Max-Planck-Institut für Mathematik
Bonn

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by

H. M. Khudaverdian
R. L. Mkrtchyan

Max-Planck-Institut für Mathematik
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H. M. Khudaverdian
R. L. Mkrtchyan

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

School of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL
UK

Yerevan Physics Institute
Yerevan
Armenia

MPIM 16-5

UNIVERSAL VOLUME OF GROUPS AND ANOMALY OF VOGEL'S SYMMETRY

H.M.KHUDAVERDIAN AND R.L.MKRTCHYAN

Abstract. We show that integral representation of universal volume function of compact simple Lie groups gives rise to six analytic functions on CP^2, which transform as two triplets under group of permutations of Vogel's projective parameters. This substitutes expected invariance under permutations of universal parameters by more complicated covariance.

We provide an analytical continuation of these functions and particularly calculate their change under permutations of parameters. This last relation is universal generalization, for an arbitrary simple Lie group and an arbitrary point in Vogel's plane, of the Kinkelin's reflection relation on Barnes' $G(1+N)$ function. Kinkelin's relation gives asymmetry of the $G(1+N)$ function (which is essentially the volume function for $SU(N)$ groups) under $N \leftrightarrow -N$ transformation (which is equivalent of the permutation of parameters, for $SU(N)$ groups), and coincides with universal relation on permutations at the $SU(N)$ line on Vogel's plane. These results are also applicable to universal partition function of Chern-Simons theory on three-dimensional sphere.

This effect is analogous to modular covariance, instead of invariance, of partition functions of appropriate gauge theories under modular transformation of couplings.

Contents

1. Introduction

1.1. **N $\leftrightarrow$ -N.** $N \leftrightarrow -N$ transformation is the symmetry of simple Lie algebras and gauge theories. E.g. dimensions of irreps of $SU(N)$ groups for a given Young diagram Y can be represented as a rational functions of N and in this form it can be uniquely continued to an arbitrary N. After change of the sign of N they are equal to dimensions of irreps with transposed Young diagram, up to the sign $(-1)^{Area(Y)}$. Similarly dimensions of $SO(N)$ transforms into those of $Sp(N)$, again with sign change and transposition of Young diagram [11]. Eigenvalues of appropriate Casimir operators, analytically continued in a similar way, have the same behavior [4, 22].

Parizi and Sourlas [25] noticed that space with N odd grassmanian coordinates can be considered in some respects as a space with negative number $(-N)$ of usual even coordinates. This is in agreement of abovementioned symmetry w.r.t. the change of the sign of the N, since transposition of Young diagram interchange symmetrization and antisymmetrization. All this became a part of the theory of superalgebras and particularly is formulated as an isomorphism of superalgebras $SU(n|m) \cong SU(m|n), OSp(n|m) \cong OSp(m|n)$. E.g., taking into account that many invariants in superalgebras depend on $n - m$, from $SU(n|m) \cong SU(m|n)$ we obtain $SU(n) \cong SU(-n)$. These dualities appear to be relevant in applications: $SU(N)$ gauge theory since first work of 't Hooft [9] is well-known to have $1/N$ expansion over even powers of $1/N$, $SO(N)$ gauge theories are dual to $Sp(N)$ theories [18], with the same correspondence in representations of matter multiplets, and similarly in many other applications.

1.2. **Universality.** In a more recent time, after work of Vogel [29], $N \leftrightarrow -N$ dualities became a part of invariance of theories under permutation of Vogel's parameters α, β, γ, in the range of applicability of both notions. Note that ranges of applicability of universality and $N \leftrightarrow -N$ duality overlap, but neither is included in other one. E.g. universality, as of now, is dealing with adjoint and its descendant representations, while $N \leftrightarrow -N$, as described above, deals practically with all representations, but of classical groups, only.

Vogel, motivated by knot theory, studied what can be called group weights of vacuum Feynman diagrams of gauge theories, but without any initially assigned Lie group. The problem he addressed was finally aimed to classify so called finite Vassiliev's invariants of knots, but during research he introduced very convenient parametrization of simple Lie algebras. These are so called universal, Vogel's, projective (i.e. relevant up to an arbitrary rescaling) parameters α, β, γ (see for details [29]). They can be defined as follows.

Let $\mathfrak{g}$ be an arbitrary simple Lie algebra. Consider symmetric square of its adjoint representation. It can be canonically decomposed [29] into three irreducible representations:

$$S^2 ad = \underline{\mathbf{1}} + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)\,.$$

Take a second Casimir operator C_2, which is uniquely defined up to a scalar multiplier. Denote by $2t$ eigenvalue of C_2 on the adjoint representation: $C_2(ad) = 2t$. Then the parameters α, β, γ are defined through values of the Casimir C_2 on these irredicble representations in the following way:

$$\begin{aligned} C_2(Y_2(\alpha)) &= 4t - 2\alpha\,, \\ C_2(Y_2(\beta)) &= 4t - 2\beta\,, \\ C_2(Y_2(\gamma)) &= 4t - 2\gamma\,. \end{aligned} \tag{1}$$

One can show that

$$\alpha + \beta + \gamma = t\,. \tag{2}$$

We see that these parameters are defined up to a rescaling. Permutation symmetry between them follows since there is no special order in these representations.

Definition 1. Parameters (α, β, γ) are called *Vogel's parameters*. They can be considered as homogeneous coordinates on projective plane $\mathbf{C}P^2$. Plane $\mathbf{C}P^2$ factorised under the action of group S_3 of permutation of homogeneous coordinates α, β, γ is called *Vogel plane*.

The values of Vogel parameters for all simple Lie algebras are given in table 1.2, where for exceptional line $Exc(n), n = -2/3, 0, 1, 2, 4, 8$ for $G_2, D_4, F_4, E_6, E_7, E_8$ respectively. (See [5, 6] for study of universality on exceptional line.) Parameter α is chosen to be equal to -2. This always can be done due to the scaling invariance. This choice ("minimal normalization") is distinguished by the fact that t becomes an integer, the dual Coxeter number of corresponding algebra. The square of long roots in this normalization is equal to 2.

TABLE 1. Vogel's parameters for simple Lie algebras

Algebra/Parameters	α	β	γ	t
sl(N)	-2	2	N	N
so(N)	-2	4	N-4	N-2
sp(N)	-2	1	N/2+2	N/2+1
Exc(n)	-2	2n+4	n+4	3n+6

Example 1.1. Duality $N \leftrightarrow -N$ is implicitly present in table 1.2 since Vogel's parameters are defined up to rescaling and permutation.

Indeed, we see from the table that transformation $N \leftrightarrow -N$ for $sl(N)$ is reduced to the switching of parameters α and β and multiplication on (-1): $(-2, 2, -N) = (-1) \cdot (2, -2, N)$. In the same way under changing of sign of N $so(N)$ transforms into $sp(N)$ since $(-2, 4, -N-4) = (-2) \cdot (1, -2, N/2+2)$.

Consider some quantity for simple Lie algebras, for example dimension of algebra, dimensions of representations $Y_2(\,.\,)$, eigenvalues of Casimir operators on irreducible representations, etc. The "reasonable" function on Vogel plane, which for points corresponding to simple Lie algebras (see table (1.2)) takes the values of that quantity on that Lie algebra, will be called universal function corresponding to this quantity. For example, dimensions of simple Lie algebras are given by the following universal dimension function

$$dim = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}\,, \quad (t=\alpha+\beta+\gamma)\,. \tag{3}$$

Examples of universal functions include dimensions of some series of representations in powers of adjoint representation[14], eigenvalues of higher Casimir operators [23], characters of some representations on Weyl line and particularly such character for adjoint representation [24, 31]:

$$\begin{aligned} f(x) &= \chi_{ad}(x\rho) = r + \sum_{\mu} e^{x(\mu,\rho)} = \\ &= \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}\,, \quad (t=\alpha+\beta+\gamma)\,, \end{aligned} \tag{4}$$

where r is the rank of simple Lie algebra, μ runs over the set of all roots of this algebra, and ρ is the Weyl vector, which is equal to the half of the sum of all positive roots:

$$\rho = \frac{1}{2}\sum_{\mu>0}\mu\,. \tag{5}$$

One can expect an existence of universal expression for quantities, related with adjoint representation, as those mentioned above. On the other hand, there is no known universal expression for e.g vectorial representations of classical groups.

