# On Liouville and Carathéodory coverings in Riemannian and complex geometry

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# Abstract

A Riemannian resp. complex manifold $X$ is called *Liouville* if it carries no nonconstant bounded harmonic resp. holomorphic functions. It is called *Carathéodory* if the points of $X$ are separated by bounded harmonic resp. holomorphic functions. We present some remarks on regular Liouville and Carathéodory coverings over a Riemannian resp. complex manifold.

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## Introduction

A Riemannian manifold resp. a complex space $X$ is called *Liouville* if it carries no nonconstant bounded harmonic resp. holomorphic functions. It is said to be *Carathéodory*, or *Carathéodory hyperbolic*, if the points of $X$ are separated by bounded harmonic resp. holomorphic functions. We present some remarks on regular Liouville and Carathéodory coverings over a Riemannian resp. complex manifold.

In §1 we start with a short survey on the Liouville property of regular, that is, Galois, coverings; some new observations are exhibited as well (see Corollary 1.8 and Proposition 1.11). It is known that a *nilpotent* Galois covering[1] over a compact manifold is always Liouville, whereas for *solvable* coverings, in general, this is not the case. In the intermediate class of *polycyclic* coverings, the situation changes drastically when passing from compact Riemannian or, in particular, Kähler manifolds to general compact complex manifolds.

In §2 we extend some known results on the Liouville property of nilpotent coverings to a more general class of coverings with FC-nilpotent Galois groups.

§3 and §4 are devoted to examples of non-Liouville and, especially, Carathéodory coverings over compact manifolds with relatively small Galois groups. Namely, in §3 for arbitrary compact Riemann surface $Y$ of genus $g \geq 2$ we construct a metabelian covering $X \to Y$ over $Y$ with a Carathéodory covering Riemann surface $X$. This is based on a construction due to Lyons and Sullivan [LySu].

In §4 we study in some details the properties of the universal covering $\pi$: $X \to Y$ of an Inoue surface [In]. This is a compact non-Kählerian complex surface $Y$ with a polycyclic fundamental group $G = \pi_1(Y)$ (= the Galois group of the above covering), and the universal covering $X$ is equivalent to the product of $\mathbb{C}$ and the upper half plane $\mathbb{H}$. Thus, $X$ is neither Liouville nor Carathéodory. Concerning the geometry of this covering, we show the following:

*a*) there are a point $x_0 \in X$ and an element $s \in G$ such that $x_0$ is not contained in the $H^\infty(X)$-convex hull of the set $\{g^{-1}sgx_0 \mid g \in G\}$; that is, there exists a bounded holomorphic function on $X$ such that $\sup_{g\in G} |f(g^{-1}sgx_0)| \leq 1$ and $|f(x_0)| > 1$;

*b*) any point $x \in X$ belongs to the $H^\infty(X)$-convex hull of its own $G$-orbit with the point $x_0$ being deleted.

In what follows, all manifolds will be smooth and connected. All complex spaces will be reduced and connected.

[1] i. e. a regular covering with a nilpotent Galois group.

## §1. Liouville-type properties of coverings: a survey

This brief survey is neither complete nor chronological; it contains only some selected results on Liouville-type properties. We do not touch on the case of harmonic functions on a discrete group with a probability measure, which is closely related to our topic (see e. g. [Av, Ma, VeKa, Ka1]).

### Coverings over a compact base

**1.1. Theorem** [LySu, Ka1]. *Let $Y$ be a compact Riemannian manifold and $X \to Y$ be a Galois covering with the Galois group $G$. Then*

*a) $X$ is Liouville whenever $G$ is polycyclic*[2] *or of subexponential growth.*

*b) If $G$ is nilpotent, then $X$ carries no nonconstant positive harmonic functions.*[3]

*c) $G$ must be amenable whenever $X$ is Liouville.*

**1.2. Remarks.** 1. In [LySu] a solvable non-Liouville covering was constructed over arbitrary hyperbolic compact Riemann surface $R$. This shows that in general Theorem 1.1($a$) does not hold for nonpolycyclic solvable coverings of compact manifolds. This example will be discussed in more details in §3.

2. A holomorphic function on a Kähler complex manifold is harmonic with respect to the Laplace-Beltrami operator related to the Kähler metric. Hence, Theorem 1.1($a$) holds also true for holomorphic functions on coverings over compact Kähler manifolds. Actually, the class of Hermitian metrics on complex manifolds with the property of harmonicity of holomorphic functions is wider; it includes, in particular, Gauduchon metrics [Ga, Ka2].

3. Theorem 1.1($a$) does not hold for holomorphic functions on general compact complex manifolds. As a counterexample one may consider the universal covering $X$ of the Inoue surface $Y$ ([Li]] see §4 below for details). The Galois group $G$ in this example is a semidirect product of $\mathbb{Z}^3$ (which is a normal subgroup in $G$) and $\mathbb{Z}$. Thus, $X$ is not Liouville whereas $G$ is a metabelian (i. e. a two-step solvable) polycyclic group. However, nilpotent coverings over compact complex spaces are Liouville ([Li]; see Theorem 1.6 below).

4. Since a finite covering of a compact manifold is compact, the statements ($a$), ($b$) of Theorem 1.1 hold also true for any finite extension of $G$ whenever $G$ is as in these statements. (Indeed, given a group extension $1 \to G \to \widetilde{G} \to K \to 1$ with a finite $K$ and a $\widetilde{G}$-covering $X \xrightarrow{\widetilde{G}} Z$, we have the corresponding tower of Galois coverings $X \xrightarrow{G} Y \xrightarrow{K} Z$, where $Y = X/G$ is compact.)

We say that a group $G$ is *almost nilpotent*[4] (resp. *almost solvable, almost polycyclic,* etc.) if it contains a nilpotent (resp. solvable, polycyclic, etc.) subgroup of finite index. Such a subgroup may clearly be assumed being normal. Thus, by Theorem 1.1(a),(b), an almost polycyclic resp. an almost nilpotent covering over a compact Riemannian manifold is Liouville resp. carries no nonconstant positive harmonic function.

[2] i. e. $G$ admits a finite normal series with cyclic quotients. An equivalent condition: $G$ is solvable and all its subgroups are finitely generated (see e.g. [Ha, Se]; in [Ha] polycyclic groups were called *supersolvable*).

[3] See also [Gui, Ma].

[4] or *virtually nilpotent*, or also *nilpotent–by–finite*.

**1.3. Definition.** A Riemannian manifold $X$ is called *recurrent* if it carries no nonconstant bounded subharmonic function. Nonrecurrent manifolds are called *transient.* Recall (see e.g. [SNWC, Gri, LySu]) that $X$ is recurrent if and only if it does not possess positive Green function; the latter property is, in turn, equivalent to the recurrence of the canonical random motion on $Y$. In [Li] a Riemannian manifold resp. a complex space $X$ is called *ultra-Liouville* if any bounded *continuous* subharmonic resp. plurisubharmonic function on $X$ is constant. Actually, for Riemannian manifolds these two properties are equivalent.

As an example of a noncompact ultra-Liouville complex space one may consider a connected Zariski open subset $Y$ of a compact complex space $\bar{Y}$ (for instance, any quasiprojective complex variety $Y$). Indeed, every bounded plurisubharmonic function on $Y$ admits a plurisubharmonic extension to $\bar{Y}$ (see e.g. [BoNa]) and, hence, by maximum principle, it is constant. Note that a smooth quasiprojective complex variety, being endowed with a Riemannian manifold structure, may be transient. For instance, this is so for $Y = \mathbb{C}^n$, $n \geq 2$, with its Euclidean metric.

The following recurrence criterion for Galois coverings of compact Riemannian manifolds was proved in [LySu] for abelian groups, and in general setting in [VSCC, X.3] (see the references therein[5]).

**1.4. Theorem** [LySu, VSCC]. *Let $X \to Y$ be a Galois covering over a compact Riemannian manifold $Y$ with the Galois group $G$. Then $X$ is recurrent if and only if $G$ is a finite extension of one of the groups $\mathbf{1}$, $\mathbb{Z}$ or $\mathbb{Z}^2$.*

In what follows we call a group $G$ as in Theorem 1.4 a *Varopoulos group.*

## Coverings over a noncompact base

**1.5. Definition[6].** Given a Riemannian manifold resp. a complex space $X$, we denote by $I(X)$ the group of all its homotheties[7] Homo $(X)$ resp. the group of all its biholomorphic automorphisms Aut $(X)$.

Let $G$ be a subgroup of $I(X)$; we say that the action of $G$ on $X$ is *ultra-Liouville* if there is no nonconstant $G$-invariant bounded continuous subharmonic resp. plurisubharmonic function on $X$. Note that in the case when the quotient $Y = X/G$ exists in the same category, the $G$-action on $X$ is ultra-Liouville if and only if $Y$ is ultra-Liouville.

