ON TWISTING OPERATORS AND NEWFORMS OF HALF-INTEGRAL WEIGHT III - SUBSPACE CORRESPONDING TO VERY-NEWFORMS -

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In this paper, we improve the previous work on a theory of newforms for Kohnen space. The notion of *very-newforms* of integral weight is introduced. Then under certain condition, we find the canonical subspace of weight $k + 1/2$ which corresponds to the space of *very-newforms* of weight $2k$.
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ON TWISTING OPERATORS AND NEWFORMS OF HALF-INTEGRAL WEIGHT III – SUBSPACE CORRESPONDING TO VERY-NEWFORMS –

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Introduction Let k and N be positive integers such that N is just divided by 4 and χ an even quadratic character modulo N.

In the previous paper [U2], we established a complete theory of newforms of Kohnen space $S(k+1/2, N, \chi)_K$. That is the following.

First of all, we defined the subspace of "oldforms" $\mathfrak{O}(k+1/2, N, \chi)_K$ in terms of Kohnen spaces of lower levels and standard operators like twisting operators, shift operators, and etc. (cf. [U2, (3.7)]).

We denote by $\mathfrak{N}(k+1/2, N, \chi)_K$ the orthogonal complement of $\mathfrak{O}(k+1/2, N, \chi)_K$ in Kohnen space $S(k+1/2, N, \chi)_K$ with respect to the Petersson inner product. Put $\Pi := \{$odd prime divisor p of N such that $p^2|N\}$. $\mathfrak{N}(k+1/2, N, \chi)_K$ is fixed by the twisting operator R_p: $\sum_{n\geq 1} a(n)\mathrm{e}(nz) \mapsto \sum_{n\geq 1} a(n)\left(\frac{n}{p}\right)\mathrm{e}(nz)$ for any $p \in \Pi$. Hence, we can decompose $\mathfrak{N}(k+1/2, N, \chi)_K$ into common eigen subspaces on these twisting operators.

$$\mathfrak{N}(k+1/2, N, \chi)_K = \bigoplus_{\kappa\in \mathrm{Map}(\Pi,\{\pm 1\})} \mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K \ ,$$

$$\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K := \{f \in \mathfrak{N}(k+1/2, N, \chi)_K \ ; \ f|R_p = \kappa(p) f \ \ (p \in \Pi)\} \ .$$

We called these subspaces $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ the space of newforms for Kohnen spaces.

These $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ satisfies the Strong Multiplicity One theorem and moreover we have an embedding as modules over Hecke algebra: (cf.[U2, §3])

$$\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K \hookrightarrow S^0(2k, N/4) \ . \tag{0.1}$$

Here, $S^0(2k, N/4)$ is the space of newforms of weight $2k$ and of level $N/4$.

We can give also explicit and exact expression of the image of this embedding. See [U2, (2.28) and (3.7)] for the details.

The purpose of this paper is some refinements of these previous works.

In the image of the above embedding (0.1), we have many liftings of cusp forms of lower levels by twisting operators. These liftings can be considered that is not

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX really new. We shall find the subspace of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ which corresponds to the spaces of 'really new' cusp forms.

We shall make a precise statement.

(0.2) **Definition** (cf. [U1, Appendix]) Let M be an odd positive integer and denote by $\nu_p := \mathrm{ord}_p(M)$ the p-adic additive valuation of M for any prime p.

Put $\Omega := \{p \mid \mathrm{ord}_p(M) \geq 1\}$ and $\Omega_2 := \{p \mid \mathrm{ord}_p(M) = 2\}$. Define

$$S^2(2k,M) := \sum_{\substack{\Omega_2 = A+B+C \\ \Omega_2 \neq A}} S^0\left(2k, \prod_{p\in\Omega-(B+C)} p^{\nu_p} \prod_{p\in B} p\right) |R_{B+C} ,$$

where $A+B+C$ etc. means the disjoint union of A, B, and C (cf. below §0). The above sum is extended over all partitions $\Omega_2 = A+B+C$ with $\Omega_2 \neq A$. $R_{B+C} = \prod_{p\in B+C} R_p$ is the twisting operator of $\prod_{p\in B+C}\left(\frac{\cdot}{p}\right)$.

We define $S^*(2k,M)$ by the orthogonal complement of $S^2(2k,M)$ in $S^0(2k,M)$ with respect to the Petersson inner product. We shall call this space $S^*(2k,M)$ the space of *very-newforms* of weight $2k$ and of level M. Also we shall call any element in $S^*(2k,M)$ a *very-newform* of weight $2k$ and of level M.

Under these notation, we set the following definitions.

(0.3) **Definition** Suppose that $\mathrm{ord}_2(N) = 2$ and put $M := N/4$. M is an odd integer. We define the subspace $\mathfrak{D}^*(k+1/2,N,\chi)_K$ as follows:

[The case of $k \geq 2$]

$$\mathfrak{D}^*(k+1/2,N,\chi)_K := \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi\left(\frac{\Delta}{\cdot}\right)=\chi}} S(k+1/2,4B,\xi)_K \,|\tilde{\delta}_A$$
$$+ \sum_{\substack{0<B|M \\ B\neq M}} \sum_{0<A|(M/B)^2} \sum_{\substack{\xi,\phi \\ \xi\left(\frac{\Delta}{\cdot}\right)\phi^2=\chi}} S(k+1/2,4B,\xi)_K \,|U(A)R_\phi \,.$$

[The case of $k = 1$]

$$\mathfrak{D}^*(3/2,N,\chi)_K := \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi\left(\frac{\Delta}{\cdot}\right)=\chi}} V(4B;\xi)_K \,|\tilde{\delta}_A$$
$$+ \sum_{\substack{0<B|M \\ B\neq M}} \sum_{0<A|(M/B)^2} \sum_{\substack{\xi,\phi \\ \xi\left(\frac{\Delta}{\cdot}\right)\phi^2=\chi}} V(4B;\xi)_K \,|U(A)R_\phi \,.$$

Here, ξ and ϕ in the $\sum_{\substack{\xi,\phi \\ \xi\left(\frac{\Delta}{\cdot}\right)\phi^2=\chi}}$ runs over the following set:

$$\left\{\begin{array}{l} (\xi,\phi)\ ;\ \xi \text{ is an even (quadratic) character defined modulo } 4B, \\ \phi \text{ is a primitive character modulo } \mathfrak{f}(\phi), \phi^4 = 1, \mathfrak{f}(\phi)|l_\Pi := \prod_{p\in\Pi} p, \\ \xi\left(\frac{\Delta}{\cdot}\right)\phi^2 = \chi \text{ as characters modulo } N \end{array}\right\}.$$

See [U1, §0 and §1] for the definition of the space $V(4B;\xi)_K$. The operator $\tilde{\delta}_A$, the shift operator $U(A)$, and the twisting operator R_ϕ are defined as follows: For $f = \sum_{n\geq 1} a(n)\mathbf{e}(nz)$,

$$(f|\tilde{\delta}_A)(z) := A^{k/2+1/4} f(Az) , \quad (f|U(A))(z) := \sum_{n\geq 1} a(An)\mathbf{e}(nz) ,$$

$$(f|R_\phi)(z) := \sum_{n\geq 1} a(n)\phi(n)\mathbf{e}(nz) .$$

This space $\mathfrak{O}^*(k+1/2, N, \chi)_K$ is a subspace of Kohnen space $S(k+1/2, N, \chi)_K$. We denote by $\mathfrak{N}^*(k+1/2, N, \chi)_K$ the orthogonal complement of $\mathfrak{O}^*(k+1/2, N, \chi)_K$ in $S(k+1/2, N, \chi)_K$.

Remark. The definition of $\mathfrak{O}^*(k+1/2, N, \chi)_K$ is much simpler than those of $\mathfrak{O}(k+1/2, N, \chi)_K$ ([U2, (3.7)]). We shall give a more simplified definition of $\mathfrak{O}(k+1/2, N, \chi)_K$ in the section 3. See below (3.7).

This orthogonal complement $\mathfrak{N}^*(k+1/2, N, \chi)_K$ is stable by the twisting operator $R_p := R_{\left(\frac{\cdot}{p}\right)}$ for any prime $p \in \Pi$. Hence we can decompose this space into common eigen subspaces.

$$\mathfrak{N}^*(k+1/2, N, \chi)_K = \bigoplus_{\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})} \mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K ,$$

$$\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K := \{f \in \mathfrak{N}^*(k+1/2, N, \chi)_K \ ; \ f|R_p = \kappa(p) f \ (p \in \Pi)\} .$$

Here $\Pi := \{\text{odd prime divisor } p \text{ of } N \text{ such that } p^2|M\}$. □

Main purpose of this paper is to prove the following theorem.

Main Theorem. Let the notation be the same as the above. Suppose the following condition (cf. below (2.19)).

(0.4) $\qquad \chi_p = 1 \quad$ for any $p \in \Pi$ such that $\mathrm{ord}_p(M) = 2$ and $p \equiv 3 \pmod 4$.

Then the subspaces $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ have the following nice properties (1)–(4).

(1) $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ has an orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p \nmid M$) and $U(p^2)$ (p:prime, $p \mid M$) which are uniquely determined up to multiplication by non-zero complex numbers. Let f be such a common eigenform and λ_p the eigenvalue of f with respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ ($p \nmid M$) resp. $U(p^2)$ ($p|M$). Then there exist a primitive (very-new)form $F \in S^*(2k, M)$ of weight $2k$ and of conductor M which is uniquely determined and satisfies the following: For a prime p,

$$F|T_{2k,M}(p) = \lambda_p F \quad \text{if } (p, M) = 1 \quad \text{and} \quad F|U(p) = \lambda_p F \quad \text{if } p|M.$$

Furthermore we can explicitly find which primitive form occurs via the above correspondence. See the trace relation (2.21-22) for the details of these.

(2) (**The Strong Multiplicity One Theorem**)

Let f, g be two non-zero elements of $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ and A a non-zero integer. Suppose that f and g are common eigenforms of all $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p: prime and $(p, A) = 1$) with the same system of eigenvalues. Then $\mathbf{C}f = \mathbf{C}g$.

Therefore from (1) and this, we have an embedding as modules over (abstract) Hecke algebra $\mathcal{H}$:

$$\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K \hookrightarrow S^*(2k, M) .$$

Here, (abstract) Hecke algebra $\mathcal{H}$ means the commutative algebra generated by all double cosets: $\Gamma_0(M)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(M)$, $a|d$, $(a, M) = 1$, a, $d \in \mathbf{Z}_+$. See [M, §4.5] for the details.

$\mathcal{H}$ operates on $S(2k, M)$ via the Hecke operators $T_{2k,M}(n)$ and $U(p)$ and also on $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ via $\tilde{T}_{k+1/2,N,\chi}(n^2)$ and $U(p^2)$. See [N] for the detail.

(3) The space of oldforms $\mathfrak{O}^*(k+1/2, N, \chi)_K$ has also an orthogonal $\mathbf{C}$-basis consisting of common eigenforms for all operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p\nmid N$). The system of eigenvalues of such a common eigenform corresponds to a primitive form F of weight $2k$ either whose conductor is a divisor of M less than M or $F \in S^2(2k, M)$, i.e., F is not a very-newform but a primitive form of conductor M. (cf. below (3.3), (3.5), and [U2, (3.5)]).

(4) The space of oldform $\mathfrak{O}^*(k+1/2, N, \chi)_K$ is generated by cusp forms of lower level. Hence, by induction, we see that the Kohnen spaces $S(k+1/2, N, \chi)_K$, $V(N; \chi)_K$ are reconstructed by the spaces of type of $\mathfrak{N}^{*,\kappa}(k+1/2, 4B, \xi)_K$ of lower level and the operators of type of $\tilde{\delta}_A$, $U(A)$, and R_ϕ.

From the above definition, $\tilde{\delta}_A$, $U(A)$, and R_ϕ (almost) preserve Fourier coefficients of cusp forms. Hence for studying Fourier coefficients of cusp forms $\in S(k+1/2, N, \chi)_K$ or $V(N; \chi)_K$, it is sufficient to study cusp forms only in the spaces of newforms $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$. □

Finally the author has some comments about the condition (0.4).