We shall not discuss here the problem of what kind of analytic continuation on entire Vogel's plane in universal formulae is implied, since in this paper we actually deal with few explicitly defined analytic functions.

1.3. **Problem.** The aim of the present paper is to study the $N \leftrightarrow -N$ duality and Vogel's permutation symmetry of parameters in more complicated cases than the case when unversal function is just a rational function of Vogel's parameters. The main objects will be the universal function of group's volume and universal partition functions of Chern-Simons theory on three dimensional sphere. Invariant volume of compact version of simple Lie group can be considered as partition function of corresponding matrix model [16]. Partition function of Chern-Simons theory on three-dimensional sphere is calculated in terms of gauge group objects in [32]. Volume function also is the part of full Chern-Simons partition function. Both are represented in the universal form in [19]. We shall see that corresponding analytic functions are not invariant w.r.t. the $N \leftrightarrow -N$ duality and permutations of Vogel's parameters (as naively expected), but instead transform according to some non-trivial representations of these permutations' groups.

For example, partition function of $SU(N)$ matrix model is essentially Barnes' G-function $G(1+N)$. At large N its asymptotic expansion indeed is a series over $1/N^2$ (see e.g. [1]), in agreement with 't Hooft perturbation theory observations [9] :

$$\begin{aligned}\log G(1+N) =\\ \left(\frac{1}{2}N^2-\frac{1}{12}\right)\log(N)+\frac{1}{2}N\log(2\pi)-\frac{3}{4}N^2+\frac{1}{12}-\log A+\\ \sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}\,.\end{aligned}$$

where B_{2g} are Bernoulli numbers.

However, small N expansion includes both even and odd powers [1]:

$$2\log G(1+N) = N\log(2\pi) - \gamma N^2 - N(N+1) + 2\sum_{k=2}^{\infty}(-1)^k\zeta(k)\frac{N^{k+1}}{k+1}\,. \tag{6}$$

The $N \leftrightarrow -N$ asymmetry is given by Kinkelin's relation [12]:

$$\log\frac{G(1+N)}{G(1-N)} = N\log(2\pi) - \int_0^N dx\,\pi x\,\cot(\pi x)\,. \tag{7}$$

So, the volume of $SU(N)$ is an analytical function $G(1+N)$, which is not invariant w.r.t. the $N \leftrightarrow -N$ duality. Functions $G(1+N)$ and $G(1-N)$ combine into doublet under duality transformation.

We are going to generalize these observations. We will present the universal formula for group's volume (and for Chern-Simons theory), will show that this universal formula defines few analytic functions, and will calculate transformation of these functions under permutations of Vogel's parameters.

In more details: we will define this universal volume function by integral representation, which turns out to be piecewise-analytical function of Vogel's parameters α, β, γ. Next we will show that this integral representation gives rise to six analytical functions on CP^2, constituting two triplet representations of group of permutations of Vogel's parameters. To study the behavior of these functions under permutation of arguments one has to analytically continue them to the range of parameters larger than initially defined by integral representation. In this way we first calculate difference between functions from different triplets and then we calculate the difference between initial function and that with parameters permuted.

As a check of this approach we specialize these results for the $SU(N)$ line on Vogel's plane. As mentioned above, in that case volume function is essentially Barnes' G-function, permutation of arguments is equivalent to changing of the sign of N, and relation between functions with permuted arguments leads to relation between $G(1 \pm N)$, which will exactly coincide with Kinkelin's reflection relation (7).

2. Universal invariant volume of simple Lie groups

Various formulae for volume of compact simple Lie groups are given by Macdonald [15] (see also [8]), Marinov [17], Kac and Peterson [10], and Fegan [7]. For few series of (super)groups volume formulae are given by Voronov [30]. In this section we derive the universal expression for volume of compact Lie groups, which generalizes these formulae for an arbitrary points on Vogel's plane, and which presents them in a uniform way.

This universal formula was obtained in [19] from the more general universal expression for perturbative part of Chern-Simons partition function. Universal volume formula is defined by a function which coincides, at points from Vogel's table, with volume of corresponding groups:

$$Vol(\alpha,\beta,\gamma) = Vol\left(G\left(\mathfrak{g}_{[\alpha,\beta,\gamma]}\right)\right)\,, \tag{8}$$

where $\mathfrak{g} = \mathfrak{g}_{(\alpha,\beta,g)}$ is simple Lie algebra $\mathfrak{g}$ with coordinates (α,β,γ) on Vogel plane (see table (1.2)); $G = G(\mathfrak{g})$ is connected, simply connected compact Lie group corresponding to Lie algebra $\mathfrak{g}$, and $Vol(G)$ is a volume of group's manifold with invariant metric. An invariant metric on group is induced by certain invariant scalar product $(\,,\,)$ on the Lie algebra. On the simple Lie algebra an invariant scalar product is proportional to Cartan-Killing form. On the other hand this scalar product defines canonically second Casimir C_2, which in its turn defines Vogel's parameters α,β,γ by equations (1), (2). If $(\mathbf{x},\mathbf{y}) = \lambda\phi(\mathbf{x},\mathbf{y})$, where $\phi(\,,\,)$ is Cartan-Killing form, then C_2 has eigenvalue $\frac{1}{\lambda} = 2t$ on the Lie algebra. Thus Vogel's parameters α,β,γ of Lie algebra define invariant scalar product by equation

$$(\mathbf{x},\mathbf{y}) = \frac{1}{2t}\phi(\mathbf{x},\mathbf{y})\,, \quad (t = \alpha+\beta+\gamma)\,. \tag{9}$$

This is a scalar product which defines a metric of the group $G = G\left(\mathfrak{g}_{(\alpha,\beta,\gamma)}\right)$ and its volume in equation(8). Under rescaling of Vogel's parameters volume changes in the following way:

$$V(\lambda\alpha,\lambda\beta,\lambda\gamma) = \lambda^{\frac{-\dim}{2}}V(\alpha,\beta,\gamma)\,.$$

2.1. **Volume function and Chern-Simons partition function.** Recall briefly construction of [19], obtained by considerations of partition function of Chern-Simons theory.

Let G be the compact Lie group. In [24] it was considered partition function $Z = Z^{(G)}(\kappa)$ for Chern-Simons theory corresponding to group G on 3-dimensional sphere with coupling constant κ. The partition function $Z^{(G)}(k)$ can be represented as a product of perturbative and not-perturbative parts $Z^{(G)}(\kappa) = Z_1^{(G)}Z_2^{(G)}$, where non-perturbative part $Z_1^{(G)}$ is shown to be equal, on the basis of Macdonald formula [15], to

$$Z_1^{(G)} = \frac{(2\pi\delta^{-1/2})^{dim}}{Vol(G)}\,, \quad \delta = \kappa+t = \kappa+\alpha+\beta+\gamma\,, \tag{10}$$

and perturbative part $Z_2^{(G)}$ is equal to

$$Z_2 = \prod_{\mu>0} \frac{\sin \frac{\pi(\mu,\rho)}{\delta}}{\frac{\pi(\mu,\rho)}{\delta}}\,, \tag{11}$$

Here α, β, γ are Vogel's parameters of simple Lie algebra which corresponds to group G.

$Vol(G)$ is the volume (8) of the corresponding compact group G, product $\prod_{\mu>0}$ is performed over all positive roots μ of Lie algebra $\mathfrak{g}$, ρ is the Weyl vector (5). and (,) is invariant scalar product(9).

