminant form technique* (see [N1]).

Let L be an arbitrary lattice (equivalently, a non-degenerate integral symmetric bilinear form). The lattice L is called *odd* if there exists $x \in L$ with odd square x^2. Otherwise, L is called *even*.

The group $A_L = L^*/L$ is called the *discriminant group of* L. Its order $\sharp A_L = |\det(L)|$. We extend the form of L to the dual lattice $L \subset L^* = Hom(L,\mathbb{Z})$. It then takes values in $\mathbb{Q}$ and defines (considering $\mod L$) a non-degenerated finite symmetric bilinear form

$$b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}. \tag{2.2.1}$$

If the lattice L is even, one similarly defines a finite quadratic form

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z} \tag{2.2.2}$$

with the symmetric bilinear form b_L. These forms are called the *discriminant bilinear and the discriminant quadratic form* of the lattice L.

An overlattice $L \subset \widetilde{L}$ is defined by an isotropic (for b_L) subgroup $H_{\widetilde{L}} = \widetilde{L}/L \subset A_L$. The discriminant bilinear form of $\widetilde{L}$ is then equal to

$$b_{\widetilde{L}} = b_L|(H_{\widetilde{L}}^{\perp}/H_{\widetilde{L}}). \tag{2.2.3}$$

If the lattice L is even and the subgroup $H_{\widetilde{L}}$ is also isotropic for the discriminant quadratic form q_L, then $\widetilde{L}$ is even and

$$q_{\widetilde{L}} = q_L|(H_{\widetilde{L}}^{\perp}/H_{\widetilde{L}}). \tag{2.2.4}$$

For a prime p, we denote as A_p the p-component of A_L, and b_p and q_p restrictions of b_L and q_L on the subgroup. Consider the p-adic lattice $L_p = L \otimes \mathbb{Z}_p$ where $\mathbb{Z}_p$ is the ring of p-adic integers and $\mathbb{Q}_p$ the field of p-adic numbers. One can similarly define the discriminant group A_{L_p}, and the discriminant forms b_{L_p}, q_{L_p} of the p-adic lattice L_p. We have natural identifications: $A_p = A_{L_p}$, $b_p = b_{L_p}$ and $q_p = q_{L_p}$.

Definition 2.2.1. A lattice L is called *elementary* if for any prime $p|det(L)$ the p-component A_{L_p} is a p-elementary Abelian group: $A_{L_p} \cong (\mathbb{Z}/p\mathbb{Z})^{r_p}$. Here $\{r_p = r_p(L) \mid p|\det(L)\}$ are invariants of the elementary lattice L. We have $|\det(L)| = \prod_p p^{r_p}$. In particular, the invariants $r_p(L)$ are defined by $\det(L)$.

We have the following simple and well-known

Proposition 2.2.2. *Any lattice L has an equivariant (and even canonical) embedding into an elementary lattice.*

Proof. Let p^{t_p} be the exponent of the group $A_p = A_{L_p}$. It means that $p^{t_p}A_p = 0$ and $p^{t_p-1}A_p \neq 0$. If L is elementary, L has an equivariant embedding into itself. If L is not elementary, one of exponents $t_p > 1$. Obviously, $b_{L_p} : A_p \times A_p \to \frac{1}{p^{t_p}}\mathbb{Z}/\mathbb{Z}$. It follows that subgroup $H_p = p^{t_p-1}A_p \subset A_p$ is isotropic. It is not zero and is invariant with respect to $O(L)$. It follows that H_p defines an equivariant embedding $L \subset \widetilde{L}$ with $\widetilde{L}/L = H_p$ and index $\sharp H_p$. If the lattice $\widetilde{L}$ is not elementary, we repeat the procedure for $\widetilde{L}$. This procedure is finite because $\det(L_1) = \det(L)[L_1 : L]^{-2}$ for an overlattice $L \subset L_1$ of finite index $[L_1 : L]$ where $\det(L)$ and $\det(L_1)$ are integers. It proves Proposition.

Let L be an elementary lattice, $m \in \mathbb{N}$ and m is square-free. The lattice

$$L^{*,m}(m) = (L^* \cap \frac{1}{m}L)(m) \tag{2.2.5}$$

is called *m-dual to the lattice L*. This lattice is also elementary and

$$r_p(L^{*,m}(m)) = \begin{cases} r_p(L), & \text{if } p \nmid m \\ \mathrm{rk}\ L - r_p(L) & \text{if } p|m \end{cases}. \tag{2.2.6}$$

One can also see that $L^{*,m}(m) \otimes \mathbb{Z}_p = (L \otimes \mathbb{Z}_p)(m)$ if $p \not| m$, and $L^{*,m}(m) \otimes \mathbb{Z}_p = (L \otimes \mathbb{Z}_p)^*(m)$ if $p|m$. Moreover,

$$(L^{*,m}(m))^{*,m}(m) = L \tag{2.2.7}$$

which shows that it really is a duality. More generally, for two square-free numbers m_1, m_2 one has

$$(L^{*,m_1}(m_1))^{*,m_2}(m_2) = L^{*,m}(m) \tag{2.2.7'}$$

where $m = \text{l.c.m.}(m_1, m_2)/\text{g.c.d}(m_1, m_2)$. It follows that one has a natural identification

$$O(L) = O(L^{*,m}(m)). \tag{2.2.8}$$

If L is a hyperbolic lattice, one can look at this equality as follows: *Hyperbolic spaces and actions of automorphism groups in these spaces of L and $L^{*,m}(m)$ are naturally identified.*

Now we consider only primitive elementary hyperbolic lattices of rank 3. Let F be a primitive elementary hyperbolic lattice of the rank 3 and $\det(F) = \prod_p p^{r_p}$. Since F is primitive, $0 \le r_p \le 2$. Let m be the product of all prime p such that $p|\det(F)$ and $r_p = 2$. Equivalently, $\det(F) = (\det(F)/m)m$ where both natural numbers $\det(F)/m$ and m are square-free. By (2.2.6), the m-dual lattice

$$S = F^{*,m}(m) \tag{2.2.9}$$

has then a square-free determinant $\det(S) = \det(F)/m$. Thus we have

Proposition 2.2.3. *Any primitive elementary hyperbolic lattice F of the rank three is m-dual to a hyperbolic lattice S with a square-free determinant $d=\det(S)$ where $m|d$. Then $\det(F)=dm$. A lattice with a square-free determinant is (obviously) elementary and primitive.*

By Proposition 2.2.3, to describe primitive elementary hyperbolic lattices of the rank three, it is sufficient to describe hyperbolic lattices of the rank three with a square-free determinant. Let S be a hyperbolic lattice with a square-free determinant $d=\det(S)$. If d is odd, then S is also odd (otherwise, the 2-adic lattice $S\otimes\mathbb{Z}_2$ is even unimodular of odd rank, which is impossible). If d is even, then the lattice S may be even or odd. If S is odd (respectively even) we say that S has *odd type* (respectively *even type*). For odd prime $p|d$ the discriminant bilinear form $b_{S_p}\cong b_{\theta_p}(p)$ where $b_{\theta_p}(p)$ is the discriminant bilinear form of the p-adic 1-dimensional lattice $\langle\theta_p p\rangle$ where $\theta_p\in\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2\ \cong\{\pm1\}$ by isomorphism $\theta_p\to\left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}$ where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and $\eta_p\in\{0,1\}$. (Here and in what follows $\langle A\rangle$ denote a lattice with the matrix A.) Thus, the lattice S with a square-free determinant d has the invariant

$$\eta=\{\eta_p\mid \text{odd } p|d\} \text{ where } \eta_p\in\{0,\ 1\} \text{ and } (-1)^{\eta_p}=\left(\frac{\theta_p}{p}\right). \tag{2.2.10}$$

Below we code this invariant by a non-negative integer η having a binary decomposition

$$\eta=\eta_{p_t}\dots.\eta_{p_1} \tag{2.2.10'}$$

where $p_1,\dots,p_t$ are all odd prime divisors of d in increasing order. For example, for $d=30=2\cdot3\cdot5$ the invariant $\eta=2$ means that $\left(\frac{\theta_3}{3}\right)=1$, $\left(\frac{\theta_5}{5}\right)=-1$.

We have

Proposition 2.2.4. *A hyperbolic lattice S of rank three and with a square-free determinant is defined up to isomorphism by the invariants $(d,\ type,\ \eta)$.*

Proof. Invariants $(d,\ type,\ \eta)$ define the discriminant bilinear form of S. By [N1], the signature, type and discriminant bilinear form of S define the genus of S. By general results of Eichler and Kneser on spinor genus of indefinite lattices of rank ≥ 3 (see [Ei], [Kn] and also [C]), elementary hyperbolic lattices of rank ≥ 3 have only one class in a genus. It proves the statement.

Definition 2.2.5. A hyperbolic lattice S of the rank three and with a square-free determinant d is called *main* if it has a square-free determinant d and *type* $\equiv d \bmod 2$. In other words, the last condition means that the lattice S should be even if the determinant d (it should be square-free) is even. If the determinant d is odd, then the lattice S will be necessarily odd. In particular, a main hyperbolic lattices of rank three is defined by the invariants $(d,\ \eta)$ where $d=\det(S)$ is square-free.

Main hyperbolic lattices of the rank three are the most important. Really, suppose that $\widetilde{S}$ has rank three and square-free determinant, but it is not main. Then the determinant $d(\widetilde{S})=2d$ is even but the lattice $\widetilde{S}$ is odd. Let $\widetilde{S}^{ev}\subset\widetilde{S}$ be the maximal even sublattice of $\widetilde{S}$. It has index two. Considering $\widetilde{S}\otimes\mathbb{Z}_2$, one finds that $\widetilde{S}^{ev}=S(2)$ where S has the odd determinant d, and is then main odd hyperbolic lattice. By the construction, $O(\widetilde{S})\subset O(S)$. Thus, *main hyperbolic lattices of the* *rank three have maximal automorphism groups.* In particular, if $\widetilde{S}$ is reflective, then S is also reflective. Moreover, we get that any non-main hyperbolic lattice $\widetilde{S}$ is the index two odd overlattice

$$S(2) \subset \widetilde{S} \tag{2.2.11}$$

where S is a main odd hyperbolic lattice. If S has invariants (d, odd, η) (where d is square-free and odd), then $\widetilde{S}$ has invariants $(2d, \text{odd}, \eta + \omega(d))$ where $\omega(d) = \{\omega(d)_p = (p^2-1)/8 \mod 2 \mid \text{odd prime } p|d\}$. We use that $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

For the fixed S, all index 2 odd overlattices $\widetilde{S}$ in (2.2.11) are isomorphic by Proposition 2.2.3. Thus, (2.2.11) defines the one-to-one correspondence $S \leftrightarrow \widetilde{S}$ between odd hyperbolic lattices of rank three and with a square-free determinant where S has odd determinant d and $\widetilde{S}$ has the even determinant $2d$.

We have mentioned that (2.2.11) defines embedding

$$O(\widetilde{S}) \subset O(S)$$

since the lattice $S(2)$ is the maximal even sublattice of $\widetilde{S}$. There are two cases.

Case when lattices S and $\widetilde{S}$ are equivariantly equivalent: By definition, it means that the odd overlattice $S(2) \subset \widetilde{S}$ is unique. Then (2.2.11) gives $O(\widetilde{S}) = O(S)$, and lattices S, $\widetilde{S}$ have the same reflective type. By the discriminant form technique, this means that the 2-component $A_{S(2)_2} \cong (\mathbb{Z}/2\mathbb{Z})^3$ of the discriminant group of $S(2)$ has an unique element $w \in A_{S(2)_2}$ with $q_{S(2)}(w) = 1 \mod 2$. Then $\widetilde{S}/S(2) = (\mathbb{Z}/2\mathbb{Z})w$, and index two odd overlattice in (2.2.11) is unique. The discriminant quadratic form $q_{S(2)_2}$ is defined by its signature $\operatorname{sign} q_{S(2)_2} \mod 8$ (see [N1] where all finite quadratic forms on 2-elementary groups were classified). One can easily check that the element w is unique if and only if $\operatorname{sign} q_{S(2)_2} \equiv \pm 1 \mod 8$. We have (e.g. see [N1])

$$\sum_p \operatorname{sign} q_{S(2)_p} \equiv \sum_{p|d} (1 - p + 4\eta_p + 4\omega(d)_p) + \operatorname{sign} q_{S(2)_2} \equiv 1 - 2 \equiv -1 \mod 8.$$

It follows that S and $\widetilde{S}$ are equivariantly equivalent if and only if

$$\sum_{p|d} (1 - p + 4\eta_p + 4\omega(d)_p) \equiv \pm 1 - 1 \equiv 0 \text{ or } 6 \mod 8.$$

Case when S and $\widetilde{S}$ are not equivariantly equivalent: by definition it means that the odd index two overlattice $\widetilde{S}$ in (2.2.11) is not unique. By consideration above, it is equivalent to

$$\sum_{p|d} (1 - p + 4\eta_p + 4\omega(d)_p) \mod 8 \notin \{0 \mod 8,\ 6 \mod 8\}.$$

One can easily check that then there are three different possibilities for the element $w \in A_{S(2)_2}$ with $q_{S(2)}(w) = 1 \mod 2$. Thus, there are three different overlattices $\widetilde{S}$ in (2.2.11). All of them give isomorphic lattices $\widetilde{S}$ but one cannon expect that automorphism groups of S and of $\widetilde{S}$ are the same, and it is never the case. Really, an isomorphism of two different overlattices $\widetilde{S}$ in (2.2.11) gives an automorphism of their common maximal even sublattice $S(2)$ which has no extension to $\widetilde{S}$. The (2.1.11) defines only embedding $O(\widetilde{S}) \subset O(S)$, and we only have that *the reflective type of $\widetilde{S}$ is dominated by the reflective type of S.* To find all reflective lattices $\widetilde{S}$, one should first find all reflective lattices S, and then check if the lattices $\widetilde{S}$ are also reflective. We summarize these considerations below.

Proposition 2.2.6. *All non-main hyperbolic lattices $\widetilde{S}$ of rank three and with a square-free determinant (remind that non-main means that $\widetilde{S}$ is odd but $\det(\widetilde{S}) = 2d$ is even) are in one-to-one correspondence $S \leftrightarrow \widetilde{S}$ with main odd hyperbolic lattices S of rank three and with a square-free determinant $d = \det(S) = \det(\widetilde{S})/2$. The correspondence is defined by the embedding of lattices*

$$S(2) \subset \widetilde{S}, \tag{2.2.12}$$

where $S(2)$ is the maximal even sublattice of $\widetilde{S}$ (it has index two).

If S has invariants $(d,\ odd,\ \eta)$, where d is odd square-free, then $\widetilde{S}$ has invariants $(2d,\ odd,\ \eta + \omega(d))$ where

$$\omega(d) = \{\omega(d)_p = (p^2-1)/8 \mod 2 \ \mid\ odd\ prime\ p|d\}. \tag{2.2.13}$$

If

$$\sum_{p|d}(1-p+4\eta_p+4\omega(d)_p) \equiv \pm 1 - 1 \equiv 0\ or\ 6 \mod 8, \tag{2.2.14}$$

the overlattice $\widetilde{S}$ in (2.2.12) is unique, (2.2.12) defines isomorphism $O(\widetilde{S}) = O(S)$, and lattices S and $\widetilde{S}$ have the same reflective type. We then say that lattices S and $\widetilde{S}$ are equivariantly equivalent.

If

$$\sum_{p|d}(1-p+4\eta_p+4\omega(d)_p) \mod 8 \notin \{0 \mod 8,\ 6 \mod 8\}, \tag{2.2.15}$$

(2.2.12) only defines the embedding $O(\widetilde{S}) \subset O(S)$, and reflective type of $\widetilde{S}$ is dominated by reflective type of S. For this case we say that lattices S and $\widetilde{S}$ are not equivariantly equivalent.

2.3. The classification of elliptically or parabolically reflective elementary hyperbolic lattices of the rank 3.

2.3.1. Notation. According to Sect. 2.2, classification of reflective (of any type) primitive elementary hyperbolic lattices of the rank three is divided in three parts:

1) Classification of main reflective hyperbolic lattices of the rank three.

2) Classification of non-main reflective hyperbolic lattices $\widetilde{S}$ of the rank three with square-free determinant where S belongs to 1) and is odd (see Proposition 2.2.6).

3) Taking of m-dual lattices to lattices L from 1) and 2) where $m|\det(L)$ (all of them will be reflective of the same type as L).

For a hyperbolic lattice S, we denote by $\mathcal{M}$ a fundamental polyhedron of $W(S)$ and by $P(\mathcal{M})_{\mathrm{pr}}$ the set of orthogonal to $\mathcal{M}$ primitive roots from S. We denote by $A(P(\mathcal{M})_{\mathrm{pr}})$ the group of symmetries of $P(\mathcal{M})_{\mathrm{pr}}$ (see (1.3.1)).

Definition 2.3.1.1. An automorphism $\phi \in A(P(\mathcal{M})_{\rm pr})$ is called a *central symmetry* if it acts as a central symmetry in the hyperbolic space $\mathcal{L}(S)$. Two central symmetries ϕ_1, $\phi_2 \in A(P(\mathcal{M})_{\rm pr})$ are equivalent if they are conjugate in $A(P(\mathcal{M})_{\rm pr})$ (i.e. $\phi_2 = \xi\phi_1\xi^{-1}$ where $\xi \in A(P(\mathcal{M})_{\rm pr})$. We denote by $h = h(S)$ the *number of classes* of central symmetries in $A(P(\mathcal{M})_{\rm pr})$.

This invariant is important because we have the following trivial

Lemma 2.3.1.2. *If a hyperbolic lattice S is reflective of elliptic type, then $h = h(S) \le 1$. If S is reflective of parabolic type, then $h = h(S) = 0$. If S is reflective of hyperbolic type and rk $S = 3$, then $h = h(S) = 0$ or 2.*

Proof. See proofs of Theorems 5.1.1 and 9.1.1.

It seems, in connection with reflective hyperbolic lattices, the invariant h was first considered by O.V. Shvartsman [Shv].

For all reflective elementary hyperbolic lattices S, we give the invariant $h = h(S)$, the set $P(\mathcal{M})_{\rm pr}$ and the Gram matrix $G(P(\mathcal{M})_{\rm pr}) = ((\alpha_i, \alpha_j))$, α_i, $\alpha_j \in P(\mathcal{M})_{\rm pr}$.

We denote by $\langle A\rangle$ the lattice with the matrix A. Thus, one has $((e_i, e_j)) = A$ for a bases $\{e_i\}$ of the lattice. This bases is called *standard.* We denote by $\oplus$ the orthogonal sum of lattices. Thus, $\langle A\rangle \oplus \langle B\rangle$ denote a lattice with the matrix $A \oplus B$. Let A be a matrix of size $n \times n$ defining a lattice $\langle A\rangle$ with the standard bases $\{e_i\}$. We denote by $\langle A\rangle(c_{11}, \dots, c_{1n}; \dots; c_{m1}, \dots, c_{mn})$ the overlattice of $\langle A\rangle$ generated by $\{e_i\}$ and by elements $c_{11}e_1 + \cdots + c_{1n}e_n$, ... , $c_{m1}e_1 + \cdots + c_{mn}e_n$ of the dual lattice $\langle A\rangle^* \subset \langle A\rangle \otimes \mathbb{Q}$ where $c_{ij} \in \mathbb{Q}$.

We denote by U the lattice

$$U = \left\langle \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\rangle. \qquad (2.3.1.1)$$

It is unimodular even of signature $(1, 1)$.

2.3.2. Classification of elliptically or parabolically reflective main hyperbolic lattices of rank three. We remind that a hyperbolic lattice S of rank three is called *main* if its determinant $d = \det(S)$ is square-free and S is even for even $d = \det(S)$. If d is odd, then S will be necessarily odd. By Proposition 2.2.4, a main hyperbolic lattice S of rank three and with square-free determinant is defined by invariants (d, η). We have the following basic result:

Basic Theorem 2.3.2.1. *Table 1 below gives the complete list (containing 122 cases numerated by N) of main hyperbolic lattices S of rank three which are reflective of elliptic or parabolic type. In Table 1, for each lattice S we give invariants (d, η) defining S up to isomorphism, number h of classes of central symmetries, matrix of S for some standard bases, the set $P(\mathcal{M})_{\rm pr}$ in the standard bases and the Gram matrix $G(P(\mathcal{M})_{\rm pr})$. All these 122 lattices are reflective of elliptic type.*

If the lattice S represents 0 (i.e. there exists a non-zero $x \in S$ with $x^2 = 0$), we give S in the form $S = U \oplus \langle -d\rangle$. Then the fundamental polygon $\mathcal{M}$ is not compact (i.e. $\mathcal{M}$ contains an infinite vertex). In particular, Table 1 contains the complete list of reflective hyperbolic lattices of elliptic or parabolic type of the form $S = U \oplus \langle -d\rangle$ where d is square-free. There are 23 these lattices corresponding to

$$d = 1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 11,\ 13,\ 14,\ 15,\ 17,\ 21,\ 22,\ 26,$$
$$30,\ 33,\ 34,\ 38,\ 42,\ 66,\ 78,\ 110.$$

If S is given in different form, the lattice S does not represent 0 *and $\mathcal{M}$ is compact. There are* 99 *these lattices.*

Idea of the Proof. Here we give a rough scheme of the proof, details will be given in Part II. First, we prove a formula for the number h of classes of central symmetries of main hyperbolic lattices S of rank 3 with a square-free determinant. This formula uses class-numbers of imaginary quadratic fields and Legendre symbol (see Theorem 3.2.1). By Lemma 2.3.1.2, elliptically or parabolically reflective S have $h \le 1$. Using the formula for h, we find all main hyperbolic lattices S of rank 3 with square-free determinant $d \le 100000$ and $h \le 1$. Their list is given in Table 3, it contains 206 lattices S. (In Table 3, we give invariants $(d,\, \eta)$, h and a lattice S with these invariants.) Conjecturally, Table 3 gives the complete (without the condition $d \le 100000$) list of main hyperbolic lattices S of the rank 3 with the invariant $h \le 1$. To avoid this conjecture, we use the same method (of narrow parts of polyhedra in hyperbolic spaces applied to fundamental polyhedra $\mathcal{M}$) which we used to prove finiteness Theorem 1.4.1 (see [N4], [N5], [N11] and also [N8]). This method gives a finite list of all possible invariants $(d,\, \eta)$ for the reflective lattices S. Calculating their invariant h (or estimating the invariant d), we find that all of them having $h \le 1$ belong to Table 3. See Theorem 5.1.1. Using Vinberg's algorithm [V2], we check reflective type of all 206 lattices S of Table 3 and calculate their fundamental polygon $\mathcal{M}$ for $W(S)$ if they are reflective. See Sect. 5. For elliptic and parabolic type, the result is given in Table 1. Actually, the method of narrow parts of polyhedra permits to find out a big part of non-reflective lattices of Table 3. In Table 3 we mark by *er*, *pr*, *hr* and *nr* elliptically, parabolically, hyperbolically reflective type and non-reflective type respectively (there are no lattices of parabolic type). It finishes the proof.

We mention an important

Corollary 2.3.2.2. *The reflection groups $W(S)$ of lattices S of Table 1 contain all maximal arithmetic reflection groups W over $\mathbb{Q}$ in hyperbolic plane.*

Proof. By É.B. Vinberg [V1], W is a reflection subgroup of finite index $W \subset O(S_1)$ of a hyperbolic lattice S_1 over $\mathbb{Z}$ of rank 3. It follows that the fundamental polygon $\mathcal{M}$ of W is elliptic, the group $A(P(\mathcal{M})_{\rm pr})$ is finite and has a generalized lattice Weyl vector ρ with $(\rho, \rho) > 0$. By considerations in Sect. 2.2 and Theorem 2.3.2.1, the group W is a reflection subgroup $W \subset W(S)$ of one of lattices S of Table 1. Since W is maximal, $W = W(S)$. It proves the statement.

Remark 2.3.2.3. Reflective type of lattices $U \oplus \langle -2k \rangle$ for all $k \le 60$ was found in [N13]. For $k = 2k_1$ where k_1 is odd, the lattice $U \oplus \langle -2k \rangle$ is equivariantly equivalent to the lattice $U \oplus \langle -k_1 \rangle$. Thus, between lattices $S = U \oplus \langle -d \rangle$ of Theorem 2.3.2.1 only the lattice $S = U \oplus \langle -33 \rangle$ was not considered in [N13].

Remark 2.3.2.4. The list of 23 elliptically reflective lattices of Theorem 2.3.2.1 of the form $U \oplus \langle -d \rangle$ where d is square-free, and calculation of their fundamental polygons $\mathcal{M}$ is contained in paper of O. Ruzmanov [R2]. Using one unpublished result of R. Scharlau (see Lemma 1 in [R2]), it is also claimed in [R2] that this list is complete. At any case, our calculations for this case and calculations in [R2] are completely agree.

2.3.3. Classification of elliptically or parabolically reflective non-main hyperbolic lattices of rank three and with square-free determinant. We remind that a lattice

S of rank three and with square-free determinant is called *non-main* if S is odd but its determinant is even. Using Basic Theorem 2.3.2.1, Proposition 2.2.6 and additional considerations and calculations, we get

Theorem 2.3.3.1. *Table 2 below gives the complete list (containing* 38 *cases numerated by* N'*) of non-main hyperbolic lattices* $\widetilde{S}$ *with square-free determinant and of rank three which are reflective of elliptic or parabolic type. In Table 2, for each lattice* $\widetilde{S}$ *we give its invariants* (d, odd, η)*, the number* h *of classes of central symmetries, matrix of* $\widetilde{S}$ *in a standard bases, invariants* $(d, \eta + \omega(d))$ *of the main odd lattice* S *of* $\widetilde{S}$ *(see Proposition 2.2.6). All these* 38 *cases have elliptic type.*

If S *and* $\widetilde{S}$ *are equivariantly equivalent (in Table 2 there are* 21 *these cases), we only give the lattice* S*. For this case, calculation of* $P(\mathcal{M})_{\rm pr}$ *and* $G(P(\mathcal{M})_{\rm pr})$ *for* $\widetilde{S}$ *follows from calculation of similar sets for* S *in Table 1 (see Remark 2.3.3.2 below). In particular, all* $\widetilde{S}$ *representing* 0 *(non-compact case) have this type, then* $\widetilde{S} = \langle 1\rangle \oplus \langle -1\rangle \oplus \langle -d\rangle$ *where*

$$d = 2,\ 6,\ 10,\ 14,\ 22,\ 26,\ 30,\ 34,\ 42,\ 66.$$

If S *and* $\widetilde{S}$ *are not equivariantly equivalent (there are 17 these cases), we (like for Table 1) give* $P(\mathcal{M})_{\rm pr}$ *and* $G(P(\mathcal{M})_{\rm pr})$ *for* $\widetilde{S}$ *in the standard bases.*

Remark 2.3.3.2. Suppose that $\widetilde{S}$ is equivariantly equivalent to S. Then $S(2) \subset \widetilde{S}$ is the unique odd overlattice of $S(2)$. Let $P(\mathcal{M})_{\rm pr}$ be the set of orthogonal to $\mathcal{M}$ primitive roots of S with the Gram matrix $((\alpha,\beta))$, α, $\beta \in P(\mathcal{M})_{\rm pr}$. We denote by $\widetilde{P}(\mathcal{M})_{\rm pr}$ the set of orthogonal to $\mathcal{M}$ primitive roots of the overlattice $S(2) \subset \widetilde{S}$. It is

$$\begin{aligned}\widetilde{P}(\mathcal{M})_{\rm pr} = \{\widetilde{\alpha} = \alpha \ \mid\ \alpha \in P(\mathcal{M})_{\rm pr} \quad \& \quad \alpha^2 \equiv 1 \mod 2\}\cup\\ \cup\{\widetilde{\alpha} = \alpha/2 \ \mid\ \alpha \in P(\mathcal{M})_{\rm pr} \quad \& \quad \alpha^2 \equiv 2 \mod 4\}.\end{aligned} \tag{2.3.3.1}$$

It follows that for $\widetilde{\alpha}$, $\widetilde{\beta} \in \widetilde{P}(\mathcal{M})_{\rm pr}$ of the lattice $\widetilde{S}$, one has

$$(\widetilde{\alpha}, \widetilde{\beta}) = \begin{cases} 2(\alpha,\beta), & \text{if } \alpha^2\beta^2 \equiv 1 \mod 2\\ (\alpha,\beta), & \text{if } \alpha^2\beta^2 \equiv 2 \mod 4\\ (\alpha,\beta)/2, & \text{if } \alpha^2 \equiv \beta^2 \equiv 2 \mod 4\end{cases} \tag{2.3.3.2}$$

which describes the Gram matrix of $\widetilde{P}(\mathcal{M})_{\rm pr}$ for the lattice $\widetilde{S}$ using the Gram matrix of $P(\mathcal{M})_{\rm pr}$ for the lattice S. Here on the left hand side of (2.3.3.2) we use the form of the lattice $\widetilde{S}$, and on the right hand side of (2.3.3.2) we use the form of the lattice S.

2.3.4. Classification of elementary hyperbolic lattices of rank three which are reflective of elliptic or parabolic type. We remind that a lattice L is called *elementary* if its discriminant group $A_L = L^*/L$ is elementary: any its p-component is a p-elementary Abelian group: $A_{L_p} \cong (\mathbb{Z}/p\mathbb{Z})^{r_p}$.

By Sect. 2.2, we get

Theorem 2.3.4.1. *A primitive elementary hyperbolic lattice* F *is reflective of elliptic or parabolic type if and only if*

$$F \cong S^{*,m}(m),\ \ m|d \tag{2.3.4.1}$$

(see (2.2.5)*) where S is a lattice of the determinant $d=\det(S)$ (it is square-free) of Tables 1 or 2. Then* $\det(F)=dm$*. In particular, for the fixed lattice S the number of lattices F is equal to 2^t where t is the number of prime divisors of d.*

Let $P(\mathcal{M})_{\mathrm{pr}}$ be the set of orthogonal to $\mathcal{M}$ primitive roots of S. Then the set $\widetilde{P}(\mathcal{M})_{\mathrm{pr}}$ of orthogonal to $\mathcal{M}$ primitive roots of F is equal to

$$\widetilde{P}(\mathcal{M})_{\mathrm{pr}}=\{\widetilde{\alpha}=\alpha/k_{\alpha,m}\ \mid\ \alpha\in P(\mathcal{M})_{\mathrm{pr}}\} \tag{2.3.4.2}$$

where $k_{\alpha,m}$ is the greatest divisor of m such that $\alpha/k_{\alpha,m}\in S^*$ (equivalently, if $(\alpha,S)=t\mathbb{Z}$, then $k_{\alpha,m}=g.c.d(t,m)$). The Gram matrix of $\widetilde{P}(\mathcal{M})_{\mathrm{pr}}$ for the lattice F is equal to

$$\Big((\widetilde{\alpha},\widetilde{\beta})\Big)=\left(\frac{(\alpha,\beta)m}{k_{\alpha,m}k_{\beta,m}}\right),\quad \widetilde{\alpha},\,\widetilde{\beta}\in\widetilde{P}(\mathcal{M})_{\mathrm{pr}}. \tag{2.3.4.3}$$

Here, on the left hand side of (2.3.4.3) *we use the form of the lattice F, and on the right hand side of* (2.3.4.3) *we use the form of the lattice S.*

Table 1.

The list of main hyperbolic lattices of the rank three which are reflective of elliptic or parabolic type

$N=1\quad d=1,\ \eta=0,\ h=0$: $U\oplus\langle -1\rangle$

$$\begin{pmatrix}1&0&-1\\0&0&1\\-1&1&0\end{pmatrix}\qquad\begin{pmatrix}-1&1&1\\1&-1&0\\1&0&-2\end{pmatrix}.$$

$N=2\quad d=2,\ \eta=0,\ h=0$: $U\oplus\langle -2\rangle$

$$\begin{pmatrix}1&0&-1\\0&0&1\\-1&1&0\end{pmatrix}\qquad\begin{pmatrix}-2&2&1\\2&-2&0\\1&0&-2\end{pmatrix}.$$

$N=3\quad d=3,\ \eta=0,\ h=0$: $\langle 3\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix}0&-1&1\\0&1&0\\1&0&-3\end{pmatrix}\qquad\begin{pmatrix}-2&1&3\\1&-1&0\\3&0&-6\end{pmatrix}.$$

$N=4\quad d=3,\ \eta=1,\ h=0$: $U\oplus\langle -3\rangle$

$$\begin{pmatrix}3&0&-1\\0&0&1\\-1&1&0\\1&1&-1\end{pmatrix}\qquad\begin{pmatrix}-3&3&3&0\\3&-3&0&3\\3&0&-2&0\\0&3&0&-1\end{pmatrix}.$$

$N=5\quad d=5,\ \eta=0,\ h=0$: $U\oplus\langle -5\rangle$

$$\begin{pmatrix}5&0&-1\\0&0&1\\-1&1&0\\2&1&-1\end{pmatrix}\qquad\begin{pmatrix}-5&5&5&0\\5&-5&0&5\\5&0&-2&1\\0&5&1&-1\end{pmatrix}.$$

$N=6\quad d=5,\ \eta=1,\ h=0$: $\langle 1\rangle\oplus\langle -10\rangle\oplus\langle -2\rangle(0,1/2,1/2)$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&-1\\1&\frac{-1}{2}&\frac{-1}{2}\end{pmatrix}\qquad\begin{pmatrix}-2&0&2&1\\0&-10&0&5\\2&0&-1&0\\1&5&0&-2\end{pmatrix}.$$

$N=7\quad d=6=2\cdot 3,\ \eta=0, h=0$: $U\oplus\langle -6\rangle$

$$\begin{pmatrix}3&0&-1\\0&0&1\\-1&1&0\end{pmatrix}\qquad\begin{pmatrix}-6&6&3\\6&-6&0\\3&0&-2\end{pmatrix}.$$

$N=8\quad d=7,\ \eta=0,\ h=1$: $\langle 7\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix}0&-1&1\\0&1&0\\1&0&-3\\1&-2&-2\end{pmatrix}\qquad\begin{pmatrix}-2&1&3&0\\1&-1&0&2\\3&0&-2&1\\0&2&1&-1\end{pmatrix}.$$

$N=9\quad d=7,\ \eta=1,\ h=0$: $U\oplus\langle -7\rangle$

$$\begin{pmatrix}7&0&-1\\0&0&1\\-1&1&0\\7&7&-4\\3&1&-1\end{pmatrix}\qquad\begin{pmatrix}-7&7&7&21&0\\7&-7&0&28&7\\7&0&-2&0&2\\21&28&0&-14&0\\0&7&2&0&-1\end{pmatrix}.$$

$N = 10 \quad d = 10 = 2 \cdot 5,\ \eta = 1, h = 0$: $U \oplus \langle -10 \rangle$

$$\begin{pmatrix} 5 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -10 & 10 & 5 & 0 \\ 10 & -10 & 0 & 10 \\ 5 & 0 & -2 & 0 \\ 0 & 10 & 0 & -2 \end{pmatrix}.$$

$N = 11 \quad d = 11,\ \eta = 0,\ h = 0$: $\langle 11 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -11 \\ 1 & -2 & -3 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 11 & 1 \\ 1 & -1 & 0 & 2 \\ 11 & 0 & -22 & 0 \\ 1 & 2 & 0 & -2 \end{pmatrix}.$$

$N = 12 \quad d = 11,\ \eta = 1,\ h = 1$: $U \oplus \langle -11 \rangle$

$$\begin{pmatrix} 11 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 7 & 7 & -3 \\ 55 & 44 & -21 \\ 66 & 44 & -23 \\ 7 & 3 & -2 \\ 5 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} -11 & 11 & 11 & 44 & 253 & 231 & 11 & 0 \\ 11 & -11 & 0 & 33 & 231 & 253 & 22 & 11 \\ 11 & 0 & -2 & 0 & 11 & 22 & 4 & 4 \\ 44 & 33 & 0 & -1 & 0 & 11 & 4 & 9 \\ 253 & 231 & 11 & 0 & -11 & 11 & 11 & 44 \\ 231 & 253 & 22 & 11 & 11 & -11 & 0 & 33 \\ 11 & 22 & 4 & 4 & 11 & 0 & -2 & 0 \\ 0 & 11 & 4 & 9 & 44 & 33 & 0 & -1 \end{pmatrix}.$$

$N = 13 \quad d = 13,\ \eta = 0,\ h = 1$: $U \oplus \langle -13 \rangle$

$$\begin{pmatrix} 13 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \\ 104 & 39 & -25 \\ 234 & 78 & -53 \\ 55 & 17 & -12 \\ 6 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} -13 & 13 & 13 & 13 & 182 & 325 & 65 & 0 \\ 13 & -13 & 0 & 13 & 325 & 689 & 156 & 13 \\ 13 & 0 & -2 & 1 & 65 & 156 & 38 & 5 \\ 13 & 13 & 1 & -1 & 0 & 13 & 5 & 2 \\ 182 & 325 & 65 & 0 & -13 & 13 & 13 & 13 \\ 325 & 689 & 156 & 13 & 13 & -13 & 0 & 13 \\ 65 & 156 & 38 & 5 & 13 & 0 & -2 & 1 \\ 0 & 13 & 5 & 2 & 13 & 13 & 1 & -1 \end{pmatrix}.$$

$N = 14 \quad d = 13,\ \eta = 1,\ h = 1$: $\langle 26 \rangle \oplus \langle -2 \rangle \oplus \langle -1 \rangle (1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -26 \\ 1 & -1 & -5 \\ \frac{1}{2} & \frac{-3}{2} & -2 \\ \frac{3}{2} & \frac{-13}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 26 & 5 & 2 & 0 \\ 0 & -2 & 0 & 2 & 3 & 13 \\ 26 & 0 & -26 & 0 & 13 & 195 \\ 5 & 2 & 0 & -1 & 0 & 26 \\ 2 & 3 & 13 & 0 & -2 & 0 \\ 0 & 13 & 195 & 26 & 0 & -26 \end{pmatrix}.$$

$N = 15 \quad d = 14 = 2 \cdot 7,\ \eta = 1,\ h = 0$: $U \oplus \langle -14 \rangle$

$$\begin{pmatrix} 7 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -14 & 14 & 7 & 0 \\ 14 & -14 & 0 & 14 \\ 7 & 0 & -2 & 1 \\ 0 & 14 & 1 & -2 \end{pmatrix}.$$

$N = 16 \quad d = 15 = 3 \cdot 5,\ \eta = 0,\ h = 0$: $\langle 3 \rangle \oplus \langle -5 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 0 \\ 0 & -5 & 0 & 5 \\ 3 & 0 & -6 & 3 \\ 0 & 5 & 3 & -2 \end{pmatrix}.$$

$N=17\quad d=15=3\cdot 5,\ \eta=1,\ h=0$: $\langle 5\rangle\oplus\langle -3\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & -5\\ 1 & -1 & -2\\ 3 & -5 & 0 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 5 & 2 & 0\\ 0 & -3 & 0 & 3 & 15\\ 5 & 0 & -5 & 0 & 30\\ 2 & 3 & 0 & -2 & 0\\ 0 & 15 & 30 & 0 & -30 \end{pmatrix}.$$

$N=18\quad d=15=3\cdot 5,\ \eta=2,\ h=0$: $U\oplus\langle -15\rangle$

$$\begin{pmatrix} 15 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 5 & 5 & -2\\ 9 & 3 & -2\\ 7 & 1 & -1 \end{pmatrix}\qquad \begin{pmatrix} -15 & 15 & 15 & 45 & 15 & 0\\ 15 & -15 & 0 & 30 & 30 & 15\\ 15 & 0 & -2 & 0 & 6 & 6\\ 45 & 30 & 0 & -10 & 0 & 10\\ 15 & 30 & 6 & 0 & -6 & 0\\ 0 & 15 & 6 & 10 & 0 & -1 \end{pmatrix}.$$

$N=19\quad d=15=3\cdot 5,\ \eta=3,\ h=0$: $\langle 15\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & -1 & 1\\ 0 & 1 & 0\\ 1 & 0 & -5\\ 1 & -3 & -3 \end{pmatrix}\qquad \begin{pmatrix} -2 & 1 & 5 & 0\\ 1 & -1 & 0 & 3\\ 5 & 0 & -10 & 0\\ 0 & 3 & 0 & -3 \end{pmatrix}.$$

$N=20\quad d=17,\ \eta=0,\ h=0$: $U\oplus\langle -17\rangle$

$$\begin{pmatrix} 17 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 17 & 17 & -6\\ 4 & 2 & -1\\ 11 & 3 & -2\\ 8 & 1 & -1 \end{pmatrix}\qquad \begin{pmatrix} -17 & 17 & 17 & 187 & 17 & 17 & 0\\ 17 & -17 & 0 & 102 & 17 & 34 & 17\\ 17 & 0 & -2 & 0 & 2 & 8 & 7\\ 187 & 102 & 0 & -34 & 0 & 34 & 51\\ 17 & 17 & 2 & 0 & -1 & 0 & 3\\ 17 & 34 & 8 & 34 & 0 & -2 & 1\\ 0 & 17 & 7 & 51 & 3 & 1 & -1 \end{pmatrix}.$$

$N=21\quad d=19,\ \eta=0,\ h=1$: $\langle 19\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & -1 & 1\\ 0 & 1 & 0\\ 13 & 0 & -57\\ 3 & -2 & -13\\ 1 & -2 & -4\\ 6 & -19 & -19 \end{pmatrix}\qquad \begin{pmatrix} -2 & 1 & 57 & 11 & 2 & 0\\ 1 & -1 & 0 & 2 & 2 & 19\\ 57 & 0 & -38 & 0 & 19 & 399\\ 11 & 2 & 0 & -2 & 1 & 57\\ 2 & 2 & 19 & 1 & -1 & 0\\ 0 & 19 & 399 & 57 & 0 & -38 \end{pmatrix}.$$

$N=22\quad d=21=3\cdot 7,\ \eta=0,\ h=0$: $\langle 14\rangle\oplus\langle -6\rangle\oplus\langle -1\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ \frac{1}{2} & \frac{-1}{2} & -2\\ \frac{3}{2} & \frac{-7}{2} & 0 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 2 & 0\\ 0 & -6 & 3 & 21\\ 2 & 3 & -2 & 0\\ 0 & 21 & 0 & -42 \end{pmatrix}.$$

$N=23\quad d=21=3\cdot 7,\ \eta=1,\ h=1$: $U\oplus\langle -21\rangle$

$$\begin{pmatrix} 21 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 3 & 3 & -1\\ 5 & 2 & -1\\ 84 & 21 & -13\\ 210 & 42 & -29\\ 61 & 11 & -8\\ 57 & 9 & -7\\ 10 & 1 & -1 \end{pmatrix}\qquad \begin{pmatrix} -21 & 21 & 21 & 42 & 21 & 168 & 273 & 63 & 42 & 0\\ 21 & -21 & 0 & 21 & 21 & 273 & 609 & 168 & 147 & 21\\ 21 & 0 & -2 & 0 & 3 & 63 & 168 & 50 & 48 & 9\\ 42 & 21 & 0 & -3 & 0 & 42 & 147 & 48 & 51 & 12\\ 21 & 21 & 3 & 0 & -1 & 0 & 21 & 9 & 12 & 4\\ 168 & 273 & 63 & 42 & 0 & -21 & 21 & 21 & 42 & 21\\ 273 & 609 & 168 & 147 & 21 & 21 & -21 & 0 & 21 & 21\\ 63 & 168 & 50 & 48 & 9 & 21 & 0 & -2 & 0 & 3\\ 42 & 147 & 48 & 51 & 12 & 42 & 21 & 0 & -3 & 0\\ 0 & 21 & 9 & 12 & 4 & 21 & 21 & 3 & 0 & -1 \end{pmatrix}.$$

$N = 24$ $d = 21 = 3 \cdot 7$, $\eta = 2$, $h = 0$: $\langle 3\rangle \oplus \langle -7\rangle \oplus \langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 7 & -4 & -7 \\ 3 & -2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 7 & 1 \\ 0 & -7 & 0 & 28 & 14 \\ 3 & 0 & -6 & 0 & 6 \\ 7 & 28 & 0 & -14 & 0 \\ 1 & 14 & 6 & 0 & -2 \end{pmatrix}.$$

$N = 25$ $d = 21 = 3 \cdot 7$, $\eta = 3$, $h = 0$: $\langle 1\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 3 & -1 & -1 \\ 7 & -3 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 3 & 0 \\ 0 & -7 & 0 & 7 & 21 \\ 3 & 0 & -2 & 0 & 7 \\ 3 & 7 & 0 & -1 & 0 \\ 0 & 21 & 7 & 0 & -14 \end{pmatrix}.$$

$N = 26$ $d = 22 = 2 \cdot 11$, $\eta = 0$, $h = 1$: $U \oplus \langle -22\rangle$

$$\begin{pmatrix} 11 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 33 & 33 & -10 \\ 88 & 66 & -23 \\ 5 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -22 & 22 & 11 & 143 & 220 & 0 \\ 22 & -22 & 0 & 220 & 506 & 22 \\ 11 & 0 & -2 & 0 & 22 & 3 \\ 143 & 220 & 0 & -22 & 22 & 11 \\ 220 & 506 & 22 & 22 & -22 & 0 \\ 0 & 22 & 3 & 11 & 0 & -2 \end{pmatrix}.$$

$N = 27$ $d = 23$, $\eta = 0$, $h = 1$: $\langle 23\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -5 \\ 12 & -17 & -55 \\ 6 & -10 & -27 \\ 1 & -3 & -4 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 5 & 38 & 17 & 1 \\ 1 & -1 & 0 & 17 & 10 & 3 \\ 5 & 0 & -2 & 1 & 3 & 3 \\ 38 & 17 & 1 & -2 & 1 & 5 \\ 17 & 10 & 3 & 1 & -1 & 0 \\ 1 & 3 & 3 & 5 & 0 & -2 \end{pmatrix}.$$

$N = 28$ $d = 26 = 2 \cdot 13$, $\eta = 1$, $h = 0$: $U \oplus \langle -26\rangle$

$$\begin{pmatrix} 13 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -1 \\ 6 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -26 & 26 & 13 & 13 & 0 \\ 26 & -26 & 0 & 26 & 26 \\ 13 & 0 & -2 & 1 & 4 \\ 13 & 26 & 1 & -2 & 0 \\ 0 & 26 & 4 & 0 & -2 \end{pmatrix}.$$

$N = 29$ $d = 29$, $\eta = 1$, $h = 1$: $\langle 1\rangle \oplus \langle -58\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 5 & \frac{-1}{2} & \frac{-5}{2} \\ 8 & -1 & -2 \\ 232 & -30 & -29 \\ 23 & -3 & -2 \\ 19 & \frac{-5}{2} & \frac{-1}{2} \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 2 & 5 & 4 & 58 & 4 & 1 \\ 0 & -58 & 0 & 29 & 58 & 1740 & 174 & 145 \\ 2 & 0 & -1 & 0 & 4 & 174 & 19 & 18 \\ 5 & 29 & 0 & -2 & 1 & 145 & 18 & 20 \\ 4 & 58 & 4 & 1 & -2 & 0 & 2 & 5 \\ 58 & 1740 & 174 & 145 & 0 & -58 & 0 & 29 \\ 4 & 174 & 19 & 18 & 2 & 0 & -1 & 0 \\ 1 & 145 & 18 & 20 & 5 & 29 & 0 & -2 \end{pmatrix}.$$

$N = 30$ $d = 30 = 2 \cdot 3 \cdot 5$, $\eta = 1$, $h = 1$: $U \oplus \langle -30\rangle$

$$\begin{pmatrix} 15 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 15 & 15 & -4 \\ 60 & 30 & -11 \\ 7 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -30 & 30 & 15 & 105 & 120 & 0 \\ 30 & -30 & 0 & 120 & 330 & 30 \\ 15 & 0 & -2 & 0 & 30 & 5 \\ 105 & 120 & 0 & -30 & 30 & 15 \\ 120 & 330 & 30 & 30 & -30 & 0 \\ 0 & 30 & 5 & 15 & 0 & -2 \end{pmatrix}.$$

$N=31\quad d=30=2\cdot 3\cdot 5,\ \eta=2,\ h=0$: $\langle 10\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & \frac{-1}{2} & \frac{1}{2}\\ 0 & 1 & 0\\ 2 & 0 & -5\\ 3 & -2 & -6 \end{pmatrix}\qquad \begin{pmatrix} -2 & 3 & 5 & 0\\ 3 & -6 & 0 & 12\\ 5 & 0 & -10 & 0\\ 0 & 12 & 0 & -6 \end{pmatrix}.$$

$N=32\quad d=33=3\cdot 11,\ \eta=0,\ h=1$: $\langle 3\rangle\oplus\langle -22\rangle\oplus\langle -2\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & -3\\ 6 & \frac{-3}{2} & \frac{-11}{2}\\ 22 & \frac{-13}{2} & \frac{-33}{2}\\ 4 & \frac{-3}{2} & \frac{-3}{2} \end{pmatrix}\qquad \begin{pmatrix} -2 & 0 & 6 & 11 & 33 & 3\\ 0 & -22 & 0 & 33 & 143 & 33\\ 6 & 0 & -6 & 3 & 33 & 15\\ 11 & 33 & 3 & -2 & 0 & 6\\ 33 & 143 & 33 & 0 & -22 & 0\\ 3 & 33 & 15 & 6 & 0 & -6 \end{pmatrix}.$$

$N=33\quad d=33=3\cdot 11,\ \eta=1,\ h=0$: $\langle 11\rangle\oplus\langle -3\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -11\\ 1 & -1 & -3\\ 15 & -22 & -33\\ 1 & -2 & -1 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 11 & 3 & 33 & 1\\ 0 & -3 & 0 & 3 & 66 & 6\\ 11 & 0 & -22 & 0 & 132 & 22\\ 3 & 3 & 0 & -1 & 0 & 2\\ 33 & 66 & 132 & 0 & -66 & 0\\ 1 & 6 & 22 & 2 & 0 & -2 \end{pmatrix}.$$

$N=34\quad d=33=3\cdot 11,\ \eta=2,\ h=1$: $U\oplus\langle -33\rangle$

$$\begin{pmatrix} 33 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 4 & 4 & -1\\ 39 & 27 & -8\\ 407 & 253 & -79\\ 68 & 41 & -13\\ 231 & 132 & -43\\ 132 & 66 & -23\\ 13 & 5 & -2\\ 8 & 2 & -1\\ 21 & 3 & -2\\ 121 & 11 & -9\\ 16 & 1 & -1 \end{pmatrix}\begin{pmatrix} -33 & 33 & 33 & 99 & 627 & 5742 & 924 & 2937 & 1419 & 99 & 33 & 33 & 66 & 0\\ 33 & -33 & 0 & 33 & 264 & 2607 & 429 & 1419 & 759 & 66 & 33 & 66 & 297 & 33\\ 33 & 0 & -2 & 0 & 12 & 154 & 27 & 99 & 66 & 8 & 6 & 18 & 110 & 15\\ 99 & 33 & 0 & -1 & 0 & 33 & 7 & 33 & 33 & 6 & 7 & 30 & 231 & 35\\ 627 & 264 & 12 & 0 & -6 & 0 & 3 & 33 & 66 & 18 & 30 & 156 & 1320 & 207\\ 5742 & 2607 & 154 & 33 & 0 & -11 & 0 & 66 & 297 & 110 & 231 & 1320 & 11627 & 1848\\ 924 & 429 & 27 & 7 & 3 & 0 & -1 & 0 & 33 & 15 & 35 & 207 & 1848 & 295\\ 2937 & 1419 & 99 & 33 & 33 & 66 & 0 & -33 & 33 & 33 & 99 & 627 & 5742 & 924\\ 1419 & 759 & 66 & 33 & 66 & 297 & 33 & 33 & -33 & 0 & 33 & 264 & 2607 & 429\\ 99 & 66 & 8 & 6 & 18 & 110 & 15 & 33 & 0 & -2 & 0 & 12 & 154 & 27\\ 33 & 33 & 6 & 7 & 30 & 231 & 35 & 99 & 33 & 0 & -1 & 0 & 33 & 7\\ 33 & 66 & 18 & 30 & 156 & 1320 & 207 & 627 & 264 & 12 & 0 & -6 & 0 & 3\\ 66 & 297 & 110 & 231 & 1320 & 11627 & 1848 & 5742 & 2607 & 154 & 33 & 0 & -11 & 0\\ 0 & 33 & 15 & 35 & 207 & 1848 & 295 & 924 & 429 & 27 & 7 & 3 & 0 & -1 \end{pmatrix}.$$

$N=35\quad d=33=3\cdot 11,\ \eta=3,\ h=1$: $\langle 1\rangle\oplus\langle -11\rangle\oplus\langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 12 & -3 & -4\\ 44 & -12 & -11\\ 3 & -1 & 0 \end{pmatrix}\qquad \begin{pmatrix} -3 & 0 & 3 & 12 & 33 & 0\\ 0 & -11 & 0 & 33 & 132 & 11\\ 3 & 0 & -2 & 0 & 11 & 3\\ 12 & 33 & 0 & -3 & 0 & 3\\ 33 & 132 & 11 & 0 & -11 & 0\\ 0 & 11 & 3 & 3 & 0 & -2 \end{pmatrix}.$$

$N=36\quad d=34=2\cdot 17,\ \eta=0,\ h=1$: $U\oplus\langle -34\rangle$

$$\begin{pmatrix} 17 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 4 & 4 & -1\\ 85 & 51 & -16\\ 204 & 102 & -35\\ 19 & 8 & -3\\ 8 & 2 & -1 \end{pmatrix}\qquad \begin{pmatrix} -34 & 34 & 17 & 34 & 323 & 544 & 34 & 0\\ 34 & -34 & 0 & 34 & 544 & 1190 & 102 & 34\\ 17 & 0 & -2 & 0 & 34 & 102 & 11 & 6\\ 34 & 34 & 0 & -2 & 0 & 34 & 6 & 6\\ 323 & 544 & 34 & 0 & -34 & 34 & 17 & 34\\ 544 & 1190 & 102 & 34 & 34 & -34 & 0 & 34\\ 34 & 102 & 11 & 6 & 17 & 0 & -2 & 0\\ 0 & 34 & 6 & 6 & 34 & 34 & 0 & -2 \end{pmatrix}.$$

$N = 37 \quad d = 35 = 5 \cdot 7$, $\eta = 0$, $h = 0$: $\langle 7 \rangle \oplus \langle -5 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 1 & -1 & -2 \\ 5 & -7 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 2 & 0 \\ 0 & -5 & 0 & 5 & 35 \\ 3 & 0 & -2 & 1 & 35 \\ 2 & 5 & 1 & -2 & 0 \\ 0 & 35 & 35 & 0 & -70 \end{pmatrix}.$$

$N = 38 \quad d = 35 = 5 \cdot 7$, $\eta = 2$, $h = 1$: $\langle 1 \rangle \oplus \langle -7 \rangle \oplus \langle -5 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \\ 5 & -1 & -2 \\ 91 & -25 & -28 \\ 140 & -40 & -41 \\ 196 & -57 & -56 \\ 54 & -16 & -15 \\ 9 & -3 & -2 \\ 7 & -3 & 0 \end{pmatrix} \begin{pmatrix} -5 & 0 & 5 & 10 & 140 & 205 & 280 & 75 & 10 & 0 \\ 0 & -7 & 0 & 7 & 175 & 280 & 399 & 112 & 21 & 21 \\ 5 & 0 & -1 & 0 & 42 & 75 & 112 & 33 & 8 & 14 \\ 10 & 7 & 0 & -2 & 0 & 10 & 21 & 8 & 4 & 14 \\ 140 & 175 & 42 & 0 & -14 & 0 & 21 & 14 & 14 & 112 \\ 205 & 280 & 75 & 10 & 0 & -5 & 0 & 5 & 10 & 140 \\ 280 & 399 & 112 & 21 & 21 & 0 & -7 & 0 & 7 & 175 \\ 75 & 112 & 33 & 8 & 14 & 5 & 0 & -1 & 0 & 42 \\ 10 & 21 & 8 & 4 & 14 & 10 & 7 & 0 & -2 & 0 \\ 0 & 21 & 14 & 14 & 112 & 140 & 175 & 42 & 0 & -14 \end{pmatrix}.$$

$N = 39 \quad d = 35 = 5 \cdot 7$, $\eta = 3$, $h = 0$: $\langle 35 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -7 \\ 3 & -10 & -15 \\ 5 & -21 & -21 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 7 & 5 & 0 \\ 1 & -1 & 0 & 10 & 21 \\ 7 & 0 & -14 & 0 & 28 \\ 5 & 10 & 0 & -10 & 0 \\ 0 & 21 & 28 & 0 & -7 \end{pmatrix}.$$

$N = 40 \quad d = 38 = 2 \cdot 19$, $\eta = 0$, $h = 1$: $U \oplus \langle -38 \rangle$

$$\begin{pmatrix} 19 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 31 & 30 & -7 \\ 171 & 152 & -37 \\ 190 & 152 & -39 \\ 6 & 3 & -1 \\ 9 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -38 & 38 & 19 & 304 & 1482 & 1406 & 19 & 0 \\ 38 & -38 & 0 & 266 & 1406 & 1482 & 38 & 38 \\ 19 & 0 & -2 & 1 & 19 & 38 & 3 & 7 \\ 304 & 266 & 1 & -2 & 0 & 38 & 7 & 66 \\ 1482 & 1406 & 19 & 0 & -38 & 38 & 19 & 304 \\ 1406 & 1482 & 38 & 38 & 38 & -38 & 0 & 266 \\ 19 & 38 & 3 & 7 & 19 & 0 & -2 & 1 \\ 0 & 38 & 7 & 66 & 304 & 266 & 1 & -2 \end{pmatrix}.$$

$N = 41 \quad d = 39 = 3 \cdot 13$, $\eta = 0$, $h = 0$: $\langle 3 \rangle \oplus \langle -13 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 2 & -1 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 1 \\ 0 & -13 & 0 & 13 \\ 3 & 0 & -6 & 3 \\ 1 & 13 & 3 & -2 \end{pmatrix}.$$

$N = 42 \quad d = 39 = 3 \cdot 13$, $\eta = 2$, $h = 1$: $\langle 3 \rangle \oplus \langle -26 \rangle \oplus \langle -2 \rangle(0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -3 \\ 5 & -1 & -5 \\ 130 & -30 & -117 \\ 12 & -3 & -10 \\ 52 & -14 & -39 \\ 10 & -3 & -6 \\ 3 & -1 & -1 \\ 26 & -9 & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 6 & 10 & 234 & 20 & 78 & 12 & 2 & 0 \\ 0 & -26 & 0 & 26 & 780 & 78 & 364 & 78 & 26 & 234 \\ 6 & 0 & -6 & 0 & 78 & 12 & 78 & 24 & 12 & 156 \\ 10 & 26 & 0 & -1 & 0 & 2 & 26 & 12 & 9 & 156 \\ 234 & 780 & 78 & 0 & -78 & 0 & 234 & 156 & 156 & 3120 \\ 20 & 78 & 12 & 2 & 0 & -2 & 0 & 6 & 10 & 234 \\ 78 & 364 & 78 & 26 & 234 & 0 & -26 & 0 & 26 & 780 \\ 12 & 78 & 24 & 12 & 156 & 6 & 0 & -6 & 0 & 78 \\ 2 & 26 & 12 & 9 & 156 & 10 & 26 & 0 & -1 & 0 \\ 0 & 234 & 156 & 156 & 3120 & 234 & 780 & 78 & 0 & -78 \end{pmatrix}.$$

$N=43$ $d=39=3\cdot 13,\ \eta=3,\ h=1$: $\langle 2\rangle\oplus\langle -26\rangle\oplus\langle -3\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 6 & -1 & -4\\ 30 & -6 & -17\\ \frac{195}{2} & \frac{-41}{2} & -52\\ \frac{13}{2} & \frac{-3}{2} & -3\\ \frac{3}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 12 & 51 & 156 & 9 & 0\\ 0 & -26 & 0 & 26 & 156 & 533 & 39 & 13\\ 3 & 0 & -1 & 0 & 9 & 39 & 4 & 3\\ 12 & 26 & 0 & -2 & 0 & 13 & 3 & 5\\ 51 & 156 & 9 & 0 & -3 & 0 & 3 & 12\\ 156 & 533 & 39 & 13 & 0 & -26 & 0 & 26\\ 9 & 39 & 4 & 3 & 3 & 0 & -1 & 0\\ 0 & 13 & 3 & 5 & 12 & 26 & 0 & -2 \end{pmatrix}.$$

$N=44$ $d=42=2\cdot 3\cdot 7,\ \eta=0,\ h=0$: $U\oplus\langle -42\rangle$

$$\begin{pmatrix} 21 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 6 & 3 & -1\\ 10 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -42 & 42 & 21 & 21 & 0\\ 42 & -42 & 0 & 42 & 42\\ 21 & 0 & -2 & 3 & 8\\ 21 & 42 & 3 & -6 & 0\\ 0 & 42 & 8 & 0 & -2 \end{pmatrix}.$$

$N=45$ $d=42=2\cdot 3\cdot 7,\ \eta=3,\ h=1$: $\langle 6\rangle\oplus\langle -14\rangle\oplus\langle -2\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -2\\ 6 & -2 & -9\\ \frac{21}{2} & \frac{-9}{2} & -14\\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 4 & 18 & 28 & 0\\ 0 & -14 & 0 & 28 & 63 & 7\\ 4 & 0 & -2 & 0 & 7 & 3\\ 18 & 28 & 0 & -2 & 0 & 4\\ 28 & 63 & 7 & 0 & -14 & 0\\ 0 & 7 & 3 & 4 & 0 & -2 \end{pmatrix}.$$

$N=46$ $d=51=3\cdot 17,\ \eta=0,\ h=0$: $\langle 3\rangle\oplus\langle -17\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -3\\ 17 & -6 & -17\\ 5 & -2 & -3\\ 7 & -3 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 17 & 3 & 0\\ 0 & -17 & 0 & 102 & 34 & 51\\ 3 & 0 & -6 & 0 & 6 & 21\\ 17 & 102 & 0 & -34 & 0 & 51\\ 3 & 34 & 6 & 0 & -2 & 3\\ 0 & 51 & 21 & 51 & 3 & -6 \end{pmatrix}.$$

$N=47$ $d=51=3\cdot 17,\ \eta=1,\ h=1$: $\langle 17\rangle\oplus\langle -3\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 4 & 0 & -17\\ 1 & -1 & -4\\ 3 & -5 & -9\\ 27 & -51 & -68\\ 4 & -8 & -9\\ 12 & -25 & -24\\ 64 & -136 & -119\\ 5 & -11 & -8\\ 3 & -7 & -3\\ 7 & -17 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 17 & 4 & 9 & 68 & 9 & 24 & 119 & 8 & 3 & 0\\ 0 & -3 & 0 & 3 & 15 & 153 & 24 & 75 & 408 & 33 & 21 & 51\\ 17 & 0 & -17 & 0 & 51 & 680 & 119 & 408 & 2329 & 204 & 153 & 476\\ 4 & 3 & 0 & -2 & 0 & 34 & 8 & 33 & 204 & 20 & 18 & 68\\ 9 & 15 & 51 & 0 & -3 & 0 & 3 & 21 & 153 & 18 & 21 & 102\\ 68 & 153 & 680 & 34 & 0 & -34 & 0 & 51 & 476 & 68 & 102 & 612\\ 9 & 24 & 119 & 8 & 3 & 0 & -1 & 0 & 17 & 4 & 9 & 68\\ 24 & 75 & 408 & 33 & 21 & 51 & 0 & -3 & 0 & 3 & 15 & 153\\ 119 & 408 & 2329 & 204 & 153 & 476 & 17 & 0 & -17 & 0 & 51 & 680\\ 8 & 33 & 204 & 20 & 18 & 68 & 4 & 3 & 0 & -2 & 0 & 34\\ 3 & 21 & 153 & 18 & 21 & 102 & 9 & 15 & 51 & 0 & -3 & 0\\ 0 & 51 & 476 & 68 & 102 & 612 & 68 & 153 & 680 & 34 & 0 & -34 \end{pmatrix}.$$

$N=48$ $d=51=3\cdot 17,\ \eta=3,\ h=1$: $\langle 51\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & -1 & 1\\ 0 & 1 & 0\\ 7 & 0 & -51\\ 1 & -2 & -7\\ 1 & -4 & -6\\ 10 & -51 & -51 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 51 & 5 & 2 & 0\\ 1 & -1 & 0 & 2 & 4 & 51\\ 51 & 0 & -102 & 0 & 51 & 969\\ 5 & 2 & 0 & -2 & 1 & 51\\ 2 & 4 & 51 & 1 & -1 & 0\\ 0 & 51 & 969 & 51 & 0 & -102 \end{pmatrix}.$$

$N=49\quad d=55=5\cdot 11,\ \eta=0,\ h=1\colon\ \langle 11\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -11\\ 3 & -2 & -9\\ 2 & -2 & -5\\ 5 & -6 & -10\\ 8 & -11 & -11\\ 2 & -3 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 11 & 9 & 5 & 10 & 11 & 1\\ 0 & -5 & 0 & 10 & 10 & 30 & 55 & 15\\ 11 & 0 & -22 & 0 & 11 & 55 & 143 & 55\\ 9 & 10 & 0 & -2 & 1 & 15 & 55 & 27\\ 5 & 10 & 11 & 1 & -1 & 0 & 11 & 9\\ 10 & 30 & 55 & 15 & 0 & -5 & 0 & 10\\ 11 & 55 & 143 & 55 & 11 & 0 & -22 & 0\\ 1 & 15 & 55 & 27 & 9 & 10 & 0 & -2 \end{pmatrix}.$$

$N=50\quad d=55=5\cdot 11,\ \eta=3,\ h=1\colon\ \langle 2\rangle\oplus\langle -11\rangle\oplus\langle -10\rangle(1/2,0,1/2)$

$$\begin{pmatrix} 66 & -28 & -3\\ 7 & -3 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{-1}{2}\\ \frac{5}{2} & -1 & \frac{-1}{2}\\ \frac{95}{2} & -20 & \frac{-7}{2}\\ 418 & -177 & -22 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 30 & 308 & 51 & 7 & 5 & 0\\ 0 & -1 & 0 & 33 & 7 & 2 & 5 & 11\\ 30 & 0 & -10 & 0 & 5 & 5 & 35 & 220\\ 308 & 33 & 0 & -11 & 0 & 11 & 220 & 1947\\ 51 & 7 & 5 & 0 & -2 & 0 & 30 & 308\\ 7 & 2 & 5 & 11 & 0 & -1 & 0 & 33\\ 5 & 5 & 35 & 220 & 30 & 0 & -10 & 0\\ 0 & 11 & 220 & 1947 & 308 & 33 & 0 & -11 \end{pmatrix}.$$

$N=51\quad d=57=3\cdot 19,\ \eta=0,\ h=1\colon\ \langle 1\rangle\oplus\langle -38\rangle\oplus\langle -6\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & -1\\ 45 & \frac{-9}{2} & \frac{-29}{2}\\ 95 & \frac{-21}{2} & \frac{-57}{2}\\ 3 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 6 & 87 & 171 & 3\\ 0 & -38 & 0 & 171 & 399 & 19\\ 6 & 0 & -2 & 3 & 19 & 3\\ 87 & 171 & 3 & -6 & 0 & 6\\ 171 & 399 & 19 & 0 & -38 & 0\\ 3 & 19 & 3 & 6 & 0 & -2 \end{pmatrix}.$$

$N=52\quad d=57=3\cdot 19,\ \eta=2,\ h=1\colon\ \langle 3\rangle\oplus\langle -19\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 223 & -84 & -123\\ 29 & -11 & -15\\ 361 & -138 & -171\\ 5 & -2 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -3\\ 3 & -1 & -3\\ 247 & -90 & -171\\ 27 & -10 & -17\\ 96 & -36 & -55\\ 1216 & -457 & -684 \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 228 & 30 & 123 & 1596 & 300 & 42 & 570 & 12 & 3 & 0\\ 0 & -1 & 0 & 2 & 15 & 209 & 42 & 7 & 114 & 4 & 3 & 19\\ 228 & 0 & -114 & 0 & 171 & 2622 & 570 & 114 & 2280 & 114 & 171 & 1710\\ 30 & 2 & 0 & -2 & 1 & 38 & 12 & 4 & 114 & 8 & 17 & 190\\ 123 & 15 & 171 & 1 & -1 & 0 & 3 & 3 & 171 & 17 & 55 & 684\\ 1596 & 209 & 2622 & 38 & 0 & -19 & 0 & 19 & 1710 & 190 & 684 & 8683\\ 300 & 42 & 570 & 12 & 3 & 0 & -6 & 0 & 228 & 30 & 123 & 1596\\ 42 & 7 & 114 & 4 & 3 & 19 & 0 & -1 & 0 & 2 & 15 & 209\\ 570 & 114 & 2280 & 114 & 171 & 1710 & 228 & 0 & -114 & 0 & 171 & 2622\\ 12 & 4 & 114 & 8 & 17 & 190 & 30 & 2 & 0 & -2 & 1 & 38\\ 3 & 3 & 171 & 17 & 55 & 684 & 123 & 15 & 171 & 1 & -1 & 0\\ 0 & 19 & 1710 & 190 & 684 & 8683 & 1596 & 209 & 2622 & 38 & 0 & -19 \end{pmatrix}.$$

$N=53\quad d=65=5\cdot 13,\ \eta=2,\ h=1\colon\ \langle 5\rangle\oplus\langle -26\rangle\oplus\langle -2\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -5\\ 3 & \frac{-1}{2} & \frac{-9}{2}\\ 20 & -5 & -26\\ 52 & -14 & -65\\ 17 & -5 & -20\\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 10 & 9 & 52 & 130 & 40 & 1\\ 0 & -26 & 0 & 13 & 130 & 364 & 130 & 13\\ 10 & 0 & -5 & 0 & 40 & 130 & 55 & 10\\ 9 & 13 & 0 & -2 & 1 & 13 & 10 & 4\\ 52 & 130 & 40 & 1 & -2 & 0 & 10 & 9\\ 130 & 364 & 130 & 13 & 0 & -26 & 0 & 13\\ 40 & 130 & 55 & 10 & 10 & 0 & -5 & 0\\ 1 & 13 & 10 & 4 & 9 & 13 & 0 & -2 \end{pmatrix}.$$

$N=54$ $d=66=2\cdot 3\cdot 11,\ \eta=1,\ h=1$: $U\oplus\langle -66\rangle$

$$\begin{pmatrix} 33 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 11 & 11 & -2\\ 8 & 4 & -1\\ 99 & 33 & -10\\ 264 & 66 & -23\\ 37 & 8 & -3\\ 121 & 22 & -9\\ 16 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -66 & 66 & 33 & 231 & 66 & 429 & 660 & 66 & 132 & 0\\ 66 & -66 & 0 & 132 & 66 & 660 & 1518 & 198 & 594 & 66\\ 33 & 0 & -2 & 0 & 4 & 66 & 198 & 29 & 99 & 14\\ 231 & 132 & 0 & -22 & 0 & 132 & 594 & 99 & 385 & 66\\ 66 & 66 & 4 & 0 & -2 & 0 & 66 & 14 & 66 & 14\\ 429 & 660 & 66 & 132 & 0 & -66 & 66 & 33 & 231 & 66\\ 660 & 1518 & 198 & 594 & 66 & 66 & -66 & 0 & 132 & 66\\ 66 & 198 & 29 & 99 & 14 & 33 & 0 & -2 & 0 & 4\\ 132 & 594 & 99 & 385 & 66 & 231 & 132 & 0 & -22 & 0\\ 0 & 66 & 14 & 66 & 14 & 66 & 66 & 4 & 0 & -2 \end{pmatrix}.$$

$N=55$ $d=66=2\cdot 3\cdot 11,\ \eta=2,\ h=1$: $\langle 22\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & \frac{-1}{2} & \frac{1}{2}\\ 0 & 1 & 0\\ 3 & 0 & -10\\ 10 & -2 & -33\\ 6 & \frac{-5}{2} & \frac{-39}{2}\\ 1 & -1 & -3 \end{pmatrix} \qquad \begin{pmatrix} -2 & 3 & 10 & 27 & 12 & 0\\ 3 & -6 & 0 & 12 & 15 & 6\\ 10 & 0 & -2 & 0 & 6 & 6\\ 27 & 12 & 0 & -2 & 3 & 10\\ 12 & 15 & 6 & 3 & -6 & 0\\ 0 & 6 & 6 & 10 & 0 & -2 \end{pmatrix}.$$

$N=56$ $d=69=3\cdot 23,\ \eta=3,\ h=0$: $\langle 1\rangle\oplus\langle -23\rangle\oplus\langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 69 & -12 & -23\\ 5 & -1 & -1\\ 23 & -5 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 69 & 3 & 0\\ 0 & -23 & 0 & 276 & 23 & 115\\ 3 & 0 & -2 & 0 & 2 & 23\\ 69 & 276 & 0 & -138 & 0 & 207\\ 3 & 23 & 2 & 0 & -1 & 0\\ 0 & 115 & 23 & 207 & 0 & -46 \end{pmatrix}.$$

$N=57$ $d=70=2\cdot 5\cdot 7,\ \eta=3,\ h=1$: $\langle 2\rangle\oplus\langle -14\rangle\oplus\langle -10\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & -1\\ 35 & -6 & -14\\ 10 & \frac{-5}{2} & \frac{-7}{2}\\ 14 & \frac{-9}{2} & \frac{-7}{2}\\ 4 & \frac{-3}{2} & \frac{-1}{2}\\ 21 & -8 & 0 \end{pmatrix} \qquad \begin{pmatrix} -10 & 0 & 10 & 140 & 35 & 35 & 5 & 0\\ 0 & -14 & 0 & 84 & 35 & 63 & 21 & 112\\ 10 & 0 & -2 & 0 & 5 & 21 & 11 & 84\\ 140 & 84 & 0 & -14 & 0 & 112 & 84 & 798\\ 35 & 35 & 5 & 0 & -10 & 0 & 10 & 140\\ 35 & 63 & 21 & 112 & 0 & -14 & 0 & 84\\ 5 & 21 & 11 & 84 & 10 & 0 & -2 & 0\\ 0 & 112 & 84 & 798 & 140 & 84 & 0 & -14 \end{pmatrix}.$$

$N=58$ $d=77=7\cdot 11,\ \eta=3,\ h=0$: $\langle 1\rangle\oplus\langle -11\rangle\oplus\langle -7\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 7 & 0 & -3\\ 21 & -3 & -7\\ 308 & -49 & -99\\ 4 & -1 & -1\\ 3 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -7 & 0 & 21 & 49 & 693 & 7 & 0\\ 0 & -11 & 0 & 33 & 539 & 11 & 11\\ 21 & 0 & -14 & 0 & 77 & 7 & 21\\ 49 & 33 & 0 & -1 & 0 & 2 & 30\\ 693 & 539 & 77 & 0 & -154 & 0 & 385\\ 7 & 11 & 7 & 2 & 0 & -2 & 1\\ 0 & 11 & 21 & 30 & 385 & 1 & -2 \end{pmatrix}.$$

$N=59$ $d=78=2\cdot 3\cdot 13,\ \eta=2, h=0$: $U\oplus\langle -78\rangle$

$$\begin{pmatrix} 39 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 6 & 6 & -1\\ 26 & 13 & -3\\ 12 & 3 & -1\\ 19 & 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -78 & 78 & 39 & 156 & 273 & 39 & 0\\ 78 & -78 & 0 & 78 & 234 & 78 & 78\\ 39 & 0 & -2 & 0 & 13 & 9 & 17\\ 156 & 78 & 0 & -6 & 0 & 12 & 48\\ 273 & 234 & 13 & 0 & -26 & 0 & 65\\ 39 & 78 & 9 & 12 & 0 & -6 & 3\\ 0 & 78 & 17 & 48 & 65 & 3 & -2 \end{pmatrix}.$$

$N = 60$ $d = 85 = 5 \cdot 17$, $\eta = 1$, $h = 1$: $\langle 1 \rangle \oplus \langle -34 \rangle \oplus \langle -10 \rangle (0, 1/2, 1/2)$

$$\begin{pmatrix} 476 & -35 & -136 \\ 25 & -2 & -7 \\ 25 & \frac{-5}{2} & \frac{-13}{2} \\ 3 & \frac{-1}{2} & \frac{-1}{2} \\ 17 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \\ 45 & \frac{-5}{2} & \frac{-27}{2} \\ 67 & \frac{-9}{2} & \frac{-39}{2} \\ 1887 & -133 & -544 \\ 560 & -40 & -161 \end{pmatrix} \qquad \begin{pmatrix} -34 & 0 & 85 & 153 & 4522 & 1360 & 1190 & 68 & 85 & 17 & 102 & 0 \\ 0 & -1 & 0 & 6 & 221 & 70 & 68 & 5 & 10 & 4 & 51 & 10 \\ 85 & 0 & -10 & 0 & 170 & 65 & 85 & 10 & 35 & 25 & 510 & 135 \\ 153 & 6 & 0 & -2 & 0 & 5 & 17 & 4 & 25 & 27 & 680 & 195 \\ 4522 & 221 & 170 & 0 & -17 & 0 & 102 & 51 & 510 & 680 & 18513 & 5440 \\ 1360 & 70 & 65 & 5 & 0 & -10 & 0 & 10 & 135 & 195 & 5440 & 1610 \\ 1190 & 68 & 85 & 17 & 102 & 0 & -34 & 0 & 85 & 153 & 4522 & 1360 \\ 68 & 5 & 10 & 4 & 51 & 10 & 0 & -1 & 0 & 6 & 221 & 70 \\ 85 & 10 & 35 & 25 & 510 & 135 & 85 & 0 & -10 & 0 & 170 & 65 \\ 17 & 4 & 25 & 27 & 680 & 195 & 153 & 6 & 0 & -2 & 0 & 5 \\ 102 & 51 & 510 & 680 & 18513 & 5440 & 4522 & 221 & 170 & 0 & -17 & 0 \\ 0 & 10 & 135 & 195 & 5440 & 1610 & 1360 & 70 & 65 & 5 & 0 & -10 \end{pmatrix}.$$

$N = 61$ $d = 87 = 3 \cdot 29$, $\eta = 3$, $h = 1$: $\langle 2 \rangle \oplus \langle -58 \rangle \oplus \langle -3 \rangle (1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ \frac{7}{2} & \frac{-1}{2} & -2 \\ 36 & -6 & -13 \\ 348 & -59 & -116 \\ 35 & -6 & -11 \\ \frac{5}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 6 & 39 & 348 & 33 & 0 \\ 0 & -58 & 0 & 29 & 348 & 3422 & 348 & 29 \\ 3 & 0 & -1 & 1 & 33 & 348 & 37 & 5 \\ 6 & 29 & 1 & -2 & 0 & 29 & 5 & 3 \\ 39 & 348 & 33 & 0 & -3 & 0 & 3 & 6 \\ 348 & 3422 & 348 & 29 & 0 & -58 & 0 & 29 \\ 33 & 348 & 37 & 5 & 3 & 0 & -1 & 1 \\ 0 & 29 & 5 & 3 & 6 & 29 & 1 & -2 \end{pmatrix}.$$

$N = 62$ $d = 91 = 7 \cdot 13$, $\eta = 3$, $h = 1$: $\langle 1 \rangle \oplus \langle -26 \rangle \oplus \langle -14 \rangle (0, 1/2, 1/2)$

$$\begin{pmatrix} 138 & \frac{-53}{2} & \frac{-15}{2} \\ 5 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 7 & 0 & -2 \\ 6 & \frac{-1}{2} & \frac{-3}{2} \\ 3 & \frac{-1}{2} & \frac{-1}{2} \\ 56 & \frac{-21}{2} & \frac{-9}{2} \\ 312 & \frac{-119}{2} & \frac{-39}{2} \\ 329 & -63 & -19 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 105 & 689 & 756 & 326 & 17 & 21 & 13 & 0 \\ 1 & -1 & 0 & 26 & 35 & 17 & 2 & 7 & 13 & 7 \\ 105 & 0 & -14 & 0 & 28 & 21 & 7 & 63 & 273 & 266 \\ 689 & 26 & 0 & -26 & 0 & 13 & 13 & 273 & 1547 & 1638 \\ 756 & 35 & 28 & 0 & -7 & 0 & 7 & 266 & 1638 & 1771 \\ 326 & 17 & 21 & 13 & 0 & -2 & 1 & 105 & 689 & 756 \\ 17 & 2 & 7 & 13 & 7 & 1 & -1 & 0 & 26 & 35 \\ 21 & 7 & 63 & 273 & 266 & 105 & 0 & -14 & 0 & 28 \\ 13 & 13 & 273 & 1547 & 1638 & 689 & 26 & 0 & -26 & 0 \\ 0 & 7 & 266 & 1638 & 1771 & 756 & 35 & 28 & 0 & -7 \end{pmatrix}.$$

$N = 63$ $d = 93 = 3 \cdot 31$, $\eta = 0$, $h = 1$: $\langle 31 \rangle \oplus \langle -6 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & -4 \\ 36 & \frac{-31}{2} & \frac{-279}{2} \\ 2 & \frac{-3}{2} & \frac{-15}{2} \\ 24 & -25 & -84 \\ 11 & -12 & -38 \\ 150 & \frac{-341}{2} & \frac{-1023}{2} \end{pmatrix} \qquad \begin{pmatrix} -2 & 3 & 4 & 93 & 3 & 9 & 2 & 0 \\ 3 & -6 & 0 & 93 & 9 & 150 & 72 & 1023 \\ 4 & 0 & -1 & 0 & 2 & 72 & 37 & 558 \\ 93 & 93 & 0 & -186 & 0 & 1023 & 558 & 8835 \\ 3 & 9 & 2 & 0 & -2 & 3 & 4 & 93 \\ 9 & 150 & 72 & 1023 & 3 & -6 & 0 & 93 \\ 2 & 72 & 37 & 558 & 4 & 0 & -1 & 0 \\ 0 & 1023 & 558 & 8835 & 93 & 93 & 0 & -186 \end{pmatrix}.$$

$N = 64$ $d = 95 = 5 \cdot 19$, $\eta = 3$, $h = 1$: $\langle 2 \rangle \oplus \langle -19 \rangle \oplus \langle -10 \rangle (1/2, 0, 1/2)$

$$\begin{pmatrix} \frac{129}{2} & -20 & \frac{-17}{2} \\ 19 & -6 & -2 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-1}{2} \\ \frac{27}{2} & -4 & \frac{-5}{2} \\ \frac{59}{2} & -9 & \frac{-9}{2} \\ \frac{585}{2} & -90 & \frac{-83}{2} \\ 1235 & -381 & -171 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 7 & 85 & 380 & 22 & 9 & 3 & 5 & 0 \\ 1 & -2 & 0 & 20 & 114 & 9 & 7 & 5 & 25 & 76 \\ 7 & 0 & -1 & 0 & 19 & 3 & 5 & 6 & 45 & 171 \\ 85 & 20 & 0 & -10 & 0 & 5 & 25 & 45 & 415 & 1710 \\ 380 & 114 & 19 & 0 & -19 & 0 & 76 & 171 & 1710 & 7239 \\ 22 & 9 & 3 & 5 & 0 & -2 & 1 & 7 & 85 & 380 \\ 9 & 7 & 5 & 25 & 76 & 1 & -2 & 0 & 20 & 114 \\ 3 & 5 & 6 & 45 & 171 & 7 & 0 & -1 & 0 & 19 \\ 5 & 25 & 45 & 415 & 1710 & 85 & 20 & 0 & -10 & 0 \\ 0 & 76 & 171 & 1710 & 7239 & 380 & 114 & 19 & 0 & -19 \end{pmatrix}.$$

$N=65 \quad d=102=2\cdot 3\cdot 17,\ \eta=0,\ h=1$: $\langle 2\rangle\oplus\langle -34\rangle\oplus\langle -6\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -2\\ 8 & -1 & -4\\ 270 & \frac{-81}{2} & \frac{-245}{2}\\ 510 & \frac{-155}{2} & \frac{-459}{2}\\ 57 & -9 & -25\\ 2 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 12 & 24 & 735 & 1377 & 150 & 3\\ 0 & -34 & 0 & 34 & 1377 & 2635 & 306 & 17\\ 12 & 0 & -6 & 0 & 150 & 306 & 42 & 6\\ 24 & 34 & 0 & -2 & 3 & 17 & 6 & 3\\ 735 & 1377 & 150 & 3 & -6 & 0 & 12 & 24\\ 1377 & 2635 & 306 & 17 & 0 & -34 & 0 & 34\\ 150 & 306 & 42 & 6 & 12 & 0 & -6 & 0\\ 3 & 17 & 6 & 3 & 24 & 34 & 0 & -2 \end{pmatrix}.$$

$N=66 \quad d=105=3\cdot 5\cdot 7,\ \eta=0, h=1$: $\langle 2\rangle\oplus\langle -42\rangle\oplus\langle -5\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -2\\ 70 & -5 & -42\\ 21 & -2 & -12\\ 90 & -10 & -49\\ \frac{315}{2} & \frac{-37}{2} & -84\\ \frac{39}{2} & \frac{-5}{2} & -10\\ \frac{175}{2} & \frac{-25}{2} & -42\\ \frac{3}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -5 & 0 & 10 & 210 & 60 & 245 & 420 & 50 & 210 & 0\\ 0 & -42 & 0 & 210 & 84 & 420 & 777 & 105 & 525 & 21\\ 10 & 0 & -2 & 0 & 6 & 50 & 105 & 17 & 105 & 9\\ 210 & 210 & 0 & -70 & 0 & 210 & 525 & 105 & 805 & 105\\ 60 & 84 & 6 & 0 & -6 & 0 & 21 & 9 & 105 & 21\\ 245 & 420 & 50 & 210 & 0 & -5 & 0 & 10 & 210 & 60\\ 420 & 777 & 105 & 525 & 21 & 0 & -42 & 0 & 210 & 84\\ 50 & 105 & 17 & 105 & 9 & 10 & 0 & -2 & 0 & 6\\ 210 & 525 & 105 & 805 & 105 & 210 & 210 & 0 & -70 & 0\\ 0 & 21 & 9 & 105 & 21 & 60 & 84 & 6 & 0 & -6 \end{pmatrix}.$$

$N=67 \quad d=105=3\cdot 5\cdot 7,\ \eta=1,\ h=0$: $\langle 1\rangle\oplus\langle -21\rangle\oplus\langle -5\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & -1\\ 8 & -1 & -3\\ 30 & -5 & -9\\ 5 & -1 & -1\\ 70 & -15 & -7\\ 9 & -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} -5 & 0 & 5 & 15 & 45 & 5 & 35 & 0\\ 0 & -21 & 0 & 21 & 105 & 21 & 315 & 42\\ 5 & 0 & -1 & 1 & 15 & 5 & 105 & 18\\ 15 & 21 & 1 & -2 & 0 & 4 & 140 & 30\\ 45 & 105 & 15 & 0 & -30 & 0 & 210 & 60\\ 5 & 21 & 5 & 4 & 0 & -1 & 0 & 3\\ 35 & 315 & 105 & 140 & 210 & 0 & -70 & 0\\ 0 & 42 & 18 & 30 & 60 & 3 & 0 & -3 \end{pmatrix}.$$

$N=68 \quad d=105=3\cdot 5\cdot 7,\ \eta=2,\ h=1$: $\langle 7\rangle\oplus\langle -15\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 108 & 360 & 53 & 15 & 57 & 15 & 3 & 10 & 0 & 0 & 1 & 3 & 69 & 75 & 87 & 1250\\ -48 & -161 & -24 & -7 & -28 & -8 & -2 & -7 & 0 & 1 & 0 & -1 & -28 & -32 & -38 & -553\\ -217 & -720 & -105 & -29 & -105 & -25 & -3 & 0 & 1 & 0 & -3 & -7 & -147 & -155 & -177 & -2520 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 3 & 7 & 147 & 155 & 177 & 2520 & 217 & 720 & 105 & 29 & 105 & 25 & 3 & 0\\ 0 & -15 & 0 & 15 & 420 & 480 & 570 & 8295 & 720 & 2415 & 360 & 105 & 420 & 120 & 30 & 105\\ 3 & 0 & -2 & 0 & 42 & 60 & 78 & 1190 & 105 & 360 & 56 & 18 & 84 & 30 & 12 & 70\\ 7 & 15 & 0 & -1 & 0 & 10 & 18 & 315 & 29 & 105 & 18 & 7 & 42 & 20 & 12 & 105\\ 147 & 420 & 42 & 0 & -42 & 0 & 42 & 1050 & 105 & 420 & 84 & 42 & 336 & 210 & 168 & 1890\\ 155 & 480 & 60 & 10 & 0 & -10 & 0 & 210 & 25 & 120 & 30 & 20 & 210 & 160 & 150 & 1890\\ 177 & 570 & 78 & 18 & 42 & 0 & -6 & 0 & 3 & 30 & 12 & 12 & 168 & 150 & 156 & 2100\\ 2520 & 8295 & 1190 & 315 & 1050 & 210 & 0 & -35 & 0 & 105 & 70 & 105 & 1890 & 1890 & 2100 & 29435\\ 217 & 720 & 105 & 29 & 105 & 25 & 3 & 0 & -1 & 0 & 3 & 7 & 147 & 155 & 177 & 2520\\ 720 & 2415 & 360 & 105 & 420 & 120 & 30 & 105 & 0 & -15 & 0 & 15 & 420 & 480 & 570 & 8295\\ 105 & 360 & 56 & 18 & 84 & 30 & 12 & 70 & 3 & 0 & -2 & 0 & 42 & 60 & 78 & 1190\\ 29 & 105 & 18 & 7 & 42 & 20 & 12 & 105 & 7 & 15 & 0 & -1 & 0 & 10 & 18 & 315\\ 105 & 420 & 84 & 42 & 336 & 210 & 168 & 1890 & 147 & 420 & 42 & 0 & -42 & 0 & 42 & 1050\\ 25 & 120 & 30 & 20 & 210 & 160 & 150 & 1890 & 155 & 480 & 60 & 10 & 0 & -10 & 0 & 210\\ 3 & 30 & 12 & 12 & 168 & 150 & 156 & 2100 & 177 & 570 & 78 & 18 & 42 & 0 & -6 & 0\\ 0 & 105 & 70 & 105 & 1890 & 1890 & 2100 & 29435 & 2520 & 8295 & 1190 & 315 & 1050 & 210 & 0 & -35 \end{pmatrix}.$$

$N=69 \quad d=105=3\cdot 5\cdot 7,\ \eta=3,\ h=0$: $\langle 14\rangle\oplus\langle -10\rangle\oplus\langle -3\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -7\\ \frac{1}{2} & \frac{-1}{2} & -1\\ \frac{5}{2} & \frac{-7}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 21 & 3 & 0\\ 0 & -10 & 0 & 5 & 35\\ 21 & 0 & -21 & 0 & 105\\ 3 & 5 & 0 & -2 & 0\\ 0 & 35 & 105 & 0 & -35 \end{pmatrix}.$$

$N = 70$ $\;d = 105 = 3 \cdot 5 \cdot 7,\ \eta = 5,\ h = 0$: $\langle 5\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -5 \\ 7 & -4 & -7 \\ 7 & -5 & -5 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 15 & 21 & 15 & 0 \\ 0 & -7 & 0 & 28 & 35 & 7 \\ 15 & 0 & -30 & 0 & 30 & 15 \\ 21 & 28 & 0 & -14 & 0 & 7 \\ 15 & 35 & 30 & 0 & -5 & 0 \\ 0 & 7 & 15 & 7 & 0 & -2 \end{pmatrix}.$$

$N = 71$ $\;d = 105 = 3 \cdot 5 \cdot 7,\ \eta = 6,\ h = 0$: $\langle 1\rangle \oplus \langle -15\rangle \oplus \langle -7\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 7 & 0 & -3 \\ 15 & -2 & -5 \\ 11 & -2 & -3 \\ 3 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -7 & 0 & 21 & 35 & 21 & 0 \\ 0 & -15 & 0 & 30 & 30 & 15 \\ 21 & 0 & -14 & 0 & 14 & 21 \\ 35 & 30 & 0 & -10 & 0 & 15 \\ 21 & 30 & 14 & 0 & -2 & 3 \\ 0 & 15 & 21 & 15 & 3 & -6 \end{pmatrix}.$$

$N = 72$ $\;d = 105 = 3 \cdot 5 \cdot 7,\ \eta = 7,\ h = 0$: $\langle 10\rangle \oplus \langle -14\rangle \oplus \langle -3\rangle(1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 42 & -15 & -70 \\ 2 & -1 & -3 \\ 8 & -5 & -10 \\ 7 & -5 & -7 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 6 & 210 & 9 & 30 & 21 & 0 \\ 0 & -14 & 0 & 210 & 14 & 70 & 70 & 7 \\ 6 & 0 & -2 & 0 & 2 & 20 & 28 & 5 \\ 210 & 210 & 0 & -210 & 0 & 210 & 420 & 105 \\ 9 & 14 & 2 & 0 & -1 & 0 & 7 & 3 \\ 30 & 70 & 20 & 210 & 0 & -10 & 0 & 5 \\ 21 & 70 & 28 & 420 & 7 & 0 & -7 & 0 \\ 0 & 7 & 5 & 105 & 3 & 5 & 0 & -1 \end{pmatrix}.$$

$N = 73$ $\;d = 110 = 2 \cdot 5 \cdot 11,\ \eta = 1,\ h = 1$: $U \oplus \langle -110\rangle$

$$\begin{pmatrix} 55 & 0 & -1 & 22 & 10 & 44 & 18 & 440 & 1980 & 361 & 3938 & 485 & 451 & 27 \\ 0 & 0 & 1 & 22 & 5 & 11 & 3 & 55 & 220 & 39 & 418 & 50 & 44 & 2 \\ -1 & 1 & 0 & -3 & -1 & -3 & -1 & -21 & -89 & -16 & -173 & -21 & -19 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -110 & 110 & 55 & 880 & 165 & 275 & 55 & 715 & 2310 & 385 & 3960 & 440 & 330 & 0 \\ 110 & -110 & 0 & 330 & 110 & 330 & 110 & 2310 & 9790 & 1760 & 19030 & 2310 & 2090 & 110 \\ 55 & 0 & -2 & 0 & 5 & 33 & 15 & 385 & 1760 & 322 & 3520 & 435 & 407 & 25 \\ 880 & 330 & 0 & -22 & 0 & 220 & 132 & 3960 & 19030 & 3520 & 38742 & 4840 & 4620 & 308 \\ 165 & 110 & 5 & 0 & -10 & 0 & 10 & 440 & 2310 & 435 & 4840 & 615 & 605 & 45 \\ 275 & 330 & 33 & 220 & 0 & -22 & 0 & 330 & 2090 & 407 & 4620 & 605 & 627 & 55 \\ 55 & 110 & 15 & 132 & 10 & 0 & -2 & 0 & 110 & 25 & 308 & 45 & 55 & 7 \\ 715 & 2310 & 385 & 3960 & 440 & 330 & 0 & -110 & 110 & 55 & 880 & 165 & 275 & 55 \\ 2310 & 9790 & 1760 & 19030 & 2310 & 2090 & 110 & 110 & -110 & 0 & 330 & 110 & 330 & 110 \\ 385 & 1760 & 322 & 3520 & 435 & 407 & 25 & 55 & 0 & -2 & 0 & 5 & 33 & 15 \\ 3960 & 19030 & 3520 & 38742 & 4840 & 4620 & 308 & 880 & 330 & 0 & -22 & 0 & 220 & 132 \\ 440 & 2310 & 435 & 4840 & 615 & 605 & 45 & 165 & 110 & 5 & 0 & -10 & 0 & 10 \\ 330 & 2090 & 407 & 4620 & 605 & 627 & 55 & 275 & 330 & 33 & 220 & 0 & -22 & 0 \\ 0 & 110 & 25 & 308 & 45 & 55 & 7 & 55 & 110 & 15 & 132 & 10 & 0 & -2 \end{pmatrix}.$$

$N = 74$ $\;d = 111 = 3 \cdot 37,\ \eta = 0,\ h = 1$: $\langle 3\rangle \oplus \langle -37\rangle \oplus \langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 21 & -5 & -20 \\ 12 & -3 & -10 \\ 148 & -38 & -111 \\ 11 & -3 & -6 \\ 7 & -2 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 3 & 20 & 10 & 111 & 6 & 1 \\ 0 & -37 & 0 & 185 & 111 & 1406 & 111 & 74 \\ 3 & 0 & -6 & 3 & 6 & 111 & 15 & 18 \\ 20 & 185 & 3 & -2 & 1 & 74 & 18 & 51 \\ 10 & 111 & 6 & 1 & -1 & 0 & 3 & 20 \\ 111 & 1406 & 111 & 74 & 0 & -37 & 0 & 185 \\ 6 & 111 & 15 & 18 & 3 & 0 & -6 & 3 \\ 1 & 74 & 18 & 51 & 20 & 185 & 3 & -2 \end{pmatrix}.$$

$N=75\quad d=130=2\cdot 5\cdot 13,\ \eta=3,\ h=1$: $\langle 10\rangle\oplus\langle -26\rangle\oplus\langle -2\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0&0&1\\ 0&1&0\\ 2&0&-5\\ \frac{3}{2}&\frac{-1}{2}&-3\\ 26&-12&-39\\ \frac{9}{2}&\frac{-5}{2}&-5\\ 5&-3&-3\\ \frac{195}{2}&\frac{-119}{2}&-39\\ 98&-60&-35\\ \frac{57}{2}&\frac{-35}{2}&-9\\ 234&-144&-65\\ 8&-5&0 \end{pmatrix}\qquad \begin{pmatrix} -2&0&10&6&78&10&6&78&70&18&130&0\\ 0&-26&0&13&312&65&78&1547&1560&455&3744&130\\ 10&0&-10&0&130&40&70&1560&1610&480&4030&160\\ 6&13&0&-2&0&5&18&455&480&146&1248&55\\ 78&312&130&0&-26&0&130&3744&4030&1248&10842&520\\ 10&65&40&5&0&-10&0&130&160&55&520&35\\ 6&78&70&18&130&0&-2&0&10&6&78&10\\ 78&1547&1560&455&3744&130&0&-26&0&13&312&65\\ 70&1560&1610&480&4030&160&10&0&-10&0&130&40\\ 18&455&480&146&1248&55&6&13&0&-2&0&5\\ 130&3744&4030&1248&10842&520&78&312&130&0&-26&0\\ 0&130&160&55&520&35&10&65&40&5&0&-10 \end{pmatrix}.$$

$N=76\quad d=141=3\cdot 47,\ \eta=3,\ h=1$: $\langle 1\rangle\oplus\langle -47\rangle\oplus\langle -3\rangle$

$$\begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&-1\\ 705&-84&-235\\ 41&-5&-13\\ 1081&-134&-329\\ 168&-21&-50\\ 1128&-142&-329\\ 23&-3&-6\\ 423&-57&-94\\ 7&-1&-1\\ 47&-7&0 \end{pmatrix}\qquad \begin{pmatrix} -3&0&3&705&39&987&150&987&18&282&3&0\\ 0&-47&0&3948&235&6298&987&6674&141&2679&47&329\\ 3&0&-2&0&2&94&18&141&5&141&4&47\\ 705&3948&0&-282&0&1128&282&2679&141&6909&282&5499\\ 39&235&2&0&-1&0&3&47&4&282&13&282\\ 987&6298&94&1128&0&-94&0&329&47&5499&282&6721\\ 150&987&18&282&3&0&-3&0&3&705&39&987\\ 987&6674&141&2679&47&329&0&-47&0&3948&235&6298\\ 18&141&5&141&4&47&3&0&-2&0&2&94\\ 282&2679&141&6909&282&5499&705&3948&0&-282&0&1128\\ 3&47&4&282&13&282&39&235&2&0&-1&0\\ 0&329&47&5499&282&6721&987&6298&94&1128&0&-94 \end{pmatrix}.$$

$N=77\quad d=155=5\cdot 31,\ \eta=3,\ h=1$: $\langle 2\rangle\oplus\langle -31\rangle\oplus\langle -10\rangle(1/2,0,1/2)$

$$\begin{pmatrix} 114&1705&12&31&0&0&\frac{1}{2}&\frac{17}{2}&\frac{4185}{2}&\frac{221}{2}&\frac{7533}{2}&\frac{7105}{2}&22785&\frac{2449}{2}\\ -28&-420&-3&-8&0&1&0&-2&-510&-27&-922&-870&-5581&-300\\ -13&-186&-1&0&1&0&\frac{-1}{2}&\frac{-3}{2}&\frac{-527}{2}&\frac{-27}{2}&\frac{-899}{2}&\frac{-843}{2}&-2697&\frac{-289}{2} \end{pmatrix}$$

$$\begin{pmatrix} -2&0&2&124&130&868&49&7&155&3&31&15&62&1\\ 0&-310&0&1550&1860&13020&775&155&5115&155&3255&2635&15810&775\\ 2&0&-1&0&10&93&7&3&155&6&155&135&837&43\\ 124&1550&0&-62&0&248&31&31&3255&155&4867&4495&28582&1519\\ 130&1860&10&0&-10&0&5&15&2635&135&4495&4215&26970&1445\\ 868&13020&93&248&0&-31&0&62&15810&837&28582&26970&173011&9300\\ 49&775&7&31&5&0&-2&1&775&43&1519&1445&9300&502\\ 7&155&3&31&15&62&1&-2&0&2&124&130&868&49\\ 155&5115&155&3255&2635&15810&775&0&-310&0&1550&1860&13020&775\\ 3&155&6&155&135&837&43&2&0&-1&0&10&93&7\\ 31&3255&155&4867&4495&28582&1519&124&1550&0&-62&0&248&31\\ 15&2635&135&4495&4215&26970&1445&130&1860&10&0&-10&0&5\\ 62&15810&837&28582&26970&173011&9300&868&13020&93&248&0&-31&0\\ 1&775&43&1519&1445&9300&502&49&775&7&31&5&0&-2 \end{pmatrix}.$$

$N=78\quad d=165=3\cdot 5\cdot 11,\ \eta=0,\ h=0$: $\langle 5\rangle\oplus\langle -6\rangle\oplus\langle -22\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&\frac{-1}{2}&\frac{-1}{2}\\ 33&\frac{-55}{2}&\frac{-15}{2}\\ 1&-1&0 \end{pmatrix}\qquad \begin{pmatrix} -22&0&11&165&0\\ 0&-6&3&165&6\\ 11&3&-2&0&2\\ 165&165&0&-330&0\\ 0&6&2&0&-1 \end{pmatrix}.$$

$N = 79 \quad d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 3, h = 1\colon\ \langle 15\rangle \oplus \langle -22\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & 0 & 1 & 44 & 5 & 22 & 4 & 2 & 22 & 13 & 286 & 27 & 88 & 6 \\ 0 & 1 & 0 & -15 & -2 & \frac{-21}{2} & \frac{-5}{2} & \frac{-3}{2} & \frac{-35}{2} & \frac{-21}{2} & \frac{-465}{2} & -22 & -72 & -5 \\ 1 & 0 & -3 & -110 & -12 & \frac{-99}{2} & \frac{-15}{2} & \frac{-5}{2} & \frac{-33}{2} & \frac{-15}{2} & \frac{-275}{2} & -12 & -33 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 6 & 220 & 24 & 99 & 15 & 5 & 33 & 15 & 275 & 24 & 66 & 0 \\ 0 & -22 & 0 & 330 & 44 & 231 & 55 & 33 & 385 & 231 & 5115 & 484 & 1584 & 110 \\ 6 & 0 & -3 & 0 & 3 & 33 & 15 & 15 & 231 & 150 & 3465 & 333 & 1122 & 90 \\ 220 & 330 & 0 & -110 & 0 & 165 & 165 & 275 & 5115 & 3465 & 81785 & 7920 & 27060 & 2310 \\ 24 & 44 & 3 & 0 & -1 & 0 & 10 & 24 & 484 & 333 & 7920 & 769 & 2640 & 230 \\ 99 & 231 & 33 & 165 & 0 & -66 & 0 & 66 & 1584 & 1122 & 27060 & 2640 & 9141 & 825 \\ 15 & 55 & 15 & 165 & 10 & 0 & -10 & 0 & 110 & 90 & 2310 & 230 & 825 & 85 \\ 5 & 33 & 15 & 275 & 24 & 66 & 0 & -2 & 0 & 6 & 220 & 24 & 99 & 15 \\ 33 & 385 & 231 & 5115 & 484 & 1584 & 110 & 0 & -22 & 0 & 330 & 44 & 231 & 55 \\ 15 & 231 & 150 & 3465 & 333 & 1122 & 90 & 6 & 0 & -3 & 0 & 3 & 33 & 15 \\ 275 & 5115 & 3465 & 81785 & 7920 & 27060 & 2310 & 220 & 330 & 0 & -110 & 0 & 165 & 165 \\ 24 & 484 & 333 & 7920 & 769 & 2640 & 230 & 24 & 44 & 3 & 0 & -1 & 0 & 10 \\ 66 & 1584 & 1122 & 27060 & 2640 & 9141 & 825 & 99 & 231 & 33 & 165 & 0 & -66 & 0 \\ 0 & 110 & 90 & 2310 & 230 & 825 & 85 & 15 & 55 & 15 & 165 & 10 & 0 & -10 \end{pmatrix}.$$