(i) If $\chi = 1$, the condition (0.4) is always satisfied for any level M. In this case, we obtain the following embedding for any odd positive integer M and any $\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})$,

$$\mathfrak{N}^{*,\kappa}(k+1/2, 4M, 1)_K \hookrightarrow S^*(2k, M) .$$

(ii) If we do not assume the condition (0.4), we have examples to which we cannot apply the method of this paper. See example (3.8) below.

This paper is composed as follows: §0 is notational preliminaries. In §1, we collect technical proposition used in later sections. In §2, we shall explicitly find the space corresponding to the space of very-newforms. In §3, we shall prove the Main Theorem. Moreover, we shall give a simplification of expression formula (or definition) of $\mathfrak{O}(k+1/2, N, \chi)_K$ in the formulation of theory of newforms [U2, (3.7)].

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§0. **Notational Preliminary.** Throughout this paper, we shall keep to the notation in the previous papers [U1] and [U2]. See [U1, §0, §1] and [U2, §1] for the details. In particular, we shall use the following.

Let A, B be subsets of a set X and $\{A_i\}_{i\in I}$ a family of subsets of X. If $A\cup B$ is a disjoint union, then we denote $A+B := A\cup B$ for simplicity. Similarly, if $\cup_{i\in I}A_i$ is a disjoint union, then we denote $\sum_{i\in I} A_i := \cup_{i\in I}A_i$.

Let A be a finite set of prime numbers and $(\alpha_p)_{p\in A}$ a system of integers. We put the following notation: $A(\alpha)_i := \{p \in A \mid \alpha_p = i\}$ and $A(\alpha)_{i+} := \{p\in A \mid \alpha_p \geq i\}$ for any $i\in \mathbf{Z}$. Also we set the notation:$l_A := \prod_{p\in A} p$.
We denote the set of positive integers by $\mathbf{Z}_+$.

Let p be a prime. We denote the additive p-adic valuation for any integer m by $\mathrm{ord}_p(m)$. $\left(\frac{a}{b}\right)$ (a, b integers with $(a,b)\neq(0,0)$) means the Kronecker symbol (cf. [M p.82]).

Let k denote a non-negative integer. If $z\in\mathbf{C}$ and $x\in\mathbf{C}$, we put $z^x = \exp(x\cdot\log(z))$ with $\log(z) = \log(|z|)+\sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z)\leq\pi$. Also we put $\mathrm{e}(z) = \exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in GL_2^+(\mathbf{R})$, $\gamma = \left(\begin{smallmatrix} u & v\\ w & x\end{smallmatrix}\right)\in \Gamma_0(4)$ and $z\in\mathfrak{H}$, we define functions $J(\alpha,z)$, $j(\gamma,z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by:$J(\alpha,z) = cz+d$, $j(\gamma,z) = \left(\frac{-1}{x}\right)^{-1/2}\left(\frac{w}{x}\right)(wz+x)^{1/2}$ and $f|[\alpha]_k(z) = (\det\alpha)^{k/2}J(\alpha,z)^{-k}f(\alpha z)$.

For $m\in\mathbf{Z}_+$ we define a shift operator $U(m)$ on formal power series in $\mathrm{e}(z)$ by

$$\sum_{n\geq 0} a(n)\mathrm{e}(nz)|U(m) := \sum_{n\geq 0} a(mn)\mathrm{e}(nz)\ .$$

Let χ be a Dirichlet character modulo N. We denote the conductor of χ by $\mathfrak{f}(\chi)$ and the local p-primary component of χ by χ_p for each prime divisor p of N.

Let V, V' be finite-dimensional vector spaces over $\mathbf{C}$. We denote the trace of a linear operator T on V by $\mathrm{tr}(T;V)$ and also the kernel of a linear map F from V to V' by $\mathrm{Ker}(F;V)$. We denote the set of all mappings from a set A to a set B by $\mathrm{Map}(A,B)$. Furthermore we use the abbreviated notation B^A $(=\mathrm{Map}(A,B))$.

As for notations of modular forms, see [U1, §0] and [U2, §1]. In particular, we shall use the following.

Let k be a positive integer. Let $N = 4M$ and M an odd natural number and let χ be an even character modulo N with $\chi^2 = 1$.
We define the **Kohnen space** $S(k+1/2,N,\chi)_K$ as follows:

$$S(k+1/2,N,\chi)_K := \left\{ \begin{array}{l} S\left(k+\frac{1}{2},N,\chi\right) \ni f(z) = \sum_{n=1}^{\infty} a(n)\mathrm{e}(nz)\ ;\\ a(n) = 0 \text{ for } \chi_2(-1)(-1)^k n \equiv 2,3 \pmod 4 \end{array}\right\}\ ,$$

where χ_2 is the 2-primary component of χ.
In the case of weight 3/2, we define $V(N;\chi)_K := V(N;\chi)\cap S(3/2,N,\chi)_K$.

These Kohnen spaces $S(k+1/2,N,\chi)_K$ and $V(N;\chi)_K$ can be considered canonical subspaces which correspond to the space of cusp forms of weight $2k$ and of odd level via Shimura correspondence. Therefore we can think these spaces are good test cases for general theory of newforms of half-integral weight.

See [U1, §0, §1] for the details.

§1 Twisting operators.

Let k be positive integer and N a positive integer divisible by 4. Let χ be an even character modulo N with $\chi^2 = 1$.

Take a primitive character ψ modulo L. We define the twisting operator of ψ, say R_ψ, by the following:

$$S(k+1/2, N, \chi) \ni f = \sum_{n\geq 1} a(n)\mathrm{e}(nz) \mapsto f|R_\psi := \sum_{n\geq 1} a(n)\psi(n)\mathrm{e}(nz) .$$

We can express this operator as follows: For $f \in S(k+1/2, N, \chi)$,

$$f|R_\psi = \mathfrak{g}(\overline{\psi})^{-1} \sum_{m \bmod L} \overline{\psi}(m) f| \left(\left(\begin{smallmatrix} L & m \\ 0 & L \end{smallmatrix} \right), 1 \right) . \tag{1.1}$$

Here, $\mathfrak{g}(\overline{\psi})$ is the gauss sum of $\overline{\psi}$, i.e., $\mathfrak{g}(\overline{\psi}) := \sum_{i=1}^{L} \overline{\psi}(i)\mathrm{e}(i/L)$.

We impose the following assumption on ψ from now on until the end of this paper.

(1.2) **Assumption.** ψ is a primitive character such that $\psi^4 = 1$. It's conductor $L := \mathfrak{f}(\psi)$ is odd and $L^2|N$.

Remark. It follows from this assumption that $L = \mathfrak{f}(\psi)$ divides l_Π. See the introduction and §1 as for the definition of Π and l_Π.

From this assumption and [Sh, Lemma (3.6)], the map $f \mapsto f|R_\psi$ define an operator from $S(k+1/2, N, \chi)$ to $S\big(k+1/2, N, \chi\psi^2\big)$.

We collect several propositions on twisting operators. We shall use them in the following sections.

(1.3) **Proposition.** Let ψ be the same as above. Let $f \in S(k+1/2, N, \chi)$. For any $n \in \mathbf{Z}_+$ with $(n, N) = 1$,

$$f|R_\psi \tilde{T}_{k+1/2, N, \chi\psi^2}(n^2) = \psi(n^2) f|\tilde{T}_{k+1/2, N, \chi}(n^2) R_\psi .$$

Proof. For any $m \in \mathbf{Z}_+$, put $\xi(m) := \left(\left(\begin{smallmatrix} L & m \\ 0 & L \end{smallmatrix} \right), 1 \right)$ and $\tau(m) := \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & m^2 \end{smallmatrix} \right), m^{k+1/2} \right)$.

From the definition, Hecke operators are linear combinations of finitely many operators of the following type: $[\Delta_0 \tau(n) \Delta_0]$ and $[\Delta_0' \tau(n) \Delta_0']$ ($n \in \mathbf{Z}_+$, $(n, N) = 1$). Here, $\Delta_0 := \Delta_0(N, \chi)$ and $\Delta_0' := \Delta_0(N, \chi\psi^2)$ (cf. [U2, §0(c)]).

We shall study a relation between the operators of these types and R_ψ.

Put $\tilde{\Delta}_0 := \tau(n)^{-1} \Delta_0 \tau(n) \cap \Delta_0$ and $\tilde{\Delta}_0' := \tau(n)^{-1} \Delta_0' \tau(n) \cap \Delta_0'$. From easy calculation, we have the following.

$$\tilde{\Delta}_0 = \left\{ \left(\gamma, \chi(d) j(\gamma, z)^{2k+1}\right) \; ; \; \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N), b \equiv 0 \pmod{n^2} \right\} .$$

$$\tilde{\Delta}_0' = \left\{ \left(\gamma, (\chi\psi^2)(d) j(\gamma, z)^{2k+1}\right) \; ; \; \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N), b \equiv 0 \pmod{n^2} \right\} .$$

Put $H := \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in SL_2(\mathbf{Z}/n^2\mathbf{Z}) \right\}$. This is a subgroup of $SL_2(\mathbf{Z}/n^2\mathbf{Z})$. For any representative $\eta \in H\backslash SL_2(\mathbf{Z}/n^2\mathbf{Z})$, we take and fix an element $\gamma'_\eta \in SL_2(\mathbf{Z})$ such that

$$\gamma'_\eta \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \bmod N , \\ \eta & \bmod n^2 . \end{cases}$$

Then we can chose a complete system of representatives:

$$\tilde{\Delta}'_0\backslash\Delta'_0 = \{(\gamma'_\eta, (\chi\psi^2)(d'_\eta)j(\gamma'_\eta, z)^{2k+1}) \ ; \ \eta \in H\backslash SL_2(\mathbf{Z}/n^2\mathbf{Z})\} ,$$

where d'_η is the $(2,2)$-entry of γ'_η.

For simplicity, put $\tilde{\gamma}'_\eta := (\gamma'_\eta, (\chi\psi^2)(d'_\eta)j(\gamma'_\eta, z)^{2k+1})$ and $\gamma'_\eta := \begin{pmatrix} a'_\eta & b'_\eta \\ c'_\eta & d'_\eta \end{pmatrix}$ for any η. Then we have for any m, $n \in \mathbf{Z}_+$ with $(n, N) = 1$,

$$\xi(mn^2)\tilde{\gamma}'_\eta\xi(mn^2)^{-1} = \left(\gamma_\eta, (\chi\psi^2)(d'_\eta)j(\gamma'_\eta, z - mn^2/L)^{2k+1}\right) ,$$

where $\gamma_\eta := \begin{pmatrix} a'_\eta + mn^2c'_\eta/L & L^{-2}\{(d'_\eta - a'_\eta)mn^2L + b'_\eta L^2 - (mn^2)^2c'_\eta\} \\ c'_\eta & d'_\eta - mn^2c'_\eta/L \end{pmatrix}$.

We have $\gamma_\eta \in \Gamma_0(N)$ because $L^2|N$ and $\gamma'_\eta \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod N. We have $4L^2|c'_\eta$ and so $\left(\frac{-1}{d'_\eta - mn^2c'_\eta/L}\right) = \left(\frac{-1}{d'_\eta}\right)$ and $\left(\frac{c'_\eta}{d'_\eta - mn^2c'_\eta/L}\right) = \left(\frac{c'_\eta/(4L^2)}{d'_\eta - mn^2c'_\eta/L}\right) = \left(\frac{c'_\eta/(4L^2)}{d'_\eta}\right) = \left(\frac{c'_\eta}{d'_\eta}\right)$. Hence $j(\gamma_\eta, z) = j(\gamma'_\eta, z - mn^2/L)$.

Since $\psi^4 = (\chi\psi^2)^2 = 1$, both conductors $\mathfrak{f}(\psi^2)$ and $\mathfrak{f}(\chi\psi^2)$ are divisors of $2^{\mathrm{ord}_2(N)}M_1l_\Pi$. From $c'_\eta \equiv 0$ (N) and $L^2|N$, $c'_\eta/L \equiv 0$ (mod $2^{\mathrm{ord}_2(N)}M_1l_\Pi$).