Note that now scaling transformation of Vogel's parameters is extended to κ: $(\alpha, \beta, \gamma, \kappa) \to (\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\kappa)$, and Chern-Simons theory is invariant with respect to exactly this transformation.

An important observation is that partition function $Z(k)$ obeys the condition

$$Z(\kappa) = 1 \text{ if } \kappa = 0\,. \tag{12}$$

This immediately implies the volume formula (taking into account that $Vol(G)$ doesn't depend on κ):

(13)
$$Vol(G) = \frac{(2\pi t^{-1/2})^{dim}}{Z_2^{(G)}} = (2\pi t^{-1/2})^{dim} \prod_{\mu>0} \left(\sin \frac{\pi(\mu,\rho)}{t} / \frac{\pi(\mu,\rho)}{t} \right)\,.$$

Remark 1. *Chern-Simons partition function.*

Partition function $Z(\kappa)$ is equal to S_{00}, where S_{00} is the $(0,0)$ element of the matrix S of modular transformations of characters of corresponding affine Kac-Moody algebra. Here κ is coupling constant in front of the Chern-Simons action, which is rescaled simultaneously with Vogel's parameters and becomes integer (the level of representation) just in normalization of table 1.2. Since there is no non-trivial representations at level zero the S matrix becomes unit in that case. This implies condition (12).

2.2. **Kac-Peterson formula.** Kac and Peterson in 1984 derived an expression for the volume of compact (connected, simply connected) simple Lie group defined with Cartan-Killing metric [10] (see also [7]):

$$Vol(G) = (2\sqrt{2}\pi)^{dim} \prod_{\mu>0} \frac{\sin 2\pi\phi(\rho,\mu)}{2\pi\phi(\rho,\mu)}\,, \tag{14}$$

where product is over positive roots of Lie algebra, and $\phi(,)$ is Cartan-Killing form.

Remark 2. Kac-Peterson formula can be immediately deduced from our formula (13) for volume. Indeed according to (9) scalar product defining metric of the group coincides with Cartan-Killing form if $t = \frac{1}{2}$. If this condition is obeyed then r.h.s. of equations (13) and (14) coincide. So, if one wishes, it is possible to completely discard Chern-Simons approach and start from equation (14), since equation (13) is obviously equivalent to equation (14).

2.3. **Universal expressions.** Now we rewrite the expressions for volume function and partition function in universal form, following [19].

It suffices to rewrite universal expressions for (11), since the universal formula for volume form (13) can be expressed via this function due to equation (3).

We have

$$\log Z_2^{(G)} = \sum_{\mu>0} \log \left(\frac{\sin(\pi(\rho,\mu)/\delta)}{\pi(\rho,\mu)/\delta} \right) .$$

$$= \sum_{\mu>0} \left(\log\left(\Gamma\left(1-(\rho,\mu)/\delta\right)\right) + \log(\Gamma(1+(\rho,\mu)/\delta))\right) , \tag{15}$$

where we use well-known representation

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{\Gamma(1-x)\Gamma(1+x)} .$$

Next using the following integral representation of gamma-function:

$$\log\Gamma(1+z) = \int_0^\infty \frac{e^{-zx} + z(1-e^{-x}) - 1}{x(1-e^{-x})} dx ,$$

we come to the integral expression for perturbative partition function:

$$\log Z_2^{(G)} = -\int_0^\infty \frac{\sum_{\mu>0}\left(e^{x\frac{(\rho,\mu)}{\delta}} + e^{-x\frac{(\rho,\mu)}{\delta}} - 2\,.\right)}{e^x - 1} dx \tag{16}$$

It follows from equations (3) and (4) that:

$$\sum_{\mu>0}\left(e^{x\frac{(\rho,\mu)}{\delta}} + e^{-x\frac{(\rho,\mu)}{\delta}} - 2\right) = f(x/\delta|\alpha,\beta,\gamma) - dim(\alpha,\beta,\gamma) .$$

where we stress dependence of $f(x)$ and dim from universal parameters.

We use special notation [24] for universal function in r.h.s.:

$$F(x) = F(x|\alpha,\beta,\gamma) = f(x|\alpha,\beta,\gamma) - dim(\alpha,\beta,\gamma) ,$$

and arrive at universal formula

$$\log Z_2(\alpha,\beta,\gamma) = -\int_0^\infty \frac{F(x/\delta)}{x(e^x-1)} dx . \tag{17}$$

In the same way as in equation (8) we denote by $Z_2(\alpha,\beta,\gamma)$ a function on Vogel parameters (α,β,γ) such that it coincides with the function $Z_2^{(G)}$ if (α,β,γ) are Vogel's parameters of Lie algebra $\mathfrak{g}$ corresponding to Lie group G.

Now using equation (13) for volume function we come to final universal expression for volume function $Vol(\alpha,\beta,\gamma)$:

$$Vol(\alpha,\beta,\gamma) = (2\pi t^{-1/2})^{dim} \exp\left(-\int_0^\infty \frac{F(x/t)}{(e^x-1)}\frac{dx}{x}\right), \tag{18}$$

and to universal expression for Chern-Simons partition function on S^3:

$$\log Z = \int_0^\infty \frac{F(x/t)-F(x/\delta)}{(e^x-1)}\frac{dx}{x} = \int_0^\infty \frac{f(x/t)-f(x/\delta)}{(e^x-1)}\frac{dx}{x}. \tag{19}$$

We would like to emphasize again that universal volume formula(18) can be deduced straightforwardly from Kac-Peterson formula (14) disregarding all considerations related with Chern-Symon partition function (see remark 2).

3. Volume function as Barnes' quadruple gamma-functions

Our main aim is to study properties of analytical volume functions. It seems reasonable to establish connection of these functions with known functions such as Barnes' multiple gamma functions. In this section we will express volume function through Barnes' multiple gamma functions, following [20]. We first recall definition of Barnes' multiple gamma functions, then we will formulate a proposition. Using this proposition we express universal formulae (17) and (18) for perturbative partition function Z_2 and volume of group in terms of Barnes's multiple gamma functions.

However, for further progress we need a developed theory of that functions as analytic functions of both argument and parameters, which is seemingly absent.

Barnes' multiple (N-tuple) gamma function $\Gamma_N = \Gamma_N(w|a_1,\dots,a_N)$ can be defined via Barnes' multiple zeta-function $\zeta_N = \zeta_N(s,w|a_1,\dots,a_N)$ in the following way [2, 26]:

$$\Gamma_N(w|a_1,\dots,a_N) = \exp\left(\frac{\partial}{\partial s}\zeta_N(s,w|a_1,\dots,a_N)\Big|_{s=0}\right), \tag{20}$$

where multiple zeta-function $\zeta_N(s,w|a_1,\dots,a_N)$ is a function on complex variables s,w such that it is defined for $\operatorname{Re} s > N$ by power series

$$\zeta_N(s,w|a_1,\dots,a_N) = \sum \frac{1}{(w+k_1a_1+\cdots+k_Na_N)^s}, \tag{21}$$

where summation goes over all non-negative integers $k_1, \ldots, k_N$. It has meromorphic continuation in s with simple poles only at $s = 1, 2, \ldots, N$.

Parameters $\{a_1, \ldots a_N\}$ are complex numbers which obey the following condition: there exist a line passing through the origin, such that all parameters are on the same side of this line.