$N = 80 \quad d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 4,\ h = 1\colon\ \langle 3\rangle \oplus \langle -5\rangle \oplus \langle -11\rangle$

$$\begin{pmatrix} 0 & 0 & 11 & 4 & 440 & 91 & 55 & 27 & 154 & 70 & 143 & 10 & 330 & 49 & 15 & 1 \\ 0 & 1 & 0 & -1 & -154 & -33 & -21 & -11 & -66 & -31 & -66 & -5 & -176 & -27 & -9 & -1 \\ 1 & 0 & -6 & -2 & -205 & -42 & -25 & -12 & -67 & -30 & -60 & -4 & -125 & -18 & -5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -11 & 0 & 66 & 22 & 2255 & 462 & 275 & 132 & 737 & 330 & 660 & 44 & 1375 & 198 & 55 & 0 \\ 0 & -5 & 0 & 5 & 770 & 165 & 105 & 55 & 330 & 155 & 330 & 25 & 880 & 135 & 45 & 5 \\ 66 & 0 & -33 & 0 & 990 & 231 & 165 & 99 & 660 & 330 & 759 & 66 & 2640 & 429 & 165 & 33 \\ 22 & 5 & 0 & -1 & 0 & 3 & 5 & 5 & 44 & 25 & 66 & 7 & 330 & 57 & 25 & 7 \\ 2255 & 770 & 990 & 0 & -55 & 0 & 55 & 110 & 1375 & 880 & 2640 & 330 & 18205 & 3300 & 1595 & 550 \\ 462 & 165 & 231 & 3 & 0 & -6 & 0 & 12 & 198 & 135 & 429 & 57 & 3300 & 606 & 300 & 108 \\ 275 & 105 & 165 & 5 & 55 & 0 & -5 & 0 & 55 & 45 & 165 & 25 & 1595 & 300 & 155 & 60 \\ 132 & 55 & 99 & 5 & 110 & 12 & 0 & -2 & 0 & 5 & 33 & 7 & 550 & 108 & 60 & 26 \\ 737 & 330 & 660 & 44 & 1375 & 198 & 55 & 0 & -11 & 0 & 66 & 22 & 2255 & 462 & 275 & 132 \\ 330 & 155 & 330 & 25 & 880 & 135 & 45 & 5 & 0 & -5 & 0 & 5 & 770 & 165 & 105 & 55 \\ 660 & 330 & 759 & 66 & 2640 & 429 & 165 & 33 & 66 & 0 & -33 & 0 & 990 & 231 & 165 & 99 \\ 44 & 25 & 66 & 7 & 330 & 57 & 25 & 7 & 22 & 5 & 0 & -1 & 0 & 3 & 5 & 5 \\ 1375 & 880 & 2640 & 330 & 18205 & 3300 & 1595 & 550 & 2255 & 770 & 990 & 0 & -55 & 0 & 55 & 110 \\ 198 & 135 & 429 & 57 & 3300 & 606 & 300 & 108 & 462 & 165 & 231 & 3 & 0 & -6 & 0 & 12 \\ 55 & 45 & 165 & 25 & 1595 & 300 & 155 & 60 & 275 & 105 & 165 & 5 & 55 & 0 & -5 & 0 \\ 0 & 5 & 33 & 7 & 550 & 108 & 60 & 26 & 132 & 55 & 99 & 5 & 110 & 12 & 0 & -2 \end{pmatrix}.$$

$N = 81 \quad d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 5,\ h = 1\colon\ \langle 1\rangle \oplus \langle -55\rangle \oplus \langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 9 & -1 & -3 \\ 75 & -9 & -20 \\ 242 & -30 & -55 \\ 120 & -15 & -26 \\ 1320 & -166 & -275 \\ 71 & -9 & -14 \\ 39 & -5 & -7 \\ 45 & -6 & -5 \\ 22 & -3 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 9 & 60 & 165 & 78 & 825 & 42 & 21 & 15 & 0 \\ 0 & -55 & 0 & 55 & 495 & 1650 & 825 & 9130 & 495 & 275 & 330 & 165 \\ 3 & 0 & -2 & 0 & 15 & 77 & 42 & 495 & 29 & 18 & 30 & 22 \\ 9 & 55 & 0 & -1 & 0 & 33 & 21 & 275 & 18 & 13 & 30 & 33 \\ 60 & 495 & 15 & 0 & -30 & 0 & 15 & 330 & 30 & 30 & 105 & 165 \\ 165 & 1650 & 77 & 33 & 0 & -11 & 0 & 165 & 22 & 33 & 165 & 374 \\ 78 & 825 & 42 & 21 & 15 & 0 & -3 & 0 & 3 & 9 & 60 & 165 \\ 825 & 9130 & 495 & 275 & 330 & 165 & 0 & -55 & 0 & 55 & 495 & 1650 \\ 42 & 495 & 29 & 18 & 30 & 22 & 3 & 0 & -2 & 0 & 15 & 77 \\ 21 & 275 & 18 & 13 & 30 & 33 & 9 & 55 & 0 & -1 & 0 & 33 \\ 15 & 330 & 30 & 30 & 105 & 165 & 60 & 495 & 15 & 0 & -30 & 0 \\ 0 & 165 & 22 & 33 & 165 & 374 & 165 & 1650 & 77 & 33 & 0 & -11 \end{pmatrix}.$$

$N = 82 \quad d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 6,\ h = 0\ \langle 1\rangle \oplus \langle -15\rangle \oplus \langle -11\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \\ 99 & -11 & -27 \\ 20 & -3 & -5 \\ 5 & -1 & -1 \\ 3 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -11 & 0 & 11 & 297 & 55 & 11 & 0 \\ 0 & -15 & 0 & 165 & 45 & 15 & 15 \\ 11 & 0 & -2 & 0 & 5 & 4 & 9 \\ 297 & 165 & 0 & -33 & 0 & 33 & 132 \\ 55 & 45 & 5 & 0 & -10 & 0 & 15 \\ 11 & 15 & 4 & 33 & 0 & -1 & 0 \\ 0 & 15 & 9 & 132 & 15 & 0 & -6 \end{pmatrix}.$$

$N=83\quad d=170=2\cdot 5\cdot 17,\ \eta=1,\ h=1$: $\langle 2\rangle\oplus\langle -10\rangle\oplus\langle -34\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 205 & \frac{147}{2} & \frac{1}{2} & 0 & 0 & 4 & \frac{153}{2} & 65 & \frac{123}{2} & \frac{89}{2} & 3060 & 720 & 176 & \frac{1377}{2}\\ -39 & \frac{-29}{2} & \frac{-1}{2} & 0 & 1 & 0 & \frac{-17}{2} & -9 & \frac{-19}{2} & \frac{-15}{2} & -544 & -129 & -32 & \frac{-255}{2}\\ -45 & -16 & 0 & 1 & 0 & -1 & -18 & -15 & -14 & -10 & -681 & -160 & -39 & -152 \end{pmatrix}$$

$$\begin{pmatrix} -10 & 0 & 10 & 1530 & 390 & 110 & 510 & 190 & 90 & 20 & 510 & 90 & 10 & 0\\ 0 & -2 & 1 & 544 & 145 & 44 & 221 & 90 & 47 & 14 & 476 & 95 & 16 & 34\\ 10 & 1 & -2 & 0 & 5 & 4 & 34 & 20 & 14 & 7 & 340 & 75 & 16 & 51\\ 1530 & 544 & 0 & -34 & 0 & 34 & 612 & 510 & 476 & 340 & 23154 & 5440 & 1326 & 5168\\ 390 & 145 & 5 & 0 & -10 & 0 & 85 & 90 & 95 & 75 & 5440 & 1290 & 320 & 1275\\ 110 & 44 & 4 & 34 & 0 & -2 & 0 & 10 & 16 & 16 & 1326 & 320 & 82 & 340\\ 510 & 221 & 34 & 612 & 85 & 0 & -34 & 0 & 34 & 51 & 5168 & 1275 & 340 & 1479\\ 190 & 90 & 20 & 510 & 90 & 10 & 0 & -10 & 0 & 10 & 1530 & 390 & 110 & 510\\ 90 & 47 & 14 & 476 & 95 & 16 & 34 & 0 & -2 & 1 & 544 & 145 & 44 & 221\\ 20 & 14 & 7 & 340 & 75 & 16 & 51 & 10 & 1 & -2 & 0 & 5 & 4 & 34\\ 510 & 476 & 340 & 23154 & 5440 & 1326 & 5168 & 1530 & 544 & 0 & -34 & 0 & 34 & 612\\ 90 & 95 & 75 & 5440 & 1290 & 320 & 1275 & 390 & 145 & 5 & 0 & -10 & 0 & 85\\ 10 & 16 & 16 & 1326 & 320 & 82 & 340 & 110 & 44 & 4 & 34 & 0 & -2 & 0\\ 0 & 34 & 51 & 5168 & 1275 & 340 & 1479 & 510 & 221 & 34 & 612 & 85 & 0 & -34 \end{pmatrix}.$$

$N=84\quad d=195=3\cdot 5\cdot 13,\ \eta=3,\ h=0$: $\langle 15\rangle\oplus\langle -13\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -5\\ 13 & -9 & -39\\ 1 & -1 & -2\\ 13 & -15 & 0 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 5 & 39 & 2 & 0\\ 0 & -13 & 0 & 117 & 13 & 195\\ 5 & 0 & -10 & 0 & 5 & 195\\ 39 & 117 & 0 & -39 & 0 & 780\\ 2 & 13 & 5 & 0 & -2 & 0\\ 0 & 195 & 195 & 780 & 0 & -390 \end{pmatrix}.$$

$N=85\quad d=195=3\cdot 5\cdot 13,\ \eta=5, h=1$: $\langle 6\rangle\oplus\langle -26\rangle\oplus\langle -5\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 5 & 0 & -6\\ 5 & -1 & -5\\ \frac{11}{2} & \frac{-3}{2} & -5\\ \frac{15}{2} & \frac{-5}{2} & -6\\ \frac{39}{2} & \frac{-15}{2} & -13\\ \frac{35}{2} & \frac{-15}{2} & -9\\ \frac{11}{2} & \frac{-5}{2} & -2\\ 2 & -1 & 0 \end{pmatrix}\qquad \begin{pmatrix} -5 & 0 & 30 & 25 & 25 & 30 & 65 & 45 & 10 & 0\\ 0 & -26 & 0 & 26 & 39 & 65 & 195 & 195 & 65 & 26\\ 30 & 0 & -30 & 0 & 15 & 45 & 195 & 255 & 105 & 60\\ 25 & 26 & 0 & -1 & 1 & 10 & 65 & 105 & 50 & 34\\ 25 & 39 & 15 & 1 & -2 & 0 & 26 & 60 & 34 & 27\\ 30 & 65 & 45 & 10 & 0 & -5 & 0 & 30 & 25 & 25\\ 65 & 195 & 195 & 65 & 26 & 0 & -26 & 0 & 26 & 39\\ 45 & 195 & 255 & 105 & 60 & 30 & 0 & -30 & 0 & 15\\ 10 & 65 & 105 & 50 & 34 & 25 & 26 & 0 & -1 & 1\\ 0 & 26 & 60 & 34 & 27 & 25 & 39 & 15 & 1 & -2 \end{pmatrix}.$$

$N=86\quad d=195=3\cdot 5\cdot 13,\ \eta=6,\ h=0$: $\langle 3\rangle\oplus\langle -65\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -3\\ 28 & -5 & -27\\ 115 & -21 & -105\\ 91 & -17 & -78\\ 15 & -3 & -10\\ 13 & -3 & 0 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 3 & 27 & 105 & 78 & 10 & 0\\ 0 & -65 & 0 & 325 & 1365 & 1105 & 195 & 195\\ 3 & 0 & -6 & 3 & 30 & 39 & 15 & 39\\ 27 & 325 & 3 & -2 & 0 & 13 & 15 & 117\\ 105 & 1365 & 30 & 0 & -15 & 0 & 30 & 390\\ 78 & 1105 & 39 & 13 & 0 & -26 & 0 & 234\\ 10 & 195 & 15 & 15 & 30 & 0 & -10 & 0\\ 0 & 195 & 39 & 117 & 390 & 234 & 0 & -78 \end{pmatrix}.$$

$N = 87 \quad d = 205 = 5 \cdot 41,\ \eta = 1,\ h = 1$: $\langle 2\rangle \oplus \langle -41\rangle \oplus \langle -10\rangle (1/2, 0, 1/2)$

$$\begin{pmatrix} 107 & \frac{185}{2} & \frac{533}{2} & 32 & 820 & 9 & 0 & 0 & \frac{1}{2} & 5 & \frac{95}{2} & \frac{1763}{2} & 248 & 10660 & 327 & 840 & 6888 & \frac{727}{2} \\ -23 & -20 & -58 & -7 & -180 & -2 & 0 & 1 & 0 & -1 & -10 & -188 & -53 & -2280 & -70 & -180 & -1477 & -78 \\ -11 & \frac{-17}{2} & \frac{-41}{2} & -2 & -41 & 0 & 1 & 0 & \frac{-1}{2} & -1 & \frac{-13}{2} & \frac{-205}{2} & -28 & -1189 & -36 & -91 & -738 & \frac{-77}{2} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 & 82 & 27 & 1230 & 40 & 110 & 943 & 52 \\ 0 & -10 & 0 & 10 & 615 & 25 & 85 & 820 & 50 \\ 82 & 0 & -82 & 0 & 615 & 41 & 205 & 2378 & 164 \\ 27 & 10 & 0 & -1 & 0 & 2 & 20 & 287 & 22 \\ 1230 & 615 & 615 & 0 & -410 & 0 & 410 & 7380 & 615 \\ 40 & 25 & 41 & 2 & 0 & -2 & 0 & 82 & 9 \\ 110 & 85 & 205 & 20 & 410 & 0 & -10 & 0 & 5 \\ 943 & 820 & 2378 & 287 & 7380 & 82 & 0 & -41 & 0 \\ 52 & 50 & 164 & 22 & 615 & 9 & 5 & 0 & -2 \\ 17 & 20 & 82 & 13 & 410 & 8 & 10 & 41 & 0 \\ 20 & 35 & 205 & 40 & 1435 & 35 & 65 & 410 & 15 \\ 82 & 205 & 1763 & 410 & 16195 & 451 & 1025 & 7708 & 369 \\ 13 & 40 & 410 & 101 & 4100 & 118 & 280 & 2173 & 108 \\ 410 & 1435 & 16195 & 4100 & 168510 & 4920 & 11890 & 93480 & 4715 \\ 8 & 35 & 451 & 118 & 4920 & 146 & 360 & 2870 & 147 \\ 10 & 65 & 1025 & 280 & 11890 & 360 & 910 & 7380 & 385 \\ 41 & 410 & 7708 & 2173 & 93480 & 2870 & 7380 & 60557 & 3198 \\ 0 & 15 & 369 & 108 & 4715 & 147 & 385 & 3198 & 171 \end{pmatrix}.$$

$N = 88 \quad d = 210 = 2 \cdot 3 \cdot 5 \cdot 7,\ \eta = 2,\ h = 0$: $\langle 2\rangle \oplus \langle -10\rangle \oplus \langle -42\rangle (1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 9 & 0 & -2 \\ 70 & -6 & -15 \\ \frac{189}{2} & \frac{-21}{2} & -20 \\ 5 & -1 & -1 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -42 & 0 & 84 & 630 & 840 & 42 & 0 \\ 0 & -10 & 0 & 60 & 105 & 10 & 5 \\ 84 & 0 & -6 & 0 & 21 & 6 & 9 \\ 630 & 60 & 0 & -10 & 0 & 10 & 40 \\ 840 & 105 & 21 & 0 & -42 & 0 & 42 \\ 42 & 10 & 6 & 10 & 0 & -2 & 0 \\ 0 & 5 & 9 & 40 & 42 & 0 & -2 \end{pmatrix}.$$

$N = 89 \quad d = 210 = 2 \cdot 3 \cdot 5 \cdot 7,\ \eta = 4,\ h = 1$: $\langle 30\rangle \oplus \langle -14\rangle \oplus \langle -2\rangle (0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -4 \\ 8 & -3 & -30 \\ 7 & \frac{-7}{2} & \frac{-51}{2} \\ 7 & \frac{-9}{2} & \frac{-49}{2} \\ 1 & -1 & -3 \\ 1 & \frac{-3}{2} & \frac{-3}{2} \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 8 & 60 & 51 & 49 & 6 & 3 \\ 0 & -14 & 0 & 42 & 49 & 63 & 14 & 21 \\ 8 & 0 & -2 & 0 & 6 & 14 & 6 & 18 \\ 60 & 42 & 0 & -6 & 3 & 21 & 18 & 87 \\ 51 & 49 & 6 & 3 & -2 & 0 & 8 & 60 \\ 49 & 63 & 14 & 21 & 0 & -14 & 0 & 42 \\ 6 & 14 & 6 & 18 & 8 & 0 & -2 & 0 \\ 3 & 21 & 18 & 87 & 60 & 42 & 0 & -6 \end{pmatrix}.$$

$N = 90 \quad d = 219 = 3 \cdot 73,\ \eta = 0,\ h = 0$: $\langle 3\rangle \oplus \langle -73\rangle \oplus \langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 6 & -1 & -6 \\ 28 & -5 & -23 \\ 49 & -9 & -36 \\ 219 & -41 & -146 \\ 5 & -1 & -2 \\ 73 & -15 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 3 & 6 & 23 & 36 & 146 & 2 & 0 \\ 0 & -73 & 0 & 73 & 365 & 657 & 2993 & 73 & 1095 \\ 3 & 0 & -6 & 0 & 15 & 39 & 219 & 9 & 219 \\ 6 & 73 & 0 & -1 & 1 & 9 & 73 & 5 & 219 \\ 23 & 365 & 15 & 1 & -2 & 3 & 73 & 9 & 657 \\ 36 & 657 & 39 & 9 & 3 & -6 & 0 & 6 & 876 \\ 146 & 2993 & 219 & 73 & 73 & 0 & -146 & 0 & 3066 \\ 2 & 73 & 9 & 5 & 9 & 6 & 0 & -2 & 0 \\ 0 & 1095 & 219 & 219 & 657 & 876 & 3066 & 0 & -438 \end{pmatrix}.$$

$N=91\quad d=231=3\cdot 7\cdot 11,\ \eta=1,\ h=1$: $\langle 14\rangle\oplus\langle -22\rangle\oplus\langle -3\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -7\\ \frac{3}{2} & \frac{-1}{2} & -3\\ \frac{99}{2} & \frac{-45}{2} & -88\\ \frac{35}{2} & \frac{-17}{2} & -30\\ 60 & -30 & -101\\ 132 & -67 & -220\\ 81 & -42 & -133\\ \frac{9}{2} & \frac{-5}{2} & -7\\ \frac{33}{2} & \frac{-21}{2} & -22\\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix}\qquad \begin{pmatrix} -3 & 0 & 21 & 9 & 264 & 90 & 303 & 660 & 399 & 21 & 66 & 0\\ 0 & -22 & 0 & 11 & 495 & 187 & 660 & 1474 & 924 & 55 & 231 & 11\\ 21 & 0 & -21 & 0 & 231 & 105 & 399 & 924 & 609 & 42 & 231 & 21\\ 9 & 11 & 0 & -1 & 0 & 4 & 21 & 55 & 42 & 4 & 33 & 5\\ 264 & 495 & 231 & 0 & -66 & 0 & 66 & 231 & 231 & 33 & 429 & 99\\ 90 & 187 & 105 & 4 & 0 & -2 & 0 & 11 & 21 & 5 & 99 & 29\\ 303 & 660 & 399 & 21 & 66 & 0 & -3 & 0 & 21 & 9 & 264 & 90\\ 660 & 1474 & 924 & 55 & 231 & 11 & 0 & -22 & 0 & 11 & 495 & 187\\ 399 & 924 & 609 & 42 & 231 & 21 & 21 & 0 & -21 & 0 & 231 & 105\\ 21 & 55 & 42 & 4 & 33 & 5 & 9 & 11 & 0 & -1 & 0 & 4\\ 66 & 231 & 231 & 33 & 429 & 99 & 264 & 495 & 231 & 0 & -66 & 0\\ 0 & 11 & 21 & 5 & 99 & 29 & 90 & 187 & 105 & 4 & 0 & -2 \end{pmatrix}.$$

$N=92\quad d=231=3\cdot 7\cdot 11,\ \eta=7,\ h=0$: $\langle 6\rangle\oplus\langle -14\rangle\oplus\langle -11\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 4 & 0 & -3\\ \frac{77}{2} & \frac{-11}{2} & -28\\ \frac{3}{2} & \frac{-1}{2} & -1\\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix}\qquad \begin{pmatrix} -11 & 0 & 33 & 308 & 11 & 0\\ 0 & -14 & 0 & 77 & 7 & 7\\ 33 & 0 & -3 & 0 & 3 & 12\\ 308 & 77 & 0 & -154 & 0 & 77\\ 11 & 7 & 3 & 0 & -1 & 1\\ 0 & 7 & 12 & 77 & 1 & -2 \end{pmatrix}.$$

$N=93\quad d=255=3\cdot 5\cdot 17,\ \eta=0,\ h=1$: $\langle 3\rangle\oplus\langle -17\rangle\oplus\langle -5\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 7 & -2 & -4\\ 34 & -11 & -17\\ 8 & -3 & -3\\ 25 & -10 & -6\\ 170 & -69 & -34\\ 49 & -20 & -9\\ 78 & -32 & -13\\ 221 & -91 & -34\\ 7 & -3 & 0 \end{pmatrix}\qquad \begin{pmatrix} -5 & 0 & 5 & 20 & 85 & 15 & 30 & 170 & 45 & 65 & 170 & 0\\ 0 & -17 & 0 & 34 & 187 & 51 & 170 & 1173 & 340 & 544 & 1547 & 51\\ 5 & 0 & -2 & 1 & 17 & 9 & 45 & 340 & 102 & 169 & 493 & 21\\ 20 & 34 & 1 & -1 & 0 & 6 & 65 & 544 & 169 & 290 & 867 & 45\\ 85 & 187 & 17 & 0 & -34 & 0 & 170 & 1547 & 493 & 867 & 2635 & 153\\ 15 & 51 & 9 & 6 & 0 & -6 & 0 & 51 & 21 & 45 & 153 & 15\\ 30 & 170 & 45 & 65 & 170 & 0 & -5 & 0 & 5 & 20 & 85 & 15\\ 170 & 1173 & 340 & 544 & 1547 & 51 & 0 & -17 & 0 & 34 & 187 & 51\\ 45 & 340 & 102 & 169 & 493 & 21 & 5 & 0 & -2 & 1 & 17 & 9\\ 65 & 544 & 169 & 290 & 867 & 45 & 20 & 34 & 1 & -1 & 0 & 6\\ 170 & 1547 & 493 & 867 & 2635 & 153 & 85 & 187 & 17 & 0 & -34 & 0\\ 0 & 51 & 21 & 45 & 153 & 15 & 15 & 51 & 9 & 6 & 0 & -6 \end{pmatrix}.$$

$N=94\quad d=255=3\cdot 5\cdot 17,\ \eta=6, h=0$: $\langle 3\rangle\oplus\langle -85\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -3\\ 6 & -1 & -5\\ 85 & -15 & -51\\ 5 & -1 & 0 \end{pmatrix}\qquad \begin{pmatrix} -1 & 0 & 3 & 5 & 51 & 0\\ 0 & -85 & 0 & 85 & 1275 & 85\\ 3 & 0 & -6 & 3 & 102 & 15\\ 5 & 85 & 3 & -2 & 0 & 5\\ 51 & 1275 & 102 & 0 & -51 & 0\\ 0 & 85 & 15 & 5 & 0 & -10 \end{pmatrix}.$$

$N=95\quad d=273=3\cdot 7\cdot 13,\ \eta=0,\ h=1$: $\langle 14\rangle\oplus\langle -13\rangle\oplus\langle -6\rangle(1/2,0,1/2)$

$$\begin{pmatrix} 93 & -84 & -70\\ 2 & -2 & -1\\ 13 & -14 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ \frac{3}{2} & 0 & \frac{-7}{2}\\ \frac{5}{2} & -2 & \frac{-5}{2}\\ \frac{793}{2} & -350 & \frac{-637}{2}\\ \frac{189}{2} & -84 & \frac{-149}{2}\\ 234 & -209 & -182 \end{pmatrix}\qquad \begin{pmatrix} -42 & 0 & 1638 & 420 & 1092 & 483 & 21 & 273 & 21 & 0\\ 0 & -2 & 0 & 6 & 26 & 21 & 3 & 91 & 15 & 26\\ 1638 & 0 & -182 & 0 & 182 & 273 & 91 & 8463 & 1911 & 4550\\ 420 & 6 & 0 & -6 & 0 & 21 & 15 & 1911 & 447 & 1092\\ 1092 & 26 & 182 & 0 & -13 & 0 & 26 & 4550 & 1092 & 2717\\ 483 & 21 & 273 & 21 & 0 & -42 & 0 & 1638 & 420 & 1092\\ 21 & 3 & 91 & 15 & 26 & 0 & -2 & 0 & 6 & 26\\ 273 & 91 & 8463 & 1911 & 4550 & 1638 & 0 & -182 & 0 & 182\\ 21 & 15 & 1911 & 447 & 1092 & 420 & 6 & 0 & -6 & 0\\ 0 & 26 & 4550 & 1092 & 2717 & 1092 & 26 & 182 & 0 & -13 \end{pmatrix}.$$

$N=96\;\; d=273=3\cdot 7\cdot 13,\ \eta=6,\ h=1\colon\ \langle 26\rangle\oplus\langle -7\rangle\oplus\langle -6\rangle(1/2,0,1/2)$

$$\begin{pmatrix} \frac{147}{2} & \frac{261}{2} & \frac{35}{2} & \frac{23}{2} & \frac{1}{2} & 0 & 0 & 6 & 7 & 34 & 17 & \frac{273}{2} \\ -64 & -117 & -17 & -13 & -1 & 0 & 1 & 0 & -4 & -26 & -14 & -117 \\ \frac{-273}{2} & \frac{-481}{2} & \frac{-63}{2} & \frac{-39}{2} & \frac{-1}{2} & 1 & 0 & -13 & -14 & -65 & -32 & \frac{-509}{2} \end{pmatrix}$$

$$\begin{pmatrix} -7 & 0 & 28 & 182 & 98 & 819 & 448 & 819 & 119 & 91 & 7 & 0 \\ 0 & -78 & 0 & 234 & 156 & 1443 & 819 & 1599 & 273 & 273 & 39 & 78 \\ 28 & 0 & -14 & 0 & 14 & 189 & 119 & 273 & 63 & 91 & 21 & 84 \\ 182 & 234 & 0 & -26 & 0 & 117 & 91 & 273 & 91 & 195 & 65 & 390 \\ 98 & 156 & 14 & 0 & -2 & 3 & 7 & 39 & 21 & 65 & 27 & 192 \\ 819 & 1443 & 189 & 117 & 3 & -6 & 0 & 78 & 84 & 390 & 192 & 1527 \\ 448 & 819 & 119 & 91 & 7 & 0 & -7 & 0 & 28 & 182 & 98 & 819 \\ 819 & 1599 & 273 & 273 & 39 & 78 & 0 & -78 & 0 & 234 & 156 & 1443 \\ 119 & 273 & 63 & 91 & 21 & 84 & 28 & 0 & -14 & 0 & 14 & 189 \\ 91 & 273 & 91 & 195 & 65 & 390 & 182 & 234 & 0 & -26 & 0 & 117 \\ 7 & 39 & 21 & 65 & 27 & 192 & 98 & 156 & 14 & 0 & -2 & 3 \\ 0 & 78 & 84 & 390 & 192 & 1527 & 819 & 1443 & 189 & 117 & 3 & -6 \end{pmatrix}.$$

$N=97\;\; d=285=3\cdot 5\cdot 19,\ \eta=3,\ h=1\colon\ \langle 10\rangle\oplus\langle -38\rangle\oplus\langle -3\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 57 & -12 & -95 \\ \frac{19}{2} & \frac{-5}{2} & -15 \\ 13 & -4 & -19 \\ 266 & -85 & -380 \\ 18 & -6 & -25 \\ 57 & -20 & -76 \\ 8 & -3 & -10 \\ 114 & -45 & -133 \\ \frac{11}{2} & \frac{-5}{2} & -5 \\ 2 & -1 & -1 \\ 19 & -10 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 6 & 285 & 45 & 57 & 1140 & 75 & 228 & 30 & 399 & 15 & 3 & 0 \\ 0 & -38 & 0 & 456 & 95 & 152 & 3230 & 228 & 760 & 114 & 1710 & 95 & 38 & 380 \\ 6 & 0 & -2 & 0 & 5 & 16 & 380 & 30 & 114 & 20 & 342 & 25 & 14 & 190 \\ 285 & 456 & 0 & -57 & 0 & 171 & 4560 & 399 & 1710 & 342 & 6555 & 570 & 399 & 6270 \\ 45 & 95 & 5 & 0 & -10 & 0 & 95 & 15 & 95 & 25 & 570 & 60 & 50 & 855 \\ 57 & 152 & 16 & 171 & 0 & -1 & 0 & 3 & 38 & 14 & 399 & 50 & 51 & 950 \\ 1140 & 3230 & 380 & 4560 & 95 & 0 & -190 & 0 & 380 & 190 & 6270 & 855 & 950 & 18240 \\ 75 & 228 & 30 & 399 & 15 & 3 & 0 & -3 & 0 & 6 & 285 & 45 & 57 & 1140 \\ 228 & 760 & 114 & 1710 & 95 & 38 & 380 & 0 & -38 & 0 & 456 & 95 & 152 & 3230 \\ 30 & 114 & 20 & 342 & 25 & 14 & 190 & 6 & 0 & -2 & 0 & 5 & 16 & 380 \\ 399 & 1710 & 342 & 6555 & 570 & 399 & 6270 & 285 & 456 & 0 & -57 & 0 & 171 & 4560 \\ 15 & 95 & 25 & 570 & 60 & 50 & 855 & 45 & 95 & 5 & 0 & -10 & 0 & 95 \\ 3 & 38 & 14 & 399 & 50 & 51 & 950 & 57 & 152 & 16 & 171 & 0 & -1 & 0 \\ 0 & 380 & 190 & 6270 & 855 & 950 & 18240 & 1140 & 3230 & 380 & 4560 & 95 & 0 & -190 \end{pmatrix}.$$

$N=98\;\; d=285=3\cdot 5\cdot 19,\ \eta=5,\ h=0\colon\ \langle 1\rangle\oplus\langle -95\rangle\oplus\langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 285 & -24 & -95 \\ 11 & -1 & -3 \\ 30 & -3 & -5 \\ 19 & -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} -3 & 0 & 3 & 285 & 9 & 15 & 0 \\ 0 & -95 & 0 & 2280 & 95 & 285 & 190 \\ 3 & 0 & -2 & 0 & 2 & 15 & 19 \\ 285 & 2280 & 0 & -570 & 0 & 285 & 855 \\ 9 & 95 & 2 & 0 & -1 & 0 & 19 \\ 15 & 285 & 15 & 285 & 0 & -30 & 0 \\ 0 & 190 & 19 & 855 & 19 & 0 & -19 \end{pmatrix}.$$

$N=99\;\; d=291=3\cdot 97,\ \eta=0,\ h=1\colon\ \langle 3\rangle\oplus\langle -97\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 0 & 0 & 1 & 97 & 13 & 25 & 2134 & 80 & 73 & 18 & 582 & 53 & 485 & 29 & 23 & 1358 & 40 & 17 \\ 0 & 1 & 0 & -14 & -2 & -4 & -345 & -13 & -12 & -3 & -98 & -9 & -83 & -5 & -4 & -237 & -7 & -3 \\ 1 & 0 & -3 & -97 & -11 & -18 & -1455 & -53 & -45 & -10 & -291 & -24 & -194 & -10 & -6 & -291 & -7 & 0 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 & 97 & 11 & 18 & 1455 & 53 & 45 \\ 0 & -97 & 0 & 1358 & 194 & 388 & 33465 & 1261 & 1164 \\ 3 & 0 & -6 & 0 & 6 & 21 & 2037 & 81 & 84 \\ 97 & 1358 & 0 & -194 & 0 & 97 & 11349 & 485 & 582 \\ 11 & 194 & 6 & 0 & -2 & 1 & 291 & 15 & 24 \\ 18 & 388 & 21 & 97 & 1 & -1 & 0 & 2 & 9 \\ 1455 & 33465 & 2037 & 11349 & 291 & 0 & -582 & 0 & 291 \\ 53 & 1261 & 81 & 485 & 15 & 2 & 0 & -2 & 3 \\ 45 & 1164 & 84 & 582 & 24 & 9 & 291 & 3 & -6 \\ 10 & 291 & 24 & 194 & 10 & 6 & 291 & 7 & 0 \\ 291 & 9506 & 873 & 8051 & 485 & 388 & 22989 & 679 & 291 \\ 24 & 873 & 87 & 873 & 57 & 51 & 3201 & 99 & 51 \\ 194 & 8051 & 873 & 9603 & 679 & 679 & 45105 & 1455 & 873 \\ 10 & 485 & 57 & 679 & 51 & 55 & 3783 & 125 & 81 \\ 6 & 388 & 51 & 679 & 55 & 65 & 4656 & 158 & 111 \\ 291 & 22989 & 3201 & 45105 & 3783 & 4656 & 339306 & 11640 & 8439 \\ 7 & 679 & 99 & 1455 & 125 & 158 & 11640 & 402 & 297 \\ 0 & 291 & 51 & 873 & 81 & 111 & 8439 & 297 & 231 \end{pmatrix}.$$

$N = 100 \quad d = 330 = 2 \cdot 3 \cdot 5 \cdot 11,\ \eta = 3,\ h = 0$: $\langle 6\rangle \oplus \langle -110\rangle \oplus \langle -2\rangle(1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 154 & -18 & -231 \\ \frac{55}{2} & \frac{-7}{2} & -40 \\ \frac{33}{2} & \frac{-5}{2} & -22 \\ 5 & -1 & -5 \\ \frac{11}{2} & \frac{-3}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 4 & 462 & 80 & 44 & 10 & 0 \\ 0 & -110 & 0 & 1980 & 385 & 275 & 110 & 165 \\ 4 & 0 & -2 & 0 & 5 & 11 & 10 & 33 \\ 462 & 1980 & 0 & -66 & 0 & 132 & 330 & 2112 \\ 80 & 385 & 5 & 0 & -10 & 0 & 40 & 330 \\ 44 & 275 & 11 & 132 & 0 & -22 & 0 & 132 \\ 10 & 110 & 10 & 330 & 40 & 0 & -10 & 0 \\ 0 & 165 & 33 & 2112 & 330 & 132 & 0 & -66 \end{pmatrix}.$$

$N = 101 \quad d = 345 = 3 \cdot 5 \cdot 23,\ \eta = 6,\ h = 0$: $\langle 1\rangle \oplus \langle -23\rangle \oplus \langle -15\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \\ 6 & -1 & -1 \\ 207 & -39 & -23 \\ 25 & -5 & -2 \\ 23 & -5 & 0 \end{pmatrix} \qquad \begin{pmatrix} -15 & 0 & 15 & 15 & 345 & 30 & 0 \\ 0 & -23 & 0 & 23 & 897 & 115 & 115 \\ 15 & 0 & -6 & 3 & 276 & 45 & 69 \\ 15 & 23 & 3 & -2 & 0 & 5 & 23 \\ 345 & 897 & 276 & 0 & -69 & 0 & 276 \\ 30 & 115 & 45 & 5 & 0 & -10 & 0 \\ 0 & 115 & 69 & 23 & 276 & 0 & -46 \end{pmatrix}.$$

$N = 102 \quad d = 357 = 3 \cdot 7 \cdot 17,\ \eta = 3,\ h = 1$: $\langle 17\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$

$$\begin{pmatrix} 13 & 55 & 7 & 0 & 0 & 7 & 3 & 105 & 1 & 13 & 7 & 6 & 42 & 367 & 117 & 3465 \\ -20 & -85 & -11 & 0 & 1 & 0 & -1 & -51 & -1 & -17 & -10 & -9 & -64 & -561 & -179 & -5304 \\ -5 & -17 & 0 & 1 & 0 & -17 & -7 & -238 & -2 & -17 & -7 & -4 & -21 & -170 & -53 & -1547 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 & 0 & 7 & 15 & 140 & 1292 & 418 & 12495 \\ 0 & -17 & 0 & 51 & 595 & 5678 & 1853 & 55692 \\ 7 & 0 & -14 & 0 & 77 & 833 & 280 & 8568 \\ 15 & 51 & 0 & -3 & 0 & 51 & 21 & 714 \\ 140 & 595 & 77 & 0 & -7 & 0 & 7 & 357 \\ 1292 & 5678 & 833 & 51 & 0 & -34 & 0 & 357 \\ 418 & 1853 & 280 & 21 & 7 & 0 & -1 & 0 \\ 12495 & 55692 & 8568 & 714 & 357 & 357 & 0 & -714 \\ 51 & 238 & 42 & 6 & 7 & 17 & 2 & 0 \\ 238 & 1173 & 238 & 51 & 119 & 680 & 187 & 4998 \\ 42 & 238 & 63 & 21 & 70 & 476 & 140 & 3927 \\ 6 & 51 & 21 & 12 & 63 & 510 & 159 & 4641 \\ 7 & 119 & 70 & 63 & 448 & 3927 & 1253 & 37128 \\ 17 & 680 & 476 & 510 & 3927 & 35003 & 11220 & 333438 \\ 2 & 187 & 140 & 159 & 1253 & 11220 & 3601 & 107100 \\ 0 & 4998 & 3927 & 4641 & 37128 & 333438 & 107100 & 3186939 \end{pmatrix}.$$

$N = 103 \quad d = 357 = 3 \cdot 7 \cdot 17,\ \eta = 5,\ h = 1$: $\langle 1\rangle \oplus \langle -119\rangle \oplus \langle -3\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 561 & -42 & -187 \\ 145 & -11 & -47 \\ 2975 & -227 & -952 \\ 156 & -12 & -49 \\ 1547 & -120 & -476 \\ 12 & -1 & -3 \\ 102 & -9 & -17 \\ 11 & -1 & -1 \\ 119 & -11 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 3 & 561 & 141 & 2856 & 147 & 1428 & 9 & 51 & 3 & 0 \\ 0 & -119 & 0 & 4998 & 1309 & 27013 & 1428 & 14280 & 119 & 1071 & 119 & 1309 \\ 3 & 0 & -2 & 0 & 4 & 119 & 9 & 119 & 3 & 51 & 8 & 119 \\ 561 & 4998 & 0 & -102 & 0 & 357 & 51 & 1071 & 51 & 2703 & 612 & 11781 \\ 141 & 1309 & 4 & 0 & -1 & 0 & 3 & 119 & 8 & 612 & 145 & 2856 \\ 2856 & 27013 & 119 & 357 & 0 & -238 & 0 & 1309 & 119 & 11781 & 2856 & 56882 \\ 147 & 1428 & 9 & 51 & 3 & 0 & -3 & 0 & 3 & 561 & 141 & 2856 \\ 1428 & 14280 & 119 & 1071 & 119 & 1309 & 0 & -119 & 0 & 4998 & 1309 & 27013 \\ 9 & 119 & 3 & 51 & 8 & 119 & 3 & 0 & -2 & 0 & 4 & 119 \\ 51 & 1071 & 51 & 2703 & 612 & 11781 & 561 & 4998 & 0 & -102 & 0 & 357 \\ 3 & 119 & 8 & 612 & 145 & 2856 & 141 & 1309 & 4 & 0 & -1 & 0 \\ 0 & 1309 & 119 & 11781 & 2856 & 56882 & 2856 & 27013 & 119 & 357 & 0 & -238 \end{pmatrix}.$$

$N=104\quad d=385=5\cdot 7\cdot 11,\ \eta=6,\ h=1\colon\ \langle 5\rangle\oplus\langle -11\rangle\oplus\langle -7\rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 77 & -28 & -55 \\ 37 & -15 & -25 \\ 49 & -21 & -32 \\ 88 & -40 & -55 \\ 14 & -7 & -8 \\ 22 & -12 & -11 \\ 5 & -3 & -2 \\ 77 & -49 & -22 \\ 23 & -15 & -5 \\ 21 & -14 & -3 \\ 22 & -15 & 0 \end{pmatrix} \qquad \begin{pmatrix} -7 & 0 & 7 & 385 & 175 & 224 & 385 & 56 & 77 & 14 & 154 & 35 & 21 & 0 \\ 0 & -11 & 0 & 308 & 165 & 231 & 440 & 77 & 132 & 33 & 539 & 165 & 154 & 165 \\ 7 & 0 & -2 & 0 & 10 & 21 & 55 & 14 & 33 & 11 & 231 & 80 & 84 & 110 \\ 385 & 308 & 0 & -154 & 0 & 77 & 385 & 154 & 539 & 231 & 6083 & 2310 & 2618 & 3850 \\ 175 & 165 & 10 & 0 & -5 & 0 & 55 & 35 & 165 & 80 & 2310 & 905 & 1050 & 1595 \\ 224 & 231 & 21 & 77 & 0 & -14 & 0 & 21 & 154 & 84 & 2618 & 1050 & 1239 & 1925 \\ 385 & 440 & 55 & 385 & 55 & 0 & -55 & 0 & 165 & 110 & 3850 & 1595 & 1925 & 3080 \\ 56 & 77 & 14 & 154 & 35 & 21 & 0 & -7 & 0 & 7 & 385 & 175 & 224 & 385 \\ 77 & 132 & 33 & 539 & 165 & 154 & 165 & 0 & -11 & 0 & 308 & 165 & 231 & 440 \\ 14 & 33 & 11 & 231 & 80 & 84 & 110 & 7 & 0 & -2 & 0 & 10 & 21 & 55 \\ 154 & 539 & 231 & 6083 & 2310 & 2618 & 3850 & 385 & 308 & 0 & -154 & 0 & 77 & 385 \\ 35 & 165 & 80 & 2310 & 905 & 1050 & 1595 & 175 & 165 & 10 & 0 & -5 & 0 & 55 \\ 21 & 154 & 84 & 2618 & 1050 & 1239 & 1925 & 224 & 231 & 21 & 77 & 0 & -14 & 0 \\ 0 & 165 & 110 & 3850 & 1595 & 1925 & 3080 & 385 & 440 & 55 & 385 & 55 & 0 & -55 \end{pmatrix}.$$

$N=105\quad d=390=2\cdot 3\cdot 5\cdot 13,\ \eta=6, h=1\colon\ \langle 10\rangle\oplus\langle -26\rangle\oplus\langle -6\rangle(1/2,1/2,0)$

$$\begin{pmatrix} 45 & \frac{65}{2} & \frac{9}{2} & \frac{11}{2} & \frac{39}{2} & \frac{5}{2} & 78 & 8 & 0 & 0 & 3 & 32 & 273 & 60 & 2652 & \frac{659}{2} \\ -9 & \frac{-15}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-21}{2} & \frac{-3}{2} & -48 & -5 & 0 & 1 & 0 & -5 & -48 & -11 & -498 & \frac{-125}{2} \\ -55 & -39 & -5 & -5 & -13 & -1 & -13 & 0 & 1 & 0 & -4 & -40 & -338 & -74 & -3263 & -405 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -6 & 0 & 24 & 240 & 2028 & 444 & 19578 & 2430 \\ 0 & -26 & 0 & 130 & 1248 & 286 & 12948 & 1625 \\ 24 & 0 & -6 & 0 & 78 & 24 & 1248 & 165 \\ 240 & 130 & 0 & -10 & 0 & 10 & 780 & 115 \\ 2028 & 1248 & 78 & 0 & -78 & 0 & 1092 & 195 \\ 444 & 286 & 24 & 10 & 0 & -2 & 0 & 5 \\ 19578 & 12948 & 1248 & 780 & 1092 & 0 & -78 & 0 \\ 2430 & 1625 & 165 & 115 & 195 & 5 & 0 & -10 \\ 330 & 234 & 30 & 30 & 78 & 6 & 78 & 0 \\ 234 & 195 & 39 & 65 & 273 & 39 & 1248 & 130 \\ 30 & 39 & 15 & 45 & 273 & 51 & 2028 & 240 \\ 30 & 65 & 45 & 235 & 1755 & 365 & 15600 & 1910 \\ 78 & 273 & 273 & 1755 & 13767 & 2925 & 126672 & 15600 \\ 6 & 39 & 51 & 365 & 2925 & 627 & 27300 & 3370 \\ 78 & 1248 & 2028 & 15600 & 126672 & 27300 & 1192542 & 147420 \\ 0 & 130 & 240 & 1910 & 15600 & 3370 & 147420 & 18235 \end{pmatrix}.$$

$N=106\quad d=399=3\cdot 7\cdot 19,\ \eta=4,\ h=0\colon\ \langle 14\rangle\oplus\langle -19\rangle\oplus\langle -6\rangle(1/2,0,1/2)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{3}{2} & 0 & \frac{-7}{2} \\ \frac{3}{2} & -1 & \frac{-3}{2} \\ \frac{5}{2} & -2 & \frac{-3}{2} \\ \frac{57}{2} & -24 & \frac{-19}{2} \\ \frac{7}{2} & -3 & \frac{-1}{2} \\ 57 & -49 & 0 \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 21 & 9 & 9 & 57 & 3 & 0 \\ 0 & -19 & 0 & 19 & 38 & 456 & 57 & 931 \\ 21 & 0 & -42 & 0 & 21 & 399 & 63 & 1197 \\ 9 & 19 & 0 & -1 & 1 & 57 & 12 & 266 \\ 9 & 38 & 21 & 1 & -2 & 0 & 4 & 133 \\ 57 & 456 & 399 & 57 & 0 & -114 & 0 & 399 \\ 3 & 57 & 63 & 12 & 4 & 0 & -1 & 0 \\ 0 & 931 & 1197 & 266 & 133 & 399 & 0 & -133 \end{pmatrix}.$$

$N=107\quad d=429=3\cdot 11\cdot 13,\ \eta=3,\ h=1\colon\ \langle 1\rangle\oplus\langle -39\rangle\oplus\langle -11\rangle$

$$\begin{pmatrix} 0 & 0 & 3 & 1287 & 41 & 1573 & 126 & 132 & 156 & 9 & 429 & 7 & 143 & 6 \\ 0 & 1 & 0 & -88 & -3 & -121 & -10 & -11 & -14 & -1 & -55 & -1 & -22 & -1 \\ 1 & 0 & -1 & -351 & -11 & -416 & -33 & -34 & -39 & -2 & -78 & -1 & -13 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -11 & 0 & 11 & 3861 & 121 & 4576 & 363 & 374 & 429 & 22 & 858 & 11 & 143 & 0 \\ 0 & -39 & 0 & 3432 & 117 & 4719 & 390 & 429 & 546 & 39 & 2145 & 39 & 858 & 39 \\ 11 & 0 & -2 & 0 & 2 & 143 & 15 & 22 & 39 & 5 & 429 & 10 & 286 & 18 \\ 3861 & 3432 & 0 & -858 & 0 & 3003 & 429 & 858 & 2145 & 429 & 62205 & 1716 & 58344 & 4290 \\ 121 & 117 & 2 & 0 & -1 & 0 & 3 & 11 & 39 & 10 & 1716 & 49 & 1716 & 129 \\ 4576 & 4719 & 143 & 3003 & 0 & -286 & 0 & 143 & 858 & 286 & 58344 & 1716 & 61633 & 4719 \\ 363 & 390 & 15 & 429 & 3 & 0 & -3 & 0 & 39 & 18 & 4290 & 129 & 4719 & 366 \\ 374 & 429 & 22 & 858 & 11 & 143 & 0 & -11 & 0 & 11 & 3861 & 121 & 4576 & 363 \\ 429 & 546 & 39 & 2145 & 39 & 858 & 39 & 0 & -39 & 0 & 3432 & 117 & 4719 & 390 \\ 22 & 39 & 5 & 429 & 10 & 286 & 18 & 11 & 0 & -2 & 0 & 2 & 143 & 15 \\ 858 & 2145 & 429 & 62205 & 1716 & 58344 & 4290 & 3861 & 3432 & 0 & -858 & 0 & 3003 & 429 \\ 11 & 39 & 10 & 1716 & 49 & 1716 & 129 & 121 & 117 & 2 & 0 & -1 & 0 & 3 \\ 143 & 858 & 286 & 58344 & 1716 & 61633 & 4719 & 4576 & 4719 & 143 & 3003 & 0 & -286 & 0 \\ 0 & 39 & 18 & 4290 & 129 & 4719 & 366 & 363 & 390 & 15 & 429 & 3 & 0 & -3 \end{pmatrix}.$$

$N=108\quad d=435=3\cdot5\cdot29,\ \eta=0,\ h=0$: $\langle 3\rangle\oplus\langle -29\rangle\oplus\langle -5\rangle$

$$\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 145 & -30 & -87\\ 4 & -1 & -2\\ 145 & -40 & -58\\ 10 & -3 & -3\\ 3 & -1 & 0 \end{pmatrix}\qquad \begin{pmatrix} -5 & 0 & 5 & 435 & 10 & 290 & 15 & 0\\ 0 & -29 & 0 & 870 & 29 & 1160 & 87 & 29\\ 5 & 0 & -2 & 0 & 2 & 145 & 15 & 9\\ 435 & 870 & 0 & -870 & 0 & 3045 & 435 & 435\\ 10 & 29 & 2 & 0 & -1 & 0 & 3 & 7\\ 290 & 1160 & 145 & 3045 & 0 & -145 & 0 & 145\\ 15 & 87 & 15 & 435 & 3 & 0 & -6 & 3\\ 0 & 29 & 9 & 435 & 7 & 145 & 3 & -2 \end{pmatrix}.$$

$N=109\quad d=435=3\cdot5\cdot29,\ \eta=6,\ h=1$: $\langle 435\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$

$$\begin{pmatrix} 8 & 24 & 11 & 3 & 1 & 2 & 84 & 8 & 125 & 3 & 5 & 1 & 0 & 0 & 1 & 119 & 13 & 223\\ -25 & -87 & -45 & -14 & -6 & -15 & -725 & -71 & -1131 & -30 & -58 & -15 & -1 & 1 & 0 & -290 & -34 & -609\\ -165 & -493 & -225 & -61 & -20 & -39 & -1595 & -151 & -2349 & -55 & -87 & -15 & 1 & 0 & -21 & -2465 & -269 & -4611 \end{pmatrix}$$

$$\begin{pmatrix} A & B\\ B & A \end{pmatrix}\ \text{where}\ \begin{pmatrix} A\\ B \end{pmatrix}= \begin{pmatrix} -10 & 0 & 30 & 25 & 30 & 150 & 11020 & 1150 & 19140\\ 0 & -58 & 0 & 29 & 58 & 348 & 27550 & 2900 & 48546\\ 30 & 0 & -15 & 0 & 15 & 120 & 10440 & 1110 & 18705\\ 25 & 29 & 0 & -2 & 1 & 21 & 2175 & 235 & 4002\\ 30 & 58 & 15 & 1 & -1 & 0 & 290 & 34 & 609\\ 150 & 348 & 120 & 21 & 0 & -6 & 0 & 6 & 174\\ 11020 & 27550 & 10440 & 2175 & 290 & 0 & -290 & 0 & 870\\ 1150 & 2900 & 1110 & 235 & 34 & 6 & 0 & -2 & 0\\ 19140 & 48546 & 18705 & 4002 & 609 & 174 & 870 & 0 & -87\\ 615 & 1595 & 630 & 140 & 25 & 15 & 145 & 5 & 0\\ 1595 & 4263 & 1740 & 406 & 87 & 87 & 1885 & 145 & 1914\\ 630 & 1740 & 735 & 180 & 45 & 60 & 1740 & 150 & 2175\\ 140 & 406 & 180 & 47 & 14 & 24 & 870 & 80 & 1218\\ 25 & 87 & 45 & 14 & 6 & 15 & 725 & 71 & 1131\\ 15 & 87 & 60 & 24 & 15 & 51 & 3045 & 309 & 5046\\ 145 & 1885 & 1740 & 870 & 725 & 3045 & 206335 & 21315 & 352350\\ 5 & 145 & 150 & 80 & 71 & 309 & 21315 & 2207 & 36540\\ 0 & 1914 & 2175 & 1218 & 1131 & 5046 & 352350 & 36540 & 605607 \end{pmatrix}.$$

$N=110\quad d=455=5\cdot7\cdot13,\ \eta=5,\ h=1$: $\langle 7\rangle\oplus\langle -26\rangle\oplus\langle -10\rangle(0,1/2,1/2)$

$$\begin{pmatrix} 325 & 325 & 16 & 65 & 2 & 1 & 13 & 0 & 0 & 25 & 9 & 260 & 48 & 199 & 9087 & 1025\\ \frac{-67}{2} & -35 & -2 & -10 & \frac{-1}{2} & \frac{-1}{2} & -7 & 0 & 1 & 0 & \frac{-1}{2} & \frac{-45}{2} & \frac{-9}{2} & \frac{-39}{2} & -903 & \frac{-205}{2}\\ \frac{-533}{2} & -266 & -13 & -52 & \frac{-3}{2} & \frac{-1}{2} & 0 & 1 & 0 & -21 & \frac{-15}{2} & \frac{-429}{2} & \frac{-79}{2} & \frac{-327}{2} & -7462 & \frac{-1683}{2} \end{pmatrix}$$

$$\begin{pmatrix} A & B\\ B & A \end{pmatrix}\ \text{where}\ \begin{pmatrix} A\\ B \end{pmatrix}= \begin{pmatrix} -26 & 0 & 13 & 585 & 117 & 507 & 23478 & 2665\\ 0 & -35 & 0 & 455 & 105 & 490 & 23205 & 2660\\ 13 & 0 & -2 & 0 & 3 & 21 & 1092 & 130\\ 585 & 455 & 0 & -65 & 0 & 65 & 4095 & 520\\ 117 & 105 & 3 & 0 & -1 & 0 & 91 & 15\\ 507 & 490 & 21 & 65 & 0 & -2 & 0 & 5\\ 23478 & 23205 & 1092 & 4095 & 91 & 0 & -91 & 0\\ 2665 & 2660 & 130 & 520 & 15 & 5 & 0 & -10\\ 871 & 910 & 52 & 260 & 13 & 13 & 182 & 0\\ 910 & 1015 & 70 & 455 & 35 & 70 & 2275 & 210\\ 52 & 70 & 7 & 65 & 7 & 19 & 728 & 75\\ 260 & 455 & 65 & 910 & 130 & 455 & 19565 & 2145\\ 13 & 35 & 7 & 130 & 21 & 80 & 3549 & 395\\ 13 & 70 & 19 & 455 & 80 & 322 & 14560 & 1635\\ 182 & 2275 & 728 & 19565 & 3549 & 14560 & 662571 & 74620\\ 0 & 210 & 75 & 2145 & 395 & 1635 & 74620 & 8415 \end{pmatrix}.$$

$N = 111 \quad d = 465 = 3 \cdot 5 \cdot 31,\ \eta = 5,\ h = 1$: $\langle 5 \rangle \oplus \langle -31 \rangle \oplus \langle -3 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -5 \\ 31 & -8 & -31 \\ 17 & -5 & -15 \\ 16 & -5 & -13 \\ 403 & -130 & -310 \\ 18 & -6 & -13 \\ 93 & -32 & -62 \\ 42 & -15 & -25 \\ 62 & -23 & -31 \\ 13 & -5 & -5 \\ 5 & -2 & -1 \\ 62 & -25 & 0 \end{pmatrix} \begin{pmatrix} -3 & 0 & 15 & 93 & 45 & 39 & 930 & 39 & 186 & 75 & 93 & 15 & 3 & 0 \\ 0 & -31 & 0 & 248 & 155 & 155 & 4030 & 186 & 992 & 465 & 713 & 155 & 62 & 775 \\ 15 & 0 & -30 & 0 & 30 & 45 & 1395 & 75 & 465 & 255 & 465 & 120 & 60 & 930 \\ 93 & 248 & 0 & -62 & 0 & 31 & 1395 & 93 & 713 & 465 & 1023 & 310 & 186 & 3410 \\ 45 & 155 & 30 & 0 & -5 & 0 & 155 & 15 & 155 & 120 & 310 & 105 & 70 & 1395 \\ 39 & 155 & 45 & 31 & 0 & -2 & 0 & 3 & 62 & 60 & 186 & 70 & 51 & 1085 \\ 930 & 4030 & 1395 & 1395 & 155 & 0 & -155 & 0 & 775 & 930 & 3410 & 1395 & 1085 & 24180 \\ 39 & 186 & 75 & 93 & 15 & 3 & 0 & -3 & 0 & 15 & 93 & 45 & 39 & 930 \\ 186 & 992 & 465 & 713 & 155 & 62 & 775 & 0 & -31 & 0 & 248 & 155 & 155 & 4030 \\ 75 & 465 & 255 & 465 & 120 & 60 & 930 & 15 & 0 & -30 & 0 & 30 & 45 & 1395 \\ 93 & 713 & 465 & 1023 & 310 & 186 & 3410 & 93 & 248 & 0 & -62 & 0 & 31 & 1395 \\ 15 & 155 & 120 & 310 & 105 & 70 & 1395 & 45 & 155 & 30 & 0 & -5 & 0 & 155 \\ 3 & 62 & 60 & 186 & 70 & 51 & 1085 & 39 & 155 & 45 & 31 & 0 & -2 & 0 \\ 0 & 775 & 930 & 3410 & 1395 & 1085 & 24180 & 930 & 4030 & 1395 & 1395 & 155 & 0 & -155 \end{pmatrix}.$$

$N = 112 \quad d = 483 = 3 \cdot 7 \cdot 23,\ \eta = 7, h = 1$: $\langle 6 \rangle \oplus \langle -46 \rangle \oplus \langle -7 \rangle (1/2, 1/2, 0)$

$$\begin{pmatrix} 7 & 46 & \frac{21}{2} & \frac{23}{2} & 0 & 0 & 1 & \frac{5}{2} & 391 & \frac{511}{2} & \frac{2599}{2} & 399 & \frac{2185}{2} & \frac{283}{2} \\ -2 & -14 & \frac{-7}{2} & \frac{-9}{2} & 0 & 1 & 0 & \frac{-1}{2} & -101 & \frac{-133}{2} & \frac{-681}{2} & -105 & \frac{-577}{2} & \frac{-75}{2} \\ -4 & -23 & -4 & 0 & 1 & 0 & -1 & -2 & -253 & -164 & -828 & -253 & -690 & -89 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 7 & 69 & 28 & 92 & 14 & 3 & 46 & 21 & 69 & 14 & 23 & 1 \\ 0 & -23 & 0 & 276 & 161 & 644 & 115 & 46 & 2139 & 1288 & 6072 & 1771 & 4646 & 575 \\ 7 & 0 & -14 & 0 & 28 & 161 & 35 & 21 & 1288 & 798 & 3864 & 1148 & 3059 & 385 \\ 69 & 276 & 0 & -138 & 0 & 207 & 69 & 69 & 6072 & 3864 & 19182 & 5796 & 15663 & 2001 \\ 28 & 161 & 28 & 0 & -7 & 0 & 7 & 14 & 1771 & 1148 & 5796 & 1771 & 4830 & 623 \\ 92 & 644 & 161 & 207 & 0 & -46 & 0 & 23 & 4646 & 3059 & 15663 & 4830 & 13271 & 1725 \\ 14 & 115 & 35 & 69 & 7 & 0 & -1 & 1 & 575 & 385 & 2001 & 623 & 1725 & 226 \\ 3 & 46 & 21 & 69 & 14 & 23 & 1 & -2 & 0 & 7 & 69 & 28 & 92 & 14 \\ 46 & 2139 & 1288 & 6072 & 1771 & 4646 & 575 & 0 & -23 & 0 & 276 & 161 & 644 & 115 \\ 21 & 1288 & 798 & 3864 & 1148 & 3059 & 385 & 7 & 0 & -14 & 0 & 28 & 161 & 35 \\ 69 & 6072 & 3864 & 19182 & 5796 & 15663 & 2001 & 69 & 276 & 0 & -138 & 0 & 207 & 69 \\ 14 & 1771 & 1148 & 5796 & 1771 & 4830 & 623 & 28 & 161 & 28 & 0 & -7 & 0 & 7 \\ 23 & 4646 & 3059 & 15663 & 4830 & 13271 & 1725 & 92 & 644 & 161 & 207 & 0 & -46 & 0 \\ 1 & 575 & 385 & 2001 & 623 & 1725 & 226 & 14 & 115 & 35 & 69 & 7 & 0 & -1 \end{pmatrix}.$$

$N = 113 \quad d = 570 = 2 \cdot 3 \cdot 5 \cdot 19,\ \eta = 6,\ h = 1$: $\langle 38 \rangle \oplus \langle -10 \rangle \oplus \langle -6 \rangle (0, 1/2, 1/2)$

$$\begin{pmatrix} 0 & 0 & 15 & 1 & 6 & 5 & 42 & 7 & 9 & 15 & 213 & 7 & 15 & 5 & 15 & 1 \\ 0 & 1 & 0 & \frac{-1}{2} & -6 & -6 & -57 & -10 & \frac{-27}{2} & \frac{-47}{2} & -342 & \frac{-23}{2} & \frac{-51}{2} & -9 & \frac{-57}{2} & -2 \\ 1 & 0 & -38 & \frac{-5}{2} & -13 & -10 & -76 & -12 & \frac{-29}{2} & \frac{-45}{2} & -304 & \frac{-19}{2} & \frac{-37}{2} & -5 & \frac{-19}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 0 & 228 & 15 & 78 & 60 & 456 & 72 & 87 & 135 & 1824 & 57 & 111 & 30 & 57 & 0 \\ 0 & -10 & 0 & 5 & 60 & 60 & 570 & 100 & 135 & 235 & 3420 & 115 & 255 & 90 & 285 & 20 \\ 228 & 0 & -114 & 0 & 456 & 570 & 6612 & 1254 & 1824 & 3420 & 52098 & 1824 & 4332 & 1710 & 6384 & 570 \\ 15 & 5 & 0 & -2 & 3 & 10 & 171 & 36 & 57 & 115 & 1824 & 66 & 165 & 70 & 285 & 28 \\ 78 & 60 & 456 & 3 & -6 & 0 & 228 & 60 & 111 & 255 & 4332 & 165 & 447 & 210 & 969 & 108 \\ 60 & 60 & 570 & 10 & 0 & -10 & 0 & 10 & 30 & 90 & 1710 & 70 & 210 & 110 & 570 & 70 \\ 456 & 570 & 6612 & 171 & 228 & 0 & -114 & 0 & 57 & 285 & 6384 & 285 & 969 & 570 & 3363 & 456 \\ 72 & 100 & 1254 & 36 & 60 & 10 & 0 & -2 & 0 & 20 & 570 & 28 & 108 & 70 & 456 & 66 \\ 87 & 135 & 1824 & 57 & 111 & 30 & 57 & 0 & -6 & 0 & 228 & 15 & 78 & 60 & 456 & 72 \\ 135 & 235 & 3420 & 115 & 255 & 90 & 285 & 20 & 0 & -10 & 0 & 5 & 60 & 60 & 570 & 100 \\ 1824 & 3420 & 52098 & 1824 & 4332 & 1710 & 6384 & 570 & 228 & 0 & -114 & 0 & 456 & 570 & 6612 & 1254 \\ 57 & 115 & 1824 & 66 & 165 & 70 & 285 & 28 & 15 & 5 & 0 & -2 & 3 & 10 & 171 & 36 \\ 111 & 255 & 4332 & 165 & 447 & 210 & 969 & 108 & 78 & 60 & 456 & 3 & -6 & 0 & 228 & 60 \\ 30 & 90 & 1710 & 70 & 210 & 110 & 570 & 70 & 60 & 60 & 570 & 10 & 0 & -10 & 0 & 10 \\ 57 & 285 & 6384 & 285 & 969 & 570 & 3363 & 456 & 456 & 570 & 6612 & 171 & 228 & 0 & -114 & 0 \\ 0 & 20 & 570 & 28 & 108 & 70 & 456 & 66 & 72 & 100 & 1254 & 36 & 60 & 10 & 0 & -2 \end{pmatrix}.$$

$N=114\quad d=615=3\cdot 5\cdot 41,\ \eta=0,\ h=1\colon\ \langle 3\rangle\oplus\langle -41\rangle\oplus\langle -5\rangle$

$$\begin{pmatrix}
0&0&1&16&82&9&410&25&20&164&27&80&246&15&410&11\\
0&1&0&-3&-17&-2&-95&-6&-5&-42&-7&-21&-65&-4&-110&-3\\
1&0&-1&-9&-41&-4&-164&-9&-6&-41&-6&-15&-41&-2&-41&0
\end{pmatrix}$$

$$\begin{pmatrix}
-5&0&5&45&205&20&820&45&30&205&30&75&205&10&205&0\\
0&-41&0&123&697&82&3895&246&205&1722&287&861&2665&164&4510&123\\
5&0&-2&3&41&7&410&30&30&287&51&165&533&35&1025&33\\
45&123&3&-6&0&6&615&57&75&861&165&582&1968&138&4305&159\\
205&697&41&0&-82&0&1025&123&205&2665&533&1968&6806&492&15785&615\\
20&82&7&6&0&-1&0&3&10&164&35&138&492&37&1230&51\\
820&3895&410&615&1025&0&-205&0&205&4510&1025&4305&15785&1230&42230&1845\\
45&246&30&57&123&3&0&-6&0&123&33&159&615&51&1845&87\\
30&205&30&75&205&10&205&0&-5&0&5&45&205&20&820&45\\
205&1722&287&861&2665&164&4510&123&0&-41&0&123&697&82&3895&246\\
30&287&51&165&533&35&1025&33&5&0&-2&3&41&7&410&30\\
75&861&165&582&1968&138&4305&159&45&123&3&-6&0&6&615&57\\
205&2665&533&1968&6806&492&15785&615&205&697&41&0&-82&0&1025&123\\
10&164&35&138&492&37&1230&51&20&82&7&6&0&-1&0&3\\
205&4510&1025&4305&15785&1230&42230&1845&820&3895&410&615&1025&0&-205&0\\
0&123&33&159&615&51&1845&87&45&246&30&57&123&3&0&-6
\end{pmatrix}.$$

$N=115\quad d=645=3\cdot 5\cdot 43,\ \eta=3,\ h=1\colon\ \langle 10\rangle\oplus\langle -86\rangle\oplus\langle -3\rangle(1/2,1/2,0)$

$$\begin{pmatrix}
\frac{165}{2}&\frac{645}{2}&\frac{43}{2}&516&22&4&\frac{387}{2}&\frac{3}{2}&\frac{43}{2}&0&0&1&129&\frac{31}{2}&11&1419&36&1591\\
\frac{-33}{2}&\frac{-131}{2}&\frac{-9}{2}&-111&-5&-1&\frac{-105}{2}&\frac{-1}{2}&\frac{-15}{2}&0&1&0&-18&\frac{-5}{2}&-2&-270&-7&-315\\
-122&-473&-31&-731&-30&-5&-215&-1&0&1&0&-2&-215&-25&-17&-2150&-54&-2365
\end{pmatrix}$$

$$\begin{pmatrix}A&B\\B&A\end{pmatrix}\ \text{where}\ \begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}
-3&0&6&645&75&51&6450&162&7095\\
0&-86&0&1548&215&172&23220&602&27090\\
6&0&-2&0&5&8&1290&36&1720\\
645&1548&0&-129&0&129&25800&774&39345\\
75&215&5&0&-10&0&645&25&1505\\
51&172&8&129&0&-1&0&2&215\\
6450&23220&1290&25800&645&0&-1290&0&7740\\
162&602&36&774&25&2&0&-2&0\\
7095&27090&1720&39345&1505&215&7740&0&-215\\
366&1419&93&2193&90&15&645&3&0\\
1419&5633&387&9546&430&86&4515&43&645\\
93&387&29&774&40&10&645&9&215\\
2193&9546&774&22317&1290&387&29670&516&16125\\
90&430&40&1290&85&30&2580&50&1720\\
15&86&10&387&30&13&1290&28&1075\\
645&4515&645&29670&2580&1290&139965&3225&130935\\
3&43&9&516&50&28&3225&77&3225\\
0&645&215&16125&1720&1075&130935&3225&138890
\end{pmatrix}.$$

$N=116\quad d=651=3\cdot 7\cdot 31,\ \eta=4,\ h=1\colon\ \langle 14\rangle\oplus\langle -31\rangle\oplus\langle -6\rangle(1/2,0,1/2)$

$$\begin{pmatrix}
129&992&13&\frac{3}{2}&31&0&0&\frac{3}{2}&\frac{31}{2}&\frac{21}{2}&\frac{899}{2}&\frac{67}{2}&\frac{183}{2}&\frac{25885}{2}&\frac{2511}{2}&5766&2091&1426\\
-78&-602&-8&-1&-21&0&1&0&-8&-6&-266&-20&-55&-7791&-756&-3473&-1260&-860\\
-86&-651&-8&\frac{-1}{2}&0&1&0&\frac{-7}{2}&\frac{-31}{2}&\frac{-17}{2}&\frac{-651}{2}&\frac{-47}{2}&\frac{-125}{2}&\frac{-17577}{2}&\frac{-1703}{2}&-3906&-1414&-961
\end{pmatrix}$$

$$\begin{pmatrix}A&B\\B&A\end{pmatrix}\ \text{where}\ \begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}
-6&0&6&33&5208&516&2418&903&651\\
0&-434&0&217&38626&3906&18662&7161&5425\\
6&0&-2&1&434&48&248&105&93\\
33&217&1&-1&0&3&31&21&31\\
5208&38626&434&0&-217&0&651&651&1519\\
516&3906&48&3&0&-6&0&21&93\\
2418&18662&248&31&651&0&-31&0&248\\
903&7161&105&21&651&21&0&-42&0\\
651&5425&93&31&1519&93&248&0&-62\\
69&651&15&9&651&51&186&42&0\\
651&7161&217&217&21917&1953&8246&2604&1302\\
15&217&9&13&1519&141&620&210&124\\
9&217&13&29&3906&375&1705&609&403\\
651&21917&1519&3906&545104&52731&241521&87234&59024\\
51&1953&141&375&52731&5109&23436&8484&5766\\
186&8246&620&1705&241521&23436&107663&39060&26660\\
42&2604&210&609&87234&8484&39060&14217&9765\\
0&1302&124&403&59024&5766&26660&9765&6789
\end{pmatrix}.$$

$N = 117 \quad d = 795 = 3 \cdot 5 \cdot 53,\ \eta = 6,\ h = 1\colon\ \langle 3\rangle \oplus \langle -106\rangle \oplus \langle -10\rangle(0, 1/2, 1/2)$

$$\begin{pmatrix} 143 & 101 & 1060 & 4 & 265 & 3 & 53 & 0 & 0 & 5 & 7 & 49 & 1590 & 46 & 23585 & 747 & 31747 & 2200 & 5300 & 895 \\ \frac{-29}{2} & \frac{-21}{2} & \frac{-225}{2} & \frac{-1}{2} & \frac{-75}{2} & \frac{-1}{2} & -9 & 0 & 1 & 0 & \frac{-1}{2} & \frac{-9}{2} & \frac{-305}{2} & \frac{-9}{2} & \frac{-4695}{2} & \frac{-149}{2} & -3171 & -220 & -531 & -90 \\ \frac{-125}{2} & \frac{-87}{2} & \frac{-901}{2} & \frac{-3}{2} & \frac{-159}{2} & \frac{-1}{2} & 0 & 1 & 0 & -3 & \frac{-7}{2} & \frac{-45}{2} & \frac{-1431}{2} & \frac{-41}{2} & \frac{-20829}{2} & \frac{-659}{2} & -13992 & -969 & -2332 & -393 \end{pmatrix}$$

$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 & 3 & 265 & 10 & 6360 & 206 & 8904 & 625 & 1537 & 270 \\ 3 & -6 & 0 & 3 & 3975 & 135 & 6042 & 435 & 1113 & 210 \\ 265 & 0 & -265 & 0 & 37365 & 1325 & 61215 & 4505 & 11925 & 2385 \\ 10 & 3 & 0 & -1 & 0 & 2 & 159 & 15 & 53 & 15 \\ 6360 & 3975 & 37365 & 0 & -1590 & 0 & 6360 & 795 & 3975 & 1590 \\ 206 & 135 & 1325 & 2 & 0 & -2 & 0 & 5 & 53 & 30 \\ 8904 & 6042 & 61215 & 159 & 6360 & 0 & -159 & 0 & 954 & 795 \\ 625 & 435 & 4505 & 15 & 795 & 5 & 0 & -10 & 0 & 30 \\ 1537 & 1113 & 11925 & 53 & 3975 & 53 & 954 & 0 & -106 & 0 \\ 270 & 210 & 2385 & 15 & 1590 & 30 & 795 & 30 & 0 & -15 \\ 47 & 42 & 530 & 5 & 795 & 19 & 636 & 35 & 53 & 0 \\ 42 & 51 & 795 & 12 & 3180 & 90 & 3498 & 225 & 477 & 60 \\ 530 & 795 & 14310 & 265 & 89040 & 2650 & 107325 & 7155 & 16165 & 2385 \\ 5 & 12 & 265 & 6 & 2385 & 73 & 3021 & 205 & 477 & 75 \\ 795 & 3180 & 89040 & 2385 & 1139235 & 35775 & 1510500 & 104145 & 248835 & 41340 \\ 19 & 90 & 2650 & 73 & 35775 & 1127 & 47700 & 3295 & 7897 & 1320 \\ 636 & 3498 & 107325 & 3021 & 1510500 & 47700 & 2022639 & 139920 & 336126 & 56445 \\ 35 & 225 & 7155 & 205 & 104145 & 3295 & 139920 & 9690 & 23320 & 3930 \\ 53 & 477 & 16165 & 477 & 248835 & 7897 & 336126 & 23320 & 56286 & 9540 \\ 0 & 60 & 2385 & 75 & 41340 & 1320 & 56445 & 3930 & 9540 & 1635 \end{pmatrix}.$$

$N = 118 \quad d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 2, h = 0\colon\ \langle 14\rangle \oplus \langle -22\rangle \oplus \langle -15\rangle(1/2, 1/2, 0)$

$$\begin{pmatrix} 0 & 0 & 1 & \frac{165}{2} & \frac{9}{2} & \frac{55}{2} & \frac{39}{2} & \frac{25}{2} & 132 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{-35}{2} & \frac{-3}{2} & \frac{-25}{2} & \frac{-21}{2} & \frac{-15}{2} & -84 & \frac{-1}{2} \\ 1 & 0 & -1 & -77 & -4 & -22 & -14 & -8 & -77 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -15 & 0 & 15 & 1155 & 60 & 330 & 210 & 120 & 1155 & 0 \\ 0 & -22 & 0 & 385 & 33 & 275 & 231 & 165 & 1848 & 11 \\ 15 & 0 & -1 & 0 & 3 & 55 & 63 & 55 & 693 & 7 \\ 1155 & 385 & 0 & -385 & 0 & 1540 & 2310 & 2310 & 31185 & 385 \\ 60 & 33 & 3 & 0 & -6 & 0 & 42 & 60 & 924 & 15 \\ 330 & 275 & 55 & 1540 & 0 & -110 & 0 & 110 & 2310 & 55 \\ 210 & 231 & 63 & 2310 & 42 & 0 & -42 & 0 & 462 & 21 \\ 120 & 165 & 55 & 2310 & 60 & 110 & 0 & -10 & 0 & 5 \\ 1155 & 1848 & 693 & 31185 & 924 & 2310 & 462 & 0 & -231 & 0 \\ 0 & 11 & 7 & 385 & 15 & 55 & 21 & 5 & 0 & -2 \end{pmatrix}.$$

$N = 119$ $\;d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 8,\ h = 1$: $\langle 1155 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$

$$\begin{pmatrix} 0 & 0 & 1 & 4 & 2 & 17 & 43 & 5 & 61 & 1 & 1 & 6 & 7 & 2 & 11 & 23 & 2 & 16 \\ -1 & 1 & 0 & -33 & -20 & -189 & -495 & -60 & -770 & -14 & -16 & -105 & -132 & -40 & -231 & -495 & -45 & -385 \\ 1 & 0 & -35 & -132 & -65 & -546 & -1375 & -159 & -1925 & -31 & -30 & -175 & -198 & -55 & -294 & -605 & -51 & -385 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2 & 1 & 35 & 99 & 45 & 357 & 880 & 99 & 1155 \\ 1 & -1 & 0 & 33 & 20 & 189 & 495 & 60 & 770 \\ 35 & 0 & -70 & 0 & 35 & 525 & 1540 & 210 & 3080 \\ 99 & 33 & 0 & -33 & 0 & 231 & 825 & 132 & 2310 \\ 45 & 20 & 35 & 0 & -5 & 0 & 55 & 15 & 385 \\ 357 & 189 & 525 & 231 & 0 & -42 & 0 & 21 & 1155 \\ 880 & 495 & 1540 & 825 & 55 & 0 & -55 & 0 & 1540 \\ 99 & 60 & 210 & 132 & 15 & 21 & 0 & -6 & 0 \\ 1155 & 770 & 3080 & 2310 & 385 & 1155 & 1540 & 0 & -770 \\ 17 & 14 & 70 & 66 & 15 & 63 & 110 & 6 & 0 \\ 14 & 16 & 105 & 132 & 40 & 231 & 495 & 45 & 385 \\ 70 & 105 & 805 & 1155 & 385 & 2415 & 5390 & 525 & 5005 \\ 66 & 132 & 1155 & 1848 & 660 & 4389 & 10065 & 1023 & 10395 \\ 15 & 40 & 385 & 660 & 245 & 1680 & 3905 & 405 & 4235 \\ 63 & 231 & 2415 & 4389 & 1680 & 11802 & 27720 & 2919 & 31185 \\ 110 & 495 & 5390 & 10065 & 3905 & 27720 & 65395 & 6930 & 74690 \\ 6 & 45 & 525 & 1023 & 405 & 2919 & 6930 & 741 & 8085 \\ 0 & 385 & 5005 & 10395 & 4235 & 31185 & 74690 & 8085 & 89705 \end{pmatrix}.$$

$N = 120$ $\;d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 14,\ h = 0$: $\langle 22 \rangle \oplus \langle -15 \rangle \oplus \langle -14 \rangle (1/2, 0, 1/2)$

$$\begin{pmatrix} 0 & 0 & \frac{7}{2} & \frac{5}{2} & 7 & \frac{15}{2} & \frac{7}{2} & \frac{525}{2} & \frac{5}{2} & 9 \\ 0 & 1 & 0 & -2 & -7 & -8 & -4 & -308 & -3 & -11 \\ 1 & 0 & \frac{-11}{2} & \frac{-5}{2} & -5 & \frac{-9}{2} & \frac{-3}{2} & \frac{-165}{2} & \frac{-1}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -14 & 0 & 77 & 35 & 70 & 63 & 21 & 1155 & 7 & 0 \\ 0 & -15 & 0 & 30 & 105 & 120 & 60 & 4620 & 45 & 165 \\ 77 & 0 & -154 & 0 & 154 & 231 & 154 & 13860 & 154 & 693 \\ 35 & 30 & 0 & -10 & 0 & 15 & 20 & 2310 & 30 & 165 \\ 70 & 105 & 154 & 0 & -7 & 0 & 14 & 2310 & 35 & 231 \\ 63 & 120 & 231 & 15 & 0 & -6 & 3 & 1155 & 21 & 165 \\ 21 & 60 & 154 & 20 & 14 & 3 & -2 & 0 & 2 & 33 \\ 1155 & 4620 & 13860 & 2310 & 2310 & 1155 & 0 & -2310 & 0 & 1155 \\ 7 & 45 & 154 & 30 & 35 & 21 & 2 & 0 & -1 & 0 \\ 0 & 165 & 693 & 165 & 231 & 165 & 33 & 1155 & 0 & -33 \end{pmatrix}.$$

$N = 121$ $\;d = 1365 = 3 \cdot 5 \cdot 7 \cdot 13,\ \eta = 5,\ h = 1$: $\langle 5 \rangle \oplus \langle -39 \rangle \oplus \langle -7 \rangle$

$$\begin{pmatrix} 90 & 49 & 33 & 3 & 13 & 0 & 0 & 1 & 182 & 9 & 28 & 132 & 63 & 1417 & 462 & 1287 & 142 & 4823 \\ -25 & -14 & -10 & -1 & -5 & 0 & 1 & 0 & -35 & -2 & -7 & -35 & -17 & -385 & -126 & -352 & -39 & -1330 \\ -48 & -25 & -15 & -1 & 0 & 1 & 0 & -1 & -130 & -6 & -17 & -75 & -35 & -780 & -253 & -702 & -77 & -2600 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ where } \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 & 0 & 60 & 39 & 975 & 336 & 975 & 114 & 4095 \\ 0 & -14 & 0 & 14 & 455 & 175 & 546 & 70 & 2730 \\ 60 & 0 & -30 & 0 & 195 & 105 & 390 & 60 & 2730 \\ 39 & 14 & 0 & -1 & 0 & 7 & 39 & 8 & 455 \\ 975 & 455 & 195 & 0 & -130 & 0 & 195 & 65 & 5005 \\ 336 & 175 & 105 & 7 & 0 & -7 & 0 & 7 & 910 \\ 975 & 546 & 390 & 39 & 195 & 0 & -39 & 0 & 1365 \\ 114 & 70 & 60 & 8 & 65 & 7 & 0 & -2 & 0 \\ 4095 & 2730 & 2730 & 455 & 5005 & 910 & 1365 & 0 & -455 \\ 84 & 63 & 75 & 15 & 195 & 42 & 78 & 3 & 0 \\ 63 & 63 & 105 & 28 & 455 & 119 & 273 & 21 & 455 \\ 75 & 105 & 255 & 90 & 1755 & 525 & 1365 & 135 & 4095 \\ 15 & 28 & 90 & 37 & 780 & 245 & 663 & 70 & 2275 \\ 195 & 455 & 1755 & 780 & 17030 & 5460 & 15015 & 1625 & 54145 \\ 42 & 119 & 525 & 245 & 5460 & 1771 & 4914 & 539 & 18200 \\ 78 & 273 & 1365 & 663 & 15015 & 4914 & 13728 & 1521 & 51870 \\ 3 & 21 & 135 & 70 & 1625 & 539 & 1521 & 171 & 5915 \\ 0 & 455 & 4095 & 2275 & 54145 & 18200 & 51870 & 5915 & 207480 \end{pmatrix}.$$

$N = 122 \quad d = 1365 = 3 \cdot 5 \cdot 7 \cdot 13,\ \eta = 15,\ h = 1\text{: } \langle 6\rangle \oplus \langle -65\rangle \oplus \langle -14\rangle(1/2, 0, 1/2)$

$$\left(\begin{matrix} \frac{223}{2} & \frac{5005}{2} & 37 & 140 & 13 & 0 & 0 & \frac{1}{2} & 91 & \frac{15}{2} & \frac{49}{2} & 31 & 1820 & 58 & \frac{1715}{2} & \frac{1209}{2} & \frac{1995}{2} & 8645 & \frac{1709}{2} & 25844 & 515 & 308 \\ -33 & -742 & -11 & -42 & -4 & 0 & 1 & 0 & -21 & -2 & -7 & -9 & -532 & -17 & -252 & -178 & -294 & -2549 & -252 & -7623 & -152 & -91 \\ \frac{-33}{2} & \frac{-715}{2} & -5 & -15 & 0 & 1 & 0 & \frac{-1}{2} & -39 & \frac{-5}{2} & \frac{-11}{2} & -6 & -325 & -10 & \frac{-285}{2} & \frac{-195}{2} & \frac{-317}{2} & -1365 & \frac{-269}{2} & -4056 & -80 & -47 \end{matrix}\right)$$

$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix}
-3 & 0 & 3 & 105 & 117 & 231 & 2145 & 219 & 6825 & 150 & 105 \\
0 & -910 & 0 & 1365 & 2275 & 5005 & 48230 & 5005 & 158340 & 3640 & 2730 \\
3 & 0 & -1 & 0 & 26 & 70 & 715 & 76 & 2457 & 60 & 49 \\
105 & 1365 & 0 & -210 & 0 & 210 & 2730 & 315 & 10920 & 315 & 315 \\
117 & 2275 & 26 & 0 & -26 & 0 & 260 & 39 & 1638 & 65 & 91 \\
231 & 5005 & 70 & 210 & 0 & -14 & 0 & 7 & 546 & 35 & 77 \\
2145 & 48230 & 715 & 2730 & 260 & 0 & -65 & 0 & 1365 & 130 & 455 \\
219 & 5005 & 76 & 315 & 39 & 7 & 0 & -2 & 0 & 5 & 35 \\
6825 & 158340 & 2457 & 10920 & 1638 & 546 & 1365 & 0 & -273 & 0 & 819 \\
150 & 3640 & 60 & 315 & 65 & 35 & 130 & 5 & 0 & -10 & 0 \\
105 & 2730 & 49 & 315 & 91 & 77 & 455 & 35 & 819 & 0 & -7 \\
48 & 1365 & 27 & 210 & 78 & 84 & 585 & 51 & 1365 & 15 & 0 \\
1365 & 42315 & 910 & 8190 & 3640 & 4550 & 34580 & 3185 & 90090 & 1365 & 455 \\
27 & 910 & 21 & 210 & 104 & 140 & 1105 & 104 & 3003 & 50 & 21 \\
210 & 8190 & 210 & 2415 & 1365 & 1995 & 16380 & 1575 & 46410 & 840 & 420 \\
78 & 3640 & 104 & 1365 & 871 & 1365 & 11570 & 1131 & 33852 & 650 & 364 \\
84 & 4550 & 140 & 1995 & 1365 & 2219 & 19110 & 1883 & 56784 & 1120 & 658 \\
585 & 34580 & 1105 & 16380 & 11570 & 19110 & 165685 & 16380 & 495495 & 9880 & 5915 \\
51 & 3185 & 104 & 1575 & 1131 & 1883 & 16380 & 1622 & 49140 & 985 & 595 \\
1365 & 90090 & 3003 & 46410 & 33852 & 56784 & 495495 & 49140 & 1490853 & 30030 & 18291 \\
15 & 1365 & 50 & 840 & 650 & 1120 & 9880 & 985 & 30030 & 615 & 385 \\
0 & 455 & 21 & 420 & 364 & 658 & 5915 & 595 & 18291 & 385 & 252
\end{pmatrix}.$$

Table 2.
The list of odd hyperbolic lattices of rank three
and with even square-free determinant (i.e. non-main)
which are reflective of elliptic or parabolic type.

$N'=1$ $d=2$, odd, $\eta=0$, $h=0$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -2\rangle$. It is equivariantly equivalent to $d=1$, $\eta=0$, $h=0$: $U\oplus\langle -1\rangle$.

$N'=2$ $d=6=2\cdot 3$, odd, $\eta=0$, $h=0$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -6\rangle$. It is equivariantly equivalent to $d=3$, $\eta=1$, $h=0$: $U\oplus\langle -3\rangle$.

$N'=3$ $d=6=2\cdot 3$, odd, $\eta=1$, $h=0$: $\langle 6\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$ $(=\widetilde{(3,0,h=0)})$

$$\begin{pmatrix}0&-1&1\\0&1&0\\1&0&-3\\1&-2&-2\end{pmatrix}\qquad\begin{pmatrix}-2&1&3&0\\1&-1&0&2\\3&0&-3&0\\0&2&0&-2\end{pmatrix}.$$

$N'=4$ $d=10=2\cdot 5$, odd, $\eta=0$, $h=1$: $\langle 2\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$ $(=\widetilde{(5,1,h=0)})$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&-2\\2&-1&-2\\10&-6&-5\\3&-2&0\end{pmatrix}\qquad\begin{pmatrix}-1&0&2&2&5&0\\0&-5&0&5&30&10\\2&0&-2&0&10&6\\2&5&0&-1&0&2\\5&30&10&0&-5&0\\0&10&6&2&0&-2\end{pmatrix}.$$

$N'=5$ $d=10=2\cdot 5$, odd, $\eta=1$, $h=0$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -10\rangle$. It is equivariantly equivalent to $d=5$, $\eta=0$, $h=0$: $U\oplus\langle -5\rangle$.

$N'=6$ $d=14=2\cdot 7$, odd, $\eta=0$, $h=1$: $\langle 14\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$ $(=\widetilde{(7,0,h=1)})$

$$\begin{pmatrix}0&-1&1\\0&1&0\\1&0&-4\\2&-3&-7\\4&-9&-12\\3&-8&-8\end{pmatrix}\qquad\begin{pmatrix}-2&1&4&4&3&0\\1&-1&0&3&9&8\\4&0&-2&0&8&10\\4&3&0&-2&1&4\\3&9&8&1&-1&0\\0&8&10&4&0&-2\end{pmatrix}.$$

$N'=7$ $d=14=2\cdot 7$, odd, $\eta=1$, $h=0$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -14\rangle$. It is equivariantly equivalent to $d=7$, $\eta=1$, $h=0$: $U\oplus\langle -7\rangle$.

$N'=8$ $d=22=2\cdot 11$, odd, $\eta=0$, $h=1$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -22\rangle$. It is equivariantly equivalent to $d=11$, $\eta=1$, $h=1$: $U\oplus\langle -11\rangle$.

$N'=9$ $d=22=2\cdot 11$, odd, $\eta=1$, $h=1$: $\langle 22\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$ $(=\widetilde{(11,0,h=0)})$

$$\begin{pmatrix}0&-1&1\\0&1&0\\7&0&-33\\3&-2&-14\\2&-3&-9\\2&-5&-8\\15&-44&-55\\3&-10&-10\end{pmatrix}\qquad\begin{pmatrix}-2&1&33&12&6&3&11&0\\1&-1&0&2&3&5&44&10\\33&0&-11&0&11&44&495&132\\12&2&0&-2&0&10&132&38\\6&3&11&0&-2&1&33&12\\3&5&44&10&1&-1&0&2\\11&44&495&132&33&0&-11&0\\0&10&132&38&12&2&0&-2\end{pmatrix}.$$

$N'=10$ $d=26=2\cdot 13$, odd, $\eta=1$, $h=1$: $\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -26\rangle$. It is equivariantly equivalent to $d = 13$, $\eta = 0$, $h = 1$: $U \oplus \langle -13 \rangle$.

$N' = 11$ $d = 30 = 2 \cdot 3 \cdot 5$, odd, $\eta = 0$, $h = 0$: $\langle 30 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$ $(=(15, \widetilde{3, h} = 0))$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -6 \\ 2 & -5 & -10 \\ 1 & -4 & -4 \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 & 6 & 5 & 0 \\ 1 & -1 & 0 & 5 & 4 \\ 6 & 0 & -6 & 0 & 6 \\ 5 & 5 & 0 & -5 & 0 \\ 0 & 4 & 6 & 0 & -2 \end{pmatrix}.$$

$N' = 12$ $d = 30 = 2 \cdot 3 \cdot 5$, odd, $\eta = 1$, $h = 0$: $\langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -30 \rangle$. It is equivariantly equivalent to $d = 15$, $\eta = 2$, $h = 0$: $U \oplus \langle -15 \rangle$.

$N' = 13$ $d = 30 = 2 \cdot 3 \cdot 5$, odd, $\eta = 2$, $h = 0$: $\langle 10 \rangle \oplus \langle -6 \rangle \oplus \langle -2 \rangle (1/2, 1/2, 0)$. It is equivariantly equivalent to $d = 15$, $\eta = 1$, $h = 0$: $\langle 5 \rangle \oplus \langle -3 \rangle \oplus \langle -1 \rangle$.

$N' = 14$ $d = 30 = 2 \cdot 3 \cdot 5$, odd, $\eta = 3$, $h = 0$: $\langle 6 \rangle \oplus \langle -10 \rangle \oplus \langle -2 \rangle (1/2, 1/2, 0)$ $(= (15, \widetilde{0, h} = 0))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 10 & -4 & -15 \\ \frac{5}{2} & \frac{-3}{2} & -3 \\ \frac{1}{2} & \frac{-1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 4 & 30 & 6 & 0 \\ 0 & -10 & 0 & 40 & 15 & 5 \\ 4 & 0 & -2 & 0 & 3 & 3 \\ 30 & 40 & 0 & -10 & 0 & 10 \\ 6 & 15 & 3 & 0 & -3 & 0 \\ 0 & 5 & 3 & 10 & 0 & -1 \end{pmatrix}.$$

$N' = 15$ $d = 34 = 2 \cdot 17$, odd, $\eta = 0$, $h = 0$: $\langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -34 \rangle$. It is equivariantly equivalent to $d = 17$, $\eta = 0$, $h = 0$: $U \oplus \langle -17 \rangle$.

$N' = 16$ $d = 42 = 2 \cdot 3 \cdot 7$, odd, $\eta = 0$, $h = 1$: $\langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -42 \rangle$. It is equivariantly equivalent to $d = 21$, $\eta = 1$, $h = 1$: $U \oplus \langle -21 \rangle$.

$N' = 17$ $d = 42 = 2 \cdot 3 \cdot 7$, odd, $\eta = 1$, $h = 1$: $\langle 7 \rangle \oplus \langle -3 \rangle \oplus \langle -2 \rangle$ $(=(21, \widetilde{0, h} = 0))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 12 & -7 & -21 \\ 2 & -2 & -3 \\ 3 & -4 & -3 \\ 2 & -3 & -1 \\ 9 & -14 & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 4 & 42 & 6 & 6 & 2 & 0 \\ 0 & -3 & 0 & 21 & 6 & 12 & 9 & 42 \\ 4 & 0 & -1 & 0 & 2 & 9 & 10 & 63 \\ 42 & 21 & 0 & -21 & 0 & 42 & 63 & 462 \\ 6 & 6 & 2 & 0 & -2 & 0 & 4 & 42 \\ 6 & 12 & 9 & 42 & 0 & -3 & 0 & 21 \\ 2 & 9 & 10 & 63 & 4 & 0 & -1 & 0 \\ 0 & 42 & 63 & 462 & 42 & 21 & 0 & -21 \end{pmatrix}.$$

$N' = 18$ $d = 42 = 2 \cdot 3 \cdot 7$, odd, $\eta = 2, h = 0$: $\langle 2 \rangle \oplus \langle -14 \rangle \oplus \langle -6 \rangle (0, 1/2, 1/2)$ $(=(21, \widetilde{3, h} = 0))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \\ 7 & \frac{-3}{2} & \frac{-7}{2} \\ 3 & -1 & -1 \\ 4 & \frac{-3}{2} & \frac{-1}{2} \\ 21 & -8 & 0 \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 12 & 21 & 6 & 3 & 0 \\ 0 & -14 & 0 & 21 & 14 & 21 & 112 \\ 12 & 0 & -6 & 0 & 6 & 18 & 126 \\ 21 & 21 & 0 & -7 & 0 & 14 & 126 \\ 6 & 14 & 6 & 0 & -2 & 0 & 14 \\ 3 & 21 & 18 & 14 & 0 & -1 & 0 \\ 0 & 112 & 126 & 126 & 14 & 0 & -14 \end{pmatrix}.$$

$N' = 19$ $d = 42 = 2 \cdot 3 \cdot 7$, odd, $\eta = 3$, $h = 0$: $\langle 6 \rangle \oplus \langle -14 \rangle \oplus \langle -2 \rangle (1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 21$, $\eta = 2$, $h = 0$: $\langle 3 \rangle \oplus \langle -7 \rangle \oplus \langle -1 \rangle$.

$N' = 20$ $d = 66 = 2 \cdot 3 \cdot 11$, odd, $\eta = 0$, $h = 1$: $\langle 2\rangle \oplus \langle -22\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$ $(=(33, \widetilde{3, h} = 1))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \\ 66 & -10 & -33 \\ \frac{17}{2} & \frac{-3}{2} & -4 \\ 30 & -6 & -13 \\ 55 & -12 & -22 \\ 12 & -3 & -4 \\ 44 & -12 & -11 \\ \frac{3}{2} & \frac{-1}{2} & 0 \end{pmatrix} \begin{pmatrix} -6 & 0 & 12 & 198 & 24 & 78 & 132 & 24 & 66 & 0 \\ 0 & -22 & 0 & 220 & 33 & 132 & 264 & 66 & 264 & 11 \\ 12 & 0 & -6 & 0 & 3 & 24 & 66 & 24 & 132 & 9 \\ 198 & 220 & 0 & -22 & 0 & 66 & 264 & 132 & 990 & 88 \\ 24 & 33 & 3 & 0 & -1 & 0 & 11 & 9 & 88 & 9 \\ 78 & 132 & 24 & 66 & 0 & -6 & 0 & 12 & 198 & 24 \\ 132 & 264 & 66 & 264 & 11 & 0 & -22 & 0 & 220 & 33 \\ 24 & 66 & 24 & 132 & 9 & 12 & 0 & -6 & 0 & 3 \\ 66 & 264 & 132 & 990 & 88 & 198 & 220 & 0 & -22 & 0 \\ 0 & 11 & 9 & 88 & 9 & 24 & 33 & 3 & 0 & -1 \end{pmatrix}.$$

$N' = 21$ $d = 66 = 2 \cdot 3 \cdot 11$, odd, $\eta = 1$, $h = 1$: $\langle 1\rangle \oplus \langle -1\rangle \oplus \langle -66\rangle$. It is equivariantly equivalent to $d = 33$, $\eta = 2$, $h = 1$: $U \oplus \langle -33\rangle$.

$N' = 22$ $d = 66 = 2 \cdot 3 \cdot 11$, odd, $\eta = 2$, $h = 0$: $\langle 22\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 33$, $\eta = 1$, $h = 0$: $\langle 11\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$.

$N' = 23$ $d = 70 = 2 \cdot 5 \cdot 7$, odd, $\eta = 1$, $h = 1$: $\langle 14\rangle \oplus \langle -10\rangle \oplus \langle -2\rangle(1/2, 1/2, 0)$ $(=(35, \widetilde{0, h} = 0))$

$$\begin{pmatrix} 50 & -11 & -130 \\ 7 & -2 & -18 \\ 10 & -4 & -25 \\ \frac{1}{2} & \frac{-1}{2} & -1 \\ \frac{5}{2} & \frac{-7}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -8 \\ 40 & -6 & -105 \\ \frac{39}{2} & \frac{-7}{2} & -51 \\ \frac{1395}{2} & \frac{-273}{2} & -1820 \\ 130 & -26 & -339 \end{pmatrix} \quad \begin{pmatrix} -10 & 0 & 60 & 35 & 1365 & 260 & 110 & 20 & 40 & 5 & 35 & 0 \\ 0 & -2 & 0 & 3 & 175 & 36 & 20 & 6 & 20 & 5 & 105 & 16 \\ 60 & 0 & -10 & 0 & 210 & 50 & 40 & 20 & 110 & 40 & 1190 & 210 \\ 35 & 3 & 0 & -1 & 0 & 2 & 5 & 5 & 40 & 17 & 560 & 102 \\ 1365 & 175 & 210 & 0 & -35 & 0 & 35 & 105 & 1190 & 560 & 19635 & 3640 \\ 260 & 36 & 50 & 2 & 0 & -2 & 0 & 16 & 210 & 102 & 3640 & 678 \\ 110 & 20 & 40 & 5 & 35 & 0 & -10 & 0 & 60 & 35 & 1365 & 260 \\ 20 & 6 & 20 & 5 & 105 & 16 & 0 & -2 & 0 & 3 & 175 & 36 \\ 40 & 20 & 110 & 40 & 1190 & 210 & 60 & 0 & -10 & 0 & 210 & 50 \\ 5 & 5 & 40 & 17 & 560 & 102 & 35 & 3 & 0 & -1 & 0 & 2 \\ 35 & 105 & 1190 & 560 & 19635 & 3640 & 1365 & 175 & 210 & 0 & -35 & 0 \\ 0 & 16 & 210 & 102 & 3640 & 678 & 260 & 36 & 50 & 2 & 0 & -2 \end{pmatrix}.$$

$N' = 24$ $d = 70 = 2 \cdot 5 \cdot 7$, odd, $\eta = 2$, $h = 1$: $\langle 70\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$ $(=(35, \widetilde{3, h} = 0))$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -42 \\ 6 & -5 & -50 \\ 11 & -14 & -91 \\ 9 & -14 & -74 \\ 6 & -11 & -49 \\ 2 & -5 & -16 \\ 9 & -28 & -70 \\ 4 & -15 & -30 \\ 3 & -14 & -21 \\ 1 & -6 & -6 \end{pmatrix} \quad \begin{pmatrix} -2 & 1 & 42 & 45 & 77 & 60 & 38 & 11 & 42 & 15 & 7 & 0 \\ 1 & -1 & 0 & 5 & 14 & 14 & 11 & 5 & 28 & 15 & 14 & 6 \\ 42 & 0 & -14 & 0 & 28 & 42 & 42 & 28 & 210 & 140 & 168 & 98 \\ 45 & 5 & 0 & -5 & 0 & 10 & 15 & 15 & 140 & 105 & 140 & 90 \\ 77 & 14 & 28 & 0 & -7 & 0 & 7 & 14 & 168 & 140 & 203 & 140 \\ 60 & 14 & 42 & 10 & 0 & -2 & 0 & 6 & 98 & 90 & 140 & 102 \\ 38 & 11 & 42 & 15 & 7 & 0 & -2 & 1 & 42 & 45 & 77 & 60 \\ 11 & 5 & 28 & 15 & 14 & 6 & 1 & -1 & 0 & 5 & 14 & 14 \\ 42 & 28 & 210 & 140 & 168 & 98 & 42 & 0 & -14 & 0 & 28 & 42 \\ 15 & 15 & 140 & 105 & 140 & 90 & 45 & 5 & 0 & -5 & 0 & 10 \\ 7 & 14 & 168 & 140 & 203 & 140 & 77 & 14 & 28 & 0 & -7 & 0 \\ 0 & 6 & 98 & 90 & 140 & 102 & 60 & 14 & 42 & 10 & 0 & -2 \end{pmatrix}.$$

$N' = 25$ $d = 70 = 2 \cdot 5 \cdot 7$, odd, $\eta = 3$, $h = 1$: $\langle 2\rangle \oplus \langle -14\rangle \oplus \langle -10\rangle(1/2, 1/2, 0)$. It is equivariantly equivalent to $d = 35$, $\eta = 2$, $h = 1$: $\langle 1\rangle \oplus \langle -7\rangle \oplus \langle -5\rangle$.

$N' = 26$ $d = 78 = 2 \cdot 3 \cdot 13$, odd, $\eta = 1$, $h = 1$: $\langle 6\rangle \oplus \langle -13\rangle \oplus \langle -1\rangle$. It is equivariantly equivalent to $d = 39$, $\eta = 2$, $h = 1$: $\langle 3\rangle \oplus \langle -26\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$.

$N' = 27$ $d = 78 = 2 \cdot 3 \cdot 13$, odd, $\eta = 3$, $h = 0$: $\langle 6\rangle \oplus \langle -26\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$ $(=(39, 0, h = 0))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 4 & -1 & -6 \\ 39 & -12 & -52 \\ 4 & \frac{-3}{2} & \frac{-9}{2} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 4 & 12 & 104 & 9 & 1 \\ 0 & -26 & 0 & 26 & 312 & 39 & 13 \\ 4 & 0 & -2 & 0 & 26 & 6 & 4 \\ 12 & 26 & 0 & -2 & 0 & 3 & 5 \\ 104 & 312 & 26 & 0 & -26 & 0 & 26 \\ 9 & 39 & 6 & 3 & 0 & -3 & 0 \\ 1 & 13 & 4 & 5 & 26 & 0 & -1 \end{pmatrix}.$$

$N' = 28$ $d = 102 = 2 \cdot 3 \cdot 17$, odd, $\eta = 0$, $h = 1$: $\langle 34\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(1/2, 1/2, 0)$. It is equivariantly equivalent to $d = 51$, $\eta = 1$, $h = 1$: $\langle 17\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$.

$N' = 29$ $d = 114 = 2 \cdot 3 \cdot 19$, $\eta = 1$, $h = 1$: $\langle 6\rangle \oplus \langle -38\rangle \oplus \langle -2\rangle(1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 57$, $\eta = 2$, $h = 1$: $\langle 3\rangle \oplus \langle -19\rangle \oplus \langle -1\rangle$.

$N' = 30$ $d = 138 = 2 \cdot 3 \cdot 23$, odd, $\eta = 2$, $h = 1$: $\langle 2\rangle \oplus \langle -46\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$ $(=(69, 3, h = 0))$

$$\begin{pmatrix} 72 & 276 & 69 & 115 & 5 & 138 & 12 & 115 & 0 & 0 & 3 & 23 & 7 & 1104 & 126 & 1541 \\ -12 & -47 & -12 & \frac{-41}{2} & -1 & \frac{-57}{2} & \frac{-5}{2} & -24 & 0 & 1 & 0 & \frac{-5}{2} & -1 & \frac{-357}{2} & \frac{-41}{2} & -252 \\ -25 & -92 & -22 & \frac{-69}{2} & -1 & \frac{-23}{2} & \frac{-1}{2} & 0 & 1 & 0 & -2 & \frac{-23}{2} & -3 & \frac{-805}{2} & \frac{-91}{2} & -552 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 0 & 12 & 69 & 18 & 2415 & 273 & 3312 & 150 & 552 & 132 & 207 & 6 & 69 & 3 & 0 \\ 0 & -46 & 0 & 115 & 46 & 8211 & 943 & 11592 & 552 & 2162 & 552 & 943 & 46 & 1311 & 115 & 1104 \\ 12 & 0 & -6 & 0 & 6 & 1794 & 210 & 2622 & 132 & 552 & 150 & 276 & 18 & 690 & 66 & 690 \\ 69 & 115 & 0 & -23 & 0 & 2484 & 299 & 3818 & 207 & 943 & 276 & 552 & 46 & 2277 & 230 & 2530 \\ 18 & 46 & 6 & 0 & -2 & 0 & 2 & 46 & 6 & 46 & 18 & 46 & 6 & 414 & 44 & 506 \\ 2415 & 8211 & 1794 & 2484 & 0 & -69 & 0 & 276 & 69 & 1311 & 690 & 2277 & 414 & 42918 & 4761 & 56856 \\ 273 & 943 & 210 & 299 & 2 & 0 & -1 & 0 & 3 & 115 & 66 & 230 & 44 & 4761 & 530 & 6348 \\ 3312 & 11592 & 2622 & 3818 & 46 & 276 & 0 & -46 & 0 & 1104 & 690 & 2530 & 506 & 56856 & 6348 & 76222 \\ 150 & 552 & 132 & 207 & 6 & 69 & 3 & 0 & -6 & 0 & 12 & 69 & 18 & 2415 & 273 & 3312 \\ 552 & 2162 & 552 & 943 & 46 & 1311 & 115 & 1104 & 0 & -46 & 0 & 115 & 46 & 8211 & 943 & 11592 \\ 132 & 552 & 150 & 276 & 18 & 690 & 66 & 690 & 12 & 0 & -6 & 0 & 6 & 1794 & 210 & 2622 \\ 207 & 943 & 276 & 552 & 46 & 2277 & 230 & 2530 & 69 & 115 & 0 & -23 & 0 & 2484 & 299 & 3818 \\ 6 & 46 & 18 & 46 & 6 & 414 & 44 & 506 & 18 & 46 & 6 & 0 & -2 & 0 & 2 & 46 \\ 69 & 1311 & 690 & 2277 & 414 & 42918 & 4761 & 56856 & 2415 & 8211 & 1794 & 2484 & 0 & -69 & 0 & 276 \\ 3 & 115 & 66 & 230 & 44 & 4761 & 530 & 6348 & 273 & 943 & 210 & 299 & 2 & 0 & -1 & 0 \\ 0 & 1104 & 690 & 2530 & 506 & 56856 & 6348 & 76222 & 3312 & 11592 & 2622 & 3818 & 46 & 276 & 0 & -46 \end{pmatrix}.$$

$N' = 31$ $d = 210 = 2\cdot 3\cdot 5\cdot 7$, odd, $\eta = 0$, $h = 0$: $\langle 7\rangle \oplus \langle -5\rangle \oplus \langle -6\rangle$ $(=(105, 3, h = 0))$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 6 & 0 & -7 \\ 2 & -1 & -2 \\ 40 & -28 & -35 \\ 3 & -3 & -2 \\ 5 & -7 & 0 \end{pmatrix} \qquad \begin{pmatrix} -6 & 0 & 42 & 12 & 210 & 12 & 0 \\ 0 & -5 & 0 & 5 & 140 & 15 & 35 \\ 42 & 0 & -42 & 0 & 210 & 42 & 210 \\ 12 & 5 & 0 & -1 & 0 & 3 & 35 \\ 210 & 140 & 210 & 0 & -70 & 0 & 420 \\ 12 & 15 & 42 & 3 & 0 & -6 & 0 \\ 0 & 35 & 210 & 35 & 420 & 0 & -70 \end{pmatrix}.$$

$N' = 32$ $d = 210 = 2 \cdot 3 \cdot 5 \cdot 7$, odd, $\eta = 1$, $h = 1$: $\langle 14\rangle \oplus \langle -30\rangle \oplus \langle -2\rangle(1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 105, \eta = 2$, $h = 1$: $\langle 7\rangle \oplus \langle -15\rangle \oplus \langle -1\rangle$.

