HURWITZ SPACES AND FROBENIUS MANIFOLDS

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1. Introduction

In this paper we give an exposition of Dubrovin's theorem [D1, D2, D3] stating that there is a hierarchy of Frobenius structures on Hurwitz spaces.

1.1. A Frobenius manifold is a complex manifold M with a flat metric and a commutative, associative multiplication on the tangent spaces such that its structure constants with respect to flat vector fields are third order derivatives of a function Φ on M; Φ thus satisfies a system of non-linear PDE, the associativity or WDVV-equations.

Besides the A- and B-model of mirror symmetry, i.e. quantum cohomology and the Barannikov-Kontsevich construction, the third large class of Frobenius manifolds comes from *Saito frameworks* which consist of a family of holomorphic functions $F : N \to \mathbb{C}$ parametrized by M, $N \to M$, and a relative volume form $\omega \in \Omega^{\max}_{N/M}$, e.g. the versal deformation of an isolated hypersurface singularity and a Saito primitive form. This third construction is also of B-model type in the sense that contrary to the A-model the multiplication is easily obtained from the Milnor rings $\mathcal{O}_C$ of F, in particular is independent of ω, whereas the flat metric is obtained by pulling back the Grothendieck residue pairing to $\mathcal{O}_C$ via $(f \mapsto f\omega)$ and finding ω is difficult.

The Hurwitz space $H_{g,t}, t = (n_j)$, is the moduli space of meromorphic functions on curves of genus g with poles of order $n_0 + 1, \dots, n_M + 1$ and simple branch points over $\mathbb{C}$. Adding rigidifying data to $H_{g,t}$ to obtain a covering $H'_{g,t} \to H_{g,t}$, there is thus the "universal" function $F : \mathcal{C}^\circ \to \mathbb{C}$ on the (pointed) universal curve $\mathcal{C}^\circ \to H'_{g,t}$.

Dubrovin's theorem states that if ω is a *descendant form*, i.e. a multi-valued relative 1-form on $\mathcal{C}^\circ \to H'_{g,t}$ with constant principal parts, a-periods, and increments, then (F, ω) is a Saito framework. As descendant forms form a *hierarchy*, i.e. a filtered local system $\Omega = \bigcup_{k \geq 0} \Omega_k$ on $H'_{g,t}$, $\operatorname{rk} \Omega_k = (k+1)\dim(H_{g,t})$, together with surjective maps $D : \Omega_k \to \Omega_{k-1}$, we obtain a hierarchy of (semisimple) Frobenius structures on $H'_{g,t}$. This hierarchy descends to $H_{g,t}$ and consists exactly of all (finite) flat metrics with the same rotation coefficients. We give an overview in section 2.

1.2. Certain Frobenius structures on Hurwitz spaces may be realized on orbit spaces of their monodromy groups: $H_{0,n}, H_{0,(n,m)}, H_{1,n}$ correspond in this way to the Coxeter group A_n [D3], the extended affine Weyl group $\tilde{W}^{(n+1)}(A_{n+m+1})$ [DZ], and the Jacobi group $\mathbb{J}(A_n)$ [B], resp.

Krichever showed [K] that there are Frobenius structures on moduli spaces of curves with a marked abelian differential of the second kind with fixed type of pole divisor, vanishing a-periods, and fixed b-periods. Hurwitz spaces are the special case where the b-periods are zero. Except for this case Krichever's Frobenius manifolds do not have an Euler field.

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2. Overview

We first recall some basic statements about Frobenius manifolds and Saito frameworks from [M], Chap.I and Chap.III, §8, resp.

2.1. **Definition.** A **Frobenius manifold** is a complex manifold M with a commutative, associative multiplication $\circ : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M$ with identity e on the holomorphic tangent sheaf and a non-degenerate, symmetric pairing $g : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M$ (called metric) which is *invariant*, i.e. $g(X \circ Y, Z) = g(X, Y \circ Z)$ for any $X, Y, Z \in \mathcal{T}_M$, such that the Levi-Civita connection of g is flat and locally there exists $\Phi \in \mathcal{O}_M$ (called Frobenius potential) such that if $\mathcal{T}_M^g$ denotes the sheaf of g-flat vector fields we have

$$g(\partial \circ \partial', \partial'') \;=\; \partial\partial'\partial''\,\Phi, \qquad \partial, \partial', \partial'' \in \mathcal{T}_M^g.$$

$E \in \mathcal{T}_M(M)$ is called an **Euler field** on M if $\mathrm{Lie}_E(\circ) = d_0 \circ$ and $\mathrm{Lie}_E(g) = Dg$ for some $d_0, D \in \mathbb{C}$. An Euler field E is called **semisimple** if the adjoint action of E on $\mathcal{T}_M^g$ is semisimple.

A Frobenius manifold is called **semisimple** if $\circ$ is semisimple, i.e. locally there exists a basis of *idempotent* vector fields (e_n), $e_n \circ e_m = \delta_{nm} e_n$. A **semisimple weak Frobenius manifold** is a complex manifold M with a multiplication $\circ$ such that locally there exist coordinates (u_n) (called canonical coordinates) such that $e_n := \frac{\partial}{\partial u_n}$ are idempotents. A metric g on a semisimple weak Frobenius manifold M is called **compatible** if $(M, \circ, g)$ is a Frobenius manifold and **admissible** if in addition $e \in \mathcal{T}_M^g$.

2.2. **Proposition.** *If $(M, \circ, g)$ is a semisimple Frobenius manifold then $(M, \circ)$ is a semisimple weak Frobenius manifold.*

Let $(M, \circ)$ be a semisimple weak Frobenius manifold. A metric g on M is invariant iff $g = \sum \eta_n (du_n)^2$ is diagonal. A diagonal metric $g = \sum \eta_n (du_n)^2$ on M is flat if and only if locally there exists $\eta \in \mathcal{O}_M$ (called metric potential) such that $e_n \eta = \eta_n$ for any n and the rotation coefficients

$$\gamma_{nm} \;:=\; \frac{e_n \sqrt{\eta_m}}{\sqrt{\eta_n}} \;=\; \frac{1}{2}\,\frac{e_n e_m\, \eta}{\sqrt{\eta_n \eta_m}} \tag{2.1}$$

satisfy the Darboux-Egoroff equations $e_k\, \gamma_{nm} = \gamma_{nk}\gamma_{km}$ and $e\gamma_{nm} = 0$ for any $n \neq m \neq k \neq n$. A diagonal metric g on M is flat if and only if g is compatible. In this case, $e = \sum e_n \in \mathcal{T}_M^g$ iff $e_n\, \gamma_{nm} = 0$ for any $n \neq m$ and $E := d_0 \sum u_n e_n$ is an Euler field iff $E\,\eta_n = (D - 2d_0)\eta_n$ for any n.

If $g = \sum (a_n)^2 (du_n)^2, g' = \sum (a_n')^2 (du_n)^2$ are metrics on M, g is admissible, and $\partial := \sum \frac{a_n'}{a_n} e_n$ (thus $g'(X, Y) = g(\partial \circ X, \partial \circ Y)$ for any $X, Y \in \mathcal{T}_M$) then g' is admissible if and only if $\partial \in \mathcal{T}_M^g$ and the rotation coefficients agree, i.e. $\frac{e_n\, a_m}{a_n} = \frac{e_n\, a_m'}{a_n'}$ for any $n \neq m$.

2.3. **Definition.** A **Saito framework** consists of submersions $p : N \to M, p' : N' \to M$ of complex manifolds, morphisms $\pi : N' \to N, F : N \to \mathbb{C}$, and a relative volume form $\omega \in \Omega^{\max}_{N'/M}$ such that

(0) $p \circ \pi = p'$ and $\pi : (p')^{-1}(m) \to p^{-1}(m)$ is a universal covering for any $m \in M$;

(i) if $C := \mathrm{Spec}(\mathcal{O}_N/J_F), J_F := \{XF \in \mathcal{O}_N \mid X \in \mathcal{T}_{N/M} \subset \mathcal{T}_N\}$, is the critical set of F then the map $s : \mathcal{T}_M \to p_*(\mathcal{O}_C), X \mapsto \bar{X}F|_C$, where $\bar{X} \in \mathcal{T}_N$ is a lift of X, is an isomorphism;

(ii) if $D_C \subset C$ is the zero-set of the Hessian $\mathrm{Hess}(F) \in (\Omega^{\max}_{N/M})^{\otimes 2}(C)$ defined by

$$\mathrm{Hess}(F) \ := \ \det\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right)(dz_1 \wedge \cdots \wedge dz_{\max})^2|_C,$$

$z_1, \ldots, z_{\max}$ being vertical coordinates, then the discriminant $D := p(D_C)$ is a divisor and $p : C \setminus p^{-1}(D) \to M \setminus D$ is étale;

(iii) if $\rho_1, \ldots, \rho_\mu : U \to C, U \subset M \setminus D$, are sections of p, $\mu := \dim(M)$, and $u_n := F \circ \rho_n$ then

$$g_\omega \ := \ \sum_{n=1}^{\mu} \frac{\omega^2}{\mathrm{Hess}(F)}(\rho_n)\,(du_n)^2 \ = \ \sum_{n=1}^{\mu} \operatorname*{res}_{\rho_n}\left[\ \frac{\omega}{\frac{\partial F}{\partial z_1} \cdots \frac{\partial F}{\partial z_{\max}}}\ \right](du_n)^2, \tag{2.2}$$

$z_1, \ldots, z_{\max}$ being vertical coordinates such that $dz_1 \wedge \cdots \wedge dz_{\max} = \omega$ (cf.[GH], p.650), is a well-defined (i.e. independent of lifts of ρ_n to N') flat metric on $M \setminus D$ that extends regularly to M.

It follows that M with multiplication $\circ$ induced by $s : \mathcal{T}_M \xrightarrow{\sim} p_*(\mathcal{O}_C)$ and metric g_ω is a generically semisimple Frobenius manifold.

2.4. **Overview.** In section 6 we construct in the category of comlex manifolds the Hurwitz space $H_{g,t}$, its rigidification $H'_{g,t}$, the universal curve $\mathcal{C} \to H'_{g,t}$, the fibrewise universal covering $\tilde{\mathcal{C}} \to \mathcal{C}^\circ$ of the pointed universal curve with the fibres over infinity removed, and the universal morphism $F \equiv \Lambda : \mathcal{C}^\circ \to \mathbb{C}$. We thus have all the data of a Saito framework except for ω. Note that the disriminant D is empty.

In order to obtain ω we discuss in sections 3,4,5 multi-valued forms on Riemann surfaces, in particular we define a pairing between multi-valued forms which is used in a reciprocity law and we give a basis of the vector space of multi-valued abelian differentials.

In section 7 we define *descendant forms* as relative 1-forms on $\tilde{\mathcal{C}} \to H'_{g,t}$ that are fibrewise multi-valued abelian differentials with poles only at the poles of Λ and with constant principal parts, a-periods, and increments. Let Ω denote the sheaf on $H'_{g,t}$ of descendant forms, $\tilde{e}, \tilde{E}$ be the (meromorphic) lifts to $\tilde{\mathcal{C}}$ of $e := \sum e_n, E := \sum u_n e_n$ such that $\tilde{e}(\Lambda) = \tilde{E}(\Lambda) = 0$, and define $D := \mathrm{Lie}_{\tilde{e}}$. We show that $\Omega_k := \ker(D^{k+1} : \Omega \to \Omega)$ is a local system of rank $(k+1)\dim(H_{g,t})$ and $D : \Omega_k \to \Omega_{k-1}$ is surjective. ω is called *primary* if $\omega \in \Omega_0$ and ω is called *eigenprimary* if in addition ω is an eigenvector of $\mathrm{Lie}_{\tilde{E}}$. We show that $\omega \in \Omega$ is primary if and only if the order of poles of ω is smaller or equal to one plus the order of Λ at the respective poles and the inrements of ω are at most linear in Λ. We also show that the natural basis of primary forms consists of eigenprimary forms.

In section 8 we prove Dubrovin's theorem: if $H_\omega \subset H'_{g,t}, \omega \in \Omega$, denotes the open subset where g_ω is non-degenerate, i.e. all residues of (2.2) are non-zero, then $(H_\omega, \circ, g_\omega)$ is a Frobenius manifold. In Theorem 8.1 we give flat coordinates, calculate the metric in flat coordinates, show that there is a natural isomorphism between Ω_0 and g_ω-flat vector fields which respects the pairings, characterize primary forms by the flatness of the identity vector field, give the metric potential and the Frobenius potential, prove that the hierarchy of flat metrics descends to $H_{g,t}$ and consists exactly of all (finite) metrics with the same rotation coefficients, prove that the Frobenius structures defined by eigenprimary forms have a semisimple Euler field and compute its spectrum.

The distinguished feature of Saito frameworks of Hurwitz spaces as compared to those of isolated hypersurface singularities and primitive forms consists in the *global geometry* of the fibres of $\mathcal{C} \to H'_{g,t}$.

3. Multi-valued forms on Riemann surfaces

In this section we define multi-valued abelian differentials on a Riemann surface with local coordinates at marked points and prove that they are in one-to-one correspondence with multi-valued Laurent tails and a-periods.

Let C be a compact Riemann surface of genus g, $Q_0, \ldots, Q_M \in C$, and $\pi : \tilde{C} \to C^\circ$ be the universal covering of $C^\circ := C \setminus \{Q_0, \ldots, Q_M\}$. Let z_j be a local coordinate at Q_j defined on a small disk D_j such that $z_j(Q_j) = 0$, $j = 0, \ldots, M$. Put $D_j^\circ := D_j \setminus \{Q_j\}$.

Denote by $\mathcal{M}_S$ and $\mathcal{M}_S^1$ the sheaves of meromorphic functions and meromorphic 1-forms on a Riemann surface S and $\mathcal{M}(S) := \mathcal{M}_S(S), \mathcal{M}^1(S) := \mathcal{M}_S^1(S)$.

$\pi_1(C^\circ)$ acts on $\mathcal{M}^1_{\tilde{C}}(\pi^{-1}(U)), U \subset C^\circ$. The difference $\mathrm{incr}_\gamma(\omega) := \gamma^*\omega - \omega, \gamma \in \pi_1(C^\circ)$, is called the **increment** of ω along γ. The map $\mathrm{incr}(\omega) : \pi_1(C^\circ) \to \mathcal{M}^1_{\tilde{C}}(\pi^{-1}(U)), \gamma \mapsto \mathrm{incr}_\gamma(\omega)$, is a crossed homomorphism.

As a substitute for the notion of "a point in $\tilde{C}$ over Q_j" we introduce the notion of a *ray* over Q_j. Everything explicit of what follows depends on choices of rays.

3.1. **Definition.** A **ray** over Q_j (with respect to z_j) is a connected subset $\tilde{Q}_j \subset \tilde{C}$ such that there exists a representative $z'_j : D'_j \to \mathbb{C}$ of the germ $z_j|_{Q_j} \in \mathcal{O}_{C,Q_j}$ such that $\pi(\tilde{Q}_j) \subset D'_j, z'_j(\pi(\tilde{Q}_j)) = (0, r) \subset \mathbb{R}, r > 0$, is an interval and such that $\tilde{Q}_j$ is maximal (wrt inclusion) among all connected subsets of $\tilde{C}$ with this property.

