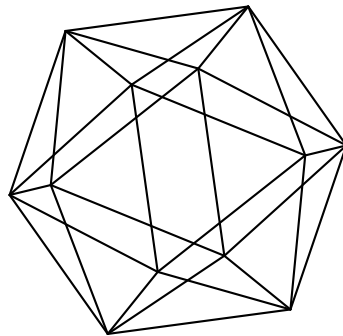


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The Einstein condition on nearly Kähler six-manifolds

by

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Abstract

We review basic facts on the structure of nearly Kähler manifolds, focussing in particular on the six-dimensional case. A self-contained proof that nearly Kähler six-manifolds are Einstein is given by combining different known results. We finally rephrase the definition of nearly Kähler six-manifold in terms of a pair of partial differential equations.

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1 Introduction

An almost Hermitian geometry is a triple (M, g, J) , where M is a $2n$ -dimensional manifold equipped with a Riemannian metric g and an orthogonal almost complex structure J . Denote by ∇ the Levi-Civita connection on M . Lowering the upper index of J yields the fundamental two-form $\sigma := g(J\cdot, \cdot)$. Each tangent space is then a $U(n)$ -module isomorphic to a copy of \mathbb{C}^n with its standard $U(n)$ -structure. In a paper published in 1980, Gray and Hervella [13] showed how to classify such geometries. Take a Euclidean, $2n$ -dimensional vector space (V, g_0) equipped with an orthogonal complex structure J_0 . The triple (V, g_0, J_0) models each tangent space of M . Define \mathcal{W} as the vector space of the type $(3, 0)$ -tensors on V satisfying the same symmetries of $\nabla\sigma$. Using the notation as in Salamon's book [20, Chapter 3],

$$\mathcal{W} := \Lambda^1 \otimes \llbracket \Lambda^{2,0} \rrbracket,$$

where $\llbracket \Lambda^{2,0} \rrbracket$ is the eigenspace of J_0 in $\Lambda^2 V^*$ associated with the eigenvalue -1 and Λ^1 stands for V^* . In general the space \mathcal{W} splits under the action of the unitary group $U(n)$ into the orthogonal direct sum of four irreducible submodules:

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

Consequently, $\nabla\sigma \in \mathcal{W}$ may be decomposed accordingly. Different combinations of its components determine sixteen classes of geometries. A trivial example are Kähler manifolds, which are obtained when $\nabla\sigma = 0$, or equivalently $\nabla J = 0$.

In this work we focus on \mathcal{W}_1 , the class of *nearly Kähler* manifolds. Their formal definition was given by Gray in the 1970s.

Definition 1.1 (Gray [11]). Let (M, g, J) be an almost Hermitian manifold with Riemannian metric g and almost complex structure J compatible with g . Let ∇ denote the Levi-Civita connection on M . Then M is called *nearly Kähler* if $(\nabla_X J)X = 0$ for every vector field X on M .

In the literature one often finds the expression *strict* nearly Kähler for nearly Kähler manifolds that are not Kähler, namely $\nabla J \neq 0$. When we write “nearly Kähler” we mean in fact “strict nearly Kähler”, so as to simplify the terminology. Our manifolds will be always assumed to be connected.

An important ingredient in the general structure theory is the six-dimensional case, as shown by Nagy [17, Theorem 1.1]. On the other hand the classification by Gray and Hervella tells us \mathcal{W}_1 is trivial in dimension two and four (see also the note by Gray [12, Lemma 3]). Therefore we focus on the case where M has dimension six. The lack of explicit six-dimensional, compact examples and the outstanding difficulty in finding new ones has made this geometry particularly exotic and appealing. There are only four homogeneous, compact examples (see [8], [9] on the homogeneous nearly Kähler structure on the six-sphere, and [14], [3] for a classification of the homogeneous examples). In 2017, Foscolo and Haskins [6] proved the existence of the first non-homogeneous nearly Kähler structures in dimension six. Progress in the theory of nearly Kähler six-manifolds with two-torus symmetry was made quite recently in [19], where a new explicit, non-compact example is also given.