$N' = 33$ $d = 210 = 2 \cdot 3 \cdot 5 \cdot 7$, odd, $\eta = 2$, $h = 0$: $\langle 2\rangle \oplus \langle -42\rangle \oplus \langle -10\rangle(0, 1/2, 1/2)$. It is equivariantly equivalent to $d = 105 = 3 \cdot 5 \cdot 7$, $\eta = 1$, $h = 0$: $\langle 1\rangle \oplus \langle -21\rangle \oplus \langle -5\rangle$.

$N' = 34$ $d = 210 = 2 \cdot 3 \cdot 5 \cdot 7$, odd, $\eta = 4$, $h = 0$: $\langle 5\rangle \oplus \langle -7\rangle \oplus \langle -6\rangle$. It is equivariantly equivalent to $d = 105$, $\eta = 7$, $h = 0$: $\langle 10\rangle \oplus \langle -14\rangle \oplus \langle -3\rangle(1/2, 1/2, 0)$.

$N' = 35 \;\; d = 210 = 2.3.5.7$, odd, $\eta = 6$, $h = 1$: $\langle 10\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$ r
$(=(\widetilde{105, 5, h} = 0))$

$$\begin{pmatrix} 0 & 0 & 3 & 2 & 13 & 35 & 81 & 105 & 24 & 28 & 21 & 4 & 7 & 7 & 9 & 7 \\ 0 & 1 & 0 & \frac{-1}{2} & -5 & \frac{-31}{2} & \frac{-75}{2} & -50 & -12 & -15 & -12 & \frac{-5}{2} & -5 & \frac{-11}{2} & \frac{-15}{2} & -6 \\ 1 & 0 & -4 & \frac{-5}{2} & -15 & \frac{-77}{2} & \frac{-175}{2} & -112 & -25 & -28 & -20 & \frac{-7}{2} & -5 & \frac{-7}{2} & \frac{-5}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 0 & 24 & 15 & 90 & 231 & 525 & 672 & 150 & 168 & 120 & 21 & 30 & 21 & 15 & 0 \\ 0 & -14 & 0 & 7 & 70 & 217 & 525 & 700 & 168 & 210 & 168 & 35 & 70 & 77 & 105 & 84 \\ 24 & 0 & -6 & 0 & 30 & 126 & 330 & 462 & 120 & 168 & 150 & 36 & 90 & 126 & 210 & 210 \\ 15 & 7 & 0 & -1 & 0 & 14 & 45 & 70 & 21 & 35 & 36 & 10 & 30 & 49 & 90 & 98 \\ 90 & 70 & 30 & 0 & -10 & 0 & 30 & 70 & 30 & 70 & 90 & 30 & 110 & 210 & 420 & 490 \\ 231 & 217 & 126 & 14 & 0 & -7 & 0 & 28 & 21 & 77 & 126 & 49 & 210 & 448 & 945 & 1148 \\ 525 & 525 & 330 & 45 & 30 & 0 & -15 & 0 & 15 & 105 & 210 & 90 & 420 & 945 & 2040 & 2520 \\ 672 & 700 & 462 & 70 & 70 & 28 & 0 & -14 & 0 & 84 & 210 & 98 & 490 & 1148 & 2520 & 3150 \\ 150 & 168 & 120 & 21 & 30 & 21 & 15 & 0 & -6 & 0 & 24 & 15 & 90 & 231 & 525 & 672 \\ 168 & 210 & 168 & 35 & 70 & 77 & 105 & 84 & 0 & -14 & 0 & 7 & 70 & 217 & 525 & 700 \\ 120 & 168 & 150 & 36 & 90 & 126 & 210 & 210 & 24 & 0 & -6 & 0 & 30 & 126 & 330 & 462 \\ 21 & 35 & 36 & 10 & 30 & 49 & 90 & 98 & 15 & 7 & 0 & -1 & 0 & 14 & 45 & 70 \\ 30 & 70 & 90 & 30 & 110 & 210 & 420 & 490 & 90 & 70 & 30 & 0 & -10 & 0 & 30 & 70 \\ 21 & 77 & 126 & 49 & 210 & 448 & 945 & 1148 & 231 & 217 & 126 & 14 & 0 & -7 & 0 & 28 \\ 15 & 105 & 210 & 90 & 420 & 945 & 2040 & 2520 & 525 & 525 & 330 & 45 & 30 & 0 & -15 & 0 \\ 0 & 84 & 210 & 98 & 490 & 1148 & 2520 & 3150 & 672 & 700 & 462 & 70 & 70 & 28 & 0 & -14 \end{pmatrix}.$$

$N' = 36 \;\; d = 330 = 2 \cdot 3 \cdot 5 \cdot 11$, odd, $\eta = 3$, $h = 1$: $\langle 6\rangle \oplus \langle -10\rangle \oplus \langle -22\rangle(1/2, 1/2, 0)$.
It is equivariantly equivalent to $d = 165$, $\eta = 4$, $h = 1$: $\langle 3\rangle \oplus \langle -5\rangle \oplus \langle -11\rangle$.
$N' = 37 \;\; d = 390 = 2 \cdot 3 \cdot 5 \cdot 13$, odd, $\eta = 4$, $h = 1$: $\langle 30\rangle \oplus \langle -26\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$
$(=(\widetilde{195, 3, h} = 0))$

$$\begin{pmatrix} 0 & 0 & 1 & 26 & 1 & 13 & 3 & 13 & 2 & 26 & 21 & 364 & 9 & 65 & 7 & 13 \\ 0 & 1 & 0 & \frac{-15}{2} & \frac{-1}{2} & -9 & \frac{-5}{2} & -12 & -2 & -27 & -22 & \frac{-765}{2} & \frac{-19}{2} & -69 & \frac{-15}{2} & -14 \\ 1 & 0 & -4 & \frac{-195}{2} & \frac{-7}{2} & -39 & \frac{-15}{2} & -26 & -3 & -26 & -18 & \frac{-585}{2} & \frac{-13}{2} & -39 & \frac{-5}{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 8 & 195 & 7 & 78 & 15 & 52 & 6 & 52 & 36 & 585 & 13 & 78 & 5 & 0 \\ 0 & -26 & 0 & 195 & 13 & 234 & 65 & 312 & 52 & 702 & 572 & 9945 & 247 & 1794 & 195 & 364 \\ 8 & 0 & -2 & 0 & 2 & 78 & 30 & 182 & 36 & 572 & 486 & 8580 & 218 & 1638 & 190 & 390 \\ 195 & 195 & 0 & -195 & 0 & 780 & 390 & 2730 & 585 & 9945 & 8580 & 152295 & 3900 & 29640 & 3510 & 7410 \\ 7 & 13 & 2 & 0 & -1 & 0 & 5 & 52 & 13 & 247 & 218 & 3900 & 101 & 780 & 95 & 208 \\ 78 & 234 & 78 & 780 & 0 & -78 & 0 & 234 & 78 & 1794 & 1638 & 29640 & 780 & 6162 & 780 & 1794 \\ 15 & 65 & 30 & 390 & 5 & 0 & -5 & 0 & 5 & 195 & 190 & 3510 & 95 & 780 & 105 & 260 \\ 52 & 312 & 182 & 2730 & 52 & 234 & 0 & -26 & 0 & 364 & 390 & 7410 & 208 & 1794 & 260 & 702 \\ 6 & 52 & 36 & 585 & 13 & 78 & 5 & 0 & -2 & 0 & 8 & 195 & 7 & 78 & 15 & 52 \\ 52 & 702 & 572 & 9945 & 247 & 1794 & 195 & 364 & 0 & -26 & 0 & 195 & 13 & 234 & 65 & 312 \\ 36 & 572 & 486 & 8580 & 218 & 1638 & 190 & 390 & 8 & 0 & -2 & 0 & 2 & 78 & 30 & 182 \\ 585 & 9945 & 8580 & 152295 & 3900 & 29640 & 3510 & 7410 & 195 & 195 & 0 & -195 & 0 & 780 & 390 & 2730 \\ 13 & 247 & 218 & 3900 & 101 & 780 & 95 & 208 & 7 & 13 & 2 & 0 & -1 & 0 & 5 & 52 \\ 78 & 1794 & 1638 & 29640 & 780 & 6162 & 780 & 1794 & 78 & 234 & 78 & 780 & 0 & -78 & 0 & 234 \\ 5 & 195 & 190 & 3510 & 95 & 780 & 105 & 260 & 15 & 65 & 30 & 390 & 5 & 0 & -5 & 0 \\ 0 & 364 & 390 & 7410 & 208 & 1794 & 260 & 702 & 52 & 312 & 182 & 2730 & 52 & 234 & 0 & -26 \end{pmatrix}.$$

$N' = 38$ $d = 570 = 2\cdot 3\cdot 5\cdot 19$, odd, $\eta = 2$, $h = 1$: $\langle 2\rangle \oplus \langle -190\rangle \oplus \langle -6\rangle (0, 1/2, 1/2)$
$(=(285, 5, h = 0))$

$$\begin{pmatrix} 158 & 2850 & 11 & 15 & 19 & 0 & 0 & 3 & 38 & 48 & 855 \\ \frac{-27}{2} & \frac{-489}{2} & -1 & \frac{-3}{2} & -2 & 0 & 1 & 0 & -2 & -3 & -56 \\ \frac{-101}{2} & \frac{-1805}{2} & -3 & \frac{-5}{2} & 0 & 1 & 0 & -2 & -19 & -22 & -380 \end{pmatrix.$$
$$\begin{matrix} 22 & 570 & 13 & 165 & 893 & 288 & 2280 & 285 & 874 & 600 & 8265 \\ \frac{-3}{2} & \frac{-81}{2} & -1 & \frac{-27}{2} & -74 & -24 & -191 & -24 & -74 & -51 & -704 \\ \frac{-19}{2} & \frac{-475}{2} & -5 & \frac{-115}{2} & -304 & -97 & -760 & -94 & -285 & -194 & -2660 \end{matrix}\right)$$

$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 135 & 874 & 303 & 2565 & 342 & 1121 & 807 & 11400 \\ 0 & -285 & 0 & 2280 & 15390 & 5415 & 46455 & 6270 & 20805 & 15105 & 214320 \\ 2 & 0 & -2 & 0 & 38 & 18 & 190 & 30 & 114 & 90 & 1330 \\ 135 & 2280 & 0 & -15 & 0 & 15 & 285 & 60 & 285 & 255 & 3990 \\ 874 & 15390 & 38 & 0 & -38 & 0 & 380 & 114 & 684 & 684 & 11210 \\ 303 & 5415 & 18 & 15 & 0 & -6 & 0 & 12 & 114 & 132 & 2280 \\ 2565 & 46455 & 190 & 285 & 380 & 0 & -190 & 0 & 380 & 570 & 10640 \\ 342 & 6270 & 30 & 60 & 114 & 12 & 0 & -6 & 0 & 24 & 570 \\ 1121 & 20805 & 114 & 285 & 684 & 114 & 380 & 0 & -38 & 0 & 380 \\ 807 & 15105 & 90 & 255 & 684 & 132 & 570 & 24 & 0 & -6 & 0 \\ 11400 & 214320 & 1330 & 3990 & 11210 & 2280 & 10640 & 570 & 380 & 0 & -190 \\ 226 & 4275 & 28 & 90 & 266 & 57 & 285 & 18 & 19 & 3 & 0 \\ 4275 & 81510 & 570 & 1995 & 6270 & 1425 & 7695 & 570 & 855 & 285 & 2280 \\ 28 & 570 & 6 & 30 & 114 & 30 & 190 & 18 & 38 & 18 & 190 \\ 90 & 1995 & 30 & 240 & 1140 & 345 & 2565 & 300 & 855 & 555 & 7410 \\ 266 & 6270 & 114 & 1140 & 5814 & 1824 & 14060 & 1710 & 5092 & 3420 & 46550 \\ 57 & 1425 & 30 & 345 & 1824 & 582 & 4560 & 564 & 1710 & 1164 & 15960 \\ 285 & 7695 & 190 & 2565 & 14060 & 4560 & 36290 & 4560 & 14060 & 9690 & 133760 \\ 18 & 570 & 18 & 300 & 1710 & 564 & 4560 & 582 & 1824 & 1272 & 17670 \\ 19 & 855 & 38 & 855 & 5092 & 1710 & 14060 & 1824 & 5814 & 4104 & 57380 \\ 3 & 285 & 18 & 555 & 3420 & 1164 & 9690 & 1272 & 4104 & 2922 & 41040 \\ 0 & 2280 & 190 & 7410 & 46550 & 15960 & 133760 & 17670 & 57380 & 41040 & 577790 \end{pmatrix}.$$

Table 3

The list of main hyperbolic lattices of the rank 3 with $d \le 100000$ and $h \le 1$

$n=1,\ d=1,\ \eta=0,\ h=0$	$U \oplus \langle -1\rangle$	*er*
$n=2,\ d=2,\ \eta=0,\ h=0$	$U \oplus \langle -2\rangle$	*er*
$n=3,\ d=3,\ \eta=0,\ h=0$	$\langle 3\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=4,\ d=3,\ \eta=1,\ h=0$	$U \oplus \langle -3\rangle$	*er*
$n=5,\ d=5,\ \eta=0,\ h=0$	$U \oplus \langle -5\rangle$	*er*
$n=6,\ d=5,\ \eta=1,\ h=0$	$\langle 1\rangle \oplus \langle -10\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*er*
$n=7,\ d=6=2\cdot 3,\ \eta=0,\ h=0$	$U \oplus \langle -6\rangle$	*er*
$n=8,\ d=7,\ \eta=0,\ h=1$	$\langle 7\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=9,\ d=7,\ \eta=1,\ h=0$	$U \oplus \langle -7\rangle$	*er*
$n=10,\ d=10=2\cdot 5,\ \eta=1,\ h=0$	$U \oplus \langle -10\rangle$	*er*
$n=11,\ d=11,\ \eta=0,\ h=0$	$\langle 11\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=12,\ d=11,\ \eta=1,\ h=1$	$U \oplus \langle -11\rangle$	*er*
$n=13,\ d=13,\ \eta=0,\ h=1$	$U \oplus \langle -13\rangle$	*er*
$n=14,\ d=13,\ \eta=1,\ h=1$	$\langle 26\rangle \oplus \langle -2\rangle \oplus \langle -1\rangle(1/2,1/2,0)$	*er*
$n=15,\ d=14=2\cdot 7,\ \eta=1,\ h=0$	$U \oplus \langle -14\rangle$	*er*
$n=16,\ d=15=3\cdot 5,\ \eta=0,\ h=0$	$\langle 3\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*er*
$n=17,\ d=15=3\cdot 5,\ \eta=1,\ h=0$	$\langle 5\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$	*er*
$n=18,\ d=15=3\cdot 5,\ \eta=2,\ h=0$	$U \oplus \langle -15\rangle$	*er*
$n=19,\ d=15=3\cdot 5,\ \eta=3,\ h=0$	$\langle 15\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=20,\ d=17,\ \eta=0,\ h=0$	$U \oplus \langle -17\rangle$	*er*
$n=21,\ d=19,\ \eta=0,\ h=1$	$\langle 19\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=22,\ d=21=3\cdot 7,\ \eta=0,\ h=0$	$\langle 14\rangle \oplus \langle -6\rangle \oplus \langle -1\rangle(1/2,1/2,0)$	*er*
$n=23,\ d=21=3\cdot 7,\ \eta=1,\ h=1$	$U \oplus \langle -21\rangle$	*er*
$n=24,\ d=21=3\cdot 7,\ \eta=2,\ h=0$	$\langle 3\rangle \oplus \langle -7\rangle \oplus \langle -1\rangle$	*er*
$n=25,\ d=21=3\cdot 7,\ \eta=3,\ h=0$	$\langle 1\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$	*er*
$n=26,\ d=22=2\cdot 11,\ \eta=0,\ h=1$	$U \oplus \langle -22\rangle$	*er*
$n=27,\ d=23,\ \eta=0,\ h=1$	$\langle 23\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=28,\ d=26=2\cdot 13,\ \eta=1,\ h=0$	$U \oplus \langle -26\rangle$	*er*
$n=29,\ d=29, \eta=0,\ h=1$	$U \oplus \langle -29\rangle$	*nr*
$n=30,\ d=29,\ \eta=1,\ h=1$	$\langle 1\rangle \oplus \langle -58\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*er*
$n=31,\ d=30=2\cdot 3\cdot 5,\ \eta=1,\ h=1$	$U \oplus \langle -30\rangle$	*er*
$n=32,\ d=30=2\cdot 3\cdot 5\cdot \eta=2,\ h=0$	$\langle 10\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*er*
$n=33,\ d=33=3\cdot 11,\ \eta=0,\ h=1$	$\langle 3\rangle \oplus \langle -22\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*er*
$n=34,\ d=33=3\cdot 11,\ \eta=1,\ h=0$	$\langle 11\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$	*er*
$n=35,\ d=33=3\cdot 11,\ \eta=2,\ h=1$	$U \oplus \langle -33\rangle$	*er*
$n=36,\ d=33=3\cdot 11,\ \eta=3,\ h=1$	$\langle 1\rangle \oplus \langle -11\rangle \oplus \langle -3\rangle$	*er*
$n=37,\ d=34=2\cdot 17,\ \eta=0,\ h=1$	$U \oplus \langle -34\rangle$	*er*
$n=38,\ d=35=5\cdot 7,\ \eta=0,\ h=0$	$\langle 7\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*er*
$n=39,\ d=35=5\cdot 7,\ \eta=2,\ h=1$	$\langle 1\rangle \oplus \langle -7\rangle \oplus \langle -5\rangle$	*er*
$n=40,\ d=35=5\cdot 7,\ \eta=3,\ h=0$	$\langle 35\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n=41,\ d=38=2\cdot 19,\ \eta=0,\ h=1$	$U \oplus \langle -38\rangle$	*er*
$n=42,\ d=39=3\cdot 13,\ \eta=0,\ h=0$	$\langle 3\rangle \oplus \langle -13\rangle \oplus \langle -1\rangle$	*er*
$n=43,\ d=39=3\cdot 13,\ \eta=2,\ h=1$	$\langle 3\rangle \oplus \langle -26\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*er*
$n=44,\ d=39=3\cdot 13,\ \eta=3,\ h=1$	$\langle 2\rangle \oplus \langle -26\rangle \oplus \langle -3\rangle(1/2,1/2,0)$	*er*
$n=45,\ d=41,\ \eta=0,\ h=1$	$U \oplus \langle -41\rangle$	*nr*

$n=46,\ d=42=2\cdot3\cdot7,\ \eta=0,\ h=0$	$U\oplus\langle-42\rangle$	*er*
$n=47,\ d=42=2\cdot3\cdot7,\ \eta=3,\ h=1$	$\langle6\rangle\oplus\langle-14\rangle\oplus\langle-2\rangle(1/2,1/2,0)$	*er*
$n=48,\ d=51=3\cdot17,\ \eta=0,\ h=0$	$\langle3\rangle\oplus\langle-17\rangle\oplus\langle-1\rangle$	*er*
$n=49,\ d=51=3\cdot17,\ \eta=1,\ h=1$	$\langle17\rangle\oplus\langle-3\rangle\oplus\langle-1\rangle$	*er*
$n=50,\ d=51=3\cdot17,\ \eta=3,\ h=1$	$\langle51\rangle\oplus\langle-1\rangle\oplus\langle-1\rangle$	*er*
$n=51,\ d=55=5\cdot11,\ \eta=0,\ h=1$	$\langle11\rangle\oplus\langle-5\rangle\oplus\langle-1\rangle$	*er*
$n=52,\ d=55=5\cdot11,\ \eta=3,\ h=1$	$\langle2\rangle\oplus\langle-11\rangle\oplus\langle-10\rangle(1/2,0,1/2)$	*er*
$n=53,\ d=57=3\cdot19,\ \eta=0,\ h=1$	$\langle1\rangle\oplus\langle-38\rangle\oplus\langle-6\rangle(0,1/2,1/2)$	*er*
$n=54,\ d=57=3\cdot19,\ \eta=1,\ h=1$	$U\oplus\langle-57\rangle$	*nr*
$n=55,\ d=57=3\cdot19,\ \eta=2,\ h=1$	$\langle3\rangle\oplus\langle-19\rangle\oplus\langle-1\rangle$	*er*
$n=56,\ d=65=5\cdot13,\ \eta=0,\ h=0$	$\langle1\rangle\oplus\langle-13\rangle\oplus\langle-5\rangle$	*hr*
$n=57,\ d=65=5\cdot13,\ \eta=2,\ h=1$	$\langle5\rangle\oplus\langle-26\rangle\oplus\langle-2\rangle(0,1/2,1/2)$	*er*
$n=58,\ d=65=5\cdot13,\ \eta=3,\ h=1$	$U\oplus\langle-65\rangle$	*nr*
$n=59,\ d=66=2\cdot3\cdot11,\ \eta=1,\ h=1$	$U\oplus\langle-66\rangle$	*er*
$n=60,\ d=66=2\cdot3\cdot11,\ \eta=2,\ h=1$	$\langle22\rangle\oplus\langle-6\rangle\oplus\langle-2\rangle(0,1/2,1/2)$	*er*
$n=61,\ d=69=3\cdot23,\ \eta=3,\ h=0$	$\langle1\rangle\oplus\langle-23\rangle\oplus\langle-3\rangle$	*er*
$n=62,\ d=70=2\cdot5\cdot7,\ \eta=3,\ h=1$	$\langle2\rangle\oplus\langle-14\rangle\oplus\langle-10\rangle(0,1/2,1/2)$	*er*
$n=63,\ d=71,\ \eta=0,\ h=1$	$\langle71\rangle\oplus\langle-1\rangle\oplus\langle-1\rangle$	*nr*
$n=64,\ d=74=2\cdot37,\ \eta=1,\ h=0$	$U\oplus\langle-74\rangle$	*hr*
$n=65,\ d=77=7\cdot11,\ \eta=1,\ h=0$	$U\oplus\langle-77\rangle$	*nr*
$n=66,\ d=77=7\cdot11,\ \eta=3,\ h=0$	$\langle1\rangle\oplus\langle-11\rangle\oplus\langle-7\rangle$	*er*
$n=67,\ d=78=2\cdot3\cdot13,\ \eta=2,\ h=0$	$U\oplus\langle-78\rangle$	*er*
$n=68,\ d=85=5\cdot17,\ \eta=0,\ h=1$	$\langle1\rangle\oplus\langle-17\rangle\oplus\langle-5\rangle$	*nr*
$n=69,\ d=85=5\cdot17,\ \eta=1,\ h=1$	$\langle1\rangle\oplus\langle-34\rangle\oplus\langle-10\rangle(0,1/2,1/2)$	*er*
$n=70,\ d=86=2\cdot43,\ \eta=0,\ h=1$	$U\oplus\langle-86\rangle$	*nr*
$n=71,\ d=87=3\cdot29,\ \eta=3,\ h=1$	$\langle2\rangle\oplus\langle-58\rangle\oplus\langle-3\rangle(1/2,1/2,0)$	*er*
$n=72,\ d=91=7\cdot13,\ \eta=3,\ h=1$	$\langle1\rangle\oplus\langle-26\rangle\oplus\langle-14\rangle(0,1/2,1/2)$	*er*
$n=73,\ d=93=3\cdot31,\ \eta=0,\ h=1$	$\langle31\rangle\oplus\langle-6\rangle\oplus\langle-2\rangle(0,1/2,1/2)$	*er*
$n=74,\ d=95=5\cdot19,\ \eta=0,\ h=0$	$\langle19\rangle\oplus\langle-5\rangle\oplus\langle-1\rangle$	*hr*
$n=75,\ d=95=5\cdot19,\ \eta=3,\ h=1$	$\langle2\rangle\oplus\langle-19\rangle\oplus\langle-10\rangle(1/2,0,1/2)$	*er*
$n=76,\ d=102=2\cdot3\cdot17,\ \eta=0,\ h=1$	$\langle2\rangle\oplus\langle-34\rangle\oplus\langle-6\rangle(0,1/2,1/2)$	*er*
$n=77,\ d=105=3\cdot5\cdot7,\ \eta=0,\ h=1$	$\langle2\rangle\oplus\langle-42\rangle\oplus\langle-5\rangle(1/2,1/2,0)$	*er*
$n=78,\ d=105=3\cdot5\cdot7,\ \eta=1,\ h=0$	$\langle1\rangle\oplus\langle-21\rangle\oplus\langle-5\rangle$	*er*
$n=79,\ d=105=3\cdot5\cdot7,\ \eta=2,\ h=1$	$\langle7\rangle\oplus\langle-15\rangle\oplus\langle-1\rangle$	*er*
$n=80,\ d=105=3\cdot5\cdot7,\ \eta=3,\ h=0$	$\langle14\rangle\oplus\langle-10\rangle\oplus\langle-3\rangle(1/2,1/2,0)$	*er*
$n=81,\ d=105=3\cdot5\cdot7,\ \eta=4,\ h=1$	$U\oplus\langle-105\rangle$	*nr*
$n=82,\ d=105=3\cdot5\cdot7,\ \eta=5,\ h=0$	$\langle5\rangle\oplus\langle-7\rangle\oplus\langle-3\rangle$	*er*
$n=83,\ d=105=3\cdot5\cdot7,\ \eta=6,\ h=0$	$\langle1\rangle\oplus\langle-15\rangle\oplus\langle-7\rangle$	*er*
$n=84,\ d=105=3\cdot5\cdot7,\ \eta=7,\ h=0$	$\langle10\rangle\oplus\langle-14\rangle\oplus\langle-3\rangle(1/2,1/2,0)$	*er*
$n=85,\ d=110=2\cdot5\cdot11,\ \eta=1,\ h=1$	$U\oplus\langle-110\rangle$	*er*
$n=86,\ d=111=3\cdot37,\ \eta=0,\ h=1$	$\langle3\rangle\oplus\langle-37\rangle\oplus\langle-1\rangle$	*er*
$n=87,\ d=114=2\cdot3\cdot19,\ \eta=2,\ h=0$	$U\oplus\langle-114\rangle$	*hr*
$n=88,\ d=119=7\cdot17,\ \eta=3,\ h=1$	$\langle119\rangle\oplus\langle-1\rangle\oplus\langle-1\rangle$	*nr*
$n=89,\ d=130=2\cdot5\cdot13,\ \eta=0,\ h=1$	$U\oplus\langle-130\rangle$	*nr*
$n=90,\ d=130=2\cdot5\cdot13,\ \eta=3,\ h=1$	$\langle10\rangle\oplus\langle-26\rangle\oplus\langle-2\rangle(1/2,1/2,0)$	*er*
$n=91,\ d=134=2\cdot67,\ \eta=0,\ h=1$	$U\oplus\langle-134\rangle$	*nr*
$n=92,\ d=141=3\cdot47,\ \eta=3,\ h=1$	$\langle1\rangle\oplus\langle-47\rangle\oplus\langle-3\rangle$	*er*
$n=93,\ d=143=11\cdot13,\ \eta=0,\ h=1$	$\langle11\rangle\oplus\langle-13\rangle\oplus\langle-1\rangle$	*nr*
$n=94,\ d=146=2\cdot73,\ \eta=0,\ h=1$	$U\oplus\langle-146\rangle$	*nr*

$n = 95,\ d = 154 = 2 \cdot 7 \cdot 11,\ \eta = 3,\ h = 1$	$U \oplus \langle -154 \rangle$	*nr*
$n = 96,\ d = 155 = 5 \cdot 31,\ \eta = 3,\ h = 1$	$\langle 2 \rangle \oplus \langle -31 \rangle \oplus \langle -10 \rangle (1/2, 0, 1/2)$	*er*
$n = 97,\ d = 161 = 7 \cdot 23,\ \eta = 1,\ h = 1$	$U \oplus \langle -161 \rangle$	*nr*
$n = 98,\ d = 161 = 7 \cdot 23,\ \eta = 3,\ h = 1$	$\langle 1 \rangle \oplus \langle -23 \rangle \oplus \langle -7 \rangle$	*nr*
$n = 99,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 0,\ h = 0$	$\langle 5 \rangle \oplus \langle -6 \rangle \oplus \langle -22 \rangle (0, 1/2, 1/2)$	*er*
$n = 100,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 1,\ h = 1$	$\langle 11 \rangle \oplus \langle -5 \rangle \oplus \langle -3 \rangle$	*nr*
$n = 101,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 3,\ h = 1$	$\langle 15 \rangle \oplus \langle -22 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$	*er*
$n = 102,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 4,\ h = 1$	$\langle 3 \rangle \oplus \langle -5 \rangle \oplus \langle -11 \rangle$	*er*
$n = 103,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 5,\ h = 1$	$\langle 1 \rangle \oplus \langle -55 \rangle \oplus \langle -3 \rangle$	*er*
$n = 104,\ d = 165 = 3 \cdot 5 \cdot 11,\ \eta = 6,\ h = 0$	$\langle 1 \rangle \oplus \langle -15 \rangle \oplus \langle -11 \rangle$	*er*
$n = 105,\ d = 170 = 2 \cdot 5 \cdot 17,\ \eta = 1,\ h = 1$	$\langle 2 \rangle \oplus \langle -10 \rangle \oplus \langle -34 \rangle (1/2, 1/2, 0)$	*er*
$n = 106,\ d = 170 = 2 \cdot 5 \cdot 17,\ \eta = 2,\ h = 0$	$U \oplus \langle -170 \rangle$	*nr*
$n = 107,\ d = 182 = 2 \cdot 7 \cdot 13,\ \eta = 3,\ h = 1$	$\langle 2 \rangle \oplus \langle -26 \rangle \oplus \langle -14 \rangle (0, 1/2, 1/2)$	*nr*
$n = 108,\ d = 186 = 2 \cdot 3 \cdot 31,\ \eta = 0,\ h = 0$	$U \oplus \langle -186 \rangle$	*nr*
$n = 109,\ d = 194 = 2 \cdot 97,\ \eta = 0,\ h = 1$	$U \oplus \langle -194 \rangle$	*nr*
$n = 110,\ d = 195 = 3 \cdot 5 \cdot 13,\ \eta = 3,\ h = 0$	$\langle 15 \rangle \oplus \langle -13 \rangle \oplus \langle -1 \rangle$	*er*
$n = 111,\ d = 195 = 3 \cdot 5 \cdot 13,\ \eta = 5,\ h = 1$	$\langle 6 \rangle \oplus \langle -26 \rangle \oplus \langle -5 \rangle (1/2, 1/2, 0)$	*er*
$n = 112,\ d = 195 = 3 \cdot 5 \cdot 13,\ \eta = 6,\ h = 0$	$\langle 3 \rangle \oplus \langle -65 \rangle \oplus \langle -1 \rangle$	*er*
$n = 113,\ d = 203 = 7 \cdot 29,\ \eta = 3,\ h = 1$	$\langle 2 \rangle \oplus \langle -58 \rangle \oplus \langle -7 \rangle (1/2, 1/2, 0)$	*nr*
$n = 114,\ d = 205 = 5 \cdot 41,\ \eta = 1,\ h = 1$	$\langle 2 \rangle \oplus \langle -41 \rangle \oplus \langle -10 \rangle (1/2, 0, 1/2)$	*er*
$n = 115,\ d = 210 = 2 \cdot 3 \cdot 5 \cdot 7,\ \eta = 2,\ h = 0$	$\langle 2 \rangle \oplus \langle -10 \rangle \oplus \langle -42 \rangle (1/2, 1/2, 0)$	*er*
$n = 116,\ d = 210 = 2 \cdot 3 \cdot 5 \cdot 7,\ \eta = 4,\ h = 1$	$\langle 30 \rangle \oplus \langle -14 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$	*er*
$n = 117,\ d = 215 = 5 \cdot 43,\ \eta = 0,\ h = 1$	$\langle 43 \rangle \oplus \langle -5 \rangle \oplus \langle -1 \rangle$	*nr*
$n = 118,\ d = 219 = 3 \cdot 73,\ \eta = 0,\ h = 0$	$\langle 3 \rangle \oplus \langle -73 \rangle \oplus \langle -1 \rangle$	*er*
$n = 119,\ d = 231 = 3 \cdot 7 \cdot 11,\ \eta = 1,\ h = 1$	$\langle 14 \rangle \oplus \langle -22 \rangle \oplus \langle -3 \rangle (1/2, 1/2, 0)$	*er*
$n = 120,\ d = 231 = 3 \cdot 7 \cdot 11,\ \eta = 2,\ h = 0$	$\langle 3 \rangle \oplus \langle -22 \rangle \oplus \langle -14 \rangle (0, 1/2, 1/2)$	*hr*
$n = 121,\ d = 231 = 3 \cdot 7 \cdot 11,\ \eta = 7,\ h = 0$	$\langle 6 \rangle \oplus \langle -14 \rangle \oplus \langle -11 \rangle (1/2, 1/2, 0)$	*er*
$n = 122,\ d = 246 = 2 \cdot 3 \cdot 41,\ \eta = 0,\ h = 1$	$\langle 2 \rangle \oplus \langle -82 \rangle \oplus \langle -6 \rangle (0, 1/2, 1/2)$	*nr*
$n = 123,\ d = 255 = 3 \cdot 5 \cdot 17,\ \eta = 0,\ h = 1$	$\langle 3 \rangle \oplus \langle -17 \rangle \oplus \langle -5 \rangle$	*er*
$n = 124,\ d = 255 = 3 \cdot 5 \cdot 17,\ \eta = 5,\ h = 1$	$\langle 51 \rangle \oplus \langle -5 \rangle \oplus \langle -1 \rangle$	*nr*
$n = 125,\ d = 255 = 3 \cdot 5 \cdot 17,\ \eta = 6,\ h = 0$	$\langle 3 \rangle \oplus \langle -85 \rangle \oplus \langle -1 \rangle$	*er*
$n = 126,\ d = 259 = 7 \cdot 37,\ \eta = 3,\ h = 1$	$\langle 2 \rangle \oplus \langle -74 \rangle \oplus \langle -7 \rangle (1/2, 1/2, 0)$	*nr*
$n = 127,\ d = 266 = 2 \cdot 7 \cdot 19,\ \eta = 0,\ h = 1$	$U \oplus \langle -266 \rangle$	*nr*
$n = 128,\ d = 266 = 2 \cdot 7 \cdot 19,\ \eta = 3,\ h = 1$	$\langle 38 \rangle \oplus \langle -14 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$	*nr*
$n = 129,\ d = 273 = 3 \cdot 7 \cdot 13,\ \eta = 0,\ h = 1$	$\langle 14 \rangle \oplus \langle -13 \rangle \oplus \langle -6 \rangle (1/2, 0, 1/2)$	*er*
$n = 130,\ d = 273 = 3 \cdot 7 \cdot 13,\ \eta = 6,\ h = 1$	$\langle 26 \rangle \oplus \langle -7 \rangle \oplus \langle -6 \rangle (1/2, 0, 1/2)$	*er*
$n = 131,\ d = 285 = 3 \cdot 5 \cdot 19,\ \eta = 1,\ h = 1$	$\langle 19 \rangle \oplus \langle -5 \rangle \oplus \langle -3 \rangle$	*nr*
$n = 132,\ d = 285 = 3 \cdot 5 \cdot 19,\ \eta = 3,\ h = 1$	$\langle 10 \rangle \oplus \langle -38 \rangle \oplus \langle -3 \rangle (1/2, 1/2, 0)$	*er*
$n = 133,\ d = 285 = 3 \cdot 5 \cdot 19,\ \eta = 5,\ h = 0$	$\langle 1 \rangle \oplus \langle -95 \rangle \oplus \langle -3 \rangle$	*er*
$n = 134,\ d = 285 = 3 \cdot 5 \cdot 19,\ \eta = 6,\ h = 1$	$\langle 57 \rangle \oplus \langle -10 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$	*nr*
$n = 135,\ d = 291 = 3 \cdot 97,\ \eta = 0,\ h = 1$	$\langle 3 \rangle \oplus \langle -97 \rangle \oplus \langle -1 \rangle$	*er*
$n = 136,\ d = 299 = 13 \cdot 23,\ \eta = 3,\ h = 1$	$\langle 2 \rangle \oplus \langle -26 \rangle \oplus \langle -23 \rangle (1/2, 1/2, 0)$	*nr*
$n = 137,\ d = 326 = 2 \cdot 163,\ \eta = 0,\ h = 1$	$U \oplus \langle -326 \rangle$	*nr*
$n = 138,\ d = 330 = 2 \cdot 3 \cdot 5 \cdot 11,\ \eta = 0,\ h = 1$	$U \oplus \langle -330 \rangle$	*nr*
$n = 139,\ d = 330 = 2 \cdot 3 \cdot 5 \cdot 11,\ \eta = 3,\ h = 0$	$\langle 6 \rangle \oplus \langle -110 \rangle \oplus \langle -2 \rangle (1/2, 1/2, 0)$	*er*
$n = 140,\ d = 335 = 5 \cdot 67,\ \eta = 0,\ h = 1$	$\langle 67 \rangle \oplus \langle -5 \rangle \oplus \langle -1 \rangle$	*nr*
$n = 141,\ d = 345 = 3 \cdot 5 \cdot 23,\ \eta = 6,\ h = 0$	$\langle 1 \rangle \oplus \langle -23 \rangle \oplus \langle -15 \rangle$	*er*
$n = 142,\ d = 354 = 2 \cdot 3 \cdot 59,\ \eta = 2,\ h = 1$	$\langle 118 \rangle \oplus \langle -6 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$	*nr*
$n = 143,\ d = 357 = 3 \cdot 7 \cdot 17,\ \eta = 0,\ h = 1$	$\langle 14 \rangle \oplus \langle -17 \rangle \oplus \langle -6 \rangle (1/2, 0, 1/2)$	*nr*

$n = 144,\ d = 357 = 3 \cdot 7 \cdot 17,\ \eta = 3,\ h = 1$	$\langle 17\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$	*er*
$n = 145,\ d = 357 = 3 \cdot 7 \cdot 17,\ \eta = 5,\ h = 1$	$\langle 1\rangle \oplus \langle -119\rangle \oplus \langle -3\rangle$	*er*
$n = 146,\ d = 371 = 7 \cdot 53,\ \eta = 3,\ h = 1$	$\langle 2\rangle \oplus \langle -106\rangle \oplus \langle -7\rangle(1/2, 1/2, 0)$	*nr*
$n = 147,\ d = 385 = 5 \cdot 7 \cdot 11,\ \eta = 3,\ h = 1$	$\langle 11\rangle \oplus \langle -14\rangle \oplus \langle -10\rangle(0, 1/2, 1/2)$	*nr*
$n = 148,\ d = 385 = 5 \cdot 7 \cdot 11,\ \eta = 6,\ h = 1$	$\langle 5\rangle \oplus \langle -11\rangle \oplus \langle -7\rangle$	*er*
$n = 149,\ d = 386 = 2 \cdot 193,\ \eta = 0,\ h = 1$	$U \oplus \langle -386\rangle$	*nr*
$n = 150,\ d = 390 = 2 \cdot 3 \cdot 5 \cdot 13,\ \eta = 6,\ h = 1$	$\langle 10\rangle \oplus \langle -26\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$	*er*
$n = 151,\ d = 399 = 3 \cdot 7 \cdot 19,\ \eta = 2,\ h = 1$	$\langle 3\rangle \oplus \langle -38\rangle \oplus \langle -14\rangle(0, 1/2, 1/2)$	*nr*
$n = 152,\ d = 399 = 3 \cdot 7 \cdot 19,\ \eta = 4,\ h = 0$	$\langle 14\rangle \oplus \langle -19\rangle \oplus \langle -6\rangle(1/2, 0, 1/2)$	*er*
$n = 153,\ d = 407 = 11 \cdot 37,\ \eta = 0,\ h = 1$	$\langle 11\rangle \oplus \langle -37\rangle \oplus \langle -1\rangle$	*nr*
$n = 154,\ d = 429 = 3 \cdot 11 \cdot 13,\ \eta = 3,\ h = 1$	$\langle 1\rangle \oplus \langle -39\rangle \oplus \langle -11\rangle$	*er*
$n = 155,\ d = 435 = 3 \cdot 5 \cdot 29,\ \eta = 0,\ h = 0$	$\langle 3\rangle \oplus \langle -29\rangle \oplus \langle -5\rangle$	*er*
$n = 156,\ d = 435 = 3 \cdot 5 \cdot 29,\ \eta = 6,\ h = 1$	$\langle 435\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n = 157,\ d = 455 = 5 \cdot 7 \cdot 13,\ \eta = 0,\ h = 1$	$\langle 7\rangle \oplus \langle -13\rangle \oplus \langle -5\rangle$	*nr*
$n = 158,\ d = 455 = 5 \cdot 7 \cdot 13,\ \eta = 5,\ h = 1$	$\langle 7\rangle \oplus \langle -26\rangle \oplus \langle -10\rangle(0, 1/2, 1/2)$	*er*
$n = 159,\ d = 465 = 3 \cdot 5 \cdot 31,\ \eta = 5,\ h = 1$	$\langle 5\rangle \oplus \langle -31\rangle \oplus \langle -3\rangle$	*er*
$n = 160,\ d = 483 = 3 \cdot 7 \cdot 23,\ \eta = 7,\ h = 1$	$\langle 6\rangle \oplus \langle -46\rangle \oplus \langle -7\rangle(1/2, 1/2, 0)$	*er*
$n = 161,\ d = 506 = 2 \cdot 11 \cdot 23,\ \eta = 0,\ h = 1$	$U \oplus \langle -506\rangle$	*nr*
$n = 162,\ d = 530 = 2 \cdot 5 \cdot 53,\ \eta = 0,\ h = 1$	$U \oplus \langle -530\rangle$	*nr*
$n = 163,\ d = 534 = 2 \cdot 3 \cdot 89,\ \eta = 0,\ h = 1$	$\langle 2\rangle \oplus \langle -178\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$	*nr*
$n = 164,\ d = 546 = 2 \cdot 3 \cdot 7 \cdot 13,\ \eta = 2,\ h = 0$	$U \oplus \langle -546\rangle$	*nr*
$n = 165,\ d = 546 = 2 \cdot 3 \cdot 7 \cdot 13,\ \eta = 4,\ h = 0$	$\langle 14\rangle \oplus \langle -26\rangle \oplus \langle -6\rangle(1/2, 0, 1/2)$	*nr*
$n = 166,\ d = 561 = 3 \cdot 11 \cdot 17,\ \eta = 2,\ h = 1$	$\langle 3\rangle \oplus \langle -11\rangle \oplus \langle -17\rangle$	*nr*
$n = 167,\ d = 570 = 2 \cdot 3 \cdot 5 \cdot 19,\ \eta = 6,\ h = 1$	$\langle 38\rangle \oplus \langle -10\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$	*er*
$n = 168,\ d = 602 = 2 \cdot 7 \cdot 43,\ \eta = 3,\ h = 1$	$U \oplus \langle -602\rangle$	*nr*
$n = 169,\ d = 615 = 3 \cdot 5 \cdot 41,\ \eta = 0,\ h = 1$	$\langle 3\rangle \oplus \langle -41\rangle \oplus \langle -5\rangle$	*er*
$n = 170,\ d = 645 = 3 \cdot 5 \cdot 43,\ \eta = 3,\ h = 1$	$\langle 10\rangle \oplus \langle -86\rangle \oplus \langle -3\rangle(1/2, 1/2, 0)$	*er*
$n = 171,\ d = 645 = 3 \cdot 5 \cdot 43,\ \eta = 6,\ h = 1$	$\langle 10\rangle \oplus \langle -6\rangle \oplus \langle -43\rangle(1/2, 1/2, 0)$	*nr*
$n = 172,\ d = 651 = 3 \cdot 7 \cdot 31,\ \eta = 2,\ h = 1$	$\langle 62\rangle \oplus \langle -7\rangle \oplus \langle -6\rangle(1/2, 0, 1/2)$	*nr*
$n = 173,\ d = 651 = 3 \cdot 7 \cdot 31,\ \eta = 4,\ h = 1$	$\langle 14\rangle \oplus \langle -31\rangle \oplus \langle -6\rangle(1/2, 0, 1/2)$	*er*
$n = 174,\ d = 714 = 2 \cdot 3 \cdot 7 \cdot 17,\ \eta = 0,\ h = 1$	$\langle 14\rangle \oplus \langle -34\rangle \oplus \langle -6\rangle(1/2, 0, 1/2)$	*nr*
$n = 175,\ d = 714 = 2 \cdot 3 \cdot 7 \cdot 17,\ \eta = 6,\ h = 1$	$\langle 102\rangle \oplus \langle -14\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*nr*
$n = 176,\ d = 759 = 3 \cdot 11 \cdot 23,\ \eta = 4,\ h = 1$	$\langle 11\rangle \oplus \langle -69\rangle \oplus \langle -1\rangle$	*nr*
$n = 177,\ d = 777 = 3 \cdot 7 \cdot 37,\ \eta = 6,\ h = 1$	$\langle 74\rangle \oplus \langle -6\rangle \oplus \langle -7\rangle(1/2, 1/2, 0)$	*nr*
$n = 178,\ d = 795 = 3 \cdot 5 \cdot 53,\ \eta = 6,\ h = 1$	$\langle 3\rangle \oplus \langle -106\rangle \oplus \langle -10\rangle(0, 1/2, 1/2)$	*er*
$n = 179,\ d = 805 = 5 \cdot 7 \cdot 23,\ \eta = 3,\ h = 1$	$\langle 1\rangle \oplus \langle -115\rangle \oplus \langle -7\rangle$	*nr*
$n = 180,\ d = 854 = 2 \cdot 7 \cdot 61,\ \eta = 3,\ h = 1$	$\langle 2\rangle \oplus \langle -122\rangle \oplus \langle -14\rangle(0, 1/2, 1/2)$	*nr*
$n = 181,\ d = 897 = 3 \cdot 13 \cdot 23,\ \eta = 5,\ h = 1$	$\langle 13\rangle \oplus \langle -23\rangle \oplus \langle -3\rangle$	*nr*
$n = 182,\ d = 897 = 3 \cdot 13 \cdot 23,\ \eta = 7,\ h = 1$	$\langle 6\rangle \oplus \langle -26\rangle \oplus \langle -23\rangle(1/2, 1/2, 0)$	*nr*
$n = 183,\ d = 930 = 2 \cdot 3 \cdot 5 \cdot 31,\ \eta = 2,\ h = 1$	$\langle 2\rangle \oplus \langle -186\rangle \oplus \langle -10\rangle(1/2, 0, 1/2)$	*nr*
$n = 184,\ d = 935 = 5 \cdot 11 \cdot 17,\ \eta = 6,\ h = 1$	$\langle 187\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*nr*
$n = 185,\ d = 966 = 2 \cdot 3 \cdot 7 \cdot 23,\ \eta = 2,\ h = 1$	$\langle 46\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$	*nr*
$n = 186,\ d = 1106 = 2 \cdot 7 \cdot 79,\ \eta = 1,\ h = 1$	$U \oplus \langle -1106\rangle$	*nr*
$n = 187,\ d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 2,\ h = 0$	$\langle 14\rangle \oplus \langle -22\rangle \oplus \langle -15\rangle(1/2, 1/2, 0)$	*er*
$n = 188,\ d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 4,\ h = 1$	$\langle 3\rangle \oplus \langle -77\rangle \oplus \langle -5\rangle$	*nr*
$n = 189,\ d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 8,\ h = 1$	$\langle 1155\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*er*
$n = 190,\ d = 1155 = 3 \cdot 5 \cdot 7 \cdot 11,\ \eta = 14,\ h = 0$	$\langle 22\rangle \oplus \langle -15\rangle \oplus \langle -14\rangle(1/2, 0, 1/2)$	*er*
$n = 191,\ d = 1254 = 2 \cdot 3 \cdot 11 \cdot 19,\ \eta = 4,\ h = 1$	$\langle 38\rangle \oplus \langle -22\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$	*nr*
$n = 192,\ d = 1365 = 3 \cdot 5 \cdot 7 \cdot 13,\ \eta = 1,\ h = 1$	$\langle 455\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$	*nr*

$n=193,\ d=1365=3\cdot5\cdot7\cdot13,\ \eta=5,\ h=1$	$\langle 5\rangle\oplus\langle -39\rangle\oplus\langle -7\rangle$	er
$n=194,\ d=1365=3\cdot5\cdot7\cdot13,\ \eta=15,\ h=1$	$\langle 6\rangle\oplus\langle -65\rangle\oplus\langle -14\rangle(1/2,0,1/2)$	er
$n=195,\ d=1394=2\cdot17\cdot41,\ \eta=3,\ h=1$	$U\oplus\langle -1394\rangle$	nr
$n=196,\ d=1659=3\cdot7\cdot79,\ \eta=2,\ h=1$	$\langle 79\rangle\oplus\langle -14\rangle\oplus\langle -6\rangle(0,1/2,1/2)$	nr
$n=197,\ d=1785=3\cdot5\cdot7\cdot17,\ \eta=4,\ h=1$	$\langle 30\rangle\oplus\langle -17\rangle\oplus\langle -14\rangle(1/2,0,1/2)$	nr
$n=198,\ d=1785=3\cdot5\cdot7\cdot17,\ \eta=6,\ h=1$	$\langle 17\rangle\oplus\langle -15\rangle\oplus\langle -7\rangle$	nr
$n=199,\ d=2145=3\cdot5\cdot11\cdot13,\ \eta=10,\ h=1$	$\langle 1\rangle\oplus\langle -143\rangle\oplus\langle -15\rangle$	nr
$n=200,\ d=2210=2\cdot5\cdot13\cdot17,\ \eta=6,\ h=1$	$\langle 2\rangle\oplus\langle -170\rangle\oplus\langle -26\rangle(1/2,0,1/2)$	nr
$n=201,\ d=2310=2\cdot3\cdot5\cdot7\cdot11,\ \eta=1,\ h=1$	$\langle 6\rangle\oplus\langle -70\rangle\oplus\langle -22\rangle(1/2,0,1/2)$	nr
$n=202,\ d=2730=2\cdot3\cdot5\cdot7\cdot13,\ \eta=6,\ h=0$	$\langle 2\rangle\oplus\langle -546\rangle\oplus\langle -10\rangle(1/2,0,1/2)$	nr
$n=203,\ d=3311=7\cdot11\cdot43,\ \eta=1,\ h=1$	$\langle 43\rangle\oplus\langle -77\rangle\oplus\langle -1\rangle$	nr
$n=204,\ d=3570=2\cdot3\cdot5\cdot7\cdot17,\ \eta=7,\ h=1$	$\langle 2\rangle\oplus\langle -714\rangle\oplus\langle -10\rangle(1/2,0,1/2)$	nr
$n=205,\ d=3990=2\cdot3\cdot5\cdot7\cdot19,\ \eta=4,\ h=1$	$\langle 30\rangle\oplus\langle -38\rangle\oplus\langle -14\rangle(1/2,1/2,0)$	nr
$n=206,\ d=4466=2\cdot7\cdot11\cdot29,\ \eta=1,\ h=1$	$U\oplus\langle -4466\rangle$	nr

3. The number h of classes of central symmetries of main hyperbolic lattices of the rank 3.

3.1. Reminding of some classical results about binary positive lattices.

Here we remind some classical results about binary positive quadratic forms (e. g. see [B-Sh] and [C]).

We consider binary (i.e. of the rank two) positive definite lattices K with square-free determinant $d=\det K$. They will be called *fundamental*. If d is odd, the fundamental lattice K is unimodular over $\mathbb{Z}_2$. Considering K over $\mathbb{Z}_2$, it is easy to see that K is odd if $d\equiv 1,2 \mod 4$. If $d\equiv -1 \mod 4$, the lattice K may be odd or even. The number

$$D=\begin{cases} -\det K, & \text{if } K \text{ is even,}\\ -4\det K, & \text{if } K \text{ is odd}\end{cases} \tag{3.1.1}$$

is called *discriminant* of a binary lattice K. Discriminants of fundamental binary lattices are called *fundamental discriminants.* Thus, for a fundamental discriminant D either $D\equiv 1 \mod 4$ or $D\equiv \pm 4,8 \mod 16$. A binary positive lattice K is called *classical fundamental* if $d=\det K$ is square-free and additionally K is even if $d\equiv -1 \mod 4$. The discriminant D of a classical fundamental binary lattice K is called the *classical fundamental discriminant.* Thus, for a classical fundamental discriminant D either $D\equiv 1 \mod 4$ (when K is even) or $D\equiv -4,8 \mod 16$ (when K is odd).

It is known

Theorem 3.1.1. *The number $h(D)$ of proper classes (i. e. classes of oriented lattices up to preserving orientation isomorphisms) of classical fundamental binary lattices K of the discriminant $D<0$ is equal to the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ of the discriminant D and is given by the Dirichlet's formula*

$$h(D)=\frac{w}{2|D|}\sum_{0<r<|D|}\left(\frac{D}{r}\right)r \tag{3.1.2}$$

where

$$w=\begin{cases} 6, & \text{if } D=-3,\\ 4, & \text{if } D=-4,\\ 2, & \text{otherwise,}\end{cases} \tag{3.1.3}$$

and $\left(\frac{D}{r}\right)$ *is the Kronecker (i.e. generalized Legendre) symbol.*

Let K be an odd binary lattice of an odd square-free determinant d where $d \equiv -1 \mod 4$ (i. e. it is fundamental, but it is not classical fundamental). The maximal even sublattice of K is the lattice $T(2)$ where T is the fundamental (even) lattice of the determinant d. Thus, $T(2) \subset K$ is an odd overlattice of index 2 of the lattice $T(2)$. If $d \equiv -1 \mod 8$, then the overlattice K is unique. If $d \equiv -5 \mod 8$, then there are 3 overlattices $T(2) \subset K$ (which we should consider up to proper automorphisms of T to get the number of proper classes of K). We have $\operatorname{discr} K = -4d$ and denote $h(-4d)$ the number of proper classes of lattices K of that discriminant. By considerations above, we get

$$h(4D) = \begin{cases} h(D), & \text{if } D \equiv 1 \mod 8, \\ 3h(D), & \text{if } D \equiv 5 \mod 8 \text{ and } D < -3, \\ 1 = h(D), & \text{if } D = -3. \end{cases} \tag{3.1.4}$$

Let K be a fundamental positive binary lattice of the discriminant D. Here $D \equiv 0, 1 \mod 4$. If $D \equiv 0 \mod 4$, then $D \equiv \pm 4, 8 \mod 16$. We use the discriminant form technique [N1]. The genus of K is defined by the discriminant form of K (it is quadratic if K is even, and is bilinear if K is odd) which is defined by the map $\mu : p \mapsto \mu_p \in \{0, 1\}$, p runs through all odd prime $p|D$. For odd (prime) $p|D$, the discriminant form $b_{K_p} = b_{\theta_p}^{(p)}(p)$ where $\left(\frac{\theta_p}{p}\right) = (-1)^{\mu_p}$. The form b_{K_2} is trivial if either $D \equiv 1 \mod 4$ or $D \equiv 4 \mod 8$. If $D \equiv 8 \mod 16$, then $b_{K_2} = b_1^{(2)}(2)$.

We denote by *t the number of prime divisors of D*. If $D \equiv 1 \mod 4$ (equivalently, K is even), existence of K with the invariants (D, μ) is equivalent to

$$\sum_{p|d} [(1-p) + 4\mu_p] \equiv 2 \mod 8, \tag{3.1.5}$$

and there are 2^{t-1} genuses.

If $D \equiv -4 \mod 16$, (equivalently, $d \equiv 1 \mod 4$) existence of K with the invariants (D, μ) is equivalent to $d \equiv 1 \mod 4$ which is given. Thus, number of genuses is equal to 2^{t-1}.

If $D \equiv 4 \mod 16$ (equivalently, $d \equiv -1 \mod 4$ and the lattice K is odd), existence of K is equivalent to existence of the corresponding even lattice of the discriminant $D = -d$ (by the construction above), and number of genuses is equal to 2^{t-2}.

If $D \equiv 8 \mod 16$, there are no conditions on (D, μ) for existence of K, and number of genuses is equal to 2^{t-1}.

Thus, *the number of different genuses of the discriminant D is equal to* $2^{\tau(D)}$ where

$$\tau(D) = \begin{cases} t-1, & \text{if } D \equiv 1 \mod 4, \\ t-1, & \text{if } D \equiv -4, 8 \mod 16, \\ t-2, & \text{if } D \equiv 4 \mod 16, \end{cases} \tag{3.1.6}$$

and t is the number of all different prime divisors of D. It is known that each genus contains the same number of classes. Thus, number of classes $h(D)_g$ in a genus of a fundamental lattice K is equal to

$$h(D)_g = \frac{h(D)}{2^{\tau(D)}}. \tag{3.1.7}$$

A fundamental binary lattice K is called *ambiguous* if $O(K)$ contains a reflection (equivalently, the oriented lattice K is proper equivalent to the lattice K with the opposite orientation). If $O(K)$ contains a reflection in $\delta_1 \in K$ where δ_1 is primitive, then it contains reflection in a primitive $\delta_2 \in K$ which is orthogonal to δ_1. We can suppose that $n_1 = \delta_1^2 \le n_2 = \delta_2^2$. It follows that a reflective lattice K is generated by either $\{\delta_1, \delta_2\}$ or $\{\delta_1, \delta_2, (\delta_1 + \delta_2)/2\}$. We say that the *type is I* if we have the first possibility, and the *type is II* if we have the second one. It is easy to see (using classification of 2-dimensional reflection groups) that $n_1 \le n_2$ and type are invariants of the ambiguous class K except $K = (1,1;I) = (2,2;II)$ of the discriminant -4. Except this case, two ambiguous binary lattices are isomorphic iff they have the same invariants $(n_1, n_2; \text{type})$. We denote by $K(n_1, n_2; \text{type})$ the ambiguous positive lattice with invariants $(n_1, n_2; \text{type})$. A lattice $K(n_1, n_2; I)$ exists for any natural $n_1,\ n_2 \in \mathbb{N}$. A lattice $K(n_1, n_2; II)$ exists iff $n_1, n_2 \in \mathbb{N}$, $n_1 \equiv n_2 \equiv 0 \mod 2$ and $n_1 + n_2 \equiv 0 \mod 4$.

We have $\det K(n_1, n_2; I) = n_1 n_2$, and the ambiguous binary lattice $K(n_1,\ n_2; I)$ is fundamental if and only if both n_1 and n_2 are square-free and $(n_1,\ n_2) = 1$. This lattice is odd of the discriminant $D = -4n_1 n_2$.

We have $\det K(n_1, n_2; II) = n_1 n_2/4$, and $K(n_1, n_2; II)$ is fundamental iff both n_1, n_2 are square-free and $(n_1, n_2) = 2$. The lattice $K(n_1, n_2; II)$ is even iff $n_1 + n_2 \equiv 0 \mod 8$. Thus, we have

$$D(K(n_1, n_2; II)) = \begin{cases} -n_1 n_2/4, & \text{if } n_1 + n_2 \equiv 0 \mod 8 \\ -n_1 n_2, & \text{otherwise.} \end{cases}$$

It follows that the number $hr(D)$ of ambiguous classes (proper or improper, it does not matter because they are improper equivalent to itself) of the discriminant D is equal to

$$hr(D) = 2^{\tau(D)} = \begin{cases} 2^{t-1}, \text{if } D \equiv 1 \mod 4, \\ 2^{t-1}, \text{if } D \equiv -4 \mod 16, \\ 2^{t-2}, \text{if } D \equiv 4 \mod 16, \\ 2^{t-1}, \text{if } D \equiv 8 \mod 16. \end{cases} \tag{3.1.8}$$

This number is equal to the number $2^{\tau(D)}$ of genuses because ambiguous lattices correspond to elements of order 2 in the group A of classes of discriminant D. For a class $a \in A$, the class $-a \in A$ is the class which is improper equivalent to a. We denote the group of ambiguous classes as A_2. The group of genuses is $A/2A$, and natural homomorphism $A \to A/2A$ is the genus map.

Further we will be especially interesting in genuses which do not contain more than one class of non-ambiguous lattices with respect to the general (i.e. proper or improper) equivalence. A genus is called *ambiguous* if it contains an ambiguous lattice. Otherwise it is called *non-ambiguous.*

Using the genus homomorphism, it is easy to see that a discriminant D contains a non-ambiguous genus iff $2^{\tau(D)+1} | h(D)$. If $2^{\tau(D)+1} | h(D)$, the number of classes of general equivalence (i.e. proper or improper) in a non-ambiguous genus is equal to $\frac{h(D)}{2^{\tau(D)+1}}$. Thus, we have

Lemma 3.1.2. *There exists a non-ambiguous genus of the discriminant D iff $2^{\tau(D)+1} | h(D)$. The number of classes of the general equivalence (i.e. proper or improper) in a non-ambiguous genus is equal to $h(D)/2^{\tau(D)+1}$. In particular, a* *non-ambiguous genus contains exactly one class of general equivalence iff* $h(D) = 2^{\tau(D)+1}$.

Number of ambiguous classes in an ambiguous genus is equal to $\sharp 2A \cap A_2$. By elementary considerations with finite Abelian groups, we get

Lemma 3.1.3. *An ambiguous genus contains only ambiguous classes iff* $h(D) = 2^m$ *where* $m \le 2\tau(D)$, *and the class group* $A \cong (\mathbb{Z}/4)^{m-\tau(D)} \oplus (\mathbb{Z}/2)^{2\tau(D)-m}$. *Equivalently,* $h(D) = 2^m$ *where* $m \le 2\tau(D)$, *and the number of ambiguous classes in the principal genus* $2A$ *is equal to* $2^{m-\tau(D)}$.

Lemma 3.1.4. *An ambiguous genus contains exactly one non-ambiguous class iff either* $h(D) = 2^{\tau(D)} \cdot 3$ *or* $h(D) = 2^{\tau(D)+2}$ *and* $A \cong (\mathbb{Z}/8) \oplus (\mathbb{Z}/2)^{\tau(D)-1}$. *We remark that if* $h(D) = 2^{\tau(D)+2}$, *then either* D *is of the type of Lemma 3.1.3 (when an ambiguous genus contains ambiguous classes only) or* $A \cong (\mathbb{Z}/8) \oplus (\mathbb{Z}/2)^{\tau(D)-1}$ *and an ambiguous genus contains only one class of general equivalence of non-ambiguous lattices. The last case is characterized by the property that the principal genus* $2A$ *contains exactly two ambiguous classes.*

Below we calculate numbers $hnr(D,\mu)$, $hr_I(D,\mu)$, $hr_{II}(D,\mu)$ and $hr(D,\mu)$ of non-ambiguous classes of general equivalence, ambiguous classes of type I, ambiguous classes of type II and ambiguous classes respectively of the genus (D,μ). We have

$$hr(-4,\mu) = hr_I(-4,\mu) = hr_{II}(-4,\mu) = 1 \text{ and } hnr(-4,\mu) = 0. \tag{3.1.9}$$

If $D \ne -4$, we have

$$hr(D,\mu) = hr_I(D,\mu) + hr_{II}(D,\mu) \tag{3.1.10}$$

and

$$hnr(D,\mu) = (h(D)/2^{\tau(D)} - hr_I(D,\mu) - hr_{II}(D,\mu))/2. \tag{3.1.11}$$

Below we calculate $hr_I(D,\mu)$ and $hr_{II}(D,\mu)$. Since the principal genus pr is ambiguous, for the number of non-ambiguous classes in an ambiguous genus $ambig$ we get

$$hnr(D, ambig) = (h(D)/2^{\tau(D)} - hr_I(D,pr) - hr_{II}(D,pr))/2. \tag{3.1.12}$$

We remind (Lemma 3.1.2) that for a non-ambiguous genus $nambig$ we have

$$hnr(D, nambig) = \begin{cases} 0, & \text{if } 2^{\tau(D)+1} \not| h(D), \\ \frac{h(D)}{2^{\tau(D)+1}}, & \text{if } 2^{\tau(D)+1} | h(D). \end{cases} \tag{3.1.13}$$

Case $D \equiv 1 \mod 4$. Then the determinant $d = -D \equiv -1 \mod 4$. All ambiguous classes of the genus (D,μ) are given by $K(2d_1, 2d_2; II)$ such that $d_1 d_2 = d$ and $\left(\frac{2d_1/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_2$. Thus,

$$\begin{aligned} hr_{II}(D,\mu) = \sharp\{d_1|d \ \mid\ d_1 \le d/d_1 \ \&\ \left(\frac{2d_1/p}{p}\right) &= (-1)^{\mu_p}\ \forall p|d_1 \\ \&\ \left(\frac{2(d/d_1)/p}{p}\right) &= (-1)^{\mu_p}\ \forall p|(d/d_1)\}, \end{aligned} \tag{3.1.14}$$

and

$$hr_I(D,\mu)=0. \tag{3.1.15}$$

The principal genus pr is given by the ambiguous lattice $K(2,2d;II)$. All ambiguous classes of that genus are $K(2d_1,2d_2;II)$ such that $d_1d_2=d$ and $\left(\frac{2d_1/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right)=\left(\frac{2d_1d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_{II}(D,pr)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1\ \&\left(\frac{d_1}{p}\right)=1\ \forall p|(d/d_1)\} \tag{3.1.16}$$

and

$$hr_I(D,pr)=0. \tag{3.1.17}$$

Case $D\equiv 4 \mod 16$. Then $d=-D/4\equiv -1 \mod 4$. All ambiguous classes of the genus (D,μ) are given by $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$\begin{aligned}hr_I(D,\mu)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\&\ &\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1\\ &\&\left(\frac{(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\},\end{aligned} \tag{3.1.18}$$

and

$$hr_{II}(D,\mu)=0. \tag{3.1.19}$$

The principal genus is given by the ambiguous lattice $K(1,d;I)$. All ambiguous classes of that genus are $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=\left(\frac{d_1d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_I(D,pr)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ \left(\frac{d/d_1}{p}\right)=1\ \forall p|d_1\ \&\left(\frac{d_1}{p}\right)=1\ \forall p|(d/d_1)\}, \tag{3.1.20}$$

and

$$hr_{II}(D,pr)=0. \tag{3.1.21}$$

Case $D\equiv -4 \mod 16$. Then $d=-D/4\equiv 1 \mod 4$. All ambiguous classes of type I of the genus (D,μ) are given by $K(d_1,d_2;I)$ such that $d_1d_2=d$ and $\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right)=(-1)^{\mu_p}$ if $p|d_2$. Thus,

$$\begin{aligned}hr_I(D,\mu)=\sharp\{d_1|d\ \mid\ d_1\le d/d_1\ \&\ &\left(\frac{d_1/p}{p}\right)=(-1)^{\mu_p}\ \forall p|d_1\\ &\&\left(\frac{(d/d_1)/p}{p}\right)=(-1)^{\mu_p}\ \forall p|(d/d_1)\}.\end{aligned} \tag{3.1.22}$$

All ambiguous classes of the type II of (D, μ) are given by $K(2d_1, 2d_2; II)$ such that $d_1 d_2 = d$ and $\left(\frac{2d_1/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_{II}(D, \mu) = \sharp\{d_1|d \mid d_1 \leq d/d_1 \ \& \ \left(\frac{2d_1/p}{p}\right) = (-1)^{\mu_p} \ \forall p|d_1 \ \& \left(\frac{2(d/d_1)/p}{p}\right) = (-1)^{\mu_p} \ \forall p|(d/d_1)\}. \tag{3.1.23}$$

The principal genus is given by the ambiguous lattice $K(1, d; I)$. All ambiguous classes of the type I of that genus are $K(d_1, d_2; 1)$ such that $d_1 d_2 = d$ and $\left(\frac{d_1/p}{p}\right) = \left(\frac{d_1 d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{d_2/p}{p}\right) = \left(\frac{d_1 d_2/p}{p}\right)$ if $p|d_2$. Thus

$$hr_I(D, pr) = \sharp\{d_1|d \mid d_1 \leq d/d_1 \ \& \ \left(\frac{d/d_1}{p}\right) = 1 \ \forall p|d_1 \& \left(\frac{d_1}{p}\right) = 1 \ \forall p|(d/d_1)\}. \tag{3.1.24}$$

All ambiguous classes of the type II of the principal genus are $K(2d_1, 2d_2; II)$ such that $d_1 d_2 = d$ and $\left(\frac{2d_1/p}{p}\right) = \left(\frac{d_1 d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right) = \left(\frac{d_1 d_2/p}{p}\right)$ if $p|d_2$. Thus

$$hr_{II}(D, pr) = \sharp\{d_1|d \mid d_1 \leq d/d_1 \ \& \ \left(\frac{d/d_1}{p}\right) = \left(\frac{2}{p}\right) \ \forall p|d_1 \ \& \left(\frac{d_1}{p}\right) = \left(\frac{2}{p}\right) \ \forall p|(d/d_1)\}. \tag{3.1.25}$$

Case $D \equiv 8 \mod 16$. Then $d = -D/4 \equiv 2 \mod 4$. All ambiguous classes of the genus (D, μ) are given by $K(d_1, 2d_2; I)$ such that $d_1 d_2 = d/2$ and $\left(\frac{d_1/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right) = (-1)^{\mu_p}$ if $p|d_2$. Thus,

$$hr_I(D, \mu) = \sharp\{d_1|d/2 \mid \& \ \left(\frac{d_1/p}{p}\right) = (-1)^{\mu_p} \ \forall p|d_1 \ \& \left(\frac{(d/d_1)/p}{p}\right) = (-1)^{\mu_p} \ \forall p|(d/2d_1)\}, \tag{3.1.26}$$

and

$$hr_{II}(D, \mu) = 0. \tag{3.1.27}$$

The principal genus is given by the ambiguous lattice $K(1, d; 1)$. All ambiguous classes of that genus are $K(d_1, 2d_2; I)$ such that $d_1 d_2 = d/2$ and $\left(\frac{d_1/p}{p}\right) = \left(\frac{2d_1 d_2/p}{p}\right)$ if $p|d_1$ and $\left(\frac{2d_2/p}{p}\right) = \left(\frac{2d_1 d_2/p}{p}\right)$ if $p|d_2$. Thus,

$$hr_I(D, pr) = \sharp\{d_1|(d/2) \mid \left(\frac{d/d_1}{p}\right) = 1 \ \forall p|d_1 \ \& \left(\frac{d_1}{p}\right) = 1 \ \forall p|(d/2d_1)\} \tag{3.1.28}$$

and

$$hr_{II}(D, pr) = 0. \tag{3.1.29}$$

Using considerations above, In Appendix 1 (to Part I) we give Program 1: $h2$ for "GP/PARI" calculator which for a fundamental discriminant $D<0$ and the genus (D,μ) calculates the vector

$$(hr_I(D,\mu),\ hr_{II}(D,\mu),\ hnr(D,\mu)). \tag{3.1.30}$$

The "GP/PARI" calculator uses the Shanks method [Sha] to calculate the class numbers $h(D)$ of the discriminant D. It is very fast: $O(|D|^{1/4})$ operations. We code the invariant μ by the non-negative integer μ having the binary decomposition

$$\mu=\mu_{p_k}\mu_{p_{k-1}}\dots\mu_{p_1} \tag{3.1.31}$$

where $p_1,\ \dots,\ p_k$ are all odd prime divisors of D in increasing order.

3.2. The number h of non-reflective classes of central symmetries of main hyperbolic lattices of the rank 3.

We denote by S a main hyperbolic (i.e. of the signature $(1,k)$) lattice of the rank 3. Remind that *main* means that the lattice S has square-free determinant $d=\det(S)$ and is even if the determinant d is even. If d is odd, then S is necessarily odd.

By Proposition 2.2.4, the lattice S is defined by its invariants (d,η) where $d=\det(S)$ and the invariant η is the map $\eta: p\mapsto \eta_p\in\{0,\ 1\}$ of the set of all odd prime divisors of d. The η is defined by the condition

$$b_{S_p}\cong b_{\theta_p}^{(p)}(p),\quad \left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}. \tag{3.2.1}$$

Here q_S and b_S denote discriminant forms of the lattice S. Here and in what follows we use discriminant forms technique and notations in [N1]. We will especially often using theorems of existence of a lattice with a given discriminant form (Theorems 1.10.1 and 1.16.5 in [N1]).

For even d the lattice S is even and the discriminant quadratic form $q_{S_2}=q_{\theta_2}^{(2)}(2)$ where $\theta_2\equiv\pm1\mod 4$. We denote

$$\theta_2\mod 8\equiv\pm1\mod 8\ \text{ if }\ \theta_2\equiv\pm1\mod 4. \tag{3.2.2}$$

We have

$$\sum_{odd\ p|d}[(1-p)+4\eta_p]+\theta_2\mod 8\equiv -1\mod 8.$$

Here sign $q_{\theta_p}^{(p)}(p)\equiv(1-p)+4\eta_p\mod 8$ and sign $q_{\theta_2}^{(2)}(2)\equiv\theta_2\mod 8$. Since $\theta_2\mod 8\equiv\pm1\mod 8$, we get

$$\sum_{odd\ p|d}[(1-p)+4\eta_p]\equiv 0,-2\mod 8 \tag{3.2.3}$$

and the invariant θ_2 is defined by

$$\theta_2\mod 8\equiv-\sum_{odd\ p|d}[(1-p)+4\eta_p]-1\mod 8. \tag{3.2.4}$$

Here (3.2.3) is the condition of existence of a main hyperbolic lattice S with the invariants (d,η) if d is even. If d is odd, the lattice S is odd, and there are no conditions of existence of S. It always does exist. Thus, *for any square-free natural number d and any map η of the set of all odd prime divisors of d into $\{0,\,1\}$, there exists a main hyperbolic lattice S of the rank 3 with invariants $(d,\,\eta)$ if and only if*

$$\sum_{odd\ p|d}[(1-p)+4\eta_p]\equiv 0,\,-2 \mod 8 \tag{3.2.5}$$

if d is even.

We consider primitive elements $f\in S$ such that $n=f^2>0$. We consider them up to $\pm f$, thus we can suppose that $f\in V^+(S)$ where $V^+(S)$ is the light-cone of S (see Sect. 1.1). We consider the elements f such that there exists an automorphism u_f of S which is identical on f and is -1 on the orthogonal negative definite lattice $K=f_S^\perp$. Then

$$\text{either } S=\mathbb{Z}f\oplus K \text{ or } S=[\mathbb{Z}f\oplus K,\ (f\oplus k)/2],\ k\in K. \tag{3.2.6}$$

This automorphism is called a *central symmetry* of S. Geometrically, it is the central symmetry at the point $\mathbb{R}_{++}f$ of the hyperbolic space $\mathcal{L}(S)=V^+(S)/\mathbb{R}_{++}$ defined by S. Vice versa, any $\phi\in O^+(S)$ acting as a central symmetry in $\mathcal{L}(S)$, has the form $\phi=u_f$. By considerations over $\mathbb{Z}_2$, one can see (using (3.2.6)) that either n is odd or $n\equiv 2 \mod 4$. (One can also use that $u_f: x\mapsto -x+(2(x,f)/f^2)f$, $x\in S$, and $f^2|2(S,f)$.) Let $n=2^kn_1$ where $(n_1,2)=1$, $k=0,\,1$. Then by (3.2.6),

$$n_1|d, \text{ and } k=1 \text{ if } d \text{ is even, and } \left(\frac{n/p}{p}\right)=(-1)^{\eta_p}\ \forall \text{ odd } p|n. \tag{3.2.7}$$

The central symmetry u_f is called *reflective* if there exists a reflection s_δ of S in $\delta\perp f$ (i.e., $\delta\in K$). Otherwise, u_f is called *non-reflective*. Geometrically, the central symmetry u_f is reflective if and only if the center $\mathbb{R}_{++}f$ belongs to a mirror of the reflection group $W(S)$. We want to calculate the number $hnr(S)$ of non-reflective central symmetries u_f of S up to conjugation in $O(S)$ (i.e., the number of non-reflective classes of the central symmetries). Since ± 1 belongs to the center of $O(S)$, it is sufficient to consider u_f up to $O^+(S)=\{\phi\in O(S)\mid \phi(V^+(S))=V^+(S)\}$. Acting by the reflection group $W(S)$, we can always suppose that the center $\mathbb{R}_{++}f\in\mathcal{M}$ where $\mathcal{M}$ is a fundamental polyhedron of $W(S)$ (see Sect. 1.1). If u_f is non-reflective, it is equivalent that $u_f\in A(P(\mathcal{M})_{\mathrm{pr}})=\{\phi\in O^+(S)\mid \phi(\mathcal{M})=\mathcal{M}\}$. That two non-reflective central symmetries of S are conjugate by $O^+(S)$ if and only if they are conjugate by $A(P(\mathcal{M})_{\mathrm{pr}})$. Thus, the number $h=hnr(S)$ of classes of non-reflective central symmetries of S is the same invariant $h=h(S)$ of S which we have introduced in Sect. 2.3. The central symmetry u_f is defined by the element f (it is defined by u_f up to $\pm f$). Thus, it is sufficient to study all these f up to action of $O(S)$. Obviously, (n,K) are invariants of f. Here $K=f_S^\perp$ is a negative binary lattice which we consider up to isomorphism.

First, we calculate invariants of genus of the central symmetries u_f, $f\in S$ above. Here two elements $f_1\in S$ and $f_2\in S$ have the same *genus* if they are conjugate over $\mathbb{Z}_p$ for all prime p. Then we calculate the number of non-reflective classes u_f using class numbers of binary positive lattices. We had considered them in Sect. 3.1.

Case I: $d \equiv 1 \mod 2$, equivalently S is odd. Then there are two cases.

Case (I, II): $K = f^\perp$ is even. Then n is odd because S is unimodular odd over $\mathbb{Z}_2$, and then f is a characteristic element of S (i.e. $(f, x) \equiv (x, x) \mod 2$ for any $x \in S$). By (3.2.6), we then have $S = \mathbb{Z}f \oplus K$. It follows that $n|d$, $\det K = d/n$, and the discriminant quadratic form q_{K_p}, $p|(d/n)$, is equal to

$$q_{K_p} = q_{S_p} = q_{\theta_p}^{(p)}(p), \quad \left(\frac{\theta_p}{p}\right) = (-1)^{\eta_p}, \quad p|(d/n).$$

By discriminant form technique [N1], existence of K is equivalent to

$$\sum_{p|(d/n)} [(1-p) + 4\eta_p] \equiv -2 \mod 8. \tag{3.2.8}$$

Conditions (3.2.7) and (3.2.8) are equivalent to existence of u_f with $f^2 = n$. The lattice $K(-1)$ is an even fundamental positive binary lattice of the discriminant $\operatorname{discr} K = -\det K = -d/n$. Thus, $-d/n \equiv 1 \mod 4$. The genus of $K(-1)$ is equal to $(-d/n, \epsilon(p) + \eta_p)$. Here $\left(\frac{-1}{p}\right) = (-1)^{\epsilon(p)}$, and it is known that $\epsilon(p) \equiv (p-1)/2 \mod 2$. To be shorter, here we denote by $\epsilon(p) + \eta_p$ the map $p \mapsto \epsilon(p) + \eta_p$ where p runs through all odd prime $p|(d/n)$. It follows that the number of non-reflective classes of central symmetries $f \in S$ of the type (I, II) is equal to

$$hnr_{II}(S) = \sum_{\substack{n|d\ \&\ (3.2.7) \\ \&\ (3.2.8)}} hnr(-d/n, \epsilon(p) + \eta_p), \tag{3.2.9}$$

where we consider the sum by all n such that $n|d$ and (3.2.7), (3.2.8) are valid.

Case (I, I): $K = f^\perp$ is odd. By (3.2.6), (3.2.7), then $n|2d$ and $\det K = (n, 2)^2 d/n$, $\operatorname{discr} K = -4(n, 2)^2 d/n$. It follows that $K(-1)$ is a fundamental positive binary lattice.

For odd $p|\det K$, we have

$$q_{K_p} = q_{S_p} = q_{\theta_p}^{(p)}(p), \quad \left(\frac{\theta_p}{p}\right) = (-1)^{\eta_p},$$

and

$$b_{K_2} = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ b_1^{(2)}(2), & \text{if } n \text{ is even} \end{cases}.$$

If n is odd, existence of K is equivalent to

$$-2 \mod 8 \in \sum_{p|(d/n)} [(1-p) + 4\eta_p] + 4\omega(d/n) + \{0, \pm 2\} \mod 8.$$

It is equivalent to

$$\sum_{p|(d/n)} [(1-p) + 4\eta_p] + 4\omega(d/n) \not\equiv 2 \mod 8. \tag{3.2.10}$$

Here $\omega(k) \equiv (k^2 - 1)/8 \mod 2$, it is known that $\left(\frac{2}{p}\right) = (-1)^{\omega(p)}$. For even n there are no conditions of existence of K. The lattice K does always exist. The genus of $K(-1)$ is equal to $(-4(n, 2)^2 d/n, \epsilon(p) + \eta_p)$.

Thus, this case is characterized by the condition: d is odd, $n|2d$, we have $\mathbb{Z}f\oplus K\subset S$ where K is an odd fundamental binary lattice of the discriminant $-4(n,2)^2d/n$. If n is odd, $\mathbb{Z}f\oplus K=S$. If n is even, the lattice S is generated by $\mathbb{Z}f\oplus K$ and $u=(f+k)/2$ where $k\in K$ is a primitive element with the property: $(k,K)\equiv 0 \mod 2$. The element $k \mod 2K$, and the overlattice S are defined uniquely. For both these cases, any automorphism of K can be extended to the automorphism of S identical on f. It follows that the number of non-reflective classes of f of the type (I,I) is equal to

$$hnr_I(S)=\sum_{\substack{n|d\ \&\ (3.2.7)\\ \&\ (3.2.10)}} hnr(-4d/n,\epsilon(p)+\eta_p)+\sum_{\substack{n=2n_1|2d\\ \&(3.2.7)}} hnr(-16d/n,\epsilon(p)+\eta_p). \tag{3.2.11}$$

As a result, we get that for the case I (equivalently, when d is odd) the full number of non-reflective classes of central symmetries $f\in S$ is equal to

$$\begin{aligned} hnr(S)=&\sum_{\substack{n|d\ \&\ (3.2.7)\ \&\\ (3.2.8)}} hnr(-d/n,\epsilon(p)+\eta_p)\\ &+\sum_{\substack{n|d\ \&\ (3.2.7)\\ \&\ (3.2.10)}} hnr(-4d/n,\epsilon(p)+\eta_p)+\sum_{\substack{n=2n_1|2d\\ \&\ (3.2.7)}} hnr(-16d/n,\epsilon(p)+\eta_p). \end{aligned} \tag{3.2.12}$$

Case II: d is even, equivalently, the lattice S is even. Then $n=2n_1$ where $n_1|(d/2)$. This case is also divided in two cases:

Case $(\mathrm{II},\mathrm{II})$: the lattice $K=f^\perp$ has an odd determinant. By (3.2.6), we have $S=\mathbb{Z}f\oplus K$. Then $q_{S_2}=q^{(2)}_{n/2}(2)$ and $n_1=n/2\equiv\theta_2 \mod 4$. By (3.2.4),

$$n/2\equiv-\sum_{odd\ p|d}[(1-p)+4\eta_p]-1 \mod 4. \tag{3.2.13}$$

We have $\det K=d/n$ and, for $p|(d/n)$,

$$q_{K_p}=q_{S_p}=q^{(p)}_{\theta_p}(p),\quad \left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}.$$

Existence of K is equivalent to $\sum_{p|(d/n)}[(1-p)+4\eta_p]\equiv-2 \mod 8$. This follows from the condition (3.2.2) (or (3.2.3)) of existence of S and (3.2.7), (3.2.13). Thus, condition of existence of K for this case is (3.2.13) together with (3.2.7).

Thus, this case is characterized by the condition: d is even, $n|d$ is even, $S=\mathbb{Z}f\oplus K$, where $K(-1)$ is an even fundamental binary lattice of the genus $(-d/n,\epsilon(p)+\eta_p)$. Thus, the number of non-reflective classes f of the type $(\mathrm{II},\mathrm{II})$ is equal to

$$hnr_{II}(S)=\sum_{\substack{n=2n_1|d\ \&\ (3.2.7)\ \&\\ (3.2.13)}} hnr(-d/n,\epsilon(p)+\eta_p). \tag{3.2.14}$$

Case (II,I): the lattice $K=f^\perp$ has even determinant. The discriminant form $q_{[f]_2}=q^{(2)}_{n/2}(2)$. By (3.2.6), we have

$$q_{K_2}=q^{(2)}_{-n/2}(2)\oplus q^{(2)}_{\theta_2}(2).$$

It follows that the lattice $K(1/2)$ is odd fundamental, K has determinant $4d/n$, and for $p|(d/n)$ one has

$$q_{K_p}=q_{S_p}=q_{\theta_p}^{(p)}(p),\quad \left(\frac{\theta_p}{p}\right)=(-1)^{\eta_p}.$$

Existence of the lattice K is equivalent to existence of the lattice S. The odd fundamental binary lattice $K(-1/2)$ has the determinant d/n, and the discriminant $-4d/n$, and for $p|(d/n)$ one has

$$b_{K(1/2)_p}=b_{2\theta_p}^{(p)}(p),\quad \left(\frac{2\theta_p}{p}\right)=(-1)^{\eta_p+\omega(p)}.$$

Thus, for this case, d is even, $f^2=n|d$ is even, $K=T(-2)$ where T is an odd fundamental positive binary lattice of the discriminant $-4d/n$ and of the genus $(-4d/n,\epsilon(p)+\omega(p)+\eta_p)$. The lattice S is an overlattice $\mathbb{Z}f\oplus K\subset S$ of the index 2 generated by $(f+k)/2$ where $k\in K$ satisfies $(k,k)\equiv -n \mod 8$. If $d/n\equiv -1 \mod 4$, then $k \mod 2K$ and S are unique. If $d/n\equiv 1 \mod 4$, there are exactly two different elements k which give the same lattice S. If discr $T=-4$, elements $k \mod 2K$ are conjugate by $O(K)$. If discr $T<-4$, elements $k \mod 2K$ are not conjugate and give different classes of $f\in S$. Thus, further we consider two cases:

Case $d/n\equiv -1 \mod 4$. The number of such non-reflective classes $f\in S$ of the type (II, I) is equal to

$$hnr_{I,-1}(S)=\sum_{\substack{n=2n_1|d\ \&\ (3.2.7)\&\\ n_1\equiv -d/2 \mod 4}} hnr(-4d/n,\epsilon(p)+\omega(p)+\eta_p). \tag{3.2.15}$$

Case $d/n\equiv 1 \mod 4$. Then the class $f\in S$ is non-reflective if and only if the lattice K does not have a reflection which can be extended identically on $\mathbb{Z}f$ to give an automorphism of S. If K is non-ambiguous, then $O(K)=\{\pm 1\}$, and we then get two classes of f corresponding to two different choices of $k \mod 2K$. If $d/n\neq 1$ and T is ambiguous of the type II, then $O(K)$ has order 4, only $\pm 1\in O(K)$ preserve the element $k \mod 2K$, and reflections of K change places two possible different elements $k \mod 2K$. Thus we get exactly one non-reflective class $f\in S$. If $d/n\neq 1$ and T is ambiguous of type I, then reflections of T are identical on $K/2K$ and can be extended on S identically on f. Thus, these T give reflective classes $f\in S$. If $d/n=1$, the lattice $T=K(1,1;I)=K(2,2;II)$, and it has reflections of both types I and II. This case does not give non-reflective classes $f\in S$. Thus, for this case, the number of non-reflective classes $f\in S$ of type (II, I) is equal to

$$hnr_{I,1}(S)=\sum_{\substack{n=2n_1|d\ \&\ (3.2.7)\ \&\\ n_1\equiv d/2 \mod 4\ \&\ n<d}}[2hnr(-4d/n,\epsilon(p)+\omega(p)+\eta_p)+hr_{II}(-4d/n,\epsilon(p)+\omega(p)+\eta_p)]. \tag{3.2.16}$$

As a result, we get for the case II (equivalently, when d is even) that the full number of non-reflective classes of central symmetries $f \in S$ is equal to

$$hnr(S) = \sum_{\substack{n=2n_1|d \ \& \ (3.2.7) \ \& \\ (3.2.13)}} hnr(-d/n, \epsilon(p)+\eta_p)$$
$$+ \sum_{\substack{n=2n_1|d \ \& \ (3.2.7)\& \\ n_1\equiv -d/2 \mod 4}} hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p)$$
$$+ \sum_{\substack{n=2n_1|d \ \& \ (3.2.7) \ \& \\ n_1\equiv d/2 \mod 4 \ \& \ n<d}} [2hnr(-4d/n, \epsilon(p)+\omega(p)+\eta_p)+hr_{II}(-4d/n, \epsilon(p)+\omega(p)+\eta_p)]. \tag{3.2.17}$$

As a final result, we get

Theorem 3.2.1. *Let d be a square-free natural number and $\eta : p \mapsto \{0, 1\}$ a map of all odd prime divisors $p|d$ into $\{0, 1\}$. There exists a main hyperbolic lattice S of the rank 3 with determinant d and the invariant η) (see (3.2.1)) if and only if for the even d the congruence (3.2.5) is valid. The number $h = hnr(S) = hnr(d, \eta)$ of classes of non-reflective central symmetries of the lattice S with the invariants (d, η) is given by (3.2.12) for the odd d and by (3.2.17) for the even d.*

Below we will code the invariant η by the non-negative integer η having the binary decomposition

$$\eta = \eta_{p_t} \dots \eta_{p_1} \tag{3.2.18}$$

where $p_1, \dots, p_t$ are all odd prime divisors of d in increasing order.

In Appendix 1 (to Part I): we give the Program 2: $h3$ for "GP/PARI" calculator which using Theorem 3.2.1 and Program 1: $h2$ (see Sect. 3.1), calculates the invariant $h = h(S) = hnr(d, \eta)$ if a main hyperbolic lattice S of the rank 3 with the invariants (d, η) does exist (otherwise, the result will be unreasonable). Using the first statement of the Theorem 3.2.1 and the Program 2, in Appendix 1 (to Part I) we give Program 3: $refh3$ which gives all pairs of invariants (d, η) such that $d \le N$, there exists a main hyperbolic lattice of the rank 3 with the invariants (d, η) and the invariant $hnr(d, \eta) \le 1$. Using Program 3, we found all these pairs (d, η) such that $d \le 100000$. The result is given in Table 3 and contains 206 lattices. Thus, we get

Theorem 3.2.2. *Table 3 gives the complete list (it has 206 lattices) of main hyperbolic lattices S of the rank 3 and with the determinant $d \le 100000$ such that the invariant $hnr(S) \le 1$. In Table 3 we give the invariants d, η, $h = hnr(S)$, the matrix of the lattice S and the reflective type of the lattice S (see Sect. 5).*

The greatest d for lattices S of the Table 3 is equal to 4466 in spite we did calculations up to $d = 100000$. Therefor, it is very likely that we have the statement

Conjecture 3.2.3. *Table 3 gives the complete list of main hyperbolic lattices S of the rank 3 with the invariant $hnr(S) \le 1$.*

In Sect. 5, we shall use Theorem 3.2.2 to find all the invariants (d, η) corresponding to elliptically or parabolically reflective main hyperbolic lattices S since for them the invariant $h \le 1$.

In Part II, Table 6, similarly we calculated all pairs (d, η) corresponding to main hyperbolic lattices of the rank three with $hnr(d, \eta) = 2$ and $d \le 100000$. (One should change in two places of Program 3 $h \le 1$ by $h = 2$.) Table 6 contains 259 pairs (d, η). The largest $d = 29526$. Again it is very likely that Table 6 contains all pairs (d, η) with $h = 2$. In Part II we use Tables 3 and 6 to find all hyperbolically reflective main hyperbolic lattices of the rank 3. All of them must have the invariant $h = 0$ or 2.

4. Narrow parts of elliptic and parabolic convex polygons on hyperbolic plane, types of polygons. Application to reflective lattices of the rank three

4.1. Narrow parts of elliptic convex polygons on the hyperbolic plane. We remind (see Sect. 1.1) that a convex polyhedron in a hyperbolic space is called *elliptic* if it is a convex envelope of a finite set of points (some of them at infinity) and it is non-degenerate. In this section we shall consider only elliptic (i.e. ordinary finite) convex polyhedra and often shall omit the word "elliptic".

Here we follow the general *method of narrow parts of polyhedra* suggested in [N4], [N5] for proving finiteness results about arithmetic reflection groups in hyperbolic spaces. On the other hand, we shall prove much more delicate and exact statements, which are important for exact classification. Estimates in [N4], [N5] were universal, they did not depend on angles of polyhedra. Here we get estimates which depend on angles of polyhedra which makes the narrow parts of polyhedra method much more efficient. Our estimates here are optimal, we believe that one cannot improve them.

We restrict by 2-dimensional case of narrow parts of elliptic polygons on the hyperbolic plane, but all results can be easily generalized (like in [N4], [N5]) on elliptic polyhedra of arbitrary dimension in hyperbolic spaces.

We shall often use the following trivial but important for us statement (certainly, it is well-known):

Lemma 4.1.1. $\frac{\sin y}{\sin x} < \frac{y}{x}$ *if* $0 < x \le \pi/2$ *and* $x < y$.

Proof. The function $sinx/x$ is decreasing if $0 \le x \le \pi/2$. In particular, $1 \ge sinx/x \ge 2/\pi$ if $0 \le x \le \pi/2$. It follows the lemma for $0 < x \le y \le \pi/2$. If $0 < x \le \pi/2 \le y$, we get

$$\frac{\sin y}{\sin x} \le \frac{1}{sinx} \le \frac{1}{x(2/\pi)} \le \frac{\pi/2}{x} \le \frac{y}{x}.$$

This proves the statement.

Lemma 4.1.2. *Let (AB) and (CD) are two lines on a hyperbolic plane with terminals A, B, C, D at infinity, and O a point on the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles AOB and COD coincide. We consider angles $\theta_1 = AOB$, $\theta_2 = COD$ and $\theta_{12} = BOC$. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$.*

Then

$$(\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1+\theta_{12}}{2}\sin\frac{\theta_2+\theta_{12}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2.$$

As a corollary, we get :

1) If lines (AB) and (CD) do not intersect each other, then

$$2\cosh\rho = (\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1+\theta_{12}}{2}\sin\frac{\theta_2+\theta_{12}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2$$

where ρ is the distance between lines (AB) and (CD) (here and in what follows we normalize the curvature $\kappa = -1$).

2) If lines (AB) and (CD) define an angle α containing O, then

$$2\cos\alpha = (\delta_1, \delta_2) = 4\frac{\sin\frac{\theta_1-\theta_{21}}{2}\sin\frac{\theta_2-\theta_{21}}{2}}{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}} - 2$$

where $\theta_{21} = -\theta_{12} = COB$.

Proof. We can correspond to two lines (AB), (CD) and a connected component (containing O) of the complement to these two lines in the hyperbolic plane two invariants of the group of motions of the hyperbolic plane. First invariant is equal to (δ_1, δ_2) and uses Klein model of the hyperbolic plane. Second invariant is equal to the cross ratio $[A : D : C : B]$ where we suppose that orientations of the triangles ABO and CDO coincide. This invariant uses Poincaré model of the hyperbolic plane. On the other hand, it is clear that any of these two invariants defines the triplet $((AB)$, (CD), the connected component to their complement (containing O)) up to motions of the hyperbolic plane. It follows that there exists a function $f(x)$ such that $(\delta_1, \delta_2) = f([A : D : C : B])$. One can check that $f(x) = 4x - 2$.

Below we shall consider an equation

$$(u - x)(v - x) = auv \tag{4.1.1}$$

where $u, v \ge 0$ and $0 \le a = \cos^2\frac{\alpha}{2} = (1 + \cos\alpha)/2 \le 1$. Equivalently we have the equation

$$x^2 - (u + v)x + uv(1 - a) = 0.$$

Its smallest root $x = g(\alpha, u, v)$ equals

$$x = g(\alpha, u, v) = \frac{u + v - \sqrt{a(u + v)^2 + (1 - a)(u - v)^2}}{2} \tag{4.1.2}$$

where $a = a(\alpha) = \cos^2\frac{\alpha}{2} = (1 + \cos\alpha)/2$. In particular, for $u = v$,

$$g(\alpha, u, u) = u(1 - \sqrt{a}) = u(1 - \cos\frac{\alpha}{2}). \tag{4.1.3}$$

The function $g(\alpha, u, v)$ has the following properties (we shall not use them further but they are important):

Proposition 4.1.3. *$u \ge g(\alpha, u, v) \ge 0$, $v \ge g(\alpha, u, v) \ge 0$, $g(\alpha, u, v)'_u \ge 0$, $g(\alpha, u, v)'_v \ge 0$, $\left(\frac{u-g(\alpha,u,v)}{u}\right)'_u \ge 0$, $\left(\frac{v-g(\alpha,u,v)}{v}\right)'_v \ge 0$.*

Proof. Suppose that $u \ge v$. The expression $(u - x)(v - x)$ equals $uv \ge auv$ if $x = 0$ and $(u - x)(v - x)$ equals 0 if $x = u$. Thus the smallest solution $g(a, u, v)$ of the equation $(u - x)(v - x) = auv$ satisfies $v \ge u \ge g(a, u, v)$.

We have $2g(a,u,v)'_u = 1-(a(u+v)+(1-a)(u-v))/\sqrt{a(u+v)^2+(1-a)(u-v)^2} \geq 0$ if $(a(u+v)+(1-a)(u-v))/\sqrt{a(u+v)^2+(1-a)(u-v)^2} \leq 1$. Equivalently, for $-1 \leq t = (u-v)/(u+v) \leq 1$ we should prove that $(a+(1-a)t)/\sqrt{a+(1-a)t^2} \leq 1$. If $-1 \leq t \leq 0$, this is obvious. For $0 \leq t \leq 1$, we have $(a+(1-a)t^2)^{\frac{3}{2}}\left((a+(1-a)t)/\sqrt{a+(1-a)t^2}\right)'_t = (1-a)(a+(1-a)t^2) - (1-a)t(a+(1-a)t) = (1-a)a(1-t) \geq 0$. It follows that $(a+(1-a)t)/\sqrt{a+(1-a)t^2} \leq (a+(1-a))/\sqrt{a+(1-a)} = 1$.

By definition, we have $(u-g(\alpha,u,v))/u = av/(v-g(\alpha,u,v)$. It follows that $((u-g(\alpha,u,v))/u)'_u \geq 0$ because $g(\alpha,u,v)'_u \geq 0$.

It finishes the proof.

Theorem 4.1.4 (about the narrow part of type (I)). *For any elliptic convex polygon $\mathcal{M}$ on a hyperbolic plane there exist its four consecutive vertices A_0, A_1, A_2 and A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1,\,\delta_2) = 2\cos\alpha_1$, $(\delta_2,\,\delta_3) = 2\cos\alpha_2$ and*

$$\text{either}\quad (\delta_1,\,\delta_3) \leq 2 \quad \text{or} \quad (\delta_1,\,\delta_3) < 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \leq 14 \tag{4.1.4}$$

where $\alpha_1 = A_0A_1A_2$ and $\alpha_2 = A_1A_2A_3$.

Moreover, the Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3\}$ is not connected (i.e. this set is union of two non-empty orthogonal subsets) if and only if $\alpha_1 = \alpha_2 = \frac{\pi}{2}$.

Proof. We take a point O inside of $\mathcal{M} = A_1A_2...A_n$. Let B_{i1} and B_{i2} are terminals at infinity of the line $l_i = (A_{i-1}A_i)$ where B_{i1}, A_{i-1}, A_i and B_{i2} are four consecutive points of the line. We introduce angles $\alpha_i = A_{i-1}A_iA_{i+1}$, $\theta_i = B_{i1}OB_{i2}$ and $\theta_{(i+1)i} = B_{(i+1)1}OB_{i2}$.

By Lemma 4.1.2,

$$2\cos\alpha_i = 4\frac{\sin\frac{\theta_i-\theta_{(i+1)i}}{2}\sin\frac{\theta_{i+1}-\theta_{(i+1)i}}{2}}{\sin\frac{\theta_i}{2}\sin\frac{\theta_{i+1}}{2}} - 2.$$

Equivalently,

$$\frac{\sin\frac{\theta_i}{2}\sin\frac{\theta_{i+1}}{2}}{\sin\frac{\theta_i-\theta_{(i+1)i}}{2}\sin\frac{\theta_{i+1}-\theta_{(i+1)i}}{2}} = \frac{2}{1+\cos\alpha_i}. \tag{4.1.5}$$

By Lemma 4.1.1, we get

$$\frac{(\theta_i-\theta_{(i+1)i})(\theta_{i+1}-\theta_{(i+1)i})}{\theta_i\theta_{i+1}} < \frac{1+\cos\alpha_i}{2} = \cos^2\frac{\alpha_i}{2}. \tag{4.1.6}$$

It follows that

$$\theta_{(i+1)i} > g(\alpha_i,\theta_i,\theta_{i+1}) \tag{4.1.7}$$

(see (4.1.2)).

To prove Theorem, we choose a line l_i with the minimal angle θ_i. Let this line be l_2. Thus

$$\theta_2 = \min_i \theta_i. \tag{4.1.8}$$

It follows that $\theta_1 \ge \theta_2$ and $\theta_3 \ge \theta_2$. We shall then prove the inequalities (4.1.4).

If lines l_1, l_3 intersect, we have $(\delta_1, \delta_3) \le 2$ and (4.1.4) is valid. Suppose that the lines l_1 and l_3 do not intersect.

There exists a line $l_1' = (B_{11}'B_{12}')$ with terminals B_{11}' and B_{12}' at infinity such that the line l_1' is contained in $\mathcal{H}_{-\delta_1}^+$, points B_{11}, B_{11}' and B_{12}, B_{12}' are contained in the same half-planes bounded by l_2, the line l_1' has the same angle $\alpha_1 = B_{11}'A_1'B_{22}$ (as l_1) with the line l_2 (we denote by A_1' their intersection point), and the angle $B_{11}'OB_{12}'$ of l_1' is equal to θ_2. We denote by δ_1' the orthogonal vector to l_1' directed outwards of $\mathcal{M}$ and with $(\delta_1')^2 = -2$, and we denote $\theta_{21}' = B_{21}OB_{12}'$.

Similarly, there exists a line $l_3' = (B_{31}'B_{32}')$ with terminals B_{31}' and B_{32}' at infinity such that the line l_3' is contained in $\mathcal{H}_{-\delta_3}^+$, points B_{31}, B_{31}' and B_{32}, B_{32}' are contained in the same half-planes bounded by l_2, the line l_3' has the same angle $\alpha_2 = B_{21}A_2'B_{32}'$ (as l_3) with the line l_2 (we denote by A_2' their point of intersection), and the angle $B_{31}'OB_{32}'$ of l_3' is equal to θ_2. We denote by δ_3' the orthogonal vector to l_3' directed outwards of $\mathcal{M}$ and with $(\delta_3')^2 = -2$, and we denote $\theta_{32}' = B_{31}'OB_{22}$.

Since the lines l_1 and l_3 do not intersect, by our construction, any interval with terminals at l_1' and l_3' intersects both lines l_1 and l_3. It follows that distance between lines l_1' and l_3' is greater than distance between lines l_1 and l_3. It follows $(\delta_1, \delta_3) \le (\delta_1', \delta_3')$. It is sufficient to prove (4.1.4) for lines l_1' and l_3'. (These geometrical considerations are related with properties in Proposition 4.1.4 of the function $g(\alpha, u, v)$.)

By Lemmas 4.1.1, 4.1.2, and (4.1.7), (4.1.3),

$$(\delta_1, \delta_3) \le (\delta_1', \delta_3') = 4\frac{\sin\frac{\theta_2+\theta_2-\theta_{21}'-\theta_{32}'}{2}\sin\frac{\theta_2+\theta_2-\theta_{21}'-\theta_{32}'}{2}}{\sin\frac{\theta_2}{2}\sin\frac{\theta_2}{2}} - 2 <$$

$$4\frac{(\theta_2+\theta_2-\theta_{21}'-\theta_{32}')(\theta_2+\theta_2-\theta_{21}'-\theta_{32}')}{\theta_2\theta_2} - 2 \ <$$

$$4\frac{(\theta_2+\theta_2 - g(\alpha_1,\theta_2,\theta_2) - g(\alpha_2,\theta_2,\theta_2))(\theta_2+\theta_2 - g(\alpha_1,\theta_2,\theta_2) - g(\alpha_2,\theta_2,\theta_2))}{\theta_2\theta_2} - 2 =$$

$$4(\sqrt{a_1}+\sqrt{a_2})^2 - 2 = 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2$$

where $a_i = cos^2\frac{\alpha_i}{2}$. It proves (4.1.4).