Denote by $Z(G)$ the center of a group $G$. Let

$$\mathbf{1} = Z_0(G) \lhd Z(G) = Z_1(G) \lhd Z_2(G) \lhd \cdots \lhd Z_n(G) \lhd \cdots \lhd G$$

be the upper central series of $G$, i. e. $Z_n(G)$ is the total preimage of $Z(G/Z_{n-1}(G))$ under the natural surjection $G \to G/Z_{n-1}(G)$, $n = 1, 2, \dots$ . The upper central series is

[5] for the case of Riemann surfaces see e.g. [My, Ne, Ro, Mo, Ts].

[6] cf. Definition 1.3 above.

[7] By a homothety of a Riemannian manifold $(X, d)$ we mean a transformation $g : X \to X$ such that $d(gx, gy) \equiv Cd(x, y)$ with some constant $C = C(g)$ which does not depend on $x, y \in X$.

continued transfinitely in the usual way, by defining $Z_\alpha(G) = \bigcup_{\beta<\alpha} Z_\beta(G)$, when $\alpha$ is a limit ordinal.

The group $G$ is called *$\omega$-nilpotent* if it coincides with the union $Z_\omega(G) = \bigcup_{n\in\mathbb{N}} Z_n(G)$. $G$ is called *hyper-nilpotent* if $G = \bigcup_\alpha Z_\alpha(G)$, where $\alpha$ runs over all the ordinals.

The following theorem was proved for $\omega$-nilpotent coverings of Riemannian manifolds in [LySu], and in its present form in [Li], by different methods.

**1.6. Theorem** [LySu, Li]. *Let $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a hyper-nilpotent subgroup of $I(X)$. The space $X$ is Liouville whenever the $G$-action on $X$ is ultra-Liouville. In particular, $X$ is Liouville if there is a hyper-nilpotent covering $X \to Y$ with the base $Y$ being an ultra-Liouville Riemannian manifold resp. an ultra-Liouville complex space.*

**1.7. Remark.** By the maximum principle, any cocompact $G$-action[8] on $X$ is ultra-Liouville. Therefore, for $\omega$-nilpotent coverings over a compact Riemannian manifold $Y$ the last assertion of Theorem 1.6 follows from Theorem 1.1($a$) (but not vice versa!). Indeed, being a quotient of a finitely generated group $\pi_1(Y)$, the Galois group of a regular covering over $Y$ is finitely generated, too. But a finitely generated $\omega$-nilpotent group is nilpotent and polycyclic (see e.g. [Ha, Se]). However, unlike Theorem 1.1, Theorem 1.6 applies also to ultra-Liouville actions which are neither free nor properly discontinuous nor cocompact.

From Theorems 1.4, 1.6 we obtain such a corollary.

**1.8. Corollary.** *Let $X \to Y$ be a Galois covering over a compact Riemannian resp. Kähler manifold $Y$ with the Galois group $G$. If $G$ is an extension of an almost hyper-nilpotent group by a Varopoulos group, then $X$ is Liouville.*

**1.9. Remarks.** 1. Corollary 1.8 does not apply to general compact complex manifolds. Indeed [Li], let $X \to \mathcal{I}$ be the universal cover over the Inoue surface $\mathcal{I}$ (see Remark 1.2.3 and §4). The semidirect decomposition $G \cong \mathbb{Z}^3 \rightthreetimes \mathbb{Z}$ provides us with the tower of Galois coverings $X \xrightarrow{\mathbb{Z}^3} Y \xrightarrow{\mathbb{Z}} \mathcal{I}$. Would $Y$ be ultra-Liouville, then, by Theorem 1.6, the abelian covering $X \xrightarrow{\mathbb{Z}^3} Y$ would be Liouville, which is wrong. Hence, $Y \xrightarrow{\mathbb{Z}} \mathcal{I}$ yields an example of a non-ultra-Liouville $\mathbb{Z}$-covering of a compact complex surface $\mathcal{I}$.

2. Even in the Riemannian setting, an analog of Corollary 1.8 does not hold any more for coverings over a noncompact base $Y$. Consider, for instance, the maximal abelian covering $X \to Y$ over the punctured Riemann sphere $Y = \mathbb{P}^1 \setminus \{3 \text{ points}\}$. The Riemann surface $X$ can be realized as the curve in $\mathbb{C}^2$ with the equation $e^x + e^y = 1$. The covering projection $X \to Y \cong \mathbb{C} \setminus \{0, 1\}$ is $(x, y) \longmapsto e^x$. The Galois group $G$ of this covering is isomorphic to $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^2$. It is known [McKSu, LyMcK] that $X$ is transient, and hence is not ultra-Liouville, whereas $G \cong \mathbb{Z}^2$ is a Varopoulos group. Note that $X$ in this example is Liouville (see [De, Wa, Sh] or Theorem 1.6).

[8] i. e. an action such that $GT = X$ for some compact set $T \subseteq X$.

**1.10. Definition.** Let $G$ be a group acting on a set $X$, and let $\mathbb{C}^X$ be the vector space of all complex valued functions on $X$. The corresponding $G$-action on $\mathbb{C}^X$ is denoted by $f \mapsto f^g$, $f^g(x) = f(gx)$. We say that an element $g \in G$ is a *period* of a function $f \in \mathbb{C}^X$, or *$f$-period*, if $f$ is $g$-invariant, i. e. $f(gx) \equiv f(x)$ for all $x \in X$. For a function $f \in \mathbb{C}^X$ the set of all its periods form a subgroup in $G$, which is denoted by $G_f$. It is a stationary subgroup of $f$ with respect to the $G$-action on $\mathbb{C}^X$. For a subspace $\mathcal{F} \subseteq \mathbb{C}^X$ denote by $G_{\mathcal{F}}$ the intersection of all the subgroups $G_f$, $f \in \mathcal{F}$. We call $G_{\mathcal{F}}$ the *group of $\mathcal{F}$-periods*, or simply the *period group*. It is easily seen that $G_{\mathcal{F}}$ is a normal subgroup of $G$ if $\mathcal{F}$ is $G$-invariant.

From now on, we denote by $\mathcal{H} = \mathcal{H}(X)$ the space $Harm^\infty(X)$ resp. $H^\infty(X)$ of all bounded complex valued harmonic resp. holomorphic functions on a Riemannian manifold resp. complex space $X$. Clearly, $\mathcal{H}(X)$ is an $I(X)$-invariant subspace of $\mathbb{C}^X$; for any subgroup $G \subseteq I(X)$ the $\mathcal{H}$-period subgroup $G_{\mathcal{H}}$ is normal in $G$.

For a subgroup $G \subseteq I(X)$ and an element $s \in I(X)$ we denote by $[s, G]$ the subgroup of $I(X)$ generated by all the commutators $[s, g] = sgs^{-1}g^{-1}$, $g \in G$.

The next proposition contains some new observations concerning harmonic (holomorphic) functions on coverings over an ultra-Liouville base.

**1.11. Proposition.** *a) Let the action of a subgroup $G \subseteq I(X)$ on a Riemannian manifold resp. on a complex space $X$ be ultra-Liouville. Then any $G$-orbit in $X$ is a uniqueness set for the function space*[9] $\mathcal{H}(X)$.

*b) If the induced diagonal $G$-action $g$: $(x, y) \longmapsto (gx, gy)$ on $X \times X$ is ultra-Liouville, then $X$ is Liouville.*

**1.12. Remark.** It follows from Proposition 1.11($b$) that a complex space $X$ is Liouville whenever the action of some subgroup $G \subseteq I(X)$ is almost doubly transitive on $X$, meaning that the induced diagonal $G$-action on $X \times X$ possesses a dense orbit. This simple observation yields yet another proof of the classical Liouville Theorem (the affine transformation group Aff $(\mathbb{C})$ = Aut $(\mathbb{C})$ is doubly transitive on $\mathbb{C}$).

The following theorem provides us with an important information about the period group of a bounded harmonic resp. holomorphic function. Actually, this more general result stays behind the proof of Theorem 1.6.

**1.13. Theorem** [Li, Thms. 2.10, 3.9]. *Let, as before, $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a subgroup of the group $I(X)$ = Homo $(X)$ resp. $I(X)$ = Aut $(X)$. Assume that one of the following two conditions is fulfilled:*

* *$G$ is amenable and its action on $X$ is ultra-Liouville;*
* *$G$-action on $X$ is cocompact.*

*Let $f$ be a bounded harmonic resp. holomorphic function on $X$. Suppose that an element $s \in I(X)$ satisfies the condition $[s, G] \subseteq I(X)_f$* (see 1.10). *Then $s \in I(X)_f$. In other words, if $f$ is invariant under all the commutators $[s, g]$, $g \in G$, then $f$ is also $s$-invariant.*

[9] i. e., if $\varphi \in \mathcal{H}$, $x_0 \in X$, and $\varphi \mid Gx_0 = 0$, then $\varphi = 0$.