Barnes' zeta function obviously obeys the scaling condition: for every complex number λ,

$$\zeta_N(s, \lambda w|\lambda a_1, \ldots, \lambda a_N) = \lambda^{-s}\zeta_N(s, w|a_1, \ldots, a_N)\,, \tag{22}$$

and recurrent relations:

(23)
$$\zeta_N(s, w+a_i|a_1, \ldots, a_N) = \zeta_N(s, w|a_1, \ldots, a_N) - \zeta_{N-1}(s, w|a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)\,.$$

It is very useful to establish integral representation for Barnes' function (20). We do it first for Barnes' zeta-function. We have

$$\zeta_N(s, w|a_1, \ldots, a_N) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} A(x) dx\,, \tag{24}$$

where

$$A(x) = \frac{e^{-wx}}{\prod_{j=1}^N (1 - e^{-a_j x})}\,. \tag{25}$$

Indeed, it is easy to see that r.h.s. of equations (24) and (21) coincide for $\operatorname{Re} s > N$, by expansion of the integrand over powers of exponents. To calculate zeta-function for other s, and in particular for $s = 0$ we consider expansion of function $A(x)$ defined by equation (25) in a vicinity of origin:

$$A(x) = \frac{e^{-wx}}{\prod_{j=1}^N (1 - e^{-a_j x})} = \frac{1}{x^N \prod_{j=1}^N a_j} + \cdots = \tag{26}$$

$$= \sum_{k=-N}^{\infty} A_k x^k = A_-(x) + A_0 + A_+(x)\,, \tag{27}$$

$$\text{where } A_-(x) = \sum_{k<0} A_k x^k\,,\ A_+(x) = \sum_{k>0} A_k x^k\,, \tag{28}$$

Remark 3. Coefficients of this expansion are *multiple Bernoulli polynomials* $B_n(w|a_1, \ldots, a_N)$:

$$A(x) = \frac{1}{x^N} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} B_{N,n}(w|a_1, \ldots, a_N) \tag{29}$$

In particular

$$A_-(x) = \sum_{n=0}^{N-1} \frac{(-1)^n x^{n-N}}{n!} B_{N,n}(w|a_1, \dots, a_N)\,, \tag{30}$$

$$A_0(x) = \frac{(-1)^N}{N!} B_{N,N}(w|a_1, \dots, a_N)\,. \tag{31}$$

Let's perform meromorphic continuation in variable s of multiple zeta-function. We use integral representation (24). It is well-defined, particularly, if

$$\operatorname{Re} w > 0, \operatorname{Re} a_i > 0\,. \tag{32}$$

Assume that condition (32) is obeyed. Then represent integral (24) as sum of integrals from 0 to 1, and from 1 till infinity. Integral $\int_1^\infty$ converges and it is an analytical function on s. Using expansion (27) , and the fact that meromorphic continuation of the function $f(s) = \int_0^1 x^{s-1+n} dx$ is equal to function $\frac{1}{s+n}$, we perform meromorphic continuation in s of integral $\int_0^1$. Thus we perform meromorphic continuation in s of integral(24). In particular for point $s = 0$ we come to the following answers. Using expansions (27) we see that for small s

$$\zeta(s, w|a_1, \dots, a_N) = \frac{1}{\Gamma(s)} \left(\sum_{k \le 0} \frac{A_k}{k+s} \right) + \tag{33}$$

$$\frac{1}{\Gamma(s)} \left(\int_0^1 x^{s-1} A_+(x) dx + \int_1^\infty x^{s-1} A(x) dx \right)\,. \tag{34}$$

$\Gamma(s) \approx \frac{1}{s}$ in the vicinity of origin, hence this equation implies that

$$\zeta(0, w|a_1, \dots, a_N) = A_0 = \frac{(-1)^N}{N!} B_{N,N}(w)\,, \tag{35}$$

where $B_{N,N}(w)$ is multiple Bernoulli polynomial defined by (31).

Performing further elementary calculations for equation (33) in the case when condition (32) is obeyed we come to the following integral representation of Barnes's function (20):

$$\Gamma_N(w|a_1, \dots, a_n) = \exp\left(\int_0^\infty \left(A(x) - A_-(x) - A_0 e^{-x} \right) \frac{dx}{x} \right)\,, \tag{36}$$

where function $A(x)$, $A_-(x)$ and A_0 are defined by equations (25) and (27). If condition (32) is not obeyed one has to use also relations (22) and (23).

Remark 4. Integral representation (36) of Barnes' function appears in [26]. Modern review of the theory of multiple Barnes' functions see in [28].

It is instructive to write down equations for transformations of Barnes gamma-functions under rescaling and under shift of argument on parameter. They follow from definition (20) of Barnes' functions and corresponding properties of zeta-function (see equations (22),(23)). Scaling property (22) implies that

$$\Gamma_N(\lambda w|\lambda a_1,\ldots,\lambda a_N) = \lambda^{-c}\Gamma_N(w|a_1,\ldots,a_N)\,, \tag{37}$$

where $c = \zeta_N(0,w|a_1,\ldots,a_N) = A_0 = \frac{(-1)^N}{N!}B_{N,N}(w|a_1,\ldots,a_N)$. This formula can be also deduced straightforwardly from integral representation (36), with the use of Frullani's integral.

Recurrent relation (23) implies that

$$\Gamma_N(w+a_i|a_1,\ldots,a_N) = \frac{\Gamma_N(w|a_1,\ldots,a_N)}{\Gamma_{N-1}(w|a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_N)\,.}$$

One can say that N-tuple Barnes' function $\Gamma(w|a_1,\ldots,a_N)$ is completely defined by function $A(x)$ in equation (25). We shall call $A(x)$ the main term in the integral representation of given multiple gamma function, or simply main term.

More formally equation (36) defines a linear map, the functional

$$A(x) \mapsto \int_0^\infty \left(A(x) - A_-(x) - A_0 e^{-x}\right)\frac{dx}{x} \tag{38}$$

on the linear space of functions which have finite Laurent series in a vicinity of origin and decrease exponentially at infinity. In particular the function $A(x)$ in equation (25)) belongs to this class (if $\operatorname{Re} w, \operatorname{Re} a_i > 0$), and the value of the functional on this function is equal to $\log\Gamma_N(w|a_1,\ldots,a_n)$. This simple observation implies the following proposition on properties of multiple gamma-functions.

Proposition 1. *Let* $\{A_p(x) = A_p(x,w_i|a_1^{(p)},\ldots,a_{N_p}^{(p)})\}$ $(p=1,\ldots,k)$ *be a finite set of functions of the form* (25)

$$A_p(x) = \frac{e^{-w_p x}}{\prod_{j=1}^{N_p}\left(1-e^{-a_j^{(p)}x}\right)}\,.$$

Consider the linear combination of these functions:

$$G(x) = \sum_{p=1}^{k} l_p A_p(x) = \sum_{p=1}^{k}\frac{l_p e^{-w_p x}}{\prod_{j=1}^{N_i}(1-e^{-a_j^{(p)}x})}\,.$$

If $G(x)/x$ is non-singular at origin, then

$$\exp\left[\int_0^\infty G(x)\frac{dx}{x}\right] = \prod_{p=1}^{k}\left(\Gamma_{N_p}(w_p|a_1^{(p)},\dots a_{N_i}^{(p)})\right)^{l_i}, \tag{39}$$

In the special case when function $G(x)$ vanishes, $G(x)\equiv 0$, we have

$$\prod_{p=1}^{M}\left(\Gamma_{N_p}(w_p|a_1,\dots,a_{N_p})\right)^{l_p}\equiv 1\,.$$

Remark 5. Proposition 1 provides the rigorous proof of all identities between multiple gamma functions used in [20, 21]. We expect that it is contained in some papers of past or current century, however we did not find an exact reference.