Certainly interest in nearly Kähler structures stems from other facts as well, e.g. links with G_2 and spin geometry (see [2], [7], [1]). One may refer to [18] for a comprehensive survey. Here we concentrate on

Theorem 1.2. *Nearly Kähler six-manifolds are Einstein with positive scalar curvature.*

This is a deep result proved first by Gray in 1976 [11, Theorem 5.2] and then studied again by Carrión [4], Morris [16]. Friedrich and Grunewald [7], [15] proved that on nearly Kähler six-manifolds there exists a Killing spinor, which implies Theorem 1.2. In [4] it is shown that in dimension six there is an equivalence between Definition 1.1 and a system of PDEs in terms of an $SU(3)$ -structure, whence

Theorem 1.3. *Let (M, g, J) be an almost Hermitian six-manifold and σ be the fundamental two-form on M . Then M is nearly Kähler if and only if there exist a constant function μ on M and a complex $(3, 0)$ -form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$ such that*

$$d\sigma = 3\mu\psi_+, \quad d\psi_- = -2\mu\sigma \wedge \sigma. \quad (1.1)$$

The function μ appears when computing the norm of any vector field of the form $(\nabla_X J)Y$, in fact we will show exactly how. That μ is constant is closely related to the Einstein condition. This seems to be a delicate issue in the literature. The main ideas behind it are scattered in essentially two works by Gray [10], [11], but the massive—though impressive—amount of technical formulas obscures the key steps. A clearer approach was pursued by Morris, who nonetheless seems to gloss over the details of a crucial step (see in particular the proof of formula (4.15), Section 4.2 in [16], where there is no explanation of the fact that β sits inside $\text{Sym}^2(\Lambda^2 TM^*)$). What is more, Carrión discusses the equivalence stated in Theorem 1.3, but does not provide any direct proof of the fact that μ is constant: this point is claimed to be a consequence of Theorem 4.20 in [4].

The goal of this paper is to go again through the proof that nearly Kähler metrics in dimension six are Einstein, hoping to provide people interested in this field with a unifying reference. We keep a basic approach, in particular we do not make use of the existence of Killing spinors. We describe symmetries and introduce useful curvature identities combining the results of Gray and Morris’, thus giving a complete proof of Theorem 1.2. Finally, we expand the outcome with a self-contained proof of Theorem 1.3, hence rephrase Definition 1.1 in terms of the PDEs (1.1).

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2 Symmetries

Let us start off with a nearly Kähler manifold (M, g, J) as in Definition 1.1. We assume throughout M to be connected. This is not restrictive and will simplify some parts of the exposition. Define the two-form $\sigma := g(J \cdot, \cdot)$ and let ∇ be the Levi-Civita connection. We work without specifying the dimension of M and switch to the six-dimensional case only when needed.

Lemma 2.1. *For each triple U, V, Z of vector fields on M we have*

$$\nabla\sigma(U, V, Z) = g((\nabla_U J)V, Z). \quad (2.1)$$

Further, we can move J across all the entries of $\nabla\sigma$:

$$\nabla\sigma(JU, V, Z) = \nabla\sigma(U, JV, Z) = \nabla\sigma(U, V, JZ). \quad (2.2)$$

Proof. Recall that g is ∇ -parallel. For each triple U, V, Z of vector fields one has

$$\begin{aligned} \nabla\sigma(U, V, Z) &= U(g(JV, Z)) - g(J\nabla_U V, Z) - g(JV, \nabla_U Z) \\ &= g(\nabla_U JV, Z) - g(J\nabla_U V, Z) \\ &= g((\nabla_U J)V, Z). \end{aligned}$$