Vectors δ_1, δ_2, δ_3 generate the hyperbolic 3-dimensional vector space defining the hyperbolic plane. Otherwise their lines either have a common point or are orthogonal to one line which is not the case. If the Gram graph of these vectors is not connected, two of these vectors generate a 2-dimensional hyperbolic vector subspace and their graph is connected. Vectors δ_1 and δ_2 cannot generate a hyperbolic 2-dimensional vector subspace because their orthogonal lines have a common point $\mathbb{R}_{++}h$ where $h^2 \ge 0$ and $h \ne 0$. Then $(h, \delta_1) = (h, \delta_2) = 0$. The same is valid for δ_2 and δ_3. Thus the Gram graph of $\{\delta_1, \delta_2, \delta_3\}$ is not connected if and only if $(\delta_1, \delta_2) = (\delta_3, \delta_2) = 0$. Equivalently, $\alpha_1 = \alpha_2 = \frac{\pi}{2}$. It finishes the proof of Theorem 4.1.4.

Theorem 4.1.5 (about narrow parts of types (II) and (III)). *For any elliptic convex polygon $\mathcal{M}$ having more than 3 vertices (i.e. it is different from a triangle) on a hyperbolic plane, one of two possibilities (II) or (III) below is valid:*

(II) There exist its five consecutive vertices A_0, A_1, A_2, A_3 and A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1,\, 2,\, 3$, and

$$\textit{either}\ \ (\delta_1,\, \delta_3) \le 2 \ \ \textit{or}\ \ (\delta_1,\, \delta_3) < 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.1.9}$$

$$(\delta_1,\, \delta_4) <$$

$$4 \max_{0\le t\le 1} \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t} - 2 =$$

$$4 \max\left(\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}\right)^2,\, \frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}}\right)^2 - \frac{1}{4}}{2}\right) - 2 \le 34 \tag{4.1.10}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1,\, 2,\, 3$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1,\, \delta_2,\, \delta_3,\, \delta_4\}$ has a connected Gram graph.

(III). There exist its six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 (where $A_0 = A_5$ if $\mathcal{M}$ is a pentagon) such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3, 4$, and

$$\textit{either}\ \ (\delta_1,\, \delta_3) \le 2 \ \ \textit{or}\ \ (\delta_1,\, \delta_3) < 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.1.11}$$

$$\textit{either}\ \ (\delta_3,\, \delta_5) \le 2 \ \ \textit{or}\ \ (\delta_3,\, \delta_5) < 4(\cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2})^2 - 2 \le 14, \tag{4.1.12}$$

and

$$(\delta_1,\, \delta_5) <$$

$$\begin{aligned}
&4 \max_{0\le t\le s\le 1} \Bigg[\Bigg(\Bigg(\sqrt{a_1 + (1-a_1)s^2} + \sqrt{a_2 + (1-a_2)s^2} + \\
&+ \sqrt{a_3 + a_3(s-t) + \frac{a_3(s-t)^2}{4} + \frac{(1-a_3)(s+t)^2}{4}} + \sqrt{a_4 + (1-a_4)t^2}\Bigg)^2 - \\
&- \frac{(s-t)^2}{4}\Bigg) \Big/ ((1+s)(1+t))\Bigg] - 2 = \\
&4 \max\Bigg[\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2}\right)^2, \\
&\frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}} + \cos\frac{\alpha_4}{2}\right)^2 - \frac{1}{4}}{2},\ 4\Bigg] - 2 \ \le\ 62
\end{aligned} \tag{4.1.13}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$*,* $i = 1, 2, 3, 4$*, and* $a_i = \cos^2\frac{\alpha_i}{2}$*. Moreover, the set* $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ *has a connected Gram graph.*

Proof. Like for the proof of Theorem 4.1.4, we take a point O inside of $\mathcal{M}$ and introduce angles θ_i, $\theta_{(i+1)i}$. We choose such a consecutive numeration of vertices of $\mathcal{M}$ that

$$\theta_2 + \theta_3 = \min_i (\theta_i + \theta_{i+1}) \text{ and } \theta_2 \le \theta_3. \tag{4.1.14}$$

Then $\theta_1 \ge \theta_3 \ge \theta_2$ and, like for the proof of Theorem 4.1.4, we get (4.1.9) and (4.1.11) (see considerations below about cases (II) and (III)).

Below we consider two cases:

Case (ii): $\theta_4 \ge (\theta_2 + \theta_3)/2$. We then prove (4.1.10).

For $1 \ge a_1, a_2, a_3 \ge 0$ and $0 \le t$ we introduce a function

$$f_{p2}(a_1, a_2, a_3, t) =$$

$$= \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t}. \tag{4.1.15}$$

We prove that

$$(\delta_1, \delta_4) < 4 \max_{0\le t\le 1} f_{p2}(a_1, a_2, a_3, t) - 2. \tag{4.1.16}$$

If $(\delta_1, \delta_4) \le 2$, it is true because $4f_{p2}(a_1, a_2, a_3, 1) - 2 = 2((2 + \sqrt{2a_3 + 1/4})^2 - 1/4) - 2 \ge 2((2 + 1/2)^2 - 1/4) - 2 = 10$.

Like for the proof of Theorem 4.1.4, we can suppose that $\theta_1 = \theta_3$ (instead of $\theta_1 \ge \theta_3$) and $\theta_4 = (\theta_2+\theta_3)/2$ (instead of $\theta_4 \ge (\theta_2+\theta_3)/2$). We denote $c = (\theta_2+\theta_3)/2$ and $z = (\theta_3 - \theta_2)/2 \ge 0$. Then $\theta_2 = c - z$ and $\theta_3 = c + z$.

Like for the proof of Theorem 4.1.4, we have

$$((\delta_1, \delta_4) + 2)/4 <$$

$$\frac{(\theta_3 + \theta_2 + \theta_3 - \theta_{21} - \theta_{32} - \theta_{43})(c + \theta_2 + \theta_3 - \theta_{21} - \theta_{32} - \theta_{43})}{\theta_3 c} <$$

$$(\theta_3 + \theta_2 + \theta_3 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, c))\times$$

$$(c + \theta_2 + \theta_3 - g(\alpha_1, \theta_3, \theta_2) - g(\alpha_2, \theta_2, \theta_3) - g(\alpha_3, \theta_3, c))/\theta_3 c =$$

$$(z/2 + \sqrt{a_1c^2 + (1-a_1)z^2} + \sqrt{a_2c^2 + (1-a_2)z^2} + \sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})\times$$

$$(-z/2+\sqrt{a_1c^2 + (1-a_1)z^2}+\sqrt{a_2c^2 + (1-a_2)z^2}+\sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})/(c+z)c =$$

$$\frac{(\sqrt{a_1c^2 + (1-a_1)z^2} + \sqrt{a_2c^2 + (1-a_2)z^2} + \sqrt{\frac{z^2}{4} + a_3cz + a_3c^2})^2 - z^2/4}{(c+z)c} =$$

$$\frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{\frac{t^2}{4} + a_3t + a_3}\right)^2 - t^2/4}{1+t}$$

where $t = z/c$ and $0 \le t \le 1$. It proves (4.1.16).

If one of angles α_1 or α_2 is not $\pi/2$, Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3\}$ is connected. Then Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3,\delta_4\}$ is connected because $\delta_1,\,\delta_2,\,\delta_3$ generate the full 3-dimensional hyperbolic vector space. If $\alpha_1=\alpha_2=\pi/2$ (equivalently, $(\delta_1,\,\delta_2)=(\delta_3,\,\delta_2)=0$), then $\delta_1,\,\delta_3$ generate a hyperbolic 2-dimensional subspace and $(\delta_1,\,\delta_3)\neq 0$. If Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3,\delta_4\}$ is not connected, $(\delta_1,\,\delta_4)=(\delta_3,\,\delta_4)=0$. It follows that $\delta_4=\lambda\delta_2$, $\lambda\in\mathbb{R}$. Then sides of $\mathcal{M}$ orthogonal to δ_2 and δ_4 should coincide. We get a contradiction.

Case (ii) gives polygons $\mathcal{M}$ which satisfy the condition (II) of Theorem 4.1.5. We mention that by Theorem 4.1.4, any quadrangle $\mathcal{M}$ satisfies the case (II) of Theorem 4.1.5. Really, for a quadrangle, $(\delta_1,\delta_4)\le 2$.

Case (iii): $\theta_4\le(\theta_2+\theta_3)/2$ and $\mathcal{M}$ has more than 4 vertices. We denote $(\theta_2+\theta_3)/2=c$.

By (4.1.14), $\theta_1\ge\theta_3$, $\theta_4+\theta_5\ge(\theta_2+\theta_3)=2c$. It follows, $\theta_5\ge c$.

Since $\theta_3\ge c$, $\theta_4\le c$ and $\theta_5\ge c$, like in the proof of Theorem 4.1.4, we get the inequality (4.1.12).

For $1\ge a_1,\,a_2,\,a_3,\,a_4\ge 0$ and $0\le s,t$, we introduce a function

$$f_{p3}(a_1,\,a_2,\,a_3,a_4,\,s,\,t)=$$

$$\Bigg(\Bigg(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+$$
$$+\sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}}+\sqrt{a_4+(1-a_4)t^2}\Bigg)^2-$$
$$-\frac{(s-t)^2}{4}\Bigg)/((1+s)(1+t)). \tag{4.1.17}$$

We prove that

$$(\delta_1,\,\delta_5)<4\max_{0\le t\le s\le 1}f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)-2. \tag{4.1.18}$$

If $(\delta_1,\,\delta_5)\le 2$, it is true because $f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,1,\,1)=4$. If $(\delta_1,\,\delta_5)>2$, like for the proof of Theorem 4.1.4, we can assume that $\theta_1=\theta_3$ (instead of $\theta_1\ge\theta_3$) and $\theta_5=2c-\theta_4$ (instead of $\theta_5\ge 2c-\theta_4$).

We denote $c=(\theta_2+\theta_3)/2$, $z=(\theta_3-\theta_2)/2$ and $w=(\theta_5-\theta_4)/2$. We have $\theta_1=\theta_3=c+z$, $\theta_2=c-z$, $\theta_4=c-w$, $\theta_5=c+w$. By definition, $c\ge 0$, $z\ge 0$, $w\ge 0$. Moreover, $z\ge w$ because $\theta_3+\theta_4=2c+z-w\ge\theta_2+\theta_3=2c$ (we use (4.1.14)).

Like for the proof ot Theorem 4.1.4, using Lemmas 4.1.1, 4.1.2, we get

$$((\delta_1,\delta_5)+2)/4<$$

$$\frac{(\theta_3+\theta_2+\theta_3+\theta_4-\theta_{21}-\theta_{32}-\theta_{43}-\theta_{54})(\theta_5+\theta_2+\theta_3+\theta_4-\theta_{21}-\theta_{32}-\theta_{43}-\theta_{54})}{\theta_3\theta_5}<$$

$$(\theta_3+\theta_2+\theta_3+\theta_4-g(\alpha_1,\theta_3,\theta_2)-g(\alpha_2,\theta_2,\theta_3)-g(\alpha_3,\theta_3,\theta_4)-g(\alpha_4,\theta_4,\theta_5))\times$$
$$(\theta_5+\theta_2+\theta_3+\theta_4-g(\alpha_1,\theta_3,\theta_2)-g(\alpha_2,\theta_2,\theta_3)-g(\alpha_3,\theta_3,\theta_4)-g(\alpha_4,\theta_4,\theta_5))/(\theta_3\theta_5)=$$

$$\Bigg(\Big(\sqrt{a_1c^2+(1-a_1)z^2}+\sqrt{a_2c^2+(1-a_2)z^2}+$$
$$\sqrt{a_3c^2+a_3c(z-w)+\frac{a_3(z-w)^2}{4}+\frac{(1-a_3)(z+w)^2}{4}}+$$
$$\sqrt{a_4c^2+(1-a_4)w^2}\Big)^2-\frac{(z-w)^2}{4}\Bigg)/((c+z)(c+w)).$$

Denoting $s=z/c$ and $t=w/c$, we get

$$((\delta_1,\delta_5)+2)/4<f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)$$

where $0\le t\le s\le 1$.

Similarly to the case (ii), one can easily prove that Gram graph of $\{\delta_1,\,\delta_2,\,\delta_3,\,\delta_4,\,\delta_5\}$ is connected. Thus, for the case (iii), we get the case (III) of Theorem.

To finish the proof of Theorem, in Lemmas 4.1.6 and 4.1.7 below, we find $\max_{0\le t\le 1} f_{p2}(a_1,\,a_2,\,a_3,\,t)$ and $\max_{0\le t\le s\le 1} f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)$.

Lemma 4.1.6. *For* $0\le a_1,\,a_2,\,a_3\le 1$ *and* $t\ge 0$ *the function*

$$f_{p2}(a_1,\,a_2,\,a_3,\,t)=$$
$$=\frac{\left(\sqrt{a_1+(1-a_1)t^2}+\sqrt{a_2+(1-a_2)t^2}+\sqrt{a_3+a_3t+t^2/4}\right)^2-\frac{t^2}{4}}{1+t}$$

has the maximum

$$\max_{0\le t\le 1} f_{p2}(a_1,\,a_2,\,a_3,\,t)=\max\left[f_{p2}(a_1,\,a_2,\,a_3,\,0),\;f_{p2}(a_1,\,a_2,\,a_3,\,1)\right]=$$
$$\max\left[(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3})^2,\;\frac{\left(2+\sqrt{2a_3+\frac{1}{4}}\right)^2-\frac{1}{4}}{2}\right].$$

Lemma 4.1.7. *For* $0\le a_1,\,a_2,\,a_3,\,a_4\le 1$ *and* $s,\,t\ge 0$ *the function*

$$f_{p3}(a_1,\,a_2,\,a_3,a_4,\,s,\,t)=$$

$$\Bigg(\Big(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+$$
$$+\sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}}+\sqrt{a_4+(1-a_4)t^2}\Big)^2-$$
$$-\frac{(s-t)^2}{4}\Bigg)/((1+s)(1+t))$$

has the maximum

$$\max_{0\le t\le s\le 1} f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,s,\,t)=$$
$$\max\left[f_{p3}(a_1,\,a_2,\,a_3,\,a_4,0,\,0),\;f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,1,\,0),\;f_{p3}(a_1,\,a_2,\,a_3,\,a_4,\,1,\,1)\right]=$$
$$\max\left[(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})^2,\;\frac{\left(2+\sqrt{2a_3+\frac{1}{4}}+a_4\right)^2-\frac{1}{4}}{2},\;4\right]$$

Proof of Lemma 4.1.6 for $1 \ge a_1,\ a_2 \ge 0.2723$, *and* $1 \ge a_3 \ge 5/32 = 0.15625$. We shall use Theorem 4.1.5 only for $1 \ge a_i \ge 1/2$ when a polygon $\mathcal{M}$ has acute angles. Therefore, we restrict proving Lemma 4.1.6 for the parameters a_i above.

To prove Lemma 4.1.6 for parameters $a_1,\ a_2,\ a_3$, it is sufficient to show that

$$f_{p2}(a_1,\ a_2,\ a_3,\ t)''_{tt} \ge 0, \quad \text{if } 0 \le t \le 1. \tag{4.1.19}$$

For $1 \ge a \ge 0$ and $0 \le t \le 1$ we consider

$$u = \frac{a + (1-a)t^2}{1+t}. \tag{4.1.20}$$

We have

$$u'_t = \frac{(1-a)t^2 + 2(1-a)t - a}{(1+t)^2} = 1 - a - \frac{1}{(1+t)^2}, \quad u''_{tt} = \frac{2}{(1+t)^3}. \tag{4.1.21}$$

It follows,

$$(u^{\frac{1}{2}})''_{tt} = u^{-\frac{3}{2}}\left(-\frac{1}{4}(u'_t)^2 + \frac{1}{2}u\,u''_{tt}\right) =$$

$$\left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \frac{-(1/4)(1-a)^2t^4 - (1-a)^2t^3 + (3/2)(a-a^2)t^2 + (a-a^2)t + a - a^2/4}{(1+t)^4} \ge$$

$$\left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \frac{-(1/4)(1-a)^2t^2 - (1-a)^2t^2 + (3/2)(a-a^2)t^2 + (a-a^2)t + a - a^2/4}{(1+t)^4} =$$

$$\left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \frac{(1-a)(\frac{11}{4}a - \frac{5}{4})t^2 + (a-a^2)t + a - \frac{1}{4}a^2}{(1+t)^4} \ge$$

$$\left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times (1+t)^{-4} \times \begin{cases} a - \frac{1}{4}a^2, & \text{if } \frac{1}{3} \le a \le 1 \\ a - \frac{1}{4}a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ 0 \le t \le \frac{4a}{5-11a} \\ -\frac{5}{4} + 6a - 4a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ \frac{4a}{5-11a} \le t \le 1. \end{cases}$$

It follows,

$$(u^{\frac{1}{2}})''_{tt} \ge$$

$$\left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times (1+t)^{-4} \times \begin{cases} a - \frac{1}{4}a^2, & \text{if } \frac{1}{3} \le a \le 1 \\ a - \frac{1}{4}a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ 0 \le t \le \frac{4a}{5-11a} \\ -\frac{5}{4} + 6a - 4a^2, & \text{if } 0 \le a \le \frac{1}{3} \ \&\ \frac{4a}{5-11a} \le t \le 1 \end{cases}$$

$$\ge \left(\frac{a+(1-a)t^2}{t+1}\right)^{-\frac{3}{2}} \times \min(a - \frac{1}{4}a^2, -\frac{5}{4} + 6a - 4a^2)/(1+t)^4. \tag{4.1.22}$$

Here $\min(a-\frac{1}{4}a^2,\ -\frac{5}{4}+6a-4a^2)\ge 0$ if $a\ge 1/4$. If follows

$$(u^{\frac{1}{2}})''_{tt}\ge \frac{\min(a-\frac{1}{4}a^2,-\frac{5}{4}+6a-4a^2)}{(1+t)^{\frac{5}{2}}}\ge 0 \tag{4.1.23}$$

if $1\ge a\ge \frac{1}{4}$ and $0\le t\le 1$.

For $1\ge a\ge 0$ and $0\le t\le 1$ we consider

$$v=\frac{t^2/4+at+a}{1+t}. \tag{4.1.24}$$

We have

$$v'_t=\frac{t^2/4+t/2}{(1+t)^2}=\frac{1}{4}-\frac{1}{4(1+t)^2},\quad v''_{tt}=\frac{1}{2(1+t)^3}. \tag{4.1.25}$$

It follows

$$(v^{\frac{1}{2}})''_{tt}=\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(1/64)t^4-(1/16)t^3+(1/4)at+(1/4)a}{(1+t)^4}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(1/64)t^3-(1/16)t^3+(1/4)at+(1/4)a}{(1+t)^4}=$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{-(5/64)t^3+(1/4)at+(1/4)a}{(1+t)^4}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\begin{cases}\frac{1}{4}a, & \text{if } \frac{5}{16}\le a\le 1\\ \frac{1}{4}a, & \text{if } 0\le a\le\frac{5}{16}\ \&\ 0\le t\le 4\sqrt{\frac{a}{5}}\\ -\frac{5}{64}+\frac{1}{2}a, & \text{if } 0\le a\le \frac{5}{16}\ \&\ 4\sqrt{\frac{a}{5}}\le t\le 1.\end{cases}$$

It follows

$$(v^{\frac{1}{2}})''_{tt}\ge$$

$$\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times(1+t)^{-4}\times\begin{cases}\frac{1}{4}a, & \text{if } \frac{5}{16}\le a\le 1\\ \frac{1}{4}a, & \text{if } 0\le a\le\frac{5}{16}\ \&\ 0\le t\le 4\sqrt{\frac{a}{5}}\\ -\frac{5}{64}+\frac{1}{2}a, & \text{if } 0\le a\le \frac{5}{16}\ \&\ 4\sqrt{\frac{a}{5}}\le t\le 1\end{cases}$$

$$\ge\left(\frac{t^2/4+at+a}{1+t}\right)^{-\frac{3}{2}}\times\frac{\min(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)}{(1+t)^4}. \tag{4.1.26}$$

Here $\min(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)\ge 0$ if $a\ge\frac{5}{32}$. If follows

$$(v^{\frac{1}{2}})''_{tt}\ge(1/4+2a)^{-\frac{3}{2}}\times\frac{\min(\frac{1}{4}a,\ -\frac{5}{64}+\frac{1}{2}a)}{(1+t)^{\frac{5}{2}}} \tag{4.1.27}$$

if $1\ge a\ge 5/32=0.15625$, $0\le t\le 1$.

We have

$$\left(\frac{t^2/4}{1+t}\right)'_t=\frac{1}{4}-\frac{1}{4(1+t)^2},\quad \left(\frac{t^2/4}{1+t}\right)''_{tt}=\frac{1}{2(1+t)^3}. \tag{4.1.28}$$

We denote for $1 \geq a_i \geq 0$ and $0 \leq t \leq 1$,

$$u_1 = \frac{a_1 + (1-a_1)t^2}{1+t}, \quad u_2 = \frac{a_2 + (1-a_2)t^2}{1+t} \quad u_3 = \frac{t^2/4 + a_3 t + a_3}{1+t}$$

and

$$w = \sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3}.$$

We have

$$(w^2)''_{tt} = (w'_t)^2 + 2ww''_{tt} \geq 2ww''_{tt}. \tag{4.1.29}$$

From (4.1.23), (4.1.27), (4.1.28) and (4.1.29), for $1 \geq a_1,\ a_2, \geq 1/4$ and $1 \geq a_3 \geq 5/32$, we get for $f_{p2} = w^2 - (1/4)t^2/(1+t)$,

$$f_{p2}(a_1,\ a_2,\ a_3,\ t)''_{tt} \geq$$

$$\Big[2\left((a_1 + (1-a_1)t^2)^{\frac{1}{2}} + (a_2 + (1-a_2)t^2)^{\frac{1}{2}} + (t^2/4 + a_3 t + a_3)^{\frac{1}{2}}\right) \times$$
$$\Big(\min(a_1 - \frac{1}{4}a_1^2,\ -\frac{5}{4} + 6a_1 - 4a_1^2) + \min(a_2 - \frac{1}{4}a_2^2,\ -\frac{5}{4} + 6a_2 - 4a_2^2) +$$
$$(1/4 + 2a_3)^{-\frac{3}{2}} \min(\frac{1}{4}a_3,\ -\frac{5}{64} + \frac{1}{2}a_3)\Big) - \frac{1}{2}\Big] /(1+t)^3 \geq$$

$$[2(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3}) \times$$
$$\Big(\min(a_1 - \frac{1}{4}a_1^2,\ -\frac{5}{4} + 6a_1 - 4a_1^2) + \min(a_2 - \frac{1}{4}a_2^2,\ -\frac{5}{4} + 6a_2 - 4a_2^2) +$$
$$(1/4 + 2a_3)^{-\frac{3}{2}} \min(\frac{1}{4}a_3,\ -\frac{5}{64} + \frac{1}{2}a_3)\Big) - \frac{1}{2}\Big] /(1+t)^3 \geq 0 \tag{4.1.30}$$

if

$$(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3}) \times$$
$$\Big(\min(a_1 - \frac{1}{4}a_1^2,\ -\frac{5}{4} + 6a_1 - 4a_1^2) + \min(a_2 - \frac{1}{4}a_2^2,\ -\frac{5}{4} + 6a_2 - 4a_2^2) +$$
$$(1/4 + 2a_3)^{-\frac{3}{2}} \min(\frac{1}{4}a_3,\ -\frac{5}{64} + \frac{1}{2}a_3)\Big) \geq \frac{1}{4}. \tag{4.1.31}$$

Thus, (4.1.19) is true if $1 \geq a_1,\ a_2 \geq 1/4$, $1 \geq a_3 \geq 5/32$ and (4.1.31) is valid. For $1 \geq a_1,\ a_2 \geq a \geq 1/4$ and arbitrary $1 \geq a_3 \geq 5/32$, (4.1.31) is valid if

$$2(2\sqrt{a} + \sqrt{\frac{5}{32}})(-\frac{5}{4} + 6a - 4a^2) \geq \frac{1}{4}. \tag{4.1.32}$$

It is true if $1 \geq a \geq 0.2723$.

It proves Lemma 4.1.6 for $1 \geq a_1,\ a_2 \geq 0.2723$ and $1 \geq a_3 \geq 5/32 = 0.15625$.

Proof of Lemma 4.1.7 for $1 \geq a_1,\ a_2,\ a_3,\ a_4 \geq 0.37646$. We shall use Theorem 4.1.5 only for $1 \geq a_i \geq 1/2$ when a polygon $\mathcal{M}$ has acute angles. Therefore, we restrict proving Lemma 4.1.7 for the parameters a_i above.

To prove Lemma 4.1.7 for parameters a_1, a_2, a_3, a_4, it is sufficient to show that

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, t)''_{tt} \geq 0, \quad \text{if } 0 \leq t \leq s \leq 1; \tag{4.1.33}$$

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0)''_{ss} \geq 0, \quad \text{if } 0 \leq s \leq 1; \tag{4.1.34}$$

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, s)''_{ss} \geq 0, \quad \text{if } 0 \leq s \leq 1. \tag{4.1.35}$$

We have

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0) =$$

$$\frac{\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+a_3s+\frac{s^2}{4}}+\sqrt{a_4}\right)^2-\frac{s^2}{4}}{1+s}. \tag{4.1.36}$$

Like for the proof of Lemma 4.1.6, it follows for $1 \geq a_1,\, a_2 \geq \frac{1}{4}$, $1 \geq a_3 \geq \frac{5}{32}$, $1 \geq a_4 \geq 0$ that

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, 0)''_{ss} \geq$$

$$\Bigg[2\left((a_1+(1-a_1)s^2)^{\frac{1}{2}}+(a_2+(1-a_2)s^2)^{\frac{1}{2}}+(s^2/4+a_3s+a_3)^{\frac{1}{2}}+a_4^{\frac{1}{2}}\right)\times$$
$$\Bigg(\min\,(a_1-\frac{1}{4}a_1^2,\; -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\; -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\; -\frac{5}{64}+\frac{1}{2}a_3)\Bigg)-\frac{1}{2}\Bigg]\,/(1+s)^3 \geq$$

$$[2\,(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\Bigg(\min\,(a_1-\frac{1}{4}a_1^2,\; -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\; -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\; -\frac{5}{64}+\frac{1}{2}a_3)\Bigg)-\frac{1}{2}\Bigg]\,/(1+s)^3 \geq 0 \tag{4.1.37}$$

if

$$(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\Bigg(\min\,(a_1-\frac{1}{4}a_1^2,\; -\frac{5}{4}+6a_1-4a_1^2)+\min\,(a_2-\frac{1}{4}a_2^2,\; -\frac{5}{4}+6a_2-4a_2^2)+$$
$$(1/4+2a_3)^{-\frac{3}{2}}\min\,(\frac{1}{4}a_3,\; -\frac{5}{64}+\frac{1}{2}a_3)\Bigg) \geq \frac{1}{4}. \tag{4.1.38}$$

Thus, (4.1.34) is true if $1 \geq a_1,\, a_2 \geq 1/4$, $1 \geq a_3 \geq 5/32$, $1 \geq a_4 \geq 0$ and (4.1.38) is valid.

For $1 \geq a_1,\, a_2 \geq a \geq 1/4$, arbitrary $1 \geq a_3 \geq 5/32$ and arbitrary $1 \geq a_4 \geq 0$, we have (4.1.38) if (4.1.32) is valid. It takes place if $1 \geq a \geq 0.2723$.

We have

$$f_{p3}(a_1,\, a_2,\, a_3,\, a_4,\, s,\, s) =$$

$$\frac{\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+(1-a_3)s^2}+\sqrt{a_4+(1-a_4)s^2}\right)^2}{(1+s)^2}. \tag{4.1.39}$$

For $1 \geq a \geq 0$ and $0 \leq s \leq 1$ we denote

$$e = \frac{a+(1-a)s^2}{(1+s)^2} \tag{4.1.40}$$

We have

$$e'_s = \frac{-2a+2s(1-a)}{(1+s)^3} = \frac{2s}{(1+s)^3} - \frac{2a}{(1+s)^2},$$
$$e''_{ss} = \frac{2+4a+(4a-4)s}{(1+s)^4} = \frac{2-4s}{(1+s)^4} + \frac{4a}{(1+s)^3}. \tag{4.1.41}$$

It follows,

$$(e^{\frac{1}{2}})''_{ss} = e^{-\frac{3}{2}}\left(-\frac{1}{4}(e'_s)^2 + \frac{1}{2}e\,e''_{ss}\right) =$$

$$e^{-\frac{3}{2}} \times \frac{(-2a^2+4a-2)s^3+(-3a^2+3a)s^2+(a^2+a)}{(1+s)^6} \geq$$

$$e^{-\frac{3}{2}} \times \frac{(-2a^2+4a-2)s^2+(-3a^2+3a)s^2+(a^2+a)}{(1+s)^6} =$$

$$e^{-\frac{3}{2}} \times \frac{(-5a^2+7a-2)s^2+(a^2+a)}{(1+s)^6} \geq$$
$$e^{-\frac{3}{2}} \times \begin{cases} a^2+a, & \text{if } 1 \geq a \geq \frac{2}{5} \\ -4a^2+8a-2, & \text{if } \frac{2}{5} \geq a \geq 0 \end{cases} \geq$$
$$e^{-\frac{3}{2}} \times \frac{\min(a^2+a,\ -4a^2+8a-2)}{(1+s)^6}. \tag{4.1.42}$$

Here $\min(a^2+a,\ -4a^2+8a-2) \geq 0$ if $a \geq (2-\sqrt{2})/2 = 0.292893...$. It follows

$$(e^{\frac{1}{2}})''_{ss} \ \geq \ \frac{\min(a^2+a,\ -4a^2+8a-2)}{(1+s)^3} \geq 0 \ \text{ if } 1 \geq a \geq (2-\sqrt{2})/2. \tag{4.1.43}$$

For $1 \geq a_1,\ a_2,\ a_3,\ a_4 \geq 0$ we denote

$$u_i = \frac{a_i+(1-a_i)s^2}{(1+s)^2}. \tag{4.1.44}$$

We get

$$f_{p3}(a_1,\ a_2,\ a_3,\ a_4,\ s,\ s)''_{ss} \geq$$

$$2(\sqrt{u_1}+\sqrt{u_2}+\sqrt{u_3}+\sqrt{u_4})((u_1^{\frac{1}{2}})''_{ss}+(u_2^{\frac{1}{2}})''_{ss}+(u_3^{\frac{1}{2}})''_{ss}+(u_4^{\frac{1}{2}})''_{ss}) \geq$$

$$2\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\sqrt{a_3+(1-a_3)s^2}+\sqrt{a_4+(1-a_4)s^2}\right)\times$$
$$\big((\min(a_1^2+a_1,\ -4a_1^2+8a_1-2)+\min(a_2^2+a_2,\ -4a_2^2+8a_2-2)+$$
$$+\min(a_3^2+a_3,\ -4a_3^2+8a_3-2)+\min(a_4^2+a_4,\ -4a_4^2+8a_4-2)\big)/(1+s)^4\geq$$

$$2\left(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4}\right)\times$$
$$\big((\min(a_1^2+a_1,\ -4a_1^2+8a_1-2)+\min(a_2^2+a_2,\ -4a_2^2+8a_2-2)+$$
$$+\min(a_3^2+a_3,\ -4a_3^2+8a_3-2)+\min(a_4^2+a_4,\ -4a_4^2+8a_4-2)\big)/(1+s)^4\geq 0 \tag{4.1.45}$$

if $1\geq a_i\geq(2-\sqrt{2})/2=0.292893...$.

Let us prove (4.1.33) for $1\geq a_1,\ a_2,\ a_3,\ a_4\geq 0.37646$. We denote

$$f=\frac{a+a(s-t)+\frac{a(s-t)^2}{4}+\frac{(1-a)(s+t)^2}{4}}{t+1}=\frac{a+as+\frac{s^2}{4}+t(-a-as+\frac{s}{2})+\frac{t^2}{4}}{t+1}.$$

We have

$$f'_t=\frac{-2a-2as+\frac{s}{2}-\frac{s^2}{4}+\frac{t}{2}+\frac{t^2}{4}}{(t+1)^2}=\frac{1}{4}-\frac{(1-s)^2+8a(s+1)}{4(t+1)^2},$$

$$f''_{tt}=\frac{(1-s)^2+8a(s+1)}{2(t+1)^3}.$$

For $1\geq a\geq 0$ and $0\leq t\leq s\leq 1$, we have

$$(f^{\frac{1}{2}})''_{tt}=f^{-\frac{3}{2}}\left(-\frac{1}{4}(f'_t)^2+\frac{1}{2}f\,f''_{tt}\right)=$$

$$f^{-\frac{3}{2}}(-\frac{1}{64}t^4-\frac{1}{16}t^3+(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a)t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))/(t+1)^4\geq$$

$$f^{-\frac{3}{2}}(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a-\frac{5}{64})t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))/(t+1)^4.$$

For $1\geq a\geq\frac{1}{4}$ and $0\leq t\leq s\leq 1$ we have

$$(\frac{3}{32}s^2+(\frac{3}{4}a-\frac{3}{16})s+\frac{3}{4}a-\frac{5}{64})t^2+$$
$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))\geq$$

$$((-\frac{1}{4}a+\frac{1}{8})s^3+(-2a^2+\frac{5}{4}a-\frac{3}{16})s^2+(-4a^2+\frac{7}{4}a)s+(-2a^2+\frac{1}{4}a))t+$$
$$(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a))\geq$$

$$\min\left(\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a),\right.$$
$$\left.(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+(a^2+\frac{1}{4}a)\right). \tag{4.1.46}$$

Here $\frac{3}{64}s^4+(\frac{1}{2}a-\frac{1}{16})s^3+(a^2+\frac{1}{2}a)s^2+(2a^2+\frac{1}{4}a)s+(a^2+\frac{1}{4}a)\geq a^2+\frac{1}{4}a$. Let us prove that $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+(a^2+\frac{1}{4}a)\geq\frac{11}{64}a^2+\frac{1}{4}a$. Equivalently, we should prove that $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq 0$.

Assume that $1\geq a\geq\frac{11}{16}$. Then $-\frac{1}{4}a+\frac{11}{64}\leq 0$ and $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq r(a,s)=(-2a^2+\frac{3}{2}a-\frac{5}{64})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2$. We have $r(a,0)\geq 0$ and $r(a,1)=-\frac{267}{64}a^2+\frac{17}{4}a-\frac{5}{64}\geq 0$. It follows that $r(a,s)\geq 0$ if $\frac{1}{2}r''_{ss}=3(-2a^2+\frac{3}{2}a-\frac{5}{64})s+(-3a^2+\frac{9}{4}a)\leq 0$. The last is valid if $\frac{1}{2}r''_{ss}(a,0)=-3a^2+\frac{9}{4}a\leq 0$ and $\frac{1}{2}r''_{ss}(a,1)=3(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\leq 0$. It is true if $1\geq a\geq\frac{3}{4}$ when $-3a^2+\frac{9}{4}a\leq 0$. Suppose that $\frac{3}{4}\geq a\geq\frac{11}{16}$. Then $-3a^2+\frac{9}{4}a\geq 0$. If also $-2a^2+\frac{3}{2}a-\frac{5}{64}\geq 0$, then $r(a,s)\geq 0$. If $-2a^2+\frac{3}{2}a-\frac{5}{64}\leq 0$, we have $r(a,s)\geq((-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a))s^2+\frac{1}{2}as+\frac{53}{64}a^2$. If $(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\leq 0$, we again have that $r(a,s)\geq\min(r(a,0),r(a,1))\geq 0$. If $(-2a^2+\frac{3}{2}a-\frac{5}{64})+(-3a^2+\frac{9}{4}a)\geq 0$, we obviously have $r(a,s)\geq 0$.

Assume that $\frac{11}{64}\geq a\geq 1/4$. Then $(-\frac{1}{4}a+\frac{11}{64})s^4+(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2\geq r(a,s)=(-2a^2+\frac{7}{4}a-\frac{1}{4})s^3+(-3a^2+\frac{9}{4}a)s^2+\frac{1}{2}as+\frac{53}{64}a^2$. Arguing with the function $r(a,s)$ like above, we prove that $r(a,s)\geq 0$. (Here one can get even better result.)

Thus, for $1\geq a\geq\frac{1}{4}$ and $0\leq t\leq s\leq 1$, we have

$$(f^{\frac{1}{2}})''_{tt}\geq$$
$$f^{-\frac{3}{2}}\times\frac{\frac{11}{64}a^2+\frac{1}{4}a}{(t+1)^4} \tag{4.1.47}$$

We have $f\leq(1+(5/4)a)/(t+1)$. It follows

$$(f^{\frac{1}{2}})''_{tt}\geq(1+\frac{5}{4}a)^{-\frac{3}{2}}\times\frac{\frac{11}{64}a^2+\frac{1}{4}a}{(t+1)^{\frac{5}{2}}} \tag{4.1.48}$$

if $1\geq a\geq\frac{1}{4}$ and $1\geq s\geq t\geq 0$.

We have $(t-s)^2/(t+1)=t+1-2(s+1)+(s+1)^2/(t+1)$, it follows

$$\left(\frac{(t-s)^2}{(t+1)(s+1)}\right)''_{tt}=\frac{2(s+1)^2}{(s+1)(t+1)^3}\leq\frac{8}{(s+1)(t+1)^3}. \tag{4.1.49}$$

Like for the proof of Lemma 4.1.6, from (4.1.23) and (4.1.48), it follows that for $1 \geq a_1,\ a_2 \geq 0,\ 1 \geq a_3,\ a_4 \geq \frac{1}{4}$,

$$f_{p3}(a_1,\ a_2,\ a_3,\ a_4,\ s,\ t)''_{tt} \geq$$

$$(1+s)^{-1}(1+t)^{-3}\left[2\left(\sqrt{a_1+(1-a_1)s^2}+\sqrt{a_2+(1-a_2)s^2}+\right.\right.$$
$$\left.+\sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}}+\sqrt{a_4+(1-a_4)t^2}\right)\times$$
$$\left.\left((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\left(a_4-\frac{1}{4}a_4^2,\ -\frac{5}{4}+6a_4-4a_4^2\right)\right)-\frac{(s+1)^2}{2}\right]\geq$$

$$(1+s)^{-1}(1+t)^{-3}\,[2\,(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\left((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\left(a_4-\frac{1}{4}a_4^2,\ -\frac{5}{4}+6a_4-4a_4^2\right)\right)-2\,]. \tag{4.1.50}$$

Thus, $f_{p3}(a_1,\ a_2,\ a_3,\ a_4,\ s,\ t)''_{tt} \geq 0$ if $1 \geq a_1,\ a_2 \geq 0,\ 1 \geq a_3,\ a_4 \geq \frac{1}{4}$ and

$$(\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4})\times$$
$$\left((1+\frac{5}{4}a_3)^{-\frac{3}{2}}(\frac{11}{64}a_3^2+\frac{1}{4}a_3)+\min\left(a_4-\frac{1}{4}a_4^2,\ -\frac{5}{4}+6a_4-4a_4^2\right)\right)\ \geq\ 1. \tag{4.1.51}$$

For arbitrary $1 \geq a_1,\ a_2,\ a_3,\ a_4 \geq a \geq \frac{1}{4}$ it is valid if

$$\sqrt{a}\left((1+\frac{5}{4}a)^{-\frac{3}{2}}(\frac{11}{64}a^2+\frac{1}{4}a)+a-\frac{1}{4}a^2\right)\ \geq \frac{1}{4}. \tag{4.1.52}$$

It is true if $a \geq 0.37646$.

It proves Lemma 4.1.7 for $1 \geq a_1,\ a_2,\ a_3,\ a_4 \geq 0.37646$ when all three conditions (4.1.33), (4.1.34) and (4.1.35) are valid.

Using Theorems 4.1.4 and 4.1.5, we can divide all convex elliptic polygons on the hyperbolic plane in three types. This subdivision is very useful for fundamental polygons of finite volume of reflection groups, and we shall use it in Sects. 4.3 and 5.

Theorem 4.1.8. *Let $\mathcal{M}$ be an elliptic convex polygon on a hyperbolic plane. Then $\mathcal{M}$ has one of types (I), (II) or (III) (or the type (I), (II) or (III) of its narrow part) below:*

Type (I): *There exist its four consecutive vertices $A_0,\ A_1,\ A_2,\ A_3$ (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) with angles $\alpha_1 = A_0A_1A_2,\ \alpha_2 = A_1A_2A_3 \neq \frac{\pi}{2}$ and such that for orthogonal vectors $\delta_1,\ \delta_2$ and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1,\ \delta_2) = 2\cos\alpha_1$, $(\delta_2,\ \delta_3) = 2\cos\alpha_2 \neq 0$ and*

$$\textit{either}\ \ (\delta_1,\ \delta_3) \leq 2\ \ \textit{or}\ \ (\delta_1,\ \delta_3) < 4(\cos\frac{\alpha_1}{2}+\cos\frac{\alpha_2}{2})^2-2 \leq 14 \tag{4.1.53}$$

The set $\{\delta_1,\,\delta_2,\,\delta_3\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Any triangle or quadrangle $\mathcal{M}$ has the type (I).

Type (II): *There exist its five consecutive vertices A_0, A_1, A_2, A_3, A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) with angles $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$ one has $(\delta_1,\,\delta_2) = (\delta_2,\,\delta_3) = 0$, $(\delta_3,\,\delta_4) = 2\,cos\alpha_3$,*

$$(\delta_1,\,\delta_3) < 6 \tag{4.1.54}$$

and

$$(\delta_1,\,\delta_4)\ <\ 4\max\left(\left(\sqrt{2}+\cos\frac{\alpha_3}{2}\right)^2,\ \frac{\left(2+\sqrt{2\cos^2\frac{\alpha_3}{2}+\frac{1}{4}}\right)^2-\frac{1}{4}}{2}\right)-2\le$$

$$\le\ 10+8\sqrt{2}=21.313708\ldots\ . \tag{4.1.55}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph.

Type (III): *There exist its six consecutive vertices A_0, A_1, A_2, A_3, A_4, A_5 (where $A_0 = A_5$ if $\mathcal{M}$ is a pentagon) with right angles $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4 = \frac{\pi}{2}$, $\alpha_4 = A_3A_4A_5 = \frac{\pi}{2}$ such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_1,\,\delta_2) = (\delta_2,\,\delta_3) = (\delta_3,\,\delta_4) = (\delta_4,\,\delta_5) = 0$ and*

$$(\delta_1,\,\delta_3) < 6, \tag{4.1.56}$$

$$(\delta_3,\,\delta_5) < 6, \tag{4.1.57}$$

and

$$(\delta_1,\,\delta_5) < 30. \tag{4.1.58}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_5\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph.

Proof. If $\mathcal{M}$ is a triangle, at least one angle of $\mathcal{M}$ is not right. It follows that $\mathcal{M}$ has a sequence of consecutive vertices of type (I). If $\mathcal{M}$ is a quadrangle, we consider its narrow part of type (I) from Theorem 4.1.4. If both angles $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, we replace δ_2 by δ_4 with $\delta_4^2 = -2$ which is orthogonal to the line (A_0A_3) and directed outwards of $\mathcal{M}$. Since a quadrangle has at least one non-right angle (we can assume that it is $\alpha_0 = A_3A_0A_1$), then the sequence of vertices A_2, A_3, A_0, A_1 with orthogonal vectors δ_3, δ_4, δ_1 to sides (A_2A_3), (A_3A_0), (A_0A_1) respectively satisfies the condition (I) of Theorem 4.1.8.

We now suppose that $\mathcal{M}$ does not have a sequence of vertices of type (I) (we always suppose that it is a sequence of consecutive vertices). Then $\mathcal{M}$ has more than 4 vertices.

By Theorem 4.1.5, $\mathcal{M}$ has a narrow part of type (II) or (III).

Suppose that $\mathcal{M}$ has the narrow part of type (II). We have $\alpha_1 = \alpha_2 = \frac{\pi}{2}$ since otherwise the sequence A_0, A_1, A_2, A_3 gives a sequence of type (I) (possibly after changing the numeration). We then get estimates (4.1.54) and (4.1.55). Obviously, the set $\{\delta_1, \delta_2, \delta_3\}$ generates the 3-dimensional hyperbolic vector space and has connected components $\{\delta_1, \delta_3\}$ and $\{\delta_2\}$ since $\alpha_1 = \alpha_2 = \frac{\pi}{2}$. If the set $\{\delta_1, \delta_3, \delta_4\}$ is not connected, it then follows that δ_4 is orthogonal to the set $\{\delta_1, \delta_3\}$ and $\delta_4 = \lambda\delta_2$ which is impossible since the lines (A_1A_2) and (A_3A_4) are different. It follows that the Gram graph of the set $\{\delta_1, \delta_3, \delta_4\}$ is connected. This set generates the 3-dimensional hyperbolic vector space because the orthogonal lines (A_0A_1), (A_2A_3), (A_3A_4) do not have a common point and are not orthogonal to one line. Thus, we have proved that the $\mathcal{M}$ has type (II) of Theorem 4.1.8.

Assume that $\mathcal{M}$ has a narrow part of type (III) of Theorem 4.1.5. If the sequence A_0, A_1, A_2, A_3, A_4, A_5 does not have a subsequence of the type (I) of Theorem 4.1.8, then all angles $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{\pi}{2}$ are right. We then get estimates (4.1.56), (4.1.57) and (4.1.58). The same consideration as above, shows that the Gram graph of the $\{\delta_1, \delta_3, \delta_5\}$ is connected and this set generates the 3-dimensional hyperbolic vector space. Thus, the $\mathcal{M}$ has type (III) of Theorem 4.1.8.

It finishes the proof.

4.2. Narrow parts of restricted parabolic convex polygons on the hyperbolic plane.

Compare [N9], [N11].

We use notation of Sect. 1.1. Fix a point $O = \mathbb{R}_{++}c$ at infinity of $\mathcal{L} = \mathcal{L}(\Phi)$. Thus, $c \in \Phi$, $(c, c) = 0$ and $(c, V^+) > 0$.

Recall that a *horosphere* $\mathcal{E}_O$ with the center O is the set of all lines in $\mathcal{L}$ containing O. The line $l = O\mathbb{R}_{++}h \in \mathcal{E}_O$, $\mathbb{R}_{++}h \in \mathcal{L}$ is the set $l = \{\mathbb{R}_{++}(tc + h) \mid t \in \mathbb{R}$ and $(tc+h, tc+h) > 0\}$. Fix a constant $R > 0$. Then there is a unique $\mathbb{R}_{++}h \in l$ such that $(h, c) = R$ and $(h, h) = 1$. Given $l_1, l_2 \in \mathcal{E}_O$, we denote the corresponding h's by h_1, h_2 and put

$$\rho(l_1, l_2) = \sqrt{-(h_1 - h_2)^2}. \tag{4.2.1}$$

Endowed with this distance, the horosphere $\mathcal{E}_O$ becomes an affine Euclidean space. If one changes R, the distance ρ is multiplied by a constant. We denote

$$\mathcal{E}_{O,R} = \{\mathbb{R}_{++}h \in \mathcal{L} \mid (h, c) = R \text{ and } (h, h) = 1\}. \tag{4.2.2}$$

The set $\mathcal{E}_{O,R} \cup \{O\}$ is a sphere in $\overline{\mathcal{L}}$, which is tangent to $\mathcal{L}_\infty$ at O. Moreover, the set $\mathcal{E}_{O,R}$ is orthogonal to every line $l \in \mathcal{E}_O$ at the point $\mathbb{R}_{++}h$, $h \in \mathcal{E}_{O,R}$ that corresponds to l. The distance in $\mathcal{L}$ induces Euclidean distance in $\mathcal{E}_{O,R}$ which is homothetic to the distance (4.2.1). The set $\mathcal{E}_{O,R}$ is identified with $\mathcal{E}_O$ and is also called a *horosphere*.

Let $K \subset \mathcal{E}_O$. The set

$$C_K = \bigcup_{\text{line } l \in K} l \tag{4.2.3}$$

is called the *cone with vertex O and base K*.

A non-degenerate convex locally finite polyhedron $\mathcal{M}$ in $\mathcal{L}$ is called *parabolic* (relative to the point $O \in \mathcal{L}_\infty$ if 1) and 2) below are valid:

1) $\mathcal{M}$ is finite at the point O, that is, the set $\{\delta \in P(\mathcal{M}) \mid (c, \delta) = 0\}$ is finite;

2) for every elliptic polyhedron $\mathcal{N} \subset \mathcal{E}_O$ (that is $\mathcal{N}$ is the convex hull of a finite subset of $\mathcal{E}_O$), the polyhedron $\mathcal{M} \cap C_{\mathcal{N}}$ is elliptic.

A parabolic polyhedron $\mathcal{M}$ is called *restricted parabolic* if

3) the set
$$r(\mathcal{M}) = \{(c, \delta/\sqrt{-\delta^2}) \mid \delta \in P(\mathcal{M})\} \text{ is finite.} \tag{4.2.4}$$

Geometrically this means that all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, of faces of $\mathcal{M}$ are tangent to a finite set of horospheres with the center O.

We remark that if $O \notin \mathcal{M}$ for a parabolic polyhedron $\mathcal{M}$, then $\mathcal{M}$ is elliptic. Thus, it is only interesting to consider parabolic polyhedra relative to a point $O \in \mathcal{L}_\infty \cap \mathcal{M}$. Moreover, if $\mathcal{M}$ is parabolic and has a finite number of faces (i. e. the set $P(\mathcal{M})$ is finite) then $\mathcal{M}$ is elliptic. Thus, it is only interesting to consider *infinite* (i.e. having infinite number of faces) parabolic polyhedra.

Theorems 4.1.4, 4.1.5 and 4.1.8 can be generalized on restricted parabolic polygons with strong inequalities replaced by non-strong ones.

Theorem 4.2.1 (about the narrow part of type (I)). *For any restricted parabolic convex polygon $\mathcal{M}$ on a hyperbolic plane there exist its four consecutive vertices A_0, A_1, A_2 and A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1, \delta_2) = 2\cos\alpha_1$, $(\delta_2, \delta_3) = 2\cos\alpha_2$ and*

$$\text{either } (\delta_1, \delta_3) \le 2 \quad \text{or} \quad (\delta_1, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14 \tag{4.2.5}$$

where $\alpha_1 = A_0A_1A_2$ and $\alpha_2 = A_1A_2A_3$.

Moreover, the Gram graph of $\{\delta_1, \delta_2, \delta_3\}$ is not connected if and only if $\alpha_1 = \alpha_2 = \frac{\pi}{2}$.

Theorem 4.2.2 (about narrow parts of types (II) and (III)). *For any restricted parabolic convex polygon $\mathcal{M}$ having more than 3 vertices (i.e. it is different from a triangle) on a hyperbolic plane, one of two possibilities (II) or (III) below is valid:*

(II) There exist its five consecutive vertices A_0, A_1, A_2 , A_3 and A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3$,

$$\text{either } (\delta_1, \delta_3) \le 2 \quad \text{or} \quad (\delta_1, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.2.6}$$

$$(\delta_1, \delta_4) \le$$

$$4\max_{0\le t\le 1} \frac{\left(\sqrt{a_1 + (1-a_1)t^2} + \sqrt{a_2 + (1-a_2)t^2} + \sqrt{a_3 + a_3t + t^2/4}\right)^2 - \frac{t^2}{4}}{1+t} - 2 =$$
$$4\max\left(\left(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2} + \cos\frac{\alpha_3}{2}\right)^2, \frac{\left(2 + \sqrt{2\cos^2\frac{\alpha_3}{2} + \frac{1}{4}}\right)^2 - \frac{1}{4}}{2}\right) - 2 \le 34 \tag{4.2.7}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1,\ 2,\ 3$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ has a connected Gram graph.

(III). There exist its six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 (where $A_0 = A_5$ if $\mathcal{M}$ is a pentagon) such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1, 2, 3, 4$,

$$\text{either } (\delta_1,\, \delta_3) \le 2 \quad \text{or } (\delta_1,\, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14, \tag{4.2.8}$$

$$\text{either } (\delta_3,\, \delta_5) \le 2 \quad \text{or } (\delta_3,\, \delta_5) \le 4(\cos\frac{\alpha_3}{2} + \cos\frac{\alpha_4}{2})^2 - 2 \le 14, \tag{4.2.9}$$

and

$$(\delta_1,\, \delta_5) \le$$

$$4 \max_{0\le t\le s\le 1} \Bigg[\Bigg(\Bigg(\sqrt{a_1+(1-a_1)s^2} + \sqrt{a_2+(1-a_2)s^2} + \sqrt{a_3+a_3(s-t)+\frac{a_3(s-t)^2}{4}+\frac{(1-a_3)(s+t)^2}{4}} + \sqrt{a_4+(1-a_4)t^2}\Bigg)^2 - \frac{(s-t)^2}{4}\Bigg)\Big/((1+s)(1+t))\Bigg] - 2 =$$

$$4 \max\left[\left(\cos\frac{\alpha_1}{2}+\cos\frac{\alpha_2}{2}+\cos\frac{\alpha_3}{2}+\cos\frac{\alpha_4}{2}\right)^2,\ \frac{\left(2+\sqrt{2\cos^2\frac{\alpha_3}{2}+\frac{1}{4}}+\cos\frac{\alpha_4}{2}\right)^2-\frac{1}{4}}{2},\ 4\right] - 2 \ \le\ 62 \tag{4.2.10}$$

where $\alpha_i = A_{i-1}A_iA_{i+1}$, $i = 1, 2, 3, 4$, and $a_i = \cos^2\frac{\alpha_i}{2}$. Moreover, the set $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ has a connected Gram graph.

Theorem 4.2.3. *Let $\mathcal{M}$ be a restricted parabolic convex polygon on a hyperbolic plane. Then $\mathcal{M}$ has one of types (I), (II) or (III) (or the type (I), (II) or (III) of its narrow part) below:*

Type (I): *There exist its four consecutive vertices A_0, A_1, A_2, A_3 (where $A_0 = A_3$ if $\mathcal{M}$ is a triangle) with angles $\alpha_1 = A_0A_1A_2$, $\alpha_2 = A_1A_2A_3 \ne \frac{\pi}{2}$ and such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ one has $(\delta_1,\, \delta_2) = 2\cos\alpha_1$, $(\delta_2,\, \delta_3) = 2\cos\alpha_2 \ne 0$ and*

$$\text{either } (\delta_1,\, \delta_3) \le 2 \quad \text{or } (\delta_1,\, \delta_3) \le 4(\cos\frac{\alpha_1}{2} + \cos\frac{\alpha_2}{2})^2 - 2 \le 14. \tag{4.2.11}$$

The set $\{\delta_1, \delta_2, \delta_3\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Any triangle or quadrangle $\mathcal{M}$ has the type (I).

Type (II): *There exist its five consecutive vertices A_0, A_1, A_2, A_3, A_4 (where $A_0 = A_4$ if $\mathcal{M}$ is a quadrangle) with angles $\alpha_1 = A_0A_1A_2 = \frac{\pi}{2}$, $\alpha_2 = A_1A_2A_3 = \frac{\pi}{2}$, $\alpha_3 = A_2A_3A_4$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$ one has $(\delta_1,\, \delta_2) = (\delta_2,\, \delta_3) = 0$, $(\delta_3,\, \delta_4) = 2cos\alpha_3$,*

$$(\delta_1,\, \delta_3) \le 6 \tag{4.2.12}$$

and

$$(\delta_1,\,\delta_4)\ \le$$

$$4\max\left(\left(\sqrt{2}+\cos\frac{\alpha_3}{2}\right)^2,\ \frac{\left(2+\sqrt{2\cos^2\frac{\alpha_3}{2}+\frac{1}{4}}\right)^2-\frac{1}{4}}{2}\right)-2\le 10+8\sqrt{2}=$$

$$=21.313708\dots\,. \tag{4.2.13}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_4\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Type (III). *There exist its six consecutive vertices* $A_0,\,A_1,\,A_2,\,A_3,\,A_4,\,A_5$ *(where* $A_0=A_5$ *if* $\mathcal{M}$ *is a pentagon) with right angles* $\alpha_1=A_0A_1A_2=\frac{\pi}{2}$, $\alpha_2=A_1A_2A_3=\frac{\pi}{2}$, $\alpha_3=A_2A_3A_4=\frac{\pi}{2}$, $\alpha_4=A_3A_4A_5=\frac{\pi}{2}$ *such that for orthogonal vectors* $\delta_1,\,\delta_2,\,\delta_3,\,\delta_4$ *and* δ_5 *to lines* (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) *and* (A_4A_5) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=\delta_5^2=-2$ *one has* $(\delta_1,\,\delta_2)=(\delta_2,\,\delta_3)=(\delta_3,\,\delta_4)=(\delta_4,\,\delta_5)=0$ *and*

$$(\delta_1,\,\delta_3)\le 6, \tag{4.2.14}$$

$$(\delta_3,\,\delta_5)\le 6, \tag{4.2.15}$$

and

$$(\delta_1,\,\delta_5)\le 30. \tag{4.2.16}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_5\}$ *generates the* 3*-dimensional hyperbolic vector space and has a connected Gram graph.*

Proofs of Theorems 4.2.1, 4.2.2, 4.2.3. If $\mathcal{M}$ is parabolic relative to $O\notin\mathcal{M}$, then $\mathcal{M}$ is finite and we can use Theorems 4.1.4, 4.1.5 and 4.1.8. Thus, we can suppose that $O\in\mathcal{M}$ and $\mathcal{M}$ is infinite.

To prove Theorems 4.1.4, 4.1.5 and 4.1.8, we were taken a point O inside of $\mathcal{M}$ and used the formulae of Lemma 4.1.2. To prove Theorems 4.2.1, 4.2.2, 4.2.3, we use the point O at infinity of $\mathcal{M}$ such that $\mathcal{M}$ is restricted parabolic relative to O. We use an analog of Lemma 4.1.2 which uses the infinite point O.

Let $O=\mathbb{R}_{++}c$ where $c^2=0$. For a line (AB) with terminals A and B at infinity and $\delta\in\Phi$ orthogonal to (AB) with $\delta^2=-2$ we introduce an 'angle'

$$\theta(\delta)=\frac{1}{(c,\,\delta)} \tag{4.2.17}$$

where one should replace δ by $-\delta$ according to the orientation of the angle AOB: e. g. one should take δ such that $\mathcal{H}^+_\delta$ contains O if AOB is oriented correctly, and one should take $-\delta$ if not.

The 'angle' $\theta(\delta)$ really behaves like an angle. For three points A, B and C at infinity and three lines (AB), (BC) and (AC) with the corresponding vectors δ_1, δ_2 and δ_3 orthogonal to lines (AB), (BC) and (AC) respectively and with $\delta_1^2=\delta_2^2=\delta_3^2=-2$, one has

$$\theta(\delta_1)+\theta(\delta_2)=\theta(\delta_3). \tag{4.2.18}$$

We leave an elementary proof of (4.2.18) to a reader.

We have the following analog of Lemma 4.1.2.

Lemma 4.2.4. *Let (AB) and (CD) are two lines on a hyperbolic plane with terminals A, B, C, D at infinity, and $O = \mathbb{R}_{++}c$ an infinite point on the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles AOB and COD coincide. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$. Let δ_{12} be the orthogonal vector with square -2 to the line (BC). Then*

$$(\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) + \theta(\delta_{12}))(\theta(\delta_2) + \theta(\delta_{12}))}{\theta(\delta_1)\theta(\delta_2)} - 2.$$

As a corollary, we get :

1) If lines (AB) and (CD) do not intersect each other, then

$$2\cosh\rho = (\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) + \theta(\delta_{12}))(\theta(\delta_2) + \theta(\delta_{12}))}{\theta(\delta_1)\theta(\delta_2)} - 2$$

where ρ is the distance between lines (AB) and (CD).

2) If lines (AB) and (CD) define an angle α containing O, then

$$2\cos\alpha = (\delta_1, \delta_2) = 4\frac{(\theta(\delta_1) - \theta(\delta_{21}))(\theta(\delta_2) - \theta(\delta_{21}))}{\theta(\delta_1)\theta(\delta_2)} - 2$$

where $\theta(\delta_{21}) = -\theta(\delta_{12}) = \theta(-\delta_{12})$.

Proof of Lemma 4.2.4. One can prove it similarly to Lemma 4.1.2. One can also prove it as follows. Take a finite point O' and move O' to the infinite point O. Lemma 4.2.4 is the limit of the Lemma 4.1.2 applied to O' when O' tends to O. We leave details to a reader.

Now the proof of Theorems 4.2.1, 4.2.2 and 4.2.3 is the same as proof of the corresponding theorems 4.1.4, 4.1.5 and 4.1.8 if one uses the 'angles' (4.2.17) and Lemma 4.2.4. One has more: almost all inequalities become equalities, and the proof is even simpler.

4.3. Description of narrow parts of fundamental polygons $\mathcal{M}$ of reflection subgroups $W \subset W(S)$ of elliptic and parabolic type where rk $S = 3$. Application to reflective lattices of elliptic and parabolic type and of the rank three.

Let S be a primitive hyperbolic lattice of rk $S = 3$ and $W \subset W(S)$ its reflection subgroup of elliptic or parabolic type with a fundamental polygon $\mathcal{M}$ (see Sects. 1.3, 1.4). Remind that it means that W and $P(\mathcal{M})_{\mathrm{pr}}$ have restricted arithmetic type and $P(\mathcal{M})_{\mathrm{pr}}$ has a generalized lattice Weyl vector ρ with $\rho^2 \geq 0$. It is known (e. g. see [N11], [N9]) that the polygon $\mathcal{M}$ is elliptic if $\rho^2 > 0$, and it is parabolic relative to $\mathbb{R}_{++}\rho$ if $\rho^2 = 0$. We remind that a lattice S having a reflection group $W(S)$ of elliptic or parabolic type is called reflective of elliptic or parabolic type. Here we apply Theorem 4.2.3 to describe narrow parts of $\mathcal{M}$. Using this description, we shall get a finite list of lattices such that any elliptically or parabolically reflective lattice S belongs to the list. Let $a(S^*/S)$ be the exponent of the discriminant group S^*/S: i. e. $a(S^*/S)$ is the least natural a such that $aS^*/S = 0$. We shall get an estimate of $a(S^*/S)$.

Let K be a lattice. We denote by K_0 the primitive lattice defined by K. Thus, $K = K_0(\lambda)$ where $\lambda \in \mathbb{N}$ and K_0 is primitive. If K is generated by elements with the Gram matrix A, then K_0 is generated by the same elements with the Gram matrix A/λ where λ is the greatest common divisor of all elements of A. We denote by $a(H)$ the exponent of a finite Abelian group H: i. e. $a(H)$ is the least natural a such that $aH = 0$.

We have the following useful statement (compare the proof of Theorem 1 in [N5, Appendix]).

Proposition 4.3.1. *Let L be a primitive lattice and $\alpha_1, \dots, \alpha_k$ are primitive roots of L which generate a sublattice $G \subset L$ of a finite index. Let G_0 be the primitive lattice defined by G. Then $a(L^*/L) | 8a(G_0^*/G_0)^2$.*

Proof. We have $G = G_0(\lambda) \subset L$ for some natural $\lambda \in \mathbb{N}$. Since G is generated by roots α_i of L and $\alpha_i^2 | 2(\alpha_i, L)$, we get $2G_0(\lambda) \subset 2L \subset \lambda G_0(\lambda)^*$. Identifying (naturally) modules of the lattices G_0 and $G_0(\lambda)$, we obviously have $\lambda G_0(\lambda)^* = G_0^*$. It follows that

$$L = M(\lambda) \text{ where } G_0 \subset M \subset \frac{1}{2} G_0^*. \tag{4.3.1}$$

Here M is some intermediate module which is invariant with respect to reflections in roots α_i defining the lattice G_0, and the roots α_i should be primitive in M. The $\lambda \in \mathbb{N}$ is the smallest natural number such that $M(\lambda)$ is a lattice (otherwise, the lattice L is not primitive). Using (4.3.1), we get

$$2G_0 \subset M^* \subset G_0^*. \tag{4.3.2}$$

If $tM \subset M^*$, $t \in \mathbb{N}$, then $M(t)$ is a lattice. Really, for any $m_1, m_2 \in M$ we have $t(m_1, m_2) = (tm_1, m_2) \in \mathbb{Z}$ because $tm_1 \in M^*$. Using (4.3.1) and (4.3.2), we get $4aM \subset M^*$ where $a = a(G_0^*/G_0)$. It follows that $\lambda | 4a$. Identifying modules of M and $M(\lambda)$, we have $M(\lambda)^* = (1/\lambda)M^* \subset (1/4a)G_0^*$. It follows that the exponent of $M(\lambda)^*/M(\lambda)$ divides the exponent of $(1/4a)G_0^*/2G_0$ which is equal to $8a^2$. This finishes the proof.

Below we describe narrow parts of the fundamental polygons $\mathcal{M}$. According to Theorem 4.2.3, a narrow part of $\mathcal{M}$ is defined by vectors $\delta_1, \dots, \delta_k$ with $\delta_i^2 = -2$ where $k \le 5$. We denote by r_i corresponding primitive roots of S such that

$$\delta_i = \frac{2r_i}{\sqrt{-2r_i^2}} \tag{4.3.3}$$

with the Gram matrix

$$\Gamma = (\gamma_{ij}) = ((\delta_i, \delta_j)). \tag{4.3.4}$$

Since r_i are primitive roots of S, we have $r_i^2 | 2(S, r_i)$. It then follows that

$$\alpha_{ij} = \gamma_{ij}^2 = (\delta_i, \delta_j)^2 = \frac{4(r_i, r_j)^2}{r_i^2 r_j^2} \tag{4.3.5}$$

are non-negative integers.

We want to describe all possible matrices

$$B = ((r_i, r_j))_{\mathrm{pr}} \tag{4.3.6}$$

where for an integral matrix T we denote by T_{pr} the corresponding primitive integral matrix T/t where $t = \text{g.c.d}(T)$ denote the greatest common divisor of all elements of T. We shall make it in three steps.

First we describe all possible symmetric $(k \times k)$-matrices

$$\mathcal{A} = (\alpha_{ij}) = \left((\delta_i,\, \delta_j)^2\right) = \left(\frac{4(r_i, r_j)^2}{r_i^2 r_j^2}\right). \tag{4.3.7}$$

The matrix $\mathcal{A}$ has non-negative integral coefficients, all its diagonal coefficients are equal to 4. All cyclic products

$$\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_{r-1} i_r}\alpha_{i_r i_1} \tag{4.3.8}$$

are perfect squares.

In the second place, we describe all possible $k \times k$-symmetrizable generalized Cartan matrices (see [Kac] about generalized Cartan matrices, but remember that we use the opposite sign)

$$A = (a_{ij}) = \left(\frac{2(r_i, r_j)}{-r_i^2}\right) \tag{4.3.9}$$

using relations

$$a_{ii} = -2,\ a_{ij} \in \mathbb{Z}_+ \text{ if } i \neq j,\ a_{ij}a_{ji} = \alpha_{ij},\ a_{ij} = 0 \text{ iff } a_{ji} = 0, \tag{4.3.10}$$

and

$$a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_{r-1} i_r}a_{i_r i_1} = a_{i_1 i_r}a_{i_r i_{r-1}}\cdots a_{i_3 i_2}a_{i_2 i_1}. \tag{4.3.11}$$

We remind (see [Kac]) that an integral matrix is *a generalized Cartan matrix,* if all its diagonal elements are equal to 2 and all non-diagonal elements are non-positive. A matrix is called *symmetrizable,* if it is symmetric after multiplication by some diagonal matrix.

In the third place, we find a diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_k) \tag{4.3.12}$$

$\lambda_i \in \mathbb{N}$, such that the matrix

$$B' = A\Lambda \tag{4.3.13}$$

is symmetric. This defines the matrix Λ uniquely up to multiplication by a scalar rational matrix (since the Gram graph of $r_1, \ldots, r_k$ is connected and the matrix A is indecomposable). The matrix $\Lambda = s\,\mathrm{diag}(-r_1^2, \ldots, -r_k^2)$ where $s \in \mathbb{Q}_{++}$. Then we calculate

$$B = \frac{B'}{\text{g.c.d}(B')} \tag{4.3.14}$$

which gives the matrix $((r_i, r_j))_{\mathrm{pr}}$, see (4.3.6). These procedure gives a finite set of possible matrices B. See the corresponding general considerations in [N5, Appendix]).

For the lattice $G_0 = [r_1, \ldots, r_k]_{\mathrm{pr}}$ defined by B we calculate the invariant $a(B) = a(G_0^*/G_0)$, the invariant $a_1(B)$ which is the product of all different odd prime divisors of $a(B)$, and the invariant $a_2(B)$ which is the greatest odd prime divisor of $a(B)$. By Proposition 4.3.1, we get an estimate of the similar invariants $a(S)$, $a_1(S)$ and $a_2(S)$:

$$a(S) \le 8a(B)^2,\ a_1(S) \le a_1(B),\ a_2(S) \le a_2(B). \tag{4.3.15}$$

Calculating the invariants $a(B)$, $a_1(B)$ and $a_2(B)$ for all possible matrices B and taking their maximum, we estimate the invariants $a(S)$, $a_1(S)$ and $a_2(S)$ for all elliptically or parabolically reflective lattices S.

Below we describe this procedure for each type of the narrow part of Theorem 4.2.3.

4.3.1. Matrices B of the narrow parts of the type (I1). It is a particular case of the type (I) of Theorem 4.2.3 when additionally the angle $\alpha_1 \ne \pi/2$. For this case $k = 3$ and δ_1, δ_2, δ_3 give a bases of the 3-dimensional hyperbolic vector space. The matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 4 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 4 \end{pmatrix} \tag{4.3.1.1}$$

with integral non-negative coefficients where from the condition of narrow part of the type (I) we have (after changing numeration if necessary) that

$$\begin{gathered} 1 \le \alpha_{12} = \alpha_{21} \le 4,\quad \alpha_{12} \le \alpha_{23} = \alpha_{32} \le 4, \\ \alpha_{23} \le \alpha_{13} = \alpha_{31} \le \left[\left(\sqrt{2+\sqrt{\alpha_{12}}} + \sqrt{2+\sqrt{\alpha_{23}}} \right)^2 - 2 \right]^2 \end{gathered} \tag{4.3.1.2}$$

(here we also use that $2\cos\frac{\alpha}{2} = \sqrt{2+2\cos\alpha}$), and

$$d = det(\Gamma) = -8 + 2\sqrt{\alpha_{12}\alpha_{23}\alpha_{31}} + 2\alpha_{12} + 2\alpha_{23} + 2\alpha_{13} > 0 \tag{4.3.1.3}$$

where

$$\alpha_{12}\alpha_{23}\alpha_{31} \ \text{ is a perfect square.} \tag{4.3.1.4}$$

It is easy to enumerate the finite set of all matrices $\mathcal{A}$ satisfying (4.3.1.2), (4.3.1.3) and (4.3.1.4). For each $\mathcal{A}$ we then find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & a_{12} & a_{13} \\ a_{21} & -2 & a_{23} \\ a_{31} & a_{32} & -2 \end{pmatrix} \tag{4.3.1.5}$$

using the relations

$$a_{12}a_{21} = \alpha_{12},\ a_{23}a_{32} = \alpha_{23},\ a_{13}a_{31} = \alpha_{13},\ a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21}. \tag{4.3.1.6}$$

The diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{32},\ a_{23}a_{31},\ a_{31}a_{32}). \tag{4.3.1.7}$$

Finally, we get that

$$B = (A\Lambda)_{\text{pr}}. \tag{4.3.1.8}$$

In Appendix 1, we give the Program 4: fund11.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nI1$ of all the matrices B, and the numbers

$$aI1 = \max_B a(B),\ aI1_1 = \max_B a_1(B),\ aI1_2 = \max_B a_2(B). \tag{4.3.1.9}$$

Calculation using this program gives

$$nI1 = 272,\ aI1 = 3528,\ aI1_1 = 543,\ aI1_2 = 181. \tag{4.3.1.10}$$

4.3.2. Matrices B of the narrow parts of the type (I0). It is a particular case of Type (I) of Theorem 4.2.3 when additionally the angle $\alpha_1 = \pi/2$. For this case $k = 3$ and δ_1, δ_2, δ_3 give a bases of the 3-dimensional hyperbolic vector space. The matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} \\ 0 & 4 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 4 \end{pmatrix} \tag{4.3.2.1}$$

with integral non-negative coefficients where from the condition of narrow part of type (I) we have (after changing numeration if necessary) that

$$1 \le \alpha_{23} = \alpha_{32} \le 4,\quad \alpha_{23} \le \alpha_{13} = \alpha_{31} \le \left[\left(\sqrt{2} + \sqrt{2 + \sqrt{\alpha_{23}}}\right)^2 - 2\right]^2, \tag{4.3.2.2}$$

and

$$d = det(\Gamma) = -8 + 2\alpha_{23} + 2\alpha_{13} > 0. \tag{4.3.2.3}$$

For each $\mathcal{A}$ we then find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} \\ 0 & -2 & a_{23} \\ a_{31} & a_{32} & -2 \end{pmatrix} \tag{4.3.2.4}$$

using the relations

$$a_{23}a_{32} = \alpha_{23},\ a_{13}a_{31} = \alpha_{13}. \tag{4.3.2.5}$$

The diagonal matrix

$$\Lambda = \operatorname{diag}(a_{13}a_{32},\ a_{23}a_{31},\ a_{31}a_{32}). \tag{4.3.2.6}$$

Finally, we get that

$$B = (A\Lambda)_{\text{pr}}. \tag{4.3.2.7}$$

In Appendix 1, we give the Program 5: fund10.gen which uses this algorithm to enumerate all the matrices B. For B it calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to defined above numbers $nI0$ and the numbers

$$aI0 = \max_B a(B),\ aI0_1 = \max_B a_1(B),\ aI0_2 = \max_B a_2(B). \tag{4.3.2.8}$$

This program gives

$$nI0 = 2998,\ aI0 = 69192,\ aI0_1 = 10209,\ aI0_2 = 89. \tag{4.3.2.9}$$

4.3.3. Matrices B of the narrow parts of the type (II1). It is a particular case of Type (II) of Theorem 4.2.3 when additionally the angle $\alpha_3 \neq \pi/2$. For this case $k = 4$ and δ_1, δ_2, δ_3, δ_4 generate the 3-dimensional hyperbolic vector space and any three of them give a bases of the space. The matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 4 & 0 & \alpha_{24} \\ \alpha_{31} & 0 & 4 & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 4 \end{pmatrix} \qquad (4.3.3.1)$$

with integral non-negative coefficients where

$1 \le \alpha_{34} = \alpha_{43} \le 4, \quad 4 < \alpha_{13} = \alpha_{31} \le 36,$

$$0 \le \alpha_{14} = \alpha_{41} \le \left[4\max\left(\left(\sqrt{2}+\sqrt{\frac{\sqrt{\alpha_{34}}}{4}+\frac{1}{2}}\right)^2, \frac{\left(2+\sqrt{\frac{\sqrt{\alpha_{34}}}{2}+\frac{5}{4}}\right)^2-\frac{1}{4}}{2} \right) - 2 \right]^2,$$

$$\left[\left(\sqrt{2}+\sqrt{2+\sqrt{\alpha_{34}}}\right)^2 - 2 \right]^2 < \alpha_{24} = \alpha_{42}. \qquad (4.3.3.2)$$

Here we use the conditions (II) of Theorem 4.2.3 and add some inequalities to avoid repeating of cases we have considered in Sect. 4.3.3. We also have

$$det(\Gamma) = 4(4 - \sqrt{\alpha_{13}\alpha_{34}\alpha_{14}} - \alpha_{14} - \alpha_{34} - \alpha_{13}) + \alpha_{24}\alpha_{13} - 4\alpha_{24} = 0 \qquad (4.3.3.3)$$

where

$$\alpha_{13}\alpha_{34}\alpha_{14} \text{ is a perfect square.} \qquad (4.3.3.4)$$

The condition (4.3.3.3) is equivalent

$$\alpha_{24} = \frac{4(\sqrt{\alpha_{13}\alpha_{34}\alpha_{14}} + \alpha_{14} + \alpha_{34} + \alpha_{13} - 4)}{\alpha_{13} - 4}. \qquad (4.3.3.5)$$

For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} \\ 0 & -2 & 0 & a_{24} \\ a_{31} & 0 & -2 & a_{34} \\ a_{41} & a_{42} & a_{43} & -2 \end{pmatrix} \qquad (4.3.3.6)$$

using relations

$$a_{34}a_{43} = \alpha_{34},\ a_{13}a_{31} = \alpha_{13},\ a_{24}a_{42} = \alpha_{24},\ a_{13}a_{34}a_{41} = \sqrt{\alpha_{13}\alpha_{34}\alpha_{14}},$$
$$a_{14}a_{41} = \alpha_{14},\ a_{14} = a_{41} = 0 \text{ if } \alpha_{14} = 0. \qquad (4.3.3.7)$$

The diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{34}a_{42},\ a_{31}a_{43}a_{24},\ a_{31}a_{34}a_{42},\ a_{31}a_{43}a_{42}). \qquad (4.3.3.8)$$

Finally, we get that

$$B = (A\Lambda)_{\mathrm{pr}}. \qquad (4.3.3.9)$$

In Appendix 1, we give the Program 6: fund21.gen which uses this algorithm to enumerate all the matrices B, to calculate $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nII1$, $aII1$, $aII1_1$, $aII1_2$. It gives

$$nII1 = 9818,\ aII1 = 47432,\ aII1_1 = 10965,\ aII1_2 = 487. \qquad (4.3.3.10)$$

4.3.4. Matrices B of the narrow parts of the type (II0). It is a particular case of Type (II) of Theorem 4.2.3 when additionally the angle $\alpha_3 = \pi/2$. For this case $k = 4$ and δ_1, δ_2, δ_3, δ_4 generate the 3-dimensional hyperbolic vector space and any three of them give a bases of the space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 4 & 0 & \alpha_{24} \\ \alpha_{31} & 0 & 4 & 0 \\ \alpha_{41} & \alpha_{42} & 0 & 4 \end{pmatrix} \tag{4.3.4.1}$$

with integral non-negative coefficients where

$$4 < \alpha_{13} = \alpha_{31} \le 36, \quad 0 < \alpha_{14} = \alpha_{41} \le (8 + 4\sqrt{5})^2 = 287.108350\ldots, \ \alpha_{13} \le \alpha_{24}. \tag{4.3.4.2}$$

($8 + 4\sqrt{5} = 16.94427190...$) Here we use the conditions (II) of Theorem 4.2.3 and that $\mathcal{M}$ is a fundamental polygon having at least 4 sides. We also have

$$det(\Gamma) = 4(4 - \alpha_{14} - \alpha_{13}) + \alpha_{24}\alpha_{13} - 4\alpha_{24} = 0 \tag{4.3.4.3}$$

which is equivalent

$$(\alpha_{13} - 4)(\alpha_{24} - 4) = 4\alpha_{14}. \tag{4.3.4.4}$$

For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} \\ 0 & -2 & 0 & a_{24} \\ a_{31} & 0 & -2 & 0 \\ a_{41} & a_{42} & 0 & -2 \end{pmatrix} \tag{4.3.4.5}$$

using relations

$$a_{13}a_{31} = \alpha_{13}, \ a_{24}a_{42} = \alpha_{24}, \ a_{14}a_{41} = \alpha_{14}. \tag{4.3.4.6}$$

The diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{14}a_{42}, \ a_{13}a_{41}a_{24}, \ a_{31}a_{14}a_{42}, \ a_{13}a_{42}a_{41}). \tag{4.3.4.7}$$

Finally, we get that

$$B = (A\Lambda)_{\mathrm{pr}}. \tag{4.3.4.8}$$

In Appendix 1, we give the Program 7: fund20.gen which uses this algorithm to enumerate all the matrices B, to calculate $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nII0$, $aII0$, $aII0_1$, $aII0_2$. It gives

$$nII0 = 376208, \ aII0 = 995316, \ aII0_1 = 238569, \ aII0_2 = 283. \tag{4.3.4.9}$$

4.3.5. Matrices B of the narrow parts of the type (III). For this case $k = 5$ and δ_1, δ_2, δ_3, δ_4, δ_5 generate the 3-dimensional hyperbolic vector space, and any three of them give a bases of the space. The matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ 0 & 4 & 0 & \alpha_{24} & \alpha_{25} \\ \alpha_{31} & 0 & 4 & 0 & \alpha_{35} \\ \alpha_{41} & \alpha_{42} & 0 & 4 & 0 \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & 0 & 4 \end{pmatrix} \tag{4.3.5.1}$$

with integral non-negative coefficients where

$$4 < \alpha_{13} = \alpha_{31} \le 36, \quad \alpha_{31} \le \alpha_{35} = \alpha_{53} \le 36, \quad 0 \le \alpha_{15} = \alpha_{51} \le 30^2 = 900,$$
$$287.108350 < \alpha_{14} = \alpha_{41}, \quad 287.108350 < \alpha_{25} = \alpha_{52}. \tag{4.3.5.2}$$

Here we use the condition (III) of Theorem 4.2.3 and that $\mathcal{M}$ is a fundamental polygon having at least 5 sides. The last two inequalities were added to avoid repetition with the previous case. The Gram matrix $\Gamma = \Gamma(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ has the rank 3. Coefficients of the matrix $\mathcal{A}$ are defined by the coefficients α_{13}, α_{15}, α_{35} defining the Gram matrix $\Gamma(\delta_1, \delta_3, \delta_5)$. Let

$$d = 4(\alpha_{13} + \alpha_{35} + \alpha_{15} + \sqrt{\alpha_{13}\alpha_{35}\alpha_{15}} - 4), \tag{4.3.5.3}$$

one can see that $d/2 = det(\Gamma(\delta_1, \delta_3, \delta_5))$. We have

$$\alpha_{14} = \frac{d}{\alpha_{35} - 4}, \quad \alpha_{25} = \frac{d}{\alpha_{13} - 4} \tag{4.3.5.4}$$

and

$$\alpha_{24} = \frac{4(\alpha_{13}\alpha_{35} + 4\sqrt{\alpha_{13}\alpha_{35}\alpha_{15}} + 4\alpha_{15})}{(\alpha_{13} - 4)(\alpha_{35} - 4)} \tag{4.3.5.5}$$

Here (4.3.5.4) follows from $\det(\Gamma(\delta_1, \delta_3, \delta_5, \delta_2)) = \det(\Gamma(\delta_1, \delta_3, \delta_5, \delta_4)) = 0$. To get (4.3.5.5), one should remark that $\delta_2 = \gamma_{25}\delta_5^*$ and $\delta_4 = \gamma_{14}\delta_1^*$. It follows that $\gamma_{24} = \gamma_{25}\gamma_{14}(g^{-1})_{13}$ where $g = \Gamma(\delta_1, \delta_3, \delta_5)$. All together, (4.3.5.4) and (4.3.5.5) are equivalent to rk $(\Gamma(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)) = 3$, or that δ_1, δ_2, δ_3, δ_4, δ_5 generate a 3-dimensional hyperbolic form. Moreover, cyclic products

$$\alpha_{13}\alpha_{35}\alpha_{52}\alpha_{24}\alpha_{41}, \quad \alpha_{13}\alpha_{35}\alpha_{51} \ \text{ are perfect squares.} \tag{4.3.5.6}$$

For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & -2 & 0 & a_{24} & a_{25} \\ a_{31} & 0 & -2 & 0 & a_{35} \\ a_{41} & a_{42} & 0 & -2 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & -2 \end{pmatrix} \tag{4.3.5.7}$$

using relations

$$a_{13}a_{31} = \alpha_{13}, \ a_{35}a_{53} = \alpha_{35}, \ a_{14}a_{41} = \alpha_{41}, \ a_{25}a_{52} = \alpha_{25}, \ a_{15}a_{51} = \alpha_{15},$$
$$a_{15} = a_{51} = 0 \text{ if } \alpha_{15} = 0, \ a_{24}a_{42} = \alpha_{24}, \ a_{25}a_{52} = \alpha_{25}, \ a_{35}a_{53} = \alpha_{35},$$
$$a_{13}a_{35}a_{52}a_{24}a_{41} = \sqrt{\alpha_{13}\alpha_{35}\alpha_{52}\alpha_{24}\alpha_{41}}, \ a_{13}a_{35}a_{51} = \sqrt{\alpha_{13}\alpha_{35}\alpha_{51}}. \tag{4.3.5.8}$$

The diagonal matrix

$$\Lambda = \text{diag}(a_{14}a_{13}a_{42}a_{25}, \ a_{41}a_{13}a_{24}a_{25}, \ a_{14}a_{31}a_{42}a_{25}, \ a_{41}a_{13}a_{42}a_{25}, \ a_{41}a_{13}a_{24}a_{52}). \tag{4.3.5.9}$$

Finally, we get that

$$B = (A\Lambda)_{\text{pr}}. \tag{4.3.5.10}$$

In Appendix 1, we give the Program 8: fund30.gen which uses this algorithm to enumerate all the matrices B, to calculate $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nIII$, $aIII$, $aIII_1$, $aIII_2$. It gives

$$nIII = 200539, \ aIII = 324900, \ aIII_1 = 26565, \ aIII_2 = 907. \tag{4.3.5.11}$$

4.3.6. The global estimate of invariants of primitive reflective hyperbolic lattices of the rank 3 having elliptic or parabolic type. For a lattice L we denote by $a(L^*/L)$ the exponent of the discriminant group L^*/L, by $a_1(L^*/L)$ the product of all different odd prime divisors of $a(L^*/L)$, and by $a_2(L^*/L)$ the greatest prime divisor of $a(L^*/L)$.

Using Proposition 4.3.1 and calculations (4.3.1.10), (4.3.2.9), (4.3.3.10), (4.3.4.9), (4.3.5.11) we get

Theorem 4.3.6.1. *For any primitive reflective hyperbolic lattice S of rk $S = 3$ having elliptic or parabolic type we have estimates:*

$$a(S^*/S) \le 8(995316)^2, \quad a_1(S^*/S) \le 238569, \quad a_2(S^*/S) \le 907.$$

Since $\det(S) \le a(S^*/S)^2$ and number of lattices with the fixed rank and determinant is finite (e. g. see [C]), Theorem 4.3.6.1 gives a finite set of lattices which contains all the reflective lattices S of the rank three and of elliptic or parabolic type.

The estimates of Theorem 4.3.6.1 are very preliminary. They can be significantly improved using similar calculations and Basic Theorem 2.3.2.1 which we prove in the next Section.

5. Classification of reflective hyperbolic lattices of the rank 3 and of elliptic or parabolic type: proofs

Now we are ready to prove classification results of Sect. 2.

5.1. Proof of Basic Theorem 2.3.2.1.

We first prove

Theorem 5.1.1. *Any elliptically or parabolically reflective main hyperbolic lattice S of rank 3 belongs to Table 3 containing all main hyperbolic lattices S of the rank 3 with $d = \det(S) \le 100000$ and $h = hnr(S) \le 1$.*

Proof. Let S be an elliptically or parabolically reflective main hyperbolic lattice of the rank 3. Let $\mathcal{M}$ be a fundamental polygon of $W(S)$.

First we will show that $h(S) \le 1$ (it gives the proof of Lemma 2.3.1.2 for elliptic and parabolic cases). If S is elliptically reflective, then $\mathcal{M}$ is an elliptic (finite and of finite volume) polygon. This polygon has ≤ 1 central symmetries. Really, otherwise, a composition of two different central symmetries gives an automorphism of infinite order of $\mathcal{M}$ which is impossible. It follows that $h(S) \le 1$. Assume that S is parabolically reflective and $\rho \in S$ is a generalized lattice Weyl vector for $\mathcal{M}$. Let $A(\mathcal{M}) \subset O^+(S)$ be the group of symmetries of $\mathcal{M}$. By definition, ρ is preserved by a subgroup $A \subset A(\mathcal{M})$ of finite index, $\rho^2 = 0$ and $\rho \ne 0$. For any $\phi \in A(\mathcal{M})$, the element $\phi(\rho)$ also has all these properties and is then a generalized lattice Weyl vector for $\mathcal{M}$. If $\phi(\rho) \ne \rho$, the group $A(\mathcal{M})$ has a subgroup of finite index which is trivial on the hyperbolic sublattice $\mathbb{Z}\rho + \mathbb{Z}\phi(\rho) \subset S$. It then follows that $A(\mathcal{M})$ is finite and S is elliptically reflective. We get a contradiction. Thus $\phi(\rho) = \rho$ for any $\phi \in A(\mathcal{M})$. If $u \in A(\mathcal{M})$ is a central symmetry, then the fixed part $S^u = \{x \in S \mid u(x) = x\}$ of u is a positive definite lattice of the rank 1 which does not have any non-zero element ρ with $\rho^2 = 0$. This shows that $h = h(S) = 0$.

Now we apply to S and $\mathcal{M}$ results of Sect. 4.3. By Proposition 4.3.1 and (4.3.1.10), (4.3.2.9), (4.3.3.10), (4.3.5.11), we have $d \le 100000$ if $\mathcal{M}$ has a narrow part of types (I1), (I0), (II1) or (III). Really, for all this cases the invariant $a_1 < 50000$ and then $d \le 2a_1(S) \le 2a_1 < 100000$. Thus, Theorem 5.1.1 is valid for these lattices S.

Only if $\mathcal{M}$ has a narrow part of type (II0), our estimate (4.3.4.9) of a_1 is not good enough:

$$a_1(S) \le aII0_1 = 238569, \quad a_2(S) \le 238. \tag{5.1.1}$$

Let us consider this case. In Sect. 4.3.4, for all primitive elliptically or parabolically reflective hyperbolic lattices S or rank 3 we considered the primitive Gram matrices $B = \Gamma_{\text{pr}}$ where $\Gamma = \Gamma(\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4)$ is the Gram matrix of primitive roots of S corresponding to a narrow part of type (II0) of $\mathcal{M}$. We now consider them only for main lattices S with square-free $d = \det(S)$. Then we should add some additional conditions to Sect. 4.3.4. We introduce these conditions below.

Let $\beta \in S$ be a primitive root and $K = \beta^\perp$. Then

$$\text{either } S = \mathbb{Z}\beta \oplus K \text{ or } S = [\beta,\ K, (\beta + k)/2] \text{ where } k \in K. \tag{5.1.2}$$

It follows (by a simple consideration over $\mathbb{Z}_2$) that

$$\beta^2 \text{ is square-free.} \tag{5.1.3}$$

If $\beta_1,\, \beta_2 \in S$ are two primitive orthogonal roots and $\mathbb{Z}h = [\beta_1,\, \beta_2]^\perp$, then for any odd p

$$S \otimes \mathbb{Z}_p = \mathbb{Z}_p\beta_1 \oplus \mathbb{Z}_p\beta_2 \oplus \mathbb{Z}_p h \text{ and } h^2 \text{ is square-free.} \tag{5.1.4}$$

It follows

$$\text{g.c.d.}(\beta_1^2, \beta_2^2) \le 2, \quad \text{g.c.d.}(\beta_1^2, h^2) \le 2, \quad \text{g.c.d.}(\beta_2^2, h^2) \le 2, \tag{5.1.5}$$

and

$$d = \beta_1^2\beta_2^2h^2/2^t, \text{ where } t \equiv 0 \mod 2. \tag{5.1.6}$$

It defines the t uniquely because d is square-free. From (5.1.6), for any odd $p|d$, we also have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{\beta_1^2/p}{p}\right) & \text{if } p|\beta_1^2, \\ \left(\frac{\beta_2^2/p}{p}\right) & \text{if } p|\beta_2^2, \\ \left(\frac{h^2/p}{p}\right) = \left(\frac{d\beta_1^2\beta_2^2/p}{p}\right) & \text{if } p|h^2, \end{cases} \tag{5.1.7}$$

where η is the invariant of S. The Gram matrix $\Gamma = \Gamma(\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4)$ has three pairs of orthogonal roots: $(\alpha_1,\, \alpha_2) = (\alpha_2,\, \alpha_3) = (\alpha_3, \alpha_4) = 0$. It then follows (it is sufficient to have only one pair) that

$$\text{either } \Gamma = B \text{ or } \Gamma = 2B. \tag{5.1.8}$$

By (5.1.3), in (5.1.8), the second case is possible only if $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$. We denote (like in Sect. 4.3) by G_0 the lattice defined by the primitive matrix B. Suppose that $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$ and $\det(G_0) = 2^km$ where m is odd and $k \equiv 1 \mod 2$. We then have the second case $\Gamma = 2B$ in (5.1.8), because otherwise S is odd but d is even. Thus, for a fixed B, we may have both cases in (5.1.8) only if $b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2$ and $\det(G_0) = 2^km$ where

m is odd and $k \equiv 0 \mod 2$. In all other cases the matrix B prescribes the Gram matrix Γ. For both cases in (5.1.8), we can calculate d using the matrix B since d is square-free. If one of α_i^2 is odd but d is even, then $\Gamma = B$, and this case is impossible for a main S (it should be even for even d).

Summarizing considerations above, we get the following additional conditions for B and calculation of the invariants (d, η) of the lattice S using the matrix B. We denote by G_0 the lattice with the matrix B. We have

$$b_{11},\ b_{22},\ b_{33},\ b_{44} \text{ are square-free} \tag{5.1.9}$$

and

$$\text{g.c.d}(b_{11}, b_{22}) \le 2,\ \text{g.c.d}(b_{22}, b_{33}) \le 2,\ \text{g.c.d}(b_{33}, b_{44}) \le 2. \tag{5.1.10}$$

We denote by $\nu_p(a)$ the order of p in factorisation of a. We have

$$\text{if odd } p | b_{11}b_{22}b_{33}b_{44},\ \text{then } \nu_p(\det(G_0)) \equiv 1 \mod 2. \tag{5.1.11}$$

We have

$$\Gamma = B,\ \text{if not all } b_{11},\ b_{22},\ b_{33},\ b_{44} \text{ are odd}; \tag{5.1.12}$$

$$\Gamma = 2B,\ \text{if } b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2 \text{ and } \nu_2(det(G_0)) \equiv 1 \mod 2; \tag{5.1.13}$$

$$\Gamma = B \text{ or } \Gamma = 2B,\ \text{if } b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv 1 \mod 2 \text{ and } \nu_2(det(G_0)) \equiv 0 \mod 2. \tag{5.1.14}$$

We denote by G the lattice generated by α_i and defined by the Gram matrix $\Gamma = ((\alpha_i,\ \alpha_j))$. Let d be the product of all prime p such that $p|\det(G)$ and $\nu_p(det(G)) \equiv 1 \mod 2$. We have

$$\nu_2(d) \equiv 0 \mod 2 \text{ if one of } \alpha_i^2 \text{ is odd}. \tag{5.1.15}$$

For odd $p|d$ we have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{\alpha_1^2/p}{p}\right) & \text{if } p|\alpha_1^2, \\ \left(\frac{\alpha_2^2/p}{p}\right) & \text{if } p|\alpha_2^2, \\ \left(\frac{d\alpha_1^2\alpha_2^2/p}{p}\right) & \text{otherwise.} \end{cases} \tag{5.1.16}$$

This defines the invariant η. At last, we have the most delicate condition which significantly drops the finite number of possibilities:

$$h = hnr(d, \eta) \le 1 \tag{5.1.17}$$

(see Theorem 3.2.1 about $hnr(d, \eta)$). To calculate $hnr(d, \eta)$ using Theorem 3.2.1, we should calculate class-numbers of imaginary quadratic fields of the discriminant D where $-4d \le D < 0$. By (5.1.1), $d \le 2\cdot 238569$ and $0 < -D \le 8\cdot 238569 = 1908552$. Moreover, by (5.1.1), prime divisors of D are not more than 238. Thus, checking the condition (5.1.17), we work with reasonable (not too big) integers. Or, if one wants, we should use a program which correctly calculates class-numbers of discriminants D where $-1908552 \le D < 0$ and D is product of primes $p \le 238$.

In Appendix: Program 9: fund20.main, we give a program for GP/PARI calculator, which enumerates all the matrices B satisfying the conditions of Sect. 4.3 (this part of Program 9 is the same as the Program 7: fund20.gen) and conditions (5.1.9) — (5.1.17), and gives all triplets of invariants (d, η, h) of the matrices B (it gives 132 triplets). We can see that all these triplets belong to the Table 3. The GP/PARI calculator calculates class numbers of negative discriminants D for $|D| < 10^{25}$ (see user's guide to the calculator). This finishes the proof of Theorem 5.1.1.

By Theorem 5.1.1, to prove Theorem 2.3.2.1, we should now check reflective type of all lattices of Table 3 and for elliptically or parabolically reflective lattices calculate the sets $P(\mathcal{M})_{\mathrm{pr}}$ and the Gram matrices $G(P(\mathcal{M})_{\mathrm{pr}})$. In Table 3, for each pair of invariants (d, η) we give a main lattice S with these invariants. If

$$S = U \oplus \langle -d\rangle, \tag{5.1.18}$$

we have

$$(-1)^{\eta_p} = \left(\frac{-d/p}{p}\right), \text{ for any odd prime } p|d. \tag{5.1.19}$$

If

$$S = \langle n_1\rangle \oplus \langle -n_2\rangle \oplus \langle -n_3\rangle(\epsilon_1/2, \epsilon_2/2, \epsilon_3/2) \tag{5.1.20}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$, we have $d = n_1n_2n_3$ if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, and $d = n_1n_2n_3/4$ otherwise. Moreover, $n_i \equiv 0 \mod 2$ if $\epsilon_i = 1$; $(n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3)/4 \in \mathbb{Z}$; $(n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3)/4 \in 2\mathbb{Z}$ if $d \equiv 0 \mod 2$; for any odd $p|d$ we have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{n_1/p}{p}\right) & \text{if } p|n_1, \\ \left(\frac{-n_2/p}{p}\right) & \text{if } p|n_2, \\ \left(\frac{-n_3/p}{p}\right) & \text{if } p|n_3. \end{cases} \tag{5.1.21}$$

Checking all these conditions, one can prove that our calculation of main lattices S corresponding to the invariants (d, η) of Table 3 are correct. Using known criteria (e. g. see [Se]), it is easy to prove (see Lemma 9.1.2 below) that $S = U \oplus \langle -d\rangle$ if and only if S represents 0. The condition (5.1.19) is equivalent to this property. If (5.1.19) is valid, we always give S in the form $S = U \oplus \langle -d\rangle$.

Thus, fortunately, all our lattices S have one of two forms (5.1.18) or (5.1.20).

We use the Vinberg's algorithm [V2] to calculate the sets $P(\mathcal{M})_{\mathrm{pr}}$ for lattices of the forms (5.1.18) and (5.1.20). Below we describe this algorithm.

First, we remark that the lattice $U \oplus \langle -d\rangle$ is equivariantly equivalent to its maximal even sublattice $U \oplus \langle -4d\rangle$ if d is odd. These lattices have naturally isomorphic groups of automorphisms and the reflection groups. Thus, it is sufficient to consider only the lattices

$$U \oplus \langle -2k\rangle, \quad k \in \mathbb{N}. \tag{5.1.22}$$

For lattices S of the form (5.1.22), we use the isotropic vector $c = (1, 0, 0)$ as the center of Vinberg's algorithm. We find $\mathcal{M}$ which contains $\mathbb{R}_{++}c$ as an infinite vertex and $v_1 = (0, 0, 1)$, $v_2 = (k, 0, -1)$ as roots orthogonal to faces of $\mathcal{M}$ containing $\mathbb{R}_{++}c$

(equivalently, they are roots of the height 0). It is easy to see that v_1, v_2 give the orthogonal primitive roots to the fundamental polygon of the stabilizer subgroup of c in the reflection group $W(S)$.

For lattices S of the form (5.1.20) we assume that either $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)/2 = 0$ or $\epsilon_i \neq 0$ at least for two different i. We take $c = (1, 0, 0)$ as the center of Vinberg's algorithm. We take as roots of the height 0
$v_1 = (0, 1, 0)$ and $v_2 = (0, 0, 1)$ if $n_2/n_3 \neq 1/3,\ 1,\ 3$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1, 1)$ if $n_2 = n_3$ and $\epsilon = (0, 0, 0)$, $(1/2, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1/2, 1/2)$ if $n_2 = n_3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2 = n_3$ and $\epsilon \neq (0, 0, 0)$, $(1/2, 1/2, 1/2)$, $(0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, -1/2, 1/2)$ if $n_2/n_3 = 3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2/n_3 = 3$ and $\epsilon \neq (0, 1/2, 1/2)$;
$v_1 = (0, 0, 1)$, $v_2 = (0, 1/2, -1/2)$ if $n_2/n_3 = 1/3$ and $\epsilon = (0, 1/2, 1/2)$;
$v_1 = (0, 1, 0)$, $v_2 = (0, 0, 1)$ if $n_2/n_3 = 1/3$ and $\epsilon \neq (0, 1/2, 1/2)$.
One can check that v_1, v_2 are the orthogonal primitive roots to the fundamental polygon of the stabilizer subgroup of c in $W(S)$.

Further steps of the Vinberg's algorithm are prescribed canonically (see [V2]). One introduces the height

$$\text{height} = \frac{2(\delta,\, c)^2}{-\delta^2} \in \mathbb{Z} \tag{5.1.23}$$

of primitive roots $\delta \in S$ with $(\delta,\, c) \geq 0$. The roots $\{v_1,\, v_2\}$ above give all roots of $P(\mathcal{M})_{\text{pr}}$ of the height 0. If one knows all roots of $P(\mathcal{M})_{\text{pr}}^{\leq n} \subset P(\mathcal{M})_{\text{pr}}$ of the height $\leq n$, all primitive roots $P(\mathcal{M})_{\text{pr}}^{n+1} \subset P(\mathcal{M})_{\text{pr}}$ of the height $n+1$ are given by the conditions

$$(\delta,\, c) \geq 0,\ \text{ and }\ \frac{2(\delta,\, c)^2}{-\delta^2} = n+1,\ \text{ and }\ (\delta,\, P(\mathcal{M})_{\text{pr}}^{\leq n}) \geq 0. \tag{5.1.24}$$

In Appendix 1, Program 10: refl0.1, we give this algorithm for lattices (5.1.22). In Appendix 1, Program 12: refl0.13, we give this algorithm for lattices (5.1.20). We also give Program 11: refl0.12 which calculates only for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ but is faster. These programs calculate a sequence v_i of all elements of the subset $P(\mathcal{M})_{\text{pr}}^{\leq n} \subset P(\mathcal{M})_{\text{pr}}$ (i. e. all roots of the height $\leq n$).

After calculation of $P(\mathcal{M})_{\text{pr}}^{\leq n}$ for a sufficiently large n, next steps of our algorithm are as follows. A sequence $r = r_1, \dots, r_k$ of different elements from $P(\mathcal{M})_{\text{pr}}$ is called a *chain* if

$$0 \leq \frac{4(r_i, r_{i+1})^2}{r_i^2 r_{i+1}^2} \leq 4 \text{ for any } 1 \leq i \leq k-1. \tag{5.1.25}$$

Geometrically it means that $\mathcal{H}_{r_1}, \dots, \mathcal{H}_{r_k}$ are lines of consecutive sides (i. e. defining vertices) of the polygon $\mathcal{M}$. We find the maximal chain $e = (e_1, \dots, e_k)$ in $P(\mathcal{M})_{\text{pr}}^{\leq n}$ containing v_1. Suppose that (for a sufficiently large n) we also have

$$\frac{4(e_1, e_k)^2}{e_1^2 e_k^2} \leq 4, \tag{5.1.26}$$

i. e. the lines $\mathcal{H}_{e_k}$ and $\mathcal{H}_{e_1}$ also define a vertex of $\mathcal{M}$. Then the polygon $\mathcal{M}$ is elliptic, $P(\mathcal{M})_{\text{pr}} = e$, and the chain e gives the orthogonal primitive roots to consecutive sides of the elliptic polygon $\mathcal{M}$. This situation takes place for 122 invariants (d, η) of the Table 3 marked by *er* (elliptically reflective cases). The result of our calculation of the chain e and the Gram matrix $((e_i, e_j))$, e_i, $e_j \in e$, for these 122 cases is given in Table 1.

For all other $206 - 122 = 84$ cases we should prove that S is not reflective of elliptic or parabolic type.

To prove that, for small determinants d we use the following arguments. For a large height n, the chain $e \subset P(\mathcal{M})_{\text{pr}}^{\leq n}$ contains a *period:* there exists $1 < q < k$ such that the Gram matrices $\Gamma(e_1, e_2)$ and $\Gamma(e_q, e_{q+1})$ coincide and there exists $C \in O^+(S)$ such that $C(e_1) = e_q$, $C(e_2) = e_{q+1}$. It follows that $C \in A(\mathcal{M})$. We find all integral (i.e. from S) eigenvectors u of C, and show that all of them have $u^2 < 0$. Clearly, these eigenvectors have the eigenvalue ± 1. It follows that S is not elliptically or parabolically reflective. Really, if S is elliptically reflective, the group $A(\mathcal{M})$ is finite, C has finite order and has an integral eigenvector with the eigenvalue 1 and positive square. If S is parabolically reflective, then A has an eigenvector with the eigenvalue 1 and with zero square (the generalized lattice Weyl vector ρ). See the proof above of Lemma 2.3.1.2. If S is hyperbolically reflective, the eigenvector u defines a generalized lattice Weyl vector ρ.