**1.14. Remark.** Under the hypothesis of Theorem 1.13, either of the above conditions $(*)$ implies:

1. The center $Z(G)$ of $G$ is always contained in the $\mathcal{H}$-period subgroup $G_{\mathcal{H}}$, where $\mathcal{H}$ is the space $Harm^\infty(X)$ resp. $H^\infty(X)$ of all bounded harmonic resp. holomorphic functions on $X$. Moreover, the transfinite induction shows that the members $Z_\alpha(G)$ of the transfinite upper central series of $G$ are contained in $G_{\mathcal{H}}$. Thus, the union $Z_{\lim}(G) = \bigcup_\alpha Z_\alpha(G)$ is contained in the $\mathcal{H}$-period subgroup $G_{\mathcal{H}}$ as well. Hence, $X$ is Liouville if $G$ is hyper-nilpotent or almost hyper-nilpotent. This proves Theorem 1.6.

2. If the center $Z(G)$ of $G$ is nontrivial, then the space $X$ is not Carathéodory [Li]. In particular, a regular covering $X$ over a quasiprojective variety cannot be Carathéodory hyperbolic whenever its Galois group is amenable and has a nontrivial center. We shall see in §3 that the latter statement may be wrong (even for solvable coverings of compact Riemann surfaces) if one omits the condition that the center is nontrivial.

3. If an element $s \in G$ is central in a finite index subgroup $S \subseteq G$ (or, more general, lies in the centralizer of such a subgroup in $G$) then $s$ is an $\mathcal{H}$-period: $s \in G_{\mathcal{H}}$ [Li, Lemma 3.3 and Th. 3.4]. That is, if the conjugacy class $s^G = \{g^{-1}sg \mid g \in G\}$ of an element $s \in G$ is finite, then any function $h \in \mathcal{H}$ is constant on the $s^G$-orbit $s^G x$ of any point $x \in X$. An element with the finite conjugacy class is called an *FC-element*; in §2 we study some analogs, generalizations, and applications of this property.

## Some proofs

Following the scheme suggested in [Li], we sketch here the proofs of Proposition 1.11 and Theorem 1.13.

We denote by $\beta G$ the Stone-Čech compactification of a discrete topological space $G$, i. e. the Gel'fand spectrum of the Banach algebra $L^\infty(G)$ of all bounded functions $G \to \mathbb{C}$. Recall that the space $\beta G$ is compact and Hausdorff, and $L^\infty(G) \cong C(\beta G)$. For $f \in L^\infty(G)$ we denote by $\hat{f}$ the unique continuous extension of $f$ to $\beta G$, and by $M(f) \subseteq \beta G$ the peak point set of the function $\hat{f}$:

$$M(f) = \left\{ \xi \in \beta G \mid \left|\hat{f}(\xi)\right| = \left\|\hat{f}\right\|_{C(\beta G)} \right\}.$$

If $G$ is a discrete group, then its right action onto itself extends to a right $G$-action on $\beta G$.

Let $X$ be a Riemannian manifold resp. a complex space, and let $G$ be a subgroup of the group $I(X)$ (see 1.5). For any function $h \in \mathcal{H} = \mathcal{H}(X)$ (see 1.10) we set $\|h\|_X = \sup_{x \in X} |h(x)|$. Let $\mathcal{K} = \mathcal{K}(X)$ denote the convex cone of all nonnegative bounded continuous subharmonic resp. plurisubharmonic functions on $X$.

**1.15. Proposition** [Li]. *Let $X, G$, and $\mathcal{H}$ be as above. Assume that*

*(i) the $G$-action on $X$ is ultra-Liouville, i. e. the cone $\mathcal{K}$ contains no nonconstant $G$-invariant function.*

*Let $h \in \mathcal{H}$. Set $h_x(g) = h(gx)$ and $\varphi_h(x) = \left\|\widehat{h_x}\right\|_{C(\beta G)}$. Then*

*a)* $\varphi_h = \text{const}$ *and* *b)* $\left\|\widehat{h_x}\right\|_{C(\beta G)} \equiv \varphi_h = \|h\|_X$ *;*

*c) the peak point set $M(h) = M(\widehat{h_x}) \subseteq \beta G$ of the function $\widehat{h_x}$ does not depend on $x \in X$ and is a $G$-invariant subset of $\beta G$;*

*d) for any $G$-invariant regular probability Borel measure $\mu$ on $\beta G$ the $L^2(\mu)$-class $\left[\widehat{h_x}\right]$ of the function $\widehat{h_x}$ does not depend on $x \in X$.*

*If, in addition, the group $G$ is amenable, then*

*e) $\beta G$ carries a $G$-invariant probability measure $\mu$ supported in $M(h)$;*

*f) $h = 0$ whenever $\left[\widehat{h_x}\right] = 0$ in $L^2(\mu)$ for a measure $\mu$ as in (e).*

*Sketch of the proof.* Note that the space $\mathcal{H}$ and the convex cone $\mathcal{K}$ satisfy the following two conditions $(ii)$, $(iii)$:

*(ii) $\mathcal{H}$ contains all the constant functions, and for any $\|\cdot\|_X$-bounded subset $\mathcal{F} \subset \mathcal{H}$ the function $k^2_{\mathcal{F}}$, $k^2_{\mathcal{F}}(x) = \sup_{f\in\mathcal{F}} |f(x)|^2$, belongs to the cone $\mathcal{K}$;*

*(iii) for any closed ball $B$ in the space $BC(X)$ of all complex valued bounded continuous functions on $X$ the sets $\mathcal{H}\cap B$ and $\mathcal{K}\cap B$ are closed in $BC(X)$ with respect to the compact open topology.*

Thus, by $(ii)$, $\varphi_h^2 \in \mathcal{K}$. Since the function $\varphi_h^2$ is $G$-invariant, $(a)$ follows from $(i)$. Clearly,

$$\varphi_h \equiv \varphi_h(x) = \left\|\widehat{h_x}\right\|_{C(\beta G)} = \|h_x\|_{L^\infty(G)} = \sup_{g\in G} |h(gx)| \le \sup_{y\in X} |h(y)| = \|h\|_X \,.$$

Would the latter inequality be strict, then for some $x_\circ \in X$ we would have

$$\begin{aligned}\varphi_h < |h(x_\circ)| &\le \sup_{g\in G} |h^g(x_\circ)| = \sup_{g\in G} |h(gx_\circ)| \\ &= \sup_{g\in G} |h_{x_\circ}(g)| = \left\|\widehat{h_{x_\circ}}\right\|_{C(\beta G)} = \varphi_h(x_\circ) = \varphi_h \,,\end{aligned}$$

which is impossible; this proves $(b)$.

Given $x_\circ \in X$ and a point $\xi_\circ$ in the peak point set $M(\widehat{h_{x_\circ}})$, consider the function $h^{\xi_\circ}(x) = \widehat{h_x}(\xi_\circ)$. It follows from $(iii)$ that this function is in $\mathcal{H}$, and the function $\left|h^{\xi_\circ}(x)\right|$ attains its maximal value $\|h\|_X$ at the point $x = x_\circ$. The Maximum Principle

*(iv)* $h = \text{const}$ *whenever* $h \in \mathcal{H}$ *and* $|h(x_\circ)| = \|h\|_X$ *for some point* $x_\circ \in X$

implies that $h^{\xi_\circ} = \text{const}$. Hence,

$$\left|\widehat{h_x}(\xi_\circ)\right| = \left|h^{\xi_\circ}(x)\right| \equiv \left|h^{\xi_\circ}(x_\circ)\right| = \|h\|_X = \left|\widehat{h_{x_\circ}}(\xi_\circ)\right| .$$

This shows that $\xi_\circ \in M(\widehat{h_x})$ for any $x \in X$, which proves the first assertion of $(c)$. The constant function $h^{\xi_\circ}$ is certainly $G$-invariant, and hence $h^{\xi_\circ g} = h^{\xi_\circ}$ for any $g \in G$. This yields $\left|\widehat{h_{x_\circ}}(\xi_\circ g)\right| = \left|h^{\xi_\circ g}(x_\circ)\right| = \left|h^{\xi_\circ}(x_\circ)\right| = \|h\|_X$ and $\xi_\circ g \in M(\widehat{h_{x_\circ}}) = M(h)$, which proves the second assertion of $(c)$.

