THE DEGREE OF MAPS BETWEEN CERTAIN 6-MANIFOLDS

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ABSTRACT. For manifolds M, M' of the form $S^2 \cup e^4 \cup e^6$ we compute the homomorphisms $H_*M \to H_*M'$ between homology groups which are realizable by a map $F : M \to M'$.

For oriented compact closed manifolds M, M' of the same dimension the degree d of a map $F : M \to M'$ is defined by the equation

$$F_*[M] = d \cdot [M']$$

Here $[M]$ denotes the fundamental class of M. In a classical paper Hopf [H] considered such degrees. In this paper we compute all possible degrees of maps $M \to M'$ where M and M' are 6-manifolds of the form $S^2 \cup e^4 \cup e^6$ and for which the cup square of a generator $x \in H^2$ is non trivial. For example for such a manifold M the degrees of maps $M \to M$ are exactly the numbers $d = k^3, k \in \mathbb{Z}$. The result in this paper answers a question of A. Van de Ven. The author is grateful to Fang Fuquan for his remarks on Pontrjagin classes.

§ 1 HOMOTOPY TYPES OF MANIFOLDS $S^2 \cup e^4 \cup e^6$ AND DEGREES OF MAPS

We consider closed differentiable manifolds M of dimension 6 which are simply connected and for which the cohomology with integral coefficients satisfies

$$(1.1) \qquad H^i(M) = \begin{cases} \mathbb{Z} & \text{for } \; i = 0,2,4,6 \\ 0 & \text{otherwise} \end{cases}$$

Moreover we assume that a generator x of $H^2(M)$ has a non-trivial cup square $x \cup x \neq 0$. We choose a generator $y \in H^4(M)$ such that $x \cup x = m\,y$ where $m \in \mathbb{N} = \{1, 2, \dots ,\}$ is a natural number; we also write $m = m(M)$. Moreover let $w = w(M) \in \mathbb{Z}/2$ be given by the second Stiefel-Whitney class. Then the Wu formulas show that $w(M) = 0$ if and only if the Steenrod square

$$(1.2) \qquad Sq^2 : H^4(M, \mathbb{Z}/2) = \mathbb{Z}/2 \to H^6(M, \mathbb{Z}/2) = \mathbb{Z}/2$$

is trivial so that (1.2) is determined by $w(M)$. Any manifold as in (1.1) admits a homotopy equivalence

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(1.3) $$M \simeq S^2 \cup_g e^4 \cup_f e^6$$

where the attaching map g represents $m\,\eta_2 \in \pi_3(S^2)$. Here η_2 is the Hopf element which generates $\pi_3(S^2) = \mathbb{Z}$. Moreover the attaching map f of the 6-cell satisfies

(1.4) $$q_* f = w\,\eta_4 \in \pi_5(S^4) \quad \text{with} \quad w = w(M)$$

where $q : S^2 \cup_g e^4 \to S^2 \cup_g e^4/S^2 = S^4$ is the quotient map. Here η_n with $n \geq 3$ denotes the generator of $\pi_{n+1}(S^n) = \mathbb{Z}/2$. Recall that $\pi_6(S^3) = \mathbb{Z}/12$ so that $\pi_6(S^3) \otimes \mathbb{Z}/4 = \mathbb{Z}/4$. We define subsets

(1.5) $$\begin{cases} \alpha(M) \subset \mathbb{Z}/4 & \text{if} \quad w(M) = 0, \\ \beta(M) \subset \mathbb{Z}/4 & \text{if} \quad m(M) \quad \text{is even} \end{cases}$$

as follows. For $w(M) = 0$ the suspension Σf of the attaching map in (1.3) admits up to homotopy a factorization

(1.6) $$\begin{array}{ccc} S^6 & \xrightarrow{\Sigma f} & \Sigma(S^2 \cup_g e^4) \\ {\scriptstyle f_0}\downarrow & & \uparrow{\scriptstyle i} \\ S^3 & = & \Sigma S^2 \end{array}$$

where i is the inclusion. Then $\alpha(M)$ consists of all elements $f_0 \otimes 1 \in \pi_6(S^3) \otimes \mathbb{Z}/4$ for which (1.6) homotopy commutes, that is $i_* f_0 = \Sigma f$ in $\pi_6(\Sigma(S^2 \cup_g e^4))$. Moreover if $m(M)$ is even then the inclusion $i : S^3 \subset \Sigma(S^2 \cup_g e^4)$ admits a retraction r. Let $\beta(M)$ be the set of all elements $(r\Sigma f) \otimes 1 \in \pi_6(S^3) \otimes \mathbb{Z}/4$ given by compositions

(1.7) $$S^6 \xrightarrow{\Sigma f} \Sigma(S^2 \cup_g e^4) \xrightarrow{r} S^3$$

where r is any retraction of i. Let $i_2 : \mathbb{Z}/2 \subset \mathbb{Z}/4$ be the inclusion which carries $1 \in \mathbb{Z}/2$ to $2 \in \mathbb{Z}/4$.