This proposition is very useful for analyzis of volume formula (18). Using the fact that integrand in equation (18) is non-singular function and Proposition (1) we obtain the following result:

$$\begin{aligned}
Vol(G) &= (2\pi t^{-1/2})^{dim}\exp\left(-\int_0^\infty\frac{dx}{x}\frac{F(x/t)}{(e^x-1)}\right) && (40)\\
&= \left(\frac{4\pi^2}{t}\right)^{\frac{dim}{2}}\frac{\Gamma_4(v_1)\Gamma_4(v_2)\Gamma_4(v_3)\Gamma_4(v_7)}{\Gamma_4(v_4)\Gamma_4(v_5)\Gamma_4(v_6)\Gamma_4(v_8)}\left(\frac{t}{\pi}\right)^{\frac{dim}{2}} && (41)\\
&= (4\pi)^{\frac{dim}{2}}\frac{\Gamma_4(v_1)\Gamma_4(v_2)\Gamma_4(v_3)\Gamma_4(v_7)}{\Gamma_4(v_4)\Gamma_4(v_5)\Gamma_4(v_6)\Gamma_4(v_8)}\,, && (42)
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= 2t-2\alpha && (43)\\
v_2 &= t+\gamma && (44)\\
v_3 &= t+\beta && (45)\\
v_4 &= 3t && (46)\\
v_5 &= 2t+2\beta+\gamma, && (47)\\
v_6 &= 2t+\beta+2\gamma, && (48)\\
v_7 &= 5t-\alpha && (49)\\
v_8 &= -\alpha\,. && (50)
\end{aligned}$$

parameters of functions Γ_4 are $(-\alpha,\beta,\gamma,2t)$, and we use equation [26]:

$$\Gamma_1(x|x) = \sqrt{\frac{x}{2\pi}}\,. \tag{51}$$

The scaling properties of volume functions now can be deduced from scaling properties of quadruple gamma functions. Direct calculation confirms that it is in agreement with (18).

Finally, we introduce multiple sine functions [27]

$$S_r(w|a_1,a_2,...) = \frac{\Gamma_r(|a|-w|a_1,a_2,...)^{(-1)^r}}{\Gamma_r(w|a_1,a_2,...)}\,, \quad (|a| = \sum_{j=1}^{r} a_j)\,. \tag{52}$$

Important feature of multiple sine functions is scaling invariance:

$$S_r(\lambda w|\lambda a_1, \lambda a_2,...) = S_r(w|a_1,a_2,...)\,. \tag{53}$$

Indeed, using linear map (38) one can define (36)-like integral representation of multiple sine function (see below equation (55)). It follows from definition (52) of multiple sine function and integral representation (36) for Barnes's functions that coefficient A_0 for multiple sine function vanishes. Thus equation (37) implies equation (53).

The reasonable question is whether simple scaling properties of volume function and its expression in terms of multiple gamma functions lead to representation of volume in terms of multiple sine functions and some simple functions with necessary (non-trivial) scaling dimension.

With that purpose and with the help of Proposition 1 we transform volume function into

$$\begin{aligned} Vol(G) = (4\pi)^{\frac{dim}{2}} \times \\ \frac{S_4(\alpha+\beta+\gamma|-\alpha,\beta,\gamma,2t)S_4(2\beta+\gamma)S_4(\beta+2\gamma)}{S_4(2\alpha+3\beta+3\gamma)} \times \\ S_3(-\alpha|-\alpha,\beta,\gamma) \times \\ \frac{\Gamma_2(\alpha+3\beta+3\gamma|\beta,\gamma)\Gamma_2(2\alpha+3\beta+3\gamma|\beta,\gamma)}{\Gamma_2(\beta+\gamma|\beta,\gamma)\Gamma_2(\alpha+\beta+\gamma|\beta,\gamma)} \times \\ \frac{\Gamma_2(-\alpha|-\alpha,\gamma)\Gamma_2(-\alpha|-\alpha,\beta)\Gamma_1(2\beta+2\gamma|-\alpha)\Gamma_1(\beta+\gamma|-\alpha)}{\Gamma_1(-\alpha|-\alpha)\Gamma_2(2\beta+2\gamma|-\alpha,\beta)\Gamma_2(2\beta+2\gamma|-\alpha,\gamma)}\,, \end{aligned} \tag{54}$$

where all functions S_4 have the same parameters $(-\alpha,\beta,\gamma,2t)$.

One can see that already double gamma functions don't combine into double sine functions.

Another important feature of multiple sines is that their (36)-like integral representation, can be transformed into the integral over entire real axis [27]:

$$\begin{aligned} \log S_r(z|\underline{\omega}) &= \\ &= (-1)^r \frac{\pi i}{r!} B_{rr}(z|\underline{\omega}) + (-1)^r \int_{R+i0} \frac{dx}{x} \frac{e^{zx}}{\prod_{k=1}^r (e^{\omega_i x}-1)} \\ &= (-1)^{r-1} \frac{\pi i}{r!} B_{rr}(z|\underline{\omega}) + (-1)^r \int_{R-i0} \frac{dx}{x} \frac{e^{zx}}{\prod_{k=1}^r (e^{\omega_i x}-1)} \end{aligned} \tag{55}$$

We shall see below, that integral representation in terms of integral over entire x axis leads to "better" analytic properties of sine function with respect to its parameters. More exactly, in this case zero is not the branch point of sine function as an analytic function of any of its parameters. We see above that volume function can't be represented as a product/ratio of multiple sine functions. However, full partition function of Chern-Simons theory can be expressed via multiple sine functions, as shown in [21, 13].

4. Analytic functions from Chern-Simons perturbative partition function and anomaly of Vogel's symmetry

Denote by $K(\alpha,\beta,\gamma)$ the integral which appears in exponents in expressions above $K(\alpha,\beta,\gamma)$

$$K(\alpha,\beta,\gamma) = \int_0^\infty \frac{dx}{x} \frac{F(x)}{(e^x-1)}\,. \tag{56}$$

This reproduces the integral in equation (17) for $\log Z_2$, if all arguments are divided on δ, or that for volume function (18), if all arguments are divided on t. Consider $K(\alpha,\beta,\gamma)$ for an arbitrary complex values of all variables, α,β,γ. We put $\delta = 1$ by scaling transformation. Next, it is evident, that one should have $\mathrm{Re}\,\alpha \neq 0, \mathrm{Re}\,\beta \neq 0, \mathrm{Re}\,\gamma \neq 0$ since otherwise there are non-integrable singularities at the poles of one of sinh in denominator in $F(x)$. This restriction divides the space of parameters into disjoint regions. Inside that regions integral can converge or diverge at large x depending on values of parameters. It converges in all regions if values of moduli of δ are sufficiently large. In our normalization ($\delta = 1$ after rescaling), this means sufficiently small values of moduli of α,β,γ.

Function K is invariant w.r.t. the permutations of parameters, but in general it is not an analytic function of parameters. To understand what happens it is instructive to consider the following toy model suggested in [19]: Consider the function f such that

$$f(z) = \int_0^\infty \frac{dx}{\cosh(zx)} = \begin{cases} \frac{\pi}{2z} & \text{if } \operatorname{Re} z > 0 \\ -\frac{\pi}{2z} & \text{if } \operatorname{Re} z < 0 \end{cases}, \tag{57}$$

or one can consider even simpler integral representation for this function (see [20]):

$$f(z) = \int_0^\infty \frac{dx}{1+(zx)^2} = \begin{cases} \frac{\pi}{2z} & \text{if } \operatorname{Re} z > 0 \\ -\frac{\pi}{2z} & \text{if } \operatorname{Re} z < 0 \end{cases}. \tag{58}$$

Both integrals define analytic function $f_+(z) = \pi/2z$ for positive real part of argument, and analytic function $f_-(z) = -\pi/2z$ for negative real part of argument. On the other hand a function $f(z)$ is not anaylitic on the whole plane. The reason is that one cannot connect two points on a complex plane of parameter z, one with positive real part and another one with negative, by a continuous path without passing through singularity of integrals (58)(57). Namely, one can't avoid crossing the line $\operatorname{Re} z = 0$, any point on which is singular for integrals. Values of integrals at $\operatorname{Re} z > 0$ and $\operatorname{Re} z < 0$ do not belong to the same analytic function. They are given instead by two different analytic functions $f_+(z)$ and $f_-(z)$ (for $z \neq 0$). Each of these functions is initially defined in the corresponding region of convergence of integral (58) or (57), i.e. corresponding open half-plane.

One can take each of these functions, for example a function $f_+(z)$, continue it analytically to the half-plane $\operatorname{Re} z < 0$, then compare the analytical continuation of function $f_+(z)$ with another function, $f_-(z)$. We see that they are related by the transformation $z \to -z$, and their difference is $f_+(z) - f_-(z) = \pi/z$. This is the simplest example of reflection relation.