Now $0 = (\nabla J^2) = (\nabla J)J + J(\nabla J)$, so J and ∇J anti-commute, whence $(\nabla_{JX} J)Y = -(\nabla_Y J)JX = J(\nabla_Y J)X = -J(\nabla_X J)Y$. Therefore $\nabla\sigma(JX, Y, Z) = g((\nabla_{JX} J)Y, Z) = -g(J(\nabla_X J)Y, Z)$. But J is orthogonal, thus the latter equals $\nabla\sigma(X, Y, JZ)$. On the other hand $-g(J(\nabla_X J)Y, Z)$ coincides with $g((\nabla_X J)JY, Z) = \nabla\sigma(X, JY, Z)$ as well. \square

Remark 2.2. Since M is nearly Kähler, $\nabla\sigma$ is a three-form, and $(\nabla_X J)Y$ is orthogonal to X, Y, JX, JY . Conversely, if we assume $\nabla\sigma$ to be skew-symmetric, then $\nabla\sigma(X, X, Y) = g((\nabla_X J)X, Y) = 0$ for every Y , so M is nearly Kähler.

For the rest of this section we assume M has dimension six. The main intention here is to provide a unifying language to describe symmetries of useful tensors. We refer to Salamon [20] for notations and ideas. Recall the identity of Lie groups

$$U(n) = SO(2n) \cap GL(n, \mathbb{C}).$$

In real dimension six this tells us $U(3)$ is the stabiliser in $GL(6, \mathbb{R})$ of an inner product and a complex structure J_0 on a copy of \mathbb{R}^6 . At the level of Lie algebras, this identity implies in particular that elements of $\mathfrak{u}(3)$ commute with J_0 . We shall always think of $U(3)$ as a subgroup of $SO(6)$. At each point of M there is a representation of $U(3)$ on the tangent space inducing the structure of $U(3)$ -module on the complexified vector space of k -forms, which we denote simply by $\Lambda^k \otimes \mathbb{C}$. Note that every orthogonal matrix coincides with the transpose of its inverse, so the $U(3)$ -modules $\Lambda^k T_p M^*$ and $\Lambda^k T_p M$ are equivalent, and one loses no information in identifying k -forms and k -vectors. This explains the choice of the symbol Λ^k for the space of real k -forms, and will allow us to identify $U(3)$ -modules and their duals in other circumstances. There is an isomorphism of vector spaces

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1}),$$

and by definition $\Lambda^{p,q} := \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1})$ is the space of complex differential forms of type (p, q) . Each $\Lambda^{p,q}$ is a $U(3)$ -invariant complex module.

For $p \neq q$ we denote by $[\Lambda^{p,q}]$ the real vector space underlying $\Lambda^{p,q}$, whose complexification is $[\Lambda^{p,q}] \otimes \mathbb{C} = \Lambda^{p,q} \oplus \Lambda^{q,p}$, whereas $[\Lambda^{p,p}]$ is the space of type (p, p) -forms α such that $\bar{\alpha} = \alpha$, hence $[\Lambda^{p,p}] \otimes \mathbb{C} = \Lambda^{p,p}$. We then have isomorphisms of $U(3)$ -modules such as

$$\Lambda^1 = [\Lambda^{1,0}], \quad \Lambda^2 = [\Lambda^{2,0}] \oplus [\Lambda^{1,1}], \quad \Lambda^3 = [\Lambda^{3,0}] \oplus [\Lambda^{2,1}], \text{ etc.}$$

Each real form of type $(p, q) + (q, p)$ satisfies a specific relation with J . To show this, we specialise to the cases $k = 2$ and $k = 3$.