(1.8) Lemma. *For $w(M) = 0$ and $m(M)$ even the sets $\alpha(M) = \beta(M)$ coincide and consist of a single element in the image of i_2. In this case let $p(M) \in \mathbb{Z}/2$ be given by*

$$i_2 p(M) = \alpha(M) = \beta(M).$$

Moreover we have

$$\begin{array}{llll} \alpha(M) = \{1,3\} & \text{if} \quad m(M) \equiv 1 \bmod 2 & \text{and} & w(M) = 0, \\ \beta(M) = \{1,3\} & \text{if} \quad m(M) \equiv 2 \bmod 4 & \text{and} & w(M) \neq 0, \\ \beta(M) = \{0,2\} & \text{if} \quad m(M) \equiv 0 \bmod 4 & \text{and} & w(M) \neq 0. \end{array}$$

For $w(M) = 0$ and $m(M)$ even the first Pontrjagin class $p_1(M) \in H^4(M) = \mathbb{Z}$ of M is divisible by 8 and hence yields by reduction mod 16 an element in $\mathbb{Z}/2$ denoted by $p_1'(M) \in \mathbb{Z}/2$; then we have in $\mathbb{Z}/2$ the formula

$$p(M) + p_1'(M) = \{m(M)/2\} \in \mathbb{Z}/2 \quad .$$

so that the element $p(M)$ in (1.8) is also determined by the Pontrjagin class $p_1(M)$. For this compare theorem 4 and the proof of theorem 7 in [W] and [Ya]. For $m \in \mathbb{N}$ and $w \in \mathbb{Z}/2$ we define the group

$$P(m, w) = \begin{cases} \mathbb{Z}/2, & \text{if } m \text{ even and } w = 0, \\ 0, & \text{otherwise} \end{cases}$$

(1.9) Proposition. *The homotopy types of manifolds (or Poincaré complexes) which satisfy the conditions in (1.1) are in 1-1 correspondence with triples (m, w, p) where $m \in \mathbb{N}, w \in \mathbb{Z}/2$ and $p \in P(m, w)$ such that $mw = 0$. The correspondence carries M to the triple $(m(M), w(M), p(M))$ defined above.*

In particular each such triple (M, w, p) is realizable by a manifold as in (1.1) and the realization is unique up to homotopy equivalence. The case of Poincaré complexes in (1.9) was proved by Unsöld [U] and by Yamaguchi [Y] and [Ya]. In fact, for Poincaré complexes proposition (1.9) can be easily derived from the proof of (1.12) below. In the case of manifolds we can use the result of Wall (theorem 8 in [W]) that each Poincaré complex with the properties in (1.1) is homotopy equivalent to a smooth manifold. Compare also the result of Zubr [Z]; according to the remark at the end of [Z] the results of Jupp [J] and Wall [W] on the homotopy classification of simply connected 6-manifold have to be modified.

We now are ready to discuss the possible degrees of maps $F : M \to M'$ where M and M' are manifolds as in (1) with generators $x \in H^2(M), x' \in H^2(M')$. We say that $k \in \mathbb{Z}$ is (M, M') -realizable if there exists a continuous map $F : M \to M'$ with $F^*(x') = k \cdot x$. Moreover we say that $k \in \mathbb{Z}$ is (M, M') -good if $k^2 \cdot m(M)$ is divisible by $m(M')$ and if

$$(1.10) \qquad w(M) \cdot \frac{k^2 \cdot m(M)}{m(M')} = w(M') \cdot k \cdot \frac{k^2 \cdot m(M)}{m(M')}$$

holds in $\mathbb{Z}/2$. One readily checks that any $k \in \mathbb{Z}$ which is (M, M') -realizable is (M, M') -good. We define the group

$$(1.11) \qquad G(M, M') = \begin{cases} \mathbb{Z}/2 & \text{if } w(M) = 0 \text{ and } m(M') \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Then we have the following result which completely determines all degrees k which are (M, M') -realizable.

(1.12) Theorem. *Let $k \in \mathbb{Z}$ be (M, M') -good then k is (M, M') -realizable if and only if an obstruction element*

$$\mathcal{O}(M, k, M') \in G(M, M')$$

is trivial. For $w(M) = 0$ and $m(M')$ even this obstruction element is given by the formula in $\mathbb{Z}/4$

$$i_2\mathcal{O}(M, k, M') = k(-\alpha + \frac{k^2 \cdot m(M)}{m(M')}\beta)$$

with $\alpha \in \alpha(M)$, $\beta \in \beta(M')$ as described in (1.8).