Let's introduce notation $K_{\pm\pm\pm}(\alpha, \beta, \gamma)$ for analytic functions which are equal to function $K(\alpha, \beta, \gamma)$ (defined by euation (56)) in the regions, where signs of real parts of parameters coincide respectively with its indices.

$$K_{\epsilon_1\epsilon_2\epsilon_3}(\alpha, \beta, \gamma) = \int_0^\infty \frac{dx}{x} \frac{F(x)}{(e^x - 1)}, \tag{59}$$
$$\epsilon_1 = sign(\operatorname{Re} \alpha), \epsilon_2 = sign(\operatorname{Re} \beta), \epsilon_3 = sign(\operatorname{Re} \gamma).$$

Functions $K_{\pm\pm\pm}$ are symmetric w.r.t. the transposition of arguments corresponding to the same signs in index, since we can interchange them smoothly by paths in the region of definition of integral. For example

$$K_{--+}(\alpha,\beta,\gamma) = K_{--+}(\beta,\alpha,\gamma) \qquad (60)$$
$$\mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta < 0, \mathrm{Re}\,\gamma > 0$$

but in general it is not symmetric w.r.t. the transposition of β,γ. From definitions we get relations:

$$K_{--+}(\alpha,\beta,\gamma) = K_{-+-}(\alpha,\gamma,\beta) = K_{+--}(\gamma,\alpha,\beta) \qquad (61)$$
$$\mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta < 0, \mathrm{Re}\,\gamma > 0$$

and

$$K_{++-}(\alpha,\beta,\gamma) = K_{+-+}(\alpha,\gamma,\beta) = K_{-++}(\gamma,\alpha,\beta) \qquad (62)$$
$$\mathrm{Re}\,\alpha > 0, \mathrm{Re}\,\beta > 0, \mathrm{Re}\,\gamma < 0$$

We shall analytically continue these functions to other regions, where they don't necessarily coincide with functions K originated from that region, and would like to calculate their difference. So, for example, we take $K_{--+}(\alpha,\beta,\gamma)$, where $\mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta < 0, \mathrm{Re}\,\gamma > 0$, analytically continue it to other region of arguments, e.g. $\mathrm{Re}\,\alpha > 0, \mathrm{Re}\,\beta < 0, \mathrm{Re}\,\gamma > 0$ and explicitly calculate difference $K_{--+}(\alpha,\beta,\gamma) - K_{+-+}(\alpha,\beta,\gamma)$. Carrying on this analytic continuation twice, w.r.t. the arguments with different signs of real part, and applying corresponding relations, we obtain the behavior of initial function K under transposition of parameters with different signs of real part.

Relations of type (60),(61) and (62) will be maintained by these analytic continuations.

So, let's consider some K in the region of convergence of its parameters, and analytically continue one parameter (say α) with $\mathrm{Re}\,\alpha > 0$ into region with negative $\mathrm{Re}\,\alpha$, assuming integral is still convergent. Integrand is regular function of α except the poles at the points where $\sinh(x\alpha/4)$ becomes zero, $x = 0$ excluded, i.e. $x = 4\pi ik/\alpha, k = \pm 1, \pm 2, ...$ Since $\mathrm{Re}\,\alpha > 0$, poles with $k > 0$ are located in upper half-plane. When we change α, keeping module non-zero and changing argument in counterclockwise direction, poles move in clockwise direction. The integral remains convergent until half of poles (with $k > 0$) reach integration contour $0 \leq x < \infty$, i.e. when some poles become real and positive. When poles reach integration contour from upper half-plane and continue to move to lower half-plane, we deform contour to prevent appearance of singularity. One can imagine that deformation as a creation of a narrow sprout of the contour, which goes from the real positive line to a pole (which is in the lower half-plane), turns around it in counterclockwise direction, and return to real positive line. Moving parameter to its new value, and simultaneously deforming the contour, we get a value of initial function, analytically continued to new value of parameter. Then we substitute new contour by equivalent one, which consists from infinite number of pieces: one piece is again real positive line from zero to positive infinity, others are small counterclockwise circles around poles in the lower half-plane. Line integral is K from new values of parameters, which is the K, originated from the region of new values of parameters, so the sum over ($2\pi i$ times residues of) poles gives difference between analytically continued function and that K.

Let's write down all this for specific K, say $K_{+-+}(\alpha,\beta,\gamma)$. Initially it is defined for $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta < 0, \operatorname{Re}\gamma > 0$. We would like to analytically continue it on the region $\operatorname{Re}\alpha < 0, \operatorname{Re}\beta < 0, \operatorname{Re}\gamma > 0$. According to above, we get:

$$K_{+-+}(\alpha,\beta,\gamma) = K_{--+}(\alpha,\beta,\gamma) + \varphi_+(\alpha|\beta,\gamma), \quad \operatorname{Re}\alpha < 0, \operatorname{Re}\beta < 0, \operatorname{Re}\gamma > 0\,, \tag{63}$$

where

$$\begin{aligned}\varphi_+(\alpha|\beta,\gamma) &= 2\pi i \sum_{k=1}^{\infty} \frac{ie^{-\frac{2ki\pi}{\alpha}} \sin\left[\frac{k\pi(\beta-2t)}{\alpha}\right] \sin\left[\frac{k\pi(\gamma-2t)}{\alpha}\right]}{2k\pi \sin\left[\frac{k\beta\pi}{\alpha}\right] \sin\left[\frac{k\gamma\pi}{\alpha}\right]} \\ &= -\sum_{k=1}^{\infty} \frac{e^{-\frac{2ki\pi}{\alpha}} \sin\left[\frac{k\pi(\beta+2\gamma)}{\alpha}\right] \sin\left[\frac{k\pi(\gamma+2\beta)}{\alpha}\right]}{k \sin\left[\frac{k\beta\pi}{\alpha}\right] \sin\left[\frac{k\gamma\pi}{\alpha}\right]}\end{aligned} \tag{64}$$

This can be further transformed into

$$\varphi_+(\alpha|\beta,\gamma) = \tag{65}$$

$$-\sum_{k=1}^{\infty}\left(\frac{2e^{-\frac{2ki\pi}{\alpha}}\sin\left[\frac{2k\pi\beta}{\alpha}\right]\cos\left[\frac{k\pi\gamma}{\alpha}\right]}{k\sin\left[\frac{k\gamma\pi}{\alpha}\right]}+\frac{2e^{-\frac{2ki\pi}{\alpha}}\sin\left[\frac{2k\pi\gamma}{\alpha}\right]\cos\left[\frac{k\pi\beta}{\alpha}\right]}{k\sin\left[\frac{k\beta\pi}{\alpha}\right]}+\right.$$

$$\left.\frac{e^{-\frac{2ki\pi}{\alpha}}}{k}\left(1+\cos\left[\frac{2k\pi\beta}{\alpha}\right]+\cos\left[\frac{2k\pi\gamma}{\alpha}\right]+2\cos\left[\frac{2k\pi\beta}{\alpha}+\frac{2k\pi\gamma}{\alpha}\right]\right)\right)=$$

$$\log\left(\left(1-e^{2ia}\right)\sqrt{1-e^{2i(a-x)}}\sqrt{1-e^{2i(a+x)}}\right.$$

$$\left.\sqrt{1-e^{2i(a-y)}}\sqrt{1-e^{2i(a+y)}}\left(1-e^{2i(a-x-y)}\right)\left(-1+e^{2i(a+x+y)}\right)\right)-$$

$$-\sum_{k=1}^{\infty}\left(\frac{2e^{2kia}\sin[2kx]\cos[ky]}{k\sin[ky]}+\frac{2e^{2kia}\sin[2ky]\cos[kx]}{k\sin[kx]}\right)$$

$$a=-\frac{\pi}{\alpha}, x=\frac{\pi\beta}{\alpha}, y=\frac{\pi\gamma}{\alpha} \tag{66}$$