At every point of M , the metric g yields a canonical isomorphism $\mathfrak{so}(6) = \Lambda^2$, which is obtained by mapping each A in $\mathfrak{so}(6)$ to the two-form $g(A \cdot, \cdot)$. Viewing $\mathfrak{so}(6)$ as the adjoint representation of $\mathrm{SO}(6) \supset \mathrm{U}(3)$, we have actually got an isomorphism of $\mathrm{U}(3)$ -modules: for $A \in \mathfrak{so}(6)$ and $B \in \mathrm{U}(3)$, the action of B on two-forms gives

$$Bg(A \cdot, \cdot) = g(AB^{-1} \cdot, B^{-1} \cdot) = g(BAB^{-1} \cdot, \cdot),$$

so the map $A \mapsto g(A \cdot, \cdot)$ is $\mathrm{U}(3)$ -equivariant and our claim follows. Now consider the splitting $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^\perp$, where $\mathfrak{u}(3)^\perp$ is the orthogonal complement of $\mathfrak{u}(3)$ as a subspace of $\mathfrak{so}(6)$. Any endomorphism A in $\mathfrak{u}(3)$ corresponds to a two-form $\alpha = g(A \cdot, \cdot)$ such that $\alpha(JX, JY) = \alpha(X, Y)$: since A and J commute

$$\alpha(JX, JY) = g(AJX, JY) = g(JAX, JY) = g(AX, Y) = \alpha(X, Y). \quad (2.3)$$

On the other hand, a two-form β in $[\Lambda^{1,1}]$ is defined so as to vanish on pairs of complex vectors of the same type, namely $\beta(X - iJX, Y - iJY) = 0$. Thus $\beta(JX, JY) = \beta(X, Y)$, and by counting dimensions the following splittings are equivalent:

$$\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^\perp, \quad \Lambda^2 = [\Lambda^{1,1}] \oplus \llbracket \Lambda^{2,0} \rrbracket.$$

We have already encountered a two-form enjoying the property of elements in $[\Lambda^{1,1}]$, that is the fundamental two-form σ . Identity (2.3) is readily checked:

$$\sigma(JX, JY) = -g(X, JY) = g(JX, Y) = \sigma(X, Y).$$

Elements of $[\Lambda^{1,1}]$ are then eigenvectors of J with eigenvalue $+1$. Likewise, elements of $\llbracket \Lambda^{2,0} \rrbracket$ are eigenvectors of J with eigenvalue -1 , which is now trivial to check.

Identity (2.2) implies

$$\nabla\sigma(U, JV, JZ) = \nabla\sigma(U, V, J^2Z) = -\nabla\sigma(U, V, Z). \quad (2.4)$$

which we may then rephrase by saying $\nabla\sigma$ sits inside $\Lambda^1 \otimes \llbracket \Lambda^{2,0} \rrbracket$. Further, since M is nearly Kähler, $\nabla\sigma$ actually takes values in $\llbracket \Lambda^{3,0} \rrbracket$. Recall that $\nabla\sigma$ is skew-symmetric, so (2.4) implies

$$\nabla\sigma(U, V, Z) = \nabla\sigma(JU, JV, Z) - \nabla\sigma(U, JV, JZ) - \nabla\sigma(JU, V, JZ).$$

On the other hand this is the characteristic property of elements in $\llbracket \Lambda^{3,0} \rrbracket$: given $\beta \in \llbracket \Lambda^{3,0} \rrbracket$ one has $\beta(X - iJX, Y - iJY, Z + iJZ) = 0$, that is

$$\beta(X, Y, Z) = \beta(JX, JY, Z) - \beta(X, JY, JZ) - \beta(JX, Y, JZ),$$

whence $\nabla\sigma \in \llbracket \Lambda^{3,0} \rrbracket$.

A last observation is motivated by Remark 2.2. Since $\nabla\sigma$ is a three-form, there is a relation between $d\sigma$ and $\nabla\sigma$. This is readily worked out, as $d\sigma = \mathcal{A}\nabla\sigma$, where $(\mathcal{A}\nabla\sigma)(X, Y, Z) := \mathfrak{S}_{X,Y,Z} \nabla\sigma(X, Y, Z)$, and $\nabla\sigma$ skew-symmetric implies $d\sigma = 3\nabla\sigma$. Conversely, if $d\sigma = 3\nabla\sigma$, then Remark 2.2 shows M is nearly Kähler. We summarise all our observations in