For any ray $\tilde{Q}_j$ over Q_j we define a logarithm $\log_{\tilde{Q}_j}$ of z_j. Assume $g > 0$ or $M > 0$ (if $g = M = 0$ the map $\pi : \tilde{C} \to C^\circ$ is an isomorphism and there is only one ray $\tilde{Q}_j$ over Q_j; we define $T_{\tilde{Q}_j} := \pi^{-1}(D_j^\circ)$ and $\log_{\tilde{Q}_j} := z_j : T_{\tilde{Q}_j} \to \mathbb{C}$). Choose $\tilde{P}_0 \in \tilde{Q}_j$ such that $P_0 := \pi(\tilde{P}_0) \in D_j$. Because $\tilde{C}$ is the universal covering of C° there exists a commutative diagram

$$\begin{tikzcd} (\tilde{D}_j^\circ, \tilde{P}_0) \arrow[r, hook, "\iota_{\tilde{Q}_j}"] \arrow[d] & (\tilde{C}, \tilde{P}_0) \arrow[d, "\pi"] \\ (D_j^\circ, P_0) \arrow[r, hook] & (C^\circ, P_0) \end{tikzcd}$$

where $\tilde{D}_j^\circ$ is the universal covering of D_j°. Put $T_{\tilde{Q}_j} := \iota_{\tilde{Q}_j}(\tilde{D}_j^\circ)$ and define $\log_{\tilde{Q}_j} : T_{\tilde{Q}_j} \to \mathbb{C}$ to be the unique holomorphic function such that $\exp \circ \log_{\tilde{Q}_j} = z_j$ and $\log_{\tilde{Q}_j}(P) = \log(z_j(P))$ for any $P \in \tilde{Q}_j \cap T_{\tilde{Q}_j}$.

The set $\mathcal{Q}_j$ of rays over Q_j is a $\pi_1(C^\circ)$-torsor. Endow $\bar{C} := \tilde{C} \sqcup \bigsqcup \mathcal{Q}_j$ with the unique (non-Hausdorff) topology such that $\tilde{C} \subset \bar{C}$ is an open subspace and a neighborhood basis of a ray $\tilde{Q}_j$ over Q_j is given by $\{\tilde{Q}_j\} \cup (\pi^{-1}(U \setminus \{Q_j\}) \cap T_{\tilde{Q}_j})$ where U is a neighborhood of Q_j. Wethe towers in $T_{\tilde{Q}_j}$ over neighborhoods of Q_j. Extend π to a continuous map $\bar{\pi} : \bar{C} \to C$ by mapping a ray over Q_j to Q_j and write $i : \tilde{C} \hookrightarrow \bar{C}$ for the inclusion.

3.2. **Definition.** (i) A 1-form $\omega \in i_*\mathcal{M}^1_{\tilde{C}}(\bar{U}), \bar{U} \subset \bar{C}$, is called **logarithmic at infinity** if for any ray $\tilde{Q}_j \in \bar{U}$ over Q_j there are $\phi, \psi \in \mathcal{M}^1_C(U)$ for some neighborhood U of Q_j such that $\omega = \phi + \psi \log_{\tilde{Q}_j}$.

(ii) A 1-form $\omega \in i_*\mathcal{M}^1_{\tilde{C}}(\bar{\pi}^{-1}(U)), U \subset C$, is called a **multi-valued abelian differential on** $(U, \bar{C})$ if ω is logarithmic at infinity and $\operatorname{im}(\operatorname{incr}_\gamma(\omega)) \subset d\mathcal{M}_C(U)$.

Denote by $\mathcal{M}^1_{\bar{C}}$ the sheaf on C of multi-valued abelian differentials on $\bar{C}$ and $\mathcal{M}^1(\bar{C}) := \mathcal{M}^1_{\bar{C}}(C)$.

The condition on $\omega \in \mathcal{M}^1_{\bar{C}}$ to be logarithmic at infinity is imposed to ensure that the following remark holds true for ω.

3.3. **Remark.** If $\omega \in \mathcal{M}^1_{\bar{C}}(U), U \subset C$, and $\operatorname{incr}(\omega) = 0$ then $\omega \in \mathcal{M}^1_C(U)$. In fact, because ω is $\pi_1(C^\circ)$-invariant $\omega \in \mathcal{M}^1_C(U \cap C^\circ)$ and because ω is logarithmic at infinity it follows that $\omega \in \mathcal{M}^1_C(U)$.

The condition on the increments of $\omega \in \mathcal{M}^1(\bar{C})$ is made in order that there is a notion of integral of ω over a homology class in $H_1(C^\circ, \mathbb{Z})$ provided that ω has no residues. This is done as follows.

3.4. **Lemma.** *Let $\omega \in \mathcal{M}^1(\bar{C})$ have no residues on $\tilde{C}$ and $P_0 \in C^\circ$ be such that ω is holomorphic at any $\tilde{P}_0 \in \pi^{-1}(P_0)$. For a closed path γ in C° at P_0 we denote by $\gamma_{\tilde{P}_0}$ the lift of γ to $\tilde{P}_0 \in \pi^{-1}(P_0)$. Then the integral*

$$\int\limits_\gamma \omega := \int\limits_{\gamma_{\tilde{P}_0}} \omega \tag{3.1}$$

only depends on P_0 and the homology class of γ.

Proof. For any $\delta \in \pi_1(C^\circ, P_0)$ we have

$$\int\limits_{\gamma_{\delta_* \tilde{P}_0}} \omega = \int\limits_{\delta_* \gamma_{\tilde{P}_0}} \omega = \int\limits_{\gamma_{\tilde{P}_0}} \delta^*\omega = \int\limits_{\gamma_{\tilde{P}_0}} \omega + \int\limits_{\gamma_{\tilde{P}_0}} \operatorname{incr}_\delta(\omega) = \int\limits_{\gamma_{\tilde{P}_0}} \omega.$$

Thus the integral is independent of the choice of $\tilde{P}_0$, in particular the integral is additive for closed paths at P_0. Because homotopic paths in C° lift to homotopic paths in $\tilde{C}$ and ω has no residues on $\tilde{C}$ the integral is well-defined on $\pi_1(C^\circ, P_0)$, hence on $H_1(C^\circ, \mathbb{Z})$. ☐

We shall always assume that we have chosen a point $P_0 \in C^\circ$ by which integrals are defined.

Let $\omega \in \mathcal{M}^1(\bar{C})$. Because the action of $\pi_1(C^\circ)$ on $d\mathcal{M}(C)$ is trivial the crossed homomorphism $\operatorname{incr}(\omega) : \pi_1(C^\circ) \to d\mathcal{M}(C)$ is a homomorphism and we get $\operatorname{incr}(\omega) : H_1(C^\circ, \mathbb{Z}) \to d\mathcal{M}(C)$.

Let $a_i, b_i \in H_1(C^\circ, \mathbb{Z}), i = 1, \ldots, g$, be such that their images in $H_1(C, \mathbb{Z})$ form a symplectic basis of $H_1(C, \mathbb{Z})$. We call $(a_i, b_i)_{i=1,\ldots,g}$ a symplectic basis of $H_1(C^\circ, \mathbb{Z})$. $\omega \in \mathcal{M}^1(\tilde{C})$ without residues on $\tilde{C}$ is called **normalized** if all a-periods of ω vanish.

Let $d_j \in H_1(C^\circ, \mathbb{Z})$ be the homology class of a small, counterclockwise oriented circle around $Q_j, j = 0, \ldots, M$. Because $a_i, b_i, d_j \in H_1(C^\circ, \mathbb{Z}), i = 1, \ldots, g, j = 1, \ldots, M$, form a basis of $H_1(C^\circ, \mathbb{Z})$, giving a homomorphism $\iota : \pi_1(C^\circ) \to d\mathcal{M}(C)$ is equivalent to giving a map $\iota : \mathcal{B} := \{a_i, b_i, d_j \mid i = 1, \ldots, g, j = 1, \ldots, M\} \to d\mathcal{M}(C)$. The 1-forms $\iota(\gamma), \gamma \in \mathcal{B}$, are called the **basic increments** of ι.

We call $P_0 \in C^\circ$ an ω-**base point** if P_0 is not a pole of $\operatorname{incr}_{d_j}(\omega), j = 0, \ldots, M$. In a neighborhood of a ray $\tilde{Q}_j$ over Q_j we write

$$\omega = \sum_{k \geq -K} c^\omega_{kj} z^k_j \, dz_j + d(r^\omega_j \log_{\tilde{Q}_j}) \tag{3.2}$$

where $c^\omega_{kj} \in \mathbb{C}$ and $r^\omega_j \in \mathcal{M}(C)$ is uniquely determined by $2\pi i \, dr^\omega_j = \operatorname{incr}_{d_j}(\omega)$ and $r^\omega_j(P_0) = 0$. We call $\sum_{k \geq -K} c^\omega_{kj} z^k_j$, $\sum_{k=-K}^{-1} c^\omega_{kj} z^k_j$, and $c^\omega_{-1,j}$ the **meromorphic part**, the **Laurent tail**, and the **residue** of ω at $\tilde{Q}_j$ (with respect to z_j and P_0) and write $\operatorname{res}_{\tilde{Q}_j}(\omega) := c^\omega_{-1,j}$.

3.5. **Definition.** A 1-form $\omega \in \mathcal{M}^1_{\bar{C}}(U), U \subset C$, is called

(i) holomorphic at a ray $\tilde{Q}_j \in \bar{\pi}^{-1}(U)$ if the Laurent tail of ω at $\tilde{Q}_j$ is zero;
(ii) holomorphic at $P \in U$ if ω is holomorphic at some $\tilde{P} \in \bar{\pi}^{-1}(P)$;
(iii) a **multi-valued holomorphic 1-form on** $(U, \bar{C})$ if ω is holomorphic at any $P \in U$.

Denote by $\Omega^1_{\bar{C}}$ the sheaf on C of multi-valued holomorphic 1-forms on $\bar{C}$ and $\Omega^1(\bar{C}) := \Omega^1_{\bar{C}}(C)$.

By Mittag-Leffler there is an isomorphism between the space of abelian differentials on C and the space of Laurent tail divisors with vanishing sum of residues and a-periods (see for example [N], p.54). We want to extend this statement to *multi-valued* abelian differentials on $\bar{C}$.

A pre-Laurent tail divisor $\mathfrak{L} = (\iota, (\omega_i))$ on $\tilde{C}$ consists of a homomorphism $\iota : \pi_1(C^\circ) \to d\mathcal{M}(C)$ and a Čech 0-cochain $\big(\omega_i \in \Gamma(U_i, \bar{\pi}_* \mathcal{M}^1_{\bar{C}})\big)_{i \in I}$ such that

(i) $\operatorname{incr}(\omega_i) = \iota$ for any $i \in I$;
(ii) the Čech differential of (ω_i) is a Čech 1-cochain of Ω^1_C.

Let $\mathfrak{L} = (\iota, (\omega_i)_{i \in I}), \mathfrak{L}' = (\iota', (\omega'_j)_{j \in J})$ be pre-Laurent tail divisors on $\tilde{C}$. $\mathfrak{L}$ is called finer than $\mathfrak{L}'$ if $\iota = \iota'$ and the Čech 0-cochain (ω_i) is finer than (ω'_j). $\mathfrak{L}, \mathfrak{L}'$ are called equivalent if they have a common refinement. A **Laurent tail divisor on** $\tilde{C}$ is an equivalence class of pre-Laurent tail divisors.

The residue at $P \in C$ of a Laurent tail divisor $(\iota, (\omega_i))$ is defined to be the residue of ω_i at some $\tilde{P} \in \bar{\pi}^{-1}(P)$ for some $i \in I$ such that $P \in U_i$. The Laurent tail divisor of $\omega \in \mathcal{M}^1(\bar{C})$ is given by $(\operatorname{incr}(\omega), \omega)$.

3.6. **Theorem.** *Let $(a_i, b_i)_{i=1,\ldots,g}$ be a symplectic basis of $H_1(C^\circ, \mathbb{Z})$. Let $\mathfrak{L}$ be a Laurent tail divisor on $\tilde{C}$ which has at most residues at $Q_0, \ldots, Q_M$ and such that the sum of the residues is zero. For any $I \in \mathbb{C}^g$ there exists a unique $\omega \in \mathcal{M}^1(\bar{C})$ with Laurent tail divisor $\mathfrak{L}$ and a-periods I.*

Proof. Let $\omega, \omega' \in \mathcal{M}^1(\bar{C})$ satisfying the above properties. By Remark 3.3 $\omega - \omega'$ is an abelian differential on C. By Mittag-Leffler $\omega - \omega'$ is zero.

Let $\mathfrak{L} = (\iota, (\omega_i))$. By Lemma 3.7 below there exists $\omega \in \Omega^1(\bar{C})$ such that $\operatorname{incr}(\omega) = \iota$. By Remark 3.3 $(\omega_i - \omega)$ is a Laurent tail divisor on C. By Mittag-Leffler there exists $\omega' \in \mathcal{M}^1(C)$ such that $\omega + \omega'$ has Laurent tail divisor $\mathfrak{L}$ and a-periods $I_1, \ldots, I_g$. $\square$

3.7. **Lemma.** *Let $\iota : \pi_1(C^\circ) \to d\mathcal{M}(C)$ be a homomorphism and $R_0, \dots, R_K \in C$ be the poles of the basic increments of ι. Let $\tilde{R}_0, \dots, \tilde{R}_K \in \bar{C}$ be lifts of $R_0, \dots, R_K$. Then there exists a unique normalized 1-form $\omega \in \Omega^1(\bar{C})$ which is holomorphic at $\tilde{R}_0, \dots, \tilde{R}_K$ and satisfies* $\operatorname{incr}(\omega) = \iota$.

Proof. Uniqueness follows in the same way as in the proof of the Theorem. The proof of existence is an adaptation of the proof of Satz (28.4) of [F].

We may assume that $R_j = Q_j, j = 0, \dots, M$, and $R_{M+1}, \dots, R_K \in C^\circ$. Let $(U_l)_{l=0,\dots,L}$, $L \geq N$, be an open covering of C such that $U_j \subset D_j, j = 0, \dots, M$, we have $R_k \in U_k \setminus \bigcup_{l \neq k} U_l, k = 0, \dots, K$, and such that there are homeomorphisms $\phi_l : \pi^{-1}(U_l) \xrightarrow{\sim} U_l \times \pi_1(C^\circ), l = M+1, \dots, L$, which are compatible with π, $\pi_1(C^\circ)$-equivariant, and satisfy $\phi_k(\tilde{R}_k) = (R_k, e), k = M+1, \dots, K$, where $e \in \pi_1(C^\circ)$ is the identity.

Consider the 1-form on $U_l \times \pi_1(C^\circ)$ which equals $\iota(\gamma)$ on $U_l \times \{\gamma\}, \gamma \in \pi_1(C^\circ), l = M+1, \dots, L$. Let ω_l be its pull-back via ϕ_l to $\pi^{-1}(U_l)$.

Define $\tilde{Q}_j := \tilde{R}_j, j = 0, \dots, M$. Choose a set of representatives Γ_j of $\pi_1(C^\circ)/\pi_1(D_j^\circ)$ such that $e \in \Gamma_j$. For any $\gamma \in \Gamma_j$ define 1-forms

$$\omega_0 := -\frac{1}{2\pi i} \left(\sum_{j'=1}^{M} \iota(d_{j'}) \right) \log_{\gamma_* \tilde{Q}_0} \qquad \omega_j := \frac{1}{2\pi i} \, \iota(d_j) \, \log_{\gamma_* \tilde{Q}_j} \qquad j = 1, \dots, M,$$

on the tower in $\gamma_* T_{\tilde{Q}_j}$ over $U_j \setminus \{Q_j\}, j = 0, \dots, M$. This gives $\omega_j \in \Omega^1_{\bar{C}}(\bar{\pi}^{-1}(U_j))$.