Would we consider movement of parameter β with initially negative $\operatorname{Re}\beta$, difference will appear in that contour of integration (positive line) will be reached by poles $x = 4\pi ik/\beta$ with negative k, $k = -1, -2, ...$ So in that case we get:

$$K_{--+}(\alpha,\beta,\gamma) = K_{-++}(\alpha,\beta,\gamma) + \varphi_-(\beta|\alpha,\gamma), \tag{67}$$

$$\varphi_-(\beta|\alpha,\gamma) = -2\pi i\sum_{k=1}^{\infty}\frac{ie^{\frac{2ki\pi}{\beta}}\sin\left[\frac{k\pi(\alpha-2t)}{\beta}\right]\sin\left[\frac{k\pi(\gamma-2t)}{\beta}\right]}{2k\pi\sin\left[\frac{k\alpha\pi}{\beta}\right]\sin\left[\frac{k\gamma\pi}{\beta}\right]} \tag{68}$$

$$\operatorname{Re}\alpha < 0, \operatorname{Re}\beta > 0, \operatorname{Re}\gamma > 0$$

Note properties of $\varphi_\pm(\alpha|\beta,\gamma)$, which follow from their definitions:

$$\varphi_\pm(\alpha|\beta,\gamma) = \varphi_\pm(\alpha|\gamma,\beta), \tag{69}$$

$$\varphi_\pm(\alpha|-\beta,-\gamma) = \varphi_\pm(\alpha|\beta,\gamma), \tag{70}$$

$$\varphi_-(\alpha|\beta,\gamma) = -\varphi_+(-\alpha|\beta,\gamma) \tag{71}$$

It is easy to check that due to these properties relations (61), (62) are maintained after analytic continuations.

Remark 6. Remaining sums over k in functions $\varphi_\pm$ (64),(68) can be further transformed due to following identity given in Appendix A1 of [13], based on Jonquière's inversion formula for polylogarithm functions:

$$\sum_{k=1}^{\infty} \frac{e^{kA}}{k \sin kB} \approx - \sum_{k=1}^{\infty} \frac{e^{-kA}}{k \sin kB} \tag{72}$$

up to the simple terms, bilinear over Bernoulli polynomials B_0, B_1, B_2. We shall not do that, since it doesn't simplify expressions strongly enough. See, however, remark below.

To obtain the change of functions K under permutation of parameters, one have to extend in (63) β from the region $\mathrm{Re}\,\beta < 0$ to $\mathrm{Re}\,\beta > 0$, i.e. apply (67), using explicit form of functions φ at all values of parameters:

$$K_{+-+}(\alpha,\beta,\gamma) = K_{--+}(\alpha,\beta,\gamma) + \varphi_+(\alpha|\beta,\gamma) = \tag{73}$$

$$K_{-++}(\alpha,\beta,\gamma) + \varphi_+(\alpha|\beta,\gamma) + \varphi_-(\beta|\alpha,\gamma) = \tag{74}$$

$$K_{+-+}(\beta,\alpha,\gamma) + \varphi_+(\alpha|\beta,\gamma) + \varphi_-(\beta|\alpha,\gamma) \tag{75}$$

$$\mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta > 0, \mathrm{Re}\,\gamma > 0\,, \tag{76}$$

where for the last equality we use (62): $K_{-++}(\alpha,\beta,\gamma) = K_{+-+}(\beta,\alpha,\gamma)$ at $\mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta > 0, \mathrm{Re}\,\gamma > 0$. So, the change of function K under transposition of two arguments with different signs is given by equation

$$K_{+-+}(\alpha,\beta,\gamma) - K_{+-+}(\beta,\alpha,\gamma) = \varphi_+(\alpha|\beta,\gamma) + \varphi_-(\beta|\alpha,\gamma) \qquad \mathrm{Re}\,\alpha < 0, \mathrm{Re}\,\beta > 0, \mathrm{Re}\,\gamma > 0\,. \tag{77}$$

In the next section we shall show that on $sl(N)$ line this formula recovers Kinkelin's relation on Barnes' G-function.

Remark 7. In this section we consider some properties of volume function(s) as an analytic functions of parameters. This is necessary for calculation of result of permutation of volume function arguments at an arbitrary point of Vogel's plane. However, a lot of questions remain untouched. E.g. one can ask on an analytic continuation along the circular path around an origin. It is easy to show that for multiple sine functions, due to integral representation (55) with integration on entire x axis, the similar (to above) deformation of contour of integration leads to the same value of function, i.e. zero is not a branch point of parameter(s). This remark is relevant for full Chern-Simons partition function on three dimensional sphere, since it is expressed purely in terms of multiple sine functions [21, 13]. For a general multiple gamma functions one will obtain finite bilinear combination of Bernoulli polynomials, due to the abovementioned identities in [13]. All that require separate study.

5. Volume analytic functions for SU(N) and Kinkelin's reflection relation for Barnes' G-function

For the case $\delta = t (= 1)$, it is easy to establish that integral converges when parameters $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta, \mathrm{Re}\,\gamma, (\alpha+\beta+\gamma = 1)$ are of different signs, and diverges otherwise (i.e. when they all are positive). On the plane $(\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta)$ line $\mathrm{Re}\,\gamma = 0$ corresponds to line $\mathrm{Re}\,\alpha + \mathrm{Re}\,\beta = 1$. So, lines of zero real parts of parameters divide $(\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta)$ plane on 7 regions. Similarly hyperplanes $\mathrm{Re}\,\alpha = 0, \mathrm{Re}\,\beta = 0$ and $\mathrm{Re}\,\gamma = 0$ divide projective space of α, β, γ (i.e. CP^2) into seven disconnected pieces. It is easy to deduce that integral doesn't converge in one region only, namely in the region where all real parts of parameters are positive.

Next we would like to make contact with Kinkelin's functional equation [12] for Barnes' G-function (which is essentially Barnes' double gamma-function). For that purpose we shall apply our equation (77) to the case when volume function is expressed via G-function, which happens for groups $SU(N)$. Group $SU(N)$ corresponds to parameters $(\alpha, \beta, \gamma) = (-2, 2, N), \delta = t = N$. We remove constraint $t = 1$, and explicitly leave $t = N$ in equations, for easier comparison with known results. Besides that, since some contributions are singular at these values, we take $\alpha = -2, \beta = 2 + x, \gamma = N - x, t = N$, and take a limit $x \to 0$. Then we have

$$\varphi_+(\alpha|\beta,\gamma) = 2\pi i \sum_{k=1}^{\infty} \frac{i e^{-\frac{2kit\pi}{\alpha}} \sin\left[\frac{k\pi(\beta-2t)}{\alpha}\right] \sin\left[\frac{k\pi(\gamma-2t)}{\alpha}\right]}{2k\pi \sin\left[\frac{k\beta\pi}{\alpha}\right] \sin\left[\frac{k\gamma\pi}{\alpha}\right]} = \tag{78}$$

$$2\pi i \sum_{k=1}^{\infty} \left(\frac{-1 + e^{2ki\pi t}}{2k^2\pi^2 x} + \frac{i\left(1 + 4e^{ki\pi t} + e^{2ki\pi t}\right)}{4k\pi} \right) + O[x]\,,$$

$$\varphi_-(\beta|\alpha,\gamma) = -2\pi i \sum_{k=1}^{\infty} \frac{i e^{\frac{2ki\pi}{\beta}} \sin\left[\frac{k\pi(\alpha-2t)}{\beta}\right] \sin\left[\frac{k\pi(\gamma-2t)}{\beta}\right]}{2k\pi \sin\left[\frac{k\alpha\pi}{\beta}\right] \sin\left[\frac{k\gamma\pi}{\beta}\right]} = \tag{79}$$

$$2\pi i \sum_{k=1}^{\infty} \left(-\frac{-1 + e^{2ki\pi t}}{2k^2\pi^2 x} + \right. \tag{80}$$

$$\left. + \frac{1 - ki\pi - 4kie^{ki\pi t}\pi + ie^{2ki\pi t}(i + k\pi(-1+2t))}{4k^2\pi^2} \right) + O[x]\,. \tag{81}$$