Proposition 2.3. *Assume (M, g, J) is an almost Hermitian six-manifold and let $\sigma = g(J \cdot, \cdot)$ be the fundamental two-form. Then the following are equivalent:*

1. M is nearly Kähler.
2. $\nabla\sigma \in \llbracket \Lambda^{3,0} \rrbracket$.
3. $d\sigma = 3\nabla\sigma$.

3 Curvature identities

By Remark 2.2, each tangent space of M splits as the orthogonal direct sum of three J -invariant planes

$$\langle X, JX \rangle \oplus \langle Y, JY \rangle \oplus \langle (\nabla_X J)Y, J(\nabla_X J)Y \rangle,$$

where angular brackets denote the real vector space spanned by a pair of vectors and Y is orthogonal to the span of X and JX . The following technical lemma states the existence of a special function on M relating the norm of $(\nabla_X J)Y$ with $\|X\|$, $\|Y\|$, $g(X, Y)$, and $\sigma(X, Y)$.

Lemma 3.1. *There exists a non-negative function μ on M such that*

$$\|(\nabla_X J)Y\|^2 = \mu^2(\|X\|^2\|Y\|^2 - g(X, Y)^2 - \sigma(X, Y)^2) \quad (3.1)$$

for every pair of vector fields X, Y on M .

Proof. We define μ in terms of a local frame, then we extend it to a global function. Given X and Y in a neighbourhood of a point there exists an orthonormal frame $\{E_i, JE_i\}, i = 1, 2, 3$, such that $X = aE_1$ and $Y = bE_1 + cJE_1 + dE_2$ for local functions a, b, c, d . Define μ by $(\nabla_{E_1} J)E_2 =: \mu E_3$. We may assume μ non-negative up to changing the orientation of the basis. Then $(\nabla_X J)Y = a(\nabla_{E_1} J)dE_2 = ad\mu E_3$, which implies $\|(\nabla_X J)Y\|^2 = a^2d^2\mu^2$. On the other hand, $\|X\|^2\|Y\|^2 = a^2(b^2 + c^2 + d^2)$, $g(X, Y)^2 = a^2b^2$, and $\sigma(X, Y)^2 = a^2c^2$. So $\mu^2(\|X\|^2\|Y\|^2 - g(X, Y)^2 - \sigma(X, Y)^2) = a^2d^2\mu^2$, and the formula is proved locally.

We can finally extend μ to a global function by imposing that (3.1) be satisfied for all pairs of vector fields X, Y on M . \square

We shall now study how μ is related to the Riemannian and the Ricci tensors on M , and finally prove that μ is *constant*. A first step in this direction is to consider second order covariant derivatives of σ and study their symmetries. We follow [11] and [16] for this part. We will sometimes use the notation $\mathfrak{X}(M)$ for the Lie algebra of vector fields on M . The Riemannian curvature tensors of type (3, 1) and (4, 0) will be denoted by the same letter.

Lemma 3.2. *Let $R \in \Lambda^2 \otimes \mathfrak{so}(2n)$ be the type (3, 1) Riemannian curvature tensor of the Levi-Civita connection on M , given by $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$. The following identities hold for every quadruple of vector fields W, X, Y, Z on M :*

1. $\nabla^2 \sigma(W, X, Y, Z) - \nabla^2 \sigma(X, W, Y, Z) = \sigma(R(X, W)Y, Z) + \sigma(Y, R(X, W)Z)$.
2. $\nabla^2 \sigma(X, X, JY, Y) = \|(\nabla_X J)Y\|^2$.