Let us consider an example of these calculations for $(d, \eta) = (114, 2)$. Then $S = U \oplus \langle -114 \rangle$. Using Program 10, we calculate up to the height $n = 50000$ and find the chain e in $P(\mathcal{M})_{\text{pr}}^{\leq n}$ which is equal to

$$\begin{aligned} &e_1 = (321, 30, -13),\ e_2 = (28, 2, -1),\ e_3 = (57, 0, -1),\ e_4 = (0, 0, 1),\\ &e_5 = (-1, 1, 0),\ e_6 = (9, 6, -1),\ e_7 = (292, 122, -25) \end{aligned} \tag{5.1.27}$$

with the Gram matrix

$$ge = \begin{pmatrix} -6 & 0 & 228 & 1482 & 291 & 714 & 10872\\ 0 & -2 & 0 & 114 & 26 & 72 & 1150\\ 228 & 0 & -114 & 114 & 57 & 228 & 4104\\ 1482 & 114 & 114 & -114 & 0 & 114 & 2850\\ 291 & 26 & 57 & 0 & -2 & 3 & 170\\ 714 & 72 & 228 & 114 & 3 & -6 & 0\\ 10872 & 1150 & 4104 & 2850 & 170 & 0 & -2 \end{pmatrix}. \tag{5.1.28}$$

We see that the Gram matrices of pairs e_1, e_2 and e_6, e_7 coincide. We calculate that

$$C = \begin{pmatrix} 2209 & 22800 & 107160\\ 912 & 9409 & 44232\\ -188 & -1940 & -9119 \end{pmatrix} \tag{5.1.29}$$

belongs to $O^+(S)$ and $C(e_1, e_2) = (e_6, e_7)$. It follows that $C \in A(\mathcal{M})$. The matrix C has the only integral eigenvector $w = (95, 19, -6)$, it has the eigenvalue 1. We have $w^2 = -494$. It follows that S is not elliptically or parabolically reflective: the automorphism C has infinite order and does not have integral eigenvectors with non-negative square.

Since for $(d, \eta) = (114, 2)$ the invariant $h = 0$, it is possible that S is hyperbolically reflective with the generalized lattice Weyl vector $\rho = w$. Let us prove that this is the case. We find another chain f in $P(\mathcal{M})_{\mathrm{pr}}^{\le n}$ which is equal to

$$\begin{gathered} f_1=(1766,172,-73),\ f_2=(6384,627,-265),\ f_3=(4560,456,-191),\\ f_4=(283,29,-12),\ f_5=(18,3,-1),\ f_6=(14,4,-1),\\ f_7=(456,171,-37),\ f_8=(2280,912,-191),\ f_9=(427,173,-36) \end{gathered} \tag{5.1.30}$$

with the Gram matrix

$$gf=\begin{pmatrix} -2 & 0 & 114 & 26 & 72 & 1150 & 72504 & 413250 & 79370\\ 0 & -114 & 114 & 57 & 228 & 4104 & 259806 & 1481658 & 284601\\ 114 & 114 & -114 & 0 & 114 & 2850 & 182058 & 1039566 & 199728\\ 26 & 57 & 0 & -2 & 3 & 170 & 11001 & 62928 & 12094\\ 72 & 228 & 114 & 3 & -6 & 0 & 228 & 1482 & 291\\ 1150 & 4104 & 2850 & 170 & 0 & -2 & 0 & 114 & 26\\ 72504 & 259806 & 182058 & 11001 & 228 & 0 & -114 & 114 & 57\\ 413250 & 1481658 & 1039566 & 62928 & 1482 & 114 & 114 & -114 & 0\\ 79370 & 284601 & 199728 & 12094 & 291 & 26 & 57 & 0 & -2 \end{pmatrix}. \tag{5.1.31}$$

We see that the Gram matrix of the pair f_1, f_2 is equal to the Gram matrix of the pair f_6, f_7. We calculate that $C(f_1, f_2) = (f_6, f_7)$. Moreover, we see that $(e_i, w) > 0$ and $(f_i, w) < 0$.

It follows that $P(\mathcal{M})_{\mathrm{pr}} = H(e) \cup H(f)$ where $H = [C]$ is the infinite cyclic group generated by the C. The infinite polygon $\mathcal{M}$ contains the line (the axis) $\mathcal{H}_w$ which is preserved by H. All sides of $\mathcal{M}$ define two infinite chains $\mathcal{H}_\delta$, $\delta \in H(e)$, and $\mathcal{H}_\delta$, $\delta \in H(f)$, these two chains are contained in two different half-planes bounded by the axis $\mathcal{H}_w$. The polygon $\mathcal{M}$ is finite in every orthogonal cylinder over a compact base in $\mathcal{H}_w$ (it is restricted hyperbolic relative to $\mathcal{H}_w$). The polygon $\mathcal{M}$ has an infinite symmetry group $H = [C]$. But it is not the full symmetry group $A(\mathcal{M})$ of $\mathcal{M}$. From our considerations, it follows that H is the full symmetry group of $\mathcal{M}$ which leaves each chain $H(e)$ and $H(f)$ invariant. But $\mathcal{M}$ has also symmetries which change places the chains $H(e)$ and $H(f)$. It is sufficient to add any of them to the group H to get the full group of symmetries of $\mathcal{M}$. The $A(\mathcal{M})$ is the cyclic group generated by the glide reflection

$$S_1=\begin{pmatrix} 48 & 475 & 2280\\ 19 & 192 & 912\\ -4 & -40 & -191 \end{pmatrix}. \tag{5.1.32}$$

with the axis $\mathcal{H}_w$. The vector w is the eigenvector of S_1 with the eigenvalue -1. We have $S_1^2 = C$. The glide reflection S_1 changes places the infinite chains $H(e)$ and $H(f)$. The lattice S is hyperbolically reflective with the generalized lattice Weyl vector $\rho = w$. To describe $P(\mathcal{M})_{\mathrm{pr}}$, it is sufficient to give $\rho = w = (95, 19, -6)$, the glide reflection S_1 generating $A(\mathcal{M})$, and the part e of the chain $H(e)$ because $P(\mathcal{M}) = A(\mathcal{M})(e)$. The result of this calculations is given in Part II, Table 4.

For large d following arguments are very useful. We calculate the set $P(\mathcal{M})_{\mathrm{pr}}^{\le n}$ for a sufficiently large height n and find two pairs of elements $r_1, r_2 \in P(\mathcal{M})_{\mathrm{pr}}^{\le n}$, $s_1, s_2 \in P(\mathcal{M})_{\mathrm{pr}}^{\le n}$ such that $4(r_1,r_2)^2/r_1^2r_2^2 \le 4$, $4(s_1,s_2)^2/s_1^2s_2^2 \le 4$ and Gram matrices of these pairs coincide. In many cases it is sufficient to consider pairs with $(r_1, r_2) = (s_1, s_2) = 0$ and $r_1^2 = s_1^2$, $r_2^2 = s_2^2$. Thus, orthogonal sides of these pairs define two vertices of $\mathcal{M}$. We find such pairs that there exists an automorphism

$B \in O^+(S)$ such that $B(r_i) = s_i$. Then $B \in A(\mathcal{M})$. Similarly, considering two other pairs in $P(\mathcal{M})_{\text{pr}}^{\le n}$, we find another $C \in A(\mathcal{M})$. If $B^2C^2 \ne C^2B^2$, the lattice S is not reflective of any type: elliptic, parabolic or hyperbolic. In many cases it is sufficient to find only one $B \in A(\mathcal{M})$ and calculate that $B^{12} \ne E$. Really, then B has infinite order and the lattice S cannot be elliptically reflective. If $h = 1$, the lattice S cannot be parabolically or hyperbolically reflective either.

We consider an example of these calculations for $(d, \eta) = (3990, 4)$. Then $S = \langle 30\rangle \oplus \langle -38\rangle \oplus \langle -14\rangle(1/2, 1/2, 0)$. Using Program 12 from Appendix, we calculate up to the height $n = 500000$. It gives 33 elements $v_i \in P(\mathcal{M})_{\text{pr}}^{\le n}$. They are

$$
\begin{aligned}
&v_1 = (0,1,0),\ v_2 = (0,0,1),\ v_3 = (1/2,-1/2,0),\\
&v_4 = (2,0,-3),\ v_5 = (13/2,-9/2,-6),\ v_6 = (7,-3,-9),\\
&v_7 = (57/2,-21/2,-38),\ v_8 = (17/2,-9/2,-10),\ v_9 = (28,-21,-22),\\
&v_{10} = (63/2,-35/2,-36),\ v_{11} = (57,-37,-57),\ v_{12} = (25,-15,-27),\\
&v_{13} = (42,-14,-57),\ v_{14} = (17,-4,-24),\ v_{15} = (76,-58,-57),\\
&v_{16} = (19,-6,-26),\ v_{17} = (52,-15,-72),\ v_{18} = (84,-21,-118),\\
&v_{19} = (1729/2,-1015/2,-950),\ v_{20} = (119/2,-69/2,-66),\\
&v_{21} = (69/2,-43/2,-36),\\
&v_{22} = (73,-57,-51),\ v_{23} = (74,-57,-54),\ v_{24} = (231/2,-147/2,-118),\\
&v_{25} = (2261/2,-1645/2,-950),\ v_{26} = (101,-27,-141),\ v_{27} = (266,-132,-323),\\
&v_{28} = (119,-87,-99),\ v_{29} = (128,-63,-156),\ v_{30} = (399/2,-315/2,-134),\\
&v_{31} = (342,-96,-475),\ v_{32} = (361,-77,-513),\ v_{33} = (238,-119,-288).
\end{aligned}
\tag{5.1.33}
$$

We look at all the pairs v_i, v_j with $(v_i, v_j) = 0$, and we find that there are five such pairs v_1, v_2; v_7, v_{13}; v_{15}, v_9; v_{11}, v_{24}; v_{27}, v_{33} having squares $v_i^2 = -38$, $v_j^2 = -14$. We calculate a primitive orthogonal element $w \in S$ to each of these five pairs v_i, v_j, and we find that between these five pairs there are two pairs v_{11}, v_{24} and v_{27}, v_{33} such that $(w + v_j)/2 \in S$. We then find the matrix

$$
C = \begin{pmatrix} 6863/2 & 5339/2 & 1694 \\ -3345/2 & -2601/2 & -826 \\ -4200 & -3268 & -2073 \end{pmatrix}
\tag{5.1.34}
$$

such that $C \in O^+(S)$ and $C(v_{11}, v_{24}) = (v_{27}, v_{33})$. It follows that $C \in A(\mathcal{M})$. We have $C^{12} \ne E$. It follows that C has infinite order. Thus, the lattice S is not elliptically reflective. For $(d, \eta) = (3990, 4)$, the invariant $h = 1$, thus the lattice S cannot be parabolically or hyperbolically reflective either. It follows that S is not reflective.

These arguments permit to prove that the rest 84 cases of Table 3 which are not contained in the Table 1, are not reflective of elliptic or parabolic type. Between these 84 cases, there are 5 hyperbolically reflective cases marked *hr* in Table 3. Results of calculations for these 5 cases are given in Table 4 of Part II. This finishes the proof.

Remark 5.1.2. Here we want to outline another way which helps to study reflective type of lattices of Table 3. We can write similar programs as Program 9:

fund20.main for all types of narrow parts of $\mathcal{M}$ of main lattices S having the invariant $h \le 1$. They would be programs fund11.main, fund10.main, fund21.main and fund 30.main specializing the programs fund11.gen, fund10.gen, fund21.gen and fund30.gen (see Part II where we need to use similar programs). Together with Program 9: fund20.main, they give a list L of invariants (d, η, h) containing in Table 3. All invariants (d, η, h) of Table 3 which are not in the list L cannot be elliptically or parabolically reflective. If $h = 1$, they cannot be hyperbolically reflective either.

We did this calculations and we found that the list L is much smaller than Table 3 (e. g. Program 9: fund20.main gives only 132 triplets (d, η, h)). It shows that the narrow parts of polyhedra geometric arguments are sometimes stronger than arithmetic arguments of studying the invariant h (even if we forget about the problem with infinity). Both these arguments surprisingly fit together.

Here is the list of invariants (d, η, h) which are contained in Table 3 but are not in the list L (39 triplets):

$$
\begin{aligned}
&(d,\eta,h) = \\
&(57,1,1);\ (65,3,1);\ (71,0,1);\ (119,3,1);\ (161,1,1);\ (182,3,1);\\
&(194,0,1);\ (246,0,1);\ (259,3,1);\ (266,0,1);\ (266,3,1);\ (285,1,1);\\
&(299,3,1);\ (326,0,1);\ (335,0,1);\ (354,2,1);\ (386,0,1);\ (407,0,1);\\
&(506,0,1);\ (530,0,1);\ (534,0,1);\ (546,2,0);\ (602,3,1);\ (645,6,1);\\
&(714,6,1);\ (777,6,1);\ (854,3,1);\ (897,5,1);\ (897,7,1);\ (935,6,1);\\
&(966,2,1);\ (1106,1,1);\ (1254,4,1);\ (1394,3,1);\ (1659,2,1);\ (2210,6,1);\\
&(3311,1,1);\ (3990,4,1);\ (4466,1,1).
\end{aligned} \tag{5.1.35}
$$

These (d, η, h) give lattices S which are not elliptically or parabolically reflective: their fundamental polygon $\mathcal{M}$ does not have a narrow part satisfying Theorem 4.2.3. Of course, the list (5.1.35) is in complete agreement with calculations above using Vinberg's algorithm.

5.2. Proof of Theorem 2.3.3.1.

We should check reflective type of non-main lattices $\widetilde{S}$ corresponding to main lattices S of Table 1 with odd d (there are 97 these cases). See Proposition 2.2.6. Let S be one of lattices of Table 1 (or of Table 3 marked by er) with invariants (d, η) where d is odd. Then $\widetilde{S}$ has invariants $(2d, odd, \eta + \omega(p))$. If S and $\widetilde{S}$ are equivariantly equivalent, i. e.

$$
\sum_{p|d} (1 - p + 4\eta_p + 4\omega(p)) \equiv 0 \text{ or } 6 \mod 8, \tag{5.2.1}
$$

then lattices S and $\widetilde{S}$ have the same reflective type and calculation of $P(\mathcal{M})_{\mathrm{pr}}$ and its Gram matrix for the lattice $\widetilde{S}$ follows from similar calculation for S (see Remark 2.3.3.2). There are 21 these cases. For example if $S = U \oplus \langle -d\rangle$, where d is odd, then $\widetilde{S} = \langle 1\rangle \oplus \langle -1\rangle \oplus \langle -2d\rangle$, and these lattices are equivariantly equivalent. Thus, we need to study only cases when (5.2.1) is not valid. There are $97 - 21 = 76$ these cases. Let S be one of these lattices.

We have: if $S = \langle n_1\rangle \oplus \langle -n_2\rangle \oplus \langle -n_3\rangle$, where all n_i are odd, then $\widetilde{S} = \langle 2n_1\rangle \oplus \langle -2n_2\rangle \oplus \langle -2n_3\rangle(\epsilon_1, \epsilon_2, \epsilon_3)$ where one of ϵ_i is equal to 0, two of them are 1/2 and

$(\epsilon_1 n_1 - \epsilon_2 n_2 - \epsilon_3 n_3)/2$ is odd. If $S = \langle n_1\rangle \oplus \langle -n_2\rangle \oplus \langle -n_3\rangle(\epsilon_1, \epsilon_2, \epsilon_3)$ where for example $\epsilon_1 = \epsilon_2 = 1/2$ and $\epsilon_3 = 0$ (it follows that $n_1 \equiv n_2 \equiv 2 \mod 4$ and n_3 is odd), then $\widetilde{S} = \langle n_1/2\rangle \oplus \langle -n_2/2\rangle \oplus \langle -2n_3\rangle$. We see that the lattice $\widetilde{S}$ which we should check for reflective type, has the form (5.1.20). Thus, we need to make similar calculations as in Sect. 5.1. They are of the same difficulty. These calculations finish the proof.

6. Appendix 1: Programs for GP/PARI calculator

Program 1: h2

```
\\hclass(d,muu) calculates h=(hrI(d,muu),hrII(d,muu),hnr(d,muu))
\\here d<0 and 0\le muu<2^k are integers,
\\k is the number of all odd prime divisors of d.
\\Assume d is a fundamental discriminant of fundamental binary
\\positive lattices (i.e. with a square free determinant),
\\then d\equiv 1 \mod 4 or
\\d \equiv \pm 4, 8\mod 16.
\\ Assume (d,muu) is the genus of them. Then
\\hrI(d,muu), hrII(d,muu), hnr(d,muu) are numbers of ambiguous
\\classes of the types I, II and
\\hnr(d,muu)=(h(d)/2^{tau(d)}- hrI(d,muu)-hrII(d,muu))/2
\\of non-ambiguous classes of the general
\\equivalence respectively of the genus (d,muu);
\\hclass(d,muu)=[0,0,0] if
\\d is not a fundamental discriminant.
\\if d is a fundamental discriminant but 0\le muu <2^k
\\does not correspond to a
\\genus, then hnr(d,muu)=[0,0,h(d)/2^{tau(d)+1}]
hclass(d,muu,dd,fdd,dd1,beta,alpha,k,hr,hrI,hrII,t,h)=\
h=[0,0,0];hr=0;hrI=0;hrII=0;\
if(mod(d,4)!=mod(1,4),,dd=-d;\
fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(2*dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1), \
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro(2*(dd/dd1)/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta));t=matsize(fdd)[1]-1;\
h=[0,hr,(classno(d)/2^t-hr)/2]);\
if(mod(d,16)!=mod(4,16),,dd=-d/4;\
fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
```

```
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1), \
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta));t=matsize(fdd)[1]-1;\
h=[hr,0,(classno(d)/2^t-hr)/2]);\
if(mod(d,16)!=mod(8,16),,dd=-d/4;\
if(dd/2==1,hr=1;h=[1,0,0],fdd=factor(dd/2);\
fordiv(dd/2,dd1,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hr=hr+beta);t=matsize(fdd)[1];\
h=[hr,0,(classno(d)/2^t-hr)/2]));\
if(mod(d,16)!=mod(-4,16),,dd=-d/4;\
if(dd==1,h=[1,1,0],fdd=factor(dd);\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro((dd/dd1)/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hrI=hrI+beta));\
fordiv(dd,dd1,if(dd1>dd/dd1,,\
beta=1;alpha=1;k=1;\
while(alpha,\
if(type(dd1/fdd[k,1])==1,\
if(kro(2*dd1/fdd[k,1],fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0),\
if(kro(2*(dd/dd1)/fdd[k,1], fdd[k,1])==(-1)^bittest(muu,k-1),\
if(k>=matsize(fdd)[1],alpha=0,k=k+1),beta=0;alpha=0)));\
hrII=hrII+beta));\
hr=hrI+hrII;t=matsize(fdd)[1];\
h=[hrI,hrII,(classno(d)/2^t-hr)/2]));h;
```

Program 2: h3

```
\\the program h3
\\hnr(d,et) calculates the number of classes of
\\non-reflective central symmetries of a 3-dimensional main
\\hyperbolic lattice with the square-free determinant
\\d and the invariant et (a non-negative integer)
\\whose binary decomposition et_{p_k}....et_{p_1}
\\gives the map of all odd prime divisors p_1,...,p_k
\\of the d in increasing order to {0,1}
```

```
\r h2

\\checking the condition (5)=(3.2.7)
beta5(d,et,n,fd,sfd,alpha,k,b,etap)=\
fd=factor(d);sfd=matsize(fd)[1];alpha=1;k=1;\
while(alpha,\
if(k>sfd,b=1;alpha=0,\
if(type(n/fd[k,1])!=1,k=k+1,\
if(fd[k,1]==2,k=k+1,\
if(fd[1,1]==2,etap=bittest(et,k-2),etap=bittest(et,k-1));\
if(kro(n/fd[k,1],fd[k,1])==(-1)^etap,k=k+1,b=0;alpha=0)))));b;

\\checking the condition (6)=(3.2.8)
beta6(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,8);\
for(k=1,sfd,\
if(type((d/n)/fd[k,1])!=1,,u=u+mod(1-fd[k,1]+4*bittest(et,k-1),8)));\
if(u!=mod(-2,8),b=0,b=1);b;

\\checking the condition (8)=(3.2.10)
beta8(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,8);\
for(k=1,sfd,\
if(type((d/n)/fd[k,1])!=1,,u=u+mod(1-fd[k,1]+ 4*bittest(et,k-1),8)));\
u=u+mod(((d/n)^2-1)/2,8);\
if(u!=mod(2,8),b=1,b=0);b;

\\checking the condition (11)=(3.2.13)
beta11(d,et,n,fd,sfd,u,k,b)=\
fd=factor(d);sfd=matsize(fd)[1];\
u=mod(0,4);\
for(k=2,sfd,u=u+mod(1-fd[k,1]+4*bittest(et,k-2),4));\
u=-u-mod(1,4);\
if(u!=mod(n/2,4),b=0,b=1);b;

\\calculation of the numbers et_p+epsilon(p) if odd p|d
eps(d,et,e,fd,sfd,sfd1,e1,et1,eet1,k)=\
if(d==1,e=0,\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=sfd,sfd1=sfd-1);\
e1=vector(sfd1,k,mod((fd[sfd-k+1,1]-1)/2,2));\
et1=vector(sfd1,k,mod(bittest(et,sfd1-k),2));\
eet1=e1+et1;eet1=lift(eet1);\
e=0;for(k=0,sfd1-1,e=e+2^k*eet1[sfd1-k]));e;

\\calculation of the numbers et_p+epsilon(p)+omega(p) if odd p|d
epsomeg(d,et,e,fd,sfd,sfd1,e1,et1,eet1,k)=\
if(d==1,e=0,\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=sfd,sfd1=sfd-1);\
e1=vector(sfd1,k,mod((fd[sfd-k+1,1]-1)/2+(fd[sfd-k+1,1]^2-1)/8,2));\
```

```
et1=vector(sfd1,k,mod(bittest(et,sfd1-k),2));\
eet1=e1+et1;eet1=lift(eet1);\
e=0;for(k=0,sfd1-1,e=e+2^k*eet1[sfd1-k]));e;
\\calculation of the numbers mu_p=eta_p if odd p|t|2d
muuu(d,et,t,m,fd,sfd,sfd1,k1,k)=\
if(d<=2,m=0,m=0;fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]>2,sfd1=1,sfd1=2);k1=0;\
for(k=0,sfd-sfd1,\
if(type(t/fd[k+sfd1,1])==1,m=m+2^k1*bittest(et,k);k1=k1+1,)));m;
\\number of classes of non-reflective central symmetries
hnr(d,et,h,n,n1)=\
h=0;\
if(mod(d,2)==mod(1,2),\
fordiv(d,n,\
if(beta5(d,et,n)==1&&beta6(d,et,n)==1,\
h=h+hclass(-d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d,n,\
if(beta5(d,et,n)==1&&beta8(d,et,n)==1,\
h=h+hclass(-4*d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d,n1,\
n=2*n1;\
if(beta5(d,et,n)==1,\
h=h+hclass(-16*d/n,muuu(d,eps(d,et),d/n1))[3],)),\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&beta11(d,et,n)==1,\
h=h+hclass(-d/n,muuu(d,eps(d,et),d/n))[3],));\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&mod(n1,4)==mod(-d/2,4),\
h=h+hclass(-4*d/n,muuu(d,epsomeg(d,et),d/n))[3],));\
fordiv(d/2,n1,\
n=2*n1;\
if(beta5(d,et,n)==1&&mod(n1,4)==mod(d/2,4)&&n<d,\
h=h+2*hclass(-4*d/n,muuu(d,epsomeg(d,et),d/n))[3]+\
hclass(-4*d/n,muuu(d,epsomeg(d,et),d/n))[2],)));h;
```

Program 3: refh3

```
\\refh3(N) gives the list of invariants (d,et) of
\\main hyperbolic lattices with the square-free
\\determinant d\le N and of the rank 3 having the
\\number h=hnr(d,et)\le 1 of classes of non-reflective
\\central symmetries. Here et is a non-negative integer
\\having the binary decomposition et=p_k,...,p_1 where
\\p_1,...,p_k are all odd prime divisors of d in icreasing order
\r h2
\r h3
refh3(m,n,d,fd,sfd,sfd1,sig,n,et,h)=\
n=0;\
```

```
for(d=1,m,\
if(issqfree(d)!=1,,\
if(d<=2,n=n+1;h=0;et=0;pprint("n=",n," d=",d," et=",et," h=",h),\
fd=factor(d);sfd=matsize(fd)[1];\
if(fd[1,1]==2,sfd1=sfd-1,sfd1=sfd);\
for(et=0,2^sfd1-1,\
if(fd[1,1]!=2,,\
sig=mod(0,8);for(k=2,sfd,sig=sig+mod(1-fd[k,1]+4*bittest(et,k-2),8)));\
if(fd[1,1]==2&&(sig==mod(0,8)||sig==mod(-2,8)),\
h=hnr(d,et);if(h<=1,n=n+1;\
pprint("n=",n," d=",d," et=",et," h=",h),),\
if(fd[1,1]!=2,h=hnr(d,et);\
if(h<=1,n=n+1;pprint("n=",n," d=",d," et=",et," h=",h),),))))))
```

Program 4: fund11.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
for(alpha12=1,4,for(alpha23=alpha12,4,\
u=((sqrt(2+sqrt(alpha12))+sqrt(2+sqrt(alpha23)))^2- 2)^2+0.000001;\
for(alpha13=alpha23,u,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))));pprint("nI1=",n);pprint("aI1=",epsilon);\
pprint("aI1_1=",epsilon1);pprint("aI1_2=",epsilon2);
```

Program 5: fund10.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;\
n=0;\
alpha12=0;for(alpha23=1,4,\
w=((sqrt(2)+sqrt(2+sqrt(alpha23)))^2-2)^2+0.000001;\
for(alpha13=alpha23,w,\
```

```
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);a12=0;a21=0;\
fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))));pprint("nI0=",n);pprint("aI0=",epsilon);\
pprint("aI0_1=",epsilon1);pprint("aI0_2=",epsilon2);
```

Program 6: fund21.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
alpha12=alpha23=0;\
for(alpha34=1,4,\
w=(4*max((sqrt(2)+sqrt(sqrt(alpha34)/4+1/2))^2,\
((2+sqrt(sqrt(alpha34)/2+5/4))^2-1/4)/2)-2)^2+0.000001;\
for(alpha14=0,w,\
for(alpha13=5,36,\
if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1||\
alpha24<=((sqrt(2)+sqrt(2+sqrt(alpha34)))^2-2)^2-0.0000001,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha[3,1]=alpha13;alpha[3,4]=alpha[4,3]=alpha34;\
alpha[2,4]=alpha[4,2]=alpha24;alpha[1,4]=alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,\
fordiv(alpha[1,3],a13,\
fordiv(alpha[2,4],a24,\
if(type(a41=isqrt(u)/(a13*a34))!=1,,\
a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[4,1]=a41;if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);\
if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
```

```
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))))));pprint("nII1=",n);pprint("aII1=",epsilon);\
pprint("aII1_1=",epsilon1);pprint("aII1_2=",epsilon2);
```

Program 7: fund20.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
alpha12=0;alpha23=0;alpha34=0;\
for(alpha14=1,287.10,\
fordiv(4*alpha14,aa,\
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
if(alpha13>36,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4]*a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
)))))));pprint("nII0=",n);pprint("aII0=",epsilon);\
pprint("aII0_1=",epsilon1);pprint("aII0_2=",epsilon2);
```

Program 8: fund30.gen

```
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
al12=al23=al34=al45=0;\
for(al15=0,900,for(al13=5,36,for(al35=al13,36,\
if(issquare(q=al13*al35*al15)==0,,d=(al13+al35+al15-4+isqrt(q))*4;\
if(type(al14=d/(al35-4))!=1||al14<=287.108350||type(al25=d/(al13-4))!=1||\
al25<=287.108350||\
type(al24=(al13*al35+4*al15+4*isqrt(q))*4/((al35-4)*(al13-4)))!=1||\
```

```
issquare(q1=al13*al35*al25*al24*al14)==0,,\
al=idmat(5)*4;al[1,5]=al[5,1]=al15;al[1,3]=al[3,1]=al13;\
al[1,4]=al[4,1]=al14;al[2,4]=al[4,2]=al24;al[2,5]=al[5,2]=al25;\
al[3,5]=al[5,3]=al35;\
fordiv(al13,a13,\
fordiv(al35,a35,a51=isqrt(q)/a13/a35;\
if(a51==0,a15==0,if(type(a15=al15/a51)!=1,,\
fordiv(al14,a14,\
fordiv(al24,a24,a52=isqrt(q1)/a13/a35/a24*a14/al14;\
if(type(a25=al25/a52)!=1,,\
a=idmat(5)*-2;\
a[1,3]=a13;a[3,1]=al13/a13;\
a[1,4]=a14;a[4,1]=al14/a14;\
a[1,5]=a15;a[5,1]=a51;\
a[2,4]=a24;a[4,2]=al24/a24;\
a[2,5]=a25;a[5,2]=a52;\
a[3,5]=a35;a[5,3]=al35/a35;\
diag=idmat(5);diag[1,1]=a[1,4]*a[1,3]*a[4,2]*a[2,5];\
diag[2,2]=a[4,1]*a[1,3]*a[2,4]*a[2,5];\
diag[3,3]=a[1,4]*a[3,1]*a[4,2]*a[2,5];\
diag[4,4]=a[4,1]*a[1,3]*a[4,2]*a[2,5];\
diag[5,5]=a[4,1]*a[1,3]*a[2,4]*a[5,2];\
b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[3];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],);\
))))))))))));pprint("nIII0=",n);\
pprint("aIII0=",epsilon);pprint("aIII0_1=",epsilon1);\
pprint("aIII0_2=",epsilon2);
```

Program 9: fund20.main

```
\r h2
\r h3
\l;\
n=0;\
alpha12=0;alpha23=0;alpha34=0;\
for(alpha14=1,287.10,\
fordiv(4*alpha14,aa,\
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
```

```
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4]*a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);\
m=m+1;db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||content([-b[3,3],-b[4,4]])>2,,\
detb=db[2]*db[3]*db[4];\
if(detb==1,,\
fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;\
for(j=1,matsize(fdetb)[1],\
if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));\
if(gam==1,,\
for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1=1,,\
detb=db[2]*db[3]*db[4];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));\
if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],\
if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],\
aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);\
for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],\
u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1));\
```

```
))))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 10: refl0.1

```
\\main(n,h) calculates for U+<-2n> (it is given by the matrix g)
\\the vectors v of the height \le h from P(\M)_\pr
\\and calculates chains e and f and their Gram matrices ge, gf
refl(n,h,m,m1,h11,h1,j,k,v1,y1,y2,y3,z,u,divdd,mdivdd)=\
g=[0,1,0;1,0,0;0,0,-2*n];\
m=3;v=matrix(m,3,j,k,0);v[1,]=[0,0,1];v[2,]=[n,0,-1];v[3,]=[-1,1,0];\
dd=4*n;divdd=divisors(dd);mdivdd=matsize(divdd)[2];\
for(h1=1,h,\
for(j1=1,mdivdd,d=divdd[j1];h11=d*h1/2;\
if(type(h11)!=1,,\
if(issquare(h11)!=1,,y1t=isqrt(h11);\
if(type(2*y1t/d)!=1,,y2=y1t;\
for(z=floor(sqrt((2*y2^2+d)/(2*n))),y2,\
y3=-z;y1=(2*n*y3^2-d)/(2*y2);\
if(type(2*n*y3/d)!=1,,\
if(type(y1)!=1,,\
if(type(2*y1/d)!=1||content([y1,y2,y3])!=1,,\
u=[y1,y2,y3];\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u >=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
)))))))))));v;
\\**
\\getting e1-matrix from v
e1fromv(v,s,s1,alpha,s2,ex)=\
s=matsize(v)[1];\
e1=matrix(s,3,j,k,0);e1[1,]=v[2,];e1[2,]=v[1,];\
alpha=1;s1=2;\
while(alpha,\
if(s1==s,alpha=0,\
for(j=1,s,\
if(v[j,]==e1[s1-1,]||v[j,]==e1[s1,]||\
(v[j,]*g*e1[s1,]~)^2/((v[j,]*g*v[j,]~)*(e1[s1,]*g*e1[s1,]~))>1,,\
s2=s1+1;e1[s2,]=v[j,]));if(s2>s1,s1=s2,alpha=0)));\
ex=matrix(s1,3,j,k,e1[j,k]);e1=ex;
\\getting e2-matrix from v
e2fromv(v,s,s1,alpha,s2,ex)=\
s=matsize(v)[1];\
e2=matrix(s,3,j,k,0);e2[1,]=v[1,];e2[2,]=v[2,];\
alpha=1;s1=2;\
while(alpha,\
if(s1==s,alpha=0,\
for(j=1,s,\
if(v[j,]==e2[s1-1,]||v[j,]==e2[s1,]||\
(v[j,]*g*e2[s1,]~)^2/((v[j,]*g*v[j,]~)*(e2[s1,]*g*e2[s1,]~))>1,,\
```

```
s2=s1+1;e2[s2,]=v[j,]));if(s2>s1,s1=s2,alpha=0)));\
ex=matrix(s1,3,j,k,e2[j,k]);e2=ex;
\\getting e-matrix from v
efromv(v,e1,e2,s,s1,s2)=\
s=matsize(v)[1];s1=matsize(e1)[1];s2=matsize(e2)[1];\
if(s1==s&&s2==s,e=e1,\
e=matrix(s1+s2-2,3,j,k,0);\
for(j=1,s2,e[j,]=e2[s2+1-j,]);\
for(j=1,s1-2,e[s2+j,]=e1[2+j,]));\
e;ge=e*g*e~;
\\getting f1-matrix from v
f1fromv(v,e,s,s1,s2,q,alpha,f1x)=\
s=matsize(v)[1];s1=matsize(e)[1];\
if(s==s1,f1=0;v1=0,\
f1=matrix(s-s1,3,j,k,0);v1=v;\
for(j=1,s,\
for(k=1,s1,\
if(v1[j,]!=e[k,],,v1[j,]=[0,0,0])));\
q=1;s2=0;alpha=1;\
while(alpha,\
if(q>s,alpha=0,\
if(v1[q,]==[0,0,0],q=q+1,s2=s2+1;f1[s2,]=v1[q,];\
v1[q,]=[0,0,0];alpha=0)));\
if(s2==s-s1,,\
for(t=1,s,\
for(q=1,s,\
if(v1[q,]==[0,0,0],,\
if((v1[q,]*g*f1[s2,]~)^2/((v1[q,]*g*v1[q,]~)*(f1[s2,]*g*f1[s2,]~))>1,,\
s2=s2+1;f1[s2,]=v1[q,];v1[q,]=[0,0,0]))))));\
if(s2==0,,f1x=matrix(s2,3,j,k,f1[j,k]);f1=f1x);f1);
\\getting f2-matrix from v
f2fromv(f1,s,s1,s2,s3,q,f2x)=\
if(f1==0,f2=0;s3=0,\
s=matsize(v1)[1];s1=matsize(e)[1];s2=matsize(f1)[1];\
f2=matrix(s-s1-s2+1,3,j,k,0);\
s3=1;f2[1,]=f1[1,];\
for(t=1,s,\
for(q=1,s,\
if(v1[q,]==[0,0,0],,\
if((v1[q,]*g*f2[s3,]~)^2/((v1[q,]*g*v1[q,]~)*(f2[s3,]*g*f2[s3,]~))>1,,\
s3=s3+1;f2[s3,]=v1[q,];v1[q,]=[0,0,0]))))));\
if(s3==0,,f2x=matrix(s3,3,j,k,f2[j,k]);f2=f2x));f2;
\\getting f-matrix from v
ffromv(f1,f2,s1,s2)=\
if(f2==0,f=f1,\
s1=matsize(f1)[1];s2=matsize(f2)[1];\
if(s2==0,f=f1,\
f=matrix(s1+s2-1,3,j,k,0);\
for(j=1,s2,f[j,]=f2[s2+1-j,]);\
```

```
for(j=1,s1-1,f[s2+j,]=f1[1+j,])));\
if(f==0,gf=0,gf=f*g*f~);
\\***
\l;\
main(n,h)=refl(n,h);e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

Program 11: refl0.12

```
\\main(n1,n2,n3,h) calculates for <n1>\oplus <-n2>\oplus <-n3>
\\(it is given by the matrix g)
\\the vectors v of the height \le h from P(\M)_\pr
\\and calculates chains e and f and their Gram matrices ge, gf
refl(n1,n2,n3,h,m,m1,j,k,v1,y1,y2,y3,z,d,dd,h1,u,u1,w,w1,\
divdd,mdivdd,h11)=\
g=[n1,0,0;0,-n2,0;0,0,-n3];\
m=2;v=matrix(m,3,j,k,0);v[1,]=[0,1,0];\
if(n2==n3,v[2,]=[0,-1,1],v[2,]=[0,0,1]);\
dd=2*lcm(lcm(n1,n2),n3);\
divdd=divisors(dd);mdivdd=matsize(divdd)[2];\
forstep(h1=n1,h,n1,\
for(j1=1,mdivdd,d=divdd[j1];h11=d*h1/2;\
if(type(h11)!=1,,\
if(issquare(h11)!=1,,y1t=isqrt(h11);\
if(type(2*y1t/d)!=1,,y1=y1t/n1;\
if(type(y1)!=1,,\
for(z=0,floor(sqrt((w=n1*y1^2+d)/n2)+0.000001),\
if(type(2*n2*z/d)!=1||type(w1=(w-n2*z^2)/n3)!=1,,\
if(issquare(w1)!=1,,\
y2=-z;y3=-isqrt(w1);if(type(2*n3*y3/d)!=1||content([y1,y2,y3])!=1,,\
u=[y1,y2,y3];\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u >=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
))))))))))));v;
\\**
This part is the same as in Program 10: refl0.1 between \\** and \\***
\\***
\l;\
main(n1,n2,n3,h)=\
refl(n1,n2,n3,h);e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

Program 12: refl0.13

```
\\main(n1,n2,n3,eps1,eps2,eps3,h) calculates for
\\<n1>\oplus <-n2>\oplus <-n3>(eps1/2,eps2/2,eps3/2)
\\(it is given by the matrix g) where ni\ge 0 and epsi=0 or 1 and
\\either all of them are 0 or at least two of them are not zero,
\\the vectors v of the height \le h from P(\M)_\pr
\\and calculates chains e and f and their Gram matrices ge, gf
```

```
refl(n1,n2,n3,eps1,eps2,eps3,h,m,m1,j,k,v1,y1,y2,y3,y1t,y2t,y3t,\
z,d,dt,dd,h1,u,u1,w,w1,divdd,mdivdd,h11,s)=\
g=[n1,0,0;0,-n2,0;0,0,-n3];eps=[eps1,eps2,eps3]/2;\
m=2;v=matrix(m,3,j,k,0);\
if(n2/n3!=1&&n2/n3!=3&&n2/n3!=1/3,v[1,]=[0,1,0];v[2,]=[0,0,1],\
if(n2==n3,if(eps==[0,0,0]||eps==[1/2,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1,1],\
if(eps==[0,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1/2,1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1])),\
if(n2/n3==3,if(eps==[0,1/2,1/2],v[1,]=[0,1,0];v[2,]=[0,-1/2,1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1]),\
if(n2/n3==1/3,if(eps==[0,1/2,1/2],v[1,]=[0,0,1];v[2,]=[0,1/2,-1/2],\
v[1,]=[0,1,0];v[2,]=[0,0,1]),))));\
dd=2*lcm(lcm(n1,n2),n3);\
divdd=divisors(dd);mdivdd=matsize(divdd)[2];\
if(eps1==1,s=n1/2,s=n1);\
forstep(h1=s,h,s,\
for(j1=1,mdivdd,d=divdd[j1];h11=d*h1/2;\
if(type(h11)!=1,,\
if(issquare(h11)!=1,,y1t=isqrt(h11);\
if(type(2*y1t/d)!=1,,y1=y1t/n1;y1t=2*y1;\
if(type(y1t)!=1,,dt=4*d;\
for(z=0,floor(sqrt((w=n1*y1t^2+dt)/n2)+0.000001),\
if(type(n2*z/d)!=1||type(w1=(w-n2*z^2)/n3)!=1,,\
if(issquare(w1)!=1,,\
y2t=-z;y3t=-isqrt(w1);if(type(n3*y3t/d)!=1,,\
y1=y1t/2;y2=y2t/2;y3=y3t/2;u=[y1,y2,y3];\
if(type(2*u*g*eps /d)!=1,,\
if(mod(2*u,2)!=mod([0,0,0],2)&&mod(2*u,2)!=mod(2*eps,2),,\
if((mod(2*u,2)==mod([0,0,0],2)&&(content(u)>1||mod(2*u,4)==mod(4*\
eps,4)))||(eps!=[0,0,0]&&mod(2*u,2)==mod(2*eps,2)&&content(2*u)>1),,\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u >=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
)))))))))))))));v;
\\**
This part is the same as in Program 10: refl0.1 between \\** and \\***
\\***
\l;\
main(n1,n2,n3,eps1,eps2,eps3,h)=\
refl(n1,n2,n3,eps1,eps2,eps3,h);\
e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

Part II. Classification of maximal hyperbolic root systems of the rank three and of hyperbolic type

Here we extend results of Part I to hyperbolic type.

The main problem for the hyperbolic type is to develop an appropriate method of narrow parts of polyhedra of restricted hyperbolic type in hyperbolic spaces. This is much more complicated than for polyhedra of elliptic and restricted parabolic types when it is known long time ago (see [N4], [N5], [N11]). In Part I we considered this method for polygons of elliptic or parabolic type in hyperbolic plane. In Sect. 8 we develop the method of narrow parts for polygons of restricted hyperbolic type in a hyperbolic plane. These results are crucial for further developing of this method in higher dimension. We mention that first results about the method of narrow parts for hyperbolic type were obtained in [N13].

Except of hyperbolic root systems of the rank three which is the main our subject (see results in Sect. 7), we here also apply the method of narrow parts of polyhedra of restricted hyperbolic type to get finiteness of the set of reflective hyperbolic lattices of hyperbolic type over purely real algebraic number field (Theorem 8.4.1.1) and to get boundedness of families of algebraic surfaces with almost finite polyhedral Mori cone (Theorem 8.4.2.1).

We continue numeration of sections and statements from Part I.

7. Classification of reflective hyperbolic lattices of the rank 3 and of hyperbolic type: formulations

Like for elliptic or parabolic type (see Sect. 2), we reduce (or restrict) the classification to hyperbolically reflective hyperbolic lattices S of the rank three with square-free determinant d. All main (i. e. with square-free determinant d and even when d is even) these reflective lattices are given in Table 4 below. We give the invariants d, μ of S which define the S up to isomorphism. We also give the lattice S itself in some standard bases (see Sect. 2.3.1). It may have one of three forms: $U \oplus \langle -n\rangle$ or $\langle n_1\rangle \oplus \langle n_2\rangle \oplus \langle n_3\rangle(\epsilon_1/2,\ \epsilon_2/2,\ \epsilon_3/2)$ where $\epsilon_i = 0$ or 1, or $\langle n\rangle \oplus A_2(1/3,\ 1/3,\ -1/3)$ where

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \tag{7.1}$$

We also give the invariant h. For hyperbolic type it is equal to 0 or 2. In Table 4 we also describe the fundamental polygon $\mathcal{M}$ for $W(S)$ (equivalently, we describe the set $P(\mathcal{M})_{\mathrm{pr}}$ in the standard bases of S). To describe $P(\mathcal{M})_{\mathrm{pr}}$, we give the following data. We give the primitive generalized lattice Weyl vector ρ (for the fixed $\mathcal{M}$, the ρ is unique up to replacing by $-\rho$) and its square $\rho^2 < 0$. The set $P(\mathcal{M})_{\mathrm{pr}}$ is divided in two subsets:

$$P(\mathcal{M})_{\mathrm{pr}_+} = \{\alpha \in P(\mathcal{M})_{\mathrm{pr}} \mid (\alpha,\ \rho) > 0\} \tag{7.2}$$

and

$$P(\mathcal{M})_{\mathrm{pr}_-} = \{\alpha \in P(\mathcal{M})_{\mathrm{pr}} \mid (\alpha,\ \rho) < 0\}. \tag{7.3}$$

Both these subsets give two infinite chains of consecutive sides of $\mathcal{M}$ and are naturally ordered (up to changing the order to the opposite one). To describe these chains, we give the part Γ_+ (and Γ_- if necessary) of the chain $P(\mathcal{M})_{\mathrm{pr}_+}$ (respectively $P(\mathcal{M})_{\mathrm{pr}_-}$) and the Gram matrix $G(\Gamma_+)$ (respectively $G(\Gamma_-)$). We also give generators of the group $A(P(\mathcal{M})_{\mathrm{pr}})$ of symmetries of $\mathcal{M}$. These data are sufficient to find the set $P(\mathcal{M})_{\mathrm{pr}}$ because we have

$$P(\mathcal{M})_{\mathrm{pr}} = A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_+) \tag{7.4}$$

(or we have

$$P(\mathcal{M})_{\mathrm{pr}} = A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_+) \cup A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_-) \tag{7.5}$$

if we also give the set Γ_-). If the invariant $h = 2$, then the group $A(P(\mathcal{M})_{\mathrm{pr}})$ is generated by two central symmetries C_1 and C_2, and it is sufficient to give only Γ_+ because $C_1(\rho) = C_2(\rho) = -\rho$. If $h = 0$ (there are five these cases) and the group $A(P(\mathcal{M})_{\mathrm{pr}})$ is generated by a glide reflection S_1 along the line orthogonal to ρ (there are three these cases), then it also is sufficient to give only the set Γ_+ because $S_1(\rho) = -\rho$. Only for two cases (when $(d,\, \mu,\, h) = (65,\, 0,\, 0),\ (74,\, 1,\, 0)$), the group $A(P(\mathcal{M})_{\mathrm{pr}})$ is generated by the translation T along the line orthogonal to ρ. Then we have to give also the set Γ_-, and we have (7.5).

We have the main result:

Basic Theorem 7.1. *Table 4 below gives the complete list (containing 66 cases numerated by N) of main (i.e. with square-free determinant and even for even determinant) hyperbolic lattices S of rank 3 which are reflective of hyperbolic type. In Table 4, for the lattice S we give: invariants $(d,\, \eta)$ defining S up to isomorphism; the number h of classes of central symmetries of the group $A(P(\mathcal{M})_{\mathrm{pr}})$ of symmetries of the fundamental polygon $\mathcal{M}$ for $W(S)$; the matrix of S for some standard bases; the generalized lattice Weyl vector ρ and its square $\rho^2 < 0$; a subset $\Gamma_+ \subset A(P(\mathcal{M})_{\mathrm{pr}})$ (or two subsets $\Gamma_+ \subset A(P(\mathcal{M})_{\mathrm{pr}})$ and $\Gamma_- \subset A(P(\mathcal{M})_{\mathrm{pr}})$, if $(d,\, \mu,\, h) = (65,\, 0,\, 0)$, $(74,\, 1,\, 0)$) of orthogonal vectors to consecutive sides of $\mathcal{M}$ such that $P(\mathcal{M})_{\mathrm{pr}} = A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_+)$ (respectively $P(\mathcal{M})_{\mathrm{pr}} = A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_+) \cup A(P(\mathcal{M})_{\mathrm{pr}})(\Gamma_-)$); generators of the group $A(P(\mathcal{M})_{\mathrm{pr}})$.*

As generators of $A(P(\mathcal{M})_{\mathrm{pr}})$, we give central symmetries C_1 and C_2, if $h = 2$; the glide reflection S_1, if $h = 0$ and $(d,\, \mu) = (95,\, 0),\ (114,\, 2),\ (231,\, 2)$; the translation T, if $h = 0$ and $(d,\, \mu) = (65,\, 0),\ (74,\, 1)$.

If the lattice S represents 0 (i.e. there exists a non-zero $x \in S$ with $x^2 = 0$), we give S in the form $S = U \oplus \langle -d \rangle$. Then the fundamental polygon $\mathcal{M}$ contains an infinite vertex. In particular, Table 4 contains the complete list of reflective hyperbolic lattices of hyperbolic type which have the form $S = U \oplus \langle -d \rangle$ where d is square-free. There are 11 these cases corresponding to

$$d = 19,\ 23,\ 35,\ 39,\ 46,\ 58,\ 62,\ 70,\ 74,\ 114,\ 210.$$

If S is given in different form, the lattice S does not represent 0 and $\mathcal{M}$ does not have infinite vertices. There are 55 these cases.

Idea of Proof. We use the same method as for the proof of Basic Theorem 2.3.2.1. If S is hyperbolically reflective and has the rank three, then the invariant h is equal to 0 or 2. Table 3 gives the list of all main hyperbolic lattices with $h = 0$ or $h = 1$ which have $d \le 100000$. It has 61 lattices with $h = 0$. Using Vinberg's algorithm [V2], we check reflective type of them. Only five of them are hyperbolically reflective and are given in Table 4 (they are marked by "*hr*" in Table 3. All other are either elliptically reflective and are given in Table 1 (they are marked by er in Table 3) or are not reflective (they are marked by nr in Table 3). Using the formula for h of Theorem 3.2.1, we obtain the list of all main lattices S with $d \leq 100000$ and $h = 2$. It contains 259 lattices S given in Table 6. Checking reflective type of these 259 lattices by Vinberg's algorithm, we find the rest 61 hyperbolically reflective lattices with $h = 2$ of Table 4.

Thus, Table 4 contains all hyperbolically reflective main hyperbolic lattices with $d \leq 100000$. Conjecturally the tables 3 and 6 contain all main hyperbolic lattices with $h \leq 2$. To avoid this conjecture, in Sect. 8 we develop the "method of narrow parts of polyhedra" for polygons of restricted hyperbolic type in hyperbolic plane. It gives finiteness of the set of all hyperbolically reflective hyperbolic lattices of rank three. Applying this method to main hyperbolic lattices S, we get a finite list of their invariants (d, η), when they can be hyperbolically reflective. We calculate the invariant h for all of them and we find that $d \leq 100000$ if $h = 0$ or $h = 2$. It proves that the Tables 4 and 6 contain all main hyperbolically reflective hyperbolic lattices of the rank three and finishes the proof of Basic Theorem 7.1. See details in Sect. 9.1.

From Basic Theorems 2.3.2.1 and 7.1, we get (similarly to Theorem 2.3.3.1) classification of non-main hyperbolically reflective hyperbolic lattices of rank three and with square-free determinant.

Theorem 7.2. *Table 5 below gives the complete list (containing 21 cases numerated by N') of non-main (i. e. odd with even determinant) hyperbolic lattices $\widetilde{S}$ with square-free determinant and of rank three which are hyperbolically reflective. In Table 7, for each lattice $\widetilde{S}$ we give its invariants (d, odd, η), the number h of classes of central symmetries, matrix of $\widetilde{S}$ in the standard bases, invariants $(d, \eta + \omega(d))$ of the main odd lattice S of $\widetilde{S}$ (see Proposition 2.2.6).*

If S and $\widetilde{S}$ are equivariantly equivalent (in Table 5 there are 11 these cases), we only give the lattice S. For this case, calculation of $P(\mathcal{M})_{\mathrm{pr}}$ and $G(P(\mathcal{M})_{\mathrm{pr}})$ for $\widetilde{S}$ follows from calculation of similar sets for S in Table 4 (see Remark 2.3.3.2). In particular, all $\widetilde{S}$ representing 0 (non-compact case) have this type, then $\widetilde{S} = \langle 1\rangle \oplus \langle -1\rangle \oplus \langle -d\rangle$ where

$$d = 38,\ 46,\ 70,\ 78.$$

If S and $\widetilde{S}$ are not equivariantly equivalent (there are 10 these cases), we (like for Table 4) give $P(\mathcal{M})_{\mathrm{pr}}$ and $G(P(\mathcal{M})_{\mathrm{pr}})$ for $\widetilde{S}$ in the standard bases.

Proof. The proof is similar to the proof of Theorem 2.3.3.1. See details in Sect. 9.1.2.

From Theorems 7.1 and 7.2 we get (like in Sect. 2.3.4) classification of all elementary hyperbolic lattices of rank three which are hyperbolically reflective.

Theorem 7.3. *A primitive elementary hyperbolic lattice F of the rank three is hyperbolically reflective if and only if*

$$F \cong S^{*,m}(m),\ \ m|d \tag{7.6}$$

(see (2.2.5)) where S is a lattice of the determinant $d = \det(S)$ (it is square-free) of Tables 4 or 5. Then $\det(F) = dm$. In particular, for the fixed lattice S the number of lattices F is equal to 2^t where t is the number of prime divisors of d.

Let $P(\mathcal{M})_{\rm pr}$ be the set of orthogonal to $\mathcal{M}$ primitive roots of S. Then the set $\widetilde{P}(\mathcal{M})_{\rm pr}$ of orthogonal to $\mathcal{M}$ primitive roots of F is equal to

$$\widetilde{P}(\mathcal{M})_{\rm pr} = \{\widetilde{\alpha} = \alpha/k_{\alpha,m} \ \mid\ \alpha \in P(\mathcal{M})_{\rm pr}\} \tag{7.7}$$

where $k_{\alpha,m}$ is the greatest divisor of m such that $\alpha/k_{\alpha,m} \in S^*$ (equivalently, if $(\alpha, S) = t\mathbb{Z}$, then $k_{\alpha,m} = g.c.d(t, m)$). The Gram matrix of $\widetilde{P}(\mathcal{M})_{\rm pr}$ for the lattice F is equal to

$$\Big((\widetilde{\alpha}, \widetilde{\beta})\Big) = \left(\frac{(\alpha, \beta)m}{k_{\alpha,m}k_{\beta,m}}\right), \quad \widetilde{\alpha},\, \widetilde{\beta} \in \widetilde{P}(\mathcal{M})_{\rm pr}. \tag{7.8}$$

Here, on the left hand side of (7.8) *we use the form of the lattice F, and on the right hand side of* (7.8) *we use the form of the lattice S.*

By Proposition 2.2.2, any hyperbolic lattice has an equivariant (and even canonical) embedding to an elementary hyperbolic lattice (of the same rank). It follows that any reflective lattice has an equivariant embedding to a reflective elementary lattice. Thus all reflective lattices are equivariant sublattices of reflective elementary lattices. We have the following interesting statement which permits to describe many non-elementary hyperbolically reflective lattices of the rank three.

Theorem 7.4. *Suppose that $S \subset F$ is an equivariant sublattice of a hyperbolically reflective elementary hyperbolic lattice F of the rank 3. (All the lattices F and their root systems are described in Theorems 7.1 — 7.3.) Then S is reflective (it is then automatically hyperbolically reflective), if and only if $W(S) = W(F)$ (equivalently, S is $W(F)$-invariant). Equivalently, for any root $r \in F$, there exists $\lambda \in \mathbb{N}$ such that λr is a root of S.*

Proof. See Sect. 9.3.

All other (different from Theorem 7.4) hyperbolically reflective hyperbolic lattices S of the rank three have an equivariant embedding to reflective lattices F classified in Theorem 2.3.4.1 (they are elliptically reflective). Classification of these lattices S is more difficult.

Table 4
The list of main hyperbolic lattices of the rank three
which are reflective of hyperbolic type

$N=1$ $d=17$, $\eta=1$, $h=2$: $S=\langle 51\rangle\oplus A_2(1/3,1/3,-1/3)$;
$\rho=(3,-17,-17)$, $\rho^2=-119$;

$$\Gamma_+=\begin{pmatrix}0&0&1\\0&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&1\\1&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}\frac{31}{3}&\frac{4}{3}&\frac{2}{3}\\ \frac{-170}{3}&\frac{-23}{3}&\frac{-10}{3}\\ \frac{-136}{3}&\frac{-16}{3}&\frac{-11}{3}\end{pmatrix},\qquad C_2=\begin{pmatrix}\frac{65}{3}&\frac{2}{3}&\frac{10}{3}\\ \frac{-238}{3}&\frac{-10}{3}&\frac{-35}{3}\\ \frac{-374}{3}&\frac{-11}{3}&\frac{-58}{3}\end{pmatrix}.$$

$N=2$ $d=19$, $\eta=1$, $h=2$: $S=U\oplus\langle -19\rangle$;
$\rho=(57,19,-11)$, $\rho^2=-133$;

$$\Gamma_+=\begin{pmatrix}9&1&-1\\19&0&-1\\0&0&1\\-1&1&0\\3&3&-1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&19&8&11\\0&-19&19&19&38\\19&19&-19&0&19\\8&19&0&-2&0\\11&38&19&0&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}38&169&494\\9&38&114\\-6&-26&-77\end{pmatrix},\qquad C_2=\begin{pmatrix}19&50&190\\8&19&76\\-4&-10&-39\end{pmatrix}.$$

$N=3$ $d=23$, $\eta=1$, $h=2$: $S=U\oplus\langle -23\rangle$;
$\rho=(46,23,-10)$, $\rho^2=-184$;

$$\Gamma_+=\begin{pmatrix}15&3&-2\\161&23&-18\\11&1&-1\\23&0&-1\\0&0&1\\-1&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&2&23&46&12\\0&-46&0&115&414&138\\2&0&-1&0&23&10\\23&115&0&-23&23&23\\46&414&23&23&-23&0\\12&138&10&23&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}23&72&276\\8&23&92\\-4&-12&-47\end{pmatrix},\qquad C_2=\begin{pmatrix}23&32&184\\18&23&138\\-6&-8&-47\end{pmatrix}.$$

$N=4$ $d=31$, $\eta=0$, $h=2$: $S=\langle 31\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(17,-31,-93)$, $\rho^2=-651$;

$$\Gamma_+=\begin{pmatrix}1&-4&-4\\0&-1&1\\0&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&4\\0&-2&1\\4&1&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}30&2&5\\-62&-5&-10\\-155&-10&-26\end{pmatrix},\qquad C_2=\begin{pmatrix}123&2&22\\-62&-2&-11\\-682&-11&-122\end{pmatrix}.$$

$N=5$ $d=35=5\cdot 7$, $\eta=1$, $h=2$: $S=U\oplus\langle -35\rangle$;
$\rho=(21,7,-3)$, $\rho^2=-21$;

$$\Gamma_+=\begin{pmatrix} 17 & 1 & -1\\ 35 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 25 & 25 & -6\\ 23 & 19 & -5 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -1 & 0 & 35 & 16 & 240 & 171\\ 0 & -35 & 35 & 35 & 665 & 490\\ 35 & 35 & -35 & 0 & 210 & 175\\ 16 & 35 & 0 & -2 & 0 & 4\\ 240 & 665 & 210 & 0 & -10 & 0\\ 171 & 490 & 175 & 4 & 0 & -1 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 35 & 162 & 630\\ 8 & 35 & 140\\ -4 & -18 & -71 \end{pmatrix},\qquad C_2=\begin{pmatrix} 14 & 45 & 210\\ 5 & 14 & 70\\ -2 & -6 & -29 \end{pmatrix}.$$

$N=6$ $d=37$, $\eta=1$, $h=2$: $S=\langle 1\rangle\oplus\langle -74\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(74,-17/2,-37/2)$, $\rho^2=-555$;

$$\Gamma_+=\begin{pmatrix} 43 & -5 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -1 & 0 & 370 & 43\\ 0 & -2 & 0 & 2\\ 370 & 0 & -74 & 0\\ 43 & 2 & 0 & -1 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 168 & 1443 & 13\\ \frac{-39}{2} & \frac{-335}{2} & \frac{-3}{2}\\ \frac{-13}{2} & \frac{-111}{2} & \frac{-3}{2} \end{pmatrix},\qquad C_2=\begin{pmatrix} 24 & 185 & 15\\ \frac{-5}{2} & \frac{-39}{2} & \frac{-3}{2}\\ \frac{-15}{2} & \frac{-111}{2} & \frac{-11}{2} \end{pmatrix}.$$

$N=7$ $d=39=3\cdot 13$, $\eta=1$, $h=2$: $S=U\oplus\langle -39\rangle$;
$\rho=(91,13,-8)$, $\rho^2=-130$;

$$\Gamma_+=\begin{pmatrix} 19 & 1 & -1\\ 39 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 13 & 13 & -3\\ 6 & 3 & -1\\ 25 & 7 & -3 \end{pmatrix},\quad G(\Gamma_+)=\begin{pmatrix} -1 & 0 & 39 & 18 & 143 & 24 & 41\\ 0 & -39 & 39 & 39 & 390 & 78 & 156\\ 39 & 39 & -39 & 0 & 117 & 39 & 117\\ 18 & 39 & 0 & -2 & 0 & 3 & 18\\ 143 & 390 & 117 & 0 & -13 & 0 & 65\\ 24 & 78 & 39 & 3 & 0 & -3 & 0\\ 41 & 156 & 117 & 18 & 65 & 0 & -1 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 351 & 3872 & 10296\\ 32 & 351 & 936\\ -24 & -264 & -703 \end{pmatrix},\qquad C_2=\begin{pmatrix} 26 & 243 & 702\\ 3 & 26 & 78\\ -2 & -18 & -53 \end{pmatrix}.$$

$N=8$ $d=43$, $\eta=0$, $h=2$: $S=\langle 43\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(47,-129,-301)$, $\rho^2=-12255$;

$$\Gamma_+=\begin{pmatrix} 0 & -1 & 1\\ 0 & 1 & 0\\ 59 & 0 & -387\\ 9 & -2 & -59 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2 & 1 & 387 & 57\\ 1 & -1 & 0 & 2\\ 387 & 0 & -86 & 0\\ 57 & 2 & 0 & -2 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 42 & 4 & 5\\ -172 & -17 & -20\\ -215 & -20 & -26 \end{pmatrix},\qquad C_2=\begin{pmatrix} 171 & 2 & 26\\ -86 & -2 & -13\\ -1118 & -13 & -170 \end{pmatrix}.$$

$N = 9$ $d = 46 = 2 \cdot 23$, $\eta = 1$, $h = 2$: $S = U \oplus \langle -46 \rangle$;
$\rho = (46, 23, -7)$, $\rho^2 = -138$;

$$\Gamma_+ = \begin{pmatrix} 11 & 2 & -1 \\ 23 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 0 & 46 & 9 \\ 0 & -46 & 46 & 23 \\ 46 & 46 & -46 & 0 \\ 9 & 23 & 0 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 23 & 64 & 368 \\ 9 & 23 & 138 \\ -3 & -8 & -47 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 23 & 36 & 276 \\ 16 & 23 & 184 \\ -4 & -6 & -47 \end{pmatrix}.$$

$N = 10$ $d = 47$, $\eta = 0$, $h = 2$: $S = \langle 47 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$;
$\rho = (39, -141, -235)$, $\rho^2 = -3619$;

$$\Gamma_+ = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -7 \\ 3 & -5 & -20 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 1 & 7 & 15 \\ 1 & -1 & 0 & 5 \\ 7 & 0 & -2 & 1 \\ 15 & 5 & 1 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 422 & 42 & 45 \\ -1974 & -197 & -210 \\ -2115 & -210 & -226 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 46 & 3 & 6 \\ -141 & -10 & -18 \\ -282 & -18 & -37 \end{pmatrix}.$$

$N = 11$ $d = 53$, $\eta = 1$, $h = 2$: $S = \langle 1 \rangle \oplus \langle -106 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$;
$\rho = (477, -41, -159)$, $\rho^2 = -1219$;

$$\Gamma_+ = \begin{pmatrix} 5 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 7 & \frac{-1}{2} & \frac{-7}{2} \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 1 & 53 & 4 & 5 \\ 1 & -2 & 0 & 2 & 7 \\ 53 & 0 & -106 & 0 & 53 \\ 4 & 2 & 0 & -1 & 0 \\ 5 & 7 & 53 & 0 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 647 & 5724 & 468 \\ -54 & -478 & -39 \\ -234 & -2067 & -170 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1155 & 10812 & 680 \\ -102 & -955 & -60 \\ -340 & -3180 & -201 \end{pmatrix}.$$

$N = 12$ $d = 55 = 5 \cdot 11$, $\eta = 1$, $h = 2$: $S = \langle 11 \rangle \oplus \langle -10 \rangle \oplus \langle -2 \rangle (0, 1/2, 1/2)$;
$\rho = (25, -11, -55)$, $\rho^2 = -385$;

$$\Gamma_+ = \begin{pmatrix} 1 & -1 & -1 \\ 10 & -11 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 14 & 0 & -33 \\ 9 & -1 & -21 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -1 & 0 & 2 & 10 & 88 & 47 \\ 0 & -110 & 0 & 110 & 1540 & 880 \\ 2 & 0 & -2 & 0 & 66 & 42 \\ 10 & 110 & 0 & -10 & 0 & 10 \\ 88 & 1540 & 66 & 0 & -22 & 0 \\ 47 & 880 & 42 & 10 & 0 & -1 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 43 & 20 & 16 \\ -22 & -11 & -8 \\ -88 & -40 & -33 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 87 & 20 & 36 \\ -22 & -6 & -9 \\ -198 & -45 & -82 \end{pmatrix}.$$

$N = 13$ $d = 57 = 3 \cdot 19$, $\eta = 3$, $h = 2$: $S = \langle 1 \rangle \oplus \langle -19 \rangle \oplus \langle -3 \rangle$;
$\rho = (95, -21, -19)$, $\rho^2 = -437$;

$$\Gamma_+ = \begin{pmatrix} 13 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 0 & 57 & 13 \\ 0 & -3 & 0 & 3 \\ 57 & 0 & -19 & 0 \\ 13 & 3 & 0 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 74 & 285 & 60 \\ -15 & -58 & -12 \\ -20 & -76 & -17 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1295 & 4788 & 1188 \\ -252 & -932 & -231 \\ -396 & -1463 & -364 \end{pmatrix}.$$

$N=14\quad d=58=2\cdot 29,\ \eta=1,\ h=2$: $S=U\oplus\langle -58\rangle$;
$\rho=(58,29,-8),\ \rho^2=-348$;

$$\Gamma_+=\begin{pmatrix}23&5&-2\\14&2&-1\\29&0&-1\\0&0&1\\-1&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&29&116&18\\0&-2&0&58&12\\29&0&-58&58&29\\116&58&58&-58&0\\18&12&29&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}29&100&580\\9&29&174\\-3&-10&-59\end{pmatrix},\qquad C_2=\begin{pmatrix}29&36&348\\25&29&290\\-5&-6&-59\end{pmatrix}.$$

$N=15\quad d=59,\ \eta=0,\ h=2$: $S=\langle 59\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(24,-59,-177),\ \rho^2=-826$;

$$\Gamma_+=\begin{pmatrix}1&-5&-6\\0&-1&1\\0&1&0\\23&0&-177\\3&-2&-23\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&1&5&295&29\\1&-2&1&177&21\\5&1&-1&0&2\\295&177&0&-118&0\\29&21&2&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}117&6&14\\-354&-19&-42\\-826&-42&-99\end{pmatrix},\qquad C_2=\begin{pmatrix}235&6&30\\-354&-10&-45\\-1770&-45&-226\end{pmatrix}.$$

$N=16\quad d=62=2\cdot 31,\ \eta=1,\ h=2$: $S=U\oplus\langle -62\rangle$;
$\rho=(93,31,-10),\ \rho^2=-434$;

$$\Gamma_+=\begin{pmatrix}15&2&-1\\31&0&-1\\0&0&1\\-1&1&0\\6&5&-1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&62&13&25\\0&-62&62&31&93\\62&62&-62&0&62\\13&31&0&-2&1\\25&93&62&1&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}124&625&3100\\25&124&620\\-10&-50&-249\end{pmatrix},\qquad C_2=\begin{pmatrix}31&64&496\\16&31&248\\-4&-8&-63\end{pmatrix}.$$

$N=17\quad d=65=5\cdot 13,\ \eta=0,\ h=0$: $S=\langle 1\rangle\oplus\langle -13\rangle\oplus\langle -5\rangle$;
$\rho=(15,-4,-4),\ \rho^2=-63$;

$$\Gamma_+=\begin{pmatrix}18&-5&0\\0&0&1\\0&1&0\\2&0&-1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&65&36\\0&-5&0&5\\65&0&-13&0\\36&5&0&-1\end{pmatrix};$$

$$\Gamma_-=\begin{pmatrix}19&-3&-7\\260&-45&-91\\4&-1&-1\\11&-3&-1\end{pmatrix},\qquad G(\Gamma_-)=\begin{pmatrix}-1&0&2&57\\0&-130&0&650\\2&0&-2&0\\57&650&0&-1\end{pmatrix};$$

$$T=\begin{pmatrix}289&1040&40\\-40&-144&-5\\-112&-403&-16\end{pmatrix}.$$

$N=18$ $d=65=5\cdot 13$, $\eta=1$, $h=2$: $S=\langle 2\rangle\oplus\langle -13\rangle\oplus\langle -10\rangle(1/2,0,1/2)$;
$\rho=(91/2,-17,-13/2)$, $\rho^2=-39$;

$$\Gamma_+=\begin{pmatrix}5&-2&0\\0&0&1\\0&1&0\\\frac{1}{2}&0&\frac{-1}{2}\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&26&5\\0&-10&0&5\\26&0&-13&0\\5&5&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}255&624&160\\-96&-235&-60\\-32&-78&-21\end{pmatrix},\qquad C_2=\begin{pmatrix}\frac{603}{2}&715&\frac{495}{2}\\-110&-261&-90\\\frac{-99}{2}&-117&\frac{-83}{2}\end{pmatrix}.$$

$N=19$ $d=69=3\cdot 23$, $\eta=0$, $h=2$: $S=\langle 46\rangle\oplus\langle -6\rangle\oplus\langle -1\rangle(1/2,1/2,0)$;
$\rho=(21/2,-23/2,-69)$, $\rho^2=-483$;

$$\Gamma_+=\begin{pmatrix}\frac{1}{2}&\frac{-3}{2}&0\\0&0&1\\0&1&0\\\frac{23}{2}&\frac{-1}{2}&-78\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&9&260\\0&-1&0&78\\9&0&-6&3\\260&78&3&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}22&3&3\\-23&-4&-3\\-138&-18&-19\end{pmatrix},\qquad C_2=\begin{pmatrix}\frac{205}{2}&\frac{9}{2}&15\\\frac{-69}{2}&\frac{-5}{2}&-5\\-690&-30&-101\end{pmatrix}.$$

$N=20$ $d=70=2\cdot 5\cdot 7$, $\eta=0$, $h=2$: $S=U\oplus\langle -70\rangle$;
$\rho=(30,10,-3)$, $\rho^2=-30$;

$$\Gamma_+=\begin{pmatrix}35&0&-1\\0&0&1\\-1&1&0\\35&35&-6\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-70&70&35&805\\70&-70&0&420\\35&0&-2&0\\805&420&0&-70\end{pmatrix};$$

$$C_1=\begin{pmatrix}20&63&420\\7&20&140\\-2&-6&-41\end{pmatrix},\qquad C_2=\begin{pmatrix}35&81&630\\16&35&280\\-4&-9&-71\end{pmatrix}.$$

$N=21$ $d=74=2\cdot 37$, $\eta=1$, $h=0$: $S=U\oplus\langle -74\rangle$;
$\rho=(30,10,-3)$, $\rho^2=-66$;

$$\Gamma_+=\begin{pmatrix}37&0&-1\\0&0&1\\-1&1&0\\6&6&-1\\259&185&-36\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-74&74&37&148&4181\\74&-74&0&74&2664\\37&0&-2&0&74\\148&74&0&-2&0\\4181&2664&74&0&-74\end{pmatrix};$$

$$\Gamma_-=\begin{pmatrix}12&3&-1\\9&4&-1\end{pmatrix},\qquad G(\Gamma_-)=\begin{pmatrix}-2&1\\1&-2\end{pmatrix};$$

$$T=\begin{pmatrix}37&225&1110\\25&148&740\\-5&-30&-149\end{pmatrix}.$$

$N=22$ $d=77=7\cdot 11,\ \eta=0,\ h=2$: $S=\langle 14\rangle\oplus\langle -22\rangle\oplus\langle -1\rangle(1/2,1/2,0)$;
$\rho=(99/2,-49/2,-154),\ \rho^2=-2618$;

$$\Gamma_+=\begin{pmatrix}\frac{1}{2}&\frac{-1}{2}&0\\0&0&1\\0&1&0\\1&0&-4\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&11&7\\0&-1&0&4\\11&0&-22&0\\7&4&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}\frac{173}{2}&\frac{165}{2}&15\\\frac{-105}{2}&\frac{-101}{2}&-9\\-210&-198&-37\end{pmatrix},\qquad C_2=\begin{pmatrix}62&33&15\\-21&-12&-5\\-210&-110&-51\end{pmatrix}.$$

$N=23$ $d=85=5\cdot 17,\ \eta=2,\ h=2$: $S=\langle 1\rangle\oplus\langle -170\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(34,-5/2,-17/2),\ \rho^2=-51$;

$$\Gamma_+=\begin{pmatrix}13&-1&0\\0&0&1\\0&1&0\\1&0&-1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&170&13\\0&-2&0&2\\170&0&-170&0\\13&2&0&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}48&595&21\\\frac{-7}{2}&\frac{-87}{2}&\frac{-3}{2}\\\frac{-21}{2}&\frac{-255}{2}&\frac{-11}{2}\end{pmatrix},\qquad C_2=\begin{pmatrix}89&1020&60\\-6&-69&-4\\-30&-340&-21\end{pmatrix}.$$

$N=24$ $d=87=3\cdot 29,\ \eta=0,\ h=2$: $S=\langle 3\rangle\oplus\langle -29\rangle\oplus\langle -1\rangle$;
$\rho=(116,-34,-87),\ \rho^2=-725$;

$$\Gamma_+=\begin{pmatrix}68&-21&-33\\3&-1&0\\0&0&1\\0&1&0\\1&0&-3\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-6&3&33&609&105\\3&-2&0&29&9\\33&0&-1&0&3\\609&29&0&-29&0\\105&9&3&0&-6\end{pmatrix};$$

$$C_1=\begin{pmatrix}199&580&40\\-60&-175&-12\\-120&-348&-25\end{pmatrix},\qquad C_2=\begin{pmatrix}362&957&110\\-99&-262&-30\\-330&-870&-101\end{pmatrix}.$$

$N=25$ $d=91=7\cdot 13,\ \eta=0,\ h=2$: $S=\langle 7\rangle\oplus\langle -13\rangle\oplus\langle -1\rangle$;
$\rho=(13,-9,-13),\ \rho^2=-39$;

$$\Gamma_+=\begin{pmatrix}0&0&1\\0&1&0\\1&0&-3\\39&-14&-91\\2&-1&-4\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&3&91&4\\0&-13&0&182&13\\3&0&-2&0&2\\91&182&0&-182&0\\4&13&2&0&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}62&78&9\\-42&-53&-6\\-63&-78&-10\end{pmatrix},\qquad C_2=\begin{pmatrix}120&143&22\\-77&-92&-14\\-154&-182&-29\end{pmatrix}.$$

$N=26$ $d=93=3\cdot 31$, $\eta=3$, $h=2$: $S=\langle 1\rangle\oplus\langle -31\rangle\oplus\langle -3\rangle$;
$\rho=(155,-27,-31)$, $\rho^2=-1457$;

$$\Gamma_+=\begin{pmatrix} 67 & -12 & -3\\ 39 & -7 & -1\\ 217 & -39 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2 & 0 & 31 & 9 & 372 & 58\\ 0 & -1 & 0 & 3 & 217 & 36\\ 31 & 0 & -62 & 0 & 1209 & 217\\ 9 & 3 & 0 & -3 & 0 & 3\\ 372 & 217 & 1209 & 0 & -31 & 0\\ 58 & 36 & 217 & 3 & 0 & -2\end{pmatrix};$$

$$C_1=\begin{pmatrix} 35 & 186 & 18\\ -6 & -32 & -3\\ -6 & -31 & -4\end{pmatrix},\qquad C_2=\begin{pmatrix} 1199 & 5580 & 1140\\ -180 & -838 & -171\\ -380 & -1767 & -362\end{pmatrix}.$$

$N=27$ $d=95=5\cdot 19$, $\eta=0$, $h=0$: $S=\langle 19\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$;
$\rho=(5,-6,-20)$, $\rho^2=-105$;

$$\Gamma_+=\begin{pmatrix} 1 & -2 & -1\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 13 & 0 & -57\\ 9 & -2 & -39\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2 & 1 & 10 & 190 & 112\\ 1 & -1 & 0 & 57 & 39\\ 10 & 0 & -5 & 0 & 10\\ 190 & 57 & 0 & -38 & 0\\ 112 & 39 & 10 & 0 & -2\end{pmatrix};$$

$$S_1=\begin{pmatrix} 115 & 10 & 26\\ -190 & -16 & -43\\ -266 & -25 & -60\end{pmatrix}.$$

$N=28$ $d=114=2\cdot 3\cdot 19$, $\eta=1$, $h=2$: $S=\langle 6\rangle\oplus\langle -38\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(19,-7,-19)$, $\rho^2=-418$;

$$\Gamma_+=\begin{pmatrix} 190 & -75 & -38\\ 5 & -2 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -38 & 0 & 76 & 2850\\ 0 & -2 & 0 & 76\\ 76 & 0 & -2 & 0\\ 2850 & 76 & 0 & -38\end{pmatrix};$$

$$C_1=\begin{pmatrix} 31 & 76 & 4\\ -12 & \frac{-59}{2} & \frac{-3}{2}\\ -12 & \frac{-57}{2} & \frac{-5}{2}\end{pmatrix},\qquad C_2=\begin{pmatrix} 23 & 38 & 10\\ -6 & \frac{-21}{2} & \frac{-5}{2}\\ -30 & \frac{-95}{2} & \frac{-27}{2}\end{pmatrix}.$$

$N=29$ $d=114=2\cdot 3\cdot 19$, $\eta=2$, $h=0$: $S=U\oplus\langle -114\rangle$;
$\rho=(95,19,-6)$, $\rho^2=-494$;

$$\Gamma_+=\begin{pmatrix} 321 & 30 & -13\\ 28 & 2 & -1\\ 57 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & 0\\ 9 & 6 & -1\end{pmatrix},\quad G(\Gamma_+)=\begin{pmatrix} -6 & 0 & 228 & 1482 & 291 & 714\\ 0 & -2 & 0 & 114 & 26 & 72\\ 228 & 0 & -114 & 114 & 57 & 228\\ 1482 & 114 & 114 & -114 & 0 & 114\\ 291 & 26 & 57 & 0 & -2 & 3\\ 714 & 72 & 228 & 114 & 3 & -6\end{pmatrix};$$

$$S_1=\begin{pmatrix} 48 & 475 & 2280\\ 19 & 192 & 912\\ -4 & -40 & -191\end{pmatrix}.$$

$N = 30$ $d = 115 = 5 \cdot 23,\ \eta = 0,\ h = 2$: $S = \langle 23\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$;
$\rho = (35, -69, -115),\ \rho^2 = -8855$;

$$\Gamma_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -5 \\ 75 & -46 & -345 \\ 4 & -3 & -18 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -1 & 0 & 5 & 345 & 18 \\ 0 & -5 & 0 & 230 & 15 \\ 5 & 0 & -2 & 0 & 2 \\ 345 & 230 & 0 & -230 & 0 \\ 18 & 15 & 2 & 0 & -1 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 22 & 10 & 1 \\ -46 & -21 & -2 \\ -23 & -10 & -2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 22 & 5 & 4 \\ -23 & -6 & -4 \\ -92 & -20 & -17 \end{pmatrix}.$$

$N = 31$ $d = 115 = 5 \cdot 23,\ \eta = 3,\ h = 2$: $S = \langle 115\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$;
$\rho = (97, -115, -1035),\ \rho^2 = -2415$;

$$\Gamma_+ = \begin{pmatrix} 61 & -115 & -644 \\ 1 & -5 & -10 \\ 3 & -23 & -23 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 15 & 0 & -161 \end{pmatrix}, \quad G(\Gamma_+) = \begin{pmatrix} -46 & 0 & 3588 & 529 & 115 & 1541 \\ 0 & -10 & 0 & 5 & 5 & 115 \\ 3588 & 0 & -23 & 0 & 23 & 1472 \\ 529 & 5 & 0 & -2 & 1 & 161 \\ 115 & 5 & 23 & 1 & -1 & 0 \\ 1541 & 115 & 1472 & 161 & 0 & -46 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 2874 & 40 & 265 \\ -4600 & -65 & -424 \\ -30475 & -424 & -2810 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1034 & 9 & 96 \\ -1035 & -10 & -96 \\ -11040 & -96 & -1025 \end{pmatrix}.$$

$N = 32$ $d = 119 = 7 \cdot 17,\ \eta = 0,\ h = 2$: $S = \langle 7\rangle \oplus \langle -17\rangle \oplus \langle -1\rangle$;
$\rho = (13, -7, -21),\ \rho^2 = -91$;

$$\Gamma_+ = \begin{pmatrix} 8 & -5 & -5 \\ 11 & -7 & -4 \\ 17 & -11 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 1 & 17 & 5 & 85 & 41 \\ 1 & -2 & 0 & 4 & 119 & 65 \\ 17 & 0 & -34 & 0 & 187 & 119 \\ 5 & 4 & 0 & -1 & 0 & 3 \\ 85 & 119 & 187 & 0 & -17 & 0 \\ 41 & 65 & 119 & 3 & 0 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 118 & 170 & 17 \\ -70 & -101 & -10 \\ -119 & -170 & -18 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 27 & 34 & 6 \\ -14 & -18 & -3 \\ -42 & -51 & -10 \end{pmatrix}.$$

$N = 33$ $d = 123 = 3 \cdot 41,\ \eta = 0,\ h = 2$: $S = \langle 3\rangle \oplus \langle -41\rangle \oplus \langle -1\rangle$;
$\rho = (369, -98, -123),\ \rho^2 = -410$;

$$\Gamma_+ = \begin{pmatrix} 71 & -19 & -18 \\ 369 & -99 & -82 \\ 11 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 4 & -1 & -3 \end{pmatrix}, \ G(\Gamma_+) = \begin{pmatrix} -2 & 0 & 6 & 18 & 779 & 159 & 19 \\ 0 & -82 & 0 & 82 & 4059 & 861 & 123 \\ 6 & 0 & -6 & 0 & 123 & 33 & 9 \\ 18 & 82 & 0 & -1 & 0 & 3 & 3 \\ 779 & 4059 & 123 & 0 & -41 & 0 & 41 \\ 159 & 861 & 33 & 3 & 0 & -6 & 3 \\ 19 & 123 & 9 & 3 & 41 & 3 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 6074 & 22140 & 585 \\ -1620 & -5905 & -156 \\ -1755 & -6396 & -170 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 2311 & 8364 & 272 \\ -612 & -2215 & -72 \\ -816 & -2952 & -97 \end{pmatrix}.$$

$N=34$ $d=123=3\cdot 41,\ \eta=3,\ h=2$: $S=\langle 123\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(21,-123,-205),\ \rho^2=-2911$;

$$\Gamma_+=\begin{pmatrix}298&-2337&-2337\\0&-1&1\\0&1&0\\11&0&-123\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-246&0&2337&115743\\0&-2&1&123\\2337&1&-1&0\\115743&123&0&-246\end{pmatrix};$$

$$C_1=\begin{pmatrix}40&2&3\\-246&-13&-18\\-369&-18&-28\end{pmatrix},\qquad C_2=\begin{pmatrix}491&14&42\\-1722&-50&-147\\-5166&-147&-442\end{pmatrix}.$$

$N=35$ $d=129=3\cdot 43,\ \eta=0,\ h=2$: $S=\langle 3\rangle\oplus\langle -86\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(86,-27/2,-129/2),\ \rho^2=-1806$;

$$\Gamma_+=\begin{pmatrix}8&\frac{-3}{2}&\frac{-3}{2}\\0&0&1\\0&1&0\\2&0&-3\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-6&3&129&39\\3&-2&0&6\\129&0&-86&0\\39&6&0&-6\end{pmatrix};$$

$$C_1=\begin{pmatrix}53&258&18\\-9&-44&-3\\-27&-129&-10\end{pmatrix},\qquad C_2=\begin{pmatrix}47&172&28\\-6&\frac{-45}{2}&\frac{-7}{2}\\-42&\frac{-301}{2}&\frac{-51}{2}\end{pmatrix}.$$

$N=36$ $d=133=7\cdot 19,\ \eta=3,\ h=2$: $S=\langle 1\rangle\oplus\langle -19\rangle\oplus\langle -7\rangle$;
$\rho=(1197,-259,-247),\ \rho^2=-268793$;

$$\Gamma_+=\begin{pmatrix}13&-3&0\\0&0&1\\0&1&0\\7&0&-3\\9&-1&-3\\68&-9&-21\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&57&91&60&371\\0&-7&0&21&21&147\\57&0&-19&0&19&171\\91&21&0&-14&0&35\\60&21&19&0&-1&0\\371&147&171&35&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}483&2090&154\\-110&-476&-35\\-22&-95&-8\end{pmatrix},\qquad C_2=\begin{pmatrix}48&133&98\\-7&-20&-14\\-14&-38&-29\end{pmatrix}.$$

$N=37$ $d=138=2\cdot 3\cdot 23,\ \eta=0,\ h=2$: $S=\langle 46\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(117,-92,-552),\ \rho^2=-30498$;

$$\Gamma_+=\begin{pmatrix}54&\frac{-145}{2}&\frac{-453}{2}\\13&-18&-54\\0&\frac{-1}{2}&\frac{1}{2}\\0&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-6&0&9&435\\0&-2&0&108\\9&0&-2&3\\435&108&3&-6\end{pmatrix};$$

$$C_1=\begin{pmatrix}183&30&34\\-230&\frac{-77}{2}&\frac{-85}{2}\\-782&\frac{-255}{2}&\frac{-291}{2}\end{pmatrix},\qquad C_2=\begin{pmatrix}183&6&38\\-46&\frac{-5}{2}&\frac{-19}{2}\\-874&\frac{-57}{2}&\frac{-363}{2}\end{pmatrix}.$$

$N=38$ $d=143=11\cdot 13$, $\eta=3$, $h=2$: $S=\langle 143\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(19,-117,-195)$, $\rho^2=-91$;

$$\Gamma_+=\begin{pmatrix}1&-8&-9\\0&-1&1\\0&1&0\\1&0&-13\\7&-33&-77\\4&-21&-43\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&1&8&26&44&17\\1&-2&1&13&44&22\\8&1&-1&0&33&21\\26&13&0&-26&0&13\\44&44&33&0&-11&0\\17&22&21&13&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}1871&84&132\\-12012&-540&-847\\-18876&-847&-1332\end{pmatrix},\qquad C_2=\begin{pmatrix}1286&54&93\\-7722&-325&-558\\-13299&-558&-962\end{pmatrix}.$$

$N=39$ $d=159=3\cdot 53$, $\eta=3$, $h=2$: $S=\langle 159\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(13,-53,-159)$, $\rho^2=-1219$;

$$\Gamma_+=\begin{pmatrix}1&-9&-9\\0&-1&1\\0&1&0\\21&0&-265\\5&-3&-63\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-3&0&9&954&201\\0&-2&1&265&60\\9&1&-1&0&3\\954&265&0&-106&0\\201&60&3&0&-3\end{pmatrix};$$

$$C_1=\begin{pmatrix}52&1&4\\-159&-4&-12\\-636&-12&-49\end{pmatrix},\qquad C_2=\begin{pmatrix}635&6&50\\-954&-10&-75\\-7950&-75&-626\end{pmatrix}.$$

$N=40$ $d=165=3\cdot 5\cdot 11$, $\eta=2$, $h=2$: $S=\langle 165\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$;
$\rho=(7,-22,-88)$, $\rho^2=-143$;

$$\Gamma_+=\begin{pmatrix}6&-30&-71\\23&-120&-270\\18&-99&-209\\4&-25&-45\\2&-15&-21\\6&-55&-55\\0&-1&1\\0&1&0\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&11&15&39&385&41&30\\0&-15&0&30&120&1320&150&120\\11&0&-22&0&66&880&110&99\\15&30&0&-10&0&110&20&25\\39&120&66&0&-6&0&6&15\\385&1320&880&110&0&-110&0&55\\41&150&110&20&6&0&-2&1\\30&120&99&25&15&55&1&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}439&12&32\\-1980&-55&-144\\-5280&-144&-385\end{pmatrix},\qquad C_2=\begin{pmatrix}131&2&10\\-330&-6&-25\\-1650&-25&-126\end{pmatrix}.$$

$N=41$ $d=174=2\cdot 3\cdot 29$, $\eta=2$, $h=2$: $S=\langle 2\rangle\oplus\langle -58\rangle\oplus\langle -6\rangle(0,1/2,1/2)$;
$\rho=(58,-21/2,-29/2)$, $\rho^2=-928$;

$$\Gamma_+=\begin{pmatrix}35&\frac{-13}{2}&\frac{-1}{2}\\0&0&1\\0&1&0\\3&0&-2\\5&\frac{-1}{2}&\frac{-5}{2}\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&3&377&204&154\\3&-6&0&12&15\\377&0&-58&0&29\\204&12&0&-6&0\\154&15&29&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}17&87&9\\-3&\frac{-31}{2}&\frac{-3}{2}\\-3&\frac{-29}{2}&\frac{-5}{2}\end{pmatrix},\qquad C_2=\begin{pmatrix}449&1740&540\\-60&-233&-72\\-180&-696&-217\end{pmatrix}.$$

$N = 42$ $d = 183 = 3 \cdot 61$, $\eta = 0$, $h = 2$: $S = \langle 3\rangle \oplus \langle -61\rangle \oplus \langle -1\rangle$;
$\rho = (305, -66, -122)$, $\rho^2 = -1525$;

$$\Gamma_+ = \begin{pmatrix} 9 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 5 & -1 & -4 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 1 & 122 & 24 & 9 \\ 1 & -1 & 0 & 3 & 4 \\ 122 & 0 & -61 & 0 & 61 \\ 24 & 3 & 0 & -6 & 3 \\ 9 & 4 & 61 & 3 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 391 & 1708 & 56 \\ -84 & -367 & -12 \\ -168 & -732 & -25 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 8111 & 34892 & 1404 \\ -1716 & -7382 & -297 \\ -4212 & -18117 & -730 \end{pmatrix}.$$

$N = 43$ $d = 195 = 3 \cdot 5 \cdot 13$, $\eta = 0$, $h = 2$: $S = \langle 3\rangle \oplus \langle -13\rangle \oplus \langle -5\rangle$;
$\rho = (65, -27, -26)$, $\rho^2 = -182$;

$$\Gamma_+ = \begin{pmatrix} 13 & -6 & -3 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 65 & -20 & -39 \\ 17 & -6 & -9 \end{pmatrix}, \quad G(\Gamma_+) = \begin{pmatrix} -6 & 0 & 15 & 78 & 24 & 390 & 60 \\ 0 & -1 & 0 & 13 & 6 & 130 & 24 \\ 15 & 0 & -5 & 0 & 5 & 195 & 45 \\ 78 & 13 & 0 & -13 & 0 & 260 & 78 \\ 24 & 6 & 5 & 0 & -2 & 0 & 6 \\ 390 & 130 & 195 & 260 & 0 & -130 & 0 \\ 60 & 24 & 45 & 78 & 6 & 0 & -6 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 244 & 455 & 140 \\ -105 & -196 & -60 \\ -84 & -156 & -49 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 506 & 845 & 390 \\ -195 & -326 & -150 \\ -234 & -390 & -181 \end{pmatrix}.$$

$N = 44$ $d = 195 = 3 \cdot 5 \cdot 13$, $\eta = 1$, $h = 2$: $S = \langle 5\rangle \oplus \langle -13\rangle \oplus \langle -3\rangle$;
$\rho = (39, -21, -26)$, $\rho^2 = -156$;

$$\Gamma_+ = \begin{pmatrix} 207 & -120 & -95 \\ 17 & -10 & -7 \\ 15 & -9 & -5 \\ 247 & -150 & -65 \\ 18 & -11 & -4 \\ 78 & -48 & -13 \\ 8 & -5 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -5 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -30 & 0 & 60 & 3120 & 330 & 2145 & 480 & 285 & 1560 & 1680 \\ 0 & -2 & 0 & 130 & 16 & 117 & 30 & 21 & 130 & 150 \\ 60 & 0 & -3 & 0 & 3 & 39 & 15 & 15 & 117 & 150 \\ 3120 & 130 & 0 & -130 & 0 & 195 & 130 & 195 & 1950 & 2730 \\ 330 & 16 & 3 & 0 & -1 & 0 & 5 & 12 & 143 & 210 \\ 2145 & 117 & 39 & 195 & 0 & -39 & 0 & 39 & 624 & 975 \\ 480 & 30 & 15 & 130 & 5 & 0 & -5 & 0 & 65 & 120 \\ 285 & 21 & 15 & 195 & 12 & 39 & 0 & -3 & 0 & 15 \\ 1560 & 130 & 117 & 1950 & 143 & 624 & 65 & 0 & -13 & 0 \\ 1680 & 150 & 150 & 2730 & 210 & 975 & 120 & 15 & 0 & -30 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 161 & 234 & 54 \\ -90 & -131 & -30 \\ -90 & -130 & -31 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 159 & 208 & 72 \\ -80 & -105 & -36 \\ -120 & -156 & -55 \end{pmatrix}.$$

$N = 45$ $d = 195 = 3 \cdot 5 \cdot 13$, $\eta = 7$, $h = 2$: $S = \langle 1\rangle \oplus \langle -65\rangle \oplus \langle -3\rangle$;
$\rho = (780, -93, -130)$, $\rho^2 = -4485$;

$$\Gamma_+ = \begin{pmatrix} 123 & -15 & -13 \\ 65 & -8 & -5 \\ 8 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 39 & -4 & -13 \\ 27 & -3 & -7 \end{pmatrix}, \quad G(\Gamma_+) = \begin{pmatrix} -3 & 0 & 9 & 39 & 975 & 84 & 390 & 123 \\ 0 & -10 & 0 & 15 & 520 & 50 & 260 & 90 \\ 9 & 0 & -1 & 0 & 65 & 8 & 52 & 21 \\ 39 & 15 & 0 & -3 & 0 & 3 & 39 & 21 \\ 975 & 520 & 65 & 0 & -65 & 0 & 260 & 195 \\ 84 & 50 & 8 & 3 & 0 & -2 & 0 & 6 \\ 390 & 260 & 52 & 39 & 260 & 0 & -26 & 0 \\ 123 & 90 & 21 & 21 & 195 & 6 & 0 & -3 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 2177 & 17160 & 792 \\ -264 & -2081 & -96 \\ -264 & -2080 & -97 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 288 & 2210 & 153 \\ -34 & -261 & -18 \\ -51 & -390 & -28 \end{pmatrix}.$$

$N=46$ $d=210=2\cdot 3\cdot 5\cdot 7$, $\eta=1$, $h=2$: $S=\langle 14\rangle\oplus\langle -30\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(15,-7,-35)$, $\rho^2=-770$;

$$\Gamma_+=\begin{pmatrix}10&-7&0\\0&0&1\\0&1&0\\3&0&-8\\80&-7&-210\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-70&0&210&420&9730\\0&-2&0&16&420\\210&0&-30&0&210\\420&16&0&-2&0\\9730&420&210&0&-70\end{pmatrix};$$

$$C_1=\begin{pmatrix}13&15&3\\-7&\frac{-17}{2}&\frac{-3}{2}\\-21&\frac{-45}{2}&\frac{-11}{2}\end{pmatrix},\qquad C_2=\begin{pmatrix}41&15&15\\-7&\frac{-7}{2}&\frac{-5}{2}\\-105&\frac{-75}{2}&\frac{-77}{2}\end{pmatrix}.$$

$N=47$ $d=210=2\cdot 3\cdot 5\cdot 7$, $\eta=7$, $h=2$: $S=U\oplus\langle -210\rangle$;
$\rho=(182,14,-5)$, $\rho^2=-154$;

$$\Gamma_+=\begin{pmatrix}425&20&-9\\52&2&-1\\105&0&-1\\0&0&1\\-1&1&0\\10&10&-1\\14&7&-1\\20&5&-1\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-10&0&210&1890&405&2560&1365&635\\0&-2&0&210&50&330&182&90\\210&0&-210&210&105&840&525&315\\1890&210&210&-210&0&210&210&210\\405&50&105&0&-2&0&7&15\\2560&330&840&210&0&-10&0&40\\1365&182&525&210&7&0&-14&0\\635&90&315&210&15&40&0&-10\end{pmatrix};$$

$$C_1=\begin{pmatrix}84&1445&7140\\5&84&420\\-2&-34&-169\end{pmatrix},\qquad C_2=\begin{pmatrix}35&432&2520\\3&35&210\\-1&-12&-71\end{pmatrix}.$$

$N=48$ $d=213=3\cdot 71$, $\eta=3$, $h=2$: $S=\langle 1\rangle\oplus\langle -71\rangle\oplus\langle -3\rangle$;
$\rho=(355,-41,-71)$, $\rho^2=-8449$;

$$\Gamma_+=\begin{pmatrix}59&-7&-1\\497&-59&0\\0&0&1\\0&1&0\\1&0&-1\\5325&-516&-1775\\257&-25&-85\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-1&0&3&497&56&52398&2483\\0&-142&0&4189&497&485001&23004\\3&0&-3&0&3&5325&255\\497&4189&0&-71&0&36636&1775\\56&497&3&0&-2&0&2\\52398&485001&5325&36636&0&-426&0\\2483&23004&255&1775&2&0&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}288&2414&51\\-34&-285&-6\\-17&-142&-4\end{pmatrix},\qquad C_2=\begin{pmatrix}99&710&90\\-10&-72&-9\\-30&-213&-28\end{pmatrix}.$$

$N=49$ $d=221=13\cdot 17$, $\eta=1$, $h=2$: $S=\langle 2\rangle\oplus\langle -26\rangle\oplus\langle -17\rangle(1/2,1/2,0)$;
$\rho=(91/2,-15/2,-13)$, $\rho^2=-195$;

$$\Gamma_+=\begin{pmatrix}51&-11&-11\\\frac{77}{2}&\frac{-17}{2}&-8\\\frac{5967}{2}&\frac{-1343}{2}&-598\\106&-24&-21\\\frac{273}{2}&\frac{-63}{2}&-26\\\frac{3}{2}&\frac{-1}{2}&0\\0&0&1\\0&1&0\\17&0&-6\\15&-1&-5\end{pmatrix},G(\Gamma_+)=\begin{pmatrix}-1&0&442&21&52&10&187&286&612&309\\0&-2&0&2&13&5&136&221&493&254\\442&0&-442&0&221&221&10166&17459&40443&21216\\21&2&0&-1&0&6&357&624&1462&771\\52&13&221&0&-26&0&442&819&1989&1066\\10&5&221&6&0&-2&0&13&51&32\\187&136&10166&357&442&0&-17&0&102&85\\286&221&17459&624&819&13&0&-26&0&26\\612&493&40443&1462&1989&51&102&0&-34&0\\309&254&21216&771&1066&32&85&26&0&-1\end{pmatrix};$$

$$C_1=\begin{pmatrix}\frac{423}{2}&\frac{1105}{2}&425\\\frac{-85}{2}&\frac{-223}{2}&-85\\-50&-130&-101\end{pmatrix},\qquad C_2=\begin{pmatrix}48&91&119\\-7&-14&-17\\-14&-26&-35\end{pmatrix}.$$

$N = 50$ $d = 231 = 3 \cdot 7 \cdot 11$, $\eta = 2$, $h = 0$: $S = \langle 3\rangle \oplus \langle -22\rangle \oplus \langle -14\rangle(0, 1/2, 1/2)$;
$\rho = (21, -7, -6)$, $\rho^2 = -259$;

$$\Gamma_+ = \begin{pmatrix} 8 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \\ 7 & -1 & -3 \\ 308 & \frac{-105}{2} & \frac{-253}{2} \\ 41 & \frac{-15}{2} & \frac{-33}{2} \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -6 & 0 & 66 & 48 & 102 & 3927 & 489 \\ 0 & -14 & 0 & 14 & 42 & 1771 & 231 \\ 66 & 0 & -22 & 0 & 22 & 1155 & 165 \\ 48 & 14 & 0 & -2 & 0 & 77 & 15 \\ 102 & 42 & 22 & 0 & -1 & 0 & 3 \\ 3927 & 1771 & 1155 & 77 & 0 & -77 & 0 \\ 489 & 231 & 165 & 15 & 3 & 0 & -6 \end{pmatrix};$$

$$S_1 = \begin{pmatrix} 112 & 297 & 49 \\ \frac{-45}{2} & \frac{-119}{2} & \frac{-21}{2} \\ \frac{-87}{2} & \frac{-231}{2} & \frac{-37}{2} \end{pmatrix}.$$

$N = 51$ $d = 231 = 3 \cdot 7 \cdot 11$, $\eta = 3$, $h = 2$: $S = \langle 11\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$;
$\rho = (105, -99, -154)$, $\rho^2 = -18480$;

$$\Gamma_+ = \begin{pmatrix} 14 & -17 & -7 \\ 4 & -5 & -1 \\ 35 & -44 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 30 & -11 & -55 \\ 35 & -15 & -63 \\ 216 & -99 & -385 \\ 154 & -73 & -273 \end{pmatrix}, \ G(\Gamma_+) = \begin{pmatrix} -14 & 0 & 154 & 21 & 119 & 112 & 2156 & 2282 & 13398 & 9296 \\ 0 & -2 & 0 & 3 & 35 & 38 & 770 & 826 & 4884 & 3402 \\ 154 & 0 & -77 & 0 & 308 & 385 & 8162 & 8855 & 52668 & 36806 \\ 21 & 3 & 0 & -3 & 0 & 6 & 165 & 189 & 1155 & 819 \\ 119 & 35 & 308 & 0 & -7 & 0 & 77 & 105 & 693 & 511 \\ 112 & 38 & 385 & 6 & 0 & -1 & 0 & 7 & 66 & 56 \\ 2156 & 770 & 8162 & 165 & 77 & 0 & -22 & 0 & 132 & 154 \\ 2282 & 826 & 8855 & 189 & 105 & 7 & 0 & -7 & 0 & 28 \\ 13398 & 4884 & 52668 & 1155 & 693 & 66 & 132 & 0 & -66 & 0 \\ 9296 & 3402 & 36806 & 819 & 511 & 56 & 154 & 28 & 0 & -14 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 21 & 14 & 6 \\ -22 & -15 & -6 \\ -22 & -14 & -7 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 351 & 112 & 168 \\ -176 & -57 & -84 \\ -616 & -196 & -295 \end{pmatrix}.$$

$N = 52$ $d = 231 = 3 \cdot 7 \cdot 11$, $\eta = 4$, $h = 2$: $S = \langle 7\rangle \oplus \langle -33\rangle \oplus \langle -1\rangle$;
$\rho = (16, -7, -14)$, $\rho^2 = -21$;

$$\Gamma_+ = \begin{pmatrix} 42 & -19 & -21 \\ 77 & -35 & -33 \\ 15 & -7 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \\ 3 & -1 & -6 \end{pmatrix}, \ G(\Gamma_+) = \begin{pmatrix} -6 & 0 & 21 & 21 & 627 & 231 & 129 \\ 0 & -11 & 0 & 33 & 1155 & 440 & 264 \\ 21 & 0 & -42 & 0 & 231 & 105 & 84 \\ 21 & 33 & 0 & -1 & 0 & 3 & 6 \\ 627 & 1155 & 231 & 0 & -33 & 0 & 33 \\ 231 & 440 & 105 & 3 & 0 & -2 & 3 \\ 129 & 264 & 84 & 6 & 33 & 3 & -6 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 377 & 792 & 36 \\ -168 & -353 & -16 \\ -252 & -528 & -25 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 230 & 462 & 33 \\ -98 & -197 & -14 \\ -231 & -462 & -34 \end{pmatrix}.$$

$N = 53$ $d = 255 = 3 \cdot 5 \cdot 17$, $\eta = 2$, $h = 2$: $S = \langle 1\rangle \oplus \langle -17\rangle \oplus \langle -15\rangle$;
$\rho = (765, -135, -136)$, $\rho^2 = -2040$;

$$\Gamma_+ = \begin{pmatrix} 265 & -50 & -43 \\ 323 & -62 & -51 \\ 75 & -15 & -11 \\ 9 & -2 & -1 \\ 255 & -60 & -17 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \\ 65 & -10 & -13 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -10 & 0 & 30 & 40 & 5610 & 210 & 645 & 850 & 150 & 340 \\ 0 & -34 & 0 & 34 & 6120 & 238 & 765 & 1054 & 204 & 510 \\ 30 & 0 & -15 & 0 & 1020 & 45 & 165 & 255 & 60 & 180 \\ 40 & 34 & 0 & -2 & 0 & 2 & 15 & 34 & 12 & 50 \\ 5610 & 6120 & 1020 & 0 & -510 & 0 & 255 & 1020 & 510 & 3060 \\ 210 & 238 & 45 & 2 & 0 & -1 & 0 & 17 & 12 & 90 \\ 645 & 765 & 165 & 15 & 255 & 0 & -15 & 0 & 15 & 195 \\ 850 & 1054 & 255 & 34 & 1020 & 17 & 0 & -17 & 0 & 170 \\ 150 & 204 & 60 & 12 & 510 & 12 & 15 & 0 & -6 & 0 \\ 340 & 510 & 180 & 50 & 3060 & 90 & 195 & 170 & 0 & -10 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 577 & 1734 & 1530 \\ -102 & -307 & -270 \\ -102 & -306 & -271 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 4607 & 13056 & 12960 \\ -768 & -2177 & -2160 \\ -864 & -2448 & -2431 \end{pmatrix}.$$

$N=54\quad d=255=3\cdot 5\cdot 17,\ \eta=3,\ h=2$: $S=\langle 15\rangle\oplus\langle -17\rangle\oplus\langle -1\rangle$;
$\rho=(119,-95,-255),\ \rho^2=-6035$;

$$\Gamma_+=\begin{pmatrix} 156 & -135 & -235\\ 31 & -27 & -45\\ 51 & -45 & -68\\ 1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -5 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -10 & 0 & 85 & 45 & 235 & 2295 & 1165\\ 0 & -3 & 0 & 6 & 45 & 459 & 240\\ 85 & 0 & -34 & 0 & 68 & 765 & 425\\ 45 & 6 & 0 & -2 & 0 & 17 & 15\\ 235 & 45 & 68 & 0 & -1 & 0 & 5\\ 2295 & 459 & 765 & 17 & 0 & -17 & 0\\ 1165 & 240 & 425 & 15 & 5 & 0 & -10 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 734 & 714 & 77\\ -630 & -613 & -66\\ -1155 & -1122 & -122 \end{pmatrix},\qquad C_2=\begin{pmatrix} 79 & 68 & 12\\ -60 & -52 & -9\\ -180 & -153 & -28 \end{pmatrix}.$$