Hence, for example, if k is (M, M') -good and if k is divisible by 4 then k is (M, M') -realizable. Moreover if $M = M'$ then any $k \in \mathbb{Z}$ is (M, M) -good and by (1.12) also (M, M) -realizable. The theorem computes all possible degrees of maps $F : M \to M'$. In fact, such degrees are exactly the numbers $k^3 \cdot m(M)/m(M')$ for which k is (M, M') -realizable.

§ 2 Proof of theorem (1.12)

For the proof of (1.12) and (1.8) we first consider the homotopy groups $\pi_n(C_g)$ of a mapping cone $C_g = B \cup_g CA$ of a map $g : A \to B$ where CA is the cone of A. We assume that $A = \Sigma A'$ is a suspension. Let $\pi_g : (CA, A) \to (C_g, B)$ be the canonical map and let $i : B \subset C_g$ be the inclusion. For the one point union $A \vee B$ let $r = (0, 1) : A \vee B \to B$ be the retraction and let

$$\pi_n(A \vee B)_2 = \text{kernel}\,(r_* : \pi_n(A \vee B) \to \pi_n B)$$

Then we obtain the following commutative diagram in which the bottom row is exact.

$$\begin{array}{ccccccc} & & & & \pi_n(CA \vee B, A \vee B) & \xrightarrow[\cong]{\partial} & \pi_n(A \vee B)_2 \\ (2.1) & & & & \downarrow{\scriptstyle (\pi_g, i)_*} & & \downarrow{\scriptstyle (g,1)_*} \\ \pi_n B & \xrightarrow{i_*} & \pi_n(C_g) & \xrightarrow{j} & \pi_n(C_g, B) & \xrightarrow{\partial} & \pi_{n-1} B \end{array}$$

Hence we can define the functional suspension operator

$$E_g : \text{kernel}(g, 1)_* \to \pi_n(C_g)/i_*\pi_n B$$
$$E_g(\xi) = j^{-1}(\pi_g, 1)_*\partial^{-1}(\xi)$$

where $\xi \in \pi_n(A \vee B)_2$ with $(g, 1)_*\xi = 0$; see 3.4.3 [BO] and II.11.7 [BA]. Now let $[C_g, U]$ be the set of homotopy classes of maps $C_g \to U$. Then the coaction $C_g \to C_g \vee \Sigma A$ yields an action $+$ of $\alpha \in [\Sigma A, U]$ on $G \in [C_g, U]$ so that $G + \alpha \in [C_g, U]$ is defined. For $f \in \pi_n(C_g)$ with $f \in E_g(\xi)$ we have by II.12.3 [BA] the formula in $\pi_n(U)$

$$f^*(G+\alpha) = f^*(G) + (\alpha, Gi)E\xi \tag{2.2}$$

where

$$E : \pi_{n-1}(A \vee B)_2 \to \pi_n(\Sigma A \vee B)_2$$

is the partial suspension; see [BA].

Now let C_h be the mapping cone of $h : A' \to B'$ and let $G : C_g \to C_h$ be a map associated to a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & A' \\ g\downarrow & & \downarrow h \\ B & \xrightarrow{b} & B' \end{array}$$

Then we call G a principal map; see [BA]. The functional suspension is natural in the sense that

$$G_* E_g(\xi) \subset E_h((a \vee b)_* \xi) \tag{2.3}$$

This follows from V.2.8 [BA] and diagram (2.1).

Now let $A = S^2$ and $B = S^2$ so that $C_g = S^2 \cup_g e^4$. Then we see by 3.4.7 [BO] or V.7.6 [BA] that $(\pi_g, i)_*$ in (2.1) is surjective for $n = 6$ and is an isomorphism for $n = 5$. Hence we obtain the exact sequence

$$\pi_5(S^3 \vee S^2)_2 \xrightarrow{(g,1)_*} \pi_5(S^2) \xrightarrow{i_*} \pi_5(C_g) \xrightarrow{\delta} \pi_4(S^3 \vee S^2)_2 \xrightarrow{(g,1)_*} \pi_4(S^2) \tag{2.4}$$

with $\delta(\alpha) = \xi$ if and only if $\alpha \in E_g(\xi)$. Here $\pi_5(S^2) = \mathbb{Z}/2$ is generated by η_2^3 and we have

$$\pi_4(S^3 \vee S^2)_2 = \mathbb{Z} \oplus \mathbb{Z}/2$$

where $\mathbb{Z}$ is generated by the Whitehead product $[i_3, i_2]$ of the inclusions $i_3 : S^3 \subset S^3 \vee S^2, i_2 : S^2 \subset S^3 \vee S^2$ and where $\mathbb{Z}/2$ is generated by $i_3\,\eta_3$. Using the Hilton Milnor theorem [H] we see that (2.4) induces for $g \in m\,\eta_2 \in \pi_3(S^2)$ the exact sequences

$$0 \to \pi_5 S^2 \xrightarrow{i_*} \pi_5(C_g) \xrightarrow{\delta} \pi_4(S^3 \vee S^2)_2 \to 0 \quad \text{if } m \text{ is even} \tag{2.5}$$

$$\pi_5 S^2 \xrightarrow{i_*=0} \pi_5(C_g) \underset{\cong}{\xrightarrow{\delta}} \mathbb{Z} \quad \text{if } m \text{ is odd} \tag{2.6}$$

For this we need the fact that the Whitehead product $[\eta_2, \iota_2] = 0$ is trivial where $\iota_2 \in \pi_2(S^2)$ is represented by the identity of S^2. We point out that (2.5) is non split if $m \equiv 2(4)$ and is split otherwise; compare [Ya].