Proof. To prove the first formula we start by expanding the first term:

$$\begin{aligned} \nabla^2 \sigma(W, X, Y, Z) &= W(\nabla \sigma(X, Y, Z)) - \nabla \sigma(\nabla_W X, Y, Z) \\ &\quad - \nabla \sigma(X, \nabla_W Y, Z) - \nabla \sigma(X, Y, \nabla_W Z) \\ &= g(\nabla_W((\nabla_X J)Y), Z) + g((\nabla_X J)Y, \nabla_W Z) - g((\nabla_{\nabla_W X} J)Y, Z) \\ &\quad - g((\nabla_X J)\nabla_W Y, Z) - g((\nabla_X J)Y, \nabla_W Z) \\ &= g((\nabla_W(\nabla_X J))Y, Z) - g((\nabla_{\nabla_W X} J)Y, Z). \end{aligned}$$

As an element of $\mathfrak{so}(2n)$, $R(W, X)$ is a skew-adjoint derivation. We can then rewrite the difference $\nabla^2 \sigma(W, X, Y, Z) - \nabla^2 \sigma(X, W, Y, Z)$ as

$$\begin{aligned} g((R(W, X)J)Y, Z) &= g(R(W, X)JY, Z) - g(JR(W, X)Y, Z) \\ &= -g(JY, R(W, X)Z) - g(JR(W, X)Y, Z) \\ &= \sigma(Y, R(X, W)Z) + \sigma(R(X, W)Y, Z). \end{aligned}$$

In order to prove the second formula we make use of (2.2):

$$\begin{aligned}
\nabla^2\sigma(X, X, JY, Y) &= X(\nabla\sigma(X, JY, Y)) - \nabla\sigma(\nabla_X X, JY, Y) \\
&\quad - \nabla\sigma(X, \nabla_X JY, Y) - \nabla\sigma(X, JY, \nabla_X Y) \\
&= X(\nabla\sigma(JX, Y, Y)) - \nabla\sigma(J\nabla_X X, Y, Y) \\
&\quad - g((\nabla_X J)\nabla_X JY, Y) - g((\nabla_X J)JY, \nabla_X Y) \\
&= g(\nabla_X JY, (\nabla_X J)Y) - g((\nabla_X J)JY, \nabla_X Y) \\
&= g((\nabla_X J)Y, (\nabla_X J)Y) + g(J\nabla_X Y, (\nabla_X J)Y) \\
&\quad - g((\nabla_X J)JY, \nabla_X Y) \\
&= \|(\nabla_X J)Y\|^2,
\end{aligned}$$

and the statement is proved. \square

Lemma 3.3. *Let $R \in \text{Sym}^2(\Lambda^2)$ be the Riemannian curvature (4,0)-tensor obtained by contraction with the metric: $R(W, X, Y, Z) := g(R(W, X)Y, Z)$. Then*

$$\|(\nabla_X J)Y\|^2 = R(X, Y, JX, JY) - R(X, Y, X, Y), \quad X, Y \in \mathfrak{X}(M). \quad (3.2)$$

Proof. Since $\nabla\sigma$ is a three-form, $\nabla^2\sigma(A, B, B, C) = 0$. We can then combine the results found in Lemma 3.2 to get

$$\begin{aligned}
\|(\nabla_X J)Y\|^2 &= \|(\nabla_X J)JY\|^2 = -\nabla^2\sigma(X, X, Y, JY) \\
&= \nabla^2\sigma(X, Y, X, JY) - \nabla^2\sigma(Y, X, X, JY) \\
&= \sigma(R(Y, X)X, JY) + \sigma(X, R(Y, X)JY) \\
&= g(R(Y, X)X, Y) - g(R(Y, X)JX, JY) \\
&= R(X, Y, JX, JY) - R(X, Y, X, Y),
\end{aligned}$$

which was our claim. \square

Formula (3.2) gives a way to calculate the norm of $(\nabla_X J)Y$ —hence the function μ in (3.1)—in terms of the curvature tensor. A remarkable consequence of it is that R is invariant under the action of J . To see this, define the tensor $S(W, X, Y, Z) := R(JW, JX, JY, JZ)$. Of course S inherits the properties of algebraic curvature tensors, namely $S \in \Lambda^2 \otimes \Lambda^2$ satisfies the first Bianchi identity. To show $R = S$ we can then check $R(X, Y, Y, X) = S(X, Y, Y, X)$. A straightforward calculation proves the claim:

$$\begin{aligned}
R(JX, JY, JY, JX) - R(X, Y, Y, X) &= R(JX, JY, JY, JX) - R(X, Y, JY, JX) \\
&\quad + R(X, Y, JY, JX) - R(X, Y, Y, X) \\
&= \|(\nabla_{JX} J)JY\|^2 - \|(\nabla_X J)Y\|^2 = 0.
\end{aligned}$$