$N=55\quad d=330=2\cdot 3\cdot 5\cdot 11,\ \eta=6,\ h=2$: $S=\langle 22\rangle\oplus\langle -10\rangle\oplus\langle -6\rangle(0,1/2,1/2)$;
$\rho=(63,-66,-88),\ \rho^2=-2706$;

$$\Gamma_+=\begin{pmatrix} 15 & -19 & -15\\ 1 & \frac{-3}{2} & \frac{-1}{2}\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -2\\ 3 & \frac{-3}{2} & \frac{-11}{2}\\ 15 & -11 & -25 \end{pmatrix},\quad G(\Gamma_+)=\begin{pmatrix} -10 & 0 & 90 & 190 & 150 & 210 & 610\\ 0 & -2 & 3 & 15 & 16 & 27 & 90\\ 90 & 3 & -6 & 0 & 12 & 33 & 150\\ 190 & 15 & 0 & -10 & 0 & 15 & 110\\ 150 & 16 & 12 & 0 & -2 & 0 & 30\\ 210 & 27 & 33 & 15 & 0 & -6 & 0\\ 610 & 90 & 150 & 110 & 30 & 0 & -10 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 197 & 105 & 63\\ -231 & \frac{-247}{2} & \frac{-147}{2}\\ -231 & \frac{-245}{2} & \frac{-149}{2} \end{pmatrix},\qquad C_2=\begin{pmatrix} 109 & 45 & 45\\ -99 & \frac{-83}{2} & \frac{-81}{2}\\ -165 & \frac{-135}{2} & \frac{-137}{2} \end{pmatrix}.$$

$N=56\quad d=390=2\cdot 3\cdot 5\cdot 13,\ \eta=0,\ h=2$: $S=\langle 130\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$;
$\rho=(17,-26,-130),\ \rho^2=-286$;

$$\Gamma_+=\begin{pmatrix} 3 & -7 & -21\\ 0 & \frac{-1}{2} & \frac{1}{2}\\ 0 & 1 & 0\\ 8 & 0 & -65\\ 3 & -2 & -24 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -6 & 0 & 42 & 390 & 78\\ 0 & -2 & 3 & 65 & 18\\ 42 & 3 & -6 & 0 & 12\\ 390 & 65 & 0 & -130 & 0\\ 78 & 18 & 12 & 0 & -6 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 311 & 24 & 36\\ -520 & -41 & -60\\ -2340 & -180 & -271 \end{pmatrix},\qquad C_2=\begin{pmatrix} 519 & 30 & 62\\ -650 & \frac{-77}{2} & \frac{-155}{2}\\ -4030 & \frac{-465}{2} & \frac{-963}{2} \end{pmatrix}.$$

$N=57\quad d=429=3\cdot 11\cdot 13,\ \eta=5,\ h=2$: $S=\langle 1\rangle\oplus\langle -143\rangle\oplus\langle -3\rangle$;
$\rho=(2431,-175,-715),\ \rho^2=-3289$;

$$\Gamma_+=\begin{pmatrix} 1352 & -99 & -377\\ 528 & -39 & -143\\ 13 & -1 & -3\\ 11 & -1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -1\\ 117 & -8 & -39 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -26 & 0 & 26 & 715 & 1131 & 14157 & 221 & 819\\ 0 & -66 & 0 & 231 & 429 & 5577 & 99 & 429\\ 26 & 0 & -1 & 0 & 9 & 143 & 4 & 26\\ 715 & 231 & 0 & -22 & 0 & 143 & 11 & 143\\ 1131 & 429 & 9 & 0 & -3 & 0 & 3 & 117\\ 14157 & 5577 & 143 & 143 & 0 & -143 & 0 & 1144\\ 221 & 99 & 4 & 11 & 3 & 0 & -2 & 0\\ 819 & 429 & 26 & 143 & 117 & 1144 & 0 & -26 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 9074 & 94380 & 7755\\ -660 & -6865 & -564\\ -2585 & -26884 & -2210 \end{pmatrix},\qquad C_2=\begin{pmatrix} 4899 & 50050 & 4410\\ -350 & -3576 & -315\\ -1470 & -15015 & -1324 \end{pmatrix}.$$

$N=58$ $\;d=462=2\cdot3\cdot7\cdot11$, $\eta=0$, $h=2$: $S=\langle 14\rangle\oplus\langle -22\rangle\oplus\langle -6\rangle(1/2,1/2,0)$;
$\rho=(33/2,-15/2,-22)$, $\rho^2=-330$;

$$\Gamma_+=\begin{pmatrix} \frac{81}{2} & \frac{-51}{2} & -38\\ \frac{1}{2} & \frac{-1}{2} & 0\\ 0&0&1\\ 0&1&0\\ 9&0&-14\\ 66&-10&-99\\ \frac{51}{2}&\frac{-9}{2}&-38\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -6&3&228&561&1911&9240&3270\\ 3&-2&0&11&63&352&129\\ 228&0&-6&0&84&594&228\\ 561&11&0&-22&0&220&99\\ 1911&63&84&0&-42&0&21\\ 9240&352&594&220&0&-22&0\\ 3270&129&228&99&21&0&-6\end{pmatrix};$$

$$C_1=\begin{pmatrix}349&330&150\\ -210&-199&-90\\ -350&-330&-151\end{pmatrix},\qquad C_2=\begin{pmatrix}\frac{61}{2}&\frac{33}{2}&18\\ \frac{-21}{2}&\frac{-13}{2}&-6\\ -42&-22&-25\end{pmatrix}.$$

$N=59$ $\;d=462=2\cdot3\cdot7\cdot11$, $\eta=6$, $h=2$: $S=\langle 22\rangle\oplus\langle -14\rangle\oplus\langle -6\rangle(1/2,1/2,0)$;
$\rho=(259/2,-121/2,-231)$, $\rho^2=-2464$;

$$\Gamma_+=\begin{pmatrix} 21&-13&-35\\ \frac{3}{2}&\frac{-3}{2}&-2\\ \frac{7}{2}&\frac{-11}{2}&0\\ 0&0&1\\ 0&1&0\\ 1&0&-2\\ 210&-76&-385\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -14&0&616&210&182&42&2338\\ 0&-6&0&12&21&9&714\\ 616&0&-154&0&77&77&10318\\ 210&12&0&-6&0&12&2310\\ 182&21&77&0&-14&0&1064\\ 42&9&77&12&0&-2&0\\ 2338&714&10318&2310&1064&0&-14\end{pmatrix};$$

$$C_1=\begin{pmatrix}351&112&168\\ -176&-57&-84\\ -616&-196&-295\end{pmatrix}\qquad C_2=\begin{pmatrix}\frac{13473}{2}&\frac{3185}{2}&3360\\ \frac{-5005}{2}&\frac{-1185}{2}&-1248\\ -12320&-2912&-6145\end{pmatrix}.$$

$N=60$ $\;d=555=3\cdot5\cdot37$, $\eta=3$, $h=2$: $S=\langle 15\rangle\oplus\langle -37\rangle\oplus\langle -1\rangle$;
$\rho=(37,-23,-37)$, $\rho^2=-407$;

$$\Gamma_+=\begin{pmatrix} 11&-7&-2\\ 259&-165&0\\ 0&0&1\\ 0&1&0\\ 1&0&-5\\ 37&-15&-111\\ 2&-1&-5\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2&0&2&259&155&1998&61\\ 0&-1110&0&6105&3885&52170&1665\\ 2&0&-1&0&5&111&5\\ 259&6105&0&-37&0&555&37\\ 155&3885&5&0&-10&0&5\\ 1998&52170&111&555&0&-111&0\\ 61&1665&5&37&5&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}124&185&10\\ -75&-112&-6\\ -150&-222&-13\end{pmatrix},\qquad C_2=\begin{pmatrix}866&1258&85\\ -510&-741&-50\\ -1275&-1850&-126\end{pmatrix}.$$

$N=61$ $\;d=561=3\cdot11\cdot17$, $\eta=6$, $h=2$: $S=\langle 1\rangle\oplus\langle -51\rangle\oplus\langle -11\rangle$;
$\rho=(119,-15,-17)$, $\rho^2=-493$;

$$\Gamma_+=\begin{pmatrix} 81&-11&-6\\ 561&-77&-34\\ 7&-1&0\\ 0&0&1\\ 0&1&0\\ 3&0&-1\\ 12&-1&-3\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -6&0&6&66&561&177&213\\ 0&-374&0&374&3927&1309&1683\\ 6&0&-2&0&51&21&33\\ 66&374&0&-11&0&11&33\\ 561&3927&51&0&-51&0&51\\ 177&1309&21&11&0&-2&3\\ 213&1683&33&33&51&3&-6\end{pmatrix};$$

$$C_1=\begin{pmatrix}63&408&88\\ -8&-52&-11\\ -8&-51&-12\end{pmatrix},\qquad C_2=\begin{pmatrix}296&1683&594\\ -33&-188&-66\\ -54&-306&-109\end{pmatrix}.$$

$N=62\ \ d=609=3\cdot 7\cdot 29,\ \eta=3,\ h=2$: $S=\langle 29\rangle\oplus\langle -7\rangle\oplus\langle -3\rangle$;
$\rho=(77,-87,-203)$, $\rho^2=-4669$;

$$\Gamma_+=\begin{pmatrix} 1&-2&-1\\ 14&-29&0\\ 0&0&1\\ 0&1&0\\ 9&0&-29\\ 7&-4&-21\\ 191&-145&-551\\ 40&-31&-115 \end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2&0&3&14&174&84&1856&381\\ 0&-203&0&203&3654&2030&48111&9947\\ 3&0&-3&0&87&63&1653&345\\ 14&203&0&-7&0&28&1015&217\\ 174&3654&87&0&-174&0&1914&435\\ 84&2030&63&28&0&-14&0&7\\ 1856&48111&1653&1015&1914&0&-29&0\\ 381&9947&345&217&435&7&0&-2 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 86&21&24\\ -87&-22&-24\\ -232&-56&-65 \end{pmatrix},\qquad C_2=\begin{pmatrix} 1043&210&306\\ -870&-176&-255\\ -2958&-595&-868 \end{pmatrix}.$$

$N=63\ \ d=663=3\cdot 13\cdot 17,\ \eta=3,\ h=2$: $S=\langle 6\rangle\oplus\langle -26\rangle\oplus\langle -17\rangle(1/2,1/2,0)$;
$\rho=(884,-357,-351)$, $\rho^2=-719355$;

$$\Gamma_+=\begin{pmatrix} \frac{1071}{2}&\frac{-493}{2}&-91\\ 91&-42&-15\\ 2&-1&0\\ 0&0&1\\ 0&1&0\\ 5&0&-3\\ \frac{221}{2}&\frac{-17}{2}&-65\\ \frac{7}{2}&\frac{-1}{2}&-2\\ \frac{85}{2}&\frac{-17}{2}&-23 \end{pmatrix},\ G(\Gamma_+)=\begin{pmatrix} -34&0&17&1547&6409&11424&200005&4947&46495\\ 0&-3&0&255&1092&1965&34476&855&8058\\ 17&0&-2&0&26&60&1105&29&289\\ 1547&255&0&-17&0&51&1105&34&391\\ 6409&1092&26&0&-26&0&221&13&221\\ 11424&1965&60&51&0&-3&0&3&102\\ 200005&34476&1105&1105&221&0&-442&0&884\\ 4947&855&29&34&13&3&0&-1&0\\ 46495&8058&289&391&221&102&884&0&-34 \end{pmatrix};$$

$$C_1=\begin{pmatrix} \frac{361}{2}&\frac{715}{2}&\frac{187}{2}\\ \frac{-165}{2}&\frac{-327}{2}&\frac{-85}{2}\\ -33&-65&-18 \end{pmatrix},\qquad C_2=\begin{pmatrix} 95&104&136\\ -24&-27&-34\\ -48&-52&-69 \end{pmatrix}.$$

$N=64\ \ d=705=3\cdot 5\cdot 47,\ \eta=6,\ h=2$: $S=\langle 1\rangle\oplus\langle -47\rangle\oplus\langle -15\rangle$;
$\rho=(705,-90,-94)$, $\rho^2=-16215$;

$$\Gamma_+=\begin{pmatrix} 2961&-411&-235\\ 35&-5&-2\\ 47&-7&0\\ 0&0&1\\ 0&1&0\\ 3&0&-1\\ 31&-3&-6\\ 1269&-129&-235 \end{pmatrix},\ G(\Gamma_+)=\begin{pmatrix} -141&0&3948&3525&19317&5358&12690&437241\\ 0&-10&0&30&235&75&200&7050\\ 3948&0&-94&0&329&141&470&17202\\ 3525&30&0&-15&0&15&90&3525\\ 19317&235&329&0&-47&0&141&6063\\ 5358&75&141&15&0&-6&3&282\\ 12690&200&470&90&141&3&-2&0\\ 437241&7050&17202&3525&6063&282&0&-141 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 967&6204&1320\\ -132&-847&-180\\ -88&-564&-121 \end{pmatrix},\qquad C_2=\begin{pmatrix} 63&376&120\\ -8&-48&-15\\ -8&-47&-16 \end{pmatrix}.$$

$N=65\ \ d=1365=3\cdot 5\cdot 7\cdot 13,\ \eta=9,\ h=2$: $S=\langle 5\rangle\oplus\langle -91\rangle\oplus\langle -3\rangle$;
$\rho=(117,-25,-65)$, $\rho^2=-1105$; $\Gamma_+=$

$$\begin{pmatrix} 1638&-375&-455\\ 13&-3&-3\\ 21&-5&0\\ 0&0&1\\ 0&1&0\\ 3&0&-5\\ 13&-2&-13\\ 11&-2&-9\\ 3003&-570&-2275 \end{pmatrix},G(\Gamma_+)=\begin{pmatrix} -2730&0&1365&1365&34125&17745&20475&9555&2037945\\ 0&-1&0&9&273&150&182&88&19110\\ 1365&0&-70&0&455&315&455&245&55965\\ 1365&9&0&-3&0&15&39&27&6825\\ 34125&273&455&0&-91&0&182&182&51870\\ 17745&150&315&15&0&-30&0&30&10920\\ 20475&182&455&39&182&0&-26&0&2730\\ 9555&88&245&27&182&30&0&-2&0\\ 2037945&19110&55965&6825&51870&10920&2730&0&-2730 \end{pmatrix};$$

$$C_1=\begin{pmatrix} 139&546&42\\ -30&-118&-9\\ -70&-273&-22 \end{pmatrix},\qquad C_2=\begin{pmatrix} 374&1365&150\\ -75&-274&-30\\ -250&-910&-101 \end{pmatrix}.$$

$N = 66$ $\ d = 1995 = 3\cdot 5\cdot 7\cdot 19$, $\eta = 7$, $h = 2$: $S = \langle 35\rangle \oplus \langle -57\rangle \oplus \langle -1\rangle$;
$\rho = (171, -105, -665)$, $\rho^2 = -47215$; $\Gamma_+ =$

$$\begin{pmatrix} 4 & -3 & -7 \\ 285 & -217 & -399 \\ 13 & -10 & -15 \\ 9 & -7 & -7 \\ 19 & -15 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -7 \\ 19 & -8 & -95 \\ 153 & -70 & -735 \\ 47 & -22 & -223 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 5 & 14 & 95 & 7 & 171 & 91 & 627 & 4305 & 1257 \\ 0 & -399 & 0 & 399 & 3990 & 399 & 12369 & 7182 & 52668 & 367080 & 107730 \\ 5 & 0 & -10 & 0 & 95 & 15 & 570 & 350 & 2660 & 18690 & 5500 \\ 14 & 399 & 0 & -7 & 0 & 7 & 399 & 266 & 2128 & 15120 & 4466 \\ 95 & 3990 & 95 & 0 & -190 & 0 & 855 & 665 & 5795 & 41895 & 12445 \\ 7 & 399 & 15 & 7 & 0 & -1 & 0 & 7 & 95 & 735 & 223 \\ 171 & 12369 & 570 & 399 & 855 & 0 & -57 & 0 & 456 & 3990 & 1254 \\ 91 & 7182 & 350 & 266 & 665 & 7 & 0 & -14 & 0 & 210 & 84 \\ 627 & 52668 & 2660 & 2128 & 5795 & 95 & 456 & 0 & -38 & 0 & 38 \\ 4305 & 367080 & 18690 & 15120 & 41895 & 735 & 3990 & 210 & 0 & -210 & 0 \\ 1257 & 107730 & 5500 & 4466 & 12445 & 223 & 1254 & 84 & 38 & 0 & -2 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 104 & 114 & 9 \\ -70 & -77 & -6 \\ -315 & -342 & -28 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 139 & 114 & 18 \\ -70 & -58 & -9 \\ -630 & -513 & -82 \end{pmatrix}.$$

Table 5.
The list of odd hyperbolic lattices of the rank three and with even square-free determinant (i.e. non-main) which are reflective of hyperbolic type.

$N'=1$ $d=30=2{\cdot}13$, odd, $\eta=0$, $h=2$: $S=\langle 13\rangle\oplus\langle -1\rangle\oplus\langle -2\rangle$ $(=\widetilde{(13,1,h}=1))$.

$\rho=(8,-26,-13)$, $\rho^2=-182$;

$$\Gamma_+=\begin{pmatrix}18&-65&0\\0&0&1\\0&1&0\\5&0&-13\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-13&0&65&1170\\0&-2&0&26\\65&0&-1&0\\1170&26&0&-13\end{pmatrix};$$

$$C_1=\begin{pmatrix}12&3&2\\-39&-10&-6\\-13&-3&-3\end{pmatrix},\qquad C_2=\begin{pmatrix}25&4&8\\-52&-9&-16\\-52&-8&-17\end{pmatrix}.$$

$N'=2$ $d=38=2\cdot 19$, odd, $\eta=0$, $h=2$: $S=\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -38\rangle$. It is equivariantly equivalent to $d=19$, $\eta=1$, $h=2$: $U\oplus\langle -19\rangle$.

$N'=3$ $d=38=2{\cdot}19$, odd, $\eta=1$, $h=2$: $S=\langle 38\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$ $(=\widetilde{(19,0,h}=1))$.

$\rho=(24,-95,-114)$, $\rho^2=-133$;

$$\Gamma_+=\begin{pmatrix}39&-170&-170\\0&-1&1\\0&1&0\\3&0&-19\\1&-2&-6\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix}-2&0&170&1216&122\\0&-2&1&19&4\\170&1&-1&0&2\\1216&19&0&-19&0\\122&4&2&0&-2\end{pmatrix};$$

$$C_1=\begin{pmatrix}341&36&42\\-1368&-145&-168\\-1596&-168&-197\end{pmatrix},\qquad C_2=\begin{pmatrix}1899&180&250\\-6840&-649&-900\\-9500&-900&-1251\end{pmatrix}.$$

$N'=4$ $d=46=2\cdot 23$, odd, $\eta=1$, $h=2$: $S=\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -46\rangle$. It is equivariantly equivalent to $d=23$, $\eta=1$, $h=2$: $U\oplus\langle -23\rangle$.

$N'=5$ $d=70=2\cdot 5\cdot 7$, odd, $\eta=0$, $h=2$: $S=\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -70\rangle$. It is equivariantly equivalent to $d=35$, $\eta=1$, $h=2$: $U\oplus\langle -35\rangle$.

$N'=6$ $d=78=2\cdot 3\cdot 13$, odd, $\eta=2$, $h=2$: $S=\langle 1\rangle\oplus\langle -1\rangle\oplus\langle -78\rangle$. It is equivariantly equivalent to $d=39$, $\eta=1$, $h=2$: $U\oplus\langle -39\rangle$.

$N'=7$ $d=102=2\cdot 3\cdot 17$, odd, $\eta=1$, $h=2$: $S=\langle 6\rangle\oplus\langle -34\rangle\oplus\langle -2\rangle(0,1/2,1/2)$ $(=\widetilde{(51,0,h=0)})$.
$\rho=(17,-6,-17)$, $\rho^2=-68$;

$$\Gamma_+=\begin{pmatrix} 6 & \frac{-5}{2} & \frac{-3}{2}\\ 68 & \frac{-57}{2} & \frac{-17}{2}\\ 25 & \frac{-21}{2} & \frac{-3}{2}\\ 119 & -50 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & -2\\ 7 & \frac{-3}{2} & \frac{-21}{2}\\ 14 & \frac{-7}{2} & \frac{-39}{2}\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -1 & 0 & 3 & 34 & 3 & 85 & 30 & 93 & 148\\ 0 & -17 & 0 & 102 & 17 & 969 & 374 & 1224 & 1989\\ 3 & 0 & -3 & 0 & 3 & 357 & 144 & 483 & 792\\ 34 & 102 & 0 & -34 & 0 & 1700 & 714 & 2448 & 4046\\ 3 & 17 & 3 & 0 & -2 & 0 & 4 & 21 & 39\\ 85 & 969 & 357 & 1700 & 0 & -34 & 0 & 51 & 119\\ 30 & 374 & 144 & 714 & 4 & 0 & -2 & 0 & 6\\ 93 & 1224 & 483 & 2448 & 21 & 51 & 0 & -3 & 0\\ 148 & 1989 & 792 & 4046 & 39 & 119 & 6 & 0 & -1\end{pmatrix};$$

$$C_1=\begin{pmatrix} 127 & 272 & 32\\ -48 & -103 & -12\\ -96 & -204 & -25\end{pmatrix},\qquad C_2=\begin{pmatrix} 53 & 102 & 18\\ -18 & -35 & -6\\ -54 & -102 & -19\end{pmatrix}.$$

$N'=8$ $d=102=2\cdot 3\cdot 17$, odd, $\eta=2$, $h=2$: $S=\langle 102\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$ $(=\widetilde{(51,3,h=1)})$.
$\rho=(6,-34,-51)$, $\rho^2=-85$;

$$\Gamma_+=\begin{pmatrix} 7 & -50 & -50\\ 0 & -1 & 1\\ 0 & 1 & 0\\ 5 & 0 & -51\\ 1 & -2 & -10\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2 & 0 & 50 & 1020 & 114\\ 0 & -2 & 1 & 51 & 8\\ 50 & 1 & -1 & 0 & 2\\ 1020 & 51 & 0 & -51 & 0\\ 114 & 8 & 2 & 0 & -2\end{pmatrix};$$

$$C_1=\begin{pmatrix} 1087 & 72 & 80\\ -7344 & -487 & -540\\ -8160 & -540 & -601\end{pmatrix},\qquad C_2=\begin{pmatrix} 101 & 6 & 8\\ -612 & -37 & -48\\ -816 & -48 & -65\end{pmatrix}.$$

$N'=9$ $d=110=2\cdot 5\cdot 11$, odd, $\eta=2$, $h=2$: $S=\langle 22\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$. It is equivariantly equivalent to $d=55$, $\eta=1$, $h=2$: $\langle 11\rangle\oplus\langle -10\rangle\oplus\langle -2\rangle(0,1/2,1/2)$.

$N'=10$ $d=110=2\cdot 5\cdot 11$, odd, $\eta=3$, $h=2$: $S=\langle 22\rangle\oplus\langle -10\rangle\oplus\langle -2\rangle(0,1/2,1/2)$ $(=\widetilde{(55,0,h=1)})$.
$\rho=(12,-11/2,-77/2)$, $\rho^2=-99$;

$$\Gamma_+=\begin{pmatrix} 2 & -2 & -5\\ 5 & -6 & -10\\ 4 & \frac{-11}{2} & \frac{-11}{2}\\ 1 & \frac{-3}{2} & \frac{-1}{2}\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 3 & 0 & -10\\ 20 & -3 & -66\end{pmatrix},\qquad G(\Gamma_+)=\begin{pmatrix} -2 & 0 & 11 & 9 & 10 & 20 & 32 & 160\\ 0 & -10 & 0 & 10 & 20 & 60 & 130 & 700\\ 11 & 0 & -11 & 0 & 11 & 55 & 154 & 869\\ 9 & 10 & 0 & -1 & 1 & 15 & 56 & 329\\ 10 & 20 & 11 & 1 & -2 & 0 & 20 & 132\\ 20 & 60 & 55 & 15 & 0 & -10 & 0 & 30\\ 32 & 130 & 154 & 56 & 20 & 0 & -2 & 0\\ 160 & 700 & 869 & 329 & 132 & 30 & 0 & -2\end{pmatrix};$$

$$C_1=\begin{pmatrix} 109 & 30 & 30\\ -66 & -19 & -18\\ -330 & -90 & -91\end{pmatrix},\qquad C_2=\begin{pmatrix} 175 & 20 & 52\\ -44 & -6 & -13\\ -572 & -65 & -170\end{pmatrix}.$$

$N'=11$ $d=130=2\cdot 5\cdot 13$, odd, $\eta=3$, $h=0$: $S=\langle 2\rangle\oplus\langle -26\rangle\oplus\langle -10\rangle(0,1/2,1/2)$. It is equivariantly equivalent to $d=65$, $\eta=0$, $h=0$: $\langle 1\rangle\oplus\langle -13\rangle\oplus\langle -5\rangle$.

$N' = 12$ $d = 210 = 2{\cdot}3{\cdot}5{\cdot}7$, odd, $\eta = 5$, $h = 2$: $S = \langle 2\rangle \oplus \langle -30\rangle \oplus \langle -14\rangle (0, 1/2, 1/2)$ $(= (105, 6, h = 0))$.
$\rho = (15, -2, -5)$, $\rho^2 = -20$;

$$\Gamma_+ = \begin{pmatrix} 7 & \frac{-3}{2} & \frac{-3}{2} \\ 15 & \frac{-7}{2} & \frac{-5}{2} \\ 14 & \frac{-7}{2} & \frac{-3}{2} \\ 15 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 21 & 0 & -8 \\ 12 & \frac{-1}{2} & \frac{-9}{2} \\ 37 & \frac{-5}{2} & \frac{-27}{2} \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -1 & 0 & 7 & 30 & 21 & 45 & 126 & 51 & 122 \\ 0 & -5 & 0 & 30 & 35 & 105 & 350 & 150 & 375 \\ 7 & 0 & -7 & 0 & 21 & 105 & 420 & 189 & 490 \\ 30 & 30 & 0 & -30 & 0 & 120 & 630 & 300 & 810 \\ 21 & 35 & 21 & 0 & -14 & 0 & 112 & 63 & 189 \\ 45 & 105 & 105 & 120 & 0 & -30 & 0 & 15 & 75 \\ 126 & 350 & 420 & 630 & 112 & 0 & -14 & 0 & 42 \\ 51 & 150 & 189 & 300 & 63 & 15 & 0 & -3 & 0 \\ 122 & 375 & 490 & 810 & 189 & 75 & 42 & 0 & -1 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 167 & 420 & 336 \\ -28 & -71 & -56 \\ -48 & -120 & -97 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 53 & 90 & 126 \\ -6 & -11 & -14 \\ -18 & -30 & -43 \end{pmatrix}.$$

$N' = 13$ $d = 330 = 2 \cdot 3 \cdot 5 \cdot 11$, odd, $\eta = 1$, $h = 0$: $S = \langle 2\rangle \oplus \langle -30\rangle \oplus \langle -22\rangle (0, 1/2, 1/2)$ $(= (165, 6, h = 0))$.
$\rho = (30, -11/2, -15/2)$, $\rho^2 = -345$;

$$\Gamma_+ = \begin{pmatrix} 100 & \frac{-47}{2} & \frac{-25}{2} \\ 231 & -55 & -27 \\ 6 & \frac{-3}{2} & \frac{-1}{2} \\ 15 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 33 & 0 & -10 \\ 15 & \frac{-1}{2} & \frac{-9}{2} \\ 17 & -1 & -5 \\ 320 & \frac{-47}{2} & \frac{-185}{2} \end{pmatrix}, \ G(\Gamma_+) = \begin{pmatrix} -5 & 0 & 5 & 180 & 275 & 705 & 3850 & 1410 & 1320 & 21995 \\ 0 & -66 & 0 & 330 & 594 & 1650 & 9306 & 3432 & 3234 & 54120 \\ 5 & 0 & -1 & 0 & 11 & 45 & 286 & 108 & 104 & 1765 \\ 180 & 330 & 0 & -30 & 0 & 120 & 990 & 390 & 390 & 6780 \\ 275 & 594 & 11 & 0 & -22 & 0 & 220 & 99 & 110 & 2035 \\ 705 & 1650 & 45 & 120 & 0 & -30 & 0 & 15 & 30 & 705 \\ 3850 & 9306 & 286 & 990 & 220 & 0 & -22 & 0 & 22 & 770 \\ 1410 & 3432 & 108 & 390 & 99 & 15 & 0 & -3 & 0 & 90 \\ 1320 & 3234 & 104 & 390 & 110 & 30 & 22 & 0 & -2 & 0 \\ 21995 & 54120 & 1765 & 6780 & 2035 & 705 & 770 & 90 & 0 & -5 \end{pmatrix};$$

$$S = \begin{pmatrix} 285 & 990 & 418 \\ -22 & -76 & -33 \\ -82 & -285 & -120 \end{pmatrix}.$$

$N' = 14$ $d = 330 = 2 \cdot 3 \cdot 5 \cdot 11$, odd, $\eta = 5$, $h = 2$: $S = \langle 330\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle (0, 1/2, 1/2)$. It is equivariantly equivalent to $d = 165$, $\eta = 2$, $h = 2$: $\langle 165\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$.

$N' = 15$ $d = 330 = 2 \cdot 3 \cdot 5 \cdot 11$, odd, $\eta = 7$, $h = 2$: $S = \langle 10\rangle \oplus \langle -3\rangle \oplus \langle -11\rangle$ $(= (165, 0, h = 0))$.
$\rho = (33, -44, -24)$, $\rho^2 = -1254$;

$$\Gamma_+ = \begin{pmatrix} 231 & -385 & -90 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 99 & -55 & -90 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -165 & 0 & 990 & 1155 & 1320 & 76065 \\ 0 & -2 & 0 & 6 & 10 & 660 \\ 990 & 0 & -11 & 0 & 11 & 990 \\ 1155 & 6 & 0 & -3 & 0 & 165 \\ 1320 & 10 & 11 & 0 & -1 & 0 \\ 76065 & 660 & 990 & 165 & 0 & -165 \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 39 & 18 & 22 \\ -60 & -28 & -33 \\ -20 & -9 & -12 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 71 & 18 & 66 \\ -60 & -16 & -55 \\ -60 & -15 & -56 \end{pmatrix}.$$

$N' = 16$ $d = 390 = 2 \cdot 3 \cdot 5 \cdot 13$, odd, $\eta = 6$, $h = 2$: $S = \langle 10\rangle \oplus \langle -26\rangle \oplus \langle -6\rangle (1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 195$, $\eta = 1$, $h = 2$: $\langle 5\rangle \oplus \langle -13\rangle \oplus \langle -3\rangle$.

$N' = 17$ $d = 390 = 2 \cdot 3 \cdot 5 \cdot 13$, odd, $\eta = 0$, $h = 2$: $S = \langle 2\rangle \oplus \langle -130\rangle \oplus \langle -6\rangle (1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 195$, $\eta = 7$, $h = 2$: $\langle 1\rangle \oplus \langle -65\rangle \oplus \langle -3\rangle$.

$N' = 18$ $d = 462 = 2\cdot 3\cdot 7\cdot 11$, odd, $\eta = 2$, $h = 0$: $S = \langle 3\rangle \oplus \langle -7\rangle \oplus \langle -22\rangle$ $(= (231, 7, h = 0))$.
$\rho = (22, -11, -7)$, $\rho^2 = -473$;

$$\Gamma_+ = \begin{pmatrix} 11 & -7 & -1 \\ 3 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 8 & 0 & -3 \\ 385 & -44 & -140 \\ 80 & -10 & -29 \\ 235 & -31 & -85 \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -2 & 1 & 22 & 49 & 198 & 7469 & 1512 & 4366 \\ 1 & -1 & 0 & 14 & 72 & 2849 & 580 & 1681 \\ 22 & 0 & -22 & 0 & 66 & 3080 & 638 & 1870 \\ 49 & 14 & 0 & -7 & 0 & 308 & 70 & 217 \\ 198 & 72 & 66 & 0 & -6 & 0 & 6 & 30 \\ 7469 & 2849 & 3080 & 308 & 0 & -77 & 0 & 77 \\ 1512 & 580 & 638 & 70 & 6 & 0 & -2 & 0 \\ 4366 & 1681 & 1870 & 217 & 30 & 77 & 0 & -2 \end{pmatrix};$$

$$S = \begin{pmatrix} 349 & 504 & 308 \\ -48 & -69 & -44 \\ -126 & -182 & -111 \end{pmatrix}.$$

$N' = 19$ $d = 462 = 2\cdot 3\cdot 7\cdot 11$, odd, $\eta = 6$, $h = 2$: $S = \langle 22\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$. It is equivariantly equivalent to $d = 231$, $\eta = 3$, $h = 2$: $\langle 11\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$.

$N' = 20$ $d = 510 = 2\cdot 3\cdot 5\cdot 17$, odd, $\eta = 1$, $h = 2$: $S = \langle 2\rangle \oplus \langle -34\rangle \oplus \langle -30\rangle(1/2, 0, 1/2)$. It is equivariantly equivalent to $d = 255$, $\eta = 2$, $h = 2$: $\langle 1\rangle \oplus \langle -17\rangle \oplus \langle -15\rangle$.

$N' = 21$ $d = 510 = 2\cdot 3\cdot 5\cdot 17$, odd, $\eta = 5$, $h = 0$: $S = \langle 6\rangle \oplus \langle -170\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$ $(= (255, 6, h = 0))$.
$\rho = (35, -6, -30)$, $\rho^2 = -570$;

$$\Gamma_+ = \begin{pmatrix} 19 & \frac{-7}{2} & \frac{-13}{2} \\ 8 & \frac{-3}{2} & \frac{-3}{2} \\ 85 & -16 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 5 & \frac{-1}{2} & \frac{-15}{2} \\ 119 & -15 & -153 \\ 19 & \frac{-5}{2} & \frac{-47}{2} \end{pmatrix}, \qquad G(\Gamma_+) = \begin{pmatrix} -1 & 0 & 170 & 13 & 595 & 88 & 175 & 2652 & 373 \\ 0 & -3 & 0 & 3 & 255 & 42 & 90 & 1428 & 204 \\ 170 & 0 & -170 & 0 & 2720 & 510 & 1190 & 19890 & 2890 \\ 13 & 3 & 0 & -2 & 0 & 4 & 15 & 306 & 47 \\ 595 & 255 & 2720 & 0 & -170 & 0 & 85 & 2550 & 425 \\ 88 & 42 & 510 & 4 & 0 & -2 & 0 & 102 & 20 \\ 175 & 90 & 1190 & 15 & 85 & 0 & -5 & 0 & 5 \\ 2652 & 1428 & 19890 & 306 & 2550 & 102 & 0 & -102 & 0 \\ 373 & 204 & 2890 & 47 & 425 & 20 & 5 & 0 & -1 \end{pmatrix};$$

$$S = \begin{pmatrix} 263 & 1360 & 36 \\ -36 & -186 & -5 \\ -312 & -1615 & -42 \end{pmatrix}.$$

Table 6
The list of main hyperbolic lattices of the rank 3
with the determinant $d \leq 100000$ and $h = 2$

$n = 1,\ d = 17,\ \eta = 1,\ h = 2$	$\langle 51\rangle \oplus A_2(1/3, 1/3, -1/3)$	*hr*
$n = 2,\ d = 19,\ \eta = 1,\ h = 2$	$U \oplus \langle -19\rangle$	*hr*
$n = 3,\ d = 23,\ \eta = 1,\ h = 2$	$U \oplus \langle -23\rangle$	*hr*
$n = 4,\ d = 31,\ \eta = 0,\ h = 2$	$\langle 31\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 5,\ d = 35 = 5 \cdot 7,\ \eta = 1,\ h = 2$	$U \oplus \langle -35\rangle$	*hr*
$n = 6,\ d = 37,\ \eta = 0,\ h = 2$	$U \oplus \langle -37\rangle$	*nr*
$n = 7,\ d = 37,\ \eta = 1,\ h = 2$	$\langle 1\rangle \oplus \langle -74\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 8,\ d = 39 = 3 \cdot 13,\ \eta = 1,\ h = 2$	$U \oplus \langle -39\rangle$	*hr*
$n = 9,\ d = 43,\ \eta = 0,\ h = 2$	$\langle 43\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 10,\ d = 46 = 2 \cdot 23,\ \eta = 1,\ h = 2$	$U \oplus \langle -46\rangle$	*hr*
$n = 11,\ d = 47,\ \eta = 0,\ h = 2$	$\langle 47\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 12,\ d = 51 = 3 \cdot 17,\ \eta = 2,\ h = 2$	$U \oplus \langle -51\rangle$	*nr*
$n = 13,\ d = 53,\ \eta = 0,\ h = 2$	$U \oplus \langle -53\rangle$	*nr*
$n = 14,\ d = 53,\ \eta = 1,\ h = 2$	$\langle 1\rangle \oplus \langle -106\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 15,\ d = 55 = 5 \cdot 11,\ \eta = 1,\ h = 2$	$\langle 11\rangle \oplus \langle -10\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 16,\ d = 57 = 3 \cdot 19,\ \eta = 3,\ h = 2$	$\langle 1\rangle \oplus \langle -19\rangle \oplus \langle -3\rangle$	*hr*
$n = 17,\ d = 58 = 2 \cdot 29,\ \eta = 1,\ h = 2$	$U \oplus \langle -58\rangle$	*hr*
$n = 18,\ d = 59,\ \eta = 0,\ h = 2$	$\langle 59\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 19,\ d = 62 = 2 \cdot 31,\ \eta = 1,\ h = 2$	$U \oplus \langle -62\rangle$	*hr*
$n = 20,\ d = 65 = 5 \cdot 13,\ \eta = 1,\ h = 2$	$\langle 2\rangle \oplus \langle -13\rangle \oplus \langle -10\rangle(1/2, 0, 1/2)$	*hr*
$n = 21,\ d = 69 = 3 \cdot 23,\ \eta = 0,\ h = 2$	$\langle 46\rangle \oplus \langle -6\rangle \oplus \langle -1\rangle(1/2, 1/2, 0)$	*hr*
$n = 22,\ d = 69 = 3 \cdot 23,\ \eta = 2,\ h = 2$	$U \oplus \langle -69\rangle$	*nr*
$n = 23,\ d = 70 = 2 \cdot 5 \cdot 7,\ \eta = 0,\ h = 2$	$U \oplus \langle -70\rangle$	*hr*
$n = 24,\ d = 77 = 7 \cdot 11,\ \eta = 0,\ h = 2$	$\langle 14\rangle \oplus \langle -22\rangle \oplus \langle -1\rangle(1/2, 1/2, 0)$	*hr*
$n = 25,\ d = 79,\ \eta = 0,\ h = 2$	$\langle 79\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 26,\ d = 85 = 5 \cdot 17,\ \eta = 2,\ h = 2$	$\langle 1\rangle \oplus \langle -170\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 27,\ d = 85 = 5 \cdot 17,\ \eta = 3,\ h = 2$	$U \oplus \langle -85\rangle$	*nr*
$n = 28,\ d = 87 = 3 \cdot 29,\ \eta = 0,\ h = 2$	$\langle 3\rangle \oplus \langle -29\rangle \oplus \langle -1\rangle$	*hr*
$n = 29,\ d = 91 = 7 \cdot 13,\ \eta = 0,\ h = 2$	$\langle 7\rangle \oplus \langle -13\rangle \oplus \langle -1\rangle$	*hr*
$n = 30,\ d = 91 = 7 \cdot 13,\ \eta = 1,\ h = 2$	$\langle 1\rangle \oplus \langle -13\rangle \oplus \langle -7\rangle$	*nr*
$n = 31,\ d = 93 = 3 \cdot 31,\ \eta = 3,\ h = 2$	$\langle 1\rangle \oplus \langle -31\rangle \oplus \langle -3\rangle$	*hr*
$n = 32,\ d = 106 = 2 \cdot 53,\ \eta = 1,\ h = 2$	$U \oplus \langle -106\rangle$	*nr*
$n = 33,\ d = 107,\ \eta = 0,\ h = 2$	$\langle 107\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 34,\ d = 113,\ \eta = 0,\ h = 2$	$U \oplus \langle -113\rangle$	*nr*
$n = 35,\ d = 114 = 2 \cdot 3 \cdot 19,\ \eta = 1,\ h = 2$	$\langle 6\rangle \oplus \langle -38\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 36,\ d = 115 = 5 \cdot 23,\ \eta = 0,\ h = 2$	$\langle 23\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*hr*
$n = 37,\ d = 115 = 5 \cdot 23,\ \eta = 3,\ h = 2$	$\langle 115\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 38,\ d = 119 = 7 \cdot 17,\ \eta = 0,\ h = 2$	$\langle 7\rangle \oplus \langle -17\rangle \oplus \langle -1\rangle$	*hr*
$n = 39,\ d = 122 = 2 \cdot 61,\ \eta = 1,\ h = 2$	$U \oplus \langle -122\rangle$	*nr*
$n = 40,\ d = 123 = 3 \cdot 41,\ \eta = 0,\ h = 2$	$\langle 3\rangle \oplus \langle -41\rangle \oplus \langle -1\rangle$	*hr*
$n = 41,\ d = 123 = 3 \cdot 41,\ \eta = 1,\ h = 2$	$\langle 41\rangle \oplus \langle -3\rangle \oplus \langle -1\rangle$	*nr*
$n = 42,\ d = 123 = 3 \cdot 41,\ \eta = 3,\ h = 2$	$\langle 123\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*hr*
$n = 43,\ d = 129 = 3 \cdot 43,\ \eta = 0,\ h = 2$	$\langle 3\rangle \oplus \langle -86\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$	*hr*
$n = 44,\ d = 129 = 3 \cdot 43,\ \eta = 2,\ h = 2$	$\langle 3\rangle \oplus \langle -43\rangle \oplus \langle -1\rangle$	*nr*

$n=45,\ d=133=7\cdot 19,\ \eta=1,\ h=2$	$\langle 19\rangle\oplus\langle -7\rangle\oplus\langle -1\rangle$	nr
$n=46,\ d=133=7\cdot 19,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -19\rangle\oplus\langle -7\rangle$	hr
$n=47,\ d=138=2\cdot 3\cdot 23,\ \eta=0,\ h=2$	$\langle 46\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	hr
$n=48,\ d=141=3\cdot 47,\ \eta=0,\ h=2$	$\langle 94\rangle\oplus\langle -6\rangle\oplus\langle -1\rangle(1/2,1/2,0)$	nr
$n=49,\ d=143=11\cdot 13,\ \eta=3,\ h=2$	$\langle 143\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	hr
$n=50,\ d=145=5\cdot 29,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -290\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	nr
$n=51,\ d=155=5\cdot 31,\ \eta=0,\ h=2$	$\langle 31\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$	nr
$n=52,\ d=159=3\cdot 53,\ \eta=0,\ h=2$	$\langle 3\rangle\oplus\langle -53\rangle\oplus\langle -1\rangle$	nr
$n=53,\ d=159=3\cdot 53,\ \eta=3,\ h=2$	$\langle 159\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	hr
$n=54,\ d=165=3\cdot 5\cdot 11,\ \eta=2,\ h=2$	$\langle 165\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	hr
$n=55,\ d=174=2\cdot 3\cdot 29,\ \eta=2,\ h=2$	$\langle 2\rangle\oplus\langle -58\rangle\oplus\langle -6\rangle(0,1/2,1/2)$	hr
$n=56,\ d=177=3\cdot 59,\ \eta=1,\ h=2$	$\langle 59\rangle\oplus\langle -3\rangle\oplus\langle -1\rangle$	nr
$n=57,\ d=182=2\cdot 7\cdot 13,\ \eta=0,\ h=2$	$U\oplus\langle -182\rangle$	nr
$n=58,\ d=183=3\cdot 61,\ \eta=0,\ h=2$	$\langle 3\rangle\oplus\langle -61\rangle\oplus\langle -1\rangle$	hr
$n=59,\ d=185=5\cdot 37,\ \eta=0,\ h=2$	$\langle 1\rangle\oplus\langle -37\rangle\oplus\langle -5\rangle$	nr
$n=60,\ d=191,\ \eta=0,\ h=2$	$\langle 191\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	nr
$n=61,\ d=195=3\cdot 5\cdot 13,\ \eta=0,\ h=2$	$\langle 3\rangle\oplus\langle -13\rangle\oplus\langle -5\rangle$	hr
$n=62,\ d=195=3\cdot 5\cdot 13,\ \eta=1,\ h=2$	$\langle 5\rangle\oplus\langle -13\rangle\oplus\langle -3\rangle$	hr
$n=63,\ d=195=3\cdot 5\cdot 13,\ \eta=7,\ h=2$	$\langle 1\rangle\oplus\langle -65\rangle\oplus\langle -3\rangle$	hr
$n=64,\ d=205=5\cdot 41,\ \eta=0,\ h=2$	$U\oplus\langle -205\rangle$	nr
$n=65,\ d=210=2\cdot 3\cdot 5\cdot 7,\ \eta=1,\ h=2$	$\langle 14\rangle\oplus\langle -30\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	hr
$n=66,\ d=210=2\cdot 3\cdot 5\cdot 7,\ \eta=7,\ h=2$	$U\oplus\langle -210\rangle$	hr
$n=67,\ d=213=3\cdot 71,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -71\rangle\oplus\langle -3\rangle$	hr
$n=68,\ d=217=7\cdot 31,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -31\rangle\oplus\langle -7\rangle$	nr
$n=69,\ d=218=2\cdot 109,\ \eta=1,\ h=2$	$U\oplus\langle -218\rangle$	nr
$n=70,\ d=221=13\cdot 17,\ \eta=0,\ h=2$	$U\oplus\langle -221\rangle$	nr
$n=71,\ d=221=13\cdot 17,\ \eta=1,\ h=2$	$\langle 2\rangle\oplus\langle -26\rangle\oplus\langle -17\rangle(1/2,1/2,0)$	hr
$n=72,\ d=222=2\cdot 3\cdot 37,\ \eta=2,\ h=2$	$U\oplus\langle -222\rangle$	nr
$n=73,\ d=230=2\cdot 5\cdot 23,\ \eta=0,\ h=2$	$U\oplus\langle -230\rangle$	nr
$n=74,\ d=231=3\cdot 7\cdot 11,\ \eta=3,\ h=2$	$\langle 11\rangle\oplus\langle -7\rangle\oplus\langle -3\rangle$	hr
$n=75,\ d=231=3\cdot 7\cdot 11,\ \eta=4,\ h=2$	$\langle 7\rangle\oplus\langle -33\rangle\oplus\langle -1\rangle$	hr
$n=76,\ d=235=5\cdot 47,\ \eta=0,\ h=2$	$\langle 47\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$	nr
$n=77,\ d=239,\ \eta=0,\ h=2$	$\langle 239\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	nr
$n=78,\ d=255=3\cdot 5\cdot 17,\ \eta=2,\ h=2$	$\langle 1\rangle\oplus\langle -17\rangle\oplus\langle -15\rangle$	hr
$n=79,\ d=255=3\cdot 5\cdot 17,\ \eta=3,\ h=2$	$\langle 15\rangle\oplus\langle -17\rangle\oplus\langle -1\rangle$	hr
$n=80,\ d=258=2\cdot 3\cdot 43,\ \eta=2,\ h=2$	$U\oplus\langle -258\rangle$	nr
$n=81,\ d=273=3\cdot 7\cdot 13,\ \eta=2,\ h=2$	$\langle 3\rangle\oplus\langle -13\rangle\oplus\langle -7\rangle$	nr
$n=82,\ d=282=2\cdot 3\cdot 47,\ \eta=0,\ h=2$	$\langle 94\rangle\oplus\langle -6\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	nr
$n=83,\ d=287=7\cdot 41,\ \eta=3,\ h=2$	$\langle 287\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	nr
$n=84,\ d=290=2\cdot 5\cdot 29,\ \eta=0,\ h=2$	$\langle 870\rangle\oplus A_2(1/3,1/3,-1/3)$	nr
$n=85,\ d=290=2\cdot 5\cdot 29,\ \eta=3,\ h=2$	$U\oplus\langle -290\rangle$	nr
$n=86,\ d=301=7\cdot 43,\ \eta=1,\ h=2$	$U\oplus\langle -301\rangle$	nr
$n=87,\ d=301=7\cdot 43,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -43\rangle\oplus\langle -7\rangle$	nr
$n=88,\ d=303=3\cdot 101,\ \eta=3,\ h=2$	$\langle 303)\oplus\langle -1\rangle\oplus\langle -1\rangle$	nr
$n=89,\ d=309=3\cdot 103,\ \eta=0,\ h=2$	$\langle 2\rangle\oplus\langle -618\rangle\oplus\langle -1\rangle(1/2,1/2,0)$	nr
$n=90,\ d=314=2\cdot 157,\ \eta=1,\ h=2$	$U\oplus\langle -314\rangle$	nr
$n=91,\ d=319=11\cdot 29,\ \eta=3,\ h=2$	$\langle 319\rangle\oplus\langle -1\rangle\oplus\langle -1\rangle$	nr
$n=92,\ d=323=17\cdot 19,\ \eta=0,\ h=2$	$\langle 19\rangle\oplus\langle -17\rangle\oplus\langle -1\rangle$	nr
$n=93,\ d=329=7\cdot 47,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -47\rangle\oplus\langle -7\rangle$	nr

$n = 94,\ d = 330 = 2\cdot 3\cdot 5\cdot 11,\ \eta = 6,\ h = 2$	$\langle 22\rangle \oplus \langle -10\rangle \oplus \langle -6\rangle(0,1/2,1/2)$	*hr*
$n = 95,\ d = 345 = 3\cdot 5\cdot 23,\ \eta = 2,\ h = 2$	$\langle 3\rangle \oplus \langle -115\rangle \oplus \langle -1\rangle$	*nr*
$n = 96,\ d = 345 = 3\cdot 5\cdot 23,\ \eta = 4,\ h = 2$	$\langle 1\rangle \oplus \langle -69\rangle \oplus \langle -5\rangle$	*nr*
$n = 97,\ d = 345 = 3\cdot 5\cdot 23,\ \eta = 5,\ h = 2$	$\langle 5\rangle \oplus \langle -23\rangle \oplus \langle -3\rangle$	*nr*
$n = 98,\ d = 357 = 3\cdot 7\cdot 17,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -51\rangle \oplus \langle -7\rangle$	*nr*
$n = 99,\ d = 366 = 2\cdot 3\cdot 61,\ \eta = 2,\ h = 2$	$U \oplus \langle -366\rangle$	*nr*
$n = 100,\ d = 374 = 2\cdot 11\cdot 17,\ \eta = 0,\ h = 2$	$\langle 2\rangle \oplus \langle -34\rangle \oplus \langle -22\rangle(0,1/2,1/2)$	*nr*
$n = 101,\ d = 390 = 2\cdot 3\cdot 5\cdot 13,\ \eta = 0,\ h = 2$	$\langle 130\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*hr*
$n = 102,\ d = 402 = 2\cdot 3\cdot 67,\ \eta = 2,\ h = 2$	$U \oplus \langle -402\rangle$	*nr*
$n = 103,\ d = 406 = 2\cdot 7\cdot 29,\ \eta = 3,\ h = 2$	$U \oplus \langle -406)$	*nr*
$n = 104,\ d = 410 = 2\cdot 5\cdot 41,\ \eta = 2,\ h = 2$	$\langle 1230\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 105,\ d = 426 = 2\cdot 3\cdot 71,\ \eta = 0,\ h = 2$	$\langle 142\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*nr*
$n = 106,\ d = 429 = 3\cdot 11\cdot 13,\ \eta = 0,\ h = 2$	$\langle 13\rangle \oplus \langle -22\rangle \oplus \langle -6\rangle(0,1/2,1/2)$	*nr*
$n = 107,\ d = 429 = 3\cdot 11\cdot 13,\ \eta = 5,\ h = 2$	$\langle 1\rangle \oplus \langle -143\rangle \oplus \langle -3\rangle$	*hr*
$n = 108,\ d = 429 = 3\cdot 11\cdot 13,\ \eta = 6,\ h = 2$	$\langle 2\rangle \oplus \langle -26\rangle \oplus \langle -33\rangle(1/2,1/2,0)$	*nr*
$n = 109,\ d = 431 =,\ \eta = 0,\ h = 2$	$\langle 431\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 110,\ d = 434 = 2\cdot 7\cdot 31,\ \eta = 1,\ h = 2$	$\langle 62\rangle \oplus \langle -14\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*nr*
$n = 111,\ d = 438 = 2\cdot 3\cdot 73,\ \eta = 0,\ h = 2$	$U \oplus \langle -438\rangle$	*nr*
$n = 112,\ d = 455 = 5\cdot 7\cdot 13,\ \eta = 3,\ h = 2$	$\langle 35\rangle \oplus \langle -13\rangle \oplus \langle -1\rangle$	*nr*
$n = 113,\ d = 455 = 5\cdot 7\cdot 13,\ \eta = 6,\ h = 2$	$\langle 91\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*nr*
$n = 114,\ d = 462 = 2\cdot 3\cdot 7\cdot 11,\ \eta = 0,\ h = 2$	$\langle 14\rangle \oplus \langle -22\rangle \oplus \langle -6\rangle(1/2,1/2,0)$	*hr*
$n = 115,\ d = 462 = 2\cdot 3\cdot 7\cdot 11,\ \eta = 6,\ h = 2$	$\langle 22\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(1/2,1/2,0)$	*hr*
$n = 116,\ d = 465 = 3\cdot 5\cdot 31,\ \eta = 1,\ h = 2$	$\langle 1\rangle \oplus \langle -93\rangle \oplus \langle -5\rangle$	*nr*
$n = 117,\ d = 465 = 3\cdot 5\cdot 31,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -31\rangle \oplus \langle -15\rangle$	*nr*
$n = 118,\ d = 469 = 7\cdot 67,\ \eta = 1,\ h = 2$	$U \oplus \langle -469\rangle$	*nr*
$n = 119,\ d = 469 = 7\cdot 67,\ \eta = 3,\ h = 2$	$\langle 1\rangle \oplus \langle -67\rangle \oplus \langle -7\rangle$	*nr*
$n = 120,\ d = 470 = 2\cdot 5\cdot 47,\ \eta = 0,\ h = 2$	$U \oplus \langle -470\rangle$	*nr*
$n = 121,\ d = 471 = 3\cdot 157,\ \eta = 0,\ h = 2$	$\langle 3\rangle \oplus \langle -157\rangle \oplus \langle -1\rangle$	*nr*
$n = 122,\ d = 474 = 2\cdot 3\cdot 79,\ \eta = 0,\ h = 2$	$U \oplus \langle -474\rangle$	*nr*
$n = 123,\ d = 483 = 3\cdot 7\cdot 23,\ \eta = 2,\ h = 2$	$\langle 3\rangle \oplus \langle -161\rangle \oplus \langle -1\rangle$	*nr*
$n = 124,\ d = 515 = 5\cdot 103,\ \eta = 0,\ h = 2$	$\langle 103\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*nr*
$n = 125,\ d = 518 = 2\cdot 7\cdot 37,\ \eta = 3,\ h = 2$	$U \oplus \langle -518\rangle$	*nr*
$n = 126,\ d = 527 = 17\cdot 31,\ \eta = 3,\ h = 2$	$\langle 527\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 127,\ d = 554 = 2\cdot 277,\ \eta = 1,\ h = 2$	$U \oplus \langle -554\rangle$	*nr*
$n = 128,\ d = 555 = 3\cdot 5\cdot 37,\ \eta = 3,\ h = 2$	$\langle 15\rangle \oplus \langle -37\rangle \oplus \langle -1\rangle$	*hr*
$n = 129,\ d = 555 = 3\cdot 5\cdot 37,\ \eta = 5,\ h = 2$	$\langle 555\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 130,\ d = 555 = 3\cdot 5\cdot 37,\ \eta = 6,\ h = 2$	$\langle 3\rangle \oplus \langle -185\rangle \oplus \langle -1\rangle$	*nr*
$n = 131,\ d = 561 = 3\cdot 11\cdot 17,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -51\rangle \oplus \langle -11\rangle$	*hr*
$n = 132,\ d = 582 = 2\cdot 3\cdot 97,\ \eta = 0,\ h = 2$	$U \oplus \langle -582\rangle$	*nr*
$n = 133,\ d = 595 = 5\cdot 7\cdot 17,\ \eta = 3,\ h = 2$	$\langle 35\rangle \oplus \langle -17\rangle \oplus \langle -1\rangle$	*nr*
$n = 134,\ d = 609 = 3\cdot 7\cdot 29,\ \eta = 3,\ h = 2$	$\langle 29\rangle \oplus \langle -7\rangle \oplus \langle -3\rangle$	*hr*
$n = 135,\ d = 609 = 3\cdot 7\cdot 29,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -87\rangle \oplus \langle -7\rangle$	*nr*
$n = 136,\ d = 618 = 2\cdot 3\cdot 103,\ \eta = 0,\ h = 2$	$U \oplus \langle -618\rangle$	*nr*
$n = 137,\ d = 623 = 7\cdot 89,\ \eta = 3,\ h = 2$	$\langle 623\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$	*nr*
$n = 138,\ d = 627 = 3\cdot 11\cdot 19,\ \eta = 1,\ h = 2$	$\langle 11\rangle \oplus \langle -57\rangle \oplus \langle -1\rangle$	*nr*
$n = 139,\ d = 627 = 3\cdot 11\cdot 19,\ \eta = 3,\ h = 2$	$\langle 1\rangle \oplus \langle -57\rangle \oplus \langle -11\rangle$	*nr*
$n = 140,\ d = 663 = 3\cdot 13\cdot 17,\ \eta = 3,\ h = 2$	$\langle 6\rangle \oplus \langle -26\rangle \oplus \langle -17\rangle(1/2,1/2,0)$	*hr*
$n = 141,\ d = 665 = 5\cdot 7\cdot 19,\ \eta = 0,\ h = 2$	$\langle 2\rangle \oplus \langle -226\rangle \oplus \langle -5\rangle(1/2,1/2,0)$	*nr*
$n = 142,\ d = 690 = 2\cdot 3\cdot 5\cdot 23,\ \eta = 1,\ h = 2$	$\langle 46\rangle \oplus \langle -30\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*nr*

$n = 143,\ d = 690 = 2\cdot 3\cdot 5\cdot 23,\ \eta = 2,\ h = 2$	$U \oplus \langle -690\rangle$	*nr*
$n = 144,\ d = 690 = 2\cdot 3\cdot 5\cdot 23,\ \eta = 7,\ h = 2$	$\langle 6\rangle \oplus \langle -46\rangle \oplus \langle -10\rangle(0,1/2,1/2)$	*nr*
$n = 145,\ d = 705 = 3\cdot 5\cdot 47,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -47\rangle \oplus \langle -15\rangle$	*hr*
$n = 146,\ d = 715 = 5\cdot 11\cdot 13,\ \eta = 0,\ h = 2$	$\langle 11\rangle \oplus \langle -13\rangle \oplus \langle -5\rangle$	*nr*
$n = 147,\ d = 715 = 5\cdot 11\cdot 13,\ \eta = 6,\ h = 2$	$\langle 143\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*nr*
$n = 148,\ d = 741 = 3\cdot 13\cdot 19,\ \eta = 0,\ h = 2$	$\langle 2\rangle \oplus \langle -1482\rangle \oplus \langle -1\rangle(1/2,1/2,0)$	*nr*
$n = 149,\ d = 741 = 3\cdot 13\cdot 19,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -114\rangle \oplus \langle -26\rangle(0,1/2,1/2)$	*nr*
$n = 150,\ d = 770 = 2\cdot 5\cdot 7\cdot 11,\ \eta = 2,\ h = 2$	$\langle 770\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle(1/2,1/2,0)$	*nr*
$n = 151,\ d = 770 = 2\cdot 5\cdot 7\cdot 11,\ \eta = 7,\ h = 2$	$\langle 10\rangle \oplus \langle -154\rangle \oplus \langle -2\rangle(1/2,0,1/2)$	*nr*
$n = 152,\ d = 791 = 7\cdot 113,\ \eta = 3,\ h = 2$	$\langle 2373\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 153,\ d = 794 = 2\cdot 397,\ \eta = 1,\ h = 2$	$U \oplus \langle -794\rangle$	*nr*
$n = 154,\ d = 798 = 2\cdot 3\cdot 7\cdot 19,\ \eta = 0,\ h = 2$	$\langle 2\rangle \oplus \langle -38\rangle \oplus \langle -42\rangle(0,1/2,1/2)$	*nr*
$n = 155,\ d = 798 = 2\cdot 3\cdot 7\cdot 19,\ \eta = 6,\ h = 2$	$U \oplus \langle -798\rangle$	*nr*
$n = 156,\ d = 806 = 2\cdot 13\cdot 31,\ \eta = 0,\ h = 2$	$U \oplus \langle -806\rangle$	*nr*
$n = 157,\ d = 834 = 2\cdot 3\cdot 139,\ \eta = 2,\ h = 2$	$U \oplus \langle -834\rangle$	*nr*
$n = 158,\ d = 858 = 2\cdot 3\cdot 11\cdot 13,\ \eta = 0,\ h = 2$	$\langle 858\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle(1/2,1/2,0)$	*nr*
$n = 159,\ d = 890 = 2\cdot 5\cdot 89,\ \eta = 2,\ h = 2$	$\langle 2670\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 160,\ d = 897 = 3\cdot 13\cdot 23,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -598\rangle \oplus \langle -6\rangle(0,1/2,1/2)$	*nr*
$n = 161,\ d = 903 = 3\cdot 7\cdot 43,\ \eta = 2,\ h = 2$	$\langle 3\rangle \oplus \langle -301\rangle \oplus \langle -1\rangle$	*nr*
$n = 162,\ d = 906 = 2\cdot 3\cdot 151,\ \eta = 0,\ h = 2$	$U \oplus \langle -906\rangle$	*nr*
$n = 163,\ d = 915 = 3\cdot 5\cdot 61,\ \eta = 5,\ h = 2$	$\langle 2\rangle \oplus \langle -1830\rangle \oplus \langle -1\rangle(1/2,1/2,0)$	*nr*
$n = 164,\ d = 935 = 5\cdot 11\cdot 17,\ \eta = 0,\ h = 2$	$\langle 11\rangle \oplus \langle -17\rangle \oplus \langle -5\rangle$	*nr*
$n = 165,\ d = 959 = 7\cdot 137,\ \eta = 3,\ h = 2$	$\langle 2877\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 166,\ d = 966 = 2\cdot 3\cdot 7\cdot 23,\ \eta = 4,\ h = 2$	$\langle 14\rangle \oplus \langle -46\rangle \oplus \langle -6\rangle(1/2,1/2,0)$	*nr*
$n = 167,\ d = 986 = 2\cdot 17\cdot 29,\ \eta = 1,\ h = 2$	$U \oplus \langle -986\rangle$	*nr*
$n = 168,\ d = 987 = 3\cdot 7\cdot 47,\ \eta = 7,\ h = 2$	$\langle 2\rangle \oplus \langle -282\rangle \oplus \langle -7\rangle(1/2,1/2,0)$	*nr*
$n = 169,\ d = 1001 = 7\cdot 11\cdot 13,\ \eta = 5,\ h = 2$	$\langle 1\rangle \oplus \langle -143\rangle \oplus \langle -7\rangle$	*nr*
$n = 170,\ d = 1010 = 2\cdot 5\cdot 101,\ \eta = 0,\ h = 2$	$\langle 3030\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 171,\ d = 1015 = 5\cdot 7\cdot 29,\ \eta = 6,\ h = 2$	$\langle 5\rangle \oplus \langle -406\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*nr*
$n = 172,\ d = 1023 = 3\cdot 11\cdot 31,\ \eta = 1,\ h = 2$	$\langle 11\rangle \oplus \langle -93\rangle \oplus \langle -1\rangle$	*nr*
$n = 173,\ d = 1066 = 2\cdot 13\cdot 41,\ \eta = 2,\ h = 2$	$U \oplus \langle -1066\rangle$	*nr*
$n = 174,\ d = 1085 = 5\cdot 7\cdot 31,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -155\rangle \oplus \langle -7\rangle$	*nr*
$n = 175,\ d = 1095 = 3\cdot 5\cdot 73,\ \eta = 3,\ h = 2$	$\langle 15\rangle \oplus \langle -73\rangle \oplus \langle -1\rangle$	*nr*
$n = 176,\ d = 1110 = 2\cdot 3\cdot 5\cdot 37,\ \eta = 0,\ h = 2$	$\langle 2\rangle \oplus \langle -370\rangle \oplus \langle -6\rangle(0,1/2,1/2)$	*nr*
$n = 177,\ d = 1122 = 2\cdot 3\cdot 11\cdot 17,\ \eta = 1,\ h = 2$	$\langle 6\rangle \oplus \langle -34\rangle \oplus \langle -22\rangle(0,1/2,1/2)$	*nr*
$n = 178,\ d = 1130 = 2\cdot 5\cdot 113,\ \eta = 2,\ h = 2$	$U \oplus \langle -1130\rangle$	*nr*
$n = 179,\ d = 1155 = 3\cdot 5\cdot 7\cdot 11,\ \eta = 10,\ h = 2$	$\langle 1\rangle \oplus \langle -35\rangle \oplus \langle -33\rangle$	*nr*
$n = 180,\ d = 1173 = 3\cdot 17\cdot 23,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -51\rangle \oplus \langle -23\rangle$	*nr*
$n = 181,\ d = 1194 = 2\cdot 3\cdot 199,\ \eta = 0,\ h = 2$	$U \oplus \langle -1194\rangle$	*nr*
$n = 182,\ d = 1218 = 2\cdot 3\cdot 7\cdot 29,\ \eta = 2,\ h = 2$	$\langle 174\rangle \oplus \langle -14\rangle \oplus \langle -2\rangle(0,1/2,1/2)$	*nr*
$n = 183,\ d = 1235 = 5\cdot 13\cdot 19,\ \eta = 6,\ h = 2$	$\langle 247\rangle \oplus \langle -5\rangle \oplus \langle -1\rangle$	*nr*
$n = 184,\ d = 1245 = 3\cdot 5\cdot 83,\ \eta = 6,\ h = 2$	$\langle 1\rangle \oplus \langle -83\rangle \oplus \langle -15\rangle$	*nr*
$n = 185,\ d = 1254 = 2\cdot 3\cdot 11\cdot 19,\ \eta = 2,\ h = 2$	$\langle 2\rangle \oplus \langle -418\rangle \oplus \langle -6\rangle(1/2,1/2,0)$	*nr*
$n = 186,\ d = 1271 = 31\cdot 41,\ \eta = 3,\ h = 2$	$\langle 3813\rangle \oplus A_2(1/3,1/3,-1/3)$	*nr*
$n = 187,\ d = 1295 = 5\cdot 7\cdot 37,\ \eta = 3,\ h = 2$	$\langle 35\rangle \oplus \langle -37\rangle \oplus \langle -1\rangle$	*nr*
$n = 188,\ d = 1302 = 2\cdot 3\cdot 7\cdot 31,\ \eta = 2,\ h = 2$	$U \oplus \langle -1302\rangle$	*nr*
$n = 189,\ d = 1302 = 2\cdot 3\cdot 7\cdot 31,\ \eta = 4,\ h = 2$	$\langle 2\rangle \oplus \langle -62\rangle \oplus \langle -42\rangle(1/2,0,1/2)$	*nr*
$n = 190,\ d = 1338 = 2\cdot 3\cdot 223,\ \eta = 0,\ h = 2$	$U \oplus \langle -1338\rangle$	*nr*
$n = 191,\ d = 1365 = 3\cdot 5\cdot 7\cdot 13,\ \eta = 9,\ h = 2$	$\langle 5\rangle \oplus \langle -91\rangle \oplus \langle -3\rangle$	*hr*

$n=192,\ d=1365=3\cdot5\cdot7\cdot13,\ \eta=10,\ h=2$	$\langle 1\rangle\oplus\langle -91\rangle\oplus\langle -15\rangle$	nr
$n=193,\ d=1365=3\cdot5\cdot7\cdot13,\ \eta=12,\ h=2$	$\langle 1\rangle\oplus\langle -195\rangle\oplus\langle -7\rangle$	nr
$n=194,\ d=1365=3\cdot5\cdot7\cdot13,\ \eta=13,\ h=2$	$\langle 1\rangle\oplus\langle -273\rangle\oplus\langle -5\rangle$	nr
$n=195,\ d=1370=2\cdot5\cdot137,\ \eta=2,\ h=2$	$U\oplus\langle -1370\rangle$	nr
$n=196,\ d=1446=2\cdot3\cdot241,\ \eta=0,\ h=2$	$U\oplus\langle -1446\rangle$	nr
$n=197,\ d=1463=7\cdot11\cdot19,\ \eta=1,\ h=2$	$\langle 19\rangle\oplus\langle -77\rangle\oplus\langle -1\rangle$	nr
$n=198,\ d=1479=3\cdot17\cdot29,\ \eta=6,\ h=2$	$\langle 3\rangle\oplus\langle -493\rangle\oplus\langle -1\rangle$	nr
$n=199,\ d=1482=2\cdot3\cdot13\cdot19,\ \eta=0,\ h=2$	$U\oplus\langle -1482\rangle$	nr
$n=200,\ d=1482=2\cdot3\cdot13\cdot19,\ \eta=6,\ h=2$	$\langle 38\rangle\oplus\langle -78\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	nr
$n=201,\ d=1495=5\cdot13\cdot23,\ \eta=5,\ h=2$	$\langle 115\rangle\oplus\langle -13\rangle\oplus\langle -1\rangle$	nr
$n=202,\ d=1526=2\cdot7\cdot109,\ \eta=3,\ h=2$	$U\oplus\langle -1526\rangle$	nr
$n=203,\ d=1551=3\cdot11\cdot47,\ \eta=4,\ h=2$	$\langle 11\rangle\oplus\langle -141\rangle\oplus\langle -1\rangle$	nr
$n=204,\ d=1554=2\cdot3\cdot7\cdot37,\ \eta=2,\ h=2$		nr
$n=205,\ d=1554=2\cdot3\cdot7\cdot37,\ \eta=4,\ h=2$	$U\oplus\langle -1554\rangle$	nr
$n=206,\ d=1586=2\cdot13\cdot61,\ \eta=0,\ h=2$		nr
$n=207,\ d=1586=2\cdot13\cdot61,\ \eta=3,\ h=2$	$U\oplus\langle -1586\rangle$	nr
$n=208,\ d=1605=3\cdot5\cdot107,\ \eta=6,\ h=2$	$\langle 1\rangle\oplus\langle -107\rangle\oplus\langle -15\rangle$	nr
$n=209,\ d=1626=2\cdot3\cdot271,\ \eta=0,\ h=2$	$U\oplus\langle -1626\rangle$	nr
$n=210,\ d=1635=3\cdot5\cdot109,\ \eta=5,\ h=2$	$\langle 2\rangle\oplus\langle -3270\rangle\oplus\langle -1\rangle(1/2,1/2,0)$	nr
$n=211,\ d=1677=3\cdot13\cdot43,\ \eta=6,\ h=2$	$\langle 2\rangle\oplus\langle -3354\rangle\oplus\langle -1\rangle(1/2,1/2,0)$	nr
$n=212,\ d=1751=17\cdot103,\ \eta=3,\ h=2$	$\langle 5253\rangle\oplus A_2(1/3,1/3,-1/3)$	nr
$n=213,\ d=1771=7\cdot11\cdot23,\ \eta=7,\ h=2$	$\langle 2\rangle\oplus\langle -3542\rangle\oplus\langle -1\rangle(1/2,1/2,0)$	nr
$n=214,\ d=1785=3\cdot5\cdot7\cdot17,\ \eta=10,\ h=2$	$\langle 1\rangle\oplus\langle -119\rangle\oplus\langle -15\rangle$	nr
$n=215,\ d=1794=2\cdot3\cdot13\cdot23,\ \eta=4,\ h=2$		nr
$n=216,\ d=1806=2\cdot3\cdot7\cdot43,\ \eta=6,\ h=2$	$\langle 86\rangle\oplus\langle -14\rangle\oplus\langle -6\rangle(1/2,1/2,0)$	nr
$n=217,\ d=1898=2\cdot13\cdot73,\ \eta=2,\ h=2$	$U\oplus\langle -1898\rangle$	nr
$n=218,\ d=1986=2\cdot3\cdot331,\ \eta=2,\ h=2$	$U\oplus\langle -1986\rangle$	nr
$n=219,\ d=1995=3\cdot5\cdot7\cdot19,\ \eta=7,\ h=2$	$\langle 35\rangle\oplus\langle -57\rangle\oplus\langle -1\rangle$	hr
$n=220,\ d=1995=3\cdot5\cdot7\cdot19,\ \eta=14,\ h=2$	$\langle 3\rangle\oplus\langle -665\rangle\oplus\langle -1\rangle$	nr
$n=221,\ d=2015=5\cdot13\cdot31,\ \eta=6,\ h=2$	$\langle 403\rangle\oplus\langle -5\rangle\oplus\langle -1\rangle$	nr
$n=222,\ d=2046=2\cdot3\cdot11\cdot31,\ \eta=0,\ h=2$	$\langle 62\rangle\oplus\langle -22\rangle\oplus\langle -6\rangle(1/2,1/2,0)$	nr
$n=223,\ d=2090=2\cdot5\cdot11\cdot19,\ \eta=0,\ h=2$	$\langle 6270\rangle\oplus A_2(1/3,1/3,-1/3)$	nr
$n=224,\ d=2145=3\cdot5\cdot11\cdot13,\eta=14,h=2$	$\langle 1\rangle\oplus\langle -65\rangle\oplus\langle -33\rangle$	nr
$n=225,\ d=2226=2\cdot3\cdot7\cdot53,\ \eta=2,\ h=2$	$\langle 318\rangle\oplus\langle -14\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	nr
$n=226,\ d=2415=3\cdot5\cdot7\cdot23,\ \eta=1,\ h=2$	$\langle 7\rangle\oplus\langle -345\rangle\oplus\langle -1\rangle$	nr
$n=227,\ d=2415=3\cdot5\cdot7\cdot23,\ \eta=4,\ h=2$	$\langle 3\rangle\oplus\langle -161\rangle\oplus\langle -5\rangle$	nr
$n=228,\ d=2415=3\cdot5\cdot7\cdot23,\ \eta=8,\ h=2$	$\langle 3\rangle\oplus\langle -805\rangle\oplus\langle -1\rangle$	nr
$n=229,\ d=2454=2\cdot3\cdot409,\ \eta=0,\ h=2$	$U\oplus\langle -2454\rangle$	nr
$n=230,\ d=2562=2\cdot3\cdot7\cdot61,\ \eta=2,\ h=2$	$U\oplus\langle -2562\rangle$	nr
$n=231,\ d=2570=2\cdot5\cdot257,\ \eta=2,\ h=2$	$U\oplus\langle -2570\rangle$	nr
$n=232,\ d=2639=7\cdot13\cdot29,\ \eta=5,\ h=2$	$\langle 13\rangle\oplus\langle -406\rangle\oplus\langle -2\rangle(0,1/2,1/2)$	nr
$n=233,\ d=2730=2\cdot3\cdot5\cdot7\cdot13,\eta=0,h=2$	$\langle 14\rangle\oplus\langle -130\rangle\oplus\langle -6\rangle(1/2,0,1/2)$	nr
$n=234,\ d=2730=2\cdot3\cdot5\cdot7\cdot13,\eta=10,h=2$	$\langle 14\rangle\oplus\langle -78\rangle\oplus\langle -10\rangle(0,1/2,1/2)$	nr
$n=235,\ d=2805=3\cdot5\cdot11\cdot17,\ \eta=3,\ h=2$	$\langle 1\rangle\oplus\langle -935\rangle\oplus\langle -3\rangle$	nr
$n=236,\ d=2829=3\cdot23\cdot41,\ \eta=6,\ h=2$	$\langle 1\rangle\oplus\langle -123\rangle\oplus\langle -23\rangle$	nr
$n=237,\ d=2886=2\cdot3\cdot13\cdot37,\ \eta=0,\ h=2$	$U\oplus\langle -2886\rangle$	nr
$n=238,\ d=3003=3\cdot7\cdot11\cdot13,\eta=11,h=2$	$\langle 11\rangle\oplus\langle -273\rangle\oplus\langle -1\rangle$	nr
$n=239,\ d=3045=3\cdot5\cdot7\cdot29,\ \eta=10,\ h=2$	$\langle 1\rangle\oplus\langle -87\rangle\oplus\langle -35\rangle$	nr
$n=240,\ d=3066=2\cdot3\cdot7\cdot73,\ \eta=6,\ h=2$	$U\oplus\langle -3066\rangle$	nr

$n = 241$, $d = 3135 = 3 \cdot 5 \cdot 11 \cdot 19$, $\eta = 2$, $h = 2$ $\langle 11\rangle \oplus \langle -285\rangle \oplus \langle -1\rangle$ nr
$n = 242$, $d = 3135 = 3 \cdot 5 \cdot 11 \cdot 19$, $\eta = 8$, $h = 2$ $\langle 3\rangle \oplus \langle -1045\rangle \oplus \langle -1\rangle$ nr
$n = 243$, $d = 3315 = 3 \cdot 5 \cdot 13 \cdot 17$, $\eta = 0$, $h = 2$ $\langle 3\rangle \oplus \langle -221\rangle \oplus \langle -5\rangle$ nr
$n = 244$, $d = 3318 = 2 \cdot 3 \cdot 7 \cdot 79$, $\eta = 2$, $h = 2$ $\langle 2\rangle \oplus \langle -474\rangle \oplus \langle -14\rangle(0, 1/2, 1/2)$ nr
$n = 245$, $d = 3354 = 2 \cdot 3 \cdot 13 \cdot 43$, $\eta = 0$, $h = 2$ nr
$n = 246$, $d = 3354 = 2 \cdot 3 \cdot 13 \cdot 43$, $\eta = 6$, $h = 2$ $U \oplus \langle -3354\rangle$ nr
$n = 247$, $d = 3410 = 2 \cdot 5 \cdot 11 \cdot 31$, $\eta = 4$, $h = 2$ nr
$n = 248$, $d = 4026 = 2 \cdot 3 \cdot 11 \cdot 61$, $\eta = 0$, $h = 2$ $\langle 1342\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$ nr
$n = 249$, $d = 4074 = 2 \cdot 3 \cdot 7 \cdot 97$, $\eta = 6$, $h = 2$ $U \oplus \langle -4074\rangle$ nr
$n = 250$, $d = 4290 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 13, \eta = 1, h = 2$ $\langle 286\rangle \oplus \langle -30\rangle \oplus \langle -2\rangle(0, 1/2, 1/2)$ nr
$n = 251$, $d = 4326 = 2 \cdot 3 \cdot 7 \cdot 103$, $\eta = 2$, $h = 2$ $U \oplus \langle -4326\rangle$ nr
$n = 252$, $d = 4902 = 2 \cdot 3 \cdot 19 \cdot 43$, $\eta = 4$, $h = 2$ $\langle 2\rangle \oplus \langle -1634\rangle \oplus \langle -6\rangle(0, 1/2, 1/2)$ nr
$n = 253$, $d = 4991 = 7 \cdot 23 \cdot 31$, $\eta = 7$, $h = 2$ $\langle 4991\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$ nr
$n = 254$, $d = 5226 = 2 \cdot 3 \cdot 13 \cdot 67$, $\eta = 0$, $h = 2$ $U \oplus \langle -5226\rangle$ nr
$n = 255$, $d = 5334 = 2 \cdot 3 \cdot 7 \cdot 127$, $\eta = 2$, $h = 2$ $\langle 2\rangle \oplus \langle -762\rangle \oplus \langle -14\rangle(0, 1/2, 1/2)$ nr
$n = 256$, $d = 6006 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13, \eta = 2, h = 2$ $\langle 286\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$ nr
$n = 257$, $d = 7590 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 23, \eta = 8, h = 2$ $\langle 230\rangle \oplus \langle -22\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$ nr
$n = 258$, $d = 10374 = 2 \cdot 3 \cdot 7 \cdot 13 \cdot 19, \eta = 2, h = 2$ $\langle 494\rangle \oplus \langle -14\rangle \oplus \langle -6\rangle(1/2, 1/2, 0)$ nr
$n = 259$, $d = 29526 = 2 \cdot 3 \cdot 7 \cdot 19 \cdot 37$, $\eta = 2$, $h = 2$ $U \oplus \langle -29526\rangle$ nr

8. Narrow parts of convex polygons of restricted hyperbolic type on the hyperbolic plane. Application to hyperbolically reflective hyperbolic lattices

Here we consider narrow parts of convex polygons of restricted hyperbolic type on the hyperbolic plane and apply them to hyperbolically reflective hyperbolic lattices of rank three. Like in [N4], [N5], [N9], these results can be generalized on convex polyhedra of restricted hyperbolic type in hyperbolic spaces of arbitrary dimension and on hyperbolically reflective hyperbolic lattices of rank ≥ 3.

8.1. Some formulae using cross-ratio.

We consider a hyperbolic plane and cross-ratio for points at infinity. The infinity is a non-singular rational curve of degree 2. We normalize the curvature $\kappa = -1$.

Lemma 8.1.1. *Let (AB) and (CD) are two lines in a hyperbolic plane with terminals A, B, C, D at infinity, and O a point of the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles OAB and OCD coincide. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$.*

Then

$$(\delta_1, \delta_2) = 4[A, D, C, B] - 2.$$

As a corollary, we get:

1) If lines (AB) and (CD) do not intersect each other and $\mathcal{H}^+_{\delta_1} \cap \mathcal{H}^+_{\delta_2}$ is a band between lines (AB) and (CD), then

$$2\,ch\ \rho = (\delta_1, \delta_2) = 4[A, D, C, B] - 2 \tag{8.1.1}$$

where ρ is the distance between lines (AB) and (CD).

2) If $\mathcal{H}^+_{\delta_1} \cap \mathcal{H}^+_{\delta_2}$ is an angle α, then

$$2\cos\alpha = (\delta_1, \delta_2) = 4[A, D, C, B] - 2. \tag{8.1.2}$$

Proof. One can prove this by the direct calculation (also see the proof of Lemma 4.1.2).

We have $[A,D,C,B]=[A,D,B,C]^{-1}=(1-[A,B,D,C])^{-1}=$ $(1-[A,B,C,D]^{-1})^{-1}=[A,B,C,D]/([A,B,C,D]-1)$. Thus, we have

$$[A,B,C,D]=\frac{\text{ch }\rho+1}{\text{ch }\rho-1}>1 \tag{8.1.3}$$

for the case 1) of Lemma 8.1.1, and

$$[A,B,C,D]=\frac{\cos\alpha+1}{\cos\alpha-1}<0 \tag{8.1.4}$$

for the case 2) of Lemma 8.1.1.

Further we fix a line $l=(MN)$ on the hyperbolic plane with terminals M and N at infinity. This line will be called *axis*. We shall consider pairs of points A and B at infinity. We assume that they are different from M and N (i. e. they do not belong to l) but one can also consider cases when one of them is equal to M or N taking an appropriate limit. The axis l divides the infinity in two arcs containing in two different half-planes bounded by l.

First suppose that A and B belong to one half-plane bounded by l. Additionally suppose that M, N, A and B are four consecutive points at infinity (here we consider the infinity as a circle). Considering a point O between the lines l and (AB) and using Lemma 8.1.1 and (8.1.3), we get

$$[M,N,A,B]=\frac{\text{ch }\rho(l,(AB))+1}{\text{ch }\rho(l,(AB))-1}\ge 1. \tag{8.1.5}$$

It follows

$$0<[M,N,B,A]=[M,N,A,B]^{-1}=\frac{\text{ch }\rho(l,(AB))-1}{\text{ch }\rho(l,(AB))+1}\le 1. \tag{8.1.6}$$

By definition of the cross-ratio,

$$[M,N,A,B]\cdot[M,N,B,C]=[M,N,A,C]. \tag{8.1.7}$$

Further, for any two points A, B at infinity which are contained in one half-plane bounded by the axis l, we introduce the *"angle"*

$$\theta(A,B)=\ln[M,N,A,B]\in\mathbb{R}. \tag{8.1.8}$$

The "angle" $\theta(A,B)$ does behave like an angle: by (8.1.6) and (8.1.7),

$$\theta(B,A)=-\theta(A,B),\quad \theta(A,C)=\theta(A,B)+\theta(B,C) \tag{8.1.9}$$

where we suppose that all points A, B and C belong to one half-plane bounded by the axis l.

The axis l divides the infinity on two arcs and defines the orthogonal projection $X \mapsto X'$ with respect to l of one arc onto another: by definition, the lines l and (XX') are perpendicular. By (8.1.4) and (8.1.7), we have

$$[M, N, X, X'] = -1, \text{ and } [M, N, A, B'] = -[M, N, A, B]. \tag{8.1.10}$$

For points A and B at infinity which belong to different half-planes bounded by the axis $l = (MN)$, we define the "angle" as follows:

$$\theta(A,\, B) := \theta(A,\, B') \tag{8.1.11}$$

where $\theta(A,\, B')$ was defined in (8.1.8). Remark that $\theta(A,\, B') = \theta(A',\, B)$. Thus, we defined the angle $\theta(A,\, B)$ for all points A and B at infinity. For all of them one has (8.1.9).

For points A and B at infinity we define the invariant

$$\epsilon(A, B) = \pm 1 \tag{8.1.12}$$

where $\epsilon(A, B) = 1$ if A and B belong to one half-plane bounded by l, and $\epsilon(A, B) = -1$ otherwise. Obviously,

$$\epsilon(A, B) = \epsilon(B, A) \text{ and } \epsilon(A, B)\epsilon(B, C) = \epsilon(A, C). \tag{8.1.13}$$

From definition above, it follows

$$[M, N, A, B] = \epsilon(A, B) \exp \theta(A, B) \tag{8.1.14}$$

for any two points A and B at infinity.

Now assume that A, B, C and D are points at infinity. We choose an affine coordinate at infinity such that $M = \infty$, $N = 0$, $A = 1$. Then $B = u$, $C = uv$, $D = uvw$ where $[M, N, A, B] = u = \epsilon(A, B) \exp \theta(A, B)$, $[M, N, B, C] = v = \epsilon(B, C) \exp \theta(B, C)$, $[M, N, C, D] = w = \epsilon(C, D) \exp \theta(C, D)$. It follows

$$[A, D, C, B] = \frac{uv-1}{uv-uvw} : \frac{u-1}{u-uvw} = \frac{(uv-1)(vw-1)}{(u-1)(w-1)v} =$$

$$\left[\frac{\exp\left(\frac{\theta(A,B)+\theta(B,C)}{2}\right) - \epsilon(A,C)\exp\left(-\frac{\theta(A,B)+\theta(B,C)}{2}\right)}{2} \times \right.$$

$$\left. \frac{\exp\left(\frac{\theta(B,C)+\theta(C,D)}{2}\right) - \epsilon(B,D)\exp\left(-\frac{\theta(B,C)+\theta(C,D)}{2}\right)}{2}\right] \Big/$$

$$\left[\frac{\exp\left(\frac{\theta(A,B)}{2}\right) - \epsilon(A,B)\exp\left(-\frac{\theta(A,B)}{2}\right)}{2} \times \frac{\exp\left(\frac{\theta(B,C)}{2}\right) - \epsilon(B,C)\exp\left(-\frac{\theta(B,C)}{2}\right)}{2}\right]. \tag{8.1.15}$$

We remark that functions used in this expression are hyperbolic sine or cosine. E. g.

$$\frac{\exp\left(\frac{\theta(A,B)}{2}\right) - \epsilon(A,B)\exp\left(-\frac{\theta(A,B)}{2}\right)}{2} = \begin{cases} \operatorname{sh} \frac{\theta(A,B)}{2} & \text{if } \epsilon(A,B) = 1 \\ \operatorname{ch} \frac{\theta(A,B)}{2} & \text{if } \epsilon(A,B) = -1 \end{cases}. \tag{8.1.16}$$

Using Lemma 8.1.1 and (8.1.15), we get the following analog of Lemmas 4.1.2 and 4.2.4 (for Lemmas 4.1.2 and 4.2.4, a finite or infinite point was used instead of the axis l).

Lemma 8.1.2. *Let $l = (MN)$ be a line (called axis) of a hyperbolic plane with terminals M and N at infinity. Let (AB) and (CD) are two lines of the hyperbolic plane with terminals A, B, C, D at infinity, and there exists a point O of the hyperbolic plane which does not belong to each line (AB) and (CD) and orientations of the triangles OAB and OCD coincide. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that O is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$.*

Then

$$(\delta_1,\,\delta_2) = 4[A, D, C, B] - 2$$

where

$$[A, D, C, B] =$$

$$\left[\frac{\exp\left(\frac{\theta(A,B)+\theta(B,C)}{2}\right) - \epsilon(A,C)\exp\left(-\frac{\theta(A,B)+\theta(B,C)}{2}\right)}{2}\times\right.$$

$$\left.\frac{\exp\left(\frac{\theta(B,C)+\theta(C,D)}{2}\right) - \epsilon(B,D)\exp\left(-\frac{\theta(B,C)+\theta(C,D)}{2}\right)}{2}\right]\Big/$$

$$\left[\frac{\exp\left(\frac{\theta(A,B)}{2}\right) - \epsilon(A,B)\exp\left(-\frac{\theta(A,B)}{2}\right)}{2}\times\frac{\exp\left(\frac{\theta(B,C)}{2}\right) - \epsilon(B,C)\exp\left(-\frac{\theta(B,C)}{2}\right)}{2}\right].\tag{8.1.17}$$

Here we use invariants θ and ϵ introduced in (8.1.8), (8.1.11) *and* (8.1.12) *(also see* (8.1.5)*).*

Below we give two the most important for us particular cases of Lemma 8.1.2.

Lemma 8.1.3. *Let $l = (MN)$ be a line (called axis) of a hyperbolic plane with terminals M and N at infinity. Let (AB) and (CD) are two lines of the plane which belong to one half-plane bounded by the axis l and A, B, C, D are their terminals at infinity. We suppose that M, N, A, B are four consecutive points at infininity and M, N, C, D are four consecutive points at infinity either. We consider angles $\theta_1 = \theta(A,B)$, $\theta_2 = \theta(C,D)$ and $\theta_{12} = \theta(B,C)$ defined by* (8.1.5) *and* (8.1.8)*. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that l is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$. Then*

$$(\delta_1,\,\delta_2) = 4\frac{sh\,\frac{\theta_1+\theta_{12}}{2}\,sh\,\frac{\theta_2+\theta_{12}}{2}}{sh\,\frac{\theta_1}{2}\,sh\,\frac{\theta_2}{2}} - 2.\tag{8.1.18}$$

As a corollary we get:

If lines (AB) and (CD) do not intersect and $\mathcal{H}^+_{\delta_1}\cap\mathcal{H}^+_{\delta_2}$ is the band bounded by these lines, we have

$$(\delta_1,\,\delta_2) \;=\; 2ch\,\rho \;=\; 4\frac{sh\,\frac{\theta_1+\theta_{12}}{2}\,sh\,\frac{\theta_2+\theta_{12}}{2}}{sh\,\frac{\theta_1}{2}\,sh\,\frac{\theta_2}{2}} - 2\tag{8.1.19}$$

where ρ is the distance between lines (AB) and (CD).

If lines (AB) and (CD) define an angle $\alpha=\mathcal{H}^+_{\delta_1}\cap\mathcal{H}^+_{\delta_2}$, we have

$$(\delta_1,\,\delta_2)\ =\ 2\cos\alpha=\ 4\frac{sh\ \frac{\theta_1-\theta_{21}}{2}\,sh\ \frac{\theta_2-\theta_{21}}{2}}{sh\ \frac{\theta_1}{2}\,sh\ \frac{\theta_2}{2}}-2 \tag{8.1.20}$$

where $\theta_{21}=-\theta_{12}=\theta(C,B)$.

Lemma 8.1.4. *Let $l=(MN)$ be a line (called axis) on a hyperbolic plane with terminals M and N at infinity. Let (AB) and (CD) are two lines of the plane which belong to different half-planes bounded by the axis l, and A, B, C, D are their terminals at infinity. We suppose that M, N, A, B are four consecutive points at infinity and M, N, D, C are four consecutive points at infinity either. We consider angles $\theta_1=\theta(A,B)$, $\theta_2=\theta(D,C)$ defined by* (8.1.5) *and* (8.1.8)*, and $\theta_{12}=\theta(B,C)$ defined by* (8.1.11)*. Let δ_1 and δ_2 are orthogonal vectors with square -2 to lines (AB) and (CD) respectively such that l is contained in both half-planes $\mathcal{H}^+_{\delta_1}$ and $\mathcal{H}^+_{\delta_2}$. Then*

$$(\delta_1,\,\delta_2)=2ch\ \rho=4\frac{ch\ \frac{\theta_1+\theta_{12}}{2}\,ch\ \frac{\theta_2-\theta_{12}}{2}}{sh\ \frac{\theta_1}{2}\,sh\ \frac{\theta_2}{2}}-2. \tag{8.1.21}$$

Here ρ is the distance between lines (AB) and (CD).

8.2. Narrow parts of convex polygons of restricted hyperbolic type in a hyperbolic plane: general results.

We use notation of Sect. 1 for convex locally finite polyhedra in hyperbolic spaces. Remind that $P(\mathcal{M})$ denote the set of orthogonal vectors to faces of $\mathcal{M}$ of highest dimension and directed outwards of $\mathcal{M}$. We normalize $\delta\in P(\mathcal{M})$ by the condition $\delta^2=-2$.

We use definition from [N9] of hyperbolic polyhedra (or polyhedra of hyperbolic type) in hyperbolic spaces. Let $\mathcal{T}\subset\mathcal{L}$ be a hyperbolic subspace of a hyperbolic space $\mathcal{L}$ such that $1\le\dim\mathcal{T}\le\dim\mathcal{L}-1$. For $K\subset\mathcal{T}$, we denote by C_K the orthogonal cylinder with the base K. Thus, C_K is the union of all lines orthogonal to $\mathcal{T}$ and containing a point of K.

Definition 8.2.1. A convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is called *hyperbolic, equivalently, it has hyperbolic type (relative to a subspace $\mathcal{T}\subset\mathcal{L}$, $1\le\dim\mathcal{T}\le\dim\mathcal{L}-1$, which is called axis),* if it is non-degenerate (i. e. it is not contained in a proper subspace of $\mathcal{L}$), locally finite, and the conditions (1) and (2) below hold:

(1) $\mathcal{M}$ is finite at infinite points, i. e. the set $\{\delta\in P(\mathcal{M})\ \mid(\delta,\,c)=0\}$ is finite for any infinite point $\mathbb{R}_{++}c\in\mathcal{L}_\infty$, $c^2=0$.

(2) For any compact elliptic polyhedron $\mathcal{N}\subset\mathcal{T}$, the polyhedron $C_{\mathcal{N}}\cap\mathcal{M}$ is elliptic if it is non-degenerate. (We remind that a polyhedron is called elliptic if it is a convex envelope of a finite set of points and is non-degenerate.)

A hyperbolic polyhedron is called *restricted* if additionally

(3) The set of angles and distances with the axis $\mathcal{T}$ defined by all half-spaces $\mathcal{H}^+_\delta$, $\delta\in P(\mathcal{M})$, is finite.

Further we shall consider only restricted hyperbolic convex polygons $\mathcal{M}$ in a hyperbolic plane. Then the axis is a line l.

If a polygon $\mathcal{M}$ is elliptic then $\mathcal{M}$ is restricted hyperbolic with respect to any axis l. If $\mathcal{M}$ is hyperbolic and is finite (i. e. the set $P(\mathcal{M})$ is finite) then $\mathcal{M}$ is elliptic. Narrow parts of elliptic and parabolic polygons have been considered in Sect. 4. Therefore, further we can suppose that $\mathcal{M}$ is infinite (i. e. it has infinite set of sides, equivalently the set $P(\mathcal{M})$ is infinite).

The axis l divides $\mathcal{M}$ in two half-s. We denote by A_i vertices of one half of $\mathcal{M}$, and by B_j vertices of another half of $\mathcal{M}$. We prove the following basic result about narrow parts of restricted hyperbolic convex polygons in a hyperbolic plane.

Theorem 8.2.2 (about narrow parts of types (II) and (III) of restricted hyperbolic convex polygons). *We fix a constant $\eta > 0$.*

Any infinite restricted hyperbolic convex polygon $\mathcal{M}$ on the hyperbolic plane has a narrow part of one of types (AII), (AIII), (BII$_1$), (BII$_2$), (BIII) defined below:

(AII) There exist five consecutive vertices A_0, A_1, A_2, A_3 and A_4 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1,\, 2,\, 3$, where $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$, and

$$2 < (\delta_1,\, \delta_3) \le 8\,ch\,\frac{\eta}{2} + 6, \tag{8.2.1}$$

$$2 < (\delta_1,\, \delta_4) \le 4(2\,ch\,\frac{\eta}{2} + 1)^2 - 2. \tag{8.2.2}$$

Moreover, the set $\{\delta_1,\, \delta_3,\, \delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i = 1,\, 2,\, 3,\, 4$, contain the axis l.

(AIII) There exist six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ one has $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1,\, 2,\, 3,\, 4$, where $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$, and

$$2 < (\delta_1, \delta_3) \le 8\,ch\,\frac{\eta}{2} + 6, \tag{8.2.3}$$

$$2 < (\delta_3, \delta_5) \le 8\,ch\,\frac{\eta}{2} + 6, \tag{8.2.4}$$

and

$$2 < (\delta_1, \delta_5) \le 32(ch^3\frac{\eta}{2} + ch^2\frac{\eta}{2}) - 2. \tag{8.2.5}$$

Moreover, the set $\{\delta_1,\, \delta_3,\, \delta_5\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i = 1,\, 2,\, 3,\, 4,\, 5$, contain the axis l.

(BII$_1$) There exist four consecutive vertices A_0, A_1, A_2, A_3 of one half of $\mathcal{M}$ and two consecutive vertices B_0, B_1 of another half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (B_0B_1) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, one has $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, $i = 1,\, 2$, where $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$, and

$$2 < (\delta_1,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2} - 1} + 6, \tag{8.2.6}$$

$$2 < (\delta_2,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2} - 1} + 6, \tag{8.2.7}$$

$$2 < (\delta_3,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2}-1} + 6. \tag{8.2.8}$$

Moreover, the set $\{\delta_1,\, \delta_2,\, \delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\, \delta_3,\, \delta_4\}$*. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i = 1, 2, 3$*, and closed half-plane* $\mathcal{H}^+_{\delta_4}$ *contain the axis* l*; both lines orthogonal to the axis* l *and containing* A_1 *and* A_2 *intersect the line* (B_0B_1)*.*

(BII$_2$*) There exist three consecutive vertices* A_0*,* A_1*,* A_2 *of one half of* $\mathcal{M}$ *and three consecutive vertices* B_0*,* B_1*,* B_2 *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1*,* δ_2*,* δ_3 *and* δ_4 *to lines* (A_0A_1)*,* (A_1A_2)*,* (B_0B_1) *and* (B_1B_2) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$*, one has* $(\delta_1,\, \delta_2) = 2\cos\alpha$ *where* $\alpha = \angle\, A_0A_1A_2$*,* $(\delta_3,\, \delta_4) = 2\cos\beta$ *where* $\beta = \angle\, B_0B_1B_2$*, and*

$$2 < (\delta_1,\, \delta_3) \le \frac{8}{ch\,\frac{\eta}{2}-1} + 6, \tag{8.2.9}$$

$$2 < (\delta_2,\, \delta_3) \le \frac{8}{ch\,\frac{\eta}{2}-1} + 6, \tag{8.2.10}$$

$$2 < (\delta_2,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2}-1} + 6. \tag{8.2.11}$$

Moreover, the set $\{\delta_1,\, \delta_2,\, \delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\, \delta_3,\, \delta_4\}$*. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i = 1, 2, 3, 4$*, contain the axis* l*; the line orthogonal to the axis* l *and containing* A_1 *intersects the line* (B_0B_1)*, and the line orthogonal to* l *and containing* B_1 *intersects the line* (A_1A_2)*.*

(BIII) There exist four consecutive vertices A_0*,* A_1*,* A_2*,* A_3 *of one half of* $\mathcal{M}$*, and two consecutive vertices* B_0*,* B_1 *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1*,* δ_2*,* δ_3 *and* δ_4 *to lines* (A_0A_1)*,* (A_1A_2)*,* (A_2A_3) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$*, one has* $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$ *where* $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$*,* $i = 1, 2$*, and*

$$2 < (\delta_1,\, \delta_3) \le 8ch\,\frac{\eta}{2} + 6, \tag{8.2.12}$$

$$2 < (\delta_1,\, \delta_4) \le 8ch\,\frac{\eta}{2} + 10 + \frac{8}{ch\,\frac{\eta}{2}-1}, \tag{8.2.13}$$

$$2 < (\delta_3,\, \delta_4) \le 8ch\,\frac{\eta}{2} + 10 + \frac{8}{ch\,\frac{\eta}{2}-1}. \tag{8.2.14}$$

Moreover, the set $\{\delta_1,\, \delta_2,\, \delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for the set* $\{\delta_2,\, \delta_3,\, \delta_4\}$*. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i = 1, 2, 3, 4$*, contain the axis* l*; there exists a line orthogonal to the axis* l *and intersecting the lines* (A_1A_2) *and* (B_0B_1)*.*

Proof. Compare with [N4, Appendix], [N13] and the proof of Theorem 4.1.5.

Further we shall normalize elements $\delta \in P(\mathcal{M})$ by the condition $\delta^2 = -2$.

Let us denote by $E \subset P(\mathcal{M})$ the set of all elements $e \in P(\mathcal{M})$ such that the line $\mathcal{H}_e$ intersects the axis l (in finite or infinite points). By condition (3), the set E is finite. Otherwise there exist two different elements $e_1,\, e_2 \in E$ such that angles of the half-planes $\mathcal{H}_{e_1}^+$ and $\mathcal{H}_{e_2}^+$ with l are equal. It then follows that one of two half-planes $\mathcal{H}_{e_1}^+$, $\mathcal{H}_{e_2}^+$ contains another. This contradicts the definition of $P(\mathcal{M})$. Let

$$\mathcal{M}_0 = \bigcap_{e\in E} \mathcal{H}_e^+. \tag{8.2.15}$$

The $\mathcal{M}_0$ is a finite convex polyhedron containing $\mathcal{M}$. If $l \cap \mathcal{M}_0$ is a finite or empty interval of l, then orthogonal projection (here and in what follows it is the orthogonal projection relative to the axis l) of $\mathcal{M}_0$ into l is compact. By (2), then $\mathcal{M}$ is finite. We get a contradiction. Thus $l \cap \mathcal{M}_0$ is either the axis $l = (NM)$ or a half-axis $(KM) \subset l$ where M and N are terminals of l at infinity and K is a finite point of l.

Suppose that $\delta \in P(\mathcal{M}) - E$. Then either the open half-plane $\mathcal{H}_\delta^+$ contains the axis l or the closed half-plane $\mathcal{H}_\delta^+$ does not intersect the axis l. For the last case we also obtain that $\mathcal{M}$ is finite (by (2)). It then follows that for any $\delta \in P(\mathcal{M}) - E$ the open half-plane $\mathcal{H}_\delta^+$ contains the axis l. Remark that then the line $\mathcal{H}_\delta$ has a compact orthogonal projection into l and defines by (2) a side (C_iC_{i+1}) of $\mathcal{M}$ where C_i and C_{i+1} are vertices of $\mathcal{M}$.

By the considerations above, there exists an infinite chain A_i, $i \to +\infty$ of consecutive vertices of $\mathcal{M}$ with the following properties. All lines $(A_{i-1}A_i)$ do not intersect the axis l and the open half-plane bounded by $(A_{i-1}A_i)$ contains l. Thus all A_i belong to one half of $\mathcal{M}$. The vertices A_i, $i \to +\infty$ tend to the terminal M of l at infinity.

Around the terminal M of l, another half of $\mathcal{M}$ is either another similar infinite chain B_j and $B_j \to M$, $j \to +\infty$, or a side (B_0M) of $\mathcal{M}$, or the axis l. Further we shall consider only the first case of the infinite chain B_j leaving more simple last two cases to a reader (for these cases, one should just argue with one chain A_i and replace the chain B_j by (B_0M) or l respectively if necessary).

We shall denote by $e_i \in P(\mathcal{M})$, $i \in I$, the orthogonal vector to the side $(A_{i-1}A_i)$ of $\mathcal{M}$. Respectively $f_j \in P(\mathcal{M})$, $j \in J$, denote the orthogonal vector to the side $(B_{j-1}B_j)$. We denote by A_{i1} and A_{i2} terminals of the line $(A_{i-1}A_i)$ at infinity such that A_{i1}, A_{i-1}, A_i, A_{i2} are consecutive points of the line $(A_{i-1}A_i)$. Similarly, we denote by B_{j1} and B_{j2} terminals of the line $(B_{j-1}B_j)$ at infinity such that B_{j1}, B_{j-1}, B_j, B_{j2} are consecutive points of the line $(B_{j-1}B_j)$. We denote $\theta_i^1 = \theta(A_{i1}, A_{i2})$, $i \in I$, and $\theta_j^2 = \theta(B_{j1}, B_{j2})$, $j \in J$. See (8.1.8).

In considerations below, without lost of generality, we suppose that the sets $I = \mathbb{Z}$ and $J = \mathbb{Z}$, i. e. chains A_i and B_j are infinite in both directions. One can always suppose this considering parts of these chains in a sufficiently small neighbourhood of the infinite point $M \in l$ of $\mathcal{M}$.