Given a $G$-invariant regular probability Borel measure $\mu$ on $\beta G$, define the function

$$\Phi^2\colon X \to \mathbb{R}, \qquad \Phi^2(x) = \left\| \left[\widehat{h_x}\right] \right\|^2_{L^2(\mu)} = \int_{\beta G} \left|\widehat{h_x}(\xi)\right|^2 d\mu(\xi) .$$

It is $G$-invariant, and it follows from $(ii)$ and $(iii)$ that $\Phi^2 \in \mathcal{K}$; by $(i)$, $\Phi^2 = \text{const}$. Fix a point $x_\circ \in X$, and consider the mapping $X \ni x \longmapsto F(x) = \left[\widehat{h_x}\right] \in L^2(\mu)$ and the inner product $\psi(x) = \langle F(x)\,, F(x_\circ)\rangle$. It follows from $(iii)$ that $\psi \in \mathcal{H}$. Clearly,

$$|\psi(x)| \le \|F(x)\|_{L^2(\mu)} \|F(x_\circ)\|_{L^2(\mu)} = \Phi(x)\Phi(x_\circ) \equiv \Phi^2(x_\circ) \quad \text{and} \quad |\psi(x_\circ)| = \Phi^2(x_\circ)\,;$$

hence, by the Maximum Principle $(iv)$, $\langle F(x)\,, F(x_\circ)\rangle \equiv \text{const}$. Set $a = F(x_\circ)$ and $b = F(x)$; then we have $\langle b,\, a\rangle = \|a\|^2$ and $\|b\| = \|a\|$. Since the norm in the Hilbert space $L^2(\mu)$ is strictly convex, this implies $b = a$, that is, $F = \text{const}$. This proves $(d)$.

The statement $(e)$ follows from $(c)$ and the Fixed Point Theorem for amenable groups [Gre, Thm. 3.3.5] applied to the natural $G$-action on the convex compactum of all probability measures supported in the $G$-invariant set $M(h)$.

Finally, $(f)$ follows from $(e)$. Indeed, the function $\widehat{h_x}$ is continuous on $C(\beta G)$, and hence $\left[\widehat{h_x}\right] = 0$ implies $\widehat{h_x} \mid \operatorname{supp}\mu = 0$. Since $\operatorname{supp}\mu \subseteq M(h) = M(\widehat{h_x})$, it follows that $\widehat{h_x} = 0$ for any $x \in X$. Thus, $h = 0$. ☐

**1.16. Remark.** Let $X$ be a topological space endowed with a $G$-action preserving a subspace $\mathcal{H} \subset BC(X)$ and a convex cone $\mathcal{K} \subset BC_{\mathbb{R}}(X)$. Assume that the conditions $(i)$ – $(iv)$ introduced above are fulfilled. *All the assertions of Proposition* 1.15 *hold true in this more general setting.* This yields analogs of Theorems 1.6 and 1.13 for certain equivariant second order elliptic operators on smooth manifolds and for harmonic functions on discrete groups (see [Li, 2.15]).

**1.17.** *Proof of Proposition* 1.11. $(a)$ is an immediate consequence of Proposition 1.15$(a, b)$. To prove $(b)$, fix a function $h \in \mathcal{H}(X)$. Note that the function

$$\widetilde{h}(x,\, y) = h(x) - h(y) \in \mathcal{H}(X \times X)$$

vanishes on the diagonal $\Delta \subset X \times X$, and therefore, it vanishes on any $G$-orbit contained in $\Delta$. By our assumption, the diagonal action of $G$ on $X \times X$ satisfies the condition of $(a)$ and leaves the diagonal $\Delta$ invariant. Thus, by $(a)$, $\widetilde{h} = 0$, and hence $h = \text{const}$. Therefore, $X$ is Liouville. ☐

**1.18.** *Proof of Theorem* 1.13 *for amenable $G$ and an element $s \in G$.* Actually, as in the proof of Proposition 1.15, the only important assumption about the space $\mathcal{H}$ and the convex cone $\mathcal{K}$ is that the conditions $(i)$-$(iv)$ mentioned above are fulfilled. We deal with a function $f \in \mathcal{H}$ and an element $s \in G$ such that $f^{[s,G]} = f$, where $f^{[s,g]}(x) = f([s, g]x)$. Thus,

$$f_{sx}(g) = f(gsx) = f(sgx) = f^s(gx) = (f^s)_x(g)$$

for all $g \in G$ and all $x \in X$, and hence

$$\widehat{f_{sx}} = \widehat{(f^s)_x} \qquad \text{for all } x \in X\,. \tag{$*$}$$

Set $h = f^s - f \in \mathcal{H}$. We must show that $h = 0$. Let $\mu$ be a measure as in Proposition 1.15($e$). Since it is $G$-invariant, Proposition 1.15($d$) implies that the $L^2(\mu)$-class $\left[\widehat{f_x}\right]$ does not depend on $x \in X$. In particular, $\left[\widehat{f_{sx}}\right] = \left[\widehat{f_x}\right]$ and

$$\left[\widehat{f_{sx} - f_x}\right] = \left[\widehat{f_{sx}} - \widehat{f_x}\right] = \left[\widehat{f_{sx}}\right] - \left[\widehat{f_x}\right] = 0\,.$$

Combined with ($*$) this leads to $\left[\widehat{(f^s)_x - f_x}\right] = 0$. Thus, $\widehat{h_x} = \left[\widehat{((f^s - f)_x)}\right] = 0$. Proposition 1.15($f$) implies that $h = 0$. □

If the element $s \in I(X)$ we deal with is not contained in the subgroup $G$, it can happen that there is no an amenable subgroup in $I(X)$ containing both $s$ and $G$. To treat this case one should work with actions in function spaces which are not induced by any action in $X$ (see [Li] for details).

As to the case of a cocompact $G$-action, the proof of Theorem 1.13 given in [Li] is mainly based on the compactness principle and a version of the Harnack inequality. This approach goes back to E. Dynkin, M. Malyutov, and G. Margulis who considered bounded and positive harmonic functions on nilpotent groups (see [DyMa, Ma]).

## §2. Liouville property of FC-nilpotent coverings

Here we give a generalization of Theorem 1.13 on the period subgroup, based on the notion of the *upper FC-series* of a group [Hai]. Namely, we show that the period subgroup of the bounded harmonic resp. holomorphic functions on a covering of an ultra-Liouville manifold contains the union of members of the upper FC-series of the covering group (see (2.1) and Corollary 2.5 below). This leads to a generalization of Theorem 1.6 resp. of Corollary 1.8 on the Liouville property of coverings (see Corollary 2.6 below).

**2.1. Definitions.** 1. *FC-groups and FC-series.* A group $G$ is called *FC-group* [Ba] (see also [Ku, To]) if the conjugacy class of each element of $G$ is finite. For example, any almost abelian group or a group with a finite commutator subgroup is so [Neu]. Both of the latter classes contain the proper subclass of groups with a central subgroup of finite index [Neu] (see also [Er; To, Thm.1.1]). It is known [Ba; To, Thm. 1.4] that the quotient $G/Z(G)$ of an FC-group $G$ by its center is a periodic group.

For any group $G$ the union $FC(G)$ of all finite conjugacy classes is a normal subgroup of $G$. Clearly, $FC(G)$ is an FC-group; it is called the *FC-center* of $G$ [To]. By the *upper FC-series* of $G$ [Hai] one means the normal series

$$1 \lhd FC_1(G) \lhd FC_2(G) \lhd \cdots \lhd FC_n(G) \lhd \cdots \lhd G\,,$$

where $FC_1(G) = FC(G)$ and $FC_{n+1}(G)$ is the total preimage of $FC\,(G/FC_n(G))$ under the natural surjection $G \to G/FC_n(G)$, $n \geq 1$. Clearly, $FC_n(G)$ is a normal subgroup of $G$; in fact, it is a strictly characteristic[10] subgroup [Hai]. The upper FC-series may be extended transfinitely [Du], by defining $FC_\alpha(G) = \bigcup_{\beta<\alpha} FC_\beta(G)$, when $\alpha$ is a limit ordinal. Set $FC_\omega(G) = \bigcup_{n\in\mathbb{N}} FC_n(G)$ and $FC_{\lim}(G) = \bigcup_\alpha FC_\alpha(G)$, where $\alpha$ runs over all the ordinals.

2. *FC-nilpotent and hyper-FC-nilpotent groups* [Hai, Du]. If $G = FC_n(G)$ and $G \neq FC_{n-1}(G)$ for some $n \in \mathbb{N}$, then $G$ is called *FC-nilpotent of class* $n$, or simply *FC-nilpotent.* We say that $G$ is $\omega$*-FC-nilpotent* resp. *hyper-FC-nilpotent* if $G = FC_\omega(G)$ resp. $G = FC_{\lim}(G)$. Clearly, an $\omega$-FC-nilpotent group is *locally FC-nilpotent,* i. e. any finitely generated subgroup of $G$ is FC-nilpotent.