For $f \in \pi_5(C_g)$ we obtain $\xi = \delta(f)$ with $f \in E_g(\xi)$. Let $X = S^2 \cup_g e^4 \cup_f e^6$ be the mapping cone of f. Then the cohomology ring $H^* = H^*(X)$ satisfies for appropriate generators $x \in H^2, y \in H^4, z \in H^6$ the formulas

(2.7) $$x \cup x = m\, y \quad \text{if} \quad g \in m\eta_2$$

(2.8) $$y \cup x = n\, z \quad \text{if} \quad \xi = n[i_3, i_2] + w \cdot i_3\eta_3$$

Moreover the squaring operation $Sq^2 : H^4(X, \mathbb{Z}/2) \to H^6(X, \mathbb{Z}/2)$ is determined by w; that is $Sq^2 \neq 0$ if and only if $w \neq 0$. Hence for a manifold M as in (1.3) we have $f \in E_g(\xi)$ with $g \in m(M) \cdot \eta_2$ and

(2.9) $$\xi = [i_3, i_2] + w(M) \cdot i_3\eta_3 \in \pi_4(S^3 \vee S^2)_2$$

Proof of (1.12). We consider manifolds $M = S^2 \cup_g e^4 \cup_f e^6$ and $M' = S^2 \cup_h e^4 \cup_d e^6$. Any map

(1) $$G : C_g = S^2 \cup_g e^4 \to C_h = S^2 \cup_h e^4$$

is principal and hence associated to a diagram

(2) $$\begin{array}{ccc} S^3 & \xrightarrow{a} & S^3 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ S^2 & \xrightarrow{b} & S^2 \end{array}$$

where b and a have degree k and $k^2 \cdot m(M)/m(M')$ respectively. We see this by V.7.4, ..., V.7.9 [BA]. Moreover for maps G, G' both associated to (a, b) there exists $\alpha \in \pi_4(S^2)$ such that

(3) $$G' = G + i_*\alpha \in [C_g, C_h]$$

We now consider the diagram

(4) $$\begin{array}{ccc} S^5 & \xrightarrow{a'} & S^5 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d} \\ C_g & \xrightarrow{G} & C_h \\ \cup & & \cup\, i \\ S^2 & \xrightarrow[b]{} & S^2 \end{array}$$

where f and d are the attaching maps of the 6-cell in M and M' respectively. The map G extends to a map $F : M \to M'$ if and only if the obstruction

$$\mathcal{O}(G) = -Gf + da' \in \pi_5(C_h) \tag{5}$$

vanishes in $\pi_5(C_h)$. We now assume that a' is a map of degree $k^3 \cdot m(M)/m(M')$ and that k is (M, M') -good as in the assumption of (1.12). Then we see by (2.9) and (2.3) that

$$j\,\mathcal{O}(G) = 0 \quad \text{in} \quad \pi_5(C_h, S^2) \tag{6}$$

Hence there exists an element $\mathcal{O}'(G) \in \pi_5(S^2)$ with

$$i_*\mathcal{O}'(G) = \mathcal{O}(G). \tag{7}$$

Moreover by (2.9) and (2.2) we see that for G' in (3) we have

$$\begin{aligned}\mathcal{O}(G') &= -f^*(G + i_*\alpha) + da' \\ &= -f^*(G) + da' - (\alpha, Gi)E\xi \\ &= \mathcal{O}(G) - (\alpha, ib)E(\xi)\end{aligned} \tag{8}$$

Here $E\xi$ is given by

$$\begin{aligned}E\xi &= E([i_3, i_2] + w(M) \cdot i_3\eta_3) \\ &= [i_4, i_2] + i_4 w(M)\eta_4 \in \pi_5(S^4 \vee S^2)_2\end{aligned}$$

Since the Whitehead product $[\alpha, \iota_2] \in \pi_5(S^2)$ vanishes for $\alpha \in \pi_4(S^2)$ we therefore get

$$\mathcal{O}(G') = \mathcal{O}(G) - w(M) \cdot i_*(\alpha \circ \eta_4). \tag{9}$$

We now are able to construct maps $M \to M'$ as follows. Let k be (M, M') -good. Then (2) homotopy commutes and hence there exists a map G associated to (a, b). If $m(M)$ is odd then (7) and (2.6) show that $\mathcal{O}(G) = 0$ and hence G can be extended to obtain a map $M \to M'$ associated to (a', b) in (4). If $w(M) \neq 0$ then $\mathcal{O}(G)$ might be non zero but by (9) and (7) we find G' such that $\mathcal{O}(G') = 0$ and hence G' can be extended. Hence we are allowed to put $G(M, M') = 0$ if $m(M')$ odd or $w(M) \neq 0$.