We denote

$$\theta = \min\big(\min_{i\in I} (\theta_{i-1}^1 + \theta_i^1),\ \min_{j\in J} (\theta_{j-1}^2 + \theta_j^2) \big). \tag{8.2.16}$$

We can assume (changing numeration if necessary) that

$$\theta = \theta_2^1 + \theta_3^1 \text{ and } \theta_2^1 \le \theta_3^1. \tag{8.2.17}$$

By definition of θ, we have

$$\theta_1^1 \ge \theta_3^1 \ge \theta_2^1 \text{ and } \theta_1^1 \ge \theta/2,\ \theta_2^1 \le \theta/2,\ \theta_3^1 \ge \theta/2. \tag{8.2.18}$$

By (8.1.19),

$$(e_1,\, e_3) \le 4\frac{\mathrm{sh}\ \frac{\theta_1^1+\theta_2^1}{2}\mathrm{sh}\ \frac{\theta_3^1+\theta_2^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_3^1}{2}} - 2. \tag{8.2.19}$$

Using (8.2.17) and (8.2.18), we then get

$$\frac{\mathrm{sh}\ \frac{\theta_1^1+\theta_2^1}{2}\mathrm{sh}\ \frac{\theta_3^1+\theta_2^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_3^1}{2}} \le \frac{(\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{ch}\ \frac{\theta_2^1}{2} + \mathrm{ch}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_2^1}{2})\mathrm{sh}\ \frac{\theta}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta}{4}} \le$$
$$(\mathrm{ch}\ \frac{\theta_2^1}{2} + \mathrm{sh}\ \frac{\theta_2^1}{2} \times \frac{\mathrm{ch}\ \frac{\theta_1^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}}) \times \frac{\mathrm{sh}\ \frac{\theta}{2}}{\mathrm{sh}\ \frac{\theta}{4}} \le (\mathrm{ch}\ \frac{\theta}{4} + \mathrm{sh}\ \frac{\theta}{4} \times \frac{\mathrm{ch}\ \frac{\theta}{4}}{\mathrm{sh}\ \frac{\theta}{4}}) \times \frac{\mathrm{sh}\ \frac{\theta}{2}}{\mathrm{sh}\ \frac{\theta}{4}} =$$
$$4\mathrm{ch}^2\frac{\theta}{4} = 2\mathrm{ch}\ \frac{\theta}{2} + 2.$$

It follows

$$(e_1,\, e_3) \le 8\mathrm{ch}\ \frac{\theta}{2} + 6. \tag{8.2.20}$$

We also have the inequality $2 < (e_1,\, e_3)$ because otherwise the lines (A_0A_1) and (A_2A_3) intersect each other and together with the line (A_1A_3) define a triangle containing the axis l inside, which is impossible. Similarly, in considerations below, one can prove inequalities of the type $2 < (.\,,\,.)$ of the theorem, and further we omit these considerations.

Further we consider two cases:

Case II: Assume that

$$\theta_4^1 \ge \theta/2. \tag{8.2.21}$$

Then, by (8.1.19),

$$(e_1,\, e_4) \le 4\frac{\mathrm{sh}\ \frac{\theta_1^1+\theta}{2}\mathrm{sh}\ \frac{\theta+\theta_4^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_4^1}{2}} - 2. \tag{8.2.22}$$

Using (8.2.17) and (8.2.21), we then get like above

$$\frac{\mathrm{sh}\ \frac{\theta_1^1+\theta}{2}\mathrm{sh}\ \frac{\theta+\theta_4^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_4^1}{2}} \le \frac{\mathrm{sh}\ \frac{3\theta}{4}\mathrm{sh}\ \frac{3\theta}{4}}{\mathrm{sh}\ \frac{\theta}{4}\mathrm{sh}\ \frac{\theta}{4}} = (2\mathrm{ch}\ \frac{\theta}{2} + 1)^2.$$

It follows

$$(e_1,\, e_4) \le 4(2\mathrm{ch}\ \frac{\theta}{2} + 1)^2 - 2. \tag{8.2.23}$$

Like above, one can prove that $2 < (e_1,\, e_4)$. Elements e_1, e_3 and e_4 generate the hyperbolic vector space defining the hyperbolic plane. Otherwise, the lines (A_0A_1), (A_2A_3) and (A_3A_4) either have a common point or are orthogonal to a line which is impossible. The Gram graph of these three elements is connected because $2 < (e_1,\, e_3)$ and $2 < (e_1,\, e_4)$. Similarly one can prove all statements about connectedeness of Gram graphs of the theorem, and further we omit these considerations.

Case (III): Assume that

$$\theta_4^1 \le \theta/2. \tag{8.2.24}$$

By definition of θ, then $\theta_5^1 \geq \theta/2$. Since $\theta_3^1 \geq \theta/2$, $\theta_4^1 \leq \theta/2$ and $\theta_5^1 \geq \theta/2$, like for (8.2.20), we get

$$(e_3,\, e_5) \leq 4\frac{\mathrm{sh}\ \frac{\theta_3^1+\theta_4^1}{2}\mathrm{sh}\ \frac{\theta_5^1+\theta_4^1}{2}}{\mathrm{sh}\ \frac{\theta_3^1}{2}\mathrm{sh}\ \frac{\theta_5^1}{2}} - 2 \leq 8\mathrm{ch}\ \frac{\theta}{2} + 6. \tag{8.2.25}$$

We have $\theta_2^1 + \theta_3^1 + \theta_4^1 = \theta + \theta_4^1 \leq 3\theta/2$. It then follows like above, that

$$(e_1,\, e_5) \leq 4\frac{\mathrm{sh}\ \frac{\theta_1^1+3\theta/2}{2}\mathrm{sh}\ \frac{\theta_5^1+3\theta/2}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_5^1}{2}} - 2. \tag{8.2.26}$$

Here

$$\frac{\mathrm{sh}\ \frac{\theta_1^1+3\theta/2}{2}\mathrm{sh}\ \frac{\theta_5^1+3\theta/2}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_5^1}{2}} \leq \frac{\mathrm{sh}^2\theta}{\mathrm{sh}^2\frac{\theta}{4}} = 8(\mathrm{ch}^3\frac{\theta}{2} + \mathrm{ch}^2\frac{\theta}{2}).$$

Thus,

$$(e_1,\, e_5) \leq 32(\mathrm{ch}^3\frac{\theta}{2} + \mathrm{ch}^2\frac{\theta}{2}) - 2. \tag{8.2.27}$$

Cases (A): Suppose that

$$\theta \leq \eta. \tag{8.2.28}$$

Then for the case (II) above we get the case (AII) of the theorem, and for the case (III) above we get the case (AIII) of the theorem.

Cases (B): Now assume that

$$\theta \geq \eta. \tag{8.2.29}$$

Equivalently, then

$$\theta_{i-1}^1 + \theta_i^1 \geq \eta \ \ \forall i \in I \ \text{ and } \ \theta_{j-1}^2 + \theta_j^2 \geq \eta \ \ \forall j \in J. \tag{8.2.30}$$

We consider two cases:

Case (BII): Under the condition (8.2.29) (equivalently, (8.2.30)), we additionally suppose that

$$\theta_i^1 \geq \eta/2 \ \forall\ i \in I \ \text{ and } \ \theta_j^2 \geq \eta/2\ \forall j \in J. \tag{8.2.31}$$

We then consider orthogonal projections of all vertices A_i, $i \in I$, and B_j, $j \in J$, into the axis l. The set of these projections gives a locally finite set of points in l. We choose two neighbouring points of this set and consider their preimages. We then get (after changing numeration if necessary) one of two possibilities (BII_1) or (BII_2) which we describe below.

Case (BII_1): There exist four vertices A_0, A_1, A_2 and A_3 and two vertices B_0 and B_1 such that orthogonal projections of A_1 and A_2 into l belong to the orthogonal projection of the line (B_0B_1) into l. Equivalently, two lines containing A_1 and A_2 and orthogonal to l both intersect the line (B_0B_1). We want to show that then all conditions of the case (BII_1) of the theorem are satisfied.

Using (8.1.21), we get

$$(e_1,\, f_1) = 4\frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\ \frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}} - 2. \tag{8.2.32}$$

where $-\theta_1^1 \le \theta_{12} = \theta(A_{12},\, B_{12}) \le \theta_1^2$ because there exists a line orthogonal to the axis l and intersecting both lines (A_0A_1) and (B_0B_1). It follows

$$\frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\ \frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}} = \frac{\frac{1}{2}\left[\mathrm{ch}\ (\frac{\theta_1^1+\theta_1^2}{2}) + \mathrm{ch}\ (\frac{\theta_1^1-\theta_1^2}{2}+\theta_{12})\right]}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}} \le$$

$$\frac{\frac{1}{2}\left[\mathrm{ch}\ (\frac{\theta_1^1+\theta_1^2}{2}) + \max\left(\mathrm{ch}\ (\frac{-\theta_1^1-\theta_1^2}{2}),\ \mathrm{ch}\ (\frac{\theta_1^1+\theta_1^2}{2})\right)\right]}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}} = \frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_1^2}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}} \le$$

$$\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{sh}\ \frac{\eta}{4}\mathrm{sh}\ \frac{\eta}{4}} = \frac{2\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1} = 2 + \frac{2}{\mathrm{ch}\ \frac{\eta}{2}-1}$$

because $\theta_1^1 \ge \eta/2$ and $\theta_2^1 \ge \eta/2$. It follows

$$(e_1,\, f_1) \le \frac{8}{\mathrm{ch}\ \frac{\eta}{2}-1} + 6. \tag{8.2.33}$$

Similarly one proves that

$$(e_2,\, f_1) \le \frac{8}{\mathrm{ch}\ \frac{\eta}{2}-1} + 6 \text{ and } (e_3,\, f_1) \le \frac{8}{\mathrm{ch}\ \frac{\eta}{2}-1} + 6. \tag{8.2.34}$$

It proves that we do get the case (BII_1) of the theorem.

Case (BII_2): There exist three vertices $A_0,\ A_1,\ A_2$ and three vertices $B_0,\ B_1,\ B_2$ such that orthogonal projection of A_1 into l belongs to the orthogonal projection of the line (B_0B_1) into l, and orthogonal projection of B_1 into l belongs to the orthogonal projection of the line (A_1A_2) into l. Let us show that then all conditions of the case (BII_2) of the theorem are valid. Really, like for the case (BII_1) above, one can prove that

$$(e_1,\, f_1),\ (e_2,\, f_1) \le \frac{8}{\mathrm{ch}\ \frac{\eta}{2}-1} + 6 \tag{8.2.35}$$

because there exists a line orthogonal to l (it contains A_1) which intersects the line (B_0B_1) and intersects lines (A_0A_1) and (A_1A_2). Similarly,

$$(e_2,\, f_1),\ (e_2,\, f_2) \le \frac{8}{\mathrm{ch}\ \frac{\eta}{2}-1} + 6. \tag{8.2.36}$$

We now consider the opposite case to the case (BII)):

Case (BIII): Under the condition (8.2.29) (equivalently, (8.2.30)), we assume that there exists either $i \in I$ such that $\theta_i^1 \le \eta/2$ or $j \in J$ such that $\theta_j^2 \le \eta/2$. We then define similar to the invariant θ above, an invariant

$$\theta_\eta = \max(\max_{\{i\in I\,|\,\theta_i^1\le\eta/2\}} \theta_i^1,\ \max_{\{j\in J\,|\,\theta_j^2\le\eta/2\}} \theta_j^2). \tag{8.2.37}$$

Obviously, $\theta_\eta \le \eta/2$. For any $i \in I$, if $\theta_i^1 > \theta_\eta$ then $\theta_i^1 > \eta/2$. For any $j \in J$, if $\theta_j^2 > \theta_\eta$ then $\theta_j^2 > \eta/2$.

For the case (BIII), after changing numeration if necessary, there exist vertices A_0, A_1, A_2, A_3 such that $\theta_2^1 = \theta_\eta$. Since $\theta_\eta \le \eta/2$, one has $\theta_1^1 \ge \eta/2$ because $\theta_1^1 + \theta_2^1 \ge \theta \ge \eta$. Similarly, $\theta_3^1 \ge \eta/2$. Like in (8.2.20), we then have

$$(e_1,\, e_3) \le 8\,\mathrm{ch}\,\frac{\eta}{2} + 6. \tag{8.2.38}$$

Suppose that there exists a vertex B_j such that its orthogonal projection into l is contained in the orthogonal projection of the line (A_1A_2) into l. Then either the line $(B_{j-1}B_j)$ has the angle $\theta_j^2 \ge \theta/2 \ge \eta/2$ or the line (B_jB_{j+1}) has the angle $\theta_{j+1}^2 \ge \theta/2 \ge \eta/2$. Remark that both these lines have a line (containing B_j) which is orthogonal to l and intersects the line (A_1A_2). If the vertex B_j does not exist, then there exists a line $(B_{j-1}B_j)$ such that the orthogonal projection of the line (A_1A_2) into l is contained inside the interval defined by orthogonal projections of $[B_{j-1}B_j]$. It follows that $\theta_j^2 > \theta_2^1 = \theta_\eta$. By definition of θ_η, then $\theta_j^2 > \eta/2$. Obviously, there exists a line orthogonal to l which intersects both lines (A_1A_2) and $(B_{j-1}B_j)$.

Finally (after changing numeration of B_j if necessary) we get that there exist vertices B_0 and B_1 such that $\theta_1^2 \ge \eta/2$ and there exists a line which is orthogonal to l and intersects both lines (A_1A_2) and (B_0B_1). Let us show that vertices A_0, A_1, A_2, A_3 and B_0, B_1 satisfy conditions of the case (BIII) of the theorem.

We have proved (8.2.12). By (8.1.21),

$$(e_1,\, f_1) = 4\frac{\mathrm{ch}\,\frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\,\frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\,\frac{\theta_1^1}{2}\mathrm{sh}\,\frac{\theta_1^2}{2}} - 2 \tag{8.2.39}$$

where $-\theta_2^1 \le \theta_{12} \le \theta_2^1 + \theta_1^2$ because there exists a line which is orthogonal to l and intersects both lines (A_1A_2) and (B_0B_1). We have

$$\frac{\mathrm{ch}\,\frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\,\frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\,\frac{\theta_1^1}{2}\mathrm{sh}\,\frac{\theta_1^2}{2}} = \frac{\frac{1}{2}\left[\mathrm{ch}\,(\frac{\theta_1^1+\theta_1^2}{2}) + \mathrm{ch}\,(\frac{\theta_1^1-\theta_1^2}{2}+\theta_{12})\right]}{\mathrm{sh}\,\frac{\theta_1^1}{2}\mathrm{sh}\,\frac{\theta_1^2}{2}} \le$$

$$\frac{\frac{1}{2}\left[\mathrm{ch}\,(\frac{\theta_1^1+\theta_1^2}{2}) + \max(\mathrm{ch}\,(\frac{\theta_1^1-2\theta_2^1-\theta_1^2}{2}),\ \mathrm{ch}\,(\frac{\theta_1^1+2\theta_2^1+\theta_1^2}{2}))\right]}{\mathrm{sh}\,\frac{\theta_1^1}{2}\mathrm{sh}\,\frac{\theta_1^2}{2}} =$$

$$\frac{\frac{1}{2}\left[\mathrm{ch}\,\frac{\theta_1^1+\theta_1^2}{2} + \mathrm{ch}\,(\frac{\theta_1^1+2\theta_2^1+\theta_1^2}{2})\right]}{\mathrm{sh}\,\frac{\theta_1^1}{2}\mathrm{sh}\,\frac{\theta_1^2}{2}} \le$$

$$\frac{\frac{1}{2}\left[\mathrm{ch}\,\frac{\eta}{2} + \mathrm{ch}\,(\eta)\right]}{\mathrm{sh}^2\frac{\eta}{4}} = \frac{\mathrm{ch}\,\frac{\eta}{2} + 2\mathrm{ch}^2\frac{\eta}{2} - 1}{\mathrm{ch}\,\frac{\eta}{2} - 1} = 2\mathrm{ch}\,\frac{\eta}{2} + 3 + \frac{2}{\mathrm{ch}\,\frac{\eta}{2} - 1}$$

because $\theta_1^1 \ge \eta/2$, $\theta_1^2 \ge \eta/2$ and $\theta_2^1 \le \eta/2$. It follows,

$$(e_1,\, f_1) \le 8\mathrm{ch}\,\frac{\eta}{2} + 10 + \frac{8}{\mathrm{ch}\,\frac{\eta}{2} - 1}. \tag{8.2.40}$$

Similarly, we have

$$(e_3,\, f_1) \le 8\mathrm{ch}\,\frac{\eta}{2} + 10 + \frac{8}{\mathrm{ch}\,\frac{\eta}{2} - 1}. \tag{8.2.41}$$

This finishes the proof of Theorem 8.2.2.

Applying Theorem 8.2.2 to $\eta = \beta$ where $\beta = 0.983986...$ is the solution of the equation

$$32(\mathrm{ch}^3\frac{\eta}{2} + \mathrm{ch}^2\frac{\eta}{2}) - 2 = 8\mathrm{ch}\ \frac{\eta}{2} + 10 + \frac{8}{\mathrm{ch}\ \frac{\eta}{2} - 1}, \tag{8.2.42}$$

we get the important in applications

Theorem 8.2.3 (about narrow part of restricted hyperbolic convex polygons in a hyperbolic plane). *For any restricted hyperbolic convex polygon $\mathcal{M}$ in a hyperbolic plane there are $\delta_1,\ \delta_2,\ \delta_3 \in P(\mathcal{M})$ with $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$ such that $\delta_1,\ \delta_2,\ \delta_3$ generate 3-dimensional hyperbolic vector space (defining the hyperbolic plane), they have connected Gram graph and*

$$(\delta_i,\ \delta_j) \leq 32(ch^3\frac{\beta}{2} + ch^2\frac{\beta}{2}) - 2 < 83.7706 \tag{8.2.43}$$

for all $1 \leq i, j, \leq 3$. Here $\beta = 0.983986...$ is the solution of the equation (8.2.42).

We remind that the same statement is valid for elliptic and restricted parabolic polygons in hyperbolic plane with the better constant 62 (see Theorems 4.1.5 and 4.2.2) and even 30 (see Theorems 4.1.8 and 4.2.3).

Similar method permits to prove Theorem 8.2.3 for restricted hyperbolic polyhedra in hyperbolic spaces of arbitrary dimension $n \geq 2$ (here 3 should be replaced by $n+1$).

8.3. Narrow parts of restricted hyperbolic convex polygons in a hyperbolic plane: refined results.

Here we improve Theorems 8.2.2 and 8.2.3 to get better estimates depending on the angles of the polygon $\mathcal{M}$.

We first solve the equation

$$\mathrm{sh}\ (u-x)\mathrm{sh}\ (v-x) = a\,\mathrm{sh}\ u\,\mathrm{sh}\ v \tag{8.3.1}$$

where $0 \leq a = \frac{1+\cos\alpha}{2} = \cos^2\frac{\alpha}{2} \leq 1$ and $u, v \geq 0$.

We have

$$(\mathrm{sh}\ u\,\mathrm{ch}\ x - \mathrm{ch}\ u\,\mathrm{sh}\ x)(\mathrm{sh}\ v\,\mathrm{ch}\ x - \mathrm{ch}\ v\,\mathrm{sh}\ x) = a(\mathrm{ch}^2\ x - \mathrm{sh}^2\ x)\mathrm{sh}\ u\,\mathrm{sh}\ v.$$

Equivalently

$$\mathrm{sh}\ u\,\mathrm{ch}\ x\,\mathrm{sh}\ v\,\mathrm{ch}\ x - \mathrm{sh}\ u\,\mathrm{ch}\ x\,\mathrm{ch}\ v\,\mathrm{sh}\ x - \mathrm{ch}\ u\,\mathrm{sh}\ x\,\mathrm{sh}\ v\,\mathrm{ch}\ x + \mathrm{ch}\ u\,\mathrm{sh}\ x\,\mathrm{ch}\ v\,\mathrm{sh}\ x =$$

$$= a(\mathrm{ch}^2\ x - \mathrm{sh}^2\ x)\,\mathrm{sh}\ u\,\mathrm{sh}\ v.$$

Equivalently

$$(1-a)\mathrm{sh}\ u\,\mathrm{sh}\ v\,\mathrm{ch}^2\ x - (\mathrm{sh}\ u\,\mathrm{ch}\ v + \mathrm{ch}\ u\,\mathrm{sh}\ v)\mathrm{ch}\ x\,\mathrm{sh}\ x +$$
$$+ (\mathrm{ch}\ u\,\mathrm{ch}\ v + a\,\mathrm{sh}\ u\,\mathrm{sh}\ v)\,\mathrm{sh}^2\ x = 0. \tag{8.3.2}$$

It follows that the smallest root $x = g(\alpha, u, v) = \widetilde{g}(a, u, v)$ of the equation (8.3.1) is defined as follows:

$$\text{th } x = \text{th } g(\alpha, u, v) = \frac{\text{sh } (u+v) - \sqrt{(a \,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v))^2 - 1}}{(a+1)\,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v)} \tag{8.3.3}$$

and

$$x = g(\alpha, u, v) = \widetilde{g}(a, u, v) = \frac{1}{2}\ln \frac{1 + \text{th } g(\alpha, u, v)}{1 - \text{th } g(\alpha, u, v)} =$$

$$\frac{1}{2}\ln \Bigg[(a+1)\,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v) + \text{sh } (u+v) - \\ - \sqrt{(a\,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v))^2 - 1}\Bigg] \Bigg/ \\ \Bigg[(a+1)\,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v) - \text{sh } (u+v) + \\ + \sqrt{(a\,\text{ch } (u+v) + (1-a)\,\text{ch } (u-v))^2 - 1}\Bigg]. \tag{8.3.4}$$

In particular, for $u = v$, we get

$$\text{th } x = \text{th } g(\alpha, u, u) = \frac{\text{sh } 2u - \sqrt{(a\,\text{ch } 2u + (1-a))^2 - 1}}{(a+1)\,\text{ch } 2u + (1-a)} \tag{8.3.5}$$

and

$$x = g(\alpha, u, u) = \widetilde{g}(a, u, u) = \\ \frac{1}{2}\ln \frac{(a+1)\,\text{ch } 2u + (1-a) + \text{sh } 2u - \sqrt{(a\,\text{ch } 2u + (1-a))^2 - 1}}{(a+1)\,\text{ch } 2u + (1-a) - \text{sh } 2u + \sqrt{(a\,\text{ch } 2u + (1-a))^2 - 1}}. \tag{8.3.6}$$

We introduce some functions which use the function $g(\alpha, u, v) = \widetilde{g}(a, u, v)$. We set

$$f_{h1}(\alpha_1, \alpha_2, \theta) = \widetilde{f}_{h1}(a_1, a_2, \theta) = \frac{\text{sh}^2(\frac{\theta}{2} - g(\alpha_1, \frac{\theta}{4}, \frac{\theta}{4}) - g(\alpha_2, \frac{\theta}{4}, \frac{\theta}{4}))}{\text{sh}^2(\frac{\theta}{4})}; \tag{8.3.7}$$

$$f_{h2}(\alpha_1, \alpha_2, \alpha_3, \theta, t) = \widetilde{f}_{h2}(a_1, a_2, a_3, \theta, t) = \\ \text{sh } \Big(\frac{3\theta}{4} + \frac{t}{4} - g(\alpha_1, \frac{\theta+t}{4}, \frac{\theta-t}{4}) - g(\alpha_2, \frac{\theta-t}{4}, \frac{\theta+t}{4}) - g(\alpha_3, \frac{\theta+t}{4}, \frac{\theta}{4})\Big) \times \\ \text{sh } \Big(\frac{3\theta}{4} - g(\alpha_1, \frac{\theta+t}{4}, \frac{\theta-t}{4}) - g(\alpha_2, \frac{\theta-t}{4}, \frac{\theta+t}{4}) - g(\alpha_3, \frac{\theta+t}{4}, \frac{\theta}{4})\Big) \Big/ \\ \Big(\text{sh } \frac{\theta+t}{4}\text{sh } \frac{\theta}{4}\Big)\ ; \tag{8.3.8}$$

$$f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,z,w)=\widetilde{f}_{h3}(a_1,a_2,a_3,a_4,\theta,z,w)=$$
$$\mathrm{sh}\,\Big(\theta+\frac{z-w}{4}-$$
$$-g(\alpha_1,\frac{\theta+z}{4},\frac{\theta-z}{4})-g(\alpha_2,\frac{\theta-z}{4},\frac{\theta+z}{4})-g(\alpha_3,\frac{\theta+z}{4},\frac{\theta-w}{4})-g(\alpha_4,\frac{\theta-w}{4},\frac{\theta+w}{4})\Big)\times$$
$$\mathrm{sh}\Big(\theta-g(\alpha_1,\frac{\theta+z}{4},\frac{\theta-z}{4})-g(\alpha_2,\frac{\theta-z}{4},\frac{\theta+z}{4})-g(\alpha_3,\frac{\theta+z}{4},\frac{\theta-w}{4})-g(\alpha_4,\frac{\theta-w}{4},\frac{\theta+w}{4})\Big)\Big/$$
$$\Big(\mathrm{sh}\,\frac{\theta+z}{4}\mathrm{sh}\,\frac{\theta+w}{4}\Big)\,. \tag{8.3.9}$$

We mention the following properties of the function $g(\alpha,u,v)$. We have $g(\alpha,u,v) = g(\alpha,v,u)$ and $g(\alpha,u,v)\le u,\ v$. For fixed α and u, the functions $g(\alpha,u,v)$ and $v-g(\alpha,u,v)$ increase if v increases. It follows that the number $g(\alpha,u,+\infty)$ is defined and

$$g(\alpha,u,+\infty)=u-\mathrm{arc\,sh}\,(a\,\mathrm{sh}\,u). \tag{8.3.10}$$

These properties follow from geometric interpretation below of the equation (8.3.1), see (8.3.31).

We also denote by

$$\mathrm{Ach}\ (x)=\begin{cases}0, & \text{if } x<1,\\ \mathrm{arc\,ch}\ (\mathrm{x}), & \text{if } x\ge 1.\end{cases} \tag{8.3.11}$$

We prove the following refined result (Theorem 8.3.1 below) where all estimates of Theorem 8.2.2 will be improved depending on angles of $\mathcal{M}$. In Theorem 8.3.1, we did exact calculations of maxima using computer only for angles α_i of the form $\frac{\pi}{n}$ where $n=2,\ 3,\ 4,\ 5,\ 6$ or $+\infty$. Only this result we need for reflective lattices. But we believe that the same results are valid for arbitrary $0\le\alpha_i<\pi$ (corresponding considerations should be similar to the proof of Theorem 4.1.5).

Theorem 8.3.1 (about narrow parts of types (II) and (III) of restricted hyperbolic convex polygons on the hyperbolic plane). *We fix a constant η such that $3\ge\eta>0$ (to get good estimates, one can take the constant $\eta=\eta_0=2\,arc\,ch\ \frac{3}{2}=$
$=1.92484730023841378999103565...$; equivalently, $ch\ \frac{\eta_0}{2}=\frac{3}{2}$).*

Any infinite restricted hyperbolic convex polygon $\mathcal{M}$ on the hyperbolic plane has a narrow part of one of types (AII), (AIII), (BII_1), (BII_2), (BIII) defined below:

(AII) There exist five consecutive vertices A_0, A_1, A_2, A_3 and A_4 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$, $i=1,\,2,\,3$, where $\alpha_i=\angle\ A_{i-1}A_iA_{i+1}$, and

$$2<(\delta_1,\,\delta_3)\le 4\max_{0\le\theta\le\eta} f_{h1}(\alpha_1,\alpha_2,\theta)-2=4\,f_{h1}(\alpha_1,\alpha_2,\eta)-2, \tag{8.3.12}$$

$$2<(\delta_1,\,\delta_4)\le 4\max_{0\le\theta\le\eta}\ \max_{0\le t\le\theta} f_{h2}(\alpha_1,\alpha_2,\alpha_3,\theta,t)-2=$$
$$4\max_{0\le\theta\le\eta}\big(f_{h2}(\alpha_1,\alpha_2,\alpha_3,\theta,0),f_{h2}(\alpha_1,\alpha_2,\alpha_3,\theta,\theta)\big)-2=$$
$$4\max\big(f_{h2}(\alpha_1,\alpha_2,\alpha_3,\eta,0),\ f_{h2}(\alpha_1,\alpha_2,\alpha_3,\eta,\eta)\big)-2. \tag{8.3.13}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3,\,4$, *contain the axis* l.

(AIII) There exist six consecutive vertices A_0, A_1, A_2, A_3, A_4 *and* A_5 *of one half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3, δ_4 *and* δ_5 *to lines* (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) *and* (A_4A_5) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=\delta_5^2=-2$ *one has* $(\delta_i,\delta_{i+1})=2\cos\alpha_i$, $i=1,\,2,\,3,\,4$, *where* $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$, *and*

$$2<(\delta_1,\delta_3)\le 4\max_{0\le\theta\le\eta} f_{h1}(\alpha_1,\alpha_2,\theta)-2=4\,f_{h1}(\alpha_1,\alpha_2,\eta)-2, \tag{8.3.14}$$

$$2<(\delta_3,\,\delta_5)\le 4\max_{0\le\theta\le\eta} f_{h1}(\alpha_1,\alpha_2,\theta)-2=4\,f_{h1}(\alpha_1,\alpha_2,\eta)-2, \tag{8.3.15}$$

and

$$2<(\delta_1,\,\delta_5)\le 4\max_{0\le\theta\le\eta}\ \max_{0\le w\le z\le\theta} f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,z,w)-2=$$

$$4\max_{0\le\theta\le\eta}\big(f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,0,0),\ f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,\theta,0),\ f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,\theta,\theta)\big)$$
$$-2=$$

$$4\max\big(f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta,0,0),\ f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta,\eta,0),\ f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta,\eta,\eta)\big)$$
$$-2. \tag{8.3.16}$$

Moreover, the set $\{\delta_1,\,\delta_3,\,\delta_5\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3,\,4$, 5, *contain the axis* l.

(BII$_1$) There exist four consecutive vertices A_0, A_1, A_2, A_3 *of one half of* $\mathcal{M}$ *and two consecutive vertices* B_0, B_1 *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3 *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (A_2A_3) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, *one has* $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$, $i=1,\,2$, *where* $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$, *and*

$$2<(\delta_1,\,\delta_4)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.17}$$

$$2<(\delta_2,\,\delta_4)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.18}$$

$$2<(\delta_3,\,\delta_4)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.19}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_1,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)\ \le$$
$$\frac{\eta}{2}+2g(\alpha_1,\,\frac{\eta}{4},+\infty), \tag{8.3.20}$$

$$Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_2,\ \delta_4)+2)}{4}-ch\ \frac{\eta}{2}\right)+Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_3,\ \delta_4)+2)}{4}-ch\ \frac{\eta}{2}\right)\ \leq$$
$$\frac{\eta}{2}+2g(\alpha_2,\ \frac{\eta}{4},+\infty). \tag{8.3.21}$$

Moreover, the set $\{\delta_1,\ \delta_2,\ \delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\ \delta_3,\ \delta_4\}$. *Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i=1,\ 2,\ 3$, *and closed half-plane* $\mathcal{H}^+_{\delta_4}$ *contain the axis* l; *both lines orthogonal to the axis* l *and containing* A_1 *and* A_2 *intersect the line* (B_0B_1).

(BII$_2$) There exist three consecutive vertices $A_0,\ A_1,\ A_2$ *of one half of* $\mathcal{M}$ *and three consecutive vertices* $B_0,\ B_1,\ B_2$ *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* $\delta_1,\ \delta_2,\ \delta_3$ *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (B_0B_1) *and* (B_1B_2) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, *one has* $(\delta_1,\ \delta_2)=2\cos\alpha$ *where* $\alpha=\angle\ A_0A_1A_2$, $(\delta_3,\ \delta_4)=2\cos\beta$ *where* $\beta=\angle\ B_0B_1B_2$, *and*

$$2<(\delta_1,\ \delta_3)\leq\frac{8}{ch\ \frac{\eta}{2}-1}+6, \tag{8.3.22}$$

$$2<(\delta_2,\ \delta_3)\leq\frac{8}{ch\ \frac{\eta}{2}-1}+6, \tag{8.3.23}$$

$$2<(\delta_2,\ \delta_4)\leq\frac{8}{ch\ \frac{\eta}{2}-1}+6, \tag{8.3.24}$$

$$Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_1,\ \delta_3)+2)}{4}-ch\ \frac{\eta}{2}\right)+Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_2,\ \delta_3)+2)}{4}-ch\ \frac{\eta}{2}\right)\ \leq$$
$$\frac{\eta}{2}+2g(\alpha,\ \frac{\eta}{4},+\infty), \tag{8.3.25}$$

$$Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_3,\ \delta_2)+2)}{4}-ch\ \frac{\eta}{2}\right)+Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_4,\ \delta_2)+2)}{4}-ch\ \frac{\eta}{2}\right)\ \leq$$
$$\frac{\eta}{2}+2g(\beta,\ \frac{\eta}{4},+\infty). \tag{8.3.26}$$

Moreover, the set $\{\delta_1,\ \delta_2,\ \delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\ \delta_3,\ \delta_4\}$. *Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i=1,\ 2,\ 3,\ 4$, *contain the axis* l; *the line orthogonal to the axis* l *and containing* A_1 *intersects* (B_0B_1), *and the line orthogonal to* l *and containing* B_1 *intersects* (A_1A_2).

(BIII) There exist four consecutive vertices $A_0,\ A_1,\ A_2,\ A_3$ *of one half of* $\mathcal{M}$, *and two consecutive vertices* $B_0,\ B_1$ *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* $\delta_1,\ \delta_2,\ \delta_3$ *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (A_2A_3) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, *one has* $(\delta_i,\ \delta_{i+1})=2\cos\alpha_i$ *where* $\alpha_i=\angle\ A_{i-1}A_iA_{i+1}$, $i=1,\ 2$, *and*

$$2<(\delta_1,\ \delta_3)\leq 4\max_{0\leq x\leq\eta} f_{h1}(\alpha_1,\alpha_2,x)-2=4\,f_{h1}(\alpha_1,\alpha_2,\eta)-2, \tag{8.3.27}$$

$$2<(\delta_1,\ \delta_4)\leq 4\frac{ch\ \frac{\eta}{2}+ch\ (\eta-2\,g(\alpha_1,\frac{\eta}{4},\frac{\eta}{4}))}{ch\ \frac{\eta}{2}-1}-2, \tag{8.3.28}$$

$$2 < (\delta_3,\, \delta_4) \le 4\frac{ch\ \frac{\eta}{2} + ch\ (\eta - 2\, g(\alpha_2, \frac{\eta}{4}, \frac{\eta}{4}))}{ch\ \frac{\eta}{2} - 1} - 2, \tag{8.3.29}$$

$$Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_1,\, \delta_4)+2)}{4} - ch\ \frac{\eta}{2}\right) + Ach\ \left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_3,\, \delta_4)+2)}{4} - ch\ \frac{\eta}{2}\right) \le$$
$$\frac{\eta}{2} + \max_{0\le u\le \frac{\eta}{2}} \left(u + 2\,|g(\alpha_1, \frac{\eta}{4}, \frac{u}{2}) - g(\alpha_2, \frac{\eta}{4}, \frac{u}{2})|\right). \tag{8.3.30}$$

Moreover, the set $\{\delta_1,\, \delta_2,\, \delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for the set $\{\delta_2,\, \delta_3,\, \delta_4\}$. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i = 1, 2, 3, 4$, contain the axis l; there exists a line orthogonal to the axis l and intersecting the lines (A_1A_2) and (B_0B_1).

Proof. Compare the proof of Theorem 4.1.5. The general line of the proof is the same as in Theorem 8.2.2, but all estimates should be improved. Thus we follow the proof of Theorem 8.2.2 and notations of the proof.

We introduce additional angles $\theta^1_{(i+1)i} = \theta(A_{(i+1)1},\, A_{i2})$ and $\theta^2_{(i+1)i} = \theta(B_{(i+1)1},\, B_{i2})$. We denote $\alpha_i = \angle A_{i-1}A_iA_{i+1}$, $a_i = \cos^2\frac{\alpha_i}{2}$. Similarly, we denote $\beta_j = \angle B_{j-1}B_jB_{j+1}$, $b_j = \cos^2\frac{\beta_j}{2}$. By (8.1.20), we get

$$\theta^1_{(i+1)i} = 2g(\alpha_i, \frac{\theta^1_i}{2}, \frac{\theta^1_{i+1}}{2}),\quad \theta^2_{(j+1)j} = 2g(\beta_j, \frac{\theta^2_j}{2}, \frac{\theta^2_{j+1}}{2}). \tag{8.3.31}$$

Let us consider the proof of the estimate (8.2.20). For lines $m_1 = (A_0A_1)$ and $m_3 = (A_2A_3)$ with orthogonal vectors e_1, e_3 we have (using (8.1.19))

$$(e_1,\, e_3) = 4\frac{\mathrm{sh}\ \frac{\theta^1_1+\theta^1_2-\theta^1_{21}-\theta^1_{32}}{2}\mathrm{sh}\ \frac{\theta^1_3+\theta^1_2-\theta^1_{21}-\theta^1_{32}}{2}}{\mathrm{sh}\ \frac{\theta^1_1}{2}\mathrm{sh}\ \frac{\theta^1_3}{2}} - 2 =$$

$$4\,\frac{\mathrm{sh}\left(\frac{\theta^1_1}{2}+\frac{\theta^1_2}{2}-g(\alpha_1, \frac{\theta^1_1}{2}, \frac{\theta^1_2}{2})-g(\alpha_2, \frac{\theta^1_2}{2}, \frac{\theta^1_3}{2})\right)\mathrm{sh}\left(\frac{\theta^1_3}{2}+\frac{\theta^1_2}{2}-g(\alpha_1, \frac{\theta^1_1}{2}, \frac{\theta^1_2}{2})-g(\alpha_2, \frac{\theta^1_2}{2}, \frac{\theta^1_3}{2})\right)}{\mathrm{sh}\frac{\theta^1_1}{2}\ \mathrm{sh}\ \frac{\theta^1_3}{2}}$$
$$- 2. \tag{8.3.32}$$

Here $\theta^1_1 \ge \theta/2$, $\theta^1_2 \le \theta/2$ and $\theta^1_3 \ge \theta/2$. In particular, $\theta^1_1 \ge \theta^1_2$ and $\theta^1_3 \ge \theta^1_2$. Then we shall prove that

$$(e_1,\, e_3) \le 4\, f_{h1}(\alpha_1, \alpha_2, 2\theta^1_2) - 2 \le 4 \max_{0\le x\le\theta} f_{h1}(\alpha_1, \alpha_2, x) - 2. \tag{8.3.33}$$

Equivalently, (8.3.32) increases if one puts $\theta^1_1 := \theta^1_2$ and $\theta^1_3 := \theta^1_2$.

We argue like in proof of Theorem 4.1.4. We know that $\theta^1_1 \ge \theta/2 \ge \theta^1_2$ and $\theta^1_3 \ge \theta/2 \ge \theta^1_2/2$. We consider lines $m_i = (A_{i-1}A_i) = (A_{i1}A_{i2})$. We know that lines m_1, m_3 do not intersect. Equivalently, $(e_1,\, e_3) > 2$.

Since $\theta^1_1 \ge \theta^1_2$, there exists a line $m'_1 = (A'_{11}A'_{12})$ with terminals A'_{11} and A'_{12} at infinity such that the line m'_1 is contained in $\mathcal{H}^+_{-e_1}$, points A_{11}, A'_{11} and A_{12}, A'_{12} are contained in the same half-planes bounded by m_2, the line m'_1 has the same angle $\alpha_1 = \angle A'_{11}A'_1A_{22}$ (as m_1) with the line m_2 (we denote by A'_1 their intersection point), and the angle $\theta(A'_{11}, A'_{12})$ of m'_1 is equal to θ^1_2. We denote by e'_1 the orthogonal vector to m'_1 directed outwards of $\mathcal{M}$ and with $(e'_1)^2 = -2$, and we denote $\theta'_{21} = \theta(A_{21}, A'_{12})$.

Similarly, there exists a line $m'_3 = (A'_{31}A'_{32})$ with terminals A'_{31} and A'_{32} at infinity such that the line m'_3 is contained in $\mathcal{H}^+_{-e_3}$, points A_{31}, A'_{31} and A_{32}, A'_{32} are contained in the same half-planes bounded by m_2, the line m'_3 has the same angle $\alpha_2 = \angle A_{21}A'_2A'_{32}$ (as m_3) with the line m_2 (we denote by A'_2 their point of intersection), and the angle $\theta(A'_{31}, A'_{32})$ of m'_3 is equal to θ^1_2. We denote by e'_3 the orthogonal vector to m'_3 directed outwards of $\mathcal{M}$ and with $(e'_3)^2 = -2$, and we denote $\theta'_{32} = \theta(A'_{31}, A_{22})$.

Since the lines m_1 and m_3 do not intersect, by our construction, any interval with terminals at m'_1 and m'_3 intersects both lines m_1 and m_3. It follows that distance between lines m'_1 and m'_3 is greater than distance between lines m_1 and m_3. It follows $(e_1, e_3) \le (e'_1, e'_3)$, and by (8.1.19), (8.1.20) we have

$$(e_1,\, e_3) \le (e'_1,\, e'_3) = 4\, f_{h1}(\alpha_1, \alpha_2, 2\theta^1_2) - 2.$$

Since $2\theta^1_2 \le \theta$, we get (8.3.33).

Thus, instead of (8.2.20) we get (8.3.33). Following the line of the proof of Theorem 8.2.2, using (8.3.33), we get estimates (8.3.12), (8.3.14), (8.3.15), (8.3.27).

Now we consider the proof of (8.2.23). Using (8.1.19) and (8.1.20), we have

$$(e_1,\, e_4) = 4\,\frac{\mathrm{sh}\,\frac{\theta^1_1+\theta-\theta^1_{21}-\theta^1_{32}-\theta^1_{43}}{2}\mathrm{sh}\,\frac{\theta^1_4+\theta-\theta^1_{21}-\theta^1_{32}-\theta^1_{43}}{2}}{\mathrm{sh}\,\frac{\theta^1_1}{2}\mathrm{sh}\,\frac{\theta^1_4}{2}} - 2 =$$

$$4\,\mathrm{sh}\,\Big(\frac{\theta^1_1}{2}+\frac{\theta}{2}-g(\alpha_1,\frac{\theta^1_1}{2},\frac{\theta^1_2}{2})-g(\alpha_2,\frac{\theta^1_2}{2},\frac{\theta^1_3}{2})-g(\alpha_3,\frac{\theta^1_3}{2},\frac{\theta^1_4}{2})\Big)\times$$
$$\mathrm{sh}\,\Big(\frac{\theta^1_4}{2}+\frac{\theta}{2}-g(\alpha_1,\frac{\theta^1_1}{2},\frac{\theta^1_2}{2})-g(\alpha_2,\frac{\theta^1_2}{2},\frac{\theta^1_3}{2})-g(\alpha_3,\frac{\theta^1_3}{2},\frac{\theta^1_4}{2})\Big)\Big/$$
$$\Big(\mathrm{sh}\,\frac{\theta^1_1}{2}\mathrm{sh}\,\frac{\theta^1_4}{2}\Big) - 2 \tag{8.3.34}$$

where $\theta^1_2 + \theta^1_3 = \theta$, $\theta^1_2 \le \theta^1_3$, $\theta^1_1 \ge \theta^1_3$ and $\theta^1_4 \ge \theta/2$. Like above, one can prove that (8.3.34) increases if one puts $\theta^1_1 := \theta^1_3$ and $\theta^1_4 =: \theta/2$. Thus, we get

$$(e_1,\, e_4) \le$$
$$4\,\mathrm{sh}\,\Big(\frac{\theta^1_3}{2}+\frac{\theta}{2}-g(\alpha_1,\frac{\theta^1_3}{2},\frac{\theta^1_2}{2})-g(\alpha_2,\frac{\theta^1_2}{2},\frac{\theta^1_3}{2})-g(\alpha_3,\frac{\theta^1_3}{2},\frac{\theta}{4})\Big)\times$$
$$\mathrm{sh}\,\Big(\frac{3\theta}{4}-g(\alpha_1,\frac{\theta^1_3}{2},\frac{\theta^1_2}{2})-g(\alpha_2,\frac{\theta^1_2}{2},\frac{\theta^1_3}{2})-g(\alpha_3,\frac{\theta^1_3}{2},\frac{\theta}{4})\Big)\Big/\Big(\mathrm{sh}\,\frac{\theta^1_3}{2}\mathrm{sh}\,\frac{\theta}{4}\Big) - 2\,. \tag{8.3.35}$$

We set $t = \theta^1_3 - \theta^1_2$ where $0 \le t \le \theta$. Then $\theta^1_2 = (\theta - t)/2$, $\theta^1_3 = (\theta + t)/2$. We get

$$(e_1,\, e_4) \le$$
$$4\,\mathrm{sh}\,\Big(\frac{3\theta}{4}+\frac{t}{4}-g(\alpha_1,\frac{\theta+t}{4},\frac{\theta-t}{4})-g(\alpha_2,\frac{\theta-t}{4},\frac{\theta+t}{4})-g(\alpha_3,\frac{\theta+t}{4},\frac{\theta}{4})\Big)\times$$
$$\mathrm{sh}\,\Big(\frac{3\theta}{4}-g(\alpha_1,\frac{\theta+t}{4},\frac{\theta-t}{4})-g(\alpha_2,\frac{\theta-t}{4},\frac{\theta+t}{4})-g(\alpha_3,\frac{\theta+t}{4},\frac{\theta}{4})\Big)\Big/$$
$$\Big(\mathrm{sh}\,\frac{\theta+t}{4}\mathrm{sh}\,\frac{\theta}{4}\Big) - 2\,. \tag{8.3.36}$$

Thus, instead of (8.2.23), we get (see (8.3.8))

$$(e_1,\, e_4)\ \le\ 4\ \max_{0\le t\le\theta} f_{h2}(\alpha_1,\alpha_2,\alpha_3,\theta,t)-2. \tag{8.3.37}$$

Following the proof of Theorem 8.2.2, using (8.3.37), we get (8.3.13).

Now we consider the proof of (8.2.27). Using (8.1.19) and (8.1.20), we have

$$(e_1,\, e_5)=4\,\frac{\mathrm{sh}\ \frac{\theta_1^1+\theta+\theta_4^1-\theta_{21}^1-\theta_{32}^1-\theta_{43}^1-\theta_{54}^1}{2}\mathrm{sh}\ \frac{\theta_5^1+\theta+\theta_4^1-\theta_{21}^1-\theta_{32}^1-\theta_{43}^1-\theta_{54}^1}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_5^1}{2}}-2=$$

$$4\,\mathrm{sh}\ \Big(\frac{\theta_1^1}{2}+\frac{\theta}{2}+\frac{\theta_4^1}{2}-g(\alpha_1,\frac{\theta_1^1}{2},\frac{\theta_2^1}{2})-g(\alpha_2,\frac{\theta_2^1}{2},\frac{\theta_3^1}{2})-g(\alpha_3,\frac{\theta_3^1}{2},\frac{\theta_4^1}{2})-g(\alpha_4,\frac{\theta_4^1}{2},\frac{\theta_5^1}{2})\Big)\times$$
$$\mathrm{sh}\ \Big(\frac{\theta_5^1}{2}+\frac{\theta}{2}+\frac{\theta_4^1}{2}-g(\alpha_1,\frac{\theta_1^1}{2},\frac{\theta_2^1}{2})-g(\alpha_2,\frac{\theta_2^1}{2},\frac{\theta_3^1}{2})-g(\alpha_3,\frac{\theta_3^1}{2},\frac{\theta_4^1}{2})-g(\alpha_4,\frac{\theta_4^1}{2},\frac{\theta_5^1}{2})\Big)\Big/$$
$$\Big(\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_5^1}{2}\Big)-2 \tag{8.3.38}$$

where $\theta_2^1+\theta_3^1=\theta$, $\theta_2^1\le\theta_3^1$, $\theta_1^1\ge\theta_3^1$ and $\theta_4^1\le\theta/2$. Then $\theta_5^1\ge\theta/2$ and $\theta_4^1+\theta_5^1\ge\theta$. Like above, (8.3.38) increases if one puts $\theta_1^1:=\theta_3^1$ and $\theta_5^1:=\theta-\theta_4^1$. Thus, we get

$$(e_1,\, e_5)\le$$
$$4\,\mathrm{sh}\ \Big(\frac{\theta_3^1}{2}+\frac{\theta}{2}+\frac{\theta_4^1}{2}-g(\alpha_1,\frac{\theta_3^1}{2},\frac{\theta_2^1}{2})-g(\alpha_2,\frac{\theta_2^1}{2},\frac{\theta_3^1}{2})-g(\alpha_3,\frac{\theta_3^1}{2},\frac{\theta_4^1}{2})-g(\alpha_4,\frac{\theta_4^1}{2},\frac{\theta-\theta_4^1}{2})\Big)\times$$
$$\times\mathrm{sh}\ \Big(\theta-g(\alpha_1,\frac{\theta_3^1}{2},\frac{\theta_2^1}{2})-g(\alpha_2,\frac{\theta_2^1}{2},\frac{\theta_3^1}{2})-g(\alpha_3,\frac{\theta_3^1}{2},\frac{\theta_4^1}{2})-g(\alpha_4,\frac{\theta_4^1}{2},\frac{\theta-\theta_4^1}{2})\Big)\Big/$$
$$\Big(\mathrm{sh}\ \frac{\theta_3^1}{2}\mathrm{sh}\ \frac{\theta-\theta_4^1}{2}\Big)\ -2\ . \tag{8.3.39}$$

We set $z=\theta_3^1-\theta_2^1$ and $w=\theta_5^1-\theta_4^1$ where $\theta\ge z\ge w\ge 0$ (this follows from definition). Then $\theta_1^1=\theta_3^1=\frac{\theta+z}{2}$, $\theta_2^1=\frac{\theta-z}{2}$, $\theta_4^1=\frac{\theta-w}{2}$, $\theta_5^1=\frac{\theta+w}{2}$. From (8.3.39), we then get

$$(e_1,\, e_5)\le 4\,\mathrm{sh}\ \Big(\theta+\frac{z-w}{4}-$$
$$-g(\alpha_1,\frac{\theta+z}{4},\frac{\theta-z}{4})-g(\alpha_2,\frac{\theta-z}{4},\frac{\theta+z}{4})-g(\alpha_3,\frac{\theta+z}{4},\frac{\theta-w}{4})-g(\alpha_4,\frac{\theta-w}{4},\frac{\theta+w}{4})\Big)\times$$
$$\mathrm{sh}\ \Big(\theta-g(\alpha_1,\frac{\theta+z}{4},\frac{\theta-z}{4})-g(\alpha_2,\frac{\theta-z}{4},\frac{\theta+z}{4})-g(\alpha_3,\frac{\theta+z}{4},\frac{\theta-w}{4})-g(\alpha_4,\frac{\theta-w}{4},\frac{\theta+w}{4})\Big)\Big/$$
$$\Big(\mathrm{sh}\ \frac{\theta+z}{4}\mathrm{sh}\ \frac{\theta+w}{4}\Big)-2\ . \tag{8.3.40}$$

It follows, that instead of (8.2.27) we have

$$(e_1,\, e_5)\le 4\max_{0\le w\le z\le\theta} f_{h3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\theta,z,w)-2. \tag{8.3.41}$$

Following the proof of Theorem 8.2.2, using (8.3.41) we get (8.3.16).

Now we consider the proof of (8.2.33) and (8.2.34). Using (8.1.21), we get

$$(e_1,\, f_1)=4\frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\ \frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}}-2 \tag{8.3.42}$$

and similar formula for $(e_2,\, f_1)$. Here $\theta_1^1\ge\frac{\eta}{2}$, $\theta_2^1\ge\frac{\eta}{2}$ and $\theta_1^2\ge\frac{\eta}{2}$. There exists a line (AB) which is orthogonal to the axis, contains the vertex A_1 and intersects the line (B_0B_1). Here we denote by A and B terminals of the line at infinity and we suppose that A belongs to the half-plane of the axis, which contains vertices A_i. Then B belongs to the half-plane of the axis containing vertices B_j. We denote by $u_1=\theta(A_{21},A)$, $u_2=\theta(A,A_{12})$, then $\theta_{21}^1=u_1+u_2=2g(\alpha_1,\frac{\theta_1^1}{2},\frac{\theta_2^1}{2})$ where $u_1\ge 0$ and $u_2\ge 0$. We denote by $v_1=\theta(B_{11},B)$, $v_2=\theta(B,B_{12})$. Then $\theta_1^2=v_1+v_2$ where $v_1\ge 0$ and $v_2\ge 0$. We have $\theta_{12}=v_2-u_2$ in (8.3.42).

Let us prove (8.3.20) (where $\delta_4=f_1$). Like for the proof of (8.3.33), we can suppose one of two cases (1) or (2) below:

Case (1). We can assume that $\theta_1^1=\theta_2^1=\theta_1^2=\frac{\eta}{2}$ (instead of inequalities $\theta_1^1,\,\theta_2^1,\,\theta_1^2\ge\frac{\eta}{2}$). Then $u_1=u_2=\theta_{21}^1/2$, and, by (8.3.42), we have

$$\frac{(e_1,\, f_1)+2}{4}\le\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1}+\frac{\mathrm{ch}\ (v_2-u_2)}{\mathrm{ch}\ \frac{\eta}{2}-1}.$$

Similarly, we have

$$\frac{(e_2,\, f_1)+2}{4}\le\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1}+\frac{\mathrm{ch}\ (v_1-u_1)}{\mathrm{ch}\ \frac{\eta}{2}-1}.$$

We have $|v_1-u_1|+|v_2-u_2|=|v_1-u_1|+|\frac{\eta}{2}-v_1-u_1|\le\frac{\eta}{2}$ because $u_1=u_2\le\frac{\eta}{2}$, $v_1+v_2=\frac{\eta}{2}$ and $v_1,\,v_2\ge 0$. It follows

$$\mathrm{Ach}\ \left(\frac{(\mathrm{ch}\ \frac{\eta}{2}-1)((e_1,\, f_1)+2)}{4}-\mathrm{ch}\ \frac{\eta}{2}\right)+\mathrm{Ach}\ \left(\frac{(\mathrm{ch}\ \frac{\eta}{2}-1)((e_2,\, f_1)+2)}{4}-\mathrm{ch}\ \frac{\eta}{2}\right)\le\frac{\eta}{2}.$$

It follows (8.3.20).

Case (2). We can assume that $\theta_1^1=\theta_1^2=\frac{\eta}{2}$, $\theta_2^1\ge\frac{\eta}{2}$, $v_2=0$ (or similar case $\theta_2^1=\theta_1^2=\frac{\eta}{2}$, $\theta_1^1\ge\frac{\eta}{2}$ and $v_1=0$). Then $v_1=\frac{\eta}{2}$, and by (8.3.42), one has

$$\frac{(e_1,\, f_1)+2}{4}\le\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1}+\frac{\mathrm{ch}\ u_2}{\mathrm{ch}\ \frac{\eta}{2}-1}$$

and

$$\frac{(e_2,\, f_1)+2}{4}\le\frac{\mathrm{ch}\ \frac{\theta_2^1+\eta/2}{2}}{2\mathrm{ch}\ \frac{\theta_2^1}{2}\mathrm{ch}\ \frac{\eta}{4}}+\frac{\mathrm{ch}\ (\frac{\theta_2^1}{2}+\frac{\eta}{4}-u_1)}{2\mathrm{ch}\ \frac{\theta_2^1}{2}\mathrm{ch}\ \frac{\eta}{4}}\le$$

$$\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1}+\frac{\mathrm{ch}\ (\frac{\eta}{4}+|\frac{\eta}{4}-u_1|)}{\mathrm{ch}\ \frac{\eta}{2}-1}.$$

If $u_1\ge\frac{\eta}{4}$, we have $u_2+\frac{\eta}{4}+|\frac{\eta}{4}-u_1|=u_2+u_1=\theta_{21}^1\le\frac{\eta}{2}$. If $u_1\le\frac{\eta}{4}$, we have $u_2+\frac{\eta}{4}+|\frac{\eta}{4}-u_1|=\frac{\eta}{2}+(u_2-u_1)\le\frac{\eta}{2}+\theta_{21}^1=\frac{\eta}{2}+2\max_{x\ge\frac{\eta}{2}}g(\alpha_1,\frac{\eta}{4},\frac{x}{2})=$

$\frac{\eta}{2}+2g(\alpha_1,\frac{\eta}{4},+\infty)$. It follows (8.3.20). Similarly we get (8.3.21), (8.3.25) and (8.3.26). All other inequalities of cases (BII_1) and (BII_2) of Theorem 8.3.1 are the same as in Theorem 8.2.2.

Now we consider the proof of (8.2.40) and (8.2.41). Using (8.1.21), we have

$$(e_1,\, f_1)=4\frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\ \frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}}-2 \tag{8.3.43}$$

where $\theta_1^1\ge\frac{\eta}{2}$, $\theta_2^1\le\frac{\eta}{2}$, $\theta_1^2\ge\frac{\eta}{2}$ and there exists a line (AB) which is orthogonal to the axis and intersects both lines (A_1A_2) and (B_0B_1). (One has similar formula for $(e_3,\, f_1)$, we shall consider it below.) Here we denote by A and B terminals of the line (AB) at infinity, and A belongs to the half-plane of the axis containing vertices A_i. Then B belongs to the half-plane of the axis which contains vertices B_j. We denote by $u_1=\theta(A_{21},A)$ and $u_2=\theta(A,A_{22})$. Then $\theta_2^1=u=u_1+u_2$ where $u_1,\,u_2\ge 0$. Similarly we introduce $v_1=\theta(B_{11},B)$, $v_2=\theta(B,B_{12})$. Then $\theta_1^2=v_1+v_2$ where $v_1,\,v_2\ge 0$.

We want to prove (8.3.28), (8.3.29) and (8.3.30) (where $\delta_4=f_1$). Like for the proof of (8.3.33), we can suppose that $\theta_1^1=\theta_3^1=\theta_1^2=\frac{\eta}{2}$ (instead of inequalities $\theta_1^1,\,\theta_3^1,\,\theta_1^2\ \ge\ \frac{\eta}{2}$). Using (8.3.43), we have

$$\frac{(e_1,\, f_1)+2}{4}=\frac{\mathrm{ch}\ \frac{\theta_1^1+\theta_{12}}{2}\mathrm{ch}\ \frac{\theta_1^2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\theta_1^1}{2}\mathrm{sh}\ \frac{\theta_1^2}{2}}\le$$

$$\frac{\mathrm{ch}\ \frac{\eta/2+\theta_{12}}{2}\mathrm{ch}\ \frac{\eta/2-\theta_{12}}{2}}{\mathrm{sh}\ \frac{\eta}{4}\mathrm{sh}\ \frac{\eta}{4}}=\frac{\mathrm{ch}\ \frac{\eta}{2}}{\mathrm{ch}\ \frac{\eta}{2}-1}+\frac{\mathrm{ch}\ \theta_{12}}{\mathrm{ch}\ \frac{\eta}{2}-1} \tag{8.3.44}$$

where $\theta_{12}=u_1-2g(\alpha_1,\frac{\eta}{4},\frac{u}{2})+v_2$. It follows

$$\mathrm{Ach}\ \left(\frac{(\mathrm{ch}\ \frac{\eta}{2}-1)((e_1,\, f_1)+2)}{4}-\mathrm{ch}\ \frac{\eta}{2}\right)\le|v_2+u_1-2\,g(\alpha_1,\frac{\eta}{4},\frac{u}{2})|\le$$
$$\frac{\eta}{2}+\max_{0\le u\le\eta/2}\left(u-2\,g(\alpha_1,\frac{\eta}{4},\frac{u}{2})\right)=\eta-2\,g(\alpha_1,\frac{\eta}{4},\frac{\eta}{4}).$$

To get the last inequality, we use that $0\le v_2\le\frac{\eta}{2}$, $0\le u_1\le u\le\frac{\eta}{2}$ and $0\le\theta_{21}^1=2\,g(\alpha_1,\frac{\eta}{4},\frac{u}{2})\le u\le\frac{\eta}{2}$. Similarly we have

$$\mathrm{Ach}\ \left(\frac{(\mathrm{ch}\ \frac{\eta}{2}-1)((e_3,\, f_1)+2)}{4}-\mathrm{ch}\ \frac{\eta}{2}\right)\le|u_2+v_1-2g(\alpha_2,\frac{\eta}{4},\frac{u}{2})|\le$$
$$\le\frac{\eta}{2}+\max_{0\le u\le\eta/2}\left(u-2\,g(\alpha_2,\frac{\eta}{4},\frac{u}{2})\right)=\eta-2\,g(\alpha_2,\frac{\eta}{4},\frac{\eta}{4}).$$

This proves (8.3.28) and (8.3.29). We have

$$|u_1+v_2-2\,g(\alpha_1,\frac{\eta}{4},\frac{u}{2})|+|u_2+v_1-2\,g(\alpha_2,\frac{\eta}{4},\frac{u}{2})|\le$$

$$\max_{0\le u\le\frac{\eta}{2}}\ \max_{0\le x\le\frac{\eta}{2}+u}\left(|x-2g(\alpha_1,\frac{\eta}{4},\frac{u}{2})|+|\frac{\eta}{2}+u-x-2\,g(\alpha_2,\frac{\eta}{4},\frac{u}{2})|\right)=$$

$$\max_{0\le u\le \frac{\eta}{2}}\Big(\frac{\eta}{2}+u+2\,|g(\alpha_1,\frac{\eta}{4},\frac{u}{2})-g(\alpha_2,\frac{\eta}{4},\frac{u}{2})|\Big).$$

Here we use obvious inequalities: $\theta_{21}^1=2\,g(\alpha_1,\frac{\eta}{4},\frac{u}{2})\le u\le \eta/2$, $\theta_{32}^1=2\,g(\alpha_2,\frac{\eta}{4},\frac{u}{2})\le u\le \eta/2$. Thus, we get

$$\mathrm{Ach}\,\left(\frac{(\mathrm{ch}\,\frac{\eta}{2}-1)((e_1,\,f_1)+2)}{4}-\mathrm{ch}\,\frac{\eta}{2}\right)+\mathrm{Ach}\,\left(\frac{(\mathrm{ch}\,\frac{\eta}{2}-1)((e_3,\,f_1)+2)}{4}-\mathrm{ch}\,\frac{\eta}{2}\right)\le$$

$$\frac{\eta}{2}+\max_{0\le u\le\frac{\eta}{2}}\Big(u+2\,|g(\alpha_1,\frac{\eta}{4},\frac{u}{2})-g(\alpha_2,\frac{\eta}{4},\frac{u}{2})|\Big).$$

This proves (8.3.30).

All other statements of Theorem 8.3.1 are the same as similar statements of Theorem 8.2.2.

It finishes the proof.

From Theorem 8.3.1 we get an important for application statement which is similar to Theorems 4.1.8 and 4.2.3.

Theorem 8.3.2 (about narrow parts of types (I), (II) and (III) of restricted hyperbolic convex polygons on the hyperbolic plane). *We fix a constant $0<\eta\le 3$.*

Any infinite restricted hyperbolic convex polygon $\mathcal{M}$ has a narrow part of one of types (AI), (AII), (AIII), (BI), (BII$_1$),(BII$_2$), (BIII) defined below:

(AI) There exist four consecutive vertices A_0, A_1, A_2, A_3 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2 and δ_3 to lines (A_0A_1), (A_1A_2) and (A_2A_3) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$, where $\alpha_i=\angle\ A_{i-1}A_iA_{i+1}$, $i=1,\,2$, $\alpha_2\ne\frac{\pi}{2}$, and

$$2<(\delta_1,\,\delta_3)\le 4\,f_{h1}(\alpha_1,\alpha_2,\eta)-2. \tag{8.3.45}$$

The set $\{\delta_1,\,\delta_2,\,\delta_3\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3$, contain the axis l.

(AII) There exist five consecutive vertices A_0, A_1, A_2, A_3 and A_4 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (A_3A_4) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$ where $\alpha_i=\angle\ A_{i-1}A_iA_{i+1}$, $i=1,\,2,\,3$, $\alpha_1=\alpha_2=\frac{\pi}{2}$, and

$$2<(\delta_1,\,\delta_3)\le 4\,f_{h1}(\frac{\pi}{2},\frac{\pi}{2},\eta)-2, \tag{8.3.46}$$

$$2<(\delta_1,\,\delta_4)\le 4\,\max\big(f_{h2}(\frac{\pi}{2},\frac{\pi}{2},\alpha_3,\eta,\,0),\ f_{h2}(\frac{\pi}{2},\frac{\pi}{2},\alpha_3,\eta,\eta)\big)-2. \tag{8.3.47}$$

The set $\{\delta_1,\,\delta_3,\,\delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3,\,4$, contain the axis l.

(AIII) There exist six consecutive vertices A_0, A_1, A_2, A_3, A_4 and A_5 of one half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3, δ_4 and δ_5 to lines (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) and (A_4A_5) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=$

$\delta_2^2=\delta_3^2=\delta_4^2=\delta_5^2=-2$ *one has* $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$*, where* $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$*,* $i=1,\,2,\,3,\,4$*,* $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\frac{\pi}{2}$*, and*

$$2<(\delta_1,\,\delta_3)\le 4\,f_{h1}(\frac{\pi}{2},\,\frac{\pi}{2},\,\eta)-2, \tag{8.3.48}$$

$$2<(\delta_3,\,\delta_5)\le 4\,f_{h1}(\frac{\pi}{2},\,\frac{\pi}{2},\,\eta)-2, \tag{8.3.49}$$

and

$$2<(\delta_1,\,\delta_5)\le$$
$$4\,\max\big(f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,0,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,\eta,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,\eta,\eta)\big)-2. \tag{8.3.50}$$

The set $\{\delta_1,\,\delta_3,\,\delta_5\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i=1,\,2,\,3,\,4,\,5$*, contain the axis* l*.*

(BI) There exist four consecutive vertices $A_0,\,A_1,\,A_2$ *of one half of* $\mathcal{M}$ *and two consecutive vertices* $B_0,\,B_1$ *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1*,* δ_2 *and* δ_3 *to lines* (A_0A_1)*,* (A_1A_2) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=-2$*, one has* $(\delta_1,\,\delta_2)=2\cos\alpha$*, where* $\alpha=\angle\, A_0A_1A_2\ne\frac{\pi}{2}$*, and*

$$2<(\delta_1,\,\delta_3)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.51}$$

$$2<(\delta_2,\,\delta_3)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.52}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_1,\,\delta_3)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_3)+2)}{4}-ch\,\frac{\eta}{2}\right)\le$$
$$\frac{\eta}{2}+2g(\alpha,\,\frac{\eta}{4},+\infty). \tag{8.3.53}$$

The set $\{\delta_1,\,\delta_2,\,\delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i=1,\,2$*, and closed half-plane* $\mathcal{H}^+_{\delta_3}$ *contain the axis* l*; the line orthogonal to the axis* l *and containing* A_1 *intersects the line* (B_0B_1)*.*

(BII$_1$*) There exist four consecutive vertices* $A_0,\,A_1,\,A_2,\,A_3$ *of one half of* $\mathcal{M}$ *and two consecutive vertices* $B_0,\,B_1$ *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* $\delta_1,\,\delta_2,\,\delta_3$ *and* δ_4 *to lines* (A_0A_1)*,* (A_1A_2)*,* (A_2A_3) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$*, one has* $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$*, where* $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$*,* $i=1,\,2$*,* $\alpha_1=\alpha_2=\frac{\pi}{2}$*, and*

$$2<(\delta_1,\,\delta_4)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.54}$$

$$2<(\delta_2,\,\delta_4)\le\frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.55}$$

$$2 < (\delta_3,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.56}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_1,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)\le$$
$$\frac{\eta}{2}+2g(\frac{\pi}{2},\,\frac{\eta}{4},+\infty), \tag{8.3.57}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_3,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)\le$$
$$\frac{\eta}{2}+2g(\frac{\pi}{2},\,\frac{\eta}{4},+\infty). \tag{8.3.58}$$

The set $\{\delta_1,\,\delta_2,\,\delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\,\delta_3,\,\delta_4\}$*. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i=1,\,2,\,3$*, and closed half-plane* $\mathcal{H}^+_{\delta_4}$ *contain the axis* l*; both lines orthogonal to the axis* l *and containing* A_1 *and* A_2 *intersect the line* (B_0B_1)*.*

(BII$_2$) There exist three consecutive vertices $A_0,\,A_1,\,A_2$ *of one half of* $\mathcal{M}$ *and three consecutive vertices* $B_0,\,B_1,\,B_2$ *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* $\delta_1,\,\delta_2,\,\delta_3$ *and* δ_4 *to lines* (A_0A_1)*,* (A_1A_2)*,* (B_0B_1) *and* (B_1B_2) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$*, one has* $(\delta_1,\,\delta_2)=2\cos\alpha$*,* $(\delta_3,\,\delta_4)=2\cos\beta$*, where* $\alpha=\angle\,A_0A_1A_2$*,* $\beta=\angle\,B_0B_1B_2$*,* $\alpha=\beta=\frac{\pi}{2}$*, and*

$$2 < (\delta_1,\, \delta_3) \le \frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.59}$$

$$2 < (\delta_2,\, \delta_3) \le \frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.60}$$

$$2 < (\delta_2,\, \delta_4) \le \frac{8}{ch\,\frac{\eta}{2}-1}+6, \tag{8.3.61}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_1,\,\delta_3)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_3)+2)}{4}-ch\,\frac{\eta}{2}\right)\le$$
$$\frac{\eta}{2}+2g(\frac{\pi}{2},\,\frac{\eta}{4},+\infty), \tag{8.3.62}$$

$$Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_3)+2)}{4}-ch\,\frac{\eta}{2}\right)+Ach\,\left(\frac{(ch\,\frac{\eta}{2}-1)((\delta_2,\,\delta_4)+2)}{4}-ch\,\frac{\eta}{2}\right)\le$$
$$\frac{\eta}{2}+2g(\frac{\pi}{2},\,\frac{\eta}{4},+\infty). \tag{8.3.63}$$

The set $\{\delta_1,\,\delta_2,\,\delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for* $\{\delta_2,\,\delta_3,\,\delta_4\}$*. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$*,* $i=1,\,2,\,3,\,4$*, contain the axis* l*; the line orthogonal to the axis* l *and containing* A_1 *intersects* (B_0B_1)*, and the line orthogonal to* l *and containing* B_1 *intersects* (A_1A_2)*.*

(BIII) There exist four consecutive vertices A_0, A_1, A_2, A_3 of one half of $\mathcal{M}$, and two consecutive vertices B_0, B_1 of another half of $\mathcal{M}$ such that for orthogonal vectors δ_1, δ_2, δ_3 and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (B_0B_1) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$ where $\alpha_i=\angle\ A_{i-1}A_iA_{i+1}$, $i=1,\,2$, $\alpha_1=\alpha_2=\frac{\pi}{2}$ and

$$2<(\delta_1,\,\delta_3)\le 4\,f_{h1}(\frac{\pi}{2},\frac{\pi}{2},\eta)-2, \tag{8.3.64}$$

$$2<(\delta_1,\,\delta_4)\le 4\frac{ch\ \frac{\eta}{2}+ch\ (\eta-2\,g(\frac{\pi}{2},\frac{\eta}{4},\frac{\eta}{4}))}{ch\ \frac{\eta}{2}-1}-2, \tag{8.3.65}$$

$$2<(\delta_3,\,\delta_4)\le 4\frac{ch\ \frac{\eta}{2}+ch\ (\eta-2\,g(\frac{\pi}{2},\frac{\eta}{4},\frac{\eta}{4}))}{ch\ \frac{\eta}{2}-1}-2, \tag{8.3.66}$$

$$Ach\left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_1,\,\delta_4)+2)}{4}-ch\ \frac{\eta}{2}\right)+Ach\left(\frac{(ch\ \frac{\eta}{2}-1)((\delta_3,\,\delta_4)+2)}{4}-ch\ \frac{\eta}{2}\right)\le\eta. \tag{8.3.67}$$

The set $\{\delta_1,\,\delta_2,\,\delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for the set $\{\delta_2,\,\delta_3,\,\delta_4\}$. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3,\,4$, contain the axis l; there exists a line orthogonal to the axis l and intersecting the lines (A_1A_2) and (B_0B_1).

Proof. The polygon $\mathcal{M}$ satisfies to one of cases (AII), (AIII), (BII_1), (BII_2), (BIII) of Theorem 8.3.1.

Suppose that $\mathcal{M}$ satisfies the case (AII) of Theorem 8.3.1 and one of angles α_1, α_2 is not equal to $\pi/2$. Suppose $\alpha_2\neq\pi/2$. Then δ_1, δ_2, δ_3 give the case (AI) of Theorem 8.3.2. If $\alpha_1=\alpha_2=\pi/2$, we get the case (AII) of Theorem 8.3.2.

Suppose that $\mathcal{M}$ satisfies the case (AIII) of Theorem 8.3.1. Like above, part of elements $\delta_1,\dots,\delta_5$ gives the case (AI) of Theorem 8.3.2, if at least one of angles $\alpha_i\neq\pi/2$, $i=1,\dots,4$. Otherwise, we get the case (AIII) of Theorem 8.3.2.

For cases (BII_1), (BII_2), (BIII) of Theorem 8.3.1 the proof is the same.

For special values of angles of Theorem 8.3.2, we can calculate maxima in Theorem 8.3.1 with the help of computer.

This finishes the proof of Theorem 8.3.2.

Applying Theorem 8.3.2 to $\eta=\beta=1,4134...$ where β is the root of the equation

$$\max\left(f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,0,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,\eta,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta,\eta,\eta)\right)=$$
$$=\frac{\mathrm{ch}\ \frac{\eta}{2}+\mathrm{ch}\ (\eta-2\,g(\frac{\pi}{2},\frac{\eta}{4},\frac{\eta}{4}))}{\mathrm{ch}\ \frac{\eta}{2}-1}, \tag{8.3.68}$$

we improve Theorem 8.2.2:

Theorem 8.3.3 (about narrow part of restricted hyperbolic convex polygons in a hyperbolic plane). *For any restricted hyperbolic convex polygon $\mathcal{M}$ in a hyperbolic plane there are δ_1, δ_2, $\delta_3\in P(\mathcal{M})$ with $\delta_1^2=\delta_2^2=\delta_3^2=-2$ such that δ_1, δ_2, δ_3 generate the 3-dimensional hyperbolic vector space (defining the hyperbolic plane), they have connected Gram graph and*

$$(\delta_i,\,\delta_j)\le 4\,\frac{ch\ \frac{\beta}{2}+ch\ (\beta-2\,g(\frac{\pi}{2},\frac{\beta}{4},\frac{\beta}{4}))}{ch\ \frac{\beta}{2}-1}<45,4629 \tag{8.3.69}$$

for all $1 \le i, j, \le 3$. *Here* $\beta = 1,4134...$ *is the root of the equation* (8.3.68).

By computer experiments (see Sect. 8.5), we found that for classification of hyperbolically reflective hyperbolic lattices of the rank three it is good to apply Theorem 8.3.1 to $\eta = \eta_0 = 2\,\mathrm{arc\,ch}\,\frac{3}{2}$. Thus, finally we get the result which we later will apply to classification of hyperbolically reflective hyperbolic lattices of the rank three.

Theorem 8.3.4 (about narrow parts of types (I), (II) and (III) of restricted hyperbolic convex polygons on the hyperbolic plane).

We set $\eta_0 = 2\,\mathrm{arc\,ch}\,\frac{3}{2} = 1.92484730023841378999910...$; *equivalently,* $\mathrm{ch}\,\frac{\eta_0}{2} = \frac{3}{2}$*).*

Any infinite restricted hyperbolic convex polygon $\mathcal{M}$ *on the hyperbolic plane has a narrow part of one of types (AI), (AII), (AIII), (BI), (BII*$_1$*),(BII*$_2$*), (BIII) defined below:*

(AI) There exist four consecutive vertices A_0, A_1, A_2, A_3 *of one half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2 *and* δ_3 *to lines* (A_0A_1), (A_1A_2) *and* (A_2A_3) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$, *one has* $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$, *where* $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$, $i = 1,\, 2$, $\alpha_2 \ne \frac{\pi}{2}$, *and*

$$2 < (\delta_1,\, \delta_3) \le r_{h1}(\alpha_1,\, \alpha_2) := 4\, f_{h1}(\alpha_1, \alpha_2, \eta_0) - 2 \tag{8.3.70}$$

where in particular:

$r_{h1}(\frac{\pi}{2},\, \frac{\pi}{2}) = 7$ $\quad r_{h1}(\frac{\pi}{2},\, \frac{\pi}{3}) = 9.412375826...$
$r_{h1}(\frac{\pi}{2},\, \frac{\pi}{4}) = 10.37390342...$ $\quad r_{h1}(\frac{\pi}{2},\, \frac{\pi}{6}) = 11.10113930...$
$r_{h1}(\frac{\pi}{2},\, 0) = 11.70820393...$ $\quad r_{h1}(\frac{\pi}{3},\, \frac{\pi}{3}) = 12.25;$
$r_{h1}(\frac{\pi}{3},\, \frac{\pi}{4}) = 13.37793021...$ $\quad r_{h1}(\frac{\pi}{3},\, \frac{\pi}{6}) = 14.23012096...$
$r_{h1}(\frac{\pi}{3},\, 0) = 14.94097150...$ $\quad r_{h1}(\frac{\pi}{4},\, \frac{\pi}{4}) = 14.57106781...$
$r_{h1}(\frac{\pi}{4},\, \frac{\pi}{6}) = 15.47225159...$ $\quad r_{h1}(\frac{\pi}{4},\, 0) = 16.22381115...$
$r_{h1}(\frac{\pi}{6},\, \frac{\pi}{6}) = 16.41025403...$ $\quad r_{h1}(\frac{\pi}{6},\, 0) = 17.19241152...$
$r_{h1}(0,\, 0) = 18.$

The set $\{\delta_1,\, \delta_2,\, \delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i = 1,\, 2,\, 3$, *contain the axis* l.

(AII) There exist five consecutive vertices A_0, A_1, A_2, A_3 *and* A_4 *of one half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3 *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (A_2A_3) *and* (A_3A_4) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = -2$, *one has* $(\delta_i,\, \delta_{i+1}) = 2\cos\alpha_i$ *where* $\alpha_i = \angle\, A_{i-1}A_iA_{i+1}$, $i = 1,\, 2,\, 3$, $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, *and*

$$2 < (\delta_1,\, \delta_3) \le r_{h1}(\frac{\pi}{2},\, \frac{\pi}{2}) = 7, \tag{8.3.71}$$

$$2 < (\delta_1,\, \delta_4) \le r_{h2}(\alpha_3) := 4\,\max\left(f_{h2}(\frac{\pi}{2}, \frac{\pi}{2}, \alpha_3, \eta_0, 0),\; f_{h2}(\frac{\pi}{2}, \frac{\pi}{2}, \alpha_3, \eta_0, \eta_0)\right) - 2 \tag{8.3.72}$$

where in particular:

$$r_{h2}(\frac{\pi}{2}) = 31.15549442... \quad r_{h2}(\frac{\pi}{3}) = 38.68043607... \quad r_{h2}(\frac{\pi}{4}) = 41.73090517...$$
$$r_{h2}(\frac{\pi}{6}) = 44.05297726... \quad r_{h2}(0) = 46. \tag{8.3.73}$$

The set $\{\delta_1, \delta_3, \delta_4\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i = 1, 2, 3, 4$, *contain the axis* l.

(AIII) There exist six consecutive vertices A_0, A_1, A_2, A_3, A_4 *and* A_5 *of one half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3, δ_4 *and* δ_5 *to lines* (A_0A_1), (A_1A_2), (A_2A_3), (A_3A_4) *and* (A_4A_5) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_4^2 = \delta_5^2 = -2$ *one has* $(\delta_i, \delta_{i+1}) = 2\cos\alpha_i$, *where* $\alpha_i = \angle\ A_{i-1}A_iA_{i+1}$, $i = 1, 2, 3, 4$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{\pi}{2}$, *and*

$$2 < (\delta_1, \delta_3) \le r_{h1}(\frac{\pi}{2}, \frac{\pi}{2}) = 7, \tag{8.3.74}$$

$$2 < (\delta_3, \delta_5) \le r_{h1}(\frac{\pi}{2}, \frac{\pi}{2}) = 7, \tag{8.3.75}$$

and

$$\begin{aligned} &2 < (\delta_1, \delta_5) \le r_{h3} := \\ &4\max\big(f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta_0,0,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta_0,\eta_0,0),\ f_{h3}(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\eta_0,\eta_0,\eta_0)\big) \\ &-2 = 68.1815011826... \end{aligned} \tag{8.3.76}$$

The set $\{\delta_1, \delta_3, \delta_5\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i = 1, 2, 3, 4, 5$, *contain the axis* l.

(BI) There exist four consecutive vertices A_0, A_1, A_2 *of one half of* $\mathcal{M}$ *and two consecutive vertices* B_0, B_1 *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2 *and* δ_3 *to lines* (A_0A_1), (A_1A_2) *and* (B_0B_1) *respectively directed outwards of* $\mathcal{M}$ *and with* $\delta_1^2 = \delta_2^2 = \delta_3^2 = -2$, *one has* $(\delta_1, \delta_2) = 2\cos\alpha$, *where* $\alpha = \angle\ A_0A_1A_2 \ne \frac{\pi}{2}$, *and*

$$2 < (\delta_1, \delta_3) \le 22, \tag{8.3.77}$$

$$2 < (\delta_2, \delta_3) \le 22, \tag{8.3.78}$$

$$\begin{aligned} &Ach\ \left(\frac{(\delta_1, \delta_3)}{8} - \frac{5}{4}\right) + Ach\ \left(\frac{(\delta_2, \delta_3)}{8} - \frac{5}{4}\right) \le \\ &r_{h4}(\alpha) := \frac{\eta_0}{2} + 2g(\alpha, \frac{\eta_0}{4}, +\infty) = \eta_0 - 2\,arc\,sh\,(\frac{1+\cos\alpha}{2} sh\ \frac{\eta_0}{4}) \end{aligned} \tag{8.3.79}$$

where in particular

$r_{h4}(\frac{\pi}{2}) = 1.429914377...$ $\quad r_{h4}(\frac{\pi}{3}) = 1.191398091...$ $\quad r_{h4}(\frac{\pi}{4}) = 1.095286061...$
$r_{h4}(\frac{\pi}{6}) = 1.022736500...$ $\quad r_{h4}(0) = \frac{\eta_0}{2} = 0.962423650...$.

The set $\{\delta_1, \delta_2, \delta_3\}$ *generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. Additionally, open half-planes* $\mathcal{H}^+_{\delta_i}$, $i = 1, 2$, *and closed half-plane* $\mathcal{H}^+_{\delta_3}$ *contain the axis* l; *the line orthogonal to the axis* l *and containing* A_1 *intersects the line* (B_0B_1).

(BII$_1$) There exist four consecutive vertices A_0, A_1, A_2, A_3 *of one half of* $\mathcal{M}$ *and two consecutive vertices* B_0, B_1 *of another half of* $\mathcal{M}$ *such that for orthogonal vectors* δ_1, δ_2, δ_3 *and* δ_4 *to lines* (A_0A_1), (A_1A_2), (A_2A_3) *and* (B_0B_1) *respectively*

directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$, where $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$, $i=1,\,2$, $\alpha_1=\alpha_2=\frac{\pi}{2}$, and

$$2<(\delta_1,\,\delta_4)\le 22, \tag{8.3.80}$$

$$2<(\delta_2,\,\delta_4)\le 22, \tag{8.3.81}$$

$$2<(\delta_3,\,\delta_4)\le 22, \tag{8.3.82}$$

$$Ach\,\left(\frac{(\delta_1,\,\delta_4)}{8}-\frac{5}{4}\right)+Ach\,\left(\frac{(\delta_2,\,\delta_4)}{8}-\frac{5}{4}\right)\le rh_4(\frac{\pi}{2})=1.429914377...\,, \tag{8.3.83}$$

$$Ach\,\left(\frac{(\delta_2,\,\delta_4)}{8}-\frac{5}{4}\right)+Ach\,\left(\frac{(\delta_3,\,\delta_4)}{8}-\frac{5}{4}\right)\le r_{h4}(\frac{\pi}{2})=1.429914377...\,. \tag{8.3.84}$$

The set $\{\delta_1,\,\delta_2,\,\delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for $\{\delta_2,\,\delta_3,\,\delta_4\}$. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3$, and closed half-plane $\mathcal{H}^+_{\delta_4}$ contain the axis l; both lines orthogonal to the axis l and containing A_1 and A_2 intersect the line (B_0B_1).

(BII_2) There exist three consecutive vertices $A_0,\,A_1,\,A_2$ of one half of $\mathcal{M}$ and three consecutive vertices $B_0,\,B_1,\,B_2$ of another half of $\mathcal{M}$ such that for orthogonal vectors $\delta_1,\,\delta_2,\,\delta_3$ and δ_4 to lines (A_0A_1), (A_1A_2), (B_0B_1) and (B_1B_2) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_1,\,\delta_2)=2\cos\alpha$, $(\delta_3,\,\delta_4)=2\cos\beta$, where $\alpha=\angle\, A_0A_1A_2$, $\beta=\angle\, B_0B_1B_2$, $\alpha=\beta=\frac{\pi}{2}$, and

$$2<(\delta_1,\,\delta_3)\le 22, \tag{8.3.85}$$

$$2<(\delta_2,\,\delta_3)\le 22, \tag{8.3.86}$$

$$2<(\delta_2,\,\delta_4)\le 22, \tag{8.3.87}$$

$$Ach\,\left(\frac{(\delta_1,\,\delta_3)}{8}-\frac{5}{4}\right)+Ach\,\left(\frac{(\delta_2,\,\delta_3)}{8}-\frac{5}{4}\right)\le r_{h4}(\frac{\pi}{2})=1.429914377...\,, \tag{8.3.88}$$

$$Ach\,\left(\frac{(\delta_3,\,\delta_2)}{8}-\frac{5}{4}\right)+Ach\,\left(\frac{(\delta_4,\,\delta_2)}{8}-\frac{5}{4}\right)\le r_{h4}(\frac{\pi}{2})=1.429914377...\,. \tag{8.3.89}$$

The set $\{\delta_1,\,\delta_2,\,\delta_3\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for $\{\delta_2,\,\delta_3,\,\delta_4\}$. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i=1,\,2,\,3,\,4$, contain the axis l; the line orthogonal to the axis l and containing A_1 intersects (B_0B_1), and the line orthogonal to l and containing B_1 intersects (A_1A_2).

(BIII) There exist four consecutive vertices $A_0,\,A_1,\,A_2,\,A_3$ of one half of $\mathcal{M}$, and two consecutive vertices $B_0,\,B_1$ of another half of $\mathcal{M}$ such that for orthogonal vectors $\delta_1,\,\delta_2,\,\delta_3$ and δ_4 to lines (A_0A_1), (A_1A_2), (A_2A_3) and (B_0B_1) respectively directed outwards of $\mathcal{M}$ and with $\delta_1^2=\delta_2^2=\delta_3^2=\delta_4^2=-2$, one has $(\delta_i,\,\delta_{i+1})=2\cos\alpha_i$ where $\alpha_i=\angle\, A_{i-1}A_iA_{i+1}$, $i=1,\,2$, $\alpha_1=\alpha_2=\frac{\pi}{2}$ and

$$2<(\delta_1,\,\delta_3)\le r_{h1}(\frac{\pi}{2},\,\frac{\pi}{2})=7, \tag{8.3.90}$$

$$2<(\delta_1,\,\delta_4)\le 4\frac{ch\,\frac{\eta_0}{2}+ch\,(\eta_0-2\,g(\frac{\pi}{2},\frac{\eta_0}{4},\frac{\eta_0}{4}))}{ch\,\frac{\eta_0}{2}-1}-2=31.70820393...\,, \tag{8.3.91}$$

$$2 < (\delta_3,\,\delta_4) \le 4\frac{ch\,\frac{\eta_0}{2} + ch\,(\eta_0 - 2\,g(\frac{\pi}{2},\frac{\eta_0}{4},\frac{\eta_0}{4}))}{ch\,\frac{\eta_0}{2} - 1} - 2 = 31.70820393...\,, \qquad (8.3.92)$$

$$Ach\,\left(\frac{(\delta_1,\,\delta_4)}{8} - \frac{5}{4}\right) + Ach\,\left(\frac{(\delta_3,\,\delta_4)}{8} - \frac{5}{4}\right) \le \eta_0 = 1.9248473002...\,. \qquad (8.3.93)$$

The set $\{\delta_1,\,\delta_2,\,\delta_4\}$ generates the 3-dimensional hyperbolic vector space and has a connected Gram graph. The same is valid for the set $\{\delta_2,\,\delta_3,\,\delta_4\}$. Additionally, open half-planes $\mathcal{H}^+_{\delta_i}$, $i = 1,\,2,\,3,\,4$, contain the axis l; there exists a line orthogonal to the axis l and intersecting the lines (A_1A_2) and (B_0B_1).

8.4. Some general applications.

8.4.1. Hyperbolically reflective hyperbolic lattices over purely real algebraic number fields. Let $\mathbb{K}$ be a purely real algebraic number field of finite degree $[\mathbb{K}:\mathbb{Q}]$ and $\mathbb{O}$ its ring of integers. A *lattice* S over $\mathbb{K}$ is a projective module over $\mathbb{O}$ of a finite rank equipped with a non-degenerate symmetric bilinear form with values in $\mathbb{O}$. A lattice S is called hyperbolic if there exists an embedding $\sigma^{(+)}:\mathbb{K}\to\mathbb{R}$ (it is called the *geometric embedding)* such that the real symmetric bilinear form $S\otimes\mathbb{R}$ is hyperbolic (i.e. it has exactly one positive square), and for all other embeddings $\sigma \ne \sigma^{(+)}$ the real form $S\otimes\mathbb{R}$ is negative definite. Using the geometric embedding, one can define the hyperbolic space $\mathcal{L}(S)$, the automorphism group $O^+(S)$, the reflection group $W(S)\subset O^+(S)$ of the lattice S. Here an automorphism of $O^+(S)$ is called *reflection* if it acts in $\mathcal{L}(S)$ as reflection with respect to a hyperplane. For a hyperbolic lattice S, the group $O^+(S)$ is discrete in $\mathcal{L}(S)$ and has a fundamental domain of finite volume. Let $\mathcal{M}\subset\mathcal{L}(S)$ be a fundamental polyhedron of $W(S)$ and $A(\mathcal{M}) = \{\phi\in O^+(S)\mid \phi(\mathcal{M}) = \mathcal{M}\}$ its group of symmetries. Then we have the semi-direct product $O^+(S) = W \rtimes A(\mathcal{M})$. The lattice S is called *reflective of hyperbolic type (or hyperbolically reflective)* if there exists a subgroup of finite index $A'\subset A(\mathcal{M})$ which keeps invariant a proper subspace $\mathcal{T}\subset\mathcal{L}(S)$ (here "proper" means that $0<\dim\mathcal{T}<\dim\mathcal{L}(S)$). Using standard discrete groups arguments, one can see that the polyhedron $\mathcal{M}$ is then restricted hyperbolic with respect to $\mathcal{T}$. From Theorem 8.2.3 (or 8.3.3) we get (like for usual elliptically reflective hyperbolic lattices, see [N4], [N5, Appendix])

Theorem 8.4.1.1. *For the fixed $N = [\mathbb{K}:\mathbb{Q}]$ and rk $S = \dim S\otimes\mathbb{R} = 3$, the set of hyperbolically reflective hyperbolic lattices S is finite up to multiplication of the form of S by elements of $\mathbb{K}$.*

We remark that the generalization of Theorem 8.2.3 for arbitrary dimension of hyperbolic space (using similar method) permits to prove Theorem 8.4.1.1 for arbitrary rk $S\ge 3$. This gives the partial answer to the problem posed in [N8, Sect. 3] about description of generalized arithmetic reflection groups (see also [N9], [N13]). For the fixed degree $[\mathbb{K}:\mathbb{Q}]$, finiteness of the set of fields $\mathbb{K}$ such that there exists a hyperbolically reflective hyperbolic lattice S over $\mathbb{K}$ and of rank rk $S\ge 3$ follows from results [N13].

8.4.2. Algebraic surfaces with almost finite polyhedral Mori cone. We refer to [N14] about this subject. Let X be a non-singular projective algebraic surface over algebraically closed field. Let $\rho(X) = \mathrm{rk}\ \mathrm{NS}(X)\ge 3$ be the Picard number of X. We denote by $\mathrm{Exc}(X)$ the set of all exceptional (i. e. irreducible and with negative self-intersection) curves of X.

We say that X has *almost finite polyhedral Mori cone* $\overline{NE}(X)$ if (1) and (2) below hold:

(1) There exist finite maxima:

$$\delta_E(X) = \max_{C\in \mathrm{Exc}(X)} -C^2 \quad \text{and} \quad p_E(X) = \max_{C\in \mathrm{Exc}(X)} p_a(C)$$

where $p_a(C)$ is the arithmetic genus of the curve C.

(2) There exists a non-zero $r \in \mathrm{NS}(X)$ such that any extremal ray of $\overline{\mathrm{NE}}(X)$ is either generated by an exceptional curve $E \in \mathrm{Exc}(X)$ or by $c \in \mathrm{NS}(X) \otimes \mathbb{R}$ such that $c^2 = 0$ and $c \cdot r = 0$. We remark that if the canonical class $K \not\equiv 0$, one can always put $r = K$.

By elementary considerations, one can see that the dual cone $\mathrm{NEF}(X)$ to the Mori cone $\overline{NE}(X)$ defines the polyhedron $\mathcal{M} = \mathrm{NEF}(X)/\mathbb{R}_{++} \subset \mathcal{L}(X)$ of restricted hyperbolic type with respect to the hyperplane orthogonal to r, if $r^2 < 0$. If $r^2 > 0$ then $\mathcal{M}$ will be elliptic, and if $r^2 = 0$ then $\mathcal{M}$ will be parabolic. Like in [N14] for elliptic case (when $r^2 > 0$), from Theorems 4.2.3 and 8.2.3 we get

Theorem 8.4.2.1. *For $\rho = 3$, there are constants $N(\rho, \delta_E)$ and $N'(\rho, \delta_E, p_E)$ depending only on (ρ, δ_E) and (ρ, δ_E, p_E) respectively such that for any X with almost finite polyhedral Mori cone and $\rho(X) = \rho$, $\delta_E(X) = \delta_E$ and $p_E(X) = p_E$, there exists an ample effective divisor h such that $h^2 \le N(\rho, \delta_E)$, and if the ground field is $\mathbb{C}$, there exists a very ample divisor h' such that ${h'}^2 \le N'(\rho, \delta_E, p_E)$.*

The generalization of Theorem 8.2.3 to arbitrary dimension (using the same method) permits to prove Theorem 8.4.2.1 for arbitrary $\rho \ge 3$.

8.5. Description of narrow parts of fundamental polygons $\mathcal{M}$ of reflection subgroups $W \subset W(S)$ of hyperbolic type where rk $S = 3$. Application to hyperbolically reflective hyperbolic lattices.

Here we want to make similar calculations for hyperbolic type as we did for elliptic and parabolic types in Sect. 4.3. We use the same notations.

Let S be a primitive hyperbolic lattice of rk $S = 3$ and $W \subset W(S)$ its reflection subgroup of hyperbolic type with a fundamental polygon $\mathcal{M}$ (see Sects. 1.3, 1.4). Remind that it means that W and $P(\mathcal{M})_{\mathrm{pr}}$ have restricted arithmetic type and $P(\mathcal{M})_{\mathrm{pr}}$ has a generalized lattice Weyl vector ρ with $\rho^2 < 0$ (and does not have a generalized lattice Weyl vector ρ with $\rho^2 \ge 0$). It is easy to see that the polygon $\mathcal{M}$ is then restricted hyperbolic. A lattice S having a reflection subgroups $W \subset W(S)$ of hyperbolic type will be here called reflective of hyperbolic type (or hyperbolically reflective). This definition is more general than definition in Section 1.4. Here we apply Theorem 8.3.4 to describe narrow parts of $\mathcal{M}$. Using this description, we shall get a finite set of lattices of the rank three such that any hyperbolically reflective lattice S of the rank three belongs to the set. Let $a(S^*/S)$ be the exponent of the discriminant group S^*/S: i. e. $a(S^*/S)$ is the least natural a such that $aS^*/S = 0$. We shall get an estimate of $a(S^*/S)$.

In Sect. 4.3 we gave description of narrow parts of fundamental polygons $\mathcal{M}$ for elliptic and parabolic types according to Theorems 4.1.8 and 4.2.3. Here we similarly want to describe narrow parts for hyperbolic type according to Theorem 8.3.4. We use the same notations as in Sect. 4.3. We remind that we describe all possible matrices $B = ((r_i, r_j))_{pr}$, $i = 1, \ldots, k$ where $r_i \in S$ are primitive roots of S orthogonal to $\mathcal{M}$, and $\delta_i = 2r_i/\sqrt{-2r_i^2}$, $i = 1, \ldots, k$, define the narrow part of

$\mathcal{M}$ described in Theorem 8.3.4. For the matrix B we calculate the invariant $a(B)$ which is equal to $a(K^*/K)$ for the lattice K defined by B, the invariant $a_1(B)$ which is equal to the product of all different odd prime divisors of $a(B)$, and the invariant $a_2(B)$ which is equal to the greatest odd prime divisor of $a(B)$.

8.5.1. Matrices B of narrow parts of the type (AI1). This is a particular case of the type (AI) in Theorem 8.3.4 when additionally $\alpha_1 \neq \pi/2$. This case is similar to Sect. 4.3.1. We should only change the condition (4.3.1.2) by the condition

$$\begin{gathered} 1 \leq \alpha_{12} = \alpha_{21} \leq 4, \quad \alpha_{12} \leq \alpha_{23} = \alpha_{32} \leq 4, \\ 4 < \alpha_{13} = \alpha_{31} \leq r_{h1}\big(\arccos \frac{\sqrt{\alpha_{12}}}{2}, \ \arccos \frac{\sqrt{\alpha_{23}}}{2}\big)^2. \end{gathered} \tag{8.5.1.1}$$

In Sect. 10, Appendix 2, we give the Program 13: funda11.gen which enumerates all matrices B of this case. For each B it calculates $a(B)$, $a_1(B)$ and $a_2(B)$, and finds the number $nAI1$ of these matrices and

$$aAI1 = \max_B a(B), \ aAI1_1 = \max_B a_1(B), \ aAI1_2 = \max_B a_2(B). \tag{8.5.1.2}$$

Calculation using this program gives

$$nAI1 = 332, \ aAI1 = 5832, \ aAI1_1 = 759, \ aAI1_2 = 181. \tag{8.5.1.3}$$

8.5.2. Matrices B of narrow parts of the type (AI0). It is a particular case of Type (AI) of Theorem 8.3.4 when additionally the angle $\alpha_1 = \pi/2$. This case is similar to Sect. 4.3.2. We should only change the condition (4.3.2.2) by the condition

$$1 \leq \alpha_{23} = \alpha_{32} \leq 4, \quad 4 < \alpha_{13} = \alpha_{31} \leq r_{h1}\big(\frac{\pi}{2}, \ \arccos \frac{\sqrt{\alpha_{23}}}{2}\big)^2. \tag{8.5.2.1}$$

In Sect. 10, Appendix 2, we give the Program 14: funda10.gen which enumerates all matrices B of this case. For B it calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to defined above numbers $nAI0$ and the numbers

$$aAI0 = \max_B a(B), \ aAI0_1 = \max_B a_1(B), \ aAI0_2 = \max_B a_2(B). \tag{8.5.2.2}$$

Calculation using this program gives

$$nAI0 = 4788, \ aAI0 = 154568, \ aAI0_1 = 21063, \ aAI0_2 = 139. \tag{8.5.2.3}$$

8.5.3. Matrices B of narrow parts of the type (AII1). It is a particular case of Type (AII) of Theorem 8.3.4 when additionally the angle $\alpha_3 \neq \pi/2$. This case is similar to Sect. 4.3.3. We should only change the condition (4.3.3.2) by the condition

$$\begin{gathered} 1 \leq \alpha_{34} = \alpha_{43} \leq 4, \quad 4 < \alpha_{13} = \alpha_{31} \leq r_{h1}(\pi/2, \pi/2)^2 = 7^2, \\ 4 \leq \alpha_{14} = \alpha_{41} \leq r_{h2}\big(\arccos \frac{\sqrt{\alpha_{34}}}{2}\big)^2, \\ r_{h1}\big(\frac{\pi}{2}, \ \arccos \frac{\sqrt{\alpha_{34}}}{2}\big)^2 < \alpha_{24} = \alpha_{42}. \end{gathered} \tag{8.5.3.1}$$

In Sect. 10, Appendix 2, we give the Program 15: funda21.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nAII1$, $aAII1$, $aAII1_1$, $aAII1_2$. It gives

$$nAII1 = 33488, \ aAII1 = 220116, \ aAII1_1 = 55029, \ aAII1_2 = 2113. \tag{8.5.3.2}$$

8.5.4. Matrices B of narrow parts of the type (AII0). It is a particular case of Type (AII) of Theorem 8.3.4 when additionally the angle $\alpha_3 = \pi/2$. This case is similar to Sect. 4.3.4. We should only change the condition (4.3.4.2) by the condition

$$4 < \alpha_{13} = \alpha_{31} \le 7^2, \quad 4 < \alpha_{14} = \alpha_{41} \le r_{h2}(\frac{\pi}{2})^2 = (31.15549442...)^2,$$
$$\alpha_{13} \le \alpha_{24} = \alpha_{42}. \tag{8.5.4.1}$$

In Sect. 10, Appendix 2, we give the Program 16: funda20.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nAII0$, $aAII0$, $aAII0_1$, $aAII0_2$. It gives

$$nAII0 = 2321854,\ aAII0 = 11430900,\ aAII0_1 = 2834349,\ aAII0_2 = 977. \tag{8.5.4.2}$$

8.5.5. Matrices B of narrow parts of the type (AIII). This case is similar to Sect. 4.3.5. We should only change the condition (4.3.5.2) by the condition

$$4 < \alpha_{13} = \alpha_{31} \le 7^2, \quad \alpha_{31} \le \alpha_{35} = \alpha_{53} \le 7^2,$$
$$4 < \alpha_{15} = \alpha_{51} \le (r_{h3})^2 = (68.1815011826...)^2,$$
$$r_{h2}(\frac{\pi}{2})^2 = (31.15549442...)^2 < \alpha_{14} = \alpha_{41},\ r_{h2}(\frac{\pi}{2})^2 = (31.15549442...)^2 < \alpha_{25} = \alpha_{52}. \tag{8.5.5.1}$$

In Sect. 10, Appendix 2, we give Program 17: funda3.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $aAIII$, $aAIII_1$, $aAIII_2$. It gives

$$nAIII = 802291,\ aAIII = 3150000,\ aAIII_1 = 219453,\ aAIII_2 = 4561. \tag{8.5.5.2}$$

8.5.6. Matrices B of narrow parts of the type (BI). This case is similar to Sect. 4.3.1. We should only change the condition (4.3.1.1) by the condition

$$1 \le \alpha_{12} = \alpha_{21} \le 4,$$
$$4 < \alpha_{13} = \alpha_{31} \le 22^2, \quad \alpha_{13} \le \alpha_{23} = \alpha_{32} \le 22^2,$$
$$\mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{13}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{23}}}{8} - \frac{5}{4}\right) \le r_{h4}(\arccos\frac{\sqrt{\alpha_{12}}}{2}). \tag{8.5.6.1}$$

In Sect. 10, Appendix 2, we give Program 18: fundb1.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers nBI, aBI, aBI_1, aBI_2. It gives

$$nBI = 24329,\ aBI = 1159692,\ aBI_1 = 289923,\ aBI_2 = 1429. \tag{8.5.6.2}$$

8.5.7. Matrices B of narrow parts of the type (BII_1). This case is similar to Sect. 4.3.3. We should only change the condition (4.3.3.2) by the condition

$$4 < \alpha_{14} = \alpha_{41} \le 22^2,\ 4 < \alpha_{24} = \alpha_{42} \le 22^2,\ \alpha_{14} \le \alpha_{34} = \alpha_{43} \le 22^2,$$
$$\mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{14}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{24}}}{8} - \frac{5}{4}\right) \le r_{h4}\left(\frac{\pi}{2}\right) = 1.429914377...\ ,$$
$$\mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{24}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{34}}}{8} - \frac{5}{4}\right) \le r_{h4}\left(\frac{\pi}{2}\right) = 1.429914377...\ ,$$
$$4 < \alpha_{13} = \alpha_{31}. \tag{8.5.7.1}$$

The natural numbers α_{ij} satisfy conditions (4.3.3.3) and (4.3.3.4). We denote

$$u = \sqrt{\alpha_{13}\alpha_{14}\alpha_{34}}. \tag{8.5.7.2}$$

We can rewrite the conditions (4.3.3.3) and (4.3.3.4) in the form

$$\begin{aligned}&(\alpha_{24}-4)u^2 - 4\alpha_{14}\alpha_{34}u + 4\alpha_{14}\alpha_{34}(4-\alpha_{14}-\alpha_{34}-\alpha_{24}) = 0,\\ &u \in \mathbb{N} \ \text{ and } \ \alpha_{13} = \frac{u^2}{\alpha_{14}\alpha_{34}} \in \mathbb{N}.\end{aligned} \tag{8.5.7.3}$$

It follows,

$$u = \frac{2\alpha_{14}\alpha_{34} + 2\sqrt{D}}{\alpha_{24}-4} \tag{8.5.7.4}$$

where

$$D = \alpha_{14}\alpha_{34}(\alpha_{14}\alpha_{34} + (\alpha_{24}-4)(\alpha_{14}+\alpha_{24}+\alpha_{34}-4)) \tag{8.5.7.5}$$

is a perfect square. All other considerations are the same as in Sect. 4.3.3. In Sect. 10, Appendix 2, we give the Program 19: fundb21.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nBII1$, $aBII1$, $aBII1_1$, $aBII1_2$. It gives

$$nBII1 = 1291199,\ aBII1 = 16144800,\ aBII1_1 = 1248555,\ aBII1_2 = 487. \tag{8.5.7.6}$$

8.5.8. Matrices B of narrow parts of the type (BII_2). For this case $k=4$ and δ_1, δ_2, δ_3, δ_4 generate the 3-dimensional hyperbolic vector space and any three of them give a bases of the space. We get that the matrix $\mathcal{A}$ is a symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 4 & 0 & \alpha_{13} & \alpha_{14}\\ 0 & 4 & \alpha_{23} & \alpha_{24}\\ \alpha_{31} & \alpha_{32} & 4 & 0\\ \alpha_{41} & \alpha_{42} & 0 & 4\end{pmatrix} \tag{8.5.8.1}$$

with integral non-negative coefficients where

$$\begin{aligned}&4 < \alpha_{13} = \alpha_{31} \le 22^2,\ 4 < \alpha_{23} = \alpha_{32} \le 22^2,\ \alpha_{13} \le \alpha_{24} = \alpha_{42} \le 22^2,\\ &\mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{13}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{23}}}{8} - \frac{5}{4}\right) \le r_{h4}\left(\frac{\pi}{2}\right) = 1.429914377...\ ,\\ &\mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{23}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\ \left(\frac{\sqrt{\alpha_{24}}}{8} - \frac{5}{4}\right) \le r_{h4}\left(\frac{\pi}{2}\right) = 1.429914377...\ ,\\ &4 < \alpha_{14} = \alpha_{41}.\end{aligned} \tag{8.5.8.2}$$

Here we use the conditions (BII_2) of Theorem 8.3.4 and add some inequalities to avoid symmetric cases. We also have

$$det(\Gamma) = \alpha_{14}(\alpha_{23}-4) - 2\sqrt{\alpha_{14}\alpha_{13}\alpha_{23}\alpha_{24}} + (\alpha_{13}-4)(\alpha_{24}-4) - 4\alpha_{23} = 0, \tag{8.5.8.3}$$

where

$$\alpha_{14}\alpha_{13}\alpha_{23}\alpha_{24} \text{ is a perfect square.} \tag{8.5.8.4}$$

We denote

$$u = \sqrt{\alpha_{14}\alpha_{13}\alpha_{23}\alpha_{24}}. \tag{8.5.8.5}$$

The conditions (8.5.8.3) and (8.5.8.4) are equivalent to

$$u^2\,(\alpha_{23}-4) - u\,2\alpha_{13}\alpha_{23}\alpha_{24} + \alpha_{13}\alpha_{23}\alpha_{24}\big((\alpha_{13}-4)(\alpha_{24}-4)-4\alpha_{23}\big) = 0,$$
$$u \in \mathbb{N} \text{ and } \alpha_{14} = \frac{u^2}{\alpha_{13}\alpha_{23}\alpha_{24}} \in \mathbb{N}. \tag{8.5.8.6}$$

It follows

$$u = \frac{\alpha_{13}\alpha_{23}\alpha_{24} \pm \sqrt{D}}{\alpha_{23}-4} \tag{8.5.8.7}$$

where

$$D = \alpha_{13}\alpha_{23}\alpha_{24}\Big(\alpha_{13}\alpha_{23}\alpha_{24} - (\alpha_{23}-4)\big((\alpha_{13}-4)(\alpha_{24}-4)-4\alpha_{23}\big)\Big) \tag{8.5.8.8}$$

is a perfect square. We can easily enumerate the finite set of all possible matrices $\mathcal{A}$ satisfying these conditions. For each $\mathcal{A}$ we find all symmetrizable generalized Cartan matrices

$$A = \begin{pmatrix} -2 & 0 & a_{13} & a_{14} \\ 0 & -2 & a_{23} & a_{24} \\ a_{31} & a_{32} & -2 & 0 \\ a_{41} & a_{42} & 0 & -2 \end{pmatrix} \tag{8.5.8.9}$$

using relations

$$a_{13}a_{31} = \alpha_{13},\ a_{14}a_{41} = \alpha_{14},\ a_{23}a_{32} = \alpha_{23},\ a_{24}a_{42} = \alpha_{24},$$
$$a_{13}a_{32}a_{24}a_{41} = \sqrt{\alpha_{13}\alpha_{23}\alpha_{24}\alpha_{14}}. \tag{8.5.8.10}$$

For the diagonal matrix

$$\Lambda = \mathrm{diag}(a_{13}a_{14}a_{32},\ a_{31}a_{14}a_{23},\ a_{31}a_{14}a_{32},\ a_{13}a_{41}a_{32}), \tag{8.5.8.11}$$

the matrix $A\Lambda$ is symmetric. Thus, finally we get that

$$B = (A\Lambda)_{\mathrm{pr}}. \tag{8.5.8.12}$$

In Sect. 10, Appendix 2, we give the Program 20: fundb22.gen which uses this algorithm to enumerate all the matrices B. For each of them it calculates the invariants $a(B)$, $a_1(B)$, $a_2(B)$ and finds the number $nBII2$ of all the matrices B, and similar to above numbers

$$aBII2 = \max_B a(B),\ aBII2_1 = \max_B a_1(B),\ aBII2_2 = \max_B a_2(B). \tag{8.5.8.13}$$

Calculation using this program gives

$$nBII2 = 929616,\ aBII2 = 3873744,\ aBII2_1 = 255255,\ aBII2_2 = 487. \tag{8.5.8.14}$$

8.5.9. Matrices B of narrow parts of the type (BIII). This case is similar to Sect. 4.3.3. We should only change the condition (4.3.3.2) by the condition

$$\begin{aligned}
&4 < \alpha_{13} = \alpha_{31} \le r_{h1}(\frac{\pi}{2}, \frac{\pi}{2})^2 = 7^2,\\
&4 < \alpha_{14} = \alpha_{41} \le (31.70820393...)^2, \quad \alpha_{14} \le \alpha_{34} = \alpha_{43} \le (31.70820393...)^2,\\
&\mathrm{Ach}\left(\frac{\sqrt{\alpha_{14}}}{8} - \frac{5}{4}\right) + \mathrm{Ach}\left(\frac{\sqrt{\alpha_{34}}}{8} - \frac{5}{4}\right) \le \eta_0 = 1.9248473002... \,.
\end{aligned} \tag{8.5.9.1}$$

In Sect. 10, Appendix 2, we give the Program 21: fundb3.gen which enumerates all matrices B of this case, calculates $a(B)$, $a_1(B)$, $a_2(B)$ and similar to above numbers $nBIII$, $aBIII$, $aBIII_1$, $aBIII_2$. It gives

$$nBIII = 1878630,\ aBIII = 7013160,\ aBIII_1 = 1627563,\ aBIII_2 = 2851. \tag{8.5.9.2}$$

8.5.10. The global estimate of invariants of primitive hyperbolically reflective hyperbolic lattices of the rank 3. Like in Sect. 4.3.6, we get some global estimates. For a lattice L we denote by $a(L^*/L)$ the exponent of the discriminant group L^*/L, we denote by $a_1(L^*/L)$ the product of all different odd prime divisors of $a(L^*/L)$, and we denote by $a_2(L^*/L)$ the greatest prime divisor of $a(L^*/L)$.