From these and the assumption, $d'_\eta \equiv 1$ (mod N), $(\chi\psi^2)(d'_\eta) = (\chi\psi^2)(d'_\eta - mn^2c'_\eta/L)$ and $\psi^2(d'_\eta - mn^2c'_\eta/L) = \psi^2(d'_\eta) = 1$. Hence $(\chi\psi^2)(d'_\eta) = \chi(d'_\eta - mn^2c'_\eta/L)$.

Therefore

$$\xi(mn^2)\tilde{\gamma}'_\eta\xi(mn^2)^{-1} = \left(\gamma_\eta, \chi(d_\eta)j(\gamma_\eta, z)^{2k+1}\right) ,$$

where d_η is the $(2,2)$-entry of γ_η.

We also have the following.

$$\gamma_\eta = \begin{pmatrix} L & mn^2 \\ 0 & L \end{pmatrix} \gamma'_\eta \begin{pmatrix} L & mn^2 \\ 0 & L \end{pmatrix}^{-1} \equiv \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \gamma'_\eta \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}^{-1} = \gamma'_\eta \equiv \eta \pmod{n^2} .$$

Hence the set $\{\xi(mn^2)\tilde{\gamma}'_\eta\xi(mn^2)^{-1} \ ; \ \eta \in H\backslash SL_2(\mathbf{Z}/n^2\mathbf{Z})\}$ becomes a complete system of representatives of $\tilde{\Delta}_0\backslash\Delta_0$.

From the above, for any $f \in S(k+1/2, N, \chi)$,

$$\begin{aligned} \mathfrak{g}(\overline{\psi})f|R_\psi\,[\Delta'_0\tau(n)\Delta'_0] &= \sum_{\substack{\tilde{\gamma}'_\eta \in \tilde{\Delta}'_0\backslash\Delta'_0 \\ m \bmod L}} \overline{\psi}(m)f|\xi(m)\tau(n)\tilde{\gamma}'_\eta \\ &= \sum_{\tilde{\gamma}'_\eta,\, m \bmod L} \overline{\psi}(m)f|\tau(n)\left(\xi(mn^2)\tilde{\gamma}'_\eta\xi(mn^2)^{-1}\right)\xi(mn^2) \\ &= \sum_{m \bmod L} \overline{\psi}(m)\left(f|\,[\Delta_0\tau(n)\Delta_0]\right)|\xi(mn^2) \\ &= \mathfrak{g}(\overline{\psi})\psi(n^2)f|\,[\Delta_0\tau(n)\Delta_0]\,R_\psi , \end{aligned}$$

The assertion follows from this, the definition of Hecke operators [U2, §0(c)], and $\psi^4 = 1$. □

Remark. We can prove this relation without the assumption $\psi^4 = 1$. Of course, we need to modify the definition of Hecke operators in the general case. □

(1.4) **Proposition.** The mapping $f \mapsto f|R_\psi$ maps the spaces $S(k+1/2, N, \chi)_K$, $V(N;\chi)$, and $V(N;\chi)_K$ into the spaces $S(k+1/2, N, \chi\psi^2)_K$, $V(N;\chi\psi^2)$, and $V(N;\chi\psi^2)_K$ respectively.

Proof. For any $f = \sum_{n\geq 1} a(n)\mathrm{e}(nz) \in S(k+1/2, N, \chi)_K$, we have $f|R_\psi = \sum_{n\geq 1} a(n)\psi(n)\mathrm{e}(nz) \in S(k+1/2, N, \chi\psi^2)$ ([Sh, Lemma(3.6)]).

From the assumption (1.2), we have $\chi_2 = (\chi\psi^2)_2$. Hence by the definition of Kohnen space [U2, §0(d)], we have the assertion for Kohnen space.

$V(N;\chi)$ has a $\mathbf{C}$-basis consisting of common eigenforms on all Hecke operators $\tilde{T}(n^2) = \tilde{T}_{3/2,N,\chi}(n^2)$, $(n, N) = 1$ (cf. [U2, §0(c)]).

Take any form f in such a basis. The system of eigenvalues of f on $\tilde{T}(n^2)$ corresponds to a primitive (cusp) forms F of weight $2k$ (cf. [U2, (3.5)(2)]).

Suppose that $f|R_\psi \neq 0$. From the proposition (1.3), $f|R_\psi$ is also a common eigenform on all Hecke operators $\tilde{T}(n^2) = \tilde{T}_{3/2,N,\chi\psi^2}(n^2)$, $(n, N) = 1$.

Moreover its system of eigenvalues corresponds to a certain primitive (cusp) form F' of weight $2k$. In fact, let ψ' be the primitive character associated with ψ^2. Then we can take as F' the primitive (cusp) form associated with the cusp form $F|R_{\psi'}$, where $R_{\psi'}$ is the twisting operator of ψ' ([U2, §0(b)]).

Since the space $U(N;\chi\psi^2)$ corresponds to Eisenstein series via Shimura correspondence (cf. [U1, §0]), the form $f|R_\psi$ is orthogonal to the space $U(N;\chi\psi^2)$. □

(1.5) **Proposition.** Let $\mathrm{ord}_2(N) = 2$. Let Π be the same notation as in the introduction and $\kappa \in \mathrm{Map}(\Pi, \{\pm 1\})$. If $f|R_\psi = 0$ for $f \in \mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$, then $f = 0$. In other wards, the twisting operator R_ψ induces a $\mathbf{C}$-linear isomorphism from $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ onto $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K\,|R_\psi$. Here, see [U1, (3.5)] as for the definition of the space $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$.

Proof. Take any non-zero form $f = \sum_{n\geq 1} a(n)\mathrm{e}(nz) \in \mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$. Choose a number n_0 such that $a(n_0) \neq 0$. Since $f|R_p = \kappa(p)f$ any $p \in \Pi$, we have $a(n_0)\left(\frac{n_0}{p}\right) = \kappa(p)a(n_0)$. Hence $\left(\frac{n_0}{p}\right) = \pm 1$ and so $(n_0, l_\Pi) = 1$.

The conductor of ψ is a divisor of l_Π (cf. The remark after (1.2)). Hence, we have $\psi(n_0) \neq 0$. The n_0-th Fourier coefficient of $f|R_\psi$ is non-zero number $a(n_0)\psi(n_0)$. This means $f|R_\psi \neq 0$. □

§2 The subspace corresponding to very-newforms.

In this section, we shall find the subspace of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ which corresponds to the space of very-newforms of weight $2k$.

We shall study only Kohnen space in the following two sections. Hence, we assume that $\mathrm{ord}_2(N) = 2$ from now on until the end of the paper.

We keep to the notation and the assumption in §0, §1, and [U2,§0,§2]. See those for the details of the definitions and notations. In particular, we shall use the following notations throughout this paper: Let denote $M := N/4$. M is an odd integer. $\Pi := \{\text{prime number such that } p^2|M\}$ $\nu_p := \mathrm{ord}_p(N)$ and $M_1 := \prod_{p|M,\nu_p:=1} p$, $M_{2+} := \prod_{p^2|M} p^{\nu_p}$.

(2.1) **Notation.** Let D and E be subsets of $\Pi(\nu)_2 := \{p \in \Pi \mid \nu_p = 2\}$ (cf. §0) satisfying the following conditions:

$$\text{(i)}\quad D \cap E = \emptyset\,.$$
$$\text{(ii)}\quad p \equiv 1 \pmod 4 \quad \text{for all } p \in D + E\,.$$

Put $F := \Pi - (D+E)$. We also set a system of integers $(\alpha_p^{D,E})_{p\in\Pi}$ as follows:

$$\alpha_p^{D,E} := \begin{cases} 0, & \text{if } p \in D, \\ 1, & \text{if } p \in E, \\ \nu_p, & \text{if } p \in F. \end{cases}$$

We shortly denote these by (α), (α_p) or α_p etc. if any confusion does not occur. For this (α), put $M_{2+}^{(\alpha)} := \prod_{p\in\Pi} p^{\alpha_p}$ and $\tilde{N}(\alpha) := 4M_1 M_{2+}^{(\alpha)}$. Fix $\kappa \in \{\pm 1\}^{\Pi} := \mathrm{Map}(\Pi, \{\pm 1\})$. □

(2.2) **Notation.** Put $\chi' := \chi \cdot \prod_{p\in D+E} \left(\frac{p}{}\right) = \chi \cdot \prod_{p\in D+E} \left(\frac{-}{p}\right)$ as character modulo N. We decompose this χ' into characters modulo N as follows:

$$\text{(2.3)}\qquad \chi' = \eta \cdot \eta' \quad \eta := \left(\frac{u}{-}\right),\ \eta' := \left(\frac{u'}{-}\right), \quad 0 < u | l_D,\ 0 < u' | M_1 l_{E+F}\,.$$

This character η' can be defined modulo $\tilde{N}(\alpha)$. Hence we can define the space $\mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K$ ([U2, §3, (3.7)]), where $\kappa'|_F := \kappa|_F \cdot \eta \in \{\pm 1\}^F := \mathrm{Map}(F, \{\pm 1\})$. □

From these spaces, we define a space as follows:

$$\text{(2.4)}\qquad C^{(D,E)} := \sum_{0<a|l_D} \mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K | U(ua^2)\,.$$

We use the same notation $B^{(\alpha)}$ as in the previous paper [U2, (2.2)] for the character $\chi' = \chi \cdot \prod_{p\in D+E} \left(\frac{-}{p}\right)$. Then $C^{(D,E)} \subseteq B^{(\alpha)}$.

The space $\mathfrak{S}^{\emptyset,\kappa'|_F}\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K$ has a $\mathbf{C}$-basis, say $\mathcal{B}_{\eta'}$, consisting of common eigenforms for all Hecke operators $\tilde{T}(n^2)$, $(n, \tilde{N}(\alpha)) = 1$ ([U1, (3.11)]).

$\mathfrak{N}^{\emptyset,\kappa'|_F} := \mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K$ is also fixed by all Hecke operators $\tilde{T}(n^2)$, $(n, \tilde{N}(\alpha)) = 1$ ([U2, (2.28)]). Hence from the Strong Multiplicity One Theorem for $\mathfrak{S}^{\emptyset,\kappa'|_F}\left(k+1/2, \tilde{N}(\alpha), \eta'\right)_K$ ([U1, (3.11)(2)]), we can see that $\mathfrak{N}^{\emptyset,\kappa'|_F}$ has a $\mathbf{C}$-basis $\mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F}$.

(2.5) **Claim.** We have the following decomposition.

$$C^{(D,E)} = \bigoplus_{\substack{f \in (\mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F})|U(u) \\ 0<a|l_D}} \mathbf{C} f | U(a^2)\,.$$

Proof of the claim. It is obvious that $C^{(D,E)}$ is generated by all elements in the set $\{f|U(a^2)\ ;\ f \in (\mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F})|U(u),\ 0 < a|l_D\}$. Linear independence of these generators follows from [U2, (2.2–3)] for the character χ'. □

For any element $f \in (\mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F})|U(u)$, we use the same notation $B_f^{(\alpha)}$ as in [U2, (2.5)]. We put

$$(2.6) \qquad C_f^{(D,E)} := \bigoplus_{0<a|l_D} \mathbf{C}f|U(a^2) = B_f^{(\alpha)}\ ,$$

where we shortly write $(\alpha) = (\alpha_p^{D,E})$. □

Hence, we can transfer the results on the space $B_f^{(\alpha)}$ in [U2 §2] to the space $C_f^{(D,E)}$. In particular, we have the following:

(2.7) **Proposition.** (cf. [U2, (2.18)]) Let notation be the same as above. For any $f \in (\mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F})|U(u)$,

$$C_f^{(D,E)}|e_\Pi^\kappa = B_f^{(\alpha)}|e_\Pi^\kappa = \mathbf{C}\, f_{\rho_0}|e_\Pi^\kappa\ ,$$

where $\rho_0 \in \mathrm{Map}(D, \mathbf{C})$ is defined as follows.

$$\rho_0 := \begin{cases} \kappa(p)\mathfrak{g}_p, & \text{if } p \in D \text{ and } \chi_p = \left(\frac{-}{p}\right); \\ -1, & \text{if } p \in D \text{ and } \chi_p = \mathbf{1}; \end{cases}$$

and f_{ρ_0} is a non-zero common eigenform on all Y_p's $(p|M_1 l_D l_E)$ satisfying

$$f_{\rho_0}|Y_p = c_p f_{\rho_0}\ , \quad c_p = \begin{cases} \lambda_p, & \text{for all } p|M_1 l_E; \\ \rho_0(p), & \text{for all } p \in D. \end{cases}$$

Here, λ_p is the eigen value of f on Y_p (cf. [U2, (2.19)]).