If $m(M')$ even and $w(M) = 0$ then we define the obstruction in (1.12) by $\mathcal{O}'(G)$ in (7); that is

$$\mathcal{O}(M, k, M') = \mathcal{O}'(G) \in \pi_5(S^2) = \mathbb{Z}/2. \tag{10}$$

Here $\mathcal{O}'(G)$ is well defined since the map i_* in (2.5) is injective. We are able to compute the element (10) by using the suspension of diagram (4). We know that the composite

$$i_2 : \mathbb{Z}/2 = \pi_5(S^2) \xrightarrow{\Sigma} \pi_6(S^3) = \mathbb{Z}/12 \twoheadrightarrow \pi_6(S^3) \otimes \mathbb{Z}/4 = \mathbb{Z}/4$$

coincides with the inclusion i_2; see Toda [T]. Hence $\mathcal{O}(M,k,M')$ is determined by

$$i_2\mathcal{O}(M,k,M') = (\Sigma\mathcal{O}'(G)) \otimes 1 \in \mathbb{Z}/4 \tag{11}$$

Since $m(M')$ is even we see that $\Sigma h \doteq 0$ so that there exists a retraction $r : \Sigma C_h \to S^3$ of $i : S^3 \subset \Sigma C_h$. Hence we get

$$\begin{aligned}(\Sigma\mathcal{O}'(G)) \otimes 1 &= r\,\Sigma(i_*\mathcal{O}'(G)) \otimes 1 \\ &= r\,\Sigma\mathcal{O}(G) \otimes 1 \\ &= (-r(\Sigma G)(\Sigma f) + r(\Sigma d)(\Sigma a')) \otimes 1 \in \mathbb{Z}/4\end{aligned} \tag{12}$$

Here we have by (1.6)

$$\begin{aligned}r(\Sigma G)\Sigma f \otimes 1 &= r(\Sigma G)i\,f_0 \otimes 1 \\ &= r\,ib\,f_0 \otimes 1 \\ &= b\,f_0 \otimes 1 = k\alpha \quad \text{with} \quad \alpha \in \alpha(M)\end{aligned} \tag{13}$$

On the other hand we have by (1.7)

$$(r\Sigma d)(\Sigma a') \otimes 1 = \text{degree}\,(a') \cdot \beta \quad \text{with} \quad \beta \in \beta(M') \tag{14}$$

By (12), (13), (14) the proof of the formula in (1.12) is complete. q.e.d.

It remains to prove lemma (1.8).

§ 3 Proof of lemma (1.8)

The proof of (1.8) relies on the following two propositions (3.1) and (3.2). Let $\mathbb{C}P_2$ be the complex projective space with $\mathbb{C}P_2 = S^2 \cup_g e^4, g \in \eta_2 \in \pi_3 S^2$.

(3.1) Proposition. *Let $h : S^5 \to \mathbb{C}P_2$ be the Hopf map which is the attaching map of the 6-cell in $\mathbb{C}P_3$. Then the suspension of h admits up to homotopy a factorization*

$$\begin{array}{ccc} S^6 & \xrightarrow{\Sigma h} & \Sigma\mathbb{C}P_2 \\ {\scriptstyle h'}\downarrow & & \cup i \\ S^3 & = & \Sigma S^2 \end{array}$$

where $h' \in \pi_6(S^3) = \mathbb{Z}/12$ *is a generator.*

As pointed out by the referee a short proof of (3.1) is obtained as follows. The complex projective space $\mathbb{C}P^3$ is the total space of the S^2 -bundle over S^4 with characteristic element $\xi \in \pi_3(SO_3) \cong \mathbb{Z}$ being a generator. The J-homomorphism $J : \pi_3(SO_3) \to \pi_6 S^3 = \mathbb{Z}/12 \cdot h'$ satisfies $J(\xi) = h'$. Hence by a formula of James-Whitehead we obtain $\sigma h = i \circ J(\xi) = i \circ h'$; see [Jam]. We give below a different proof of (3.1) which does not use the J-homomorphism. Our proof is related with the proofs of (3.3) and (3.4) which as well are needed for the main result in this paper.

Let J_2S^2 be the <u>second reduced product</u> of S^2 with $J_2S^2 = S^2 \cup_g e^4, g \in 2\eta_2 = [i_2, i_2] \in \pi_3 S^2$. We define a map

$$\rho : \pi_5(J_2S^2) \to \mathbb{Z}/2 \tag{3.2}$$

by $\rho(f) = (r\Sigma f) \otimes 1 \in \pi_6(S^3) \otimes \mathbb{Z}/2$. Here ρ does not depend on the choice of the retraction $r : \Sigma J_2S^2 \to \Sigma S^2$ of $i : \Sigma S^2 \subset \Sigma J_2 S^2$.