Then $\operatorname{incr}(\omega_l) = \iota, l = 0, \dots, L$, and therefore the 1-form $\rho_{ll'} := \omega_l - \omega_{l'}$ on $\pi^{-1}(U_l) \cap \pi^{-1}(U_{l'})$ is $\pi_1(C^\circ)$-invariant, more precisely $\rho_{ll'} \in \Gamma(U_l \cap U_{l'}, \Omega^1_C)$. These elements define a Čech-1-cocycle. Because $H^1(C, \Omega^1_C(R_0)) = H^0(C, \mathcal{O}_C(-R_0)) = 0$ there are $\rho_l \in \Gamma(U_l, \Omega^1_C(* \sum R_k))$ such that $\rho_l - \rho_{l'} = \rho_{ll'} = \omega_l - \omega_{l'}$ on $U_l \cap U_{l'}$. The 1-form ω on $\tilde{C}$ given on $\pi^{-1}(U_l \cap C^\circ)$ by $\omega_l - \rho_l$ is thus globally defined and satisfies $\operatorname{incr}(\omega) = \iota$.

The Čech 0-cochain (ρ_l) defines a Laurent tail divisor $\mathfrak{L}$. By Mittag-Leffler there exists $\omega' \in \mathcal{M}^1(C)$ with Laurent tail divisor $\mathfrak{L}$ and the same a-periods as $-\omega$. By construction $\omega'' := \omega + \omega' \in \Omega^1(\bar{C})$ is normalized, holomorphic at $\tilde{R}_0, \dots, \tilde{R}_K$ and satisfies $\operatorname{incr}(\omega'') = \iota$. □

Let $\omega \in \mathcal{M}^1(\bar{C})$. A subset $S \subset \bar{C}$ is called a set of poles of ω if ω is holomorphic on $\bar{C} \setminus (\pi_1(C^\circ) S)$ and $\bar{\pi}|_S$ is injective.

3.8. **Corollary.** *If S is a set of poles of ω then $\sum_{P \in S} \operatorname{res}_P(\omega) = 0$.*

Proof. By Lemma 3.7 there exists $\omega' \in \Omega^1(\bar{C})$ such that $\operatorname{incr}(\omega') = \operatorname{incr}(\omega)$. Then $\omega - \omega' \in \mathcal{M}^1_C(C)$ and $\sum_{P \in S} \operatorname{res}_P(\omega) = \sum_{P \in S} \operatorname{res}_P(\omega - \omega') = 0$. □

4. A pairing and a reciprocity law

In this section we define a pairing between multi-valued abelian differentials and prove a reciprocity law from them. Reciprocity laws for single-valued abelian differentials are discussed for example in [GH].

Let $P_1, \dots, P_N \in C^\circ$ and $C^\diamond := C^\circ \setminus \{P_1, \dots, P_N\}$. The pairing is defined on a subspace of $\Omega^1_{\bar{C}}(C^\diamond)$ and depends on a cutting of $C^\diamond$ into a simply connected domain $\check{C}$.

Let $P_0 \in C^\diamond$. Choose closed smooth paths a_i, b_i at P_0 in C° and smooth paths $c_j : [0,1] \to C$ from P_0 to Q_j in such a way that for $\epsilon > 0$ sufficiently small

(i) a_i, b_i, c_j only meet at P_0, are not self-intersecting, and do not pass through $P_1, \dots, P_N$;

(ii) a_i, b_i have homology classes $a_i, b_i \in H_1(C^\circ, \mathbb{Z})$;
(iii) $c_j([0,1)) \subset C^\circ$ and $z_j(c_j(1-\epsilon)) \in \mathbb{R}_{>0}$.

Define $c_{j\epsilon} : [0,1] \to C^\circ, t \mapsto c_j((1-\epsilon)t)$, and $s_{j\epsilon} : [0,1] \to C^\circ$ be a small *clockwise* loop around Q_j at $c_{j\epsilon}(1)$.

Choose a lift $\tilde{P}_0 \in \tilde{C}$ of P_0 and let $\tilde{a}_1$ be the lift of a_1 to $\tilde{P}_0$. By induction on $i \geq 1$ we define $\tilde{b}_i, \tilde{a}_i^-, \tilde{b}_i^-$, and $\tilde{a}_{i+1}$ to be the lifts of b_i, a_i^{-1}, b_i^{-1}, and a_{i+1} to the end-points of $\tilde{a}_i, \tilde{b}_i, \tilde{a}_i^-$, and $\tilde{b}_i^-$, resp. It follows that $\tilde{a}_i^- = (b_i)_* \tilde{a}_i^{-1}$, $\tilde{b}_i^- = (a_i^{-1})_* \tilde{b}_i^{-1}$.

Let $\tilde{c}_{0\epsilon}$ be the lift of $c_{0\epsilon}$ to the end-point of $\tilde{b}_g^-$. By induction on $j \geq 1$ we define $\tilde{s}_{j\epsilon}, \tilde{c}_{j\epsilon}^-$ and $\tilde{c}_{j+1,\epsilon}$ to be the lifts of $s_{j\epsilon}, c_{j\epsilon}^{-1}$, and $c_{j+1,\epsilon}$ to the end-points of $\tilde{c}_{j\epsilon}, \tilde{s}_{j\epsilon}$, and $\tilde{c}_{j,\epsilon}^-$, resp. We denote $\tilde{P}_{0j}$ the starting-point of $\tilde{c}_{j\epsilon}$ and $\tilde{Q}_j$ the unique ray over Q_j that is contained in $c_j([0,1])$. Let $t_j \in \pi_1(C^\circ, P_0)$ be such that $(t_j)_*(\tilde{s}_{j\epsilon}(0)) = \tilde{s}_{j\epsilon}(1)$. It follows that $\tilde{c}_{j\epsilon}^- = (t_j)_* \tilde{c}_{j\epsilon}^{-1}$.

Let $\check{C}$ be the simply connected region in $\tilde{C}$ whose oriented boundary is

$$\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^- \tilde{b}_1^- \ldots \tilde{a}_g \tilde{b}_g \tilde{a}_g^- \tilde{b}_g^- \quad \tilde{c}_{0\epsilon} \tilde{s}_{0\epsilon} \tilde{c}_{0\epsilon}^- \ldots \tilde{c}_{M\epsilon} \tilde{s}_{M\epsilon} \tilde{c}_{M\epsilon}^-.$$

(We have to choose the paths a_i, b_i, c_j in such a way that the above order is possible, namely the tangents of these paths at P_0 have to respect the same order.) $\check{C}$ is called a **basic region** over $(C, (z_j), (a_i, b_i), (P_n)_{n=0,\ldots,N})$.

In addition, the pairing involves the notion of principal value. Let $\omega \in \mathcal{M}^1(\bar{C})$ be such that ω has no residues on $\tilde{C}$. For any ray $\tilde{Q}_j$ we define a primitive of ω, called the principal value,

$$\text{p.v.} \int_{\tilde{Q}_j} \omega \ : \tilde{C} \to \mathbb{CP}^1$$

by giving its values in a pointed neighborhood of $\tilde{Q}_j$:

$$\text{p.v.} \int_{\tilde{Q}_j}^{P} \omega \ := \sum_{\substack{k \geq -K \\ k \neq -1}} \frac{c_{kj}^\omega}{k+1} z_j(P)^{k+1} + (c_{-1,j}^\omega + r_j^\omega) \log_{\tilde{Q}_j}(P) \tag{4.1}$$

where we used (3.2). We also define for $P \in \tilde{C}$

$$\text{p.v.} \int_{P}^{\tilde{Q}_j} \omega \ := -\text{p.v.} \int_{\tilde{Q}_j}^{P} \omega, \qquad \text{p.v.} \int_{\tilde{Q}_j}^{\tilde{Q}_k} \omega \ := \text{p.v.} \int_{\tilde{Q}_j}^{P} \omega \ + \ \text{p.v.} \int_{P}^{\tilde{Q}_k} \omega$$

where the second number does not depend on P.

Let $\omega, \omega' \in \Omega^1_{\bar{C}}(C^\diamond)$ be such that P_0 is an ω- and ω'-base point. Let $\sum_{k \geq -K} c_{kj} z_j^k dz_j$ be the meromorphic part of ω at $\tilde{Q}_j$ and $\text{incr}_{a_i}(\omega) = dp_i, \text{incr}_{b_i}(\omega) = dq_i, \text{incr}_{d_j}(\omega) = 2\pi i \, dr_j \in d\mathcal{M}(C), r_j(P_0) = 0$. Define $c'_{kj} \in \mathbb{C}$ and $p'_i, q'_i, r'_j \in \mathcal{M}(C)$ in the same way for ω'.

4.1. **Definition.** Let $\check{C}$ be a basic region over $(C, (z_j), (a_i, b_i), (P_n))$. Define on the subspace of $\Omega^1_{\bar{C}}(C^\diamond)$ consisting of ω such that P_0 is an ω-base point a bilinear pairing $\langle ., . \rangle_{\check{C}}$ by the formula

$$\langle \omega, \omega' \rangle_{\check{C}} \ := \ -\frac{1}{2\pi i} \sum_{i=1}^{g} \Big(\int_{a_i} q_i \omega' - \int_{b_i} p_i \omega' + \int_{b_i} \omega \int_{a_i} \omega' - \int_{a_i} \omega \int_{b_i} \omega' \Big)$$

$$+\sum_{j=0}^{M}\Big(\sum_{k=0}^{K-2}\Big(c_{-k-2,j}\frac{c'_{kj}}{k+1}\Big)\ +\ c_{-1,j}\text{p.v.}\int_{\tilde P_{0j}}^{\tilde Q_j}\omega'\ +\ \text{p.v.}\int_{\tilde P_{0j}}^{\tilde Q_j} r_j\omega'\ +\ \pi i\operatorname*{res}_{Q_j}(r_j\omega')\Big). \tag{4.2}$$

4.2. **Proposition.** *Let $\omega \in \Omega^1_{\check C}(C^\diamond)$ and ω' be an abelian differential of the second kind on C with at most poles at $P_1,\dots,P_N$ of order two. Assume that $\mathrm{incr}_{d_j}(\omega)$ vanishes at $P_1,\dots,P_N, j=0,\dots,M$. Let $\check C$ be a basic region over $(C,(z_j),(a_i,b_i),(P_n))$ such that P_0 is an ω- and ω'-base point. Let $\tilde P_1,\dots,\tilde P_N \in \check C$ be the points over $P_1,\dots,P_N$. Then*

$$\sum_{n=1}^{N}\operatorname*{res}_{\tilde P_n}(\omega\int_{\tilde P_0}\omega')\ =\ -\langle\omega,\omega'\rangle_{\check C}.$$

Proof. Put $f := \int_{\tilde P_0}\omega'$. Because $\tilde P_1,\dots,\tilde P_N$ are the only poles of ωf in $\check C$ the residue theorem gives us

$$\sum_{n=1}^{N}\operatorname*{res}_{\tilde P_n}(\omega f) = \frac{1}{2\pi i}\int_{\partial\check C}\omega f = \frac{1}{2\pi i}\sum_{i=1}^{g}\int_{\tilde a_i\tilde b_i\tilde a_i^-\tilde b_i^-}\omega f + \frac{1}{2\pi i}\sum_{j=0}^{M}\int_{\tilde c_{j\epsilon}\tilde s_{j\epsilon}\tilde c_{j\epsilon}^-}\omega f.$$

The integral over $\tilde a_i\tilde b_i\tilde a_i^-\tilde b_i^-$ is

$$\begin{aligned}\int_{\tilde a_i\tilde b_i\tilde a_i^-\tilde b_i^-}\omega f &= \int_{\tilde a_i}\omega f + \int_{\tilde b_i}\omega f - \int_{\tilde a_i} b_i^*\omega\Big(f+\int_{b_i}\omega'\Big) - \int_{\tilde b_i}(a_i^{-1})^*\omega\Big(f-\int_{a_i}\omega'\Big)\\ &= -\int_{\tilde a_i} dq_i\, f + \int_{\tilde b_i} dp_i\, f - \int_{\tilde a_i} b_i^*\omega\int_{b_i}\omega' + \int_{\tilde b_i}(a_i^{-1})^*\omega\int_{a_i}\omega'\\ &= \int_{a_i} q_i\omega' - \int_{b_i} p_i\omega' - \int_{a_i}\omega\int_{b_i}\omega' + \int_{b_i}\omega\int_{a_i}\omega'.\end{aligned}$$

We compute the integral over $\tilde c_{j\epsilon}\tilde s_{j\epsilon}\tilde c_{j\epsilon}^-$ by splitting it into four parts

$$\int_{\tilde c_{j\epsilon}\tilde s_{j\epsilon}\tilde c_{j\epsilon}^-}\omega f = \int_{\tilde c_{j\epsilon}}\omega f - \int_{\tilde c_{j\epsilon}} t_j^*(\omega f) + \int_{\tilde s_{j\epsilon}}\omega f = \tag{4.3}$$

$$= 2\pi i\int_{\tilde c_{j\epsilon}} dr_j\, f + \int_{s_{j\epsilon}}\sum_{\substack{l\geq -K\\ k\geq 0}} c_{lj}\frac{c'_{kj}}{k+1}z_j^{l+k+1}dz_j + \int_{s_{j\epsilon}}\sum_{l\geq -K} c_{lj}z_j^l dz_j(\text{p.v.}\int_{\tilde P_{0j}}^{\tilde Q_j}\omega') + \int_{\tilde s_{j\epsilon}} d(r_j\log_{\tilde Q_j})f$$

where we used $\omega - t_j^*\omega = 2\pi i\, dr_j$ ($s_{j\epsilon}$ is clockwise oriented), used (3.2) for ω, (4.1) for ω', and applied distributivity. By the residue theorem the second and third integrals of the very RHS of (4.3) give the fifth and sixth term of $-\langle\omega,\omega'\rangle_{\check C}$.