The sum is regular at $x \to 0$. It is equal to

$$2\pi i \sum_{k=1}^{\infty} \frac{1 + e^{2ki\pi N}(-1 + 2ki\pi N)}{4k^2\pi^2} \tag{82}$$

So, from equation (77) we have that

$$K_{-++}\left(-\frac{2}{N},\frac{2}{N},1\right)-K_{-++}\left(\frac{2}{N},-\frac{2}{N},1\right)= \\ -2\pi i\sum_{k=1}^{\infty}\frac{1+e^{2ki\pi N}(-1+2ki\pi N)}{4k^2\pi^2}\,. \tag{83}$$

The same expression appears, when we calculate in reverse order: first put $SU(N)$ parameters into integral (59), and then calculate its asymmetry under transposition of parameters. Integral for $SU(N)$ is [19, 20] is equal to

(84)

$$K_{-++}\left(-\frac{2}{N},\frac{2}{N},1\right)=\int_0^{\infty}\left(\frac{1-e^{-x}}{4\sinh^2(\frac{x}{2N})}-\frac{N^2}{e^x-1}\right)\frac{dx}{x}\,,\quad (\mathrm{Re}\,N>0)\,.$$

For $SU(N)$, switching of parameters α,β is equivalent to transformation $N\to -N$. So we need a change of (84) under the change of sign of N. It can be done in the same way as above. Let we have N with $\mathrm{Re}\,N>0$. N-dependent poles of integrand of 84 are in the points $x=\pm i\pi k/N, k=1,2,....$ Now let's move N to $-N$, e.g. by multiplying on phase factor, changing from 1 to -1 in counterclockwise direction. Then poles will move in clockwise direction and those with $k>0$ will touch the integration line $[0,\infty)$. To avoid singularity, we change contour as above. Finally, when N becomes $-N$, we get new contour of integration and replace that by half-line from 0 to infinity and a small circles, enclosing poles at points $x=-i\pi k/N, k=1,2,...$ in counterclockwise direction. Integral over half-line is an initial integral with $-N$ instead of N, which is the same. So, the value of analytically continued function at the point $-N$ is equal to its value at the point N plus $2\pi i$ times residues at poles. Residue in the pole at $x=-i\pi k/N$ is:

$$Res_{x=-\frac{i\pi k}{N}}\left(\frac{1}{x}\left(\frac{1-e^{-x}}{4\sinh^2(\frac{x}{2N})}-\frac{N^2}{e^x-1}\right)\right)= \tag{85}$$

$$\frac{1+e^{2ki\pi N}(-1+2ki\pi N)}{4k^2\pi^2}\,. \tag{86}$$

So the sum coincides with expression (82).

Now let's use this answer with integral representation [19] of Barnes' G-function [3] in terms of integral (84).

$$\log(G(1+N)) = \frac{1}{2}N^2\log N - \frac{1}{2}(N^2-N)\log(2\pi) + \\ +K_{-++}(-\frac{2}{N},\frac{2}{N},1)\,. \tag{87}$$

From this equation, applying the procedure of sign changing of N by counterclockwise rotation, and using (83), we get reflection relation for Barnes' G-function:

$$\log\frac{G(1+N)}{G(1-N)} = \frac{i\pi}{2}N^2 + \\ N\log(2\pi) - i\sum_{k=1}^{\infty}\frac{1+e^{2ki\pi N}(-1+2ki\pi N)}{2k^2\pi} \tag{88}$$

provided we choose appropriate branch of $\log N$.

We would like to compare this with Kinkelin's functional equation [12],

$$\log\frac{G(1+N)}{G(1-N)} = N\log(2\pi) - \int_0^N dx\,\pi x\,cot(\pi x) \tag{89}$$

in a form given in [1]:

$$\log\frac{G(1+N)}{G(1-N)} = \frac{i}{2\pi}Li_2(e^{2\pi iN}) + N\log\left(\frac{\pi}{\sin\pi N}\right) - \frac{\pi i}{2}B_2(N)\,, \tag{90}$$

where Li_2 is the dilogarithm function, $B_2(z) = z^2 - z + 1/6$ is second Bernoulli polynomial. These two forms of Kinkelin's relation are equivalent, due to the following formula for indefinite integral (antiderivative):

$$\int dx\,\pi x\,\cot(\pi x) = x\log(1-e^{2\pi ix}) - \frac{i}{2\pi}\left(\pi^2x^2 + Li_2(e^{2\pi ix})\right)\,. \tag{91}$$

Writing functions in the r.h.s. of (90) or (91), (89) as a sums over powers of $e^{2\pi iN}$:

$$Li_2(e^{2\pi iN}) = \sum_{k=1}^{\infty}\frac{e^{2\pi ikN}}{k^2} \tag{92}$$

$$N\log\frac{\pi}{\sin\pi N} = N\log 2\pi - \frac{i\pi}{2}N + i\pi N^2 + N\sum_{k=1}^{\infty}\frac{e^{2\pi ikN}}{k} \tag{93}$$

we get:

$$\log \frac{G(1+N)}{G(1-N)} = \frac{i}{2\pi}\sum_{k=1}^{\infty}\frac{e^{2\pi i k N}}{k^2} + N\sum_{k=1}^{\infty}\frac{e^{2\pi i k N}}{k} + \frac{i\pi}{2}N^2 + N\log 2\pi - \frac{i\pi}{12} \tag{94}$$

which coincides with (88) due to $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

6. Conclusion

We conclude, that volume of $SU(-N)$ isn't given by analytically continued volume of $SU(N)$. This is in correspondence with the fact, that volume of $SU(N|M)$ doesn't analytically depend on $N-M$ [30]. However, $N \leftrightarrow -N$ remains symmetry of the theory, realized in more complicated way - volume function give rise to two analytical functions, which combine into the doublet of this symmetry. Similar considerations are applicable to Vogel's symmetry with respect to permutations of parameters. Let's stress that according to this picture (and this is our general understanding), N and $-N$ are on the completely equal footing, as well as Vogel's parameters and their any permuted set. Each statement, feature, etc. for a given N (or for given set of Vogel's parameters), has its counterpart for $-N$ (or for permuted set of parameters).

For full consideration of dependence of analytical volume functions (and Chern-Simons partition functions) on its parameters one need an understanding of analytical properties of Barnes' multiple gamma functions as an analytical functions of parameters. Some initial considerations are given in Sections 4, 5, where we calculated the change of volume functions under permutation of Vogel's parameters and make contact with Kinkelin's reflection relation on Barnes' G-function. One can continue this line by considerations of branching around zero in the complex plane of each parameter, considerations of analytic properties (with respect to the parameters) of the special combinations of multiple gamma functions, such as multiple sine functions, etc. This last case is relevant for full Chern-Simons partition function on three dimensional sphere. We hope to consider these problems elsewhere.

The reasonable analogy for the anomaly of Vogel's permutation symmetry seems to be the behavior of partition functions of some gauge theories under modular transformations of their couplings, [33]. As discussed in [34] in the most simple example of Maxwell theory, there are two parameters - theta angle θ and electromagnetic coupling g. Theory is unchanged under shift of θ and electromagnetic duality $g^2 \sim 1/g^2$, which together combine into modular parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ with usual modular transformation rules. Partition function, however, behaves as modular form, i.e get an additional multiplier, besides the change of arguments. It is interesting to study how far this analogy is going, particularly, whether Vogel's anomaly restricts couplings of the theory with some other fields.

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Manchester University and Max Planck Institute for Mathematics
E-mail address: `khudian@manchester.ac.uk`

Yerevan Physics Institute and Max Planck Institute for Mathematics
E-mail address: `mrl55@list.ru`