(3.3) Proposition. *The function* ρ *coincides with the function which carries* $f \in \pi_5(J_2S^2)$ *to* $qf \in \pi_5 S^4 = \mathbb{Z}/2$ *where* $q : J_2S^2 \to S^4$ *is the quotient map.*

In addition we get the following result:

(3.4) Addendum. *For* $\epsilon = 1, 2$ *there exist* $h_\epsilon \in \pi_5(J_2S^2)$ *with* $h_1 \in E_g([i_3, i_2] + \iota_3\eta_3)$ *and* $h_2 \in E_g([i_3, i_2]), g \in 2\eta_2$, *such that for an appropriate retraction* r *the following diagram homotopy commutes.*

$$\begin{array}{ccc} S^6 & \xrightarrow{\Sigma h_\epsilon} & \Sigma J_2 S^2 \\ {\scriptstyle \epsilon \cdot h'}\downarrow & & \downarrow {\scriptstyle r} \\ S^3 & = & \Sigma S^2 \end{array}$$

Here h' *is a generator of* $\pi_6 S^3 \cong \mathbb{Z}/12$.

Proof of (1.8). Let $M = S^2 \cup_g e^4 \cup_f e^6$ as in § 1. If $m(M)$ is odd (and hence $w(M) = 0$) there is a map

$$G : S^2 \cup_g e^4 \to \mathbb{C}P_2$$

of degree $m(M)$ in H_4 and degree 1 in H_2. By (2.6) and (2.9) this map carries f to

$$G_* f = m(M) \cdot h$$

where h is the Hopf map in (3.1). Hence (3.1) shows that $\alpha(M)$ contain $\{m(M)\} \in \mathbb{Z}/4$. Hence $\alpha(M) = \{1, 3\}$ since $\alpha(M)$ is a coset of $i_2\,\mathbb{Z}/2$ and $m(M)$ odd.

Next let $m(M)$ be even. In this case we obtain a map

$$G : S^2 \cup_g e^4 \to J_2 S^2$$

of degree $t = m(M)/2$ in H_4 and degree 1 in H_2. By (2.6) and (2.9) the map G carries f to

$$G_* f \in E_{2\eta_2}(t \cdot [i_3, i_2] + t \cdot w(M) \cdot i_3\eta_3)$$

On the other hand a retraction $r : \Sigma J_2 S^2 \to S^3$ yields a retraction $r' = r(\Sigma G) : S^2 \cup_g e^4 \to S^3$ so that in $\pi_6(S^3) \otimes \mathbb{Z}/2$ we have by (3.3)

$$\begin{aligned}(r'\Sigma f) \otimes 1 &= r(\Sigma G)(\Sigma f) \otimes 1 \\ &= \rho((\Sigma G)(\Sigma f)) \\ &= q(Gf) \\ &= t \cdot w(M) \bmod 2\end{aligned}$$

This shows $\beta(M) \in i_2(\mathbb{Z}/2) \subset \mathbb{Z}/4$ if $w(M) = 0$ and it yields the formula for $\beta(M)$ in (1.8) if $w(M) \neq 0$. q.e.d.

For the proof of (3.1), (3.3) and (3.4) we need the infinite reduced product JX of James [Ja] where X is a pointed space. In fact J is a functor which carries pointed spaces to pointed spaces and one has a canonical natural transformation

(3.5) $$JX \xrightarrow{\simeq} \Omega\Sigma X$$

which is a homotopy equivalence since we assume that X is a connected CW-complex. Moreover J is a monad in the sense that there are natural maps $i = i_X : X \to JX$, $\mu : JJX \to JX$ satisfying

(1) $$\mu J(i_X) = 1 \quad \text{and} \quad \mu i_{JX} = 1.$$

By (3.5) the suspension Σ can be described by the composite

(2) $$\Sigma : [Y, X] \xrightarrow{(i_X)_*} [Y, JX] \underset{\cong}{\xrightarrow{\vartheta}} [\Sigma Y, \Sigma X]$$

where the isomorphism ϑ is obtained by (3.5).