By assumption $dr_j\, f$ has no residues on $\tilde{C}$. We write $dr_j\, f = \sum_{k\geq -K} h_{kj} z_j^k dz_j$ in a neighborhood of $\tilde{Q}_j$. By definition of the principal value, the first integral of the very RHS of (4.3) is

$$\int_{\tilde{c}_{j\epsilon}} dr_j\, f = \sum_{\substack{k\geq -K\\ k\neq -1}} \Big(\frac{h_k}{k+1} z_j^{k+1}(\tilde{c}_{j\epsilon}(1))\Big) + h_{-1}\log_{\tilde{Q}_j}(\tilde{c}_{j\epsilon}(1)) + \text{p.v.}\int_{\tilde{P}_{0j}}^{\tilde{Q}_j} dr_j\, f =$$

$$= \sum_{\substack{k\geq -K\\ k\neq -1}} \Big(\frac{h_k}{k+1} z_j(\tilde{c}_{j\epsilon}(1))^{k+1}\Big) + h_{-1}\log_{\tilde{Q}_j}(\tilde{c}_{j\epsilon}(1)) - \text{p.v.}\int_{\tilde{P}_{0j}}^{\tilde{Q}_j} r_j\omega' + \operatorname*{res}_{\tilde{Q}_j}\Big(\frac{r_j f}{z_j}\Big) - r_j(P_0) f(\tilde{P}_{0j}) \tag{4.4}$$

where we used integration by parts and took the constant terms of developments in z_j in order to rewrite the principal value. Because $r_j(P_0) = 0$ the last term vanishes. The third term of (4.4) is the seventh term of $-\langle\omega,\omega'\rangle_{\check{C}}$. Finally, the fourth integral of the very RHS of (4.3) is

$$\int_{\tilde{s}_{j\epsilon}} d(r_j \log_{\tilde{Q}_j}) f \;=\; \int_{\tilde{s}_{j\epsilon}} dr_j\, f \log_{\tilde{Q}_j} + \int_{s_{j\epsilon}} \frac{r_j f}{z_j} dz_j \;=\; -\int_{s_{j\epsilon}} \Big(\sum_{\substack{k\geq -K\\ k\neq -1}} \frac{h_k}{k+1} z_j^{k+1}\frac{dz_j}{z_j}\Big) + \tag{4.5}$$

$$+\Big[\sum_{\substack{k\geq -K\\ k\neq -1}} \frac{h_k}{k+1} z_j^{k+1} \log_{\tilde{Q}_j}\Big]_{\tilde{s}_{j\epsilon}(0)}^{\tilde{s}_{j\epsilon}(1)} \;+\; h_{-1}\int_{s_{j\epsilon}} \frac{\log_{\tilde{Q}_j}}{z_j} dz_j \;-\; 2\pi i \operatorname*{res}_{\tilde{Q}_j}\Big(\frac{r_j f}{z_j}\Big)$$

$$= -2\pi i \sum_{\substack{k\geq -K\\ k\neq -1}} \Big(\frac{h_k}{k+1} z_j(s_{j\epsilon}(0))^{k+1}\Big) \;+\; h_{-1}(-2\pi i \log_{\tilde{Q}_j}(\tilde{s}_{j\epsilon}(0)) - 2\pi^2) \;-\; 2\pi i \operatorname*{res}_{\tilde{Q}_j}\Big(\frac{r_j f}{z_j}\Big).$$

We applied integration by parts to $dr_j\, f - h_{-1}\frac{dz_j}{z_j}$ and $\log_{\tilde{Q}_j}$ and used the identities

$$\int_{\tilde{s}_{j\epsilon}} \frac{\log_{\tilde{Q}_j}}{z_j} dz_j = \Big[\frac{1}{2}\log^2_{\tilde{Q}_j}\Big]_{\tilde{s}_{j\epsilon}(0)}^{\tilde{s}_{j\epsilon}(1)} = \frac{1}{2}(-4\pi i \log_{\tilde{Q}_j}(\tilde{s}_{j\epsilon}(0)) + (2\pi i)^2).$$

Because $\tilde{c}_{j\epsilon}(1) = \tilde{s}_{j\epsilon}(0)$ three terms of (4.4) cancel with three terms of the very RHS of (4.5). We obtain the eighth term of $-\langle\omega,\omega'\rangle_{\check{C}}$ because $h_{-1} = \operatorname*{res}_{\tilde{Q}_j}(dr_j\, f) = -\operatorname*{res}_{Q_j}(r_j\omega')$. $\square$

5. Multi-valued abelian differentials on $\underline{\lambda}$

In this section we define multi-valued abelian differentials on a Riemann surface with a marked meromorphic function and write down a basis of the vector space they form.

Let $\lambda \in \mathcal{M}(C)$ with divisor of poles $\sum_{j=0}^M (n_j+1)Q_j, n_j \in \mathbb{Z}_{\geq 0}$. Let z_j be a branch of the (n_j+1)-st root of λ^{-1}, $z_i^{-(n_j+1)} = \lambda$, defined on a small disk D_j around Q_j. It follows that z_j is a local coordinate at Q_j. Let $P_1,\ldots,P_N \in \lambda^{-1}(\mathbb{C})$ be the finite branch points of λ.

Define $\underline{\lambda} := (\lambda, (Q_j), (z_j|_{Q_j}), (a_i, b_i))$ where $z_j|_{Q_j} \in \mathcal{O}_{C,Q_j}$ is the germ of z_j at Q_j and $(a_i,b_i)_{i=1,\ldots,g}$ is a symplectic basis of $H_1(C,\mathbb{Z})$. A basic region over $(C,(z_j),(a_i',b_i'),(P_n))$ is called a **basic region over** $\underline{\lambda}$ if $a_i', b_i' \in H_1(C^\circ,\mathbb{Z})$ map to a_i, b_i.

5.1. **Definition.** A 1-form $\omega \in \mathcal{M}^1_{\tilde{C}}(U)$ is called a **multi-valued abelian differential on** $(U, \underline{\lambda})$ if $\mathrm{im}(\mathrm{incr}_\gamma(\omega)) \subset d\mathbb{C}[\lambda]$. If furthermore $\omega \in \Omega^1_{\tilde{C}}(U)$ then ω is called a **multi-valued holomorphic 1-form on** $(U, \underline{\lambda})$.

Denote by $\mathcal{M}^1_{\underline{\lambda}}, \Omega^1_{\underline{\lambda}}$ the sheaves on C of multi-valued abelian differentials on $\underline{\lambda}$ and of multi-valued holomorphic 1-forms on $\underline{\lambda}$. Put $\mathcal{M}^1(\underline{\lambda}) := \mathcal{M}^1_{\underline{\lambda}}(C)$.

The condition on the increments is made in order that for any $\omega, \omega' \in \Omega^1_{\underline{\lambda}}(C^\circ)$ the residue

$$\operatorname*{res}_{\tilde{P}} \frac{\omega\omega'}{d\lambda}, \qquad \tilde{P} \in \tilde{C},$$

only depends on $P := \pi(\tilde{P}) \in C^\circ$ as one easily shows.

In the following any zero $P_0 \in C^\circ$ of λ will be taken as ω-base point for any $\omega \in \mathcal{M}^1(\underline{\lambda})$ and will be used to define integrals (3.1). Both is independent of the choice of the zero P_0.

5.2. **Lemma.** *Let $\omega \in \Omega^1_{\underline{\lambda}}(C^\circ)$ and $\gamma \in \pi_1(C^\circ, P_0)$. If $\mathrm{incr}_\gamma(\omega) = dr, r \in \lambda\mathbb{C}[\lambda]$, then*

$$\mathrm{p.v.}\int_{\tilde{Q}_j}^{\gamma_* P} \omega \;=\; \mathrm{p.v.}\int_{\tilde{Q}_j}^{P} \omega \;+\; \int_\gamma \omega \;+\; r(P). \tag{5.1}$$

Proof. Let $\tilde{\gamma}$ be the lift to $\tilde{P}_0 \in \pi^{-1}(P_0)$ of a representative of $\gamma \in \pi_1(C^\circ, P_0)$ and α be a path from $\tilde{P}_0$ to P. Then we have

$$\int_P^{\gamma_* P} \omega \;=\; \int_{\alpha^{-1}} \omega + \int_{\tilde{\gamma}} \omega + \int_{\gamma_*(\alpha)} \omega \;=\; \int_\gamma \omega + \int_\alpha \mathrm{incr}_\gamma(\omega) \;=\; \int_\gamma \omega + r(P).$$

□

Introduce a set of labels

$$\mathcal{N} := \; \{s^i, r^i, w^j \mid i = 1, \ldots, g, j = 1, \ldots, M\} \;\cup\; \{t^{kj} \mid 1 \le k \le n_j + 1, j = 0, \ldots, M\}.$$

As a special case of Theorem 3.6 we obtain a basis of $\Omega^1_{\underline{\lambda}}(C^\circ)$.

5.3. **Proposition.** *Let $\tilde{Q}_0, \ldots, \tilde{Q}_M$ be rays over $Q_0, \ldots, Q_M$. The 1-forms $\omega_{A,k}$, $A \in \mathcal{N}, k \ge 0$, form a basis of the vector space $\Omega^1_{\underline{\lambda}}(C^\circ)$ where*

(i) $\omega_{s^i,0}$ is the abelian differential of the first kind on C with a-periods

$$\int_{a_j} \omega_{s^i,0} = \delta_{ij} \qquad i, j = 1, \ldots, g.$$

(ii) $\omega_{s^i,k}, k \ge 1$, and $\omega_{r^i,k}, k \ge 0$, are the normalized multi-valued holomorphic 1-forms on (C, λ) which are holomorphic at $\tilde{Q}_0, \ldots, \tilde{Q}_M$ and have all basic increments equal to zero except for

$$\mathrm{incr}_{a_i}(\omega_{s^i,k+1}) \;=\; \mathrm{incr}_{b_i}(\omega_{r^i,k}) \;=\; -\frac{d\lambda^{k+1}}{(k+1)!}, \qquad k \ge 0.$$

(iii) $\omega_{t^{kj},k'}$ *is the normalized abelian differential of the second kind on* C *with only a pole at* Q_j *with Laurent tail equal to the Laurent tail of*

$$-\frac{(n_j+1)^{k'}dz_j^{-(k+k'(n_j+1))}}{\prod_{l=0}^{k'}(k+l(n_j+1))};$$

(iv) $\omega_{w^j,0}$ *is the normalized abelian differential of the third kind on* C *with only poles at* Q_0 *and* Q_j *with residues* -1 *and 1, resp.;*

(v) $\omega_{w^j,k}$ *is the multi-valued holomorphic 1-form on* $(C\setminus\{Q_0,Q_j\},\lambda), k\geq 1$*, which is holomorphic at* $\tilde{Q}_{j'}, j'\neq 0,j$*, has Laurent tail at* $\tilde{Q}_0,\tilde{Q}_j$ *equal to the Laurent tail at* $\tilde{Q}_0,\tilde{Q}_j$ *of*

$$-\frac{1}{n_0+1}\Big(\sum_{n=1}^{k}\frac{1}{n}\Big)\frac{d\lambda^k}{k!} \qquad \text{and} \qquad \frac{1}{n_j+1}\Big(\sum_{n=1}^{k}\frac{1}{n}\Big)\frac{d\lambda^k}{k!},$$

and has all basic increments equal to zero except for

$$\mathrm{incr}_{d_j}(\omega_{w^j,k}) \;=\; 2\pi i\frac{d\lambda^k}{k!}.$$

Note that the development of $\omega_{w^j,k}$ at $\tilde{Q}_j$ is given by

$$\omega_{w^j,k} \;=\; d\Big(\frac{\lambda^k}{k!}\Big(\log_{\tilde{Q}_j}+\frac{1}{n_j+1}\sum_{n=1}^{k}\frac{1}{n}\Big)\Big)+(\mathrm{reg}) \tag{5.2}$$

and mutatis mutandis for $\tilde{Q}_0$.

6. Hurwitz spaces

In this section we construct the Hurwitz spaces $H_{g,t}, H'_{g,t}$ as complex manifolds together with the universal curve $\mathcal{C}\to H'_{g,t}$ and the fibrewise universal covering $\tilde{\mathcal{C}}\to\mathcal{C}^\circ$ of the pointed universal curve.

6.1. **Definition.** Let $g,M\in\mathbb{Z}_{\geq 0}$, $t=(n_0,\ldots,n_M)\in\mathbb{Z}_{\geq 0}^{M+1}$ and $d:=\sum(n_j+1)$. Define $H_{g,t}$ and $H'_{g,t}$ to be the sets of equivalence classes of

$$\lambda \qquad \text{and} \qquad \underline{\lambda}:=(\lambda,\,(Q_0,\ldots,Q_M),\,(z_0,\ldots,z_M),\,(a_i,b_i)_{i=1,\ldots,g})$$

where

(i) λ is a meromorphic function on a curve C of genus g with poles of order $n_0+1,\ldots,n_M+1$ and with only *simple* branch points over $\mathbb{C}$, i.e. $\lambda^{-1}(z)$ contains at least $d-1$ elements for any $z\in\mathbb{C}$;

(ii) $Q_j\in C$ is a pole of λ of order n_j+1;

(iii) $z_j\in\mathcal{O}_{C,Q_j}$ is the germ of a branch of the n_j+1st root of λ^{-1}, $z_j^{-(n_j+1)}=\lambda$;

(iv) $(a_i,b_i)_{i=1,\ldots,g}$ is a symplectic basis of $H_1(C,\mathbb{Z})$;

and where λ,λ' (resp. $\underline{\lambda},\underline{\lambda}'$) are equivalent if there is an isomorphism $\iota: C\xrightarrow{\sim}C'$ such that $\lambda'=\lambda\circ\iota$ (and $\iota(Q_j)=Q'_j, \iota^*(z'_j)=z_j, \iota_*(a_i)=a'_i, \iota_*(b_i)=b'_i$).

We will use the notations $C(\underline{\lambda}):=C, C^\circ(\underline{\lambda}):=C^\circ, Q_j(\underline{\lambda}):=Q_j$, etc.

The Riemann-Hurwitz formula says that the number of branch points of λ is $b:=2(g+d)-2$. Let $P_n(\underline{\lambda})\in C(\underline{\lambda})$ be the branch points of λ with *finite* branch values $u_n(\lambda):=\lambda(P_n(\underline{\lambda}))\in\mathbb{C}, n=1,\ldots,N:=b-\sum n_j$. Assigning to a covering its branching locus gives a map Π from $H_{g,t}$ to the N-th symmetric power of $\mathbb{CP}^1$, $\Pi : H_{g,t} \to \mathbb{CP}^N, \underline{\lambda} \mapsto \{u_1(\lambda), \ldots, u_N(\lambda)\}$.

Define $\mathcal{C} := \bigsqcup_{\underline{\lambda} \in H'_{g,t}} C(\underline{\lambda})$ and $\mathcal{C}^\circ := \bigsqcup_{\underline{\lambda} \in H'_{g,t}} C^\circ(\underline{\lambda})$.

6.2. **Lemma.** *The sets $H_{g,t}, H'_{g,t}$ and $\mathcal{C}$ carry unique structures of complex manifolds such that*

(i) $\Pi : H_{g,t} \to \mathbb{CP}^N$ *and* $\phi : H'_{g,t} \to H_{g,t}, \underline{\lambda} \mapsto \lambda$, *are unramified coverings, i.e.* $u_1, \ldots, u_N$ *are local coordinates on* $H_{g,t}, H'_{g,t}$*;*

(ii) $\kappa : \mathcal{C} \to H'_{g,t}, P \mapsto \underline{\lambda}$ *if* $P \in C(\underline{\lambda})$*, and* $\Lambda : \mathcal{C} \to \mathbb{CP}^1$, $\Lambda|_{C(\underline{\lambda})} = \lambda$, *are holomorphic;*

(iii) *there exists a diagram*

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[r, "\pi"] \arrow[d, "\tilde\kappa"] & \mathcal{C}^\circ \arrow[d, "\kappa"'] \\ H'_{g,t} \arrow[r, equal] & H'_{g,t} \end{tikzcd}$$

such that $\pi : \tilde{C}(\underline{\lambda}) \to C^\circ(\underline{\lambda})$ is the universal covering for any $\underline{\lambda} \in H'_{g,t}$.

Proof. (i) (cf.[Fu, MPR]) Let $B = \{u_1, \ldots, u_N\} \in \Pi(H_{g,t})$. Let $\Delta_n \subset \mathbb{C}$ be small disks around u_n and $\Delta \subset \mathbb{CP}^N$ be their symmetric product considered as an open neighborhood of B. We construct a natural bijection $\beta_B : \Pi^{-1}(\Delta) \xrightarrow{\sim} \Pi^{-1}(B) \times \Delta$ as follows.