Moreover $f_{\rho_0}|e_\Pi^\kappa \neq 0$ if and only if f satisfies the following conditions:

$$(*) \qquad f|Y_p = \kappa(p)\mathfrak{g}_p f \quad \text{for all } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right).$$

□

Take the element $g \in \mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F}$ such that $f = g|U(u)$. Observing that $U(u)$ is injection on $S\left(k + 1/2, \tilde{N}(\alpha) \cdot l_D, \eta'\right)_K$, we easily seen that f satisfies the condition $(*) \Leftrightarrow g$ satisfies the following condition $(**)$ ([U2, §2]):

$$(**) \qquad g|\mathbf{w}_p = \chi'(p)\kappa(p)g \quad \text{for all } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right).$$

Here, we note $\chi'(p)$ has the meaning because $\chi'_p = \mathbf{1}$.

Let denote

$$\begin{aligned}\mathfrak{N}_{**}^{\emptyset,\kappa'|_F} &:= \mathfrak{N}_{**}^{\emptyset,\kappa'|_F}(k+1/2,\tilde{N}(\alpha),\eta')_K \\ &:= \left\{ \begin{array}{l} h \in \mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2,\tilde{N}(\alpha),\eta')_K \ ; \\ h|\mathrm{w}_p = \chi'(p)\kappa(p)h \ \text{ for all } p\in E \text{ and } \chi_p = \left(\frac{-}{p}\right) \end{array} \right\} .\end{aligned}$$

We know $g|\mathrm{w}_p = \pm g$ for any $g \in \mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F}$ ([U1, (3.9) and (3.11)]). Hence, $\{g \in \mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F}$; g satisfies the condition $(**)\}$ becomes a $\mathbf{C}$-basis of the space $\mathfrak{N}_{**}^{\emptyset,\kappa'|_F}$.

Hecke operators $\tilde{T}(n^2)$, $(n,N)=1$ commute with the operators $U(u)$, $U(a^2)$, and e_{Π}^{κ}. Hence, all of three forms g, f, and $f_{\rho_0}|e_{\Pi}^{\kappa}$ are common eigenforms on $\tilde{T}(n^2)$, $(n,N)=1$ and all of them belong to the same system of eigenvalues.

From this, (2.8) For all $n \in \mathbf{Z}_+$ with $(n,N)=1$,

$$\begin{aligned}\mathrm{tr}\left(\tilde{T}(n^2); \mathcal{C}^{(D,E)}|e_{\Pi}^{\kappa}\right) &= \mathrm{tr}\left(\tilde{T}(n^2); \bigoplus_{f\in(\mathcal{B}_{\eta'}\cap\mathfrak{N}^{\emptyset,\kappa'|_F})|U(u),\ f:(*)} \mathbf{C}\, f_{\rho_0}|e_{\Pi}^{\kappa}\right) \\ &= \mathrm{tr}\left(\tilde{T}(n^2); \bigoplus_{g\in\mathcal{B}_{\eta'}\cap\mathfrak{N}^{\emptyset,\kappa'|_F},\ g:(*)} \mathbf{C}\, g\right) \\ &= \mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{N}_{**}^{\emptyset,\kappa'|_F}(k+1/2,\tilde{N}(\alpha),\eta')_K\right)\end{aligned}$$

Here, we use the Strong Multiplicity One theorem on $\mathfrak{S}^{\emptyset,\kappa'|_F}\left(k+1/2,\tilde{N}(\alpha),\eta'\right)_K$ at first equality ([U1, (3.11)(2)]). □

We choose and fix a primitive character $\phi := \phi_{D+E}$ such that

$$\phi^2 = {\phi_{D+E}}^2 = \prod_{p\in D+E}\left(\frac{-}{p}\right) . \tag{2.9}$$

Remark. This is possible because of the assumption (2.1)(ii). The conductor of ϕ is l_{D+E}. Of course, there exist many characters which satisfy the above condition. □

By using the twisting operator R_{ϕ}, we can construct a subspace of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$.

(2.10) **Proposition.** Under the above notations, we have

$$C^{(D,E)}|e_{\Pi}^{\kappa}R_{\phi_{D+E}} \subseteq \mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K .$$

Proof. We shall deal only with the case of $k\geq 2$. We can prove the case of $k=1$ in the same way.

From the definition and [U2, (2.2)], $C^{(D,E)} \subseteq B^{(\alpha)} \subseteq S(k+1/2, N, \chi')_K$. Applying the twisting operator R_ϕ to the both sides and using the proposition (1.4), $C^{(D,E)}|R_\phi \subseteq S(k+1/2, N, \chi')_K\,|R_\phi \subseteq S(k+1/2, N, \chi)_K$. Hence,

$$C^{(D,E)}|e^\kappa_\Pi R_\phi = C^{(D,E)}|R_\phi e^\kappa_\Pi \subseteq S(k+1/2, N, \chi)_K\,|e^\kappa_\Pi =: S^{\emptyset,\kappa}(k+1/2, N, \chi)_K .$$

From the claim (2.5), $C^{(D,E)}$ has a C-basis consisting of common eigenform on $\tilde{T}(n^2)$, $(n, N) = 1$. We take any such common eigenform h and a primitive form H of weight $2k$ which corresponds to h in the sense of [U1, (3.11)].

From [U2, (2.28)], the claim (2.5), and the Strong Multiplicity One theorem of weight $2k$, $H \in S^0(2k, M_1 M_{2+}^{(\alpha)})$ follows.

Suppose $h|e^\kappa_\Pi R_\phi \neq 0$. Then we have isomorphism $\mathbf{C} \cong \mathbf{C}h|e^\kappa_\Pi R_\phi$. From this, the proposition (1.3), and [U1, (A.5)],

(2.11)

$$\begin{aligned}\operatorname{tr}\left(\tilde{T}(n^2); \mathbf{C}h|e^\kappa_\Pi R_\phi\right) &= \left(\frac{n}{l_{D+E}}\right) \operatorname{tr}\left(\tilde{T}(n^2); \mathbf{C}h\right) = \left(\frac{n}{l_{D+E}}\right) \operatorname{tr}(T(n); \mathbf{C}H) \\ &= \operatorname{tr}(T(n); \mathbf{C}H|R_{D+E}) .\end{aligned}$$

Here, $R_{D+E} := \prod_{p \in D+E} R_p$ is the twisting operator of $\left(\frac{}{l_{D+E}}\right)$. □

From [A-L, p.228, Theorem 4.1, Corollary 4.1] and the definition of D and E, $H|R_{D+E} \in S^0(2k, M_1 {l_{D+E}}^2 \prod_{p \in F} p^{\nu_p}) = S^0(2k, M_1 M_{2+}) = S^0(2k, M)$. Hence, by using [U1, (3.10)(1) and (2)], we can see $h|e^\kappa_\Pi R_\phi \in \mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$.

Moreover, from [U2, (2.28)] and the Strong Multiplicity one theorem of weight $2k$, $h|e^\kappa_\Pi R_\phi \in \mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$. If $h|e^\kappa_\Pi R_\phi = 0$, the last relation trivially holds. Therefore, we have $C^{(D,E)}|e^\kappa_\Pi R_\phi \subseteq \mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$. □

From (1.3) and the results in [U1, §1], this subspace $C^{(D,E)}|e^\kappa_\Pi R_\phi$ is fixed by all Hecke operators $\tilde{T}(n^2)$, $(n, N) = 1$. Moreover $C^{(D,E)}|e^\kappa_\Pi \subseteq B^{(\alpha)}|e^\kappa_\Pi \subseteq \mathfrak{S}^{\emptyset,\kappa}(k+1/2, N, \chi')_K$ [U2, (2.4)]. Hence, we can get the following formula from (1.5) and (2.8).

(2.12) For all $n \in \mathbf{Z}_+$ with $(n, N) = 1$,

$$\begin{aligned}\operatorname{tr}\left(\tilde{T}(n^2); C^{(D,E)}|e^\kappa_\Pi R_\phi\right) &= \operatorname{tr}\left(\left(\frac{n}{l_{D+E}}\right) \tilde{T}(n^2); C^{(D,E)}|e^\kappa_\Pi\right) \\ &= \left(\frac{n}{l_{D+E}}\right) \operatorname{tr}\left(\tilde{T}(n^2); \mathfrak{N}^{\emptyset,\kappa'|_F}_{**}(k+1/2, \tilde{N}(\alpha), \eta')_K\right) .\end{aligned}$$

□

We shall describe the right-hand side of the above in terms of cusp forms of weight $2k$.

Take any element $g \in \mathcal{B}_{\eta'} \cap \mathfrak{N}^{\emptyset,\kappa'|_F}$ and the primitive form G of weight $2k$ which corresponds to g in the sense of [U1, (3.11)]. Then we can get

(2.13)

"g satisfies the condition $(**)$"

$$\Leftrightarrow \quad g|U(p^2) = -p^{k-1}\chi'(p)\kappa(p)g \quad \text{for all } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right)$$

$$\Leftrightarrow \quad G|U(p) = -p^{k-1}\chi'(p)\kappa(p)G \quad \text{for all } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right)$$

$$\Leftrightarrow \quad G|W(p) = \chi'(p)\kappa(p)G \quad \text{for all } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right).$$

Here, we use [U1, (3.9)], [U1, (3.11)], and [M, Cor.(4.6.18)] in turn. $W(p)$ is the Atkin-Lehner operator of weight $2k$ (cf. [U2, §0(b)]). □

Now, we shall express $\mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{N}_{**}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K\right)$ in terms of the traces of Hecke operators $T(n)$ on spaces of cusp forms of weight $2k$. We have the expression formula [U2, (2.28)] of the trace on the space $\mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K$. We shall transform this expression formula.

From (2.13), the space $\mathfrak{N}_{**}^{\emptyset,\kappa'|_F}$ corresponds to all G's which satisfy the last condition of (2.13). We can take all these forms in the same way as in [U2, (2.21–22)] by using [U1, (A.2)(3) and (A.5)]. See [U1, §4] and [U2, §2] for the detail of transformations.

(2.14) For any $n \in \mathbf{Z}_+$ prime to $\tilde{N}(\alpha)$,

$$\begin{aligned}
&\mathrm{tr}\left(\tilde{T}(n^2); \mathfrak{N}_{**}^{\emptyset,\kappa'|_F}(k+1/2, \tilde{N}(\alpha), \eta')_K\right) \\
&= \sum_{F(\nu)_2 = I+J+K} \sum_{\substack{\tau \in \{\pm 1\}^{\Pi} \\ \sigma \in \{\pm 1\}^{F-(I+J)}}} \Xi'((\alpha(I,J)_l), I+J, (\tau|_F, \sigma)) \times \Xi'' \\
&\quad \times \mathrm{tr}\left(T(n); S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p \in F-(I+J)} p^{\nu_p}) | R_{I+J}\right).
\end{aligned}$$

Here, the notations are as follows: $F(\nu)_2 := \{p \in F \mid \nu_p = 2\}$, $\sum_{F(\nu)_2 = I+J+K}$ is the sum extended over all partitions such that $F(\nu)_2 = I+J+K$, $\alpha(I,J)_l$ is a constant which has a value 0, 1, and ν_l $(:= \mathrm{ord}_l(N))$ according to $l \in I$, J, $F-(I+J)$, $\Xi'((\alpha(I,J)_l), I+J, (\tau|_F, \sigma))$'s are the constant determined by the table [U1, (2.22)], $\tau|_F$ is the restriction of τ to F, and $\Xi'' := \prod_{p \in D+E} \Xi''_p$, is defined by the following table.

$$2 \times \Xi''_p := \begin{cases} 1 + \tau(p), & \text{if } p \in D, \\ 1 + 1, & \text{if } p \in E \text{ and } \chi_p = \mathbf{1}, \\ 1 + \chi'(p)\kappa(p)\tau(p)\prod_{q \in I+J}\left(\frac{p}{q}\right), & \text{if } p \in E \text{ and } \chi_p = \left(\frac{-}{p}\right). \end{cases}$$

We use the same notation $S^*(2k,N)$ (the space of very-newforms) as in the Introduction (0.2). Then,

$$S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p\in F-(I+J)} p^{\nu_p})$$
$$:= \left\{ \begin{array}{l} f \in S^*(2k, M_1 l_{E+J} \prod_{p\in F-(I+J)} p^{\nu_p}) \,; \\ f|W_p = \tau(p) f \quad \text{for all } p \in \Pi, \\ f|R_p W_p = \sigma(p) f|R_p \quad \text{for all } p \in F-(I+J) \end{array} \right\}.$$

R_{I+J} is the twisting operator of $\left(\frac{}{l_{I+J}}\right)$. □

Since $(D+E)\cap(I+J) \subseteq (D+E)\cap F = \emptyset$, the twisting operator R_{D+E} gives an isomorphism: $S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p\in F-(I+J)} p^{\nu_p})|R_{I+J} \to S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p\in F-(I+J)} p^{\nu_p})|R_{I+J+D+E}$ (cf. [U1, (A.5)]).