**2.2. Remarks.** 1. A nilpotent (resp. $\omega$-nilpotent, hyper-nilpotent, locally nilpotent) group is FC-nilpotent (resp. $\omega$-FC-nilpotent, hyper-FC-nilpotent, locally FC-nilpotent). It is easily seen that FC-nilpotence, $\omega$-FC-nilpotence, and hyper-FC-nilpotence and are preserved under finite extensions and passing to a subgroup or to a quotient group. In particular, a finite extension of a nilpotent group is FC-nilpotent. Vice versa, a finitely generated FC-nilpotent group of class $n$ is a finite extension of a nilpotent group of class at most $n$ [DuMcL, Thm. 2].

2. We say that a group $G$ is *normally generated* by its elements $g_1, \ldots, g_k$ if $G$ coincides with the minimal normal subgroup $<< g_1, \ldots, g_k >>$ containing $g_1, \ldots, g_k$, or, which is the same, if the conjugacy classes of $g_1, \ldots, g_k$ generate $G$. It is easily seen that a normally finitely generated $\omega$-FC-nilpotent group $G$ is actually FC-nilpotent. If, in addition, all the members $FC_i(G)$ of the upper FC-series of $G$ are normally finitely generated, then $G$ is a finitely generated almost nilpotent group (and hence, $G$ is an almost polycyclic group).

3. Any locally FC-nilpotent (and so any $\omega$-FC-nilpotent) group $G$ is amenable. Indeed, $G$ is a union of the direct system of its finitely generated FC-nilpotent subgroups. Therefore, by Theorem 1.2.7 in [Gre], the statement follows once we know that any finitely generated FC-nilpotent group is amenable. The latter holds since a finitely generated FC-group is almost nilpotent [DuMcL, Thm. 2] (see 1 above).

The following simple example shows that in general an FC-group that is not finitely generated my be neither almost solvable nor almost $\omega$-nilpotent.

[10]A subgroup $H \subseteq G$ is called *strictly characteristic* if $\phi(H) \subseteq H$ for any surjective endomorphism $\phi\colon G \to G$.

**2.3. Example.** Let $G = \bigoplus_{n=5}^{\infty} A_n$ be the direct sum of the alternating groups $A_n \subset S_n$, where $S_n$ stays for the symmetric group. It is an FC-group. Since the summands are simple (hence, nonsolvable) groups, $G$ is not almost solvable. To see that $G$ is not almost $\omega$-nilpotent suppose, on the contrary, that there exists a normal $\omega$-nilpotent subgroup $H \subseteq G$ of finite index. Clearly, for some $n$ we have $H \cap A_n \neq \mathbf{1}$, and thus $H \cap A_n$ is a nonunit normal subgroup of the simple group $A_n$. Hence, $H \cap A_n = A_n$, that is, $A_n \subseteq H$. However, $\omega$-nilpotent group $H$, being the increasing union of the nilpotent subgroups $Z_k(H)$ (the members of its upper central series), cannot contain a finite nonsolvable subgroup.

The concept of FC-nilpotence occurs convenient to establish the Liouville property of some coverings. The following lemma is an easy consequence of Theorem 1.13.

**2.4. Lemma.** *Let $X$ be a Riemannian manifold resp. a complex space, $\mathcal{H}$ be the space of all bounded harmonic resp. holomorphic functions on $X$, and $G \subseteq I(X)$ be a subgroup of the group $I(X) = \mathrm{Homo}(X)$ resp. $I(X) = \mathrm{Aut}(X)$. Suppose that one of the two conditions of Theorem 1.13 is fulfilled, i. e. either*

* *$G$ is amenable and its action on $X$ is ultra-Liouville, or*
* *the $G$-action on $X$ is cocompact.*

*Let $N \lhd G$ be a normal subgroup, and let $s \in G$ be an element such that its image $\bar{s}$ in the quotient group $G/N$ has a finite conjugacy class. If $N$ is contained in the $\mathcal{H}$-period subgroup $G_{\mathcal{H}}$, then $s \in G_{\mathcal{H}}$, too.*

*Proof.* By our assumption, the centralizer $\bar{C}$ of the element $\bar{s} \in G/N$ is of finite index in $G/N$. The total preimage $\bar{C}$ of $C$ in $G$ is a subgroup of finite index. Therefore (see [Li, Lemma 3.3]), $C$ satisfies the same condition $(*)$ as $G$. Furthermore, $C$ contains both $s$ and $N$. Since $\bar{s}$ is central in $\bar{C}$ we have $[s, C] \subseteq N \subseteq G_{\mathcal{H}}$. By Theorem 1.13, this implies that $s \in G_{\mathcal{H}}$. □

**2.5. Corollary.** *Suppose that one of the conditions $(*)$ of Lemma* 2.4 *is fulfilled. Then* $FC_{\lim}(G) \subseteq G_{\mathcal{H}}$.

*Proof.* Starting with the unit subgroup $\mathbf{1} \subset G_{\mathcal{H}}$, we proceed by transfinite induction. Assume that $FC_{\alpha}(G) \subseteq G_{\mathcal{H}}$. Set $N = FC_{\alpha}(G) \lhd G$. By Lemma 2.5, for any element $s \in FC_{\alpha+1}(G)$ we have $s \in G_{\mathcal{H}}$, and thus $FC_{\alpha+1}(G) \subseteq G_{\mathcal{H}}$. Furthermore, if $\alpha$ is a limit ordinal and $FC_{\beta}(G) \subseteq G_{\mathcal{H}}$ for all $\beta < \alpha$, then $FC_{\alpha}(G) = \bigcup_{\beta<\alpha} FC_{\beta}(G) \subseteq G_{\mathcal{H}}$. By induction, it follows that $FC_{\lim}(G) = \bigcup_{\alpha} FC_{\alpha}(G) \subseteq G_{\mathcal{H}}$. □

**2.6. Corollary[11].** *Let $X \to Y$ be a regular covering over a compact Riemannian resp. Kähler manifold with the Galois group $G$. If $G$ is an extension of a hyper-FC-nilpotent group by a Varopoulos group, then $X$ is Liouville.*

**2.7. Remark.** As follows from Corollary 2.5, under one of the assumptions $(*)$ of Lemma 2.4 the period subgroup $G_{\mathcal{H}} \lhd G$ has the following property: the FC-center $FC(G/G_{\mathcal{H}}$ is trivial, i. e. all the conjugacy classes of the elements of $G/G_{\mathcal{H}}$ different from $\mathbf{e}$ are infinite. Clearly, the subgroup $FC_{\lim}(G) \lhd G_{\mathcal{H}}$ has the same property. It would be interesting to find an example (if it does exist) in which $FC_{\lim}(G) \neq G_{\mathcal{H}}$.

[11] cf. Corollary 1.8 above.

## §3. On solvable Carathéodory hyperbolic coverings of a compact Riemann surface

A complex space $X$ is called *Carathéodory hyperbolic* if the algebra $H^\infty(X)$ of the bounded holomorphic functions on $X$ separates the points of $X$. In [LySu] for arbitrary compact Riemann surface $Z$ of genus $g \geq 2$ there was constructed a non-Liouville Galois covering $X \to Z$ with a metabelian (i. e. two-step solvable) Galois group. Modifying the construction of Lyons and Sullivan, we prove the following theorem.

**3.1. Theorem.** *For any compact Riemann surface $Z$ of genus $g \geq 2$ there exists a metabelian Carathéodory hyperbolic covering $X \to Z$ over $Z$.*

*Proof.* There is a natural one-to-one correspondence between abelian coverings of $Z$ and those normal subgroups of $\pi_1(Z)$ that contain the commutator subgroup $\pi_1'(Z)$ of $\pi_1(Z)$. The *maximal abelian covering* over $Z$ is the Galois covering corresponding to the commutator subgroup $\pi_1'(Z)$; it dominates any other abelian covering of $Z$.

Let $Y \to Z$ be a covering over $Z$ with a free abelian Galois group $G$ of $\operatorname{rank} G \geq 3$ (for instance, the maximal abelian covering). By a theorem of A. Mori [Mo] (see also [Ts, Theorem X.46]), for an arbitrary point $y \in Y$ there exists a unique positive Green function, say $g_y$, with pole at $y$.