Proof of (3.1). We consider $V = J\mathbb{C}P_2$ and the suspension

(1) $$\Sigma : \pi_5 \mathbb{C}P_2 \xrightarrow{i_*} \pi_5(V) \cong \pi_6(\Sigma\mathbb{C}P_2)$$

Using $g = \Sigma\eta_2$ in (2.1) we see that the sequence

(2) $$\pi_6 S^4 \xrightarrow{(\eta_3)_*} \pi_6(S^3) \xrightarrow{i_*} \pi_6\Sigma\mathbb{C}P_2 \to 0$$

is exact since $(\pi_g, i)_*$ is an isomorphism for $n = 7, 6$; compare 3.4.7 [BO] or V.7.6 [BA]. Here we have $(\eta_3)_*\pi_6 S^4 = \Sigma\pi_5 S^2$ so that the following diagram commutes

$$
(3) \qquad \begin{array}{ccccccccc}
\pi_6(S^3) & \xrightarrow{i_*} & \pi_6\Sigma\mathbb{C}P_2 & & & & & & \\
\Sigma\uparrow & & \| & & & & & & \\
\pi_5 S^2 & \xrightarrow{0} & \pi_5 V & \overset{j}{\rightarrowtail} & \pi_5(V,S^2) & \xrightarrow{\partial} & \pi_4 S^2 & \longrightarrow & 0
\end{array}
$$

The bottom row is exact. The space V is a CW-complex in which all cells have even dimension. Therefore we obtain the exact sequence

$$
(4) \qquad \pi_6(V^6,V^4) \xrightarrow{\partial} \pi_5(V^4,S^2) \to \pi_5(V,S^2) \to 0
$$

Let $S^3_W = S^3_H = S^3$ and let $A = S^3_W \vee S^3_H$ be the one point union with inclusions $i_W, i_H : S^3 \subset A$ accordingly. Then V^4 is the mapping cone of $g : A \to S^2$ with $gi_W = [\iota_2, \iota_2]$ and $gi_H = \eta_2$. This shows that

$$
(5) \qquad \begin{array}{ccc}
\pi_5(V^4,S^2) & \underset{\cong}{\xrightarrow{\partial}} & \pi_4(A \vee S^2)_2 \\
\partial\downarrow & & \downarrow(g,1)_* \\
\pi_4 S^2 & = & \mathbb{Z}/2
\end{array}
$$

commutes. The isomorphism is $\theta^{-1} = (\pi_g, i)_*\partial^{-1}$ as in (2.1). Moreover we have

$$
\pi_4(A \vee S^2)_2 = \mathbb{Z}/2\, i_W\eta_3 \oplus \mathbb{Z}/2\, i_H\eta_3 + \mathbb{Z}[i_W,i_2] + \mathbb{Z}[i_H,i_2]
$$

The space V has exactly 3 cells a, b, c of dimension 6. Let

$$
\begin{aligned}
p_a &: S^2 \times \mathbb{C}P_2 \to V \\
p_b &: \mathbb{C}P_2 \times S^2 \to V \\
p_c &: S^2 \times S^2 \times S^2 \to JS^2 \subset V
\end{aligned}
$$

be the canonical maps given by $S^2 \subset \mathbb{C}P_2$. Then $a = p_a(e^2 \times e^4), b = p_b(e^4 \times e^2)$ and $c = p_c(e^2 \times e^2 \times e^2)$ where $e^2 \cup * = S^2$ and $S^2 \cup e^4 = \mathbb{C}P_2$. We claim that $\theta\partial$ defined by (4) and (5) satisfies the formulas:

$$
(6) \qquad \begin{cases} \theta\partial(a) = \theta\partial(b) = [i_H,i_2] + [i_W,i_2] + i_W\eta_3 \\ \theta\partial(c) = 3[i_W,i_2] \end{cases}
$$

Moreover we have for ji_* defined by (1) and (3)

$$
(7) \qquad ji_*(h) = [i_H,i_2]
$$

Now (6) and (7) yield by (4) the proposition in (3.1). In fact by (3) and (5) the group

$$(8) \qquad \pi_5 V \cong (\mathbb{Z}/2 \oplus \mathbb{Z} \oplus \mathbb{Z})/\sim$$

is generated by $i_W\eta_3, [i_H, i_2], [i_W, i_2]$ with the relation $\theta\partial(a) \sim 0$ and $\theta\partial(c) = 0$ where i_*h is represented by $[i_h, i_2]$. Hence i_*h in (1) is a generator of $\pi_5 V \cong \mathbb{Z}/6$. It remains to prove the formulas in (6). Since Sq^2 is non trivial in $S^2 \times \mathbb{C}P_2$ and $\mathbb{C}P_2 \times S^2$ we see that $i_W\eta_3$ has to be a summand of $\theta\partial(a)$ and $\theta\partial(b)$. On the other hand we show below that

$$(9) \qquad 2\theta\partial(a) = 2\theta\partial(b) = 2[i_H, i_2] + 2[i_W, i_2]$$

This implies the first formula in (6).