From the formulae (2.12), (2.14), and [U1, (A.2)(1)]; we have the following.

(2.15) For all $n \in \mathbf{Z}_+$ such that $(n,N)=1$,

$$\begin{aligned} &\operatorname{tr}\left(\tilde{T}(n^2); C^{(D,E)}|e^{\kappa}_{\Pi} R_\phi\right) \\ &= \sum_{F(\nu)_2 = I+J+K} \sum_{\substack{\tau\in\{\pm1\}^{\Pi} \\ \sigma\in\{\pm1\}^{\Pi-(I+J+D+E)}}} \Xi'((\alpha(I,J)_l), I+J, (\tau|_F,\sigma)) \times \Xi'' \\ &\quad \times \operatorname{tr}\left(T(n); S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p\in\Pi-(I+J+D+E)} p^{\nu_p})|R_{I+J+D+E} \right). \end{aligned}$$

Here, $\tau|_F$ is the restriction of τ to F. We use the relation: $F-(I+J) = (D+E+F)-(I+J+D+E) = \Pi-(I+J+D+E)$. □

We shall compare this formula (2.15) with the formula [U2, (2.28)].

We have the following one-to-one correspondence between all partitions of $F(\nu)_2$ and those of $\Pi(\nu)_2$: $F(\nu)_2 = I+J+K \mapsto \Pi(\nu)_2 = (D+I)+(E+J)+K$.

By using this correspondence, the range of parameters in the formula (2.15) is considered as a part of the range of parameters in [U2, (2.28)]. Furthermore the space $S^{*(\tau,\sigma)}(2k, M_1 l_{E+J} \prod_{p\in\Pi-(I+J+D+E)} p^{\nu_p})|R_{I+J+D+E})$ occurs in common. Thus it is enough to compare the coefficients of the traces of $T(n)$ on this spaces in both formulae.

Both coefficients are defined as products of local components on $p \in \Pi$. We shall compare these local components. For simplicity, we put $\tilde{I} := D+I$ and $\tilde{J} := E+J$. For $p \in E+F$, we can verify the following identity:

$$(2.16)\quad \Xi_p((\nu(\tilde{I},\tilde{J})_p), \tilde{I}+\tilde{J}, (\tau,\sigma)) = \begin{cases} \Xi'_p((\alpha(I,J)_l), I+J, (\tau|_F,\sigma)), & \text{if } p\in F, \\ \Xi''_p, & \text{if } p \in E, \end{cases}$$

where the left-hand side is the constant with respect to $\mathfrak{S}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ determined by the table [U1, (2.22)]. The constant in the right-hand side $\Xi'_p((\alpha(I,J)_l),$

$I+J,(\tau|_F,\sigma))$ is the coefficient in the expression formula [U2, (2.28)] for $\mathfrak{N}^{\emptyset,\kappa'|_F}(k+1/2,\tilde{N}(\alpha),\eta')_K$. □

These identities are easily verified, in case by case, from the following facts (1)–(3):

(1) For any $p\in F$, we have $\alpha(I,J)_p=\nu(\tilde{I},\tilde{J})_p$ and $\eta'_p=\chi'_p=\chi_p$.

(2) We use the same notations P^0 and P^1 as in [U1, §2] for any subset P of Π, i.e., $P^0:=\{p\in P\mid \chi_p=1\}$ and $P^1:=\{p\in P\mid \chi_p\neq 1\}$. Then we have $\Pi^1-(\tilde{I}+\tilde{J})^1+(\tilde{I}+\tilde{J})^0=F^1-(I+J)^1+(I+J)^0+(D+E)^0$.

(3) $\nu(\tilde{I},\tilde{J})_p=1$ for any $p\in E$.

Finally we shall consider the case of $p\in D$. Then if $\tau(p)=-1$, any identity like (2.16) does not hold. However, since any prime $p\in D$ does not occurs in the level $M_1l_{\tilde{J}}\prod_{p\in\Pi-(\tilde{I}+\tilde{J})}p^{\nu_p}$, the Atkin-Lehner operator W_p must be the identity 1. Hence if $\tau(p)=-1$, the space $S^{*(\tau,\sigma)}(2k,M_1l_{\tilde{J}}\prod_{p\in\Pi-(\tilde{I}+\tilde{J})}p^{\nu_p})$ is equal to $\{0\}$. Therefore without a loss of validity, we can exchange the coefficient Ξ''_p with $\Xi_p((\nu(\tilde{I},\tilde{J})_l),\tilde{I}+\tilde{J},(\tau,\sigma))$ in such cases. Thus we can have the same identity as (2.16) also in the case of $p\in D$. □

From the above arguments, we obtain the following

(2.17) **Proposition.** For all $n\in\mathbf{Z}_+$ such that $(n,N)=1$, we have

$$\operatorname{tr}\left(\tilde{T}(n^2);C^{(D,E)}|e^{\kappa}_{\Pi}R_\phi\right)=\sum_{\Pi(\nu_2)=\tilde{I}+\tilde{J}+K}\ \sum_{\substack{\tau\in\{\pm1\}^{\Pi}\\ \sigma\in\{\pm1\}^{\Pi-(\tilde{I}+\tilde{J})}}}\Xi((\nu(\tilde{I},\tilde{J})_l),\tilde{I}+\tilde{J},(\tau,\sigma))$$
$$\times\operatorname{tr}\left(T(n);S^{*(\tau,\sigma)}(2k,M_1l_{\tilde{J}}\prod_{p\in\Pi-(\tilde{I}+\tilde{J})}p^{\nu_p})|R_{\tilde{I}+\tilde{J}}\right).$$

Here, $\sum_{\Pi(\nu_2)=\tilde{I}+\tilde{J}+K}$ is the sum extended over all partitions $\Pi(\nu_2)=\tilde{I}+\tilde{J}+K$. Moreover this expression formula is a part of the formula [U2, (2.28)(1)]. □

We shall study the subspace generated by all these subspaces $C^{(D,E)}|e^{\kappa}_{\Pi}R_\phi$'s. We set the following notations.

(2.18) **Definition.** For any $\kappa\in\mathrm{Map}(\Pi,\{\pm1\})=\{\pm1\}^{\Pi}$,

$$\mathfrak{N}^{2,\kappa}=\mathfrak{N}^{2,\kappa}(k+1/2,N,\chi)_K:={\sum_{D,E}}'C^{(D,E)}|e^{\kappa}_{\Pi}R_\phi\ ,$$

where the above sum $\sum'_{D,E}$ is the sum extended over all partitions $\Pi(\nu)_2=D+E+F$ such that $D+E\neq\emptyset$ and $p\equiv1\pmod 4$ for all $p\in D+E$.

This space $\mathfrak{N}^{2,\kappa}$ is a subspace of $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$. We denote by $\mathfrak{N}^{*,\kappa}=\mathfrak{N}^{*,\kappa}(k+1/2,N,\chi)_K$ the orthogonal complement of $\mathfrak{N}^{2,\kappa}$ in $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ with respect to the Petersson inner product. Therefore we have

$$\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K=\mathfrak{N}^{*,\kappa}(k+1/2,N,\chi)_K\oplus\mathfrak{N}^{2,\kappa}(k+1/2,N,\chi)_K\ .$$

From [U1, p.146] and (1.3), both spaces $\mathfrak{N}^{2,\kappa}$ and $\mathfrak{N}^{\emptyset,\kappa}$ are fixed by all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$, $(n,N)=1$. □

The aim of this paper is to find the subspace which corresponds to the space of very-newforms of weight $2k$. Unfortunately, we need the following condition in order to obtain such results as yet.

(2.19) **Condition** (= the condition (0.4) in the Introduction).

$$\chi_p = 1 \quad \text{for any } p \in \Pi(\nu)_2 \text{ such that } p \equiv 3 \pmod 4.$$

In the end of the paper, we remark this topic and give some examples. □

Take any partition $\Pi(\nu)_2 = I + J + K$ and $\tau \in \{\pm 1\}^{\Pi}$, $\sigma \in \{\pm 1\}^{\Pi-(I+J)}$. From [U2, (2.28)(1)], the Hecke submodule $S^{*(\tau,\sigma)}(2k, M_1 l_J \prod_{p\in\Pi-(I+J)} p^{\nu_p})|R_{I+J}$ occurs in $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$ with the multiplicity $\Xi((\nu(I,J)_l), I+J, (\tau,\sigma))$ (= 0, or 1).

As for this, we can see the following proposition.

(2.20) **Proposition.** Let I, J, and (τ,σ) be the same as above and suppose $I+J \neq \emptyset$. Under the condition (2.19), either the following (1) or (2) holds.

(1) $\Xi((\nu(I,J)_l), I+J, (\tau,\sigma)) = 0$.

(2) The Hecke module $S^{*(\tau,\sigma)}(2k, M_1 l_J \prod_{p\in\Pi-(I+J)} p^{\nu_p})|R_{I+J}$ occurs in the space $C^{(I,J)}|e^{\kappa}_{\Pi} R_{\phi}$ and so in $\mathfrak{N}^{2,\kappa}(k+1/2,N,\chi)_K$.

Proof. We assume that the statement (1) does not hold good. Then we have $\Xi((\nu(I,J)_l), I+J, (\tau,\sigma)) = 1$.

Suppose that there exists $p \in I+J$ such that $p \equiv 3 \pmod 4$. From the definition, $\nu(I,J)_p = 0$, or 1. Moreover from the condition (2.19), $\chi_p = 1$. Hence, this prime p belongs to the case of $Q^0_1 + Q^0_0$ in the table [U1, (2.22)]. Therefore, $\Xi((\nu(I,J)_l), I+J, (\tau,\sigma)) = \frac{1}{2}\left(1 + \left(\frac{-1}{p}\right)\right) = 0$. This is contradiction to the assumption. Thus $p \equiv 1 \pmod 4$ for any $p \in I+J$.

From this, I and J satisfy the conditions (i)–(ii) of (2.1). Hence we can define $C^{(I,J)}|e^{\kappa}_{\Pi} R_{\phi}$. The space $S^{*(\tau,\sigma)}(2k, M_1 l_J \prod_{p\in\Pi-(I+J)} p^{\nu_p})|R_{I+J}$ occurs in the expression formula (2.17) of $C^{(I,J)}|e^{\kappa}_{\Pi} R_{\phi}$ as the case of the partition $\Pi(\nu)_2 = I + J + (\Pi(\nu)_2\text{-(I+J)})$. □

From this proposition and the formula (2.17), we have the following formula.

(2.21) **Proposition.** Let the notation be same as above. Under the condition (2.19), we have the following expressions.