Let $D \subset Y \setminus \{y\}$ be a simply connected domain. Then there is a conjugate harmonic function $g_y^*$ of $g_y$ in $D$, which is defined uniquely up to an additive real constant. Therefore, the differential $\omega_y = df_y$, where $f_y = g_y + ig_y^*$, is a well-defined holomorphic 1-form on $Y\setminus\{y\}$. Its real part $\operatorname{Re}\omega_y = dg_y$ is an exact 1-form on $Y\setminus\{y\}$. Hence, the real part of each period $\int_\gamma \omega_y$ of $\omega_y$, where $\gamma \in H_1(Y \setminus \{y\}; \mathbb{Z})$, is zero. Thus, $\omega$ defines a homomorphism $H_1(Y \setminus \{y\}; \mathbb{Z}) \to i\mathbb{R}$.

Fix a point $z_0 \in Y \setminus \{y\}$. For any particular choice of $g_y^*$ consider the function

$$\varphi_y(z) = \exp\left(-2\pi f_y(z)\right) = \exp\left(-2\pi\left(f_y(z_0) + \int_{z_0}^{z} \omega_y\right)\right).$$

This is a multi-valued holomorphic function on $Y$ with values in the unit disc $\mathbb{D}$. For a given $y \in Y$ any two such functions coincide up to a constant factor $\lambda \in S^1$, where $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. For any $y \in Y$ choose, once forever, one of the functions $\varphi_y$.

Any two values of $\varphi_y$ differ by a factor of the form $\exp\left(-2\pi\int_\gamma \omega_y\right) \in S^1$, where $\gamma \in H_1(Y; \mathbb{Z})$. More precisely, we have a well-defined character

$$\alpha_y\colon H_1(Y \setminus \{y\}; \mathbb{Z}) \ni \gamma \longmapsto \exp\left(-2\pi\int_\gamma \omega_y\right) \in S^1.$$

Actually, it yields a character

$$\alpha_y\colon H_1(Y; \mathbb{Z}) \to S^1.$$

Indeed, consider the exact sequence

$$0 \to \mathbb{Z} \to H_1(Y \setminus \{y\}; \mathbb{Z}) \to H_1(Y; \mathbb{Z}) \to 0,$$

where the subgroup $\mathbb{Z} \subset H_1(Y \setminus \{y\};\, \mathbb{Z})$ is generated by a very small circle $\sigma_\epsilon$ in $Y$ centered at $y$. In a small disc $\delta_y$ around $y$ we have $g_y(z) = -\frac{1}{2\pi}\log|z-y| + h_y(z)$, where $h_y$ is a single-valued harmonic function in $\delta_y$; hence, $f_y(z) = -\frac{1}{2\pi}\log(z-y) + \widetilde{f}_y(z)$, where $\widetilde{f_y}$ is a single-valued holomorphic function in $\delta_y$. It follows that

$$2\pi \int_{\sigma_\epsilon} \omega_y = 2\pi \int_{\sigma_\epsilon} \left(-\frac{1}{2\pi} d\log(z-y) + d\widetilde{f}_y\right) \in 2\pi i\mathbb{Z}\,.$$

Thereby, $\exp\left(-2\pi \int_{\sigma_y} \omega_y\right) = 1$, the restriction of the homomorphism $\alpha_y$ to the kernel subgroup $\mathbb{Z}$ in the above exact sequence is trivial, and $\alpha_y$ can be pushed down to the quotient group.

The set of values of the function $\varphi_y$ at a point $z \in Y \setminus \{y\}$ coinsides with a coset of the subgroup Image$\,(\alpha_y)$ in the multiplicative group $\mathbb{C}^*$, whereas all its values at the point $y$ are zero.

Let $\rho\colon\ \pi_1(Y) \to H_1(Y;\,\mathbb{Z}) \cong \pi_1(Y)/\pi_1'(Y)$ be the canonical surjection. Set $\widehat{\alpha}_y = \alpha_y \circ \rho$. The covering $X_y \to Y$ over $Y$ corresponding to the subgroup Ker $\widehat{\alpha_y} \lhd \pi_1(Y)$ is the minimal one such that the function $\varphi_y$ becomes single-valued when lifted to $X_y$. Set

$$K = \bigcap_{y \in Y} \operatorname{Ker} \alpha_y \subset H_1(Y;\,\mathbb{Z})\,,$$

and $\widehat{K} = \rho^{-1}(H) \subset \pi_1(Y)$. Let $p\colon\ X \to Y$ be the abelian covering over $Y$ associated with the subgroup $\widehat{K} \lhd \pi_1(Y)$. Clearly, this is the minimal covering over $Y$ such that all the functions $\{\varphi_y\}_{y\in Y}$ become single-valued when lifted to $X$. Let $E = \{\widehat{\varphi_y}\}_{y\in Y} \subset H^\infty(X)$ be the collection of all the lifted functions. We will show that $E$ separates the points of $X$. Since $X \to Z$ is a metabelian covering this proves the theorem.

Denote by $F_y = p^{-1}(y) \subset X$ the fiber of $p$ over $y \in Y$. For any two distinct points $y, y' \in Y$ the function $\widehat{\varphi_y}$ vanishes identically on $F_y$ and does not vanish at the points of $F_{y'}$. Therefore, $E$ separates the fibers $\{F_y\}$.

Thus, it is sufficient to show that $E$ separates the points of each fiber $F_y$. It is easily seen that for $y' \neq y$ the function $\widehat{\varphi_y}$ separates the points of $F_{y'}$ if and only if $\operatorname{Ker}\alpha_y = K$. If the latter equality holds for a certain pair of distinct points $y_1,\, y_2 \in Y$, then the points of each fiber $F_y,\ \ y \in Y$, are separated by at least one of the functions $\widehat{\varphi_{y_1}},\ \widehat{\varphi_{y_2}}$. Hence, the theorem follows from the next claim.

**Claim 1.** *There exists a countable union* $C = \bigcup_{n\in\mathbb{N}} C_n \subset Y$ *of real analytic curves* $C_n$ *in* $Y$ *such that* $\operatorname{Ker}\alpha_y = K$ *for each point* $y \in Y \setminus C$.

The proof is based on the following statement[12]:

**Claim 2.** *The function of two complex variables* $g(y, y') = g_y(y')$ *is harmonic on the complement* $(Y \times Y) \setminus \Delta$, *where* $\Delta \subset Y \times Y$ *is the diagonal.*

[12]It should be well known; for the sake of completeness we give a simple proof.

*Proof of Claim* 2. By the symmetry property of Green function [Ts, Theorem I.16], we have $g_y(y') = g_{y'}(y)$ for any $y \neq y',\ y,\, y' \in Y$. Hence, $g(y,\, y')$ is a harmonic function in each argument on $(Y \times Y) \setminus \Delta$. It is sufficient to show that it is harmonic as a function of two complex variables in each bidisc $\delta \times \delta' \subset\subset (Y \times Y) \setminus \Delta$, where $\delta,\ \delta'$ are two small discs in $Y$. Being harmonic in each variable, the function $g(y,\, y')$ in the bidisc $\delta \times \delta'$ satisfies the Laplace equation $\Delta_{y,y'} g(y,\, y') = \Delta_y g(y,\, y') + \Delta_{y'} g(y,\, y') = 0$, where $\Delta_y,\ \Delta_{y'}$ are the usual Laplacians. Therefore, $g(y,\, y')$ is harmonic in $\delta \times \delta'$ as soon as it is continuous there.

Since the function $g(y,\, 0)$ is continuous in the closed disc $\bar{\delta}$, the family $g_y = g_y(y')$ of positive harmonic functions in $\delta'$ is equicontinuous in every smaller closed disc (the standard proof of this fact follows by the Harnack inequality). This implies that $g = g(y,\, y')$ is a continuous function in $\delta \times \delta'$, which completes the proof. □

*Proof of Claim* 1. It is sufficient to check our statement locally. Fix a small disc $\delta \subset Y$. We will show that $\operatorname{Ker} \alpha_y = K$ for all $y \in \delta$ outside of a countable union $C_\delta \subset \delta$ of closed real analytic curves in $\delta$.

It follows from Claim 2 that in each local chart $\Omega$ in $Y$ the coefficients of the holomorphic 1-form $\omega_y$ are real analytic functions of $y \in Y \setminus \Omega$.