Fix $u_0 \in \mathbb{C} \setminus S$, $S := \bigcup \Delta_n \subset \mathbb{CP}^1$. For $\lambda \in \Pi^{-1}(\Delta)$ giving the branched covering λ is equivalent to giving $\{u_1(\lambda), \ldots, u_N(\lambda)\} \in \Delta$ plus a representation $\pi_1(\mathbb{CP}^1 \setminus \{u_1(\lambda), \ldots, u_N(\lambda), \infty\}, u_0) \to \mathcal{S}_d$ up to conjugacy such that the permutation corresponding to a loop around ∞ decomposes into (n_0+1)–$, \ldots, (n_M+1)$–cycles. Because $\mathbb{CP}^1 \setminus (S \cup \{\infty\})$ is a deformation retract of $\mathbb{CP}^1 \setminus \{u_1(\lambda), \ldots, u_N(\lambda), \infty\}$ we get a natural isomorphism between $\pi_1(\mathbb{CP}^1 \setminus \{u_1(\lambda), \ldots, u_N(\lambda), \infty\}, u_0)$ and $\pi_1(\mathbb{CP}^1 \setminus \{u_1, \ldots, u_N, \infty\}, u_0)$. Let $\lambda' : C' \to \mathbb{CP}^1$ be the covering given by $\{u_1, \ldots, u_N\}$ and the representation of $\pi_1(\mathbb{CP}^1 \setminus \{u_1, \ldots, u_N, \infty\}, u_0)$ corresponding to that of λ via this isomorphism of fundamental groups. We define $\beta_B(\lambda) := (\lambda', \Pi(\lambda))$.

The structure of a complex manifold on $H_{g,t}$ is defined by the condition that $\Pi : H_{g,t} \to \Pi(H_{g,t}) \subset \mathbb{CP}^N$ becomes a covering of complex manifolds which is locally trivialized by the bijections β_B for $B \in \Pi(H_{g,t})$.

Let U be a small neighborhood of $\lambda' \in H_{g,t}$. Again it suffices to construct a natural bijection $\beta_\lambda : \phi^{-1}(U) \xrightarrow{\sim} \phi^{-1}(\lambda') \times U$.

For $\underline{\lambda} \in \phi^{-1}(U)$ there is a (non-canonical) isomorphism $\iota = \iota(\underline{\lambda}, \lambda') : C(\underline{\lambda}) \setminus \lambda^{-1}(S) \xrightarrow{\sim} C' \setminus (\lambda')^{-1}(S)$ of Riemann surfaces over $\mathbb{CP}^1 \setminus S$. Define $Q'_j := \iota(Q_j)$ and $z'_j := (\iota^*)^{-1}(z_j)$. Because $C \setminus \lambda^{-1}(S), C' \setminus (\lambda')^{-1}(S)$ are deformation retracts of $C^\circ(\underline{\lambda}), (C')^\circ$, we obtain from $a_i, b_i \in H_1(C(\underline{\lambda}), \mathbb{Z})$ via ι classes $a'_i, b'_i \in H_1(C', \mathbb{Z})$. Altogether we constructed $\underline{\lambda}' := (\lambda', (Q'_j), (z'_j), (a'_i, b'_i)) \in \phi^{-1}(\lambda') \subset H'_{g,t}$ and the isomorphism $\iota(\underline{\lambda}, \underline{\lambda}') := \iota(\underline{\lambda}, \lambda')$ compatible with the extra data is now canonical. We define $\beta_\lambda(\underline{\lambda}) := (\underline{\lambda}', \lambda)$.

(ii) Let U be a small neighborhood of $\underline{\lambda} \in H'_{g,t}$ and z be a local coordinate defined on $V \subset C(\underline{\lambda}) \setminus \lambda^{-1}(S)$. Then we define a map

$$\bigsqcup_{\underline{\lambda}' \in U} \iota(\underline{\lambda}, \underline{\lambda}')(V) \hookrightarrow \mathbb{C}^{N+1}, \quad P \mapsto \big(z(\iota(\underline{\lambda}, \underline{\lambda}')^{-1}(P)), u_1(\kappa(P)), \ldots, u_N(\kappa(P))\big).$$

Because $P_n(\underline{\lambda}')$ is a simple branch point of λ' there is a local coordinate $\tau_n(\lambda') : V_n(\lambda') \to \mathbb{C}$ such that $\lambda' = u_n(\lambda') + \tau_n(\lambda')^2, \tau_n(\lambda')(P_n(\underline{\lambda}')) = 0$. For any small neighborhood D of $0 \in \mathbb{C}$

we define a map

$$\bigsqcup_{\underline{\lambda}'\in U} \tau_n(\lambda')^{-1}(D) \hookrightarrow \mathbb{C}^{N+1}, \quad P \mapsto \big(\tau_n(\lambda')(P), u_1(\kappa(P)), \dots, u_N(\kappa(P))\big).$$

These two types of coordinate charts define on $\mathcal{C}$ the structure of a complex manifold.

(iii) Let $\mathcal{P}(\underline{\lambda}) := \{p : (0,1] \to C^\circ(\underline{\lambda}) \mid \exists \epsilon > 0 \;\, z_0(p((0,\epsilon)) \subset \mathbb{R}\}$ and $\mathcal{P} := \bigsqcup \mathcal{P}(\underline{\lambda})$. Define maps $\pi : \mathcal{P} \to \mathcal{C}^\circ, p \mapsto p(1)$, and $\tilde{\kappa} : \mathcal{P} \to H'_{g,t}, p \mapsto \underline{\lambda}$ if $p \in \mathcal{P}(\underline{\lambda})$. Let $\tilde{C}(\underline{\lambda})$ be the set of homotopy classes of paths $p \in \mathcal{P}(\underline{\lambda})$ with homotopies relative to $\{1\} \subset (0,1]$ and inside $\mathcal{P}(\underline{\lambda})$. Put $\tilde{\mathcal{C}} := \bigsqcup \tilde{C}(\underline{\lambda})$ and let $q : \mathcal{P} \to \tilde{\mathcal{C}}$ be the quotient map, $\pi : \tilde{\mathcal{C}} \to \mathcal{C}^\circ$ and $\tilde{\kappa} : \tilde{\mathcal{C}} \to H'_{g,t}$ be the induced maps. It follows that $\tilde{C}(\underline{\lambda})$ has a natural topology such that $\pi(\underline{\lambda}) : \tilde{C}(\underline{\lambda}) \to C^\circ(\underline{\lambda})$ is the universal covering for any $\underline{\lambda} \in H'_{g,t}$.

Let $p \in \tilde{C}(\underline{\lambda})$ and $U \subset \mathcal{C}^\circ$ be an open neighborhood of $\pi(p)$ such that $U(\underline{\lambda}) := U \cap C^\circ(\underline{\lambda})$ is a connected and simply connected neighborhood of $\pi(p)$. Let $\tilde{U}(\underline{\lambda}) \subset \tilde{C}(\underline{\lambda})$ be the connected neighborhood of p such that $\pi(\tilde{U}(\underline{\lambda})) = U(\underline{\lambda})$. Define for any open neighborhood $V \subset \kappa(U)$ of $\underline{\lambda}$ the set of V-parametrized paths

$$\mathcal{S}_{U,V} := \left\{ s : (0,1] \times V \to \mathcal{C}^\circ \;\middle|\; \begin{array}{c} \forall \;\underline{\lambda}' \in V \;\; s(.,\underline{\lambda}') \in \mathcal{P}(\underline{\lambda}'); \;\; q(s(.,\underline{\lambda})) \in \tilde{U}(\underline{\lambda}) \\ \exists \;\epsilon > 0 \;\;\, s((1-\epsilon,1] \times V) \subset U \end{array} \right\}$$

and

$$\tilde{U} := q\{p' \in \mathcal{P} \mid \exists \text{ open } V \subset \kappa(U) \; \tilde{\kappa}(p'), \underline{\lambda} \in V \; \exists \; s \in \mathcal{S}_{U,V} \;\; s(.,\tilde{\kappa}(p')) = p'\} \;\subset \tilde{\mathcal{C}}.$$

The sets $\mathcal{B}(p) := \{\tilde{U} \mid U \subset \mathcal{C}^\circ \text{ open}, p \in U, U(\tilde{\kappa}(p)) \text{ connected simply connected}\}, p \in \tilde{\mathcal{C}}$, form a system of neighborhood bases for $\tilde{\mathcal{C}}$, i.e.

(i) $\tilde{U}_1, \tilde{U}_2 \in \mathcal{B}(p) \;\Rightarrow\; \exists \, \tilde{U} \in \mathcal{B}(p) \quad \tilde{U} \subset \tilde{U}_1 \cap \tilde{U}_2$;
(ii) $\tilde{U} \in \mathcal{B}(p) \;\Rightarrow\; \exists \, \tilde{U}' \in \mathcal{B}(p) \;\; \forall \, p' \in \tilde{U}' \quad \tilde{U} \in \mathcal{B}(p')$;

and thus we obtain a topology on $\tilde{\mathcal{C}}$. The map $\pi : \tilde{\mathcal{C}} \to \mathcal{C}^\circ$ becomes an unramified covering of topological spaces. We endow $\tilde{\mathcal{C}}$ with the structure of a complex manifold induced by π. □

6.3. **Definition.** Define a multiplication on the tangent sheaves of $H_{g,t}, H'_{g,t}$ by

$$e_n \circ e_m \;:=\; \delta_{nm} e_n, \qquad e_n := \frac{\partial}{\partial u_n}.$$

By definition, $(H_{g,t}, \circ), (H'_{g,t}, \circ)$ are semisimple weak Frobenius manifolds.

Note that objects on the curve $C(\underline{\lambda})$ extend to a small neighborhood U of $\underline{\lambda}$, e.g. the point $P_n(\underline{\lambda}) \in C(\underline{\lambda})$ extends to a section $P_n : U \to \mathcal{C}, \underline{\lambda}' \mapsto P_n(\underline{\lambda}')$.

The functions $u_1, \dots, u_N, \Lambda$ form locally a coordinate system on $\mathcal{C}^\diamond := \bigsqcup C^\diamond(\underline{\lambda})$. Let $\tilde{e}_1, \dots, \tilde{e}_N, \partial_\lambda$ be the corresponding vector fields. Because ($\Lambda = u_n + \tau_n^2$ at P_n)

$$\tilde{e}_m \tau_n = -\frac{\delta_{mn}}{2\tau_n} \qquad \partial_\lambda \tau_n = \frac{1}{2\tau_n} \qquad \tilde{e}_m z_j = 0 \qquad \partial_\lambda z_j = -\frac{1}{n_j+2} z_j^{n_j+1} \tag{6.1}$$

$\tilde{e}_1, \dots, \tilde{e}_N$ are locally defined and $\tilde{e} := \sum \tilde{e}_n, \tilde{E} := \sum u_n \tilde{e}_n, \partial_\lambda$ are globally defined meromorphic vector fields on $\mathcal{C}$ with simple poles only at the finite branch points $P_1, \dots, P_N$.

7. A hierarchy of primitive forms

In this section we define descendant, primary, and eigenprimary forms, show that descendant forms form a hierarchy (Ω, D), and compute the e-derivative of the pairing $\langle\omega_1,\omega_2\rangle_{\tilde{C}}$ for primary forms ω_1,ω_2.

7.1. **Definition.** A 1-form $\omega\in\tilde\kappa_*\Omega^1_{\tilde{\mathcal{C}}/H'_{g,t}}(U), U\subset H'_{g,t}$, is called a **descendant form** on U if

(i) $\omega(\underline{\lambda}):=\omega|_{\tilde{C}(\underline{\lambda})}\in\Omega^1_{\underline{\lambda}}(C^\circ(\underline{\lambda}))$ for any $\underline{\lambda}\in U$;

(ii) the Laurent tails at Q_j with respect to $z_j, j=0,\dots,M$, the a-periods, and the increments of ω are constant, i.e. $\mathrm{incr}_\gamma(\omega)\in\mathbb{C}[\Lambda]d\Lambda$ for any $\gamma\in(R^1\kappa_*\underline{\mathbb{Z}})^*$ (where $\mathrm{incr}_\gamma(\omega)(\underline{\lambda}):=\mathrm{incr}_{\gamma(\underline{\lambda})}(\omega(\underline{\lambda})))$.

Denote by $\Omega\subset\tilde\kappa_*\Omega^1_{\tilde{\mathcal{C}}/H'_{g,t}}$ the sheaf of descendant forms and $\Omega_{\mathcal{O}}:=\Omega\otimes_{\underline{\mathbb{C}}}\mathcal{O}_{H'_{g,t}}$. For $\omega\in\Omega(U)$ let H_ω be the set of $\underline{\lambda}\in U$ such that $\omega(\underline{\lambda})$ is not vanishing at $P_1(\underline{\lambda}),\dots,P_N(\underline{\lambda})$.

Note that the condition that the Laurent tails are constant is independent of the choice of rays because the increments are constant. Also the condition that the a-periods are constant is independent of the choice of pre-images of (a_i,b_i) in $H_1(C^\circ,\mathbb{Z})$ because the residues at $Q_0,\dots,Q_M$ are constant.

Define Lie derivatives $D:=\mathrm{Lie}_{\tilde{e}+\partial_\lambda}$ and $D_E:=\mathrm{Lie}_{\tilde{E}+\Lambda\partial_\lambda}$ that act on $\tilde\kappa_*\Omega^1_{\tilde{\mathcal{C}}/H'_{g,t}}$ and $\Omega_k:=\ker(D^{k+1})\cap\Omega$, $k\ge 0$.

7.2. **Definition.** A **primary form** is a section of Ω_0. A primary form ω is called **eigenprimary** if $D_E\omega=[\omega]\omega$ for some $[\omega]\in\mathbb{C}$.

Expand ω in a neighborhood of $\tilde{P}_n\in\pi^{-1}(P_n(\underline{\lambda}))$ with respect to $\tau_n, \Lambda=u_n+\tau_n^2$,

$$\omega=\sum_{k\ge 0}a^\omega_{kn}\tau_n^k d\tau_n. \tag{7.1}$$

Note that the function a^ω_{0n} does not depend on the choice of $\tilde{P}_n$ because the increments $r(\Lambda)d\Lambda\in\mathbb{C}[\Lambda]d\Lambda$ of ω vanish at P_n.

7.3. **Lemma.** *(i) Ω_k is a local system of rank $(k+1)N$ and $D:\Omega_k\to\Omega_{k-1}$ is surjective. Ω is the union of $\Omega_k, k\ge 0$.*

(ii) The action of D_E on Ω_0 is diagonalizable.

(iii) Let $\mathcal{E}$ be the sheaf of eigenprimary forms $\omega\in\Omega_0(U), U\subset H'_{g,t}$, which satisfy $H_\omega=U$. The family $(H_\omega)_{\omega\in\mathcal{E}}$ is a covering of $H'_{g,t}$.

Proof. Let $U\subset H'_{g,t}$ be a small neighborhood of $\underline{\lambda}\in H'_{g,t}$. Choose rays $\tilde{Q}_j(\underline{\lambda})$ over $Q_j(\underline{\lambda}), j=0,\dots,M$.