(1) For all $n \in \mathbf{Z}_+$, $(n,N)=1$,

$$\begin{aligned}
&\mathrm{tr}\Big(\tilde{T}(n^2); \mathfrak{N}^{2,\kappa}(k+1/2,N,\chi)_K\Big) \\
&= \sum_{\substack{\Pi(\nu)_2 = I+J+K \\ I+J\neq\emptyset}} \ \sum_{\substack{\tau\in\{\pm1\}^{\Pi} \\ \sigma\in\{\pm1\}^{\Pi-(I+J)}}} \Xi((\nu(I,J)_l), I+J, (\tau,\sigma)) \\
&\quad \times \mathrm{tr}\left(T(n); S^{*(\tau,\sigma)}(2k, M_1 l_J \prod_{p\in\Pi-(I+J)} p^{\nu_p})|R_{I+J}\right).
\end{aligned}$$

(2) For all $n \in \mathbf{Z}_+$, $(n, N) = 1$,

$$\mathrm{tr}\Big(\tilde{T}(n^2); \mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K\Big) = \sum_{\tau,\sigma \in \{\pm 1\}^{\Pi}} \Xi(\tau, \sigma)\, \mathrm{tr}\Big(T(n); S^{*(\tau,\sigma)}(2k, M)\Big)\ .$$

Here, the coefficients $\Xi(\tau,\sigma) := \prod_{p \in \Pi} \Xi_p(\tau,\sigma)$ is determined by the following table.

(2.22) **Table.**

$$2 \times \Xi_p(\tau,\sigma)$$
$$= \begin{cases} 1+\tau(p), & \text{if } \nu_p := \mathrm{ord}_p(N) \text{ is even and } \chi_p = 1, \\ 1+\left(\frac{-1}{p}\right)\sigma(p), & \text{if } \nu_p \text{ is even and } \chi_p = \left(\frac{\cdot}{p}\right), \\ 1+\kappa(p)\left(\frac{-1}{p}\right)^{k} \tau(p) \prod_{p\neq q|N} \chi_q(p), & \text{if } \nu_p \text{ is odd and } \chi_p = 1, \\ 1+\kappa(p)\left(\frac{-1}{p}\right)^{k-1} \sigma(p) \prod_{p\neq q|N} \chi_q(p), & \text{if } \nu_p \text{ is odd and } \chi_p = \left(\frac{\cdot}{p}\right). \end{cases}$$

□

The space $\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K$ satisfies the Strong Multiplicity One theorem [U2, (3.7)(2)] and $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ is its Hecke submodule. Hence, from [U2, (2.28)(2–3)], we have the following results.

(2.23) **Proposition.** (1) $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ has an orthogonal **C**-basis consisting of common eigenforms for all Hecke operators $\tilde{T}_{k+1/2,N,\chi}(p^2)$ (p:prime, $p \nmid M$) and $U(p^2)$ (p:prime, $p|M$) which are uniquely determined up to multiplication by non-zero complex numbers. Let f be such a common eigenform and λ_p the eigenvalue of f with respect to $\tilde{T}_{k+1/2,N,\chi}(p^2)$ ($p \nmid M$) resp. $U(p^2)$ ($p|M$). Then there uniquely exists a (very-new) primitive form $F \in S^*(2k, M)$ of weight $2k$ and of conductor M which satisfied $F|T(p) = \lambda_p F$ ($p \nmid M$) and $F|U(p) = \lambda_p F$ ($p|M$).

(2) (**the Strong Multiplicity One theorem**) Let $f, g \in \mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ be two non-zero common eigenforms of all $\tilde{T}_{k+1/2,N,\chi}(p^2)$, $(p, A) = 1$. Here, A is some positive integer. Then if f and g has the same system of eigenvalues on all $\tilde{T}_{k+1/2,N,\chi}(p^2)$, $(p, A) = 1$, then $\mathbf{C}f = \mathbf{C}g$.

(3) From the above, we can see that $\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K$ is embedded into $S^*(2k, M)$ as module over Hecke algebra $\mathcal{H}$ (cf. Main theorem (2) in the Introduction). That is

$$\mathfrak{N}^{*,\kappa}(k+1/2, N, \chi)_K \hookrightarrow S^*(2k, M) \quad \text{as module over Hecke algebra } \mathcal{H}.$$

We note that we already obtained (cf. [U2, (3.7)])

$$\mathfrak{N}^{\emptyset,\kappa}(k+1/2, N, \chi)_K \hookrightarrow S^0(2k, M) \quad \text{as module over Hecke algebra } \mathcal{H}.$$

□

§3 Final results and remarks.

We found the subspaces $\mathfrak{N}^{*,\kappa}(k+1/2,N,\chi)_K$ which correspond to the space of very-newforms of weight $2k$ in the previous section. However, its definition is so complicated. In this section, we shall give another much simpler definition of this subspace and state the final formulation.

In the previous paper [U2, §3], we had given a similar result for the space of newforms $\mathfrak{N}^{\emptyset,\kappa}(k+1/2,N,\chi)_K$. Also in this case, we shall give much simpler definition later. Moreover, we study some examples which do not satisfy the condition (2.19).

We keep to the notation in §2.

(3.1) **Definition.**

$$\mathfrak{N}^*(k+1/2,N,\chi)_K := \bigoplus_{\kappa\in\{\pm1\}^\Pi} \mathfrak{N}^{*,\kappa}(k+1/2,N,\chi)_K \,.$$

$$\mathfrak{N}^2(k+1/2,N,\chi)_K := \bigoplus_{\kappa\in\{\pm1\}^\Pi} \mathfrak{N}^{2,\kappa}(k+1/2,N,\chi)_K \,.$$

From now on, we shortly denote these by $\mathfrak{N}^* := \mathfrak{N}^*(k+1/2,N,\chi)_K$ and $\mathfrak{N}^2 := \mathfrak{N}^2(k+1/2,N,\chi)_K$. From the identity in (2.18) and [U2, (3.2)], we have

$$(3.2) \qquad \mathfrak{N}(k+1/2,N,\chi)_K = \mathfrak{N}^*(k+1/2,N,\chi)_K \oplus \mathfrak{N}^2(k+1/2,N,\chi)_K \,.$$

We easily see that $\mathfrak{N}^*$ is the orthogonal complement of $\mathfrak{N}^2$ in $\mathfrak{N}(k+1/2,N,\chi)_K$. □

We shall use the same notations as in [U2] i.e., for any primitive (cusp) form F of weight $2k$,

$$S(k+1/2,N,\chi;F)_K := \left\{ \begin{array}{l} f\in S(k+1/2,N,\chi)_K \;\; ; \;\; f|\tilde{T}(n^2)=\lambda_F(n)f \\ \text{for all } n\in \mathbf{Z}_+,\ (n,N)=1 \end{array} \right\} ,$$

$$V(N,\chi;F)_K := \left\{ \begin{array}{l} f\in V(N;\chi)_K \;\; ; \;\; f|\tilde{T}(n^2)=\lambda_F(n)f \\ \text{for all } n\in \mathbf{Z}_+,\ (n,N)=1 \end{array} \right\} .$$

Here, $\lambda_F(n)$ is the eigenvalue of F on $T(n)$ ($n\in\mathbf{N}$). □

We have the following proposition under the condition (2.19).

(3.3) **Proposition.** Under the condition (2.19), we have the following decompositions.

(1) For $k\geq 2$,

$$\mathfrak{N}^*(k+1/2,N,\chi)_K = \bigoplus_{F:(*4)} S(k+1/2,N,\chi;F)_K \,.$$

$$\mathfrak{N}^2(k+1/2,N,\chi)_K = \bigoplus_{F:(*5)} S(k+1/2,N,\chi;F)_K \,.$$

(2) For $k=1$,

$$\mathfrak{N}^*(3/2,N,\chi)_K = \bigoplus_{F:(*4)} V(N,\chi;F)_K \,.$$

$$\mathfrak{N}^2(3/2,N,\chi)_K = \bigoplus_{F:(*5)} V(N,\chi;F)_K \,.$$

Here, (*4) and (*5) are the following conditions for a primitive (cusp) forms F of weight $2k$.

(*4): F is a very-newform of conductor M (i.e., $F \in S^*(2k, M)$).

(*5): F is a primitive form of conductor M, but is not a very-newform (i.e., $F \in S^2(2k, M)$).

See (0.2) in the Introduction or [U1, Appendix] for the definition of the notation $S^*(2k, M)$ and $S^2(2k, M)$.

Proof. For simplicity, we deal only with the case of $k \geq 2$. We can also prove the case of $k = 1$ in the same way.

Now we have the following: Both $\mathfrak{N}^*$ and $\mathfrak{N}^2$ have $\mathbf{C}$-basis consisting of common eigenforms on the Hecke operators $\tilde{T}(n^2)$, $(n, N) = 1$. Let f be any element in such a basis. The system of eigenvalues of f corresponds to a primitive form F of weight $2k$. Moreover $F \in S^*(2k, M)$ if $f \in \mathfrak{N}^*$, resp. $F \in S^2(2k, M)$ if $f \in \mathfrak{N}^2$. All of these follow from the trace relation (2.21). From these, $\mathfrak{N}^* \subseteq \oplus_{F:(*4)} S(k+1/2, N, \chi; F)_K$ and $\mathfrak{N}^2 \subseteq \oplus_{F:(*5)} S(k+1/2, N, \chi; F)_K$.

We can easily see that "F satisfies either (*4) or (*5)" if and only if "F satisfies the condition (*2) in [U2, (3.5)]". Hence from [U2, (3.5)], we have

$$\mathfrak{N}^* \oplus \mathfrak{N}^2 \subseteq \bigoplus_{F:(*2)} S(k+1/2, N, \chi; F)_K = \mathfrak{N}(k+1/2, N, \chi)_K = \mathfrak{N}^* \oplus \mathfrak{N}^2 \, .$$

The assertion is easily deduced from this. □

We shall shortly denote the notation in [U2] as follows:

$$\mathfrak{N} := \mathfrak{N}(k+1/2, N, \chi)_K, \qquad \mathfrak{O} := \mathfrak{O}(k+1/2, N, \chi)_K \, .$$

Hence, we have

$$\left\{ \begin{matrix} S(k+1/2, N, \chi)_K \, , & \text{if } k \geq 2 \\ V(N; \chi)_K \, , & \text{if } k = 1 \end{matrix} \right\} = \mathfrak{N} \oplus \mathfrak{O} = \mathfrak{N}^* \oplus \mathfrak{N}^2 \oplus \mathfrak{O} \, .$$

It is easily shown that $\mathfrak{N}^*$ becomes the orthogonal complement of $\mathfrak{N}^2 \oplus \mathfrak{O}$.

We shall explicitly express this space $\mathfrak{N}^2 \oplus \mathfrak{O}$ in terms of classical spaces of cusp forms of weight $k + 1/2$.

(3.4) **Definition.** We put the notation $\mathfrak{O}^*(k+1/2, N, \chi)_K$ as follows:

[The case of $k \geq 2$]

$$\mathfrak{O}^*(k+1/2, N, \chi)_K := \sum_{\substack{0<B|M \\ B \neq M}} \sum_{\substack{0<A|(M/B) \\ \xi(\frac{A}{}) = \chi}} S(k+1/2, 4B, \xi)_K \, |\tilde{\delta}_A$$

$$+ \sum_{\substack{0<B|M \\ B \neq M}} \sum_{0<A|(M/B)^2} \sum_{\substack{\xi, \phi \\ \xi(\frac{A}{})\phi^2 = \chi}} S(k+1/2, 4B, \xi)_K \, |U(A) R_\phi \, .$$

[The case of $k=1$]

$$\mathfrak{O}^*(3/2,N,\chi)_K := \sum_{\substack{0<B|M\\ B\neq M}} \sum_{\substack{0<A|(M/B)\\ \xi(\frac{A}{})=\chi}} V(4B;\xi)_K\,|\tilde{\delta}_A$$

$$+ \sum_{\substack{0<B|M\\ B\neq M}} \sum_{0<A|(M/B)^2} \sum_{\substack{\xi,\phi\\ \xi(\frac{A}{})\phi^2=\chi}} V(4B;\xi)_K\,|U(A)R_\phi\,.$$

Here, $M=N/4$ and ξ and ϕ in the $\sum_{\substack{\xi,\phi\\ \xi(\frac{A}{})\phi^2=\chi}}$ runs over the following set:

$$\left\{\begin{array}{l}(\xi,\phi)\ ;\ \xi \text{ is an even (quadratic) character defined modulo } 4B,\\ \phi \text{ is a primitive character modulo } \mathfrak{f}(\phi), \phi^4=1, \mathfrak{f}(\phi)|l_\Pi,\\ \xi\left(\frac{A}{}\right)\phi^2=\chi \text{ as characters modulo } N\end{array}\right\}.$$

□

(3.5) **Proposition.** We have $\mathfrak{N}^2\oplus\mathfrak{O}\subseteq\mathfrak{O}^*$. Furthermore under the condition (2.19), we have the identity: $\mathfrak{N}^2\oplus\mathfrak{O}=\mathfrak{O}^*$. □

We shall prepare the following general claim.