Using Proposition 4.3.1 and calculations (8.5.1.3), (8.5.2.3), (8.5.3.2), (8.5.4.2), (8.5.5.2),(8.5.6.2), (8.5.7.6), (8.5.8.13), (8.5.9.2), we get

Theorem 8.5.10.1. *For any primitive hyperbolically reflective hyperbolic lattice S of rk $S = 3$ we have estimates:*

$$a(S^*/S) \le 8(16144800)^2, \quad a_1(S^*/S) \le 2834349, \quad a_2(S^*/S) \le 4561.$$

Since $\det(S) \le a(S^*/S)^2$ and number of lattices with the fixed rank and determinant is finite (e. g. see [C]), Theorem 8.5.10.1 gives a finite set of lattices which contains all reflective lattices S of the rank three.

The estimates of Theorem 8.5.10.1 are very preliminary. They can be significantly improved using similar calculations and Basic Theorem 7.1 which we prove in the next section.

9. Classification of hyperbolically reflective elementary hyperbolic lattices of the rank 3: Proofs

We are now ready to prove classification results of Sect. 7. The proofs are very similar to Sect. 5 but their details are more complicated and require much more calculations.

9.1. Proof of Basic Theorem 7.1.

Below (see the proof of Theorem 9.1.1) we will show that if a hyperbolic lattice S of the rank 3 is hyperbolically reflective (i. e. the reflection group $W(S)$ has hyperbolic type), then its invariant h is equal to 0 or 2. Table 3 contains all main hyperbolic lattices of the rank 3 which have $h \le 1$ and the determinant $d \le 100000$. Using Theorem 3.2.1 and Program 3: refh3 (see Sect. 6, Appendix 1), we can similarly find all main hyperbolic lattices of the rank 3 with $h = 2$ and $d \le 100000$. (One should only replace $h \le 1$ by $h = 2$ in two places of Program 3.) The result of these calculations is given in Table 6. It contains 259 lattices. The maximal $d = 29526$. Conjecturally, Tables 3 and 6 contain all main hyperbolic lattices or the rank 3 with $h \le 2$ (without the additional condition $d \le 100000$). To avoid this conjecture, we apply results of Sect. 8.

We prove analog of Theorem 5.1.1.

Theorem 9.1.1. *Any hyperbolically reflective main hyperbolic lattice of the rank 3 belongs either to the list of lattices of Table 3 with $h = 0$ or to the list of lattices of Table 6 (all of them have $h = 2$). We remind that the tables 3 and 6 contain all main hyperbolic lattices S of the rank 3 with $h = hnr(S) \le 2$ and $d = \det(S) \le 100000$.*

Proof. Assume that a hyperbolic lattice S of the rank 3 is hyperbolically reflective, and $\mathcal{M} \subset \mathcal{L}(S)$ is a fundamental polygon for the reflection group $W(S)$, and $A(\mathcal{M})$ is its group of symmetries. Let us show that then either $h = 0$ or 2. Remind that h is the number of conjugacy classes of central symmetries from $A(\mathcal{M})$. Suppose that ρ_1 and ρ_2 are two primitive generalized lattice Weyl vectors for $A(\mathcal{M})$ such that $\rho_1 \ne \pm\rho_2$. Then a subgroup $G \subset A(\mathcal{M})$ of finite index acts trivially on the sublattice $[\rho_1,\ \rho_2]$ of the rank 2 generated by ρ_1 and ρ_2, and any non-zero element of $[\rho_1,\ \rho_2]$ is a generalized lattice Weyl vector. All of them should have negative square since S is hyperbolically reflective. It then follows that $[\rho_1,\ \rho_2]$ is negative definite. Its orthogonal complement is a one-dimensional hyperbolic lattice generated by an element f with $f^2 > 0$. The f is then also invariant with respect to a subgroup of finite index of $A(\mathcal{M})$ and is a generalized lattice Weyl vector. It is impossible since S is hyperbolically reflective. Thus $\rho_1 = \pm\rho_2$. It follows that a primitive generalized lattice Weyl vector ρ is unique up to replacing by $-\rho$. If $g \in A(\mathcal{M})$, then $g(\rho)$ is also a generalized lattice Weyl vector of S. Thus, $g(\rho) = \pm\rho$. Let l be the line of $\mathcal{L}(S)$ which is orthogonal to ρ. The group $A(\mathcal{M})$ preserves this line and acts as a discrete group of motions of the line. In particular, the center of any central symmetry of $A(\mathcal{M})$ belongs to the line l. The group $A(\mathcal{M})$ does not have reflections and is infinite. It follows that its action on the line l has only trivial kernel. A non-trivial element of $A(\mathcal{M})$ is a central symmetry if and only if it has a fixed point on the line l (it is a central symmetry of the line l). Any discrete group of motions of a line either has no central symmetries or is generated by two central symmetries which give all two different conjugacy classes of central symmetries of the group. Thus, the same is valid for $A(\mathcal{M})$, and the invariant h is equal to 0 or 2.

Now, to prove Theorem 9.1.1, we should show that the determinant $d \le 100000$ for hyperbolically reflective main hyperbolic lattices of the rank three.

Let $D \subset l$ be a fundamental interval for the action of $A(\mathcal{M})$ in l and C_D is its orthogonal to l cylinder. The set $C_D \cap \mathcal{M}$ is a fundamental domain for the group $O^+(S)$. By the theory of arithmetic groups or by the theory of automorphism groups of integral quadratic forms (e. g. see [R1]), the domain $C_D \cap \mathcal{M}$ is a finite polygon. It then follows that the locally finite polygon $\mathcal{M}$ is restricted hyperbolic relative to the line l which is its axis.

Now we apply to S and $\mathcal{M}$ results of Sect. 8.5. Depending on the type of the narrow part of $\mathcal{M}$, these results give estimates on d. By Proposition 4.3.1, $d \le 2a_1$ where $a_1 = \max_B a_1(B)$ and B runs through all matrices B describing possible narrow parts of the corresponding type. They are given in Sect. 8.5. Unfortunately, for all types of narrow parts (except for the types (AI1) and (A10)), the invariant $a_1 > 50000$, and we don't get the desirable estimate (≤ 100000) for d.

In Sect. 5.1 we had the same problem for the type (II0). To overcome this problem, we had considered matrices B only for main hyperbolic lattices (with square-free d). Using the matrices B, we had calculated the invariants $(d,\,\eta,\,h)$ of S and found that if $h\le 1$, than $d\le 100000$. Actually, for the type (II0) we found all triplets $(d,\,\eta,\,h)$ with $h\le 1$. There were 132 the triplets, and all of them were contained in Table 3.

Here we have to do the same for all types (AI1), (AI0), (AII1), (AII0), (AIII), (BI), (BII_1), (BII_2), (BIII) of narrow parts of $\mathcal{M}$.

The following elementary Lemma is very useful. Actually, we had used its statements (a) and (d) without proof (since they are very simple).

Lemma 9.1.2. *Let S be a main hyperbolic lattice of the rank three with determinant d (it is square-free), and p denote an odd prime $p|d$. We have:*

(a) r^2 is square-free for any primitive root $r\in S$. If odd prime $p|r^2$, then $p|d$.

(b) Let $r_1,\,r_2$ are two primitive roots of S such that $r_2\neq\lambda r_1$, $\lambda\in\mathbb{Q}$, and $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=0,\ 2$ or 3. Then g.c.d$(r_1^2,\,r_2^2)\le 2$. If $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=0$, then $S\otimes\mathbb{Z}_p\cong\langle r_1^2\rangle\oplus\langle r_2^2\rangle\oplus\langle d/(r_1^2r_2^2)\rangle$. If $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=2$, then $S\otimes\mathbb{Z}_p\cong\langle -1\rangle\oplus\langle -1\rangle\oplus\langle d\rangle$. If $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=3$, then $3|d$ and $S\otimes\mathbb{Z}_p\cong\langle -2\rangle\oplus\langle -6\rangle\oplus\langle d/3\rangle$.

(c) Let $r_1,\,r_2\in P(\mathcal{M})$ are primitive roots and $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=1$. Then g.c.d$(r_1^2,\,r_2^2)$ ≤ 2. Moreover $d\equiv 2\mod 3$ and $S\otimes\mathbb{Z}_p\cong\langle -2\rangle\oplus\langle -6\rangle\oplus\langle 3d\rangle$.

(d) Assume that S represents 0. Then $S\cong U\oplus\langle -d\rangle$. In particular, this is true if there are two different primitive roots $r_1,\,r_2\in P(\mathcal{M})$ such that $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=4$.

Proof. (a). Assume that r is a primitive root of S and K is its orthogonal complement. Then either $S=\mathbb{Z}r\oplus K$ or $S=[r,K,(r+k)/2]$ where $k\in K$ is a primitive element. Since the determinant of S is square-free, it follows that $p^2\nmid r^2$ for any odd prime p, and $2^2\nmid r^2$ if $S=\mathbb{Z}r\oplus K$. Suppose that $S=[r,K,(r+k)/2]$ where $k\in K$ is a primitive element. Suppose that $4|r^2$. Since $((r+k)/2)^2=(r^2+k^2)/4\in\mathbb{Z}$, it follows $4|k^2$. Let $k,\,s$ be a bases in K. We have $(k,s)/2=((r+k)/2,\,s)\in\mathbb{Z}$ and $2|(k,s)$. It follows $4|det(K)=k^2s^2-(k,s)^2$ and $16|det(\mathbb{Z}r\oplus K)=r^2det(K)$. Since $[S:\mathbb{Z}r\oplus K]=2$, we have $det(S)=det(\mathbb{Z}r\oplus K)/4$ and $4|det(S)$. We get the contradiction. Thus $2^2\nmid r^2$, and r^2 is square-free for any primitive root $r\in S$.

(b) $4(r_1,\,r_2)^2/(r_1^2r_2^2)\in\mathbb{Z}$ because r_1 and r_2 are primitive roots and $r_1^2|2(r_1,r_2)$, $r_2^2|2(r_1,r_2)$.

Suppose that $(r_1,\,r_2)=0$ and $\mathbb{Z}h$ is the orthogonal complement to $[r_1,r_2]$. Since r_1 and r_2 define reflections of S, we have $S\otimes\mathbb{Z}_p=\mathbb{Z}_pr_1\oplus\mathbb{Z}_pr_2\oplus\mathbb{Z}_ph$ for any odd prime p. It follows that p can divide only one of three numbers r_1^2, r_2^2, h^2. Since r_1^2 and r_2^2 are square-free, it follows g.c.d$(r_1^2,\,r_2^2)\le 2$.

Suppose that $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=2$. Then we have (up to changing numeration and replacing r_1 by $-r_1$) that $r_1^2=-k$, $r_2^2=-2k$ where $k=(r_1,\,r_2)>0$. Then $s_{r_1}(r_2)=r_2'$ is a primitive root such that $(r_2',\,r_2)=0$ and $(r_2')^2=r_2^2=-2k$. By consideration above, g.c.d$(r_2^2,\,r_2^2)\le 2$, $k=1$ and $r_2^2=-2$. It follows that g.c.d$(r_1^2,r_2^2)\le 2$. We have $r_1^2=-1$, $r_2^2=-2$, $(r_1,\,r_2)=1$. Let $r_1'=s_{r_2}(r_1)$. Then $(r_1',\,r_1)=0$ and $S\otimes\mathbb{Z}_p\cong\langle -1\rangle\oplus\langle -1\rangle\oplus\langle d\rangle$.

Suppose that $\frac{4(r_1,\,r_2)^2}{r_1^2r_2^2}=3$. Then we have (up to changing numeration and replacing r_1 by $-r_1$) $r_1^2=-2k$, $r_2^2=-6k$, $(r_1,\,r_2)=3k$ where $k\in\mathbb{N}$. Let $r_1'=$

$s_{r_2}(r_1)$, $r_2' = s_{r_1}(r_2)$. Then $(r_1', r_2') = 0$, and, as above, we get g.c.d$(r_1^2, r_2^2) \le 2$. Actually, we get $k = 1$, $r_1^2 = -2$, $r_2^2 = -6$, $(r_1, r_2) = 3$. Moreover, we get $3|d$ and $S \otimes \mathbb{Z}_p \cong \langle -2\rangle \oplus \langle -6\rangle \oplus \langle d/3\rangle$.

(c) We have $r_1^2 = r_2^2 = -2k$ and $(r_1, r_2) = k \in \mathbb{N}$ where k is odd and square-free. Suppose that $v = ar_1 + br_2 \in S$ where $a, b \in \mathbb{Q}$. Since $r_1^2|2(r_1, v)$, $r_2^2|2(r_2, v)$ and $v^2 \in \mathbb{Z}$, simple calculations show that either $v = 0$ or $3|k$ and $v = (r_1 - r_2)/3$ modulo $\mathbb{Z}r_1 + \mathbb{Z}r_2$. It follows that either r_1 and r_2 generate a primitive sublattice $[r_1, r_2]$ of S, or $3|k$ and $[r_1, r_2, (r_1 - r_2)/3]$ is a primitive sublattice of S. Let us consider the first case and suppose that r_1, r_2, u is a bases of S. Since $-2k = r_1^2|2(r_1, u)$ and $-2k = r_2^2|2(r_2, u)$, we then get that $k^2|det(S)$. It follows that $k = 1$ and g.c.d$(r_1^2, r_2^2) \le 2$. Suppose that $3|k$ and $[r_1, r_2, (r_1 - r_2)/3]$ is a primitive sublattice of S. Like above, we then get $k = 3$. Then $r_3 = (r_1 - r_2)/3$ is a root of S because $r_3^2 = -2$. We have $(r_1, r_3) = -3$ and $(r_2, r_3) = 3$. It is impossible because $r_1, r_2 \in P(\mathcal{M})$ are orthogonal to sides of $\mathcal{M}$ defining a vertex of $\mathcal{M}$ with the angle $\pi/3$. But the line orthogonal to the root r_3 contains the same vertex and divides the angle. Thus, we have proved that $k = 1$ and $r_1^2 = r_2^2 = -2$, $(r_1, r_2) = 1$. Moreover, the sublattice $[r_1, r_2]$ is primitive in S. Let $\mathbb{Z}h$ be the orthogonal complement to $[r_1, r_2]$. Suppose that $S = [r_1, r_2] \oplus \mathbb{Z}h$. Then $r_3 = (r_1 - r_2)$ is a root of S such that $(r_3, r_1) = -3$ and $(r_3, r_2) = 3$. Like above, we get a contradiction. Thus, the lattice S is bigger than its sublattice of finite index $[r_1, r_2] \oplus \mathbb{Z}h$. There are rational (not all integral) a, b, c such that $v = ar_1 + br_2 + ch \in S$. Simple calculations show that modulo $[r_1, r_2] \oplus \mathbb{Z}h$ the element $v = (r_1 - r_2)/3 \pm h/3$ where $h^2 = n \equiv 6 \mod 9$. It follows that $d = n/3 \equiv 2 \mod 3$. Simple calculations show that $S \otimes \mathbb{Z}_p \cong \langle -2\rangle \oplus \langle -6\rangle \oplus \langle d/3\rangle$.

(d) Suppose that S represents zero. Then for any prime p, the lattice $S \otimes \mathbb{Z}_p$ represents zero, and $S \otimes \mathbb{Z}_p \cong U \otimes \mathbb{Z}_p \oplus \langle -d\rangle$ because d is square-free. It follows that S has the same genus as $U \oplus \langle -d\rangle$. By Proposition 2.2.4, $S \cong U \oplus \langle -d\rangle$.

Suppose that there are two primitive roots $r_1, r_2 \in P(\mathcal{M})$ such that $\frac{4(r_1, r_2)^2}{(r_1^2 r_2^2)} = 4$. Then we have: either $r_1^2 = r_2^2 = -k$, $(r_1, r_2) = k > 0$ and $(r_1 + r_2)^2 = 0$ or $r_1^2 = -k$, $r_2^2 = -4k$, $(r_1, r_2) = 2k > 0$ and $(2r_1 + r_2)^2 = 0$. In both cases S represents 0.

This finishes the proof of the lemma.

Now suppose that S is a hyperbolically reflective main hyperbolic lattice of the rank 3. Let $r_1, \dots, r_k \in P(\mathcal{M})$ are primitive roots defining a narrow part of $\mathcal{M}$ (from Sect. 8.5) and $\Gamma = ((r_i, r_j))$, $1 \le i, j \le k$. In Sect. 8.5 we described possible matrices $B = \Gamma_{\mathrm{pr}}$ for all hyperbolically reflective lattices S of the rank 3. Here we add additional conditions which follow from the fact that S is a main hyperbolic lattice, and describe all possible matrices $\Gamma = \lambda B$, $\lambda \in \mathbb{N}$, for these lattices. We denote by G_0 a lattice with the matrix B (it is defined by the matrix B modulo its kernel). We denote $d_1 = \det(G_0)$, and d_2 the product of all different odd prime $p|d_1$ such that $\nu_p(d_1) \equiv 0 \mod 2$.

By Lemma 9.1.2, we an add to conditions of Sect. 8.5 additional conditions (1) — (10) below:

(1) b_{ii} is square-free; if odd prime $p|b_{ii}$, then $\nu_p(d_1) \equiv 1 \mod 2$, $1 \le i \le k$; if $2|(b_{11}\cdots b_{kk})$ and g.c.d$(2, b_{11}, b_{22}, ..., b_{kk}) = 1$, then $\nu_2(d_1) \equiv 0 \mod 2$.

(2) If $\alpha_{ij} \le 3$, then g.c.d$(b_{ii}, b_{jj}) \le 2$, $1 \le i < j \le k$.

(3) $\lambda|2d_2$; g.c.d$(\lambda, b_{11}b_{22}\cdots b_{kk}) = 1$; $\lambda = 1$ or 2 if there exists $1 \le i < j \le k$ such that $\alpha_{ij} \le 3$.

(4) The invariant $d=\det(S)$ is the product of all different prime p such that $\nu_p(d_1\lambda)\equiv 1 \mod 2$.

(5) If $\alpha_{ij}=0$, then the invariant η of S is defined by

$$(-1)^{\eta_p}=\begin{cases}\left(\frac{\lambda b_{ii}/p}{p}\right) & \text{if } p|b_{ii},\\ \left(\frac{\lambda b_{jj}/p}{p}\right) & \text{if } p|b_{jj},\\ \left(\frac{d b_{ii} b_{jj}/p}{p}\right) & \text{otherwise}\end{cases}$$

for odd prime $p|d$.

(6) If $\alpha_{ij}=1$, $1\le i<j\le k$, then $d\equiv 2 \mod 3$ and $(-1)^{\eta_p}=\left(\frac{3d}{p}\right)$ for odd prime $p|d$.

(7) If $\alpha_{ij}=2$, $1\le i<j\le k$, then the invariant η of S is defined by $(-1)^{\eta_p}=\left(\frac{d}{p}\right)$ for odd prime $p|d$.

(8) If $\alpha_{ij}=3$, $1\le i<j\le k$, then $\eta_3=0$ and $(-1)^{\eta_p}=\left(\frac{d/3}{p}\right)$ for odd prime $p|(d/3)$.

(9) If $\alpha_{ij}=4$, $1\le i<j\le k$, then $(-1)^{\eta_p}=\left(\frac{-d}{p}\right)$ for odd prime $p|d$.

(10) $h=hnr(S)=h(d,\eta)=0$ or 2.

Here we use that $\det(K_1)=\det(K)[K:K_1]^2$ for a sublattice $K_1\subset K$ of finite index, and $\det(K(\lambda))=\lambda^{\mathrm{rk}\ K}\det K$ where $K(\lambda)$ is a lattice K with the form multiplied by $\lambda\in\mathbb{Q}$. Moreover, we use that d is odd if the lattice S is odd. (See similar considerations for the type (II0) in Sect. 5.1.)

In Sect. 8.5.1 (the type (AI1) of the narrow part), $1\le\alpha_{12}\le\alpha_{23}\le 4$. We can use $i=1$, $j=2$ in (6)–(9) to find the invariant η of S. Moreover, $\lambda\le 2$, if $\alpha_{12}\le 3$. In Sect. 10, Appendix 2 we give the Program 22: funda11.main which enumerates all possible matrices Γ satisfying conditions of Sect. 8.5.1 and the additional conditions (1) — (10), and their invariants (d,η,h). All of them have $d\le 100000$. There are 28 these triplets given in Table 7.

In Sect. 8.5.2 (the type (AI0) of the narrow part), $\alpha_{12}=0$. Then $\lambda\le 2$, and we can use $i=1$, $j=2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 23: funda10.main which enumerates all possible matrices Γ satisfying conditions of Sect. 8.5.2 and (1) — (10), and their invariants (d,η,h). All of them have $d\le 100000$. There are 86 these triplets given in Table 7.

In Sect. 8.5.3 (the type (AII1) of the narrow part), $\alpha_{12}=0$. Then $\lambda\le 2$, and we can use $i=1$, $j=2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 24: funda21.main which enumerates all possible matrices Γ satisfying Sect. 8.5.3 and (1) — (10), and their invariants (d,η,h). All of them have $d\le 100000$. There are 44 these triplets given in Table 7.

In Sect. 8.5.4 (the type (AII0) of the narrow part), $\alpha_{12}=0$. Then $\lambda\le 2$, and we can use $i=1$, $j=2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 25: funda20.main which enumerates all possible matrices Γ satisfying Sect. 8.5.4 and (1) — (10), and their invariants (d,η,h). All of them have $d\le 100000$. There are 164 these triplets given in Table 7.

In Sect. 8.5.5 (the type (AIII) of the narrow part), $\alpha_{12}=0$. Then $\lambda\le 2$, and we can use $i=1$, $j=2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 26: funda3.main which enumerates all possible matrices Γ

satisfying Sect. 8.5.5 and (1) — (10), and their invariants (d, η, h). All of them have $d \leq 100000$. There are 69 these triplets given in Table 7.

In Sect. 8.5.6 (the type (BI) of the narrow part), $1 \leq \alpha_{12} \leq 4$. We can use $i = 1$, $j = 2$ in (6)—(9) to find the invariant η of S. Moreover, $\lambda \leq 2$ if $\alpha_{12} \leq 3$. In Sect. 10, Appendix 2 we give the Program 27: fundb1.main which enumerates all possible matrices Γ satisfying Sect. 8.5.6 and (1) — (10), and their invariants (d, η, h). All of them have $d \leq 100000$. There are 155 these triplets given in Table 7.

In Sect. 8.5.7 (the type (BII_1) of the narrow part), $\alpha_{12} = 0$. Then $\lambda \leq 2$, and we can use $i = 1$, $j = 2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 28: fundb21.main which enumerates all possible matrices Γ satisfying Sect. 8.5.7 and (1) — (10), and their invariants (d, η, h). All of them have $d \leq 100000$. There are 234 these triplets given in Table 7.

In Sect. 8.5.8 (the type (BII_2) of the narrow part), $\alpha_{12} = 0$. Then $\lambda \leq 2$, and we can use $i = 1$, $j = 2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 29: fundb22.main which enumerates all possible matrices Γ satisfying Sect. 8.5.8 and (1) — (10), and their invariants (d, η, h). All of them have $d \leq 100000$. There are 223 these triplets given in Table 7.

In Sect. 8.5.9 (the type (BIII) of the narrow part), $\alpha_{12} = 0$. Then $\lambda \leq 2$, and we can use $i = 1$, $j = 2$ in (5) to find the invariant η of S. In Sect. 10, Appendix 2 we give the Program 30: fundb3.main which enumerates all possible matrices Γ satisfying Sect. 8.5.9 and (1) — (10), and their invariants (d, η, h). All of them have $d \leq 100000$. There are 216 these triplets given in Table 7.

Since all the triplets (d, η, h) which we get from the calculations above have $h = 0,\ 2$ and $d \leq 100000$, all these triplets are contained in Tables 3 and 6. In Table 7, for each triplet (d, η, h) of Table 3 with $h = 0$ and each triplet of Table 6 we give the types (AI1) — (BIII) of narrow parts which give the triplet in calculations above.

This finishes the proof of Theorem 9.1.1.

Table 7

Possible types of narrow parts from Theorem 8.3.4
of main hyperbolic lattices of the rank 3 with $h = 0$ or 2

$n=1$ $d=1$ $\eta=0$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=2$ $d=2$ $\eta=0$ $h=0$	AI1	AI0				BI		BII_2	
$n=3$ $d=3$ $\eta=0$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=4$ $d=3$ $\eta=1$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=5$ $d=5$ $\eta=0$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=6$ $d=5$ $\eta=1$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=7$ $d=6$ $\eta=0$ $h=0$	AI1	AI0	AII1			BI	BII_1	BII_2	BIII
$n=9$ $d=7$ $\eta=1$ $h=0$	AI1	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n=10$ $d=10$ $\eta=1$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=11$ $d=11$ $\eta=0$ $h=0$	AI1	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n=15$ $d=14$ $\eta=1$ $h=0$	AI1	AI0				BI	BII_1	BII_2	
$n=16$ $d=15$ $\eta=0$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=17$ $d=15$ $\eta=1$ $h=0$				AII0	AIII		BII_1	BII_2	BIII
$n=18$ $d=15$ $\eta=2$ $h=0$		AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n=19$ $d=15$ $\eta=3$ $h=0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII

$n = 20\ d = 17\ \eta = 0\ h = 0$		AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 22\ d = 21\ \eta = 0\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 24\ d = 21\ \eta = 2\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 25\ d = 21\ \eta = 3\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 28\ d = 26\ \eta = 1\ h = 0$	AI1	AI0	AII1			BI	BII_1	BII_2	BIII
$n = 32\ d = 30\ \eta = 2\ h = 0$	AI1	AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 34\ d = 33\ \eta = 1\ h = 0$		AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 38\ d = 35\ \eta = 0\ h = 0$	AI1	AI0		AII0		BI	BII_1	BII_2	BIII
$n = 40\ d = 35\ \eta = 3\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 42\ d = 39\ \eta = 0\ h = 0$	AI1	AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 46\ d = 42\ \eta = 0\ h = 0$		AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n = 48\ d = 51\ \eta = 0\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 56\ d = 65\ \eta = 0\ h = 0$				AII0			BII_1	BII_2	BIII
$n = 61\ d = 69\ \eta = 3\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 64\ d = 74\ \eta = 1\ h = 0$		AI0				BI	BII_1	BII_2	
$n = 65\ d = 77\ \eta = 1\ h = 0$				AII0		BI	BII_1	BII_2	BIII
$n = 66\ d = 77\ \eta = 3\ h = 0$	AI1	AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 67\ d = 78\ \eta = 2\ h = 0$	AI1	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n = 74\ d = 95\ \eta = 0\ h = 0$		AI0				BI	BII_1	BII_2	BIII
$n = 78\ d = 105\ \eta = 1\ h = 0$		AI0		AII0		BI	BII_1	BII_2	BIII
$n = 80\ d = 105\ \eta = 3\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 82\ d = 105\ \eta = 5\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 83\ d = 105\ \eta = 6\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 84\ d = 105\ \eta = 7\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 87\ d = 114\ \eta = 2\ h = 0$		AI0				BI	BII_1	BII_2	
$n = 99\ d = 165\ \eta = 0\ h = 0$	AI1	AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 104\ d = 165\ \eta = 6\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 106\ d = 170\ \eta = 2\ h = 0$		AI0				BI	BII_1	BII_2	
$n = 108\ d = 186\ \eta = 0\ h = 0$		AI0	AII1			BI	BII_1	BII_2	BIII
$n = 110\ d = 195\ \eta = 3\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 112\ d = 195\ \eta = 6\ h = 0$		AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 115\ d = 210\ \eta = 2\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 118\ d = 219\ \eta = 0\ h = 0$	AI1	AI0		AII0		BI	BII_1	BII_2	BIII
$n = 120\ d = 231\ \eta = 2\ h = 0$				AII0			BII_1	BII_2	BIII
$n = 121\ d = 231\ \eta = 7\ h = 0$		AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n = 125\ d = 255\ \eta = 6\ h = 0$			AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 133\ d = 285\ \eta = 5\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 139\ d = 330\ \eta = 3\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 141\ d = 345\ \eta = 6\ h = 0$		AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 152\ d = 399\ \eta = 4\ h = 0$		AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 155\ d = 435\ \eta = 0\ h = 0$		AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n = 164\ d = 546\ \eta = 2\ h = 0$						BI			
$n = 165\ d = 546\ \eta = 4\ h = 0$				AII0			BII_1	BII_2	BIII
$n = 187\ d = 1155\ \eta = 2\ h = 0$				AII0	AIII		BII_1	BII_2	BIII
$n = 190\ d = 1155\ \eta = 14\ h = 0$		AI0	AII1	AII0	AIII	BI	BII_1	BII_2	BIII
$n = 202\ d = 2730\ \eta = 6\ h = 0$							BII_1	BII_2	BIII

$n = 1\ d = 17\ \eta = 1\ h = 2$						BI			
$n = 2\ d = 19\ \eta = 1\ h = 2$		AI0				BI	BII_1	BII_2	

$n = 3$ $d = 23$ $\eta = 1$ $h = 2$		AI0		AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 4$ $d = 31$ $\eta = 0$ $h = 2$		AI0				BI	BII$_1$	BII$_2$	
$n = 5$ $d = 35$ $\eta = 1$ $h = 2$		AI0	AII1	AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 6$ $d = 37$ $\eta = 0$ $h = 2$		AI0				BI			
$n = 7$ $d = 37$ $\eta = 1$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 8$ $d = 39$ $\eta = 1$ $h = 2$		AI0		AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 9$ $d = 43$ $\eta = 0$ $h = 2$						BI	BII$_1$	BII$_2$	BIII
$n = 10$ $d = 46$ $\eta = 1$ $h = 2$		AI0				BI	BII$_1$	BII$_2$	
$n = 11$ $d = 47$ $\eta = 0$ $h = 2$		AI0				BI			
$n = 12$ $d = 51$ $\eta = 2$ $h = 2$		AI0				BI	BII$_1$		
$n = 13$ $d = 53$ $\eta = 0$ $h = 2$		AI0				BI			
$n = 14$ $d = 53$ $\eta = 1$ $h = 2$		AI0		AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 15$ $d = 55$ $\eta = 1$ $h = 2$				AII0	AIII		BII$_1$	BII$_2$	BIII
$n = 16$ $d = 57$ $\eta = 3$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 17$ $d = 58$ $\eta = 1$ $h = 2$		AI0				BI	BII$_1$	BII$_2$	BIII
$n = 18$ $d = 59$ $\eta = 0$ $h = 2$	AI1		AII1			BI			BIII
$n = 19$ $d = 62$ $\eta = 1$ $h = 2$		AI0				BI		BII$_2$	
$n = 20$ $d = 65$ $\eta = 1$ $h = 2$				AII0	AIII		BII$_1$	BII$_2$	BIII
$n = 21$ $d = 69$ $\eta = 0$ $h = 2$		AI0		AII0		BI	BII$_1$	BII$_2$	BIII
$n = 22$ $d = 69$ $\eta = 2$ $h = 2$		AI0		AII0		BI	BII$_1$		BIII
$n = 23$ $d = 70$ $\eta = 0$ $h = 2$		AI0				BI			
$n = 24$ $d = 77$ $\eta = 0$ $h = 2$				AII0	AIII		BII$_1$	BII$_2$	BIII
$n = 25$ $d = 79$ $\eta = 0$ $h = 2$		AI0				BI	BII$_1$		
$n = 26$ $d = 85$ $\eta = 2$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 27$ $d = 85$ $\eta = 3$ $h = 2$						BI		BII$_2$	
$n = 28$ $d = 87$ $\eta = 0$ $h = 2$				AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 29$ $d = 91$ $\eta = 0$ $h = 2$				AII0	AIII		BII$_1$	BII$_2$	BIII
$n = 30$ $d = 91$ $\eta = 1$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 31$ $d = 93$ $\eta = 3$ $h = 2$				AII0	AIII		BII$_1$	BII$_2$	BIII
$n = 32$ $d = 106$ $\eta = 1$ $h = 2$		AI0				BI	BII$_1$	BII$_2$	
$n = 33$ $d = 107$ $\eta = 0$ $h = 2$			AII1			BI	BII$_1$		BIII
$n = 34$ $d = 113$ $\eta = 0$ $h = 2$				AII0		BI	BII$_1$	BII$_2$	BIII
$n = 35$ $d = 114$ $\eta = 1$ $h = 2$							BII$_1$	BII$_2$	BIII
$n = 36$ $d = 115$ $\eta = 0$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 37$ $d = 115$ $\eta = 3$ $h = 2$		AI0		AII0	AIII	BI	BII$_1$	BII$_2$	BIII
$n = 38$ $d = 119$ $\eta = 0$ $h = 2$		AI0		AII0		BI	BII$_1$	BII$_2$	BIII
$n = 39$ $d = 122$ $\eta = 1$ $h = 2$		AI0				BI	BII$_1$	BII$_2$	
$n = 40$ $d = 123$ $\eta = 0$ $h = 2$		AI0		AII0		BI	BII$_1$	BII$_2$	BIII
$n = 41$ $d = 123$ $\eta = 1$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 42$ $d = 123$ $\eta = 3$ $h = 2$			AII1	AII0		BI	BII$_1$	BII$_2$	BIII
$n = 43$ $d = 129$ $\eta = 0$ $h = 2$			AII1			BI	BII$_1$	BII$_2$	BIII
$n = 44$ $d = 129$ $\eta = 2$ $h = 2$			AII1	AII0		BI	BII$_1$	BII$_2$	BIII
$n = 45$ $d = 133$ $\eta = 1$ $h = 2$				AII0		BI			BIII
$n = 46$ $d = 133$ $\eta = 3$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 47$ $d = 138$ $\eta = 0$ $h = 2$						BI	BII$_1$	BII$_2$	BIII
$n = 48$ $d = 141$ $\eta = 0$ $h = 2$						BI	BII$_1$		
$n = 49$ $d = 143$ $\eta = 3$ $h = 2$	AI1	AI0	AII1	AII0		BI	BII$_1$	BII$_2$	BIII
$n = 50$ $d = 145$ $\eta = 3$ $h = 2$				AII0			BII$_1$	BII$_2$	BIII
$n = 51$ $d = 155$ $\eta = 0$ $h = 2$						BI	BII$_1$	BII$_2$	

$n=52$ $d=159$ $\eta=0$ $h=2$	AI0					BII_1		BIII
$n=53$ $d=159$ $\eta=3$ $h=2$	AI0		AII0		BI	BII_1		BIII
$n=54$ $d=165$ $\eta=2$ $h=2$		AII1	AII0		BI	BII_1	BII_2	BIII
$n=55$ $d=174$ $\eta=2$ $h=2$			AII0		BI	BII_1	BII_2	BIII
$n=56$ $d=177$ $\eta=1$ $h=2$					BI	BII_1	BII_2	BIII
$n=57$ $d=182$ $\eta=0$ $h=2$	AI0				BI		BII_2	
$n=58$ $d=183$ $\eta=0$ $h=2$	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n=59$ $d=185$ $\eta=0$ $h=2$			AII0			BII_1	BII_2	BIII
$n=60$ $d=191$ $\eta=0$ $h=2$					BI			
$n=61$ $d=195$ $\eta=0$ $h=2$			AII0			BII_1	BII_2	BIII
$n=62$ $d=195$ $\eta=1$ $h=2$			AII0	AIII		BII_1	BII_2	BIII
$n=63$ $d=195$ $\eta=7$ $h=2$			AII0			BII_1	BII_2	BIII
$n=64$ $d=205$ $\eta=0$ $h=2$	AI0					BII_1	BII_2	BIII
$n=65$ $d=210$ $\eta=1$ $h=2$						BII_1	BII_2	BIII
$n=66$ $d=210$ $\eta=7$ $h=2$	AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n=67$ $d=213$ $\eta=3$ $h=2$			AII0			BII_1	BII_2	BIII
$n=68$ $d=217$ $\eta=3$ $h=2$			AII0			BII_1	BII_2	BIII
$n=69$ $d=218$ $\eta=1$ $h=2$	AI0				BI		BII_2	
$n=70$ $d=221$ $\eta=0$ $h=2$							BII_2	
$n=71$ $d=221$ $\eta=1$ $h=2$			AII0			BII_1	BII_2	BIII
$n=72$ $d=222$ $\eta=2$ $h=2$	AI0				BI	BII_1	BII_2	BIII
$n=73$ $d=230$ $\eta=0$ $h=2$	AI0				BI			
$n=74$ $d=231$ $\eta=3$ $h=2$			AII0	AIII		BII_1	BII_2	BIII
$n=75$ $d=231$ $\eta=4$ $h=2$	AI0		AII0		BI	BII_1	BII_2	BIII
$n=76$ $d=235$ $\eta=0$ $h=2$			AII0			BII_1		BIII
$n=77$ $d=239$ $\eta=0$ $h=2$					BI			
$n=78$ $d=255$ $\eta=2$ $h=2$			AII0	AIII		BII_1	BII_2	BIII
$n=79$ $d=255$ $\eta=3$ $h=2$			AII0	AIII		BII_1	BII_2	BIII
$n=80$ $d=258$ $\eta=2$ $h=2$	AI0				BI		BII_2	
$n=81$ $d=273$ $\eta=2$ $h=2$						BII_1	BII_2	BIII
$n=82$ $d=282$ $\eta=0$ $h=2$	AI0				BI	BII_1	BII_2	
$n=83$ $d=287$ $\eta=3$ $h=2$					BI			
$n=84$ $d=290$ $\eta=0$ $h=2$					BI			
$n=85$ $d=290$ $\eta=3$ $h=2$			AII0		BI	BII_1	BII_2	BIII
$n=86$ $d=301$ $\eta=1$ $h=2$								
$n=87$ $d=301$ $\eta=3$ $h=2$						BII_1	BII_2	BIII
$n=88$ $d=303$ $\eta=3$ $h=2$			AII0		BI	BII_1	BII_2	BIII
$n=89$ $d=309$ $\eta=0$ $h=2$					BI	BII_1	BII_2	BIII
$n=90$ $d=314$ $\eta=1$ $h=2$					BI			
$n=91$ $d=319$ $\eta=3$ $h=2$					BI	BII_1	BII_2	BIII
$n=92$ $d=323$ $\eta=0$ $h=2$	AI0		AII0	AIII	BI	BII_1	BII_2	BIII
$n=93$ $d=329$ $\eta=3$ $h=2$			AII0			BII_1	BII_2	BIII
$n=94$ $d=330$ $\eta=6$ $h=2$	AI0		AII0		BI	BII_1	BII_2	BIII
$n=95$ $d=345$ $\eta=2$ $h=2$			AII0			BII_1	BII_2	BIII
$n=96$ $d=345$ $\eta=4$ $h=2$			AII0		BI	BII_1	BII_2	BIII
$n=97$ $d=345$ $\eta=5$ $h=2$			AII0			BII_1	BII_2	BIII
$n=98$ $d=357$ $\eta=6$ $h=2$			AII0			BII_1		BIII
$n=99$ $d=366$ $\eta=2$ $h=2$					BI	BII_1	BII_2	BIII
$n=100$ $d=374$ $\eta=0$ $h=2$					BI		BII_2	

$n = 101$ $d = 390$ $\eta = 0$ $h = 2$	AI0				BI	BII_1	BII_2	BIII
$n = 102$ $d = 402$ $\eta = 2$ $h = 2$					BI			
$n = 103$ $d = 406$ $\eta = 3$ $h = 2$			AII0		BI	BII_1		BIII
$n = 104$ $d = 410$ $\eta = 2$ $h = 2$					BI			
$n = 105$ $d = 426$ $\eta = 0$ $h = 2$					BI	BII_1	BII_2	
$n = 106$ $d = 429$ $\eta = 0$ $h = 2$						BII_1	BII_2	
$n = 107$ $d = 429$ $\eta = 5$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 108$ $d = 429$ $\eta = 6$ $h = 2$		AII1	AII0		BI	BII_1	BII_2	
$n = 109$ $d = 431$ $\eta = 0$ $h = 2$					BI			
$n = 110$ $d = 434$ $\eta = 1$ $h = 2$							BII_2	
$n = 111$ $d = 438$ $\eta = 0$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 112$ $d = 455$ $\eta = 3$ $h = 2$			AII0		BI	BII_1		BIII
$n = 113$ $d = 455$ $\eta = 6$ $h = 2$							BII_2	
$n = 114$ $d = 462$ $\eta = 0$ $h = 2$		AII1	AII0		BI	BII_1	BII_2	BIII
$n = 115$ $d = 462$ $\eta = 6$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 116$ $d = 465$ $\eta = 1$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 117$ $d = 465$ $\eta = 6$ $h = 2$						BII_1	BII_2	BIII
$n = 118$ $d = 469$ $\eta = 1$ $h = 2$						BII_1		BIII
$n = 119$ $d = 469$ $\eta = 3$ $h = 2$			AII0	AIII		BII_1		BIII
$n = 120$ $d = 470$ $\eta = 0$ $h = 2$					BI			
$n = 121$ $d = 471$ $\eta = 0$ $h = 2$					BI			BIII
$n = 122$ $d = 474$ $\eta = 0$ $h = 2$					BI			
$n = 123$ $d = 483$ $\eta = 2$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 124$ $d = 515$ $\eta = 0$ $h = 2$		AII1			BI			BIII
$n = 125$ $d = 518$ $\eta = 3$ $h = 2$					BI	BII_1	BII_2	
$n = 126$ $d = 527$ $\eta = 3$ $h = 2$					BI			
$n = 127$ $d = 554$ $\eta = 1$ $h = 2$					BI			
$n = 128$ $d = 555$ $\eta = 3$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 129$ $d = 555$ $\eta = 5$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 130$ $d = 555$ $\eta = 6$ $h = 2$			AII0		BI	BII_1	BII_2	BIII
$n = 131$ $d = 561$ $\eta = 6$ $h = 2$	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n = 132$ $d = 582$ $\eta = 0$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 133$ $d = 595$ $\eta = 3$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 134$ $d = 609$ $\eta = 3$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 135$ $d = 609$ $\eta = 6$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 136$ $d = 618$ $\eta = 0$ $h = 2$					BI			
$n = 137$ $d = 623$ $\eta = 3$ $h = 2$					BI			
$n = 138$ $d = 627$ $\eta = 1$ $h = 2$						BII_1	BII_2	BIII
$n = 139$ $d = 627$ $\eta = 3$ $h = 2$						BII_1	BII_2	BIII
$n = 140$ $d = 663$ $\eta = 3$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 141$ $d = 665$ $\eta = 0$ $h = 2$						BII_1	BII_2	BIII
$n = 142$ $d = 690$ $\eta = 1$ $h = 2$						BII_1	BII_2	BIII
$n = 143$ $d = 690$ $\eta = 2$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 144$ $d = 690$ $\eta = 7$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 145$ $d = 705$ $\eta = 6$ $h = 2$		AII1	AII0		BI	BII_1	BII_2	BIII
$n = 146$ $d = 715$ $\eta = 0$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 147$ $d = 715$ $\eta = 6$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 148$ $d = 741$ $\eta = 0$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 149$ $d = 741$ $\eta = 6$ $h = 2$	AI0				BI	BII_1	BII_2	

$n = 150$ $d = 770$ $\eta = 2$ $h = 2$						BII_1	BII_2	
$n = 151$ $d = 770$ $\eta = 7$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 152$ $d = 791$ $\eta = 3$ $h = 2$					BI			
$n = 153$ $d = 794$ $\eta = 1$ $h = 2$								
$n = 154$ $d = 798$ $\eta = 0$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 155$ $d = 798$ $\eta = 6$ $h = 2$						BII_1	BII_2	
$n = 156$ $d = 806$ $\eta = 0$ $h = 2$								
$n = 157$ $d = 834$ $\eta = 2$ $h = 2$					BI			
$n = 158$ $d = 858$ $\eta = 0$ $h = 2$					BI	BII_1		
$n = 159$ $d = 890$ $\eta = 2$ $h = 2$					BI			
$n = 160$ $d = 897$ $\eta = 6$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 161$ $d = 903$ $\eta = 2$ $h = 2$	AI0	AII1	AII0		BI	BII_1	BII_2	BIII
$n = 162$ $d = 906$ $\eta = 0$ $h = 2$							BII_2	
$n = 163$ $d = 915$ $\eta = 5$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 164$ $d = 935$ $\eta = 0$ $h = 2$						BII_1	BII_2	BIII
$n = 165$ $d = 959$ $\eta = 3$ $h = 2$								
$n = 166$ $d = 966$ $\eta = 4$ $h = 2$						BII_1	BII_2	BIII
$n = 167$ $d = 986$ $\eta = 1$ $h = 2$								
$n = 168$ $d = 987$ $\eta = 7$ $h = 2$			AII0			BII_1		BIII
$n = 169$ $d = 1001$ $\eta = 5$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 170$ $d = 1010$ $\eta = 0$ $h = 2$					BI			
$n = 171$ $d = 1015$ $\eta = 6$ $h = 2$						BII_1	BII_2	BIII
$n = 172$ $d = 1023$ $\eta = 1$ $h = 2$	AI0		AII0		BI	BII_1	BII_2	BIII
$n = 173$ $d = 1066$ $\eta = 2$ $h = 2$								
$n = 174$ $d = 1085$ $\eta = 6$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 175$ $d = 1095$ $\eta = 3$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 176$ $d = 1110$ $\eta = 0$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 177$ $d = 1122$ $\eta = 1$ $h = 2$						BII_1	BII_2	BIII
$n = 178$ $d = 1130$ $\eta = 2$ $h = 2$								
$n = 179$ $d = 1155$ $\eta = 10$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 180$ $d = 1173$ $\eta = 6$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 181$ $d = 1194$ $\eta = 0$ $h = 2$								
$n = 182$ $d = 1218$ $\eta = 2$ $h = 2$						BII_1	BII_2	
$n = 183$ $d = 1235$ $\eta = 6$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 184$ $d = 1245$ $\eta = 6$ $h = 2$			AII0	AIII	BI	BII_1	BII_2	BIII
$n = 185$ $d = 1254$ $\eta = 2$ $h = 2$						BII_1		BIII
$n = 186$ $d = 1271$ $\eta = 3$ $h = 2$								
$n = 187$ $d = 1295$ $\eta = 3$ $h = 2$			AII0			BII_1		BIII
$n = 188$ $d = 1302$ $\eta = 2$ $h = 2$					BI	BII_1		
$n = 189$ $d = 1302$ $\eta = 4$ $h = 2$						BII_1		BIII
$n = 190$ $d = 1338$ $\eta = 0$ $h = 2$								
$n = 191$ $d = 1365$ $\eta = 9$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 192$ $d = 1365$ $\eta = 10$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 193$ $d = 1365$ $\eta = 12$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 194$ $d = 1365$ $\eta = 13$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 195$ $d = 1370$ $\eta = 2$ $h = 2$								
$n = 196$ $d = 1446$ $\eta = 0$ $h = 2$					BI			
$n = 197$ $d = 1463$ $\eta = 1$ $h = 2$							BII_2	
$n = 198$ $d = 1479$ $\eta = 6$ $h = 2$					BI			BIII

$n = 199$ $d = 1482$ $\eta = 0$ $h = 2$								
$n = 200$ $d = 1482$ $\eta = 6$ $h = 2$			AII0				BII_2	BIII
$n = 201$ $d = 1495$ $\eta = 5$ $h = 2$							BII_2	BIII
$n = 202$ $d = 1526$ $\eta = 3$ $h = 2$								
$n = 203$ $d = 1551$ $\eta = 4$ $h = 2$					BI	BII_1	BII_2	BIII
$n = 204$ $d = 1554$ $\eta = 2$ $h = 2$								
$n = 205$ $d = 1554$ $\eta = 4$ $h = 2$	AI0		AII0			BII_1	BII_2	BIII
$n = 206$ $d = 1586$ $\eta = 0$ $h = 2$								
$n = 207$ $d = 1586$ $\eta = 3$ $h = 2$					BI			BIII
$n = 208$ $d = 1605$ $\eta = 6$ $h = 2$				AIII	BI	BII_1	BII_2	BIII
$n = 209$ $d = 1626$ $\eta = 0$ $h = 2$								
$n = 210$ $d = 1635$ $\eta = 5$ $h = 2$						BII_1	BII_2	BIII
$n = 211$ $d = 1677$ $\eta = 6$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 212$ $d = 1751$ $\eta = 3$ $h = 2$					BI			
$n = 213$ $d = 1771$ $\eta = 7$ $h = 2$			AII0			BII_1		
$n = 214$ $d = 1785$ $\eta = 10$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 215$ $d = 1794$ $\eta = 4$ $h = 2$								
$n = 216$ $d = 1806$ $\eta = 6$ $h = 2$						BII_1		BIII
$n = 217$ $d = 1898$ $\eta = 2$ $h = 2$								
$n = 218$ $d = 1986$ $\eta = 2$ $h = 2$								
$n = 219$ $d = 1995$ $\eta = 7$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 220$ $d = 1995$ $\eta = 14$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 221$ $d = 2015$ $\eta = 6$ $h = 2$						BII_1	BII_2	BIII
$n = 222$ $d = 2046$ $\eta = 0$ $h = 2$			AII0			BII_1		BIII
$n = 223$ $d = 2090$ $\eta = 0$ $h = 2$								
$n = 224$ $d = 2145$ $\eta = 14$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 225$ $d = 2226$ $\eta = 2$ $h = 2$								
$n = 226$ $d = 2415$ $\eta = 1$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 227$ $d = 2415$ $\eta = 4$ $h = 2$	AI0	AII1	AII0	AIII		BII_1	BII_2	BIII
$n = 228$ $d = 2415$ $\eta = 8$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 229$ $d = 2454$ $\eta = 0$ $h = 2$								
$n = 230$ $d = 2562$ $\eta = 2$ $h = 2$								
$n = 231$ $d = 2570$ $\eta = 2$ $h = 2$								
$n = 232$ $d = 2639$ $\eta = 5$ $h = 2$						BII_1	BII_2	
$n = 233$ $d = 2730$ $\eta = 0$ $h = 2$			AII0			BII_1		BIII
$n = 234$ $d = 2730$ $\eta = 10$ $h = 2$			AII0		BI	BII_1	BII_2	BIII
$n = 235$ $d = 2805$ $\eta = 3$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 236$ $d = 2829$ $\eta = 6$ $h = 2$						BII_1		BIII
$n = 237$ $d = 2886$ $\eta = 0$ $h = 2$					BI			
$n = 238$ $d = 3003$ $\eta = 11$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 239$ $d = 3045$ $\eta = 10$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 240$ $d = 3066$ $\eta = 6$ $h = 2$								
$n = 241$ $d = 3135$ $\eta = 2$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 242$ $d = 3135$ $\eta = 8$ $h = 2$			AII0	AIII		BII_1	BII_2	BIII
$n = 243$ $d = 3315$ $\eta = 0$ $h = 2$			AII0			BII_1	BII_2	BIII
$n = 244$ $d = 3318$ $\eta = 2$ $h = 2$								BIII
$n = 245$ $d = 3354$ $\eta = 0$ $h = 2$								
$n = 246$ $d = 3354$ $\eta = 6$ $h = 2$								
$n = 247$ $d = 3410$ $\eta = 4$ $h = 2$								

$n = 248$ $d = 4026$ $\eta = 0$ $h = 2$				
$n = 249$ $d = 4074$ $\eta = 6$ $h = 2$				
$n = 250$ $d = 4290$ $\eta = 1$ $h = 2$		BII_1	BII_2	BIII
$n = 251$ $d = 4326$ $\eta = 2$ $h = 2$				
$n = 252$ $d = 4902$ $\eta = 4$ $h = 2$				
$n = 253$ $d = 4991$ $\eta = 7$ $h = 2$				
$n = 254$ $d = 5226$ $\eta = 0$ $h = 2$	BI			
$n = 255$ $d = 5334$ $\eta = 2$ $h = 2$				
$n = 256$ $d = 6006$ $\eta = 2$ $h = 2$		BII_1	BII_2	
$n = 257$ $d = 7590$ $\eta = 8$ $h = 2$		BII_1		BIII
$n = 258$ $d = 10374$ $\eta = 2$ $h = 2$				
$n = 259$ $d = 29526$ $\eta = 2$ $h = 2$				

By Theorem 9.1.1, to prove Theorem 7.1, we now should check reflective type of all lattices of Table 3 with $h = 0$ and Table 6. To find reflective type of these lattices, we should write down these lattices explicitly. Suppose that the lattice S has invariants (d, η). Any lattice S of Table 3 has one of forms (9.1.1) or (9.1.3) below (see (5.1.18) — (5.1.21)):

$$S = U \oplus \langle -d \rangle, \tag{9.1.1}$$

where

$$(-1)^{\eta_p} = \left(\frac{-d/p}{p} \right), \text{ for any odd } p|d, \tag{9.1.2}$$

or

$$S = \langle n_1 \rangle \oplus \langle -n_2 \rangle \oplus \langle -n_3 \rangle (\epsilon_1/2, \epsilon_2/2, \epsilon_3/2) \tag{9.1.3}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$, $d = n_1 n_2 n_3$ if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, and $d = n_1 n_2 n_3/4$ otherwise. Moreover, $n_i \equiv 0 \mod 2$ if $\epsilon_i = 1$; $(n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3)/4 \in \mathbb{Z}$; for any odd prime $p|d$ we have

$$(-1)^{\eta_p} = \begin{cases} \left(\frac{n_1/p}{p} \right) & \text{if } p|n_1, \\ \left(\frac{-n_2/p}{p} \right) & \text{if } p|n_2, \\ \left(\frac{-n_3/p}{p} \right) & \text{if } p|n_3. \end{cases} \tag{9.1.4}$$

Not all lattices of Table 6 have these forms. Some of them have the form

$$\langle 3d \rangle \oplus A_2(1/3, 1/3, -1/3), \quad A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \tag{9.1.5}$$

where $d \equiv 2 \mod 3$. Then

$$(-1)^{\eta_p} = \left(\frac{3d}{p} \right) \tag{9.1.6}$$

for any odd prime $p|d$.

We have

Lemma 9.1.3. *Any reflective main hyperbolic lattice S of rank 3 and with determinant d (it is square-free) has one of forms* (9.1.1), (9.1.3) *or* (9.1.5).

Proof. Let us consider a vertex of a fundamental polygon $\mathcal{M}$ of $W(S)$. If the lattice S is reflective, this vertex does exist. Let $r_1,\, r_2 \in P(\mathcal{M})$ are primitive roots of S orthogonal to sides of $\mathcal{M}$ in this vertex. We have

$$0 \le \alpha_{12} = \frac{4(r_1,\, r_2)^2}{r_1^2 r_2^2} \le 4$$

where $\alpha_{12} = 4(\cos \pi/k)^2$, $k = 2,\, 3,\, 4,\, 6$ or $+\infty$. We apply Lemma 9.1.2 (see also its proof). If $k = +\infty$, then S has the form (9.1.1). If $k = 3$, then S has the form (9.1.5). If $k = 2,\, 4,\, 6$, then S has the form (9.1.3).

This finishes the proof.

By Lemma 9.1.3, if a lattice S of Table 6 has neither forms (9.1.1), (9.1.3) or (9.1.5), then S is not reflective. This is the case for $n = 204,\, 206,\, 215,\, 245,\, 247$ of Table 6.

All other lattices of Tables 3 and 6 have one of forms (9.1.1), (9.1.3) or (9.1.5). We apply to them Vinberg's algorithm (see Sect. 5). For lattices S of forms (9.1.1) and (9.1.3), we have described this algorithm in Sect. 5. For lattices S of form (9.1.5), we take $c = (1, 0, 0)$ as the center of Vinberg's algorithm, and roots $v_1 = (0, 1, 0)$ and $v_2 = (0, 0, 1)$ as roots of the height 0. In Sect. 10, Appendix 2, Program 31: refl0.14 we give Vinberg's algorithm for lattices of the form (9.1.5).

In Sect. 5 we gave several examples of calculations of reflective types of lattices of Table 3 using Vinberg's algorithm. Similar calculations we should do for all lattices of Table 6. As the result of these calculations, we get that exactly 66 lattices of Tables 3 and 6 give hyperbolically reflective lattices. 5 of them belong to Table 3 and have $h = 0$, the rest 61 belong to Table 6 and have $h = 2$. All these 66 lattices are given in Table 4.

This finishes the proof of Basic Theorem 7.1.

We remark that for some lattices of Table 6 we can prove that they are not reflective by considering Table 7. Lattices of Table 7 which have neither type (AI1) — (BIII) of narrow part are not reflective: their fundamental polygon $\mathcal{M}$ does not have a narrow part satisfying Theorem 8.3.4. This is true for 36 triplets

$$\begin{aligned}
&(d, \eta, h) = \\
&(301, 1, 2),\ (794, 1, 2),\ (806, 0, 2),\ (959, 3, 2),\ (986, 1, 2), (1066, 2, 2),\\
&(1130, 2, 2),\ (1194, 0, 2),\ (1271, 3, 2),\ (1338, 0, 2),\ (1370, 2, 2),\ (1482, 0, 2),\\
&(1526, 3, 2),\ (1554, 2, 2),\ (1586, 0, 2),\ (1626, 0, 2),\ (1794, 4, 2),\ (1898, 2, 2),\\
&(1986, 2, 2),\ (2090, 0, 2),\ (2226, 2, 2),\ (2454, 0, 2),\ (2562, 2, 2),\ (2570, 2, 2),\\
&(3066, 6, 2),\ (3354, 0, 2),\ (3354, 6, 2),\ (3410, 4, 2),\ (4026, 0, 2),\ (4074, 6, 2),\\
&(4326, 2, 2),\ (4902, 4, 2),\ (4991, 7, 2),\ (5334, 2, 2),\ (10374, 2, 2),\ (29526, 2, 2).
\end{aligned} \tag{9.1.7}$$

Thus, all triplets (9.1.7) give non-reflective lattices.

9.2. Proof of Theorem 7.2.

We use notations and results of Sect. 2.2. Especially see Proposition 2.2.6. Assume that $\widetilde{S}$ is hyperbolically reflective non-main hyperbolic lattice of the rank 3 and of determinant $2d$ where S is main hyperbolic lattice of odd determinant d (it is square-free). Since $O(\widetilde{S}) \subset O(S)$, reflective type of $\widetilde{S}$ is dominated by reflective type of S. In particular, S is reflective.

First, assume that S is elliptically or parabolically reflective. Then S belongs to Table 1 of Basic Theorem 2.3.2.1. Then we argue like in Sect. 5.2. The lattice $\widetilde{S}$ also has one of forms (9.1.1) or (9.1.3), and we can calculate its reflective type using Vinberg's algorithm (given by us exactly for lattices of these forms). These calculations are of the same difficulty as for main lattices. If by these calculations a lattice $\widetilde{S}$ is elliptically or parabolically reflective, it belongs to Table 2 of Theorem 2.3.3.1. If $\widetilde{S}$ is hyperbolically reflective, we give it in Table 5 of Theorem 7.2. There are 10 these cases corresponding to lattices S with $h = 0$ or $h = 1$ which are not equivariantly equivalent to $\widetilde{S}$.

Now suppose that the lattice S is hyperbolically reflective. Then S belongs to the Table 4 of Theorem 7.1. Let us show that $\widetilde{S}$ is also hyperbolically reflective if and only if S and $\widetilde{S}$ are equivariantly equivalent, i. e.

$$\sum_{p|d} (1 - p + 4\eta_p + 4\omega(d)_p) \equiv \pm 1 - 1 \equiv 0 \text{ or } 6 \mod 8, \tag{9.2.1}$$

see Proposition 2.2.6. Suppose that (9.2.1) is valid. Then $O(\widetilde{S}) = O(S)$ (see Proposition 2.2.6), and the lattice $\widetilde{S}$ is hyperbolically reflective if S does. All these cases are given in Table 5 as equivariantly equivalent cases.

Suppose that (9.2.1) is not valid but $\widetilde{S}$ is hyperbolically reflective. Let $\widetilde{\mathcal{M}}$ be a fundamental polygon of $W(\widetilde{S})$. Since $W(\widetilde{S}) \subset W(S)$, we can choose a fundamental polygon $\mathcal{M}$ for $W(S)$ such that $\mathcal{M} \subset \widetilde{\mathcal{M}}$. Suppose that $\mathcal{M} \neq \widetilde{\mathcal{M}}$ (equivalently $W(\widetilde{S}) \neq W(S)$). Then there exists a reflection $s_r \in W(S)$ with respect to a side of $\mathcal{M}$ such that $s_r \notin W(\widetilde{S})$. Then $s_r(\mathcal{M})$ is also contained in $\widetilde{\mathcal{M}}$. Let l be the axis of $\mathcal{M}$. For all 66 cases of Theorem 7.1 the axis l is contained inside $\mathcal{M}$. Then lines l and $s_r(l)$ are contained in $\widetilde{\mathcal{M}}$ and are different. This is impossible for a polygon $\widetilde{\mathcal{M}}$ of restricted hyperbolic type. Only its axis has this property, and it is unique. Thus, we have proved that $W(\widetilde{S}) = W(S)$. We remind that $S(2) \subset \widetilde{S}$ is an overlattice of $S(2)$ of index 2, and $S(2)$ is the maximal even sublattice of $\widetilde{S}$. If (9.2.1) is not valid, then the overlattice $\widetilde{S}$ is not unique, there are 3 of them, but all of them are conjugate by $O(S) = O(S(2))$. Since $W(\widetilde{S}) = W(S)$, then all these three overlattices are invariant with respect to $W(S)$. Simple considerations over $\mathbb{Z}_2$ show that this is true for a reflection s_r in a primitive root $r \in S$ if and only if $r^2 \equiv 1 \mod 2$. Thus, the equality $W(\widetilde{S}) = W(S)$ implies that all primitive roots $r \in S$ have odd square r^2. In Table 4 we are describing all primitive roots of the lattices S. There always exists a primitive root $r \in S$ with even r^2. We get a contradiction.

This finishes the proof of Theorem 7.2.

9.3. Proof of Theorem 7.4.

The proof is the same as in Sect. 9.2 above where we proved that $W(\widetilde{S}) = W(S)$, if S and $\widetilde{S}$ are hyperbolically reflective.

10. Appendix 2: Programs for GP/PARI calculator

Program 13: funda11.gen

```
\\funda11.gen
\\hyperbolic type, case AI1.
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
rh1=[12.25000000,13.37793021,14.23012096,14.94097150;\
13.37793021,14.57106781,15.47225159,16.22381115;\
14.23012096,15.47225159,16.41025403,17.19241152;\
14.94097150,16.22381115,17.19241152,18.00000000];\
for(alpha12=1,4,for(alpha23=alpha12,4,u=(rh1[alpha12,alpha23]+0.1)^2;\
for(alpha13=5,u,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13, \
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))));pprint("nAI1=",n);pprint("aAI1=",epsilon);\
pprint("aAI1_1=",epsilon1);pprint("aAI1_2=",epsilon2);
```

Program 14: funda10.gen

```
\\funda10.gen
\\hyperbolic type, case A10
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
rh1=[9.412375826,10.37390342,11.10113930,11.70820393];\
alpha12=0;for(alpha23=1,4,w=(rh1[alpha23]+0.1)^2; for(alpha13=5,w,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
```

```
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);a12=0;a21=0;\
fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13; a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31; dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;\
db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))));pprint("nAI0=",n);pprint("aAI0=",epsilon);\
pprint("aAI0_1=",epsilon1);pprint("aAI0_2=",epsilon2);
```

Program 15: funda21.gen

```
\\funda21.gen
\\hyperbolic type, case AII1
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
rh1=[9.412375826,10.37390342,11.10113930,11.70820393];\
rh2=[38.68043607,41.73090517,44.05297726,46];alpha12=alpha23=0;\
for(alpha34=1,4,w=(rh2[alpha34]+0.1)^2;\
for(alpha14=5,w,for(alpha13=5,(7+0.1)^2,\
if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1||\
alpha24<=(rh1[alpha34]-0.1)^2,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24; \
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,\
if(type(a41=isqrt(u)/(a13*a34))!=1,,a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[4,1]=a41;if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);\
if(type(a[1,4])!=1,,diag=idmat(4);diag[1,1]=a[1,3]*a[3,4] *a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];r1=1;for(j=1,tfr,r1=r1 *fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);if(r1>epsilon1,epsilon1=r1,); \
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],)); \
)))))))))));pprint("nAII1=",n);pprint("aAII1=",epsilon); \
pprint("aAII1_1=",epsilon1);pprint("aAII1_2=",epsilon2);
```

Program 16: funda20.gen

```
\\program funda20.gen
\\hyperbolic type, case AII0
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;alpha12=0;alpha23=0;alpha34=0;\
for(alpha14=5,(31.15549442+0.1)^2,fordiv(4*alpha14,aa,\
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
if(alpha13>(7+0.1)^2,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4] *a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
)))))));pprint("nAII0=",n);pprint("aAII0=",epsilon);\
pprint("aAII0_1=",epsilon1);pprint("aAII0_2=",epsilon2);
```

Program 17: funda3.gen

```
\\funda3.gen
\\hyperbolic type, case AIII
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;al12=al23=al34=al45=0;\
for(al15=5,(68.1815011826+0.1)^2,for(al13=5,(7+0.1)^2,\
for(al35=al13,(7+0.1)^2,\
if(issquare(q=al13*al35*al15)==0,,d=(al13+al35+al15-4+isqrt(q))*4;\
if(type(al14=d/(al35-4))!=1||al14<=(31.15549442-0.1)^2\
||type(al25=d/(al13-4))!=1||al25<=(31.15549442-0.1)^2||\
type(al24=(al13*al35+4*al15+4*isqrt(q))*4/((al35-4)*(al13-4)))!=1||\
issquare(q1=al13*al35*al25*al24*al14)==0,,\
al=idmat(5)*4;al[1,5]=al15;al[5,1]=al15;al[1,3]=al13;al[3,1]=al13;\
al[1,4]=al14;al[4,1]=al14;al[2,4]=al24;al[4,2]=al24;\
al[2,5]=al25;al[5,2]=al25;al[3,5]=al35;al[5,3]=al35;\
fordiv(al13,a13,fordiv(al35,a35,a51=isqrt(q)/a13/a35;\
if(a51==0,a15==0,if(type(a15=al15/a51)!=1,,fordiv(al14,a14,\
fordiv(al24,a24,a52=isqrt(q1)/a13/a35/a24*a14/al14;\
if(type(a25=al25/a52)!=1,,a=idmat(5)*-2;\
a[1,3]=a13;a[3,1]=al13/a13;a[1,4]=a14;a[4,1]=al14/a14;\
a[1,5]=a15;a[5,1]=a51;a[2,4]=a24;a[4,2]=al24/a24;\
a[2,5]=a25;a[5,2]=a52;a[3,5]=a35;a[5,3]=al35/a35;\
```

```
diag=idmat(5);diag[1,1]=a[1,4]*a[1,3]*a[4,2]*a[2,5];\
diag[2,2]=a[4,1]*a[1,3]*a[2,4]*a[2,5];\
diag[3,3]=a[1,4]*a[3,1]*a[4,2]*a[2,5];\
diag[4,4]=a[4,1]*a[1,3]*a[4,2]*a[2,5];\
diag[5,5]=a[4,1]*a[1,3]*a[2,4]*a[5,2];\
b=a*diag;b=b/content(b);n=n+1;db=smith(b);r=db[3];\
if(r>epsilon,epsilon=r,);fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],);\
))))))))))));pprint("nAIII=",n);\
pprint("aAIII=",epsilon);pprint("aAIII_1=",epsilon1);\
pprint("aAIII_2=",epsilon2);
```

Program 18: fundb1.gen

```
\\fundb1.gen
\\hyperbolic type, case BI
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;\
rh4=[1.191398091,1.095286061,1.022736500,0.962423650];\
for(alpha12=1,4,for(alpha23=5,22^2,for(alpha13=5,alpha23,\
if(sqrt(alpha13)/8-5/4<1+0.000001,ach1=0,\
ach1=acosh(sqrt(alpha13)/8-5/4));\
if(sqrt(alpha23)/8-5/4<1+0.000001,ach2=0,\
ach2=acosh(sqrt(alpha23)/8-5/4));\
if(ach1+ach2>rh4[alpha12]+0.1||alpha23==0||\
issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);n=n+1;db=smith(b);r=db[1];\
if(r>epsilon,epsilon=r,);fr=factor(r);tfr=matsize(fr)[1];\
r1=1;for(j=1,tfr,r1=r1*fr[j,1]);if(type(r1/2)==1,r1=r1/2,);\
if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))));pprint("nBI=",n);pprint("aBI1=",epsilon);\
pprint("aBI_1=",epsilon1);pprint("aB1_2=",epsilon2);
```

Program 19: fundb21.gen

```
\\fundb21.gen
```

```
\\hyperbolic type, case BII_1
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;alpha12=alpha23=0;\
for(alpha24=5,22^2,if(sqrt(alpha24)/8-5/4<1+0.000001,ach2=0,\
ach2=acosh(sqrt(alpha24)/8-5/4));\
vv=min((8*(cosh(1.429914377-ach2)+5/4))^2+0.1,22^2);\
for(alpha14=5,vv,for(alpha34=alpha14,vv,\
if(issquare(discrim=alpha14*alpha34*(alpha14*alpha34+(alpha24-4)*\
(alpha14+alpha24+alpha34-4)))!=1,,\
if(type(u=(2*alpha14*alpha34+2*isqrt(discrim))/(alpha24-4))!=1,,\
if(type(alpha13=u^2/(alpha14*alpha34))!=1,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;alpha[4,3]=alpha34;\
alpha[2,4]=alpha24;alpha[4,2]=alpha24;alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,\
if(type(a41=u/(a13*a34))!=1,,a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;a[4,2]=alpha[2,4]/a24; \
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[4,1]=a41;if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
)))))))))));pprint("nBII1=",n);pprint("aBII1=",epsilon);\
pprint("aBII1_1=",epsilon1);pprint("aBII1_2=",epsilon2);
```

Program 20: fundb22.gen

```
\\fundb22.gen
\\hyperbolic type, case BII_2
\\cos(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;alpha12=alpha34=0;\
for(alpha23=5,22^2,if(sqrt(alpha23)/8-5/4<1+0.000001,ach2=0,\
ach2=acosh(sqrt(alpha23)/8-5/4));\
vv=min((8*(cosh(1.429914377-ach2)+5/4))^2+0.1,22^2);\
for(alpha13=5,vv,for(alpha24=alpha13,vv,\
if(issquare(discrim=alpha13*alpha23*alpha24*\
(alpha13*alpha23*alpha24-(alpha23-4)*((alpha13-4)*(alpha24-4)-\
4*alpha23)))!=1,,forstep(tau=-1,1,2,\
if(type(u=(alpha13*alpha23*alpha24+tau*isqrt(discrim))/(alpha23-4))!=1,,\
if(type(alpha14=u^2/(alpha13*alpha23*alpha24))!=1||alpha14<=4,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,3],a32,fordiv(alpha[2,4],a24,\
if(type(a41=u/(a13*a32*a24))!=1,,a=-2*idmat(4);\
```

```
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[3,2]=a32;a[2,3]=alpha[2,3]/a32;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;a[4,1]=a41;a[1,4]=alpha[1,4]/a41;\
if(type(a[1,4])!=1,,diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[3,2];\
diag[2,2]=a[3,1]*a[1,4]*a[2,3];diag[3,3]=a[3,1]*a[1,4]*a[3,2];\
diag[4,4]=a[1,3]*a[4,1]*a[3,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))))))));pprint("nBII2=",n);pprint("aBII2=",epsilon);\
pprint("aBII2_1=",epsilon1);pprint("aBII2_2=",epsilon2);
```

Program 21: fundb3.gen

```
\\fundb3.gen
\\hyperbolic type, case BIII
\\cosh(eta/2)=3/2
\l;\
epsilon=1;epsilon1=1;epsilon2=1;n=0;alpha12=alpha23=0;\
for(alpha14=5,(31.70820393)^2+0.1,\
if(sqrt(alpha14)/8-5/4<1+0.000001,ach1=0,\
ach1=acosh(sqrt(alpha14)/8-5/4));\
vv=min((8*(cosh(1.9248473002-ach1)+5/4))^2+0.1,(31.70820393)^2+0.1);\
for(alpha34=alpha14,vv,for(alpha13=5,(7+0.1)^2,\
if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1,,\
alpha=4*idmat(4);alpha[1,3]=alpha13;alpha[3,1]=alpha13;\
alpha[3,4]=alpha34;alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;fordiv(alpha[3,4],a34,\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,\
if(type(a41=isqrt(u)/(a13*a34))!=1,,a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[4,1]=a41;\
if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);n=n+1;\
db=smith(b);r=db[2];if(r>epsilon,epsilon=r,);\
fr=factor(r);tfr=matsize(fr)[1];r1=1;for(j=1,tfr,r1=r1*fr[j,1]);\
if(type(r1/2)==1,r1=r1/2,);if(r1>epsilon1,epsilon1=r1,);\
if(tfr<=0,,if(fr[tfr,1]>epsilon2,epsilon2=fr[tfr,1],));\
))))))))));pprint("nBIII=",n);pprint("aBIII=",epsilon);\
pprint("aBIII_1=",epsilon1);pprint("aBIII_2=",epsilon2);
```

Program 22: funda11.main

```
\\funda11.mai
\\hyperbolic type, case AI1
\\cosh(eta/2)=3/2
\r h2
\r h3
```

```
\l;\
n=0;\
rh1=\
[12.25000000,13.37793021,14.23012096,14.94097150;\
13.37793021,14.57106781,15.47225159,16.22381115;\
14.23012096,15.47225159,16.41025403,17.19241152;\
14.94097150,16.22381115,17.19241152,18.00000000];\
for(alpha12=1,4,for(alpha23=alpha12,4,u1u=(rh1[alpha12,alpha23]+0.1)^2;\
for(alpha13=5,u1u,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32; \
b=a*dd;b=b/content(b);db=smith(b);fdb=factor(db[1]);\
detb=db[1]*db[2]*db[3];fdetb=factor(detb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
(content([-b[1,1]*-b[2,2]*-b[3,3],2])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3],8])<8&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2))||\
(alpha[1,2]<4&&content([-b[1,1],-b[2,2]])>2)||\
(alpha[2,3]<4&&content([-b[2,2],-b[3,3]])>2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));\
if(gam==1,,if(alpha[1,2]<4,b=b;\
d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));\
if(alpha[1,2]==1&&mod(d,3)!=mod(2,3),,if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);et=0;\
for(k=1,matsize(fd1)[1],if(alpha[1,2]==1,\
if(kro(3*d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),if(alpha[1,2]==2,\
if(kro(d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),if(alpha[1,2]==3,\
if(fd1[k,1]==3,,if(kro(3*d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))),)))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(b);pprint(db);\
pprint(fdb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1))))),\
```

```
if(alpha[1,2]==4,d2=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(0,2)&&fdetb[k,1]!=2,d2=d2*fdetb[k,1],)); \
d2=2*d2/content([2*d2,-b[1,1]*-b[2,2]*-b[3,3]]);fordiv(d2,t,\
if(mod(d2,2)==mod(0,2)&&fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2)&&\
mod(t,2)==mod(1,2),,bb=b*t;dbb=smith(bb);fdbb=factor(dbb[1]);\
detbb=dbb[1]*dbb[2]*dbb[3];fdetbb=factor(detbb);\
d=1;for(k=1,matsize(fdetbb)[1],\
if(mod(fdetbb[k,2],2)==mod(1,2),d=d*fdetbb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);et=0;\
for(k=1,matsize(fd1)[1],\
if(kro(-d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(bb);pprint(dbb);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],\
u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1))))),\
))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 23: funda10.main

```
\\funda10.mai
\\hyperbolic type, case A10
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;\
rh1=[9.412375826,10.37390342,11.10113930,11.70820393];\
alpha12=0;for(alpha23=1,4,w=(rh1[alpha23]+0.1)^2; for(alpha13=5,w,\
if(alpha23==0||issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);a12=0;a21=0;\
fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);db=smith(b);fdb=factor(db[1]);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
content([-b[1,1],-b[2,2]])>2||\
(alpha[2,3]<4&&content([-b[2,2],-b[3,3]])>2),,detb=det(b);(if(detb==1,,\
fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3],2])>1&&\
```

```
content([-b[1,1]*-b[2,2]*-b[3,3],8])<8&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3],2])==1,\
b=2*b;db=smith(b);fdb=factor(db[1]);gam1=0,gam1=1);\
if(k==1&&gam1==1,,\
detb=db[1]*db[2]*db[3];fdetb=factor(detb);d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(b);pprint(db);\
pprint(fdb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1));\
)))))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 24: funda21.main

```
\\funda21.main
\\hyperbolic type, case A21
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;\
rh1=[9.412375826,10.37390342,11.10113930,11.70820393];\
rh2=[38.68043607,41.73090517,44.05297726,46];alpha12=alpha23=0;\
for(alpha34=1,4,w=(rh2[alpha34]+0.1)^2;for(alpha14=5,w,\
for(alpha13=5,(7+0.1)^2,if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1||\
alpha24<=(rh1[alpha34]-0.1)^2,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,fordiv(alpha[1,3],a13,\
fordiv(alpha[2,4],a24,if(type(a41=isqrt(u)/(a13*a34))!=1,,\
a=-2*idmat(4);a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;\
a[4,2]=alpha[2,4]/a24;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
```

```
a[4,1]=a41;if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);\
if(type(a[1,4])!=1,,diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);\
db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||\
(alpha[3,4]<4&&content([-b[3,3],-b[4,4]])>2),,\
detb=db[2]*db[3]*db[4];if(detb==1,,fdetb=factor(detb); \
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1==1,,detb=db[2]*db[3]*db[4];\
fdetb=factor(detb);d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,uuu=matrix(1,3,j,k,0);uuu[1,]=[d,et,hhh],\
aaa=0;nnn=matsize(uuu)[1];for(j=1,nnn,if(uuu[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,uuu1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(uuu[t,1]<d||(uuu[t,1]==d&&uuu[t,2]<et),uuu1[t,]=uuu[t,],\
uuu1[t+1,]=uuu[t,]));for(t=1,nnn+1,if(uuu1[t,]==[0,0,0],\
uuu1[t,]=[d,et,hhh],));uuu=uuu1));)))))))))))))))))));\
pprint("uuu=",uuu);pprint("matsize uuu=",matsize(uuu));
```

Program 25: funda20.main

```
\\program funda20.main
\\hyperbolic type, case AII0
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;alpha12=0;alpha23=0;alpha34=0;for(alpha14=5,(31.15549442+0.1)^2,\
fordiv(4*alpha14,aa,\
```

```
if(aa^2>4*alpha14,,alpha13=4+aa;alpha24=4*alpha14/aa+4;\
if(alpha13>(7+0.1)^2,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;\
alpha[4,3]=alpha34;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,fordiv(alpha[1,4],a14,\
a=-2*idmat(4);a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[1,4]=a14;\
a[4,1]=alpha[1,4]/a14;a[1,3]=a13;a[3,1]=alpha[1,3]/a13;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[4,2];\
diag[2,2]=a[1,3]*a[4,1]*a[2,4];diag[3,3]=a[3,1]*a[1,4]*a[4,2];\
diag[4,4]=a[1,3]*a[4,2]*a[4,1];b=a*diag;b=b/content(b);\
m=m+1;db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||content([-b[3,3],-b[4,4]])>2,,\
detb=db[2]*db[3]*db[4];if(detb==1,,fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1=1,,detb=db[2]*db[3]*db[4];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;\
nnn=matsize(u)[1];for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1));\
))))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 26: funda3.main

```
\\funda3.mai
\\hyperbolic type, case AIII
```

```
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;al12=al23=al34=al45=0;for(al15=5,(68.1815011826+0.1)^2,\
for(al13=5,(7+0.1)^2,for(al35=al13,(7+0.1)^2,\
if(issquare(q=al13*al35*al15)==0,,d=(al13+al35+al15-4+isqrt(q))*4;\
if(type(al14=d/(al35-4))!=1||al14<=(31.15549442-0.1)^2\
||type(al25=d/(al13-4))!=1||al25<=(31.15549442-0.1)^2||\
type(al24=(al13*al35+4*al15+4*isqrt(q))*4/((al35-4)*(al13-4)))!=1||\
issquare(q1=al13*al35*al25*al24*al14)==0,,\
al=idmat(5)*4;al[1,5]=al15;al[5,1]=al15;al[1,3]=al13;al[3,1]=al13;\
al[1,4]=al14;al[4,1]=al14;al[2,4]=al24;al[4,2]=al24;\
al[2,5]=al25;al[5,2]=al25;al[3,5]=al35;al[5,3]=al35;\
fordiv(al13,a13,fordiv(al35,a35,a51=isqrt(q)/a13/a35;\
if(a51==0,a15==0,if(type(a15=al15/a51)!=1,,fordiv(al14,a14,\
fordiv(al24,a24,a52=isqrt(q1)/a13/a35/a24*a14/al14;\
if(type(a25=al25/a52)!=1,,a=idmat(5)*-2;\
a[1,3]=a13;a[3,1]=al13/a13;a[1,4]=a14;a[4,1]=al14/a14;\
a[1,5]=a15;a[5,1]=a51;a[2,4]=a24;a[4,2]=al24/a24;\
a[2,5]=a25;a[5,2]=a52;a[3,5]=a35;a[5,3]=al35/a35;\
diag=idmat(5);diag[1,1]=a[1,4]*a[1,3]*a[4,2]*a[2,5];\
diag[2,2]=a[4,1]*a[1,3]*a[2,4]*a[2,5];\
diag[3,3]=a[1,4]*a[3,1]*a[4,2]*a[2,5];\
diag[4,4]=a[4,1]*a[1,3]*a[4,2]*a[2,5];\
diag[5,5]=a[4,1]*a[1,3]*a[2,4]*a[5,2];\
b=a*diag;b=b/content(b);db=smith(b);dbb=db[3];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||issqfree(b[5,5])==0||\
content([-b[1,1],-b[2,2]])>2||content([-b[2,2],-b[3,3]])>2||\
content([-b[3,3],-b[4,4]])>2||content([-b[4,4],-b[5,5]])>2,,\
detb=db[3]*db[4]*db[5];if(detb==1,,fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]*-b[5,5],2])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]*-b[5,5],32])<32&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]*-b[5,5]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]*-b[5,5],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]*-b[5,5],2])==1,\
b=2*b;db=smith(b);dbb=db[3];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1==1,,detb=db[3]*db[4]*db[5];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
```

```
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1));\
))))))))))))))))))));pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 27: fundb1.main

```
\\fundb1.mai
\\hyperbolic type, case BI
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;\
rh4=[1.191398091,1.095286061,1.022736500,0.962423650];\
for(alpha12=1,4,for(alpha23=5,22^2,for(alpha13=5,alpha23,\
if(sqrt(alpha13)/8-5/4<1+0.000001,ach1=0,ach1=acosh(sqrt(alpha13)/8-5/4));\
if(sqrt(alpha23)/8-5/4<1+0.000001,ach2=0,ach2=acosh(sqrt(alpha23)/8-5/4));\
if(ach1+ach2>rh4[alpha12]+0.1||alpha23==0||\
issquare(alpha12*alpha23*alpha13)!=1||\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23<=0,,\
alpha=4*idmat(3);alpha[1,2]=alpha12;alpha[2,1]=alpha12;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[1,3]=alpha13;\
alpha[3,1]=alpha13;dalpha=\
-8+2*isqrt(alpha12*alpha23*alpha13)+2*alpha12+2*alpha13+2*alpha23;\
a=-2*idmat(3);\
fordiv(alpha[1,2],a12,fordiv(alpha[2,3],a23,fordiv(alpha[1,3],a13,\
a21=alpha[1,2]/a12;a32=alpha[2,3]/a23;a31=alpha[1,3]/a13;\
if(a12*a23*a31!=a21*a13*a32,,\
a[1,2]=a12;a[2,1]=a21;a[2,3]=a23;a[3,2]=a32;a[1,3]=a13;a[3,1]=a31;\
dd=idmat(3);dd[1,1]=a13*a32;dd[2,2]=a23*a31;dd[3,3]=a31*a32;\
b=a*dd;b=b/content(b);db=smith(b);fdb=factor(db[1]);\
detb=db[1]*db[2]*db[3];fdetb=factor(detb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
(content([-b[1,1]*-b[2,2]*-b[3,3],2])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3],8])<8&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2))||\
(alpha[1,2]<4&&content([-b[1,1],-b[2,2]])>2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,\
if(alpha[1,2]<4,b=b;d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));\
```

```
if(alpha[1,2]==1&&mod(d,3)!=mod(2,3),,if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(alpha[1,2]==1,\
if(kro(3*d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),if(alpha[1,2]==2,\
if(kro(d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(alpha[1,2]==3,if(fd1[k,1]==3,,\
if(kro(3*d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))),)))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,n=n+1;pprint("n=",n);pprint(b);\
pprint(db);pprint(fdb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1))))),\
if(alpha[1,2]==4,d2=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(0,2)&&fdetb[k,1]!=2,d2=d2*fdetb[k,1],));\
d2=2*d2/content([2*d2,-b[1,1]*-b[2,2]*-b[3,3]]);fordiv(d2,t,\
if(mod(d2,2)==mod(0,2)&&fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2)&&\
mod(t,2)==mod(1,2),,bb=b*t;dbb=smith(bb);fdbb=factor(dbb[1]);\
detbb=dbb[1]*dbb[2]*dbb[3];fdetbb=factor(detbb);d=1;\
for(k=1,matsize(fdetbb)[1],\
if(mod(fdetbb[k,2],2)==mod(1,2),d=d*fdetbb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);et=0;\
for(k=1,matsize(fd1)[1],if(kro(-d/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(bb);pprint(dbb);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,u=matrix(1,3,j,k,0);u[1,]=[d,et,hhh],aaa=0;nnn=matsize(u)[1];\
for(j=1,nnn,if(u[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,u1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(u[t,1]<d||(u[t,1]==d&&u[t,2]<et),u1[t,]=u[t,],u1[t+1,]=u[t,]));\
for(t=1,nnn+1,if(u1[t,]==[0,0,0],u1[t,]=[d,et,hhh],));u=u1))))),\
))))))))))));\
pprint("u=",u);pprint("matsize u=",matsize(u));
```

Program 28: fundb21.main

```
\\fundb21.mai
\\hyperbolic type, case BII_1
\\cosh(eta/2)=3/2
\r h2;\
\r h3;\
\l;\
n=0;alpha12=alpha23=0;for(alpha24=5,22^2,\
if(sqrt(alpha24)/8-5/4<1+0.000001,ach2=0,ach2=acosh(sqrt(alpha24)/8-5/4));\
vv=min((8*(cosh(1.429914377-ach2)+5/4))^2+0.1,22^2);\
for(alpha14=5,vv,for(alpha34=alpha14,vv,\
if(issquare(discrim=alpha14*alpha34*(alpha14*alpha34+(alpha24-4)*\
(alpha14+alpha24+alpha34-4)))!=1,,\
if(type(u=(2*alpha14*alpha34+2*isqrt(discrim))/(alpha24-4))!=1,,\
```

```
if(type(alpha13=u^2/(alpha14*alpha34))!=1,,alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[3,4]=alpha34;alpha[4,3]=alpha34;\
alpha[2,4]=alpha24;alpha[4,2]=alpha24;alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,\
if(type(a41=u/(a13*a34))!=1,,a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[4,1]=a41;\
if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);\
db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||\
(alpha[3,4]==2&&content([-b[3,3],-b[4,4]])>1)||\
(alpha[3,4]==3&&content([-b[3,3],-b[4,4]])>2),,\
detb=db[2]*db[3]*db[4];if(detb==1,,fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1==1,,detb=db[2]*db[3]*db[4];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],if(mod(fdetb[k,2],2)==mod(1,2),\
d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,uuu=matrix(1,3,j,k,0);uuu[1,]=[d,et,hhh],\
aaa=0;nnn=matsize(uuu)[1];for(j=1,nnn,if(uuu[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,uuu1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(uuu[t,1]<d||(uuu[t,1]==d&&uuu[t,2]<et),uuu1[t,]=uuu[t,],\
uuu1[t+1,]=uuu[t,]));for(t=1,nnn+1,if(uuu1[t,]==[0,0,0],\
uuu1[t,]=[d,et,hhh],));uuu=uuu1));)))))))))))))))))));\
pprint("uuu=",uuu);pprint("matsize uuu=",matsize(uuu));
```

Program 29: fundb22.main

```
\\fundb22.mai
\\hyperbolic type, case BII_2
\\cos(eta/2)=3/2
\r h2;\
\r h3;\
\l;\
n=0;alpha12=alpha34=0;for(alpha23=5,22^2,\
if(sqrt(alpha23)/8-5/4<1+0.000001,ach2=0,\
ach2=acosh(sqrt(alpha23)/8-5/4));\
vv=min((8*(cosh(1.429914377-ach2)+5/4))^2+0.1,22^2);\
for(alpha13=5,vv,for(alpha24=alpha13,vv,\
if(issquare(discrim=alpha13*alpha23*alpha24*\
(alpha13*alpha23*alpha24-(alpha23-4)*((alpha13-4)*(alpha24-4)-\
4*alpha23)))!=1,,forstep(tau=-1,1,2,\
if(type(u=(alpha13*alpha23*alpha24+tau*isqrt(discrim))/(alpha23-4))!=1,,\
if(type(alpha14=u^2/(alpha13*alpha23*alpha24))!=1||alpha14<=4,,\
alpha=4*idmat(4);\
alpha[1,3]=alpha13;alpha[3,1]=alpha13;alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
alpha[2,3]=alpha23;alpha[3,2]=alpha23;alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
fordiv(alpha[1,3],a13,fordiv(alpha[2,3],a32,fordiv(alpha[2,4],a24,\
if(type(a41=u/(a13*a32*a24))!=1,,a=-2*idmat(4);\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[3,2]=a32;a[2,3]=alpha[2,3]/a32;\
a[2,4]=a24;a[4,2]=alpha[2,4]/a24;a[4,1]=a41;a[1,4]=alpha[1,4]/a41;\
if(type(a[1,4])!=1,,diag=idmat(4);diag[1,1]=a[1,3]*a[1,4]*a[3,2];\
diag[2,2]=a[3,1]*a[1,4]*a[2,3];diag[3,3]=a[3,1]*a[1,4]*a[3,2];\
diag[4,4]=a[1,3]*a[4,1]*a[3,2];b=a*diag;b=b/content(b);\
db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[3,3],-b[4,4]])>2,,detb=db[2]*db[3]*db[4];\
if(detb==1,,fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1==1,,detb=db[2]*db[3]*db[4];fdetb=factor(detb);\
d=1;for(k=1,matsize(fdetb)[1],if(mod(fdetb[k,2],2)==mod(1,2),\
d=d*fdetb[k,1],));if(d<=2,et=0,fd=factor(d);\
if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);et=0;\
for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
```

```
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,uuu=matrix(1,3,j,k,0);uuu[1,]=[d,et,hhh],\
aaa=0;nnn=matsize(uuu)[1];\
for(j=1,nnn,if(uuu[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,uuu1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(uuu[t,1]<d||(uuu[t,1]==d&&uuu[t,2]<et),uuu1[t,]=uuu[t,],\
uuu1[t+1,]=uuu[t,]));for(t=1,nnn+1,if(uuu1[t,]==[0,0,0],\
uuu1[t,]=[d,et,hhh],));uuu=uuu1));))))))))))))))))))));\
pprint("uuu=",uuu);pprint("matsize uuu=",matsize(uuu));
```

Program 30: fundb3.main

```
\\fundb3.mai
\\hyperbolic type, case BIII
\\cosh(eta/2)=3/2
\r h2
\r h3
\l;\
n=0;alpha12=alpha23=0;for(alpha14=5,(31.70820393)^2+0.1,\
if(sqrt(alpha14)/8-5/4<1+0.000001,ach1=0,\
ach1=acosh(sqrt(alpha14)/8-5/4));\
vv=min((8*(cosh(1.9248473002-ach1)+5/4))^2+0.1,(31.70820393)^2+0.1);\
for(alpha34=alpha14,vv,for(alpha13=5,(7+0.1)^2,\
if(issquare(u=alpha13*alpha34*alpha14)!=1,,\
if(type(alpha24=4+4*(alpha14+alpha34+isqrt(u))/(alpha13-4))!=1,,\
alpha=4*idmat(4);alpha[1,3]=alpha13;alpha[3,1]=alpha13;\
alpha[3,4]=alpha34;alpha[4,3]=alpha34;\
alpha[2,4]=alpha24;alpha[4,2]=alpha24;\
alpha[1,4]=alpha14;alpha[4,1]=alpha14;\
fordiv(alpha[3,4],a34,fordiv(alpha[1,3],a13,fordiv(alpha[2,4],a24,\
if(type(a41=isqrt(u)/(a13*a34))!=1,,a=-2*idmat(4);\
a[3,4]=a34;a[4,3]=alpha[3,4]/a34;a[2,4]=a24;a[4,2]=alpha[2,4]/a24;\
a[1,3]=a13;a[3,1]=alpha[1,3]/a13;a[4,1]=a41;\
if(a41==0,a[1,4]=0,a[1,4]=alpha[1,4]/a41);if(type(a[1,4])!=1,,\
diag=idmat(4);diag[1,1]=a[1,3]*a[3,4]*a[4,2];\
diag[2,2]=a[3,1]*a[4,3]*a[2,4];diag[3,3]=a[3,1]*a[3,4]*a[4,2];\
diag[4,4]=a[3,1]*a[4,3]*a[4,2];b=a*diag;b=b/content(b);\
db=smith(b);dbb=db[2];fdbb=factor(dbb);\
if(issqfree(b[1,1])==0||issqfree(b[2,2])==0||issqfree(b[3,3])==0||\
issqfree(b[4,4])==0||content([-b[1,1],-b[2,2]])>2||\
content([-b[2,2],-b[3,3]])>2||\
(alpha[3,4]==2&&content([-b[3,3],-b[4,4]])>1)||\
(alpha[3,4]==3&&content([-b[3,3],-b[4,4]])>2),,\
detb=db[2]*db[3]*db[4];if(detb==1,,fdetb=factor(detb);\
if(content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])>1&&\
content([-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],16])<16&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
```

```
gam=0;for(j=1,matsize(fdetb)[1],if(fdetb[j,1]!=2&&\
type(-b[1,1]*-b[2,2]*-b[3,3]*-b[4,4]/fdetb[j,1])==1&&\
mod(fdetb[j,2],2)==mod(0,2),gam=1,));if(gam==1,,for(k=0,1,\
if(k==0&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1&&\
fdetb[1,1]==2&&mod(fdetb[1,2],2)==mod(1,2),,\
if(k==1&&content(-[b[1,1]*-b[2,2]*-b[3,3]*-b[4,4],2])==1,\
b=2*b;db=smith(b);dbb=db[2];fdbb=factor(dbb);gam1=0,gam1=1);\
if(k==1&&gam1==1,,detb=db[2]*db[3]*db[4];\
fdetb=factor(detb);d=1;for(k=1,matsize(fdetb)[1],\
if(mod(fdetb[k,2],2)==mod(1,2),d=d*fdetb[k,1],));if(d<=2,et=0,\
fd=factor(d);if(fd[1,1]==2,d1=d/2,d1=d);fd1=factor(d1);\
et=0;for(k=1,matsize(fd1)[1],if(type(b[1,1]/fd1[k,1])==1,\
if(kro(b[1,1]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(type(b[2,2]/fd1[k,1])==1,\
if(kro(b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1)),\
if(kro(d*b[1,1]*b[2,2]/fd1[k,1],fd1[k,1])==1,,et=et+2^(k-1))))));\
hhh=hnr(d,et);if(hhh>2||hhh==1,,\
n=n+1;pprint("n=",n);pprint(a);pprint(b);pprint(db);\
pprint(fdbb);pprint("d=",d," et=",et," h=",hhh);\
if(n==1,uuu=matrix(1,3,j,k,0);uuu[1,]=[d,et,hhh],\
aaa=0;nnn=matsize(uuu)[1];\
for(j=1,nnn,if(uuu[j,]==[d,et,hhh],aaa=1,));\
if(aaa==1,,uuu1=matrix(nnn+1,3,j,k,0);for(t=1,nnn,\
if(uuu[t,1]<d||(uuu[t,1]==d&&uuu[t,2]<et),uuu1[t,]=uuu[t,],\
uuu1[t+1,]=uuu[t,]));\
for(t=1,nnn+1,if(uuu1[t,]==[0,0,0],uuu1[t,]=[d,et,hhh],));uuu=uuu1));\
))))))))))))))))));pprint("uuu=",uuu);\
pprint("matsize uuu=",matsize(uuu));
```

Program 31: refl0.14

```
\\main(n,h) calculates for
\\<3n>\oplus A_2(1/3,1/3,-1/3) where
\\where n\equiv 2 mod 3
\\it is given by the matrix g
\\the vectors v of the height \le h from P(\M)_\pr
\\and calculates chains e and f and their Gram matrices ge, gf
\\comment:
\\possible squares d of roots divide dd
\\y1t=product of a root with f=(1,0,0)
\\(y1,y2,y3) are coordinates of a root
\\we have y1\ge 0 and y2,y3\le 0;z1=3y1;z2=-3y2;z3=-3y3;
refl(n,h,m,k,v1,y1,y2,y3,y1t,w1,w,z1,z2,z3,dd,h1,u,\
u1,w,w1,alpha,m1,divdd,mdivdd,h11)=\
g=[3*n,0,0;0,-2,1;0,1,-2];eps=[1/3,1/3,-1/3];\
m=2;v=matrix(m,3,j,k,0);\
v[1,]=[0,1,0];v[2,]=[0,0,1];\
dd=2*n;divdd=divisors(dd);mdivdd=matsize(divdd)[2];\
forstep(h1=n,h,n,\
for(j1=1,mdivdd,d=divdd[j1];h11=d*h1/2;\
```

```
if(type(h11)!=1,,\
if(issquare(h11)!=1,,y1t=isqrt(h11);\
if(type(2*y1t/d)!=1,,y1=y1t/(3*n);z1=y1*3;\
if(type(z1)!=1,,\
for(z3=0,floor(sqrt(w1=2*n*z1^2+6*d)+0.000001),\
if(issquare(w=3*w1-3*z3^2)!=1,,\
z2=(z3+isqrt(w))/2;\
if(type(z2)!=1,,\
if(mod(z1,3)!=mod(-z2,3)||mod(z1,3)!=mod(z3,3),,\
y2=-z2/3;y3=-z3/3;u=[y1,y2,y3];\
if(type(2*[1,0,0]*g*u /d)!=1||type(2*[0,1,0]*g*u /d)!=1||\
type(2*[0,0,1]*g*u /d)!=1||type(2*eps*g*u /d)!=1,,\
alpha=1;m1=1;\
while(alpha,if(m1>m,alpha=0,if(v[m1,]*g*u >=0,m1=m1+1,alpha=0));\
if(m1<=m,,m=m1;v1=matrix(m,3,j,k,0);\
for(j=1,m-1,v1[j,]=v[j,]);v1[m,]=[y1,y2,y3];v=v1;kill(v1)\
)))))))))))));v;

\\**
This part is the same as in Program 10: refl0.1 between \\** and \\***
\\***

\l;\
main(n,h)=\
refl(n,h);\
e1fromv(v);e2fromv(v,e1);efromv(v,e1,e2);\
f1fromv(v,e);f2fromv(f1);ffromv(f1,f2);
```

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