(i) Let $\omega_{A,k}\in\Omega(U)$, $A\in\mathcal{N}, k\ge 0$, be the relative 1-form such that $\omega_{A,k}(\underline{\lambda}')\in\Omega^1_{\underline{\lambda}'}(C^\circ(\underline{\lambda}'))$ is the 1-form defined in Proposition 5.3 using $\tilde{Q}_0(\underline{\lambda}'),\dots,\tilde{Q}_M(\underline{\lambda}')$. It follows that $(\omega_{A,k})$ is a local basis of Ω. So it remains to prove

$$D\omega_{A,k}=\omega_{A,k-1},\qquad D\omega_{A,0}=0,\qquad \text{for any } A\in\mathcal{N}, k\ge 1. \tag{7.2}$$

We have $\mathrm{incr}(D\omega_{A,k})=D\,\mathrm{incr}(\omega_{A,k})=\partial_\lambda\,\mathrm{incr}(\omega_{A,k})=\mathrm{incr}(\omega_{A,k-1})$ because $\mathrm{Lie}_{\tilde{e}}(\mathrm{incr}(\omega_{A,k}))=0$ and $\mathrm{Lie}_{\partial_\lambda}(\frac{d\Lambda^k}{k!})=\frac{d\Lambda^{k-1}}{(k-1)!}, k\ge 1$, $\mathrm{Lie}_{\partial_\lambda}(d\Lambda)=0$.

Put $\tilde{\mathcal{C}}^\diamond := \pi^{-1}(\mathcal{C}^\diamond)$. The 1-form $D\omega_{A,k}$ is holomorphic on $\tilde{\kappa}^{-1}(U)\cap\tilde{\mathcal{C}}^\diamond$ because $\tilde{e}, \partial_\lambda$, and $\omega_{A,k}$ are holomorphic there. From (6.1) we obtain $D(\tau_n) = (\tilde{e}+\partial_\lambda)\tau_n = -\frac{1}{2\tau_n} + \frac{1}{2\tau_n} = 0$ and therefore

$$D\omega = \sum_{k\geq 0}\big(D(a^\omega_{kn})\tau_n^k d\tau_n + a^\omega_{kn} D(\tau_n^k d\tau_n)\big) = \sum_{k\geq 0} e(a^\omega_{kn})\tau_n^k d\tau_n. \tag{7.3}$$

Hence $D\omega$ is also holomorphic on $\pi^{-1}(P_n^{-1}(U)), n = 1,\dots,N$. The Laurent tail of $D\omega_{A,k}(\underline{\lambda})$ at $\tilde{Q}_j(\underline{\lambda})$ is equal to the Laurent tail of $\mathrm{Lie}_{\partial_\lambda}(\omega_{A,k})(\underline{\lambda})$ at $\tilde{Q}_j(\underline{\lambda})$ because $\mathrm{Lie}_{\tilde{e}}(dz_j^k) = 0$. We have

$$\mathrm{Lie}_{\partial_\lambda}(\omega_{t^{kj},k'}) \;=\; -\frac{1}{n_j+1}\mathrm{Lie}_{z_j^{n_j+2}\frac{\partial}{\partial z_j}}\Big(-\frac{(n_j+1)^{k'} dz_j^{-(k+k'(n_j+1))}}{\prod_{l=0}^{k'}(k+l(n_j+1))}\Big) \;+\; (\mathrm{reg})$$

which gives the same Laurent tail at $\tilde{Q}_j(\underline{\lambda})$ as that of $\omega_{t^{kj},k'-1}(\underline{\lambda})$ and using (5.2)

$$\mathrm{Lie}_{\partial_\lambda}(\omega_{w^j,k}) = d\Big(\frac{\Lambda^{k-1}}{(k-1)!}\Big(\log_{\tilde{Q}_j} + \frac{1}{n_j+1}\sum_{n=1}^{k}\frac{1}{n}\Big) + \frac{\Lambda^k}{k!}\Big(-\frac{z_j^{n_j+2}}{n_j+1}\Big)\frac{1}{z_j}\Big) \;+\; (\mathrm{reg})$$

which gives the same Laurent tail at $\tilde{Q}_j(\underline{\lambda})$ as that of $\omega_{w^j,k-1}(\underline{\lambda})$. Thus the Laurent tails of $\mathrm{Lie}_{\partial_\lambda}(\omega_{A,k})$ and $\omega_{A,k-1}$ are the same.

Finally, the a-periods of $D\omega_{A,k}$ are

$$\int_{a_i} D\omega_{A,k} \;=\; e\int_{a_i}\omega_{A,k} \;+\; \int_{a_i}\mathrm{Lie}_{\partial_\lambda}(\omega_{A,k}) \;=\; \Big(\frac{a_i^*\omega_{A,k}-\omega_{A,k}}{d\Lambda}\Big)\Big|_{P_0} \;=\; \delta_{A,s^i}\delta_{k1}$$

because $\mathrm{Lie}_{\partial_\lambda}(\omega) = \partial_\lambda(f)d\Lambda = d(\frac{\omega}{d\Lambda})$ if $\omega = f d\Lambda$, and $\frac{d\Lambda^k}{d\Lambda} = k\Lambda^{k-1}$. Theorem 3.6 now implies (7.2).

(ii) Introduce a new set of labels

$$\begin{aligned}\mathcal{N}_p \;:=\; &\{s^i, r^i, v^j, w^j \mid i = 1,\dots,g,\; j = 1,\dots,M\}\\ &\cup\{t^{kj}\mid k = 1,\dots,n_j,\; j = 0,\dots,M\}.\end{aligned} \tag{7.4}$$

Define $\omega_A := \omega_{A,0}$ for $A\in\mathcal{N}_p\setminus\{v^1,\dots,v^M\}\subset\mathcal{N}$ and $\omega_{v^j} := (n_j+1)\omega_{t^{n_j+1,j},0}, j = 1,\dots,M$. Because $(\omega_A)_{A\in\mathcal{N}_p}$ is a local basis of Ω_0 it suffices to prove that $D_E\omega_A = [\omega_A]\omega_A$ where

$$[\omega_{t^{kj}}] = \frac{k}{n_j+1}, \qquad [\omega_{v^j}] = [\omega_{r^i}] = 1, \qquad [\omega_{w^j}] = [\omega_{s^i}] = 0. \tag{7.5}$$

The proof is analogous to that of (7.2).

We have $\mathrm{incr}(D_E\omega_A) = D_E\,\mathrm{incr}(\omega_A) = \mathrm{Lie}_{\Lambda\partial_\lambda}(\mathrm{incr}(\omega_A)) = \mathrm{incr}(\omega_A)$ because $\mathrm{Lie}_{\tilde{E}}(\mathrm{incr}(\omega_A)) = 0$ and $\mathrm{Lie}_{\Lambda\partial_\lambda}(d\Lambda) = d\Lambda$.

The 1-form $D_E\omega_A$ is holomorphic on $\tilde{\kappa}^{-1}(U)\cap\tilde{\mathcal{C}}^\diamond$ because $\tilde{E}, \Lambda\partial_\lambda$, and ω_A are holomorphic there. Because $\tilde{E}\tau_n = \sum u^m\tilde{e}_m\tau_n = -\frac{u^n}{2\tau_n}$ and $\Lambda\partial_\lambda(\tau_n) = (u^n+\tau_n^2)\frac{1}{2\tau_n}$ we have $D_E\tau_n = \frac{1}{2}\tau_n$. We obtain for any $\omega\in\Omega_{\mathcal{O}}$

$$D_E\omega = \sum_{k\geq 0}\Big(E(a^\omega_{kn})\tau_n^k d\tau_n + a^\omega_{kn}\frac{k}{2}\tau_n^k d\tau_n + a^\omega_{kn}\tau_n^k\frac{1}{2}d\tau_n\Big) = \sum_{k\geq 0}\big(E(a^\omega_{kn}) + a^\omega_{kn}\frac{k+1}{2}\big)\tau_n^k d\tau_n. \tag{7.6}$$

Hence $D_E\omega_A$ is also holomorphic on $\pi^{-1}(P_n^{-1}(U)), n = 1,\ldots,N$. The Laurent tail of $D_E\omega_A(\underline{\lambda})$ at $\tilde{Q}_j(\underline{\lambda})$ is equal to the Laurent tail of $\mathrm{Lie}_{\Lambda\partial_\lambda}(\omega_A)(\underline{\lambda})$ at $\tilde{Q}_j(\underline{\lambda})$ because $\mathrm{Lie}_{\tilde{E}}(dz_j^k) = 0$. From $\mathrm{Lie}_{\Lambda\partial_\lambda}(d\Lambda) = d\Lambda$,

$$\mathrm{Lie}_{\Lambda\partial_\lambda}\Big(-\frac{dz_j^{-k}}{k}\Big) = \frac{1}{n_j+1}\mathrm{Lie}_{z_j\frac{\partial}{\partial z_j}}\Big(\frac{dz_j^{-k}}{k}\Big) = \frac{k}{n_j+1}\Big(-\frac{dz_j^{-k}}{k}\Big)$$

and

$$\mathrm{Lie}_{\Lambda\partial_\lambda}\Big(\frac{dz_j}{z_j}\Big) = -\frac{1}{n_j+1}\mathrm{Lie}_{z_j\frac{\partial}{\partial z_j}}\Big(\frac{dz_j}{z_j}\Big) = -\frac{1}{n_j+1}\Big(\frac{dz_j}{z_j}-\frac{dz_j}{z_j}\Big) = 0$$

we see that $D_E\omega_A$ and $[\omega_A]\omega_A$ have the same Laurent tails at $\tilde{Q}_0,\ldots,\tilde{Q}_M$.

Finally, the a-periods of $D_E\omega_A$ are

$$\int_{a_i} D_E\omega_A = E\int_{a_i}\omega_A + \int_{a_i}\Lambda d\Big(\frac{\omega_A}{d\Lambda}\Big) + \int_{a_i}\omega_A = -\int_{a_i}\omega_A + \Big[\Lambda\frac{\omega_A}{d\Lambda}\Big]_{\tilde{P}_0}^{(a_i)_*\tilde{P}_0} + \int_{a_i}\omega_A = 0$$

because $\mathrm{Lie}_{\Lambda\partial_\lambda}(\omega) = \Lambda\partial_\lambda(f)d\Lambda + fd\Lambda = \Lambda d(\frac{\omega}{d\Lambda}) + \omega$ if $\omega = fd\Lambda$, and $\Lambda\circ P_0 = 0$.

By Theorem 3.6 this implies $D_E\omega_A = [\omega_A]\omega_A$.

(iii) First assume $g>0$. For any $\underline{\lambda}\in H'_{g,t}$ one has $H^0(C(\underline{\lambda}),K) \supsetneq H^0(C(\underline{\lambda}),K-P_n(\underline{\lambda}))$ because the canonical divisor K is base-point-free. Thus there is a holomorphic 1-form ω' on $C(\underline{\lambda})$ which is not vanishing at $P_1(\underline{\lambda}),\ldots,P_N(\underline{\lambda})$. If $\omega\in\Omega$ satisfies $\omega(\underline{\lambda}) = \omega'$ then $\omega\in\mathcal{E}$ and $\underline{\lambda}\in H_\omega$.

If $g=0$ and $M>0$ then $\omega_{w^1}\in\mathcal{E}$ has no zeros and if $g=M=0$ and $n_0>0$ then $\omega_{t^{1,0}}$ has no zeros. □

Let U be a small neighborhood of $\underline{\lambda}\in H'_{g,t}$ and $\omega\in\Omega(U)$. Choosing a ray $\tilde{Q}_0$ over $Q_0(\underline{\lambda})$, we define a local primitive $z_\omega:\tilde{\kappa}^{-1}(U)\to\mathbb{CP}^1, U\subset H'_{g,t}$, of ω by

$$z_\omega := \mathrm{p.v.}\int_{\tilde{Q}_0}\omega = \sum_{k=-(n_j+2)}^{-1}\frac{c_{kj}^\omega}{k+1}z_j^{k+1} + c_{-1,j}^\omega\log_{\tilde{Q}_j} + \mathrm{p.v.}\int_{\tilde{Q}_0}^{\tilde{Q}_j}\omega + \sum_{k\geq 0}\frac{c_{kj}^\omega}{k+1}z_j^{k+1} \tag{7.7}$$

where we also wrote down the expansion of z_ω at a ray $\tilde{Q}_j$ using (3.2).

For a basic domain $\check{C}$ over $\underline{\lambda}\in H'_{g,t}$ the pairing $\langle.,.\rangle_{\check{C}}$ of (4.2) extends over an open neighborhood $U\subset H'_{g,t}$ of $\underline{\lambda}$ to a pairing $\langle.,.\rangle_{\check{C}}:\Omega_{\mathcal{O}}|_U\otimes\Omega_{\mathcal{O}}|_U\to\mathcal{O}_U$.

7.4. **Lemma.** *Let U be small neighborhood of $\underline{\lambda}\in H'_{g,t}$, $\omega_1,\omega_2\in\Omega_0(U)$, and $\check{C}$ be a basic region over $\underline{\lambda}$. Write $c_{kj}^{(\alpha)} := c_{kj}^{\omega_\alpha}, \mathrm{incr}_{b_i}(\omega_\alpha) = q_i^{(\alpha)}d\Lambda, q_i^{(\alpha)}\in\mathbb{C}$, and $A_i^{(\alpha)} := \int_{a_i}\omega_\alpha$ where $\alpha=1,2$. Then we have*

$$e\langle\omega_1,\omega_2\rangle_{\check{C}} = -\sum_{j=0}^{M}\Big(\frac{1}{n_j+1}\sum_{k=0}^{n_j-1}c_{-k-2,j}^{(1)}c_{k-n_j-1,j}^{(2)}\Big)$$

$$-\sum_{j=1}^{M}\frac{1}{n_j+1}\big(c_{-1,j}^{(1)}c_{-n_j-2,j}^{(2)} + c_{-1,j}^{(2)}c_{-n_j-2,j}^{(1)}\big) - \frac{1}{2\pi i}\sum_{i=1}^{g}\big(q_i^{(1)}A_i^{(2)} + q_i^{(2)}A_i^{(1)}\big).$$

Proof. By continuity we may assume that λ has a simple zero $P_0(\underline{\lambda})$.

Let $\omega \in \Omega_0(U)$. Because $d(Dz_\omega) = D(dz_\omega) = D(\omega) = 0$ we have $Dz_\omega = c \in \mathcal{O}_{H'_{g,t}}$. For any sections $P, P' : U \to \tilde{\mathcal{C}}$ of $\tilde{\kappa}$ such that $\Lambda \circ P, \Lambda \circ P'$ are constant we obtain

$$e(z_\omega \circ P) \;=\; ((\partial_\lambda z_\omega) \circ P) e(\Lambda \circ P) + (\tilde{e} z_\omega) \circ P \;=\; (Dz_\omega - \partial_\lambda z_\omega) \circ P \;=\; c - \frac{\omega}{d\Lambda} \circ P \tag{7.8}$$

and

$$e \int_P^{P'} \omega \;=\; e\Big(\big[z_\omega \big]_P^{P'} \Big) \;=\; -\Big[\frac{\omega}{d\Lambda} \Big]_P^{P'}, \tag{7.9}$$

in particular for any $a \in \{a_i, b_i | i = 1, \ldots, g\}$ we have

$$e \int_a \omega \;=\; e \int_{\tilde{a}} \omega \;=\; -\frac{a^* \omega}{d\Lambda} \circ \tilde{P}_0 + \frac{\omega}{d\Lambda} \circ \tilde{P}_0. \tag{7.10}$$

For $\Lambda\omega \in \Omega_{\mathcal{O}}$ we have analogously $D(\Lambda\omega) = \omega, D(z_{\Lambda\omega}) = z_\omega + c', c' \in \mathcal{O}_{H'_{g,t}}, e(z_{\Lambda\omega} \circ P) = z_\omega \circ P + c' - \Lambda \frac{\omega}{d\Lambda} \circ P$ and

$$e \int_a \Lambda\omega \;=\; e \int_{\tilde{a}} \Lambda\omega \;=\; \Big[z_\omega - \Lambda \frac{\omega}{d\Lambda} \Big]_{\tilde{a}(0)}^{\tilde{a}(1)} \;=\; \int_a \omega \tag{7.11}$$

because $\Lambda \circ \tilde{a}(0) = \Lambda \circ \tilde{a}(1) = \Lambda \circ P_0 = 0$.