The identity just obtained allows us to carry out a polarisation process giving a way to measure inner products of vectors of the form $(\nabla_X J)Y$ in terms of the curvature. We work out all the details of the next essential result.

Lemma 3.4. *For every quadruple of vector fields W, X, Y, Z on M we have the formula*

$$g((\nabla_W J)X, (\nabla_Y J)Z) = R(W, X, JY, JZ) - R(W, X, Y, Z). \quad (3.3)$$

Proof. Mapping $X \mapsto A + B$ in formula (3.2) one has

$$\begin{aligned}
\|(\nabla_{A+B} J)Y\|^2 &= R(A + B, Y, JA + JB, JY) - R(A + B, Y, A + B, Y) \\
&= R(A, Y, JA, JY) - R(A, Y, A, Y) + R(B, Y, JB, JY) - R(B, Y, B, Y) \\
&\quad + R(A, Y, JB, JY) - R(A, Y, B, Y) + R(B, Y, JA, JY) - R(B, Y, A, Y).
\end{aligned}$$

The left hand side is

$$\|(\nabla_{A+B}J)Y\|^2 = \|(\nabla_AJ)Y\|^2 + \|(\nabla_BJ)Y\|^2 + 2g((\nabla_AJ)Y, (\nabla_BJ)Y),$$

so applying once again (3.2) we find

$$\begin{aligned} 2g((\nabla_AJ)Y, (\nabla_BJ)Y) &= R(A, Y, JB, JY) - R(A, Y, B, Y) \\ &\quad + R(B, Y, JA, JY) - R(B, Y, A, Y). \end{aligned}$$

Putting now $Y \mapsto C + D$, we expand $2g((\nabla_AJ)(C + D), (\nabla_BJ)(C + D))$ and obtain the expression

$$\begin{aligned} &R(A, C, JB, JC) - R(A, C, B, C) + R(A, C, JB, JD) - R(A, C, B, D) \\ &+ R(A, D, JB, JC) - R(A, D, B, C) + R(A, D, JB, JD) - R(A, D, B, D) \\ &+ R(B, C, JA, JC) - R(B, C, A, C) + R(B, C, JA, JD) - R(B, C, A, D) \\ &+ R(B, D, JA, JC) - R(B, D, A, C) + R(B, D, JA, JD) - R(B, D, A, D). \end{aligned}$$

Linearity in the various arguments implies

$$\begin{aligned} &2g((\nabla_AJ)(C + D), (\nabla_BJ)(C + D)) \\ &= 2(g((\nabla_AJ)C, (\nabla_BJ)C) + g((\nabla_AJ)C, (\nabla_BJ)D) \\ &\quad + g((\nabla_AJ)D, (\nabla_BJ)C) + g((\nabla_AJ)D, (\nabla_BJ)D)). \end{aligned}$$

Simplifying we are left with

$$\begin{aligned} &g((\nabla_AJ)C, (\nabla_BJ)D) + g((\nabla_AJ)D, (\nabla_BJ)C) \\ &= R(A, C, JB, JD) - R(A, C, B, D) + R(A, D, JB, JC) - R(A, D, B, C). \end{aligned} \quad (3.4)$$