Let a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of 1-cycles in $Y$ be a free basis of the homology group

$$H_1(Y;\, \mathbb{Z}) \cong \mathbb{Z}^\infty = \bigoplus_1^\infty \mathbb{Z}\,.$$

We may assume that they do not meet the closed disc $\bar{\delta}$. The periods $c_n(y) = \int_{\gamma_n} \omega_y$, $n \in \mathbb{N}$, are real analytic functions of $y \in \delta$. For $\gamma = \sum_{j=1}^n a_j \gamma_j \in H_1(Y;\, \mathbb{Z})$ we have

$$\langle \gamma,\, \omega_y \rangle = \int_\gamma \omega_y = \sum_{j=1}^n a_j \int_{\gamma_j} \omega_y = \sum_{j=1}^n a_j c_j(y) = \langle \bar{a},\, c(y) \rangle\,,$$

where $\bar{a} = (a_1,\, \ldots, a_n,\, 0, \ldots)$ and $c(y) = (c_j(y))_{j=1}^\infty$. By the definition of the character $\alpha_y\colon\ H_1(Y;\, \mathbb{Z}) \to S^1$, we have

$$\operatorname{Ker} \alpha_y = \{\gamma \in H_1(Y;\, \mathbb{Z}) \mid \langle \gamma,\, i\omega_y \rangle = \langle \bar{a},\, ic(y) \rangle \in \mathbb{Z}\}\,.$$

Set

$$L = \{\bar{a} \in \mathbb{Z}^\infty \mid \langle \bar{a},\, ic(y) \rangle \in \mathbb{Z} \text{ for all } y \in \delta\}\,.$$

For each $\bar{a} \in \mathbb{Z}^\infty \setminus L$ and for each $k \in \mathbb{Z}$ consider the real analytic curve

$$C_{\bar{a},\, k} = \{y \in \delta \mid \langle \bar{a},\, ic(y) \rangle = k\}\,.$$

Put

$$C_\delta = \bigcup_{\bar{a} \in \mathbb{Z}^\infty \setminus L;\ k \in \mathbb{Z}} C_{\bar{a},\, k}\,.$$

It is easily seen that for $y \in \delta \setminus C_\delta$ the subgroup $\operatorname{Ker} \alpha_y \subset H_1(Y;\, \mathbb{Z})$ does not depend on $y$ and coincides with $L$. Furthermore, for any $y \in C_\delta$ we have $\operatorname{Ker} \alpha_y \supset L$, and hence $L = K$. This proves Claim 1 and completes the proof of Theorem 3.1. □

## §4. $H^\infty$-HULLS IN A SOLVABLE COVER OF INOUE SURFACE

In this section we study in more details the universal covering $\pi$: $X \to \mathcal{I}$ over one of the Inoue surfaces $\mathcal{I}$ [In]. We start with a description of the Inoue surface.

Let $A \in \mathrm{SL}(3; \mathbb{Z})$ be a matrix with one real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta, \overline{\beta} \in \mathbb{C} \setminus \mathbb{R}$ (certainly $|\beta| < 1$). Let $\mathbf{a} = (a_1, a_2, a_3)$ resp. $\mathbf{b} = (b_1, b_2, b_3)$ be a real resp. a complex eigenvector of $A$ corresponding to the eigenvalue $\alpha$ resp. $\beta$.

Set $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ (the upper halfplane) and $X = \mathbb{H} \times \mathbb{C}$. Consider the subgroup $G \subset \mathrm{Aut}\, X$ generated by the following four automorphisms $g_j$:

$$g_0(z,w) = (\alpha z, \beta w), \qquad g_j(z,w) = (z + a_j, w + b_j), \quad 1 \le j \le 3, \quad (z,w) \in X = \mathbb{H} \times \mathbb{C}.$$

The action of the group $G$ on $X$ is free, properly discontinuous, and cocompact. The smooth compact complex surface $\mathcal{I} = X/G$ is one of the *Inoue surfaces* [In].

The subgroup $G_0 \subset G$ generated by $g_1, g_2, g_3$ is isomorphic to $\mathbb{Z}^3$; this subgroup is normal in $G$, and the quotient group $G/G_0$ is isomorphic to $\mathbb{Z}$. Thus, we have the exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow G \stackrel{\tau}{\longrightarrow} \mathbb{Z} \longrightarrow 0\,, \tag{1}$$

and the corresponding tower of the abelian coverings

$$X \stackrel{G_0 \cong \mathbb{N}^3}{\longrightarrow} Y \stackrel{\mathbb{N}}{\longrightarrow} \mathcal{I}\,, \quad \text{where } Y = X/G_0\,.$$

In particular, $G$ is a metabelian (i. e. two-step solvable) polycyclic group, and $X \to \mathcal{I}$ is a polycyclic covering with the Galois group $G$.

We intend to establish certain analytic properties of the covering $X \to \mathcal{I}$ and its fibers. To this end, we need some simple algebraic observations.

First, note that the sequence (1) splits; a splitting $\rho$: $\mathbb{Z} \to G$ $\;(\tau \circ \rho = \mathrm{id}_{\mathbb{Z}})$ may be defined by $\mathbb{Z} \ni m \mapsto g_0^m \in G$. Therefore, $G$ is a semi-direct product $\mathbb{Z}^3 \rightthreetimes \mathbb{Z}$, and any element $g \in G$ admits a unique representation of the form

$$g = g_0^m \widetilde{g} = g_0^m g_1^{r_1} g_2^{r_2} g_3^{r_3}, \quad \text{where } \; m = \tau(g) \in \mathbb{Z},\; r_1, r_2, r_3 \in \mathbb{Z},\; \widetilde{g} = g_1^{r_1} g_2^{r_2} g_3^{r_3} \in G_0.$$

Using this normal form, for any $d \in \mathbb{Z}$ we can write

$$g^{-1} g_0^d g = (g_0^m \widetilde{g})^{-1} \cdot g_0^d \cdot (g_0^m \widetilde{g})$$

$$= \widetilde{g}^{-1} g_0^{-m} \cdot g_0^d \cdot g_0^m \widetilde{g} = \widetilde{g}^{-1} g_0^d \widetilde{g} = g_3^{-r_3} g_2^{-r_2} g_1^{-r_1} \cdot g_0^d \cdot g_1^{r_1} g_2^{r_2} g_3^{r_3}.$$

**4.1. Lemma.** *The conjugacy class $s^G$ of the element $s = g_0^d$ consists of all the transformations of the form*

$$(z,w) \mapsto \left(\alpha^d z + (\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3),\, \beta^d w + (\beta^d - 1)(r_1 b_1 + r_2 b_2 + r_3 b_3)\right)\,,$$

*where $r_1$, $r_2$, and $r_3$ run over $\mathbb{Z}$.*

*Proof.* Since the elements $g_j$, $j = 1, 2, 3$, commute, the lemma follows from (2) and the formulae

$$g_j^{-r} g_0^d g_j^r(z, w) = (\alpha^d z + (\alpha^d - 1)r a_j, \beta^d w + (\beta^d - 1) r b_j). \qquad \square$$

**4.2. Lemma.** *a) The real eigenvalue $\alpha$ of the matrix $A$ is a nonquadratic irrationality.*

*b) The coordinates $a_1, a_2, a_3$ of the corresponding eigenvector* **a** *are linearly independent over $\mathbb{Q}$.*

*c) For any subgroup $L \subseteq \mathbb{Z}^3$ of rank $\mathrm{rk}\, L \geq 2$ and for any finite subset $S \subset L$ we have*

$$\inf\{|r_1 a_1 + r_2 a_2 + r_3 a_3| \mid \mathbf{r} = (r_1, r_2, r_3) \in L - S\} = 0.$$

*Proof.* $a)$ The characteristic polynomial $P(z) = t^3 + pt^2 + qt + 1$, $p, q \in \mathbb{Z}$, of the unimodular matrix $A$ has no rational root except, possibly, of $\pm 1$. Since $\alpha, \beta \neq \pm 1$, the polynomial $P$ is irreducible over $\mathbb{Q}$, which proves $(a)$.

$b)$ Assume, on the contrary, that $a_1, a_2, a_3$ are linearly dependent over $\mathbb{Q}$. Let $A = (a_{ij})_{i,j=1}^3$, where $a_{ij} \in \mathbb{Z}$. Then we have four linear relations of the form

$$\begin{aligned} r_1 a_1 + \quad & r_2 a_2 + \quad & r_3 a_3 &= 0 \\ (a_{11} - \alpha) a_1 + \quad & a_{12} a_2 + \quad & a_{13} a_3 &= 0 \\ a_{21} a_1 + \quad & (a_{22} - \alpha) a_2 + \quad & a_{23} a_3 &= 0 \\ a_{31} a_1 + \quad & a_{32} a_2 + \quad & (a_{33} - \alpha) a_3 &= 0 \end{aligned}$$

with some $(r_1, r_2, r_3) \in \mathbb{Z}^3 \setminus \{0\}$. Since $(a_1, a_2, a_3) \neq \mathbf{0}$, we obtain the following three equations (each of degree at most 2) for $\alpha$:

$$\det \begin{pmatrix} r_1 & r_2 & r_3 \\ a_{21} & a_{22} - \alpha & a_{23} \\ a_{31} & a_{32} & a_{33} - \alpha \end{pmatrix} = 0, \qquad \det \begin{pmatrix} a_{11} - \alpha & a_{12} & a_{13} \\ r_1 & r_2 & r_3 \\ a_{31} & a_{32} & a_{33} - \alpha \end{pmatrix} = 0,$$

and

$$\det \begin{pmatrix} a_{11} - \alpha & a_{12} & a_{13} \\ a_{21} & a_{22} - \alpha & a_{23} \\ r_1 & r_2 & r_3 \end{pmatrix} = 0.$$

At least one of these equations must certainly be of degree 2 (for $(r_1, r_2, r_3) \neq \mathbf{0}$), which contradicts $(a)$.