(3.6) **Claim.** Notations are the same as above. Let p be a prime such that $p^2|M$ and ξ an even quadratic character modulo $4M/p$. Then we have the following.

(1) For $k\geq 2$,

$$S(k+1/2,4M/p,\xi)_K\,|{R_p}^2$$
$$\subseteq S(k+1/2,4M/p,\xi)_K + S\Big(k+1/2,4M/p,\xi\left(\tfrac{p}{}\right)\Big)_K\,|\tilde{\delta}_p\,.$$

(2) For $k=1$,

$$V(4M/p;\xi)_K\,|{R_p}^2\subseteq V(4M/p;\xi)_K+V\Big(4M/p;\xi\left(\tfrac{p}{}\right)\Big)_K\,|\tilde{\delta}_p\,.$$

Proof of the claim. Take any $f=\sum_{n\geq1}a(n)\mathrm{e}(nz)\in S(k+1/2,4M/p,\xi)_K$. We easily see that $(f|{R_p}^2)(z)=f(z)-p^{-k/2-1/4}f|U(p)\tilde{\delta}_p(z)$. From $p|(M/p)$, $f|U(p)\in S\big(k+1/2,4M/p,\xi\left(\frac{p}{}\right)\big)_K$ (cf. [U1, §1]). The assertion (1) is easily deduced from this. By using [U1, (1.23)], we can prove the assertion (2) in the same way. □

We return to the proof of the proposition (3.5).

Proof of (3.5). We shall prove only the case of $k\geq 2$. We can prove the case of $k=1$ in the same way.

[$\mathfrak{O}\subseteq\mathfrak{O}^*$]. From the definitions of $\mathfrak{O}$ and $\mathfrak{O}^*$, both first terms coincide with each other.

We shall consider the spaces in the second term of $\mathfrak{O}$. Take any such a subspace: $S(k+1/2,4B,\xi)_K\,|U(A)\prod_{l\in\Pi}{R_l}^{e_l}$. Here, $0<B|M$, $B\neq M$, $0<A|(M/B)^2$, ξ is an even quadratic character such that $\xi\left(\frac{A}{}\right)=\chi$, and $0\leq e_l\leq 2$ $(l\in\Pi)$.

Since $0 < B|M$ and $B \neq M$, there exists a prime divisor p of M such that $B|(M/p)$. We decompose $A = A_1 p^a$, $(p, A_1) = 1$, $a \geq 0$.

Then $0 < A_1|(M/p)^2$. Hence every prime divisor of A_1 is a divisor of M/p. From these (cf. [U1, (1.22)]),

$$S(k+1/2, 4B, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} \subseteq S(k+1/2, 4M/p, \xi)_K \,|U(A_1)U(p^a) \prod_{l \in \Pi} R_l{}^{e_l}$$
$$\subseteq S\left(k+1/2, 4M/p, \xi\left(\frac{A_1}{}\right)\right)_K |U(p^a) \prod_{l \in \Pi} R_l{}^{e_l} .$$

We divides into two cases: $\mathrm{ord}_p(M) = 1$, or $\mathrm{ord}_p(M) \geq 2$ ($\Leftrightarrow p \in \Pi$).

[The case $\mathrm{ord}_p(M) = 1$]. From [U1, (1.20)(2)], $U(p^a)$ commutes with $\prod_{l \in \Pi} R_l{}^{e_l}$ up to multiplication of complex numbers. Moreover since $l^2|(M/p)$ for all $l \in \Pi$, R_l fixes the space $S\big(k+1/2, 4M/p, \xi\left(\frac{A_1}{}\right)\big)_K$. Hence, we have

$$S(k+1/2, 4B, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} \subseteq S\left(k+1/2, 4M/p, \xi\left(\frac{A_1}{}\right)\right)_K |U(p^a).$$

This last subspace contains in $\mathfrak{O}^*$. In facts, from $p^a|A|(M/B)^2$, we can see $a = 0$, 1, 2. Hence, this subspace is the space in the second term of $\mathfrak{O}^*$ whose parameter is $(M/p, p^a, \xi\left(\frac{A_1}{}\right), 1)$.

Therefore, in this case, we have $S(k+1/2, 4B, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} \subseteq \mathfrak{O}^*$.

[The case of $\mathrm{ord}_p(M) \geq 2$]. In this case, $p|(M/p)$ and so every prime divisor of A is also a divisor of M/p. Moreover, any twisting operator R_l ($p \neq l \in \Pi$) fixes the space $S(k+1/2, 4M/p, \chi)_K$. Hence,

$$S(k+1/2, 4M/p, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} \subseteq S(k+1/2, 4M/p, \chi) \,| \prod_{l \in \Pi} R_l{}^{e_l}$$
$$\subseteq S(k+1/2, 4M/p, \chi) \,|R_p{}^{e_p} .$$

When e_p is either 0 or 1, the last subspace is in the second term of $\mathfrak{O}^*$ whose parameter is $(M/p, 1, \chi, 1)$ or $(M/p, 1, \chi, \left(\frac{}{p}\right))$ respectively.

Suppose $e_p = 2$. From the claim (3.6),

$$S(k+1/2, 4M/p, \chi)_K \,|R_p{}^2 \subseteq S(k+1/2, 4M/p, \chi)_K +$$
$$S\left(k+1/2, 4M/p, \chi\left(\frac{p}{}\right)\right)_K |\tilde{\delta}_p .$$

The last subspace is a sum of the spaces in the first term of $\mathfrak{O}^*$ whose parameter are $(M/p, 1, \chi)$ and $(M/p, 1, \chi\left(\frac{p}{}\right))$. Therefore, also in this case, we have $S(k+1/2, 4B, \xi)_K \,|U(A) \prod_{l \in \Pi} R_l{}^{e_l} \subseteq \mathfrak{O}^*$. Thus we obtain that $\mathfrak{O} \subseteq \mathfrak{O}^*$.

Remark. From the same argument as above, we can also get some simplification of the definition of $\mathfrak{O}^*$ ([U2, (3.7)]). See the bellow (3.7).

[$\mathfrak{N}^2 \subseteq \mathfrak{O}^*$]. $\mathfrak{N}^2$ is generated by all subspaces $C^{(D,E)}|e_\Pi^\kappa R_{\phi_{D+E}}$, where κ runs over all elements in $\{\pm 1\}^\Pi$ and D, E runs over the subsets of $\Pi(\nu)_2$ such that

$\emptyset \neq D+E$ and $p \equiv 1 \pmod 4$ for all $p \in D+E$. $\phi = \phi_{D+E}$ is chosen in (2.9) with respect to D and E.

For this D and E, we use the same notation (2.1–2) in §2: η', u, and $\alpha = \alpha_p^{D,E}$, etc.. From the definitions of $C^{(D,E)}$ and e_{Π}^{κ}, we have

$$C^{(D,E)}|e_{\Pi}^{\kappa}R_{\phi} \subseteq \sum_{0<a|l_D} \sum_{\substack{0\le e_l\le 2 \\ l\in\Pi}} S\Big(k+1/2, \tilde{N}(\alpha), \eta'\Big)_K |U(ua^2)\left(\prod_{l\in\Pi} R_l{}^{e_l}\right) R_{\phi}$$

So, it is enough to study the space $S\Big(k+1/2, \tilde{N}(\alpha), \eta'\Big)_K |U(ua^2)\left(\prod_{l\in\Pi} R_l{}^{e_l}\right) R_{\phi}$.

Since $\tilde{N}(\alpha) = 4M_1 l_E \prod_{l\in F} l^{\nu_l}$, there exists $p \in D+E$ such that $\tilde{N}(\alpha)|4M/p$. From $p|(M/p)$, we have

$$\begin{aligned} S\Big(k+1/2, \tilde{N}(\alpha), \eta'\Big)_K |U(ua^2) &\subseteq S(k+1/2, 4M/p, \eta')_K |U(ua^2) \\ &\subseteq S\Big(k+1/2, 4M/p, \eta'\left(\tfrac{u}{-}\right)\Big)_K = S(k+1/2, 4M/p, \chi')_K \ , \end{aligned}$$

where $\chi' = \chi\left(\frac{}{l_{D+E}}\right)$.

We decompose the primitive character ϕ as follows: $\phi = \phi' \cdot \phi''$, ϕ' is the local p-primary part of ϕ. Under this notation, we have (cf. the proposition (1.4))

$$\begin{aligned} &S(k+1/2, 4M/p, \eta')_K |U(ua^2) \prod_{l\in\Pi} R_l{}^{e_l} R_{\phi} \\ &\subseteq S(k+1/2, 4M/p, \chi')_K \mid \prod_{l\in\Pi} R_l{}^{e_l} R_{\phi'\phi''} \subseteq S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K |R_p{}^{e_p} R_{\phi'}. \end{aligned}$$

From a choice of $\phi = \phi_{D+E}$ (cf. (2.9)), we have $R_p := R_{\left(\frac{-}{p}\right)} = R_{\phi'}{}^2$ and so $R_p{}^{e_p} R_{\phi'} = R_{\phi'}{}^{2e_p+1}$. Moreover we have $R_{\phi'}{}^5 = R_{\phi'}$ because $\phi'(n)^5 = \phi'(n)$ for all $n \in \mathbf{Z}$. Then if e_p is either 0 or 2, $S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K |R_p{}^{e_p} R_{\phi'} = S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K |R_{\phi'}$. This subspace is the space in the second term of $\mathfrak{O}^*$ whose parameter is $(M/p, 1, \chi\left(\frac{p}{}\right) = \chi\left(\frac{-}{p}\right), \phi')$.

Let $e_p = 1$. Denote the inverse character of ϕ' by $\tilde{\phi}'$. Then $R_{\phi'}{}^3 = R_{\tilde{\phi}'}$ and so $S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K |R_p{}^{e_p} R_{\phi'} = S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K |R_{\tilde{\phi}'}$. This subspace is the space in the second term of $\mathfrak{O}^*$ whose parameter is $(M/p, 1, \chi\left(\frac{p}{}\right),$ $\tilde{\phi}')$. Thus we obtain that $\mathfrak{N}^2 \subseteq \mathfrak{O}$ and the first assertion is proved.

Next, we shall prove the opposite inclusion under the condition (2.19).

The first term of $\mathfrak{O}^*$ is equal to the first term of $\mathfrak{O}$. It is enough to study the second term of $\mathfrak{O}^*$. Take any space $S(k+1/2, 4B, \xi)_K |U(A) R_{\phi}$ in the second term of $\mathfrak{O}^*$. Since $0 < B|M$ and $B \neq M$, there exists a prime divisor p of M such that

$B|(M/p)$. We decompose A and ϕ as follows: $A = A_1 p^a$, $(A_1, p) = 1$, $0 \le a \in \mathbf{Z}$ and $\phi = \phi'\phi''$, ϕ' is the local p-primary component.

We shall divide the cases.

[The case of $\mathrm{ord}_p(M) = 1$]. In this case, we have $(B, p) = 1$ and $p^a|p^2$. Moreover ϕ can be defined modulo $4M/p$ because of $l_\Pi|(M/p)$.

It is easily seen from computation that $U(p^a)$ commute with R_ϕ up to multiplications of complex numbers.

Hence, $S(k+1/2, 4B, \xi)_K \,|U(A)R_\phi \subseteq S\big(k+1/2, 4M/p, \xi\left(\frac{A_1}{}\right)\big)_K \,|U(p^a)R_\phi \subseteq S\big(k+1/2, 4M/p, \xi\left(\frac{A_1}{}\right)\phi^2\big)_K \,|U(p^a)$. This last space is the space in the second term of $\mathfrak{D}$ whose parameter is $(M/p, p^a, \xi\left(\frac{A_1}{}\right)\phi^2, (0)_{l\in\Pi})$.