Applying (7.10) and (7.11) we get

$$\sum_{i=1}^{g} -q_i^{(1)} \, e \int_{a_i} \Lambda\omega_2 \;+\; A_i^{(1)} \, e \int_{b_i} \omega_2$$

$$= \sum_{i=1}^{g} -q_i^{(1)} \int_{a_i} \omega_2 \;+\; A_i^{(1)} \Big(-\frac{b_i^* \omega_2}{d\Lambda} \circ \tilde{P}_0 + \frac{\omega_2}{d\Lambda} \circ \tilde{P}_0 \Big) \;=\; -\sum_{i=1}^{g} q_i^{(1)} A_i^{(2)} + A_i^{(1)} q_i^{(2)}.$$

From (7.9) and (3.2) we obtain for $P(\underline{\lambda})$ in a neighborhood of $\tilde{Q}_j(\underline{\lambda})$

$$e \int_{\tilde{P}_0}^{P} \omega \;=\; -\Big[\frac{\omega}{d\Lambda} \Big]_{\tilde{P}_0}^{P} \;=\; \sum_{k \geq -(n_j+2)} \frac{c_{kj}^\omega}{n_j + 1} (z_j \circ P)^{k+n_j+2} + \frac{\omega}{d\Lambda} \circ \tilde{P}_0.$$

On the other hand because $e(c_{kj}^\omega) = 0$ for $k < 0$

$$e \int_{\tilde{P}_0}^{P} \omega \;=\; \sum_{k \geq 0} \frac{e(c_{kj}^\omega)}{k+1} (z_j \circ P)^{k+1} + e \, \text{p.v.} \int_{\tilde{P}_0}^{\tilde{Q}_j} \omega$$

and therefore by comparison of coefficients we obtain $e(c_{ki}^\omega) = \frac{k+1}{n_j+1} c_{k-n_j-1,j}^\omega$ for $k \geq 0$, and

$$e \, \text{p.v.} \int_{\tilde{P}_0}^{\tilde{Q}_j} \omega \;=\; \frac{c_{-n_j-2}^\omega}{n_j+1} + \frac{\omega}{d\Lambda} \circ \tilde{P}_0.$$

Applying these results the assertion follows from

$$\sum_{j=0}^{M}\Big(\sum_{k=0}^{n_j} c^{(1)}_{-k-2,j}\frac{e(c^{(2)}_{kj})}{k+1}\Big) + c^{(1)}_{-1,j}\, e\,\text{p.v.}\int_{\tilde P_{0j}}^{\tilde Q_j}\omega_2$$

$$= \sum_{j=0}^{M}\Big(\sum_{k=0}^{n_j} c^{(1)}_{-k-2,j}\frac{c^{(2)}_{k-n_j-1,j}}{n_j+1}\Big) + c^{(1)}_{-1,j}\Big(\frac{c^{(2)}_{-n_j-2,j}}{n_j+1}+\frac{\omega_2}{d\Lambda}\circ\tilde P_0\Big)$$

$$= \sum_{j=0}^{M}\frac{1}{n_j+1}\sum_{k=0}^{n_j-1} c^{(1)}_{-k-2,j}c^{(2)}_{k-n_j-1,j} + \sum_{j=1}^{M}\frac{1}{n_j+1}\Big(c^{(1)}_{-n_j-2,j}c^{(2)}_{-1,j}+c^{(1)}_{-1,j}c^{(2)}_{-n_j-2,j}\Big)$$

because $c^{(2)}_{-n_j-2,0}=0$ and $\sum_{j=0}^{M} c^{(1)}_{-1,j}=0$ by Corollary 3.8. $\square$

8. A hierarchy of Frobenius manifolds

In the case of Saito frameworks with one-dimensional fibres the pairing (2.2) is given by (see (7.1))

$$g_\omega := \sum_{n=1}^{N}\Big(\operatorname*{res}_{P_n}\frac{\omega^2}{d\Lambda}\Big)(du_n)^2 = \sum_{n=1}^{N}\frac{(a^\omega_{0n})^2}{2}(du_n)^2,\qquad \omega\in\Omega. \tag{8.1}$$

8.1. **Theorem.** *$(H_\omega,\circ,g_\omega)$ is a Frobenius manifold for any $\omega\in\Omega$. More explicitly,*

(i) the functions (see (7.7))

$$x^{t^{kj}}_\omega := -\frac{(n_j+1)c^\omega_{n_j-k,j}}{n_j-k+1}\qquad x^{v^j}_\omega := -\text{p.v.}\int_{\tilde Q_0}^{\tilde Q_j}\omega\qquad x^{w^j}_\omega := -c^\omega_{n_j,j}$$

$$x^{r^i}_\omega := -\int_{b_i}\omega\qquad x^{s^i}_\omega := \int_{a_i} z_\omega d\Lambda = -\int_{a_i}\Lambda\omega \tag{8.2}$$

are local g_ω-flat coordinates and if $\partial^\omega_A := \frac{\partial}{\partial x^A_\omega}, A\in\mathcal N_p$ (cf. (7.4)), then $g_\omega(\partial^\omega_A,\partial^\omega_B)=0$ except for

$$g_\omega(\partial^\omega_{t^{kj}},\partial^\omega_{t^{n_j+1-k,j}}) = g_\omega(\partial^\omega_{v^j},\partial^\omega_{w^j}) = \frac{1}{n_j+1}\qquad g_\omega(\partial^\omega_{s^i},\partial^\omega_{r^i}) = \frac{1}{2\pi i}; \tag{8.3}$$

(ii) the morphism

$$\partial^\omega : \Omega|_{H_\omega}\to\mathcal T_{H_\omega},\quad \omega'\mapsto\partial^\omega_{\omega'} := -\sum_{n=1}^{N}\frac{a^{\omega'}_{0n}}{a^\omega_{0n}}e_n,$$

induces an isomorphism $\partial^\omega : \Omega_0|_{H_\omega}\xrightarrow{\sim}\mathcal T^{g_\omega}_{H_\omega}$ such that $\partial^\omega_{\omega_A}=\partial^\omega_A, A\in\mathcal N_p$ and for any $\omega'\in\Omega$

$$g_{\omega'}(X,Y) = g_\omega(\partial^\omega_{\omega'}\circ X,\partial^\omega_{\omega'}\circ Y),\quad X,Y\in\mathcal T_{H_\omega}; \tag{8.4}$$

for a basic region $\check C$ over $\underline\lambda\in H_\omega$ and $\omega_1,\omega_2\in\Omega_0$ we have

$$g_\omega(\partial^\omega_{\omega_1},\partial^\omega_{\omega_2}) = e\,\langle\omega_1,\omega_2\rangle_{\check C}; \tag{8.5}$$

if $\nabla := 1 \otimes d$ *is the canonical connection on* $\Omega_{\mathcal{O}} = \Omega \otimes \mathcal{O}_{H'_{g,t}}$ *then* $z_\omega d\Lambda \in \Omega_{\mathcal{O}}$ *and*

$$\nabla_{\partial^\omega_{\omega'}}(z_\omega d\Lambda) = \omega'; \tag{8.6}$$

$e \in \mathcal{T}_M^{g_\omega}$ *if and only if* $\omega \in \Omega_0$*;*

(iii) for any basic region $\check{C}$ *over* $\underline{\lambda} \in H_\omega$ *a metric potential of* g_ω *is given by*

$$\eta^\omega \ := \ \langle \omega, \omega \rangle_{\check{C}}$$

and a Frobenius potential is given by

$$\Phi_\omega := \frac{1}{2}\langle z_\omega d\Lambda, z_\omega d\Lambda \rangle_{\check{C}};$$

(iv) the rotation coefficients of the metrics g_ω

$$\gamma_{mn} \ := \ \gamma^\omega_{mn} \ := \ \frac{e_m(a^\omega_{0n})}{a^\omega_{0m}}, \qquad m \neq n,$$

are independent of ω *and for any metric* $g = \sum a_n^2 (du_n)^2$ *on* $H \subset H_{g,t}$ *such that* $\frac{e_m(a_n)}{a_m} = \gamma_{mn}, m \neq n, e^{k+1}(a_n) = 0$*, and any small open* $U \subset \phi^{-1}(H) \subset H'_{g,t}$ *there exists* $\omega \in \Omega(U)$ *such that* $g = g_\omega$*.*

(v) if ω *is eigenprimary then the vector field* E *is a semisimple Euler field on* H_ω *with spectrum* $d_0 = 1$*,* $D = 1 + 2[\omega]$ *and*

$$d_{t^{kj}} = 1 + [\omega] - [\omega_{t^{kj}}] \qquad d_{v^j} = d_{r^i} = [\omega] \qquad d_{w^j} = d_{s^i} = 1 + [\omega]$$

where $[\partial^\omega_A, E] = d_A \partial^\omega_A$*,* $A \in \mathcal{N}_p$*;*

Proof. Let U be a neighborhood of $\underline{\lambda} \in H_\omega$ and $\tilde{Q}_0, \ldots, \tilde{Q}_M$ be rays over $Q_0(\underline{\lambda}), \ldots, Q_M(\underline{\lambda})$. By continuity we may assume that λ has a simple zero $P_0(\underline{\lambda})$.

(i), (ii) Using (7.7) we see that $z_\omega d\Lambda$ has the following expansion at $\tilde{Q}_j$

$$\begin{aligned} z_\omega d\Lambda \ = \ & - \sum_{k=-K}^{-1} \frac{(n_j+1)c^\omega_{kj}}{k+1} z_j^{k-n_j-1} dz_j + c^\omega_{-1,j} \log_{\tilde{Q}_j} d\Lambda + \Big(\text{p.v.} \int_{\tilde{Q}_0}^{\tilde{Q}_j} \omega\Big) d\Lambda \\ & - \sum_{k\geq 0}^{n_j-1} \frac{(n_j+1)c^\omega_{kj}}{k+1} \frac{dz_j^{k-n_j}}{k-n_j} - c^\omega_{n_j,j}\frac{dz_j}{z_j} - \sum_{k>n_j} \frac{(n_j+1)c^\omega_{kj}}{k+1} z_j^{k-n_j-1} dz_j, \end{aligned} \tag{8.7}$$

in particular $z_\omega d\Lambda$ is logarithmic at infinity. As $d\Lambda$ is invariant the basic increments of $z_\omega d\Lambda$ are determined by the basic increments of z_ω. If $\text{incr}_{a_i}(\omega) = dp_i^\omega, \text{incr}_{b_i}(\omega) = q_i^\omega, \text{incr}_{d_j}(\omega) = dr_j^\omega$, $p_i^\omega, q_i^\omega, r_j^\omega \in \Lambda\mathbb{C}[\Lambda]$, then we have by (5.1) and (8.7)

$$\text{incr}_{a_i}(z_\omega) = \int_{a_i} \omega + p_i^\omega \qquad \text{incr}_{b_i}(z_\omega) = -x_\omega^{r^i} + q_i^\omega \qquad \text{incr}_{d_j}(z_\omega) = 2\pi i\, c_{-1} + r_j^\omega. \tag{8.8}$$

From (8.7) and (8.8) we see that $z_\omega d\Lambda \in \Omega_{\mathcal{O}}$ and that expressing $z_\omega d\Lambda$ in the local basis $\omega_{A,k} \in \Omega_{\mathcal{O}}$ gives

$$z_\omega d\Lambda \ = \ \psi^\omega \ + \sum_{A \in \mathcal{N}_p} x_\omega^A \omega_A, \qquad \psi^\omega \in \text{span}_{\mathbb{C}}\{\omega_{A,k} \mid A \in \mathcal{N}, k \geq 1\}. \tag{8.9}$$

We calculate the expansion of $\tilde{e}_m(z_\omega d\Lambda) = \tilde{e}_m(z_\omega)d\Lambda$ in a neighborhood of $\tilde{P}_n \in \pi^{-1}(P_n(\underline{\lambda}))$ using (6.1) and (7.7)

$$\tilde{e}_m(z_\omega d\Lambda) = \Big(e_m \,\text{p.v.}\int_{\tilde{Q}_0}^{\tilde{P}_n} \omega + \sum_{k\geq 0} \frac{e_m a^\omega_{kn}}{k+1}\tau_n^{k+1} + \sum_{k\geq 0}\frac{a^\omega_{kn}}{k+1}\Big(-\frac{(k+1)\delta_{nm}}{2}\Big)\tau_n^{k-1}\Big)2\tau_n d\tau_n. \tag{8.10}$$

This gives $\tilde{e}_m(z_\omega d\Lambda) = (-\delta_{nm}a^\omega_{0n} + \dots)d\tau_n$. On the other hand from (8.9) we obtain

$$\tilde{e}_m(z_\omega d\Lambda) = \sum_A \frac{\partial x^A_\omega}{\partial u^m}\omega_A + \sum_A x^A_\omega \tilde{e}_m(\omega_A) + \tilde{e}_m(\psi^\omega) \tag{8.11}$$

Put $\rho := \sum_A x^A_\omega \tilde{e}_m(\omega_A) + \tilde{e}_m(\psi^\omega)$. For any $\omega' \in \Omega$ we have

$$\langle\omega',\rho\rangle_{\check{C}} = \sum_A x^A_\omega \langle\omega', \tilde{e}_m(\omega_A)\rangle_{\check{C}} + \langle\omega', \tilde{e}_m(\psi^\omega)\rangle_{\check{C}}$$

$$= \sum_A x^A_\omega e_m\langle\omega',\omega_A\rangle_{\check{C}} + e_m\langle\omega',\psi^\omega\rangle_{\check{C}} = \frac{a^{\omega'}_{0m}a^{z_\omega d\Lambda}_{0m}}{2} \equiv 0.$$

Let $\omega_k \in \Omega, k\geq 0$, such that $\omega_k(\underline{\lambda})$ for any $\underline{\lambda} \in U$ is the normalized abelian differential of the second kind on $C(\underline{\lambda})$ with only a pole at $Q_0(\underline{\lambda})$ with Laurent tail equal to the Laurent tail of $(k+1)z_0^{-k-2}dz_0$. Then $0 \equiv \langle\omega_k,\rho\rangle_{\check{C}} = c^\rho_{k0}$ for any $k \geq 0$. Because ρ is holomorphic on $Q_0(U)$ we have $\rho = 0$.

Therefore putting (8.10) and (8.11) together we obtain

$$-\delta_{nm}a^\omega_{0n} = \sum_A \frac{\partial x^A_\omega}{\partial u^m}\, a^A_{0n}.$$

We thus see that $(x^A_\omega)_A$ are local coordinates and

$$\partial^\omega_A(x^A_\omega) = -\frac{a^A_{0n}}{a^\omega_{0n}}. \tag{8.12}$$

This shows that $\partial^\omega_{\omega_A} = \partial^\omega_A, A \in \mathcal{N}_p$ which in turn implies that ∂^ω is an isomorphism, and that (8.4) holds; (8.6) follows from (8.9).