$c)$ As follows from $(b)$, the homomorphism

$$\chi : L \ni \mathbf{r} = (r_1, r_2, r_3) \mapsto r_1 a_1 + r_2 a_2 + r_3 a_3 \in \mathbb{R}$$

is injective. Hence, $M = \chi(L) \subset \mathbb{R}$ is a free Abelian subgroup of rank $\mathrm{rk}\, M = \mathrm{rk}\, L \geq 2$. The closure $\overline{M}$ of $M$ in $\mathbb{R}$ coincides with $\mathbb{R}$ (for otherwise, $\overline{M} \cong \mathbb{Z}$ and hence $M \cong \mathbb{Z}$, which contradicts the property $\mathrm{rk}\, M \geq 2$). This implies $(c)$. $\square$

**4.3. Definition.** The $H^\infty(X)$-*hull* $\widehat{Y} = H^\infty - \text{hull}_X(Y)$ of a set $Y \subseteq X$ in a complex space $X$ is defined as follows:

$$\widehat{Y} = H^\infty - \text{hull}_X(Y) = \left\{ x \in X \mid |f(x)| \le \sup_{y \in Y} |f(y)| \quad \text{for all} \quad f \in H^\infty(Z) \right\}.$$

**4.4. Proposition** (cf. Remark 1.14.3). *Let, as in Lemma* 4.1, $s = g_0^d$. *Suppose that* $\alpha^d > 2$. *Then:*

*a) The* $s^G$*-orbit* $s^G(x_\circ) = \{g^{-1} s g x_\circ \mid g \in G\}$ *of the point* $x_\circ = (i, 0) \in X = \mathbb{H} \times \mathbb{C}$ *consists of all the points* $x = (z, w) \in X = \mathbb{H} \times \mathbb{C}$ *of the form*

$$(z, w) = \left((\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3) + i\alpha^d, \quad (\beta^d - 1)(r_1 b_1 + r_2 b_2 + r_3 b_3)\right),$$

*where* $r_1, r_2, r_3 \in \mathbb{Z}$.

*b) The bounded holomorphic function* $F(z, w) = 2(z + i)^{-1}$ *on* $X = \mathbb{H} \times \mathbb{C}$ *satisfies the inequality*

$$|F(x_\circ)| = 1 > \frac{2}{3} > \sup_{x \in s^G(x_\circ)} |F(x)|.$$

*In particular, the* $H^\infty(X)$*-hull* $\widehat{s^G(x_\circ)}$ *of the* $s^G$*-orbit* $s^G(x_\circ)$ *does not contain the point* $x_\circ$ *itself, and for any mean* $\mathfrak{m}$ *on* $L^\infty(s^G(x_\circ))$ *we have* $F(x_\circ) \ne \mathfrak{m}(F \mid s^G(x_\circ))$.

*Proof.* ($a$) follows immediately from Lemma 4.1. In view the assumption $\alpha^d > 2$, ($a$) implies that

$$\sup_{x \in s^G(x_\circ)} |F(x)| = 2 \sup_{r_1, r_2, r_3 \in \mathbf{Z}} \left|(\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3) + i(\alpha^d + 1)\right|^{-1}$$

$$= 2 \left[\inf_{r_1, r_2, r_3 \in \mathbf{Z}} \left|(\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3) + i(\alpha^d + 1)\right|\right]^{-1} \le \frac{2}{\alpha^d + 1} < \frac{2}{3},$$

which proves ($b$). □

The $H^\infty(X)$-hull $\widehat{Y}$ of a subset $Y \subseteq X$ may be found as follows:

$$\widehat{Y} = \widehat{\text{pr}_{\mathbf{H}} Y} \times \mathbb{C},$$

where $\text{pr}_{\mathbb{H}} \colon X = \mathbb{H} \times \mathbb{C} \to \mathbb{H}$ is the natural projection and $\widehat{\text{pr}_{\mathbf{H}} Y}$ is the $H^\infty(\mathbb{H})$-hull of the subset $\text{pr}_{\mathbf{H}} Y \subseteq \mathbb{H}$. In view of Lemma 4.3($b$), we would like to pose the following question.

**4.5. Question.** For which subsets $\Gamma \subseteq G \setminus \{e\}$

($*$) *any point* $x \in X$ *is contained in the* $H^\infty(X)$*-hull* $\widehat{\Gamma(x)}$ *of its* $\Gamma$*-orbit* $\Gamma(x)$?

Proposition 4.6 below provides examples of subsets $\Gamma \subseteq G \setminus \{e\}$ with the property ($*$).

Recall that the elements $g_1, g_2, g_3$ form a free basis of the normal subgroup $G_0 \cong \mathbb{Z}^3$ in $G$. Any subgroup $H \subseteq G_0$ is a free Abelian group of rank rk $H \le 3$.

**4.6. Proposition.** *Let $H \subseteq G_0$ be a subgroup of rank* rk $H \geq 2$, *and let* $\Gamma \subseteq H$ *be the complement of a finite subset*[13] $S \subset H$. *Then* $x \in \widehat{\Gamma(x)}$ *for any* $x \in X$. *In particular,* $x \in \widehat{G(x) - \{x\}}$ *for any* $x \in X$.

*Proof.* By Liouville Theorem, any function $f \in H^\infty(X) = H^\infty(\mathbb{H} \times \mathbb{C})$ is of the form

$$f = \widetilde{f} \circ \mathrm{pr}_{\mathbb{H}}, \quad \text{where } \widetilde{f} \in H^\infty(\mathbb{H}). \tag{3}$$

Hence, for any point $x = (z, w) \in \mathbb{H} \times \mathbb{C}$ and any element $h = g_1^{r_1} g_2^{r_2} g_3^{r_3} \in H$, we have

$$hx = (z + r_1 a_1 + r_2 a_2 + r_3 a_3 \, , w + r_1 b_1 + r_2 b_2 + r_3 b_3) \tag{4}$$

and

$$f(hx) = \widetilde{f}(z + r_1 a_1 + r_2 a_2 + r_3 a_3) \, . \tag{5}$$

When $h$ runs over $H$ (resp. over the complement $\Gamma = H - S$), the corresponding vector $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{Z}^3$ in (4) and (5) runs over a sublattice $\widetilde{H} \subseteq \mathbb{Z}^3$ isomorphic to $H$ (resp. over the complement $\widetilde{\Gamma} = \widetilde{H} - \widetilde{S}$ of a finite subset $\widetilde{S} \subset \widetilde{H}$); in particular, rk $\widetilde{H} =$ rk $H \geq 2$. Since $\widetilde{f}$ is a continuous function, it follows from (3), (5) and Lemma 4.3($c$) that

$$f(x) = \widetilde{f}(z) \in \overline{f(\Gamma(x))}$$

(the closure in $\mathbb{C}$). Therefore,

$$|f(x)| \leq \sup_{y \in \Gamma(x)} |f(y)| \, ,$$

and hence $x \in \widehat{\Gamma(x)}$. □

**4.7. Remark.** Despite Lemma 4.3($b$), the following fact holds[14]:

*For any integer $d \neq 0$ and for any $x \in X$, the $s^G$-orbit $s^G(x)$ is a uniqueness set for bounded holomorphic functions on $X$.*

That is, $f = 0$ whenever $f \in H^\infty(X)$ and $f \,|\, s^G(x) = 0$. Indeed, any $f \in H^\infty(X)$ is of the form (3); hence, $f \,|\, s^G(x) = 0$ implies $\widetilde{f} \,|\, \mathrm{pr}_{\mathbb{H}} \left[s^G(x)\right] = 0$. However, it follows from Lemmas 4.3 and 4.5 that the point $\alpha^d w \in \mathbb{H}$ is a limit point of the set

$$\mathrm{pr}_{\mathbb{H}} \left[s^G(x)\right] = \left\{\alpha^d z + (\alpha^d - 1)(r_1 a_1 + r_2 a_2 + r_3 a_3) \,|\, r_1, r_2, r_3 \in \mathbb{Z}\right\} \subset \mathbb{H}.$$

Thus, $\widetilde{f} = 0$, and so $f = 0$.

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[13]The statement of the lemma is trivial if $\mathbf{e} \in \Gamma$; to make it meaningful we may assume that $\mathbf{e} \in S$.

[14]cf. Proposition 1.11(a).

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