[The case of $p^2|M$]. In this case, $p|(M/p)$ and so all prime divisors of A are also prime divisors of M/p. Hence,

$$\begin{aligned} &S(k+1/2, 4B, \xi)_K \,|U(A)R_\phi \subseteq S(k+1/2, 4M/p, \xi)_K \,|U(A)R_\phi \\ &\subseteq S\left(k+1/2, 4M/p, \xi\left(\frac{A}{}\right)\phi''^2\right)_K |R_{\phi'} = S\left(k+1/2, 4M/p, \chi\phi'^2\right)_K |R_{\phi'} \,. \end{aligned}$$

If ϕ' is of order either 1 or 2, this last space is the space in the second term of $\mathfrak{D}$ whose parameter is $(M/p, 1, \chi\phi'^2, (0)_{l\in\Pi})$ or $(M/p, 1, \chi\phi'^2, (1, (0)_{l\neq p}))$ respectively.

Suppose the order of ϕ' is 4. Then $\phi'^2 = \left(\frac{-}{p}\right)$. $S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right)\right)_K$ has a $\mathbf{C}$-basis consisting of common eigenforms on $\tilde{T}(n^2)$, $(n, N) = 1$.

Take and fix a form f from such a basis. If $f|R_{\phi'} = 0$, we trivially see that $f|R_{\phi'} \in \mathfrak{N}^2 \oplus \mathfrak{D}$. Suppose that $f|R_{\phi'} \neq 0$. From [U2, (3.5)], there exists a primitive form F of weight $2k$ such that $f \in S\left(k+1/2, 4M/p, \chi\left(\frac{-}{p}\right); F\right)_K$ and the conductor of F is a divisor of M/p.

From (1.3), $f|R_{\phi'} \in S(k+1/2, N, \chi)$ becomes a common eigenform on $\tilde{T}(n^2)$, $(n, N) = 1$ and moreover its system of eigenvalues is equal to those of $F|R_p$. We note that this cusp form $F|R_p$ is not always a primitive form. we divides the case.

[The case of $p^3|M$]. In this case, we have $p^2|(M/p)$ and so the space $S(2k, M/p)$ is fixed by the twisting operator R_p. Hence, $F|R_p \in S(2k, M/p)$. We denote by G the primitive form corresponding to $F|R_p$. The conductor of G is a divisor of M/p. Since $f|R_{\phi'} \in S(k+1/2, N, \chi; G)_K$, it follows from [U2, (3.5)] that $f|R_{\phi'} \in \mathfrak{D}$.

[The case of $\mathrm{ord}_p(M) = 2$]. In this case $\mathrm{ord}_p(M/p) = 1$. Let M'' be the conductor of F. Since $0 < M''|(M/p)$, $\mathrm{ord}_p(M'') = 0$ or 1. From the results of [A-L], we have $F|R_p \in S^0(2k, M'')|R_p \subseteq S^0(2k, M')$ and so $F|R_p$ is a primitive form. Here, M' is the least common multiplier of p^2 and M''.

If $M' < M$, $F|R_p$ satisfies the condition (*3) in [U2, (3.5)]. Hence, $f|R_{\phi'} \in S(k+1/2, N, \chi; F|R_p)_K$ is contained in $\mathfrak{D}$.

If $M' = M$, $F|R_p \in S^2(2k, M)$ and so $F|R_p$ satisfies the condition (*5) in the proposition (3.3). Hence, $f|R_{\phi'} \in S(k+1/2, N, \chi; F|R_p)_K$ is contained in $\mathfrak{N}^2$.

Thus the assertion of the proposition is proved. □

Now, we can prove Main Theorem in the Introduction.

Proof of Main Theorem. Assume the condition (2.19). From the above proposition (3.5), we have $\mathfrak{D}^* = \mathfrak{N}^2 \oplus \mathfrak{D}$. Hence $\mathfrak{N}^*$ is the orthogonal complement of $\mathfrak{D}^*$ (cf. the remark before (3.4)). The statements (1) and (2) of Main Theorem follow from the definition (3.1) and the proposition (2.23).

The statement (3) of Main Theorem follows from (3.3), (3.5), and [U2, (3.5)]. □

By using the argument in the proof of the proposition (3.5), we can give much simpler expression of $\mathfrak{D}$ and $\mathfrak{D}^*$.

(3.7) **Proposition.** (1-1) A simpler expression of $\mathfrak{D}$: For $k \geq 2$,

$$\begin{aligned}\mathfrak{D} &= \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi(\frac{A}{}) = \chi}} S(k+1/2, 4B, \xi)_K \,|\tilde{\delta}_A \\ &\quad + \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B)^2 \\ \xi(\frac{A}{}) = \chi}} \sum_{\substack{\phi^2=1 \\ \mathfrak{f}(\phi)|l_\Pi}} S(k+1/2, 4B, \xi)_K \,|U(A)R_\phi \\ &= \sum_{p|M} \left\{ S(k+1/2, N/p, \chi)_K + S\left(k+1/2, N/p, \chi\left(\tfrac{p}{}\right)\right)_K |\tilde{\delta}_p \right\} \\ &\quad + \sum_{\substack{p|M_1, a=1,2 \\ \xi(\frac{p^a}{}) = \chi}} S(k+1/2, N/p, \xi)_K \,|U(p^a) + \sum_{p\in\Pi} S(k+1/2, N/p, \chi)_K \,|R_p\,.\end{aligned}$$

(1-2) A simpler expression of $\mathfrak{D}$: For $k = 1$,

$$\begin{aligned}\mathfrak{D} &= \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi(\frac{A}{}) = \chi}} V(4B;\xi)_K \,|\tilde{\delta}_A \\ &\quad + \sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B)^2 \\ \xi(\frac{A}{}) = \chi}} \sum_{\substack{\phi^2=1 \\ \mathfrak{f}(\phi)|l_\Pi}} V(4B;\xi)_K \,|U(A)R_\phi \\ &= \sum_{p|M} \left\{ V(N/p;\chi)_K + V\left(N/p;\chi\left(\tfrac{p}{}\right)\right)_K |\tilde{\delta}_p \right\} \\ &\quad + \sum_{\substack{p|M_1, a=1,2 \\ \xi(\frac{p^a}{}) = \chi}} V(N/p;\xi)_K \,|U(p^a) + \sum_{p\in\Pi} V(N/p;\chi)_K \,|R_p\,.\end{aligned}$$

(2-1) A simpler expression of $\mathfrak{D}^*$: For $k \geq 2$,

$$\begin{aligned}\mathfrak{D}^* &= \sum_{p|M} \left\{ S(k+1/2, N/p, \chi)_K + S\left(k+1/2, N/p, \chi\left(\tfrac{p}{}\right)\right)_K |\tilde{\delta}_p \right\} \\ &\quad + \sum_{\substack{p|M_1, a=1,2 \\ \xi(\frac{p^a}{}) = \chi}} S(k+1/2, N/p, \xi)_K \,|U(p^a) + \sum_{\substack{p\in\Pi, \phi^4=1 \\ \mathfrak{f}(\phi)|p}} S\left(k+1/2, N/p, \chi\phi^2\right)_K |R_\phi.\end{aligned}$$

(2-2) A simpler expression of $\mathfrak{D}^*$: For $k=1$,

$$\mathfrak{D}^* = \sum_{p|M} \left\{ V(N/p;\chi)_K + V\Big(N/p;\chi\left(\tfrac{p}{\cdot}\right)\Big)_K |\tilde{\delta}_p \right\} + \sum_{\substack{p|M_1, a=1,2 \\ \xi\left(\frac{p^a}{\cdot}\right)=\chi}} V(N/p;\xi)_K |U(p^a) + \sum_{\substack{p\in\Pi, \phi^4=1 \\ \mathfrak{f}(\phi)|p}} V\big(N/p;\chi\phi^2\big)_K |R_\phi \,.$$

Here, if χ (resp. $\chi\left(\frac{p}{\cdot}\right)$) cannot be defined modulo N/p, we consider that the spaces $S(k+1/2,N/p,\chi)_K$ and $V(N/p;\chi)_K$ (resp. $S\big(k+1/2,N/p,\chi\left(\frac{p}{\cdot}\right)\big)_K$ and $V\big(N/p;\chi\left(\frac{p}{\cdot}\right)\big)_K$) are equal to $\{0\}$.

Proof. We deal only with the case of $k \geq 2$ because we can also prove the case of $k=1$ in the same way.

The first term of $\mathfrak{D}$ is equal to those of $\mathfrak{D}^*$ and it is

$$\sum_{\substack{0<B|M \\ B\neq M}} \sum_{\substack{0<A|(M/B) \\ \xi\left(\frac{A}{\cdot}\right)=\chi}} S(k+1/2,4B,\xi)_K |\tilde{\delta}_A \,.$$

We divide the cases.

[The case of $AB \neq M$]. In this case, there exists a prime divisor p of M such that $AB|(M/p)$. Hence $S(k+1/2,4B,\xi)_K |\tilde{\delta}_A \subseteq S\big(k+1/2,4AB,\xi\left(\frac{A}{\cdot}\right)\big)_K \subseteq S(k+1/2,N/p,\chi)_K$ from [U1, (1.9)]. Here we must note the character χ. If $p^2|M$, χ can be defined modulo N/p. Otherwise if $\mathrm{ord}_p(M)=1$, we have $(p,M/p)=1$ and so $(p,AB)=1$. Hence from the relation $\xi\left(\frac{A}{\cdot}\right)=\chi$, we have $\chi_p=1$. Therefore χ can be defined modulo N/p also in this case.

[The case of $AB = M$]. In this case, from $B \neq M$, there exists a prime divisor p of M such that $p|A$. From [U1, (1.9)], we have $S(k+1/2,4B,\xi)_K |\tilde{\delta}_A \subseteq S\Big(k+1/2,N/p,\xi\left(\frac{A/p}{\cdot}\right)\Big)_K |\tilde{\delta}_p = S\big(k+1/2,N/p,\chi\left(\frac{p}{\cdot}\right)\big)_K |\tilde{\delta}_p$. Here, we must note the character $\chi\left(\frac{p}{\cdot}\right)$.

If $p^2|M$, χ can be defined modulo N/p. Otherwise if $\mathrm{ord}_p(M)=1$, we have $(p,B)=1$ and so $\xi_p=1$. From the relation $\chi=\xi\left(\frac{A}{\cdot}\right)$, we have $\chi_p=\left(\frac{\cdot}{p}\right)$. Hence, $\chi\left(\frac{p}{\cdot}\right)$ can be defined modulo N/p.

We easily make simplifications of the second terms of both $\mathfrak{D}$ and $\mathfrak{D}^*$ in the same way as in the proof of (3.5). After the above procedure, we may have the same space from both the first term and the second term. In that case, we cut out such a space from the second term. Thus we obtain the assertions. □

Finally, we give the simplest example which does not satisfy the condition (2.19).

(3.8) **Example.** Let p be an odd prime such that $p \equiv 3 \pmod 4$ and $k \in \mathbf{Z}_+$. For any $\kappa \in \mathrm{Map}(\{p\},\{\pm 1\})$, we have the following isomorphism as modules over Hecke algebra $\mathcal{H}$ (cf. [U2, (2.28)]).

$$(3.9)\quad \mathfrak{N}^{\emptyset,\kappa}\big(k+1/2,4p^2,\left(\tfrac{p}{\cdot}\right)\big)_K \cong S^{*((-1)^k\kappa,\cdot)}(2k,p)|R_p \oplus \bigoplus_{\tau\in\{\pm1\}^{\{p\}}} S^{*(\tau,1)}(2k,p^2)\,.$$

Here, we use the fact that Atkin-Lehner operator is equal to $\mathbf{1}$ on $S(2k,1)$.

Noting $S^*(2k,p) = S^0(2k,p)$, we can give examples $S^{*((-1)^k\kappa,\cdot)}(2k,p) \neq \{0\}$ for some p, k, and κ. Then we also have $S^{*((-1)^k\kappa,\cdot)}(2k,p)|R_p \neq \{0\}$.

There is no character of order 4 in the character group $(\mathbf{Z}/p\mathbf{Z})^\times$. Hence, we cannot construct the space corresponding to $S^{*((-1)^k\kappa,\cdot)}(2k,p)|R_p$ by the same method of the section 2.

I wonder that any form in this "bad"-part is of special type, for example a theta series of special type. □

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