For $i = 1, 2, 3$ let $S_i = S^2$ be the 2-sphere with 2-cell e_i, that is $S_i = * \cup e_i$. Moreover let $T = S_1 \times S_2 \times S_3$ and let

$$\xi_i : S_i \subset S_1 \vee S_2 \vee S_3 = T^2$$

be the inclusions. Then the two cell $e_i \times e_j$ in T with $i < j$ has the attaching map $[\xi_i, \xi_j]$ which is the Whitehead product of ξ_i, ξ_j. Hence T^4 is the mapping cone of

$$g : A = S_{12} \vee S_{13} \vee S_{23} \to S_1 \vee S_2 \vee S_3$$

where $S_{12} = S_{13} = S_{23} = S^3$ and $g|S_{ij} = [\xi_i, \xi_j]$. Moreover let $w \in \pi_5(T^4)$ be the attaching map of the 6-cell $e_1 \times e_2 \times e_3$ in T. Then we know

$$(10) \qquad w \in E_g([\xi_{12}, \xi_3] + [\xi_{13}, \xi_2] + [\xi_{23}, \xi_1])$$

where $\xi_{ij} : S_{ij} \subset A \subset A \vee T^2$ and $\xi_i : S^2 \subset T^2 \subset A \vee T^2$ are the inclusions. Formula (10) corresponds to the Nakaoka Toda formula [NT], see also 3.6.10 in [BO] or [BI]. Now (10) and the canonical map $T \to JS^2$ show that the second formula in (6) holds. For this we use the naturality (2.3). On the other hand we have the canonical map $\lambda : S^2 \times S^2 \to J_2S^2 \to \mathbb{C}P_2$ which is of degree 2 in H_4. Then (10) and the maps $p_a(1 \times \lambda) : T \to V, p_b(\lambda \times 1) : T \to V$ show that (9) holds. For this we again use (2.3). q.e.d.

Proof of (3.3) and (3.4). The space J_2S^2 is the 4-skeleton of JS^2; let $j : J_2S^2 \subset JS^2$ be the inclusion. Then j induces the exact sequences

$$(1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_5 J_2S^2 & \xrightarrow{j_*} & \pi_5 JS^2 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{(3,0)} & \mathbb{Z} \oplus \mathbb{Z}/2 & \xrightarrow{1 \oplus 1} & \mathbb{Z}/3 \oplus \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

Here δ is the map in (2.5) for $g = [\iota_2, \iota_2]$. In the top row $1 \in \mathbb{Z}$ is mapped to the attaching map w of the 6-cell in JS^2 for which $\delta(w) = (3,0)$ by (10) in the proof of (3.1) above. Recall that the second coordinate of $\delta(x), x \in \pi_5 J_2 S^2$, coincides with $q(x) \in \pi_5 S^4 = \mathbb{Z}/2$. The kernel of δ is given by the inlcusion $i_* : \pi_5 S^2 \subset \pi_5 J_2 S^2$. We now obtain by the maps in (3.5) (1) the following commutative diagram

$$
(2) \qquad
\begin{array}{ccccc}
\pi_6 S^3 & \overset{\vartheta}{\cong} \pi_5 J S^2 & \xrightarrow{\;1\;} & \pi_5(JS^2) \\
\downarrow{\scriptstyle i_*} & \downarrow{\scriptstyle (Ji)_*} & & \uparrow{\scriptstyle \mu_*} \\
\pi_6(\Sigma J_2 S^2) & \overset{\vartheta}{\cong} \pi_5(J J_2 S^2) & \xrightarrow{(Jj)_*} & \pi_5(JJS^2) \\
\downarrow{\scriptstyle r_*} & \uparrow{\scriptstyle u_1} & & \uparrow{\scriptstyle u_2} \\
\pi_6(S^3) & \pi_5(J_2 S^2) & \xrightarrow{\;j_*\;} & \pi_5(JS^2) \overset{\vartheta}{\cong} \pi_6 S^3
\end{array}
$$

Here u_1, resp. u_2, is induced by the inclusion $i_X : X \subset JX$ with $X = J_2 S^2$ and $X = JS^2$ respectively. We have $\vartheta u_1 x = \Sigma(x)$. Moreover we have $\mu_* u_2 = 1$. Now we get for $y = r_* \Sigma(x) \in \pi_6(S^3)$ the equation $\vartheta u_1 x = i_* y + z$ with $r_*(z) = 0$ and $2z = 0$ since kernel $(r_*) = \mathbb{Z}/2$. Now we obtain

$$
(3) \qquad u_1 x = \vartheta^{-1}(i_* y + z) = (Ji)_* \vartheta^{-1} y + \vartheta^{-1} z
$$

and hence by diagram (2)

$$
(4) \qquad \begin{aligned} j_*(x) &= \mu_*(Jj)_* u_1 x \\ &= \vartheta^{-1} y + \mu_*(Jj)_* \vartheta^{-1} z \end{aligned}
$$

Therefore we get

$$
(5) \qquad \vartheta j_*(x) = y + z' = r_* \Sigma(x) + z'
$$

where z' is an element of order at most 2. Since the kernel of δ' in (1) is the element of order 2 we thus derive from (5) the result in (3.3) and (3.4) respectively; compare the definition of δ in (2.4). q.e.d

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