Let $\check{C}$ be a basic region over $\underline{\lambda}$, $\omega_1, \omega_2 \in \Omega(U)$ and $\underline{\lambda}' \in U$. From (7.1) and $d\Lambda = 2\tau_n d\tau_n$ we get

$$\operatorname*{res}_{P_n(\underline{\lambda}')} \frac{\omega_1(\underline{\lambda}')\omega_2(\underline{\lambda}')}{d\lambda'} = \frac{a^{\omega_1}_{0n}(\underline{\lambda}')a^{\omega_2}_{0n}(\underline{\lambda}')}{2}. \tag{8.13}$$

On the other hand, from

$$\begin{aligned}\tilde{e}_m\omega(\underline{\lambda}) &= \sum_{k\geq 0}\Big(e_m(a^\omega_{kn})\tau_n^k d\tau_n + a^\omega_{kn}k\tau_n^{k-1}\Big(-\frac{\delta_{mn}}{2\tau_n}\Big)d\tau_n + a^\omega_{kn}\tau_n^k\frac{\delta_{mn}}{2\tau_n^2}d\tau_n\Big)\\ &= \sum_{k\geq -2}\Big(e_m(a^\omega_{kn}) - \delta_{mn}\frac{(k+1)a^\omega_{k+2,n}}{2}\Big)\tau_n^k d\tau_n\\ &= \Big(\delta_{mn}\frac{a^\omega_{0n}}{2}\tau_n^{-2} + e_m(a^\omega_{0n}) - \delta_{mn}\frac{a^\omega_{2n}}{2} + \dots\Big)d\tau_n\end{aligned} \tag{8.14}$$

we obtain for P in a neighborhood of $\tilde{P}_n(\underline{\lambda}') \in \check{C}$ that $\int\limits_{P_0(\underline{\lambda}')}^{P} \tilde{e}_n\omega_2(\underline{\lambda}') = -\frac{a_{0n}^{\omega_2}(\underline{\lambda}')}{2\tau_n(P)} + (\mathrm{reg})$ and therefore

$$-\frac{a_{0n}^{\omega_1}(\underline{\lambda})a_{0n}^{\omega_2}(\underline{\lambda})}{2} = \operatorname*{res}_{\tilde{P}_n(\underline{\lambda}')}\Big(\omega_1(\underline{\lambda}')\int\limits_{P_0(\underline{\lambda}')}^{P} \tilde{e}_n\omega_2(\underline{\lambda}')\Big). \tag{8.15}$$

Because $\omega_1(\underline{\lambda}')$ is holomorphic on $\tilde{C}(\underline{\lambda}')$ and $\tilde{P}_n(\underline{\lambda}')$ is the only pole of $\tilde{e}_n\omega_2(\underline{\lambda}')$ in $\check{C}(\underline{\lambda}')$ the reciprocity law together with (8.13) and (8.15) implies

$$\operatorname*{res}_{P_n(\underline{\lambda}')}\frac{\omega_1(\underline{\lambda}')\omega_2(\underline{\lambda}')}{d\lambda'} = \langle\omega_1(\underline{\lambda}'), \tilde{e}_n\omega_2(\underline{\lambda}')\rangle_{\check{C}}.$$

Because ω_1 is a descendant form one obtains from (4.2) $\langle\omega_1(\underline{\lambda}'), \tilde{e}_n\omega_2(\underline{\lambda}')\rangle_{\check{C}} = e_n\langle\omega_1(\underline{\lambda}'), \omega_2(\underline{\lambda}')\rangle_{\check{C}}$ and thus

$$\operatorname*{res}_{P_n(\underline{\lambda}')}\frac{\omega_1(\underline{\lambda}')\omega_2(\underline{\lambda}')}{d\lambda'} = e_n\langle\omega_1, \omega_2\rangle_{\check{C}}(\underline{\lambda}'). \tag{8.16}$$

This implies (8.5) and from Lemma 7.4 we obtain (8.3). It remains to prove the statement about the flat identity.

By (7.3) $D\omega = 0$ implies $ea_{0n}^{\omega} = 0$ which is equivalent to $e\eta_n^{\omega} = 0$ by (8.1) which is equivalent to the flatness of e by Proposition 2.2. Conversely, assume $ea_{0n}^{\omega} = 0, n = 1, \ldots, N$. We may assume that $D\omega \in \Omega_0$. The matrix $(a_{0n}^{A})_{A,n}$ is invertible by (8.12). Therefore $a_{0n}^{D\omega} = ea_{0n}^{\omega} = 0, n = 1, \ldots, N$, (see (7.3)) implies $D\omega = 0$.

(iii) That η^{ω} is a metric potential follows from (8.16). Because $z_\omega d\Lambda$ is vanishing on $P_1(U), \ldots, P_N(U)$ we have for any $\omega' \in \Omega, \partial \in \mathcal{T}_U$

$$\begin{aligned}\partial\langle\omega', \psi^{\omega}\rangle + \sum_A x_\omega^A\partial\langle\omega', \omega_A\rangle &= \sum_n \partial(u^n)e_n\langle\omega', \psi^{\omega}\rangle + \sum_A x_\omega^A\partial(u^n)e_n\langle\omega', \omega_A\rangle \\ &= \sum_n \partial(u^n)\frac{a_{0n}^{\omega'}}{2}(a_{0n}^{\psi^\omega} + \sum_A x_\omega^A a_{0n}^A) = 0.\end{aligned}$$

Using this we obtain from (8.9) by linearity

$$\partial_A^{\omega}\langle z_\omega d\Lambda, z_\omega d\Lambda\rangle = \langle\psi^{\omega}, \omega_A\rangle + \langle\omega_A, \psi^{\omega}\rangle + \sum_C x_\omega^C(\langle\omega_A, \omega_C\rangle + \langle\omega_C, \omega_A\rangle)$$

and

$$\partial_B^{\omega}\partial_A^{\omega}\langle z_\omega d\Lambda, z_\omega d\Lambda\rangle = \langle\omega_A, \omega_B\rangle + \langle\omega_B, \omega_A\rangle.$$

The assertion now follows from

$$\begin{aligned}\partial_A^{\omega}\langle\omega_B, \omega_C\rangle &= \sum_{n=1}^{N}\frac{\partial u^n}{\partial t_\omega^A}\, e_n\langle\omega_B, \omega_C\rangle = \sum_{n=1}^{N}\Big(-\frac{a_{0n}^A}{a_{0n}^{\omega}}\Big)\frac{a_{0n}^B a_{0n}^C}{2} \\ &= -\sum_{n=1}^{N}\operatorname*{res}_{P_n}\frac{\omega_A\omega_B\omega_C}{d\Lambda\;\omega} = g_\omega(\partial_A^{\omega}\circ\partial_B^{\omega}, \partial_C^{\omega}).\end{aligned}$$

(iv) Define two operations on the space of (possibly degenerate) metrics $g = \sum(a_n)^2 (du_n)^2$, $g_{+\omega} := \sum(a_n + a_{0n}^{\omega})^2(du_n)^2$ and $Dg := \sum(e(a_n))^2(du_n)^2$.

The second Darboux-Egoroff equation $e\gamma_{mn}^{\omega} = 0, m \neq n$, implies $e_m(e(a_n^{\omega}))a_m^{\omega} - e_m(a_n^{\omega}) e(a_m^{\omega}) = 0$, i.e. $\gamma_{mn}^{\omega} = \gamma_{mn}^{D\omega}$ on $H_\omega \cap H_{D\omega}$ because $e(a_m^{\omega}) = a_m^{D\omega}$ by (7.3). Proposition 2.2

says that (8.4) implies that $\gamma^{\omega}_{mn} = \gamma^{\omega'}_{mn}, m \neq n$, for any $\omega, \omega' \in \Omega_0$. Thus $\gamma^{\omega}_{mn} = \gamma^{\omega'}_{mn}, m \neq n$, for any $\omega, \omega' \in \Omega$.

The converse statement is proven by induction on k. The case $k = 0$ follows from Proposition of 2.2 and (8.4) because $k = 0$ means that e is flat. If $g = \sum (a_n)^2 (du_n)^2$ is such that $\frac{e_m(a_n)}{a_m} = \gamma_{mn}, m \neq n$, and $e^{k+1}(a_n) = 0$ then $g_{+\omega}, \omega \in \Omega_k$, satisfies the same properties provided that $g_{+\omega}$ is non-degenerate. There exists $\omega_1 \in \Omega_k$ such that $D(g_{+\omega_1})$ is non-degenerate and satisfies these properties for $k-1$. By induction there is $\omega_2 = \sum c_{A,l}\omega_{A,l} \in \Omega_{k-1}, c_{A,l} \in \mathbb{C}$, such that $D(g_{+\omega_1}) = g_{\omega_2}$. Thus $g_{+(\omega_1 - D^{-1}\omega_2)}$, $D^{-1}\omega_2 := \sum c_{A,l}\omega_{A,l+1} \in \Omega_k$, also satisfies these properties for $k-1$. Again by induction there is $\omega_3 \in \Omega_{k-1}$ such that $g_{+(\omega_1 - D^{-1}\omega_2)} = g_{\omega_3}$, thus $g = g_\omega$ for $\omega := -\omega_1 + D^{-1}\omega_2 + \omega_3 \in \Omega_k$.

(v) By (7.6) $D_E\omega = [\omega]\omega$ implies $Ea^{\omega}_{0n} = \big([\omega] - \frac{1}{2}\big)a^{\omega}_{0n}$. By (8.1) this implies $E\eta^{\omega}_n = (2[\omega]-1)\eta^{\omega}_n$. From Proposition 2.2 we obtain $\mathrm{Lie}_E(g_\omega) = D_\omega g_\omega$ where $D_\omega = (2[\omega]-1) + 2d_0 = 2[\omega]+1$. The rest of the proof is analogous to that of Lemma 7.4 with e replaced by E.

Since $d(D_E z_\omega) = D_E(dz_\omega) = D_E\omega = [\omega]\omega$ we have $D_E z_\omega = [\omega]z_\omega + c$ for some $c \in \mathcal{O}_{H'_{g,t}}$. As in (7.8) we obtain for any sections $P, P' : U \to \tilde{\mathcal{C}}$ of $\tilde{\kappa}$ such that $\Lambda \circ P, \Lambda \circ P'$ are constant

$$E(z_\omega \circ P) = (\tilde{E} z_\omega) \circ P = ((D_E - \mathrm{Lie}_{\Lambda\partial_\lambda}) z_\omega) \circ P = [\omega] z_\omega \circ P + c - \Lambda \frac{\omega}{d\Lambda} \circ P$$

and

$$E \int_P^{P'} \omega = [\omega] \int_P^{P'} \omega - \Big[\Lambda \frac{\omega}{d\Lambda}\Big]_P^{P'}, \tag{8.17}$$

in particular

$$E(x^{r^i}_\omega) \;=\; -E \int_{b_i} \omega \;=\; -[\omega] \int_{b_i} \omega + \Big[\Lambda \frac{\omega}{d\Lambda}\Big]_{b_i(0)}^{b_i(1)} \;=\; [\omega] x^{r^i}_\omega .$$

For $\Lambda\omega \in \Omega_{\mathcal{O}}$ we have analogously $D_E(\Lambda\omega) = ([\omega]+1)\Lambda\omega, D_E(z_{\Lambda\omega}) = ([\omega]+1)z_{\Lambda\omega} + c', c' \in \mathcal{O}_{H'_{g,t}}, E(z_{\Lambda\omega} \circ P) = ([\omega]+1)z_{\Lambda\omega} \circ P + c - \Lambda^2 \frac{\omega}{d\Lambda} \circ P$ and

$$E(x^{s^i}_\omega) \;=\; -E \int_{\tilde{a}_i} \Lambda\omega \;=\; \Big[([\omega]+1)z_{\Lambda\omega} - \Lambda^2 \frac{\omega}{d\Lambda}\Big]_{\tilde{a}_i(0)}^{\tilde{a}_i(1)} \;=\; ([\omega]+1) x^{s^i}_\omega . \tag{8.18}$$

From (8.17) and the development of ω at $\tilde{Q}_j$ we obtain

$$E \int_{\tilde{P}_0}^{P} \omega \;=\; [\omega] \int_{\tilde{P}_0}^{P} \omega \;-\; \Big[\Lambda \frac{\omega}{d\Lambda}\Big]_{\tilde{P}_0}^{P}$$

$$= \sum_{\substack{k \geq -K \\ k \neq -1}} [\omega] \frac{c^{\omega}_{kj}}{k+1} (z_j \circ P)^{k+1} + [\omega] c^{\omega}_{-1,j} \log_{\tilde{Q}_j} \circ P + [\omega]\, \mathrm{p.v.} \int_{\tilde{P}_0}^{\tilde{Q}_j} \omega + \sum_{k \geq -K} \frac{c^{\omega}_{kj}}{n_j + 1} (z_j \circ P)^{k+1}$$

On the other hand

$$E \int_{\tilde{P}_0}^{P} \omega = \sum_{\substack{k \geq -K \\ k \neq -1}} \frac{E(c^{\omega}_{kj})}{k+1} (z_j \circ P)^{k+1} + E(c^{\omega}_{-1,j}) \log_{\tilde{Q}_j} \circ P + E(\mathrm{p.v.} \int_{\tilde{P}_0}^{\tilde{Q}_j} \omega).$$

Comparing coefficients we get

$$E(c^\omega_{kj}) \;=\; \Big([\omega] + \frac{k+1}{n_j+1}\Big)c^\omega_{kj}, \qquad\qquad E\,(\mathrm{p.v.}\int\limits_{\tilde P_0}^{\tilde Q_j}\omega) \;=\; [\omega]\,\mathrm{p.v.}\int\limits_{\tilde P_0}^{\tilde Q_j}\omega \;+\; \frac{c^\omega_{-1,j}}{n_j+1}.$$

From this we obtain

$$E(x^{v^j}_\omega) = E(\mathrm{p.v.}\int\limits_{\tilde Q_0}^{\tilde Q_j}\omega) = [\omega]\mathrm{p.v.}\int\limits_{\tilde Q_0}^{\tilde Q_j}\omega + \frac{c^\omega_{-1,j}}{n_j+1} - \frac{c^\omega_{-1,0}}{n_0+1} = [\omega]x^{v^j}_\omega + \frac{c^\omega_{-1,j}}{n_j+1} - \frac{c^\omega_{-1,0}}{n_0+1},$$

$$\begin{aligned} E(x^{t^{kj}}_\omega) \;&=\; -\frac{n_j+1}{n_j+1-k}E(c_{n_j-k,j}) \;=\; -\frac{n_j+1}{n_j+1-k}\Big([\omega] + \frac{n_j-k+1}{n_j+1}\Big)c_{n_j-k,j} \\ &= \big([\omega] + 1 - \frac{k}{n_j+1}\big)x^{t^{kj}}_\omega, \end{aligned}$$

and

$$E(x^{w^j}_\omega) \;=\; -E(c^\omega_{n_j j}) \;=\; -\Big([\omega] + \frac{n_j+1}{n_j+1}\Big)c^\omega_{n_j j} \;=\; ([\omega]+1)x^{w^j}_\omega.$$

□

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