Set $L(A, B, C, D) := R(A, B, C, D) + R(A, D, C, B)$. The first Bianchi identity, together with (3.4), gives

$$\begin{aligned} 0 &= R(A, B, C, D) + R(B, C, A, D) + R(C, A, B, D) \\ &= R(A, B, C, D) - R(C, B, A, D) + (L(C, A, B, D) - R(C, D, B, A)) \\ &= R(A, B, C, D) + (L(C, A, B, D) + R(A, B, C, D)) - (L(C, B, A, D) - R(C, D, A, B)) \\ &= 3R(A, B, C, D) + L(C, A, B, D) - L(C, B, A, D) \\ &= 3R(A, B, C, D) + (R(C, A, B, D) + R(C, D, B, A)) - (R(C, B, A, D) + R(C, D, A, B)) \\ &= 3R(A, B, C, D) - 2R(C, D, JA, JB) + R(C, A, JB, JD) - R(C, B, JA, JD) \\ &\quad + 2g((\nabla_CJ)D, (\nabla_AJ)B) - g((\nabla_CJ)A, (\nabla_BJ)D) + g((\nabla_CJ)B, (\nabla_AJ)D). \end{aligned} \quad (3.5)$$

Now we set $C \mapsto JC, D \mapsto JD$:

$$\begin{aligned} 0 &= 3R(A, B, JC, JD) - 2R(JC, JD, JA, JB) - R(JC, A, JB, D) + R(JC, B, JA, D) \\ &\quad + 2g((\nabla_{JC}J)JD, (\nabla_AJ)B) - g((\nabla_{JC}J)A, (\nabla_BJ)JD) + g((\nabla_{JC}J)B, (\nabla_AJ)JD). \end{aligned}$$

Using that R is J -invariant, the difference between the latter and (3.5) becomes

$$\begin{aligned} 0 &= 3R(A, B, JC, JD) - 2R(JC, JD, JA, JB) - R(JC, A, JB, D) + R(JC, B, JA, D) \\ &\quad + 2g((\nabla_{JC}J)JD, (\nabla_AJ)B) - g((\nabla_{JC}J)A, (\nabla_BJ)JD) + g((\nabla_{JC}J)B, (\nabla_AJ)JD) \\ &\quad - 3R(A, B, C, D) + 2R(C, D, JA, JB) - R(C, A, JB, JD) + R(C, B, JA, JD) \\ &\quad - 2g((\nabla_CJ)D, (\nabla_AJ)B) + g((\nabla_CJ)A, (\nabla_BJ)D) - g((\nabla_CJ)B, (\nabla_AJ)D) \\ &= 5R(A, B, JC, JD) - 5R(A, B, C, D) - R(A, C, JD, JB) - R(A, JC, D, JB) \\ &\quad - R(A, D, JB, JC) - R(A, JD, JB, C) - 4g((\nabla_AJ)B, (\nabla_CJ)D). \end{aligned}$$

Applying the first Bianchi identity once again we have

$$4g((\nabla_A J)B, (\nabla_C J)D) = 5R(A, B, JC, JD) - 5R(A, B, C, D) + R(A, JB, C, JD) + R(A, JB, JC, D). \quad (3.6)$$

Now map $B \mapsto JB, C \mapsto JC$ and add a fifth of the result to (3.6):

$$\begin{aligned} \frac{24}{5}g((\nabla_A J)JB, (\nabla_{JC} J)D) &= -R(A, JB, C, JD) - R(A, JB, JC, D) \\ &\quad - \frac{1}{5}R(A, B, JC, JD) + \frac{1}{5}R(A, B, C, D) \\ &\quad + 5R(A, B, JC, JD) - 5R(A, B, C, D) \\ &\quad + R(A, JB, C, JD) + R(A, JB, JC, D) \\ &= \frac{24}{5}R(A, B, JC, JD) - \frac{24}{5}R(A, B, C, D). \end{aligned}$$

Since $g((\nabla_A J)JB, (\nabla_{JC} J)D) = g((\nabla_A J)B, (\nabla_C J)D)$ we are done. \square

Lemma 3.5. *Let $W, X, Y, Z \in \mathfrak{X}(M)$. The following formula holds:*

$$2\nabla^2\sigma(W, X, Y, Z) = - \sum_{X, Y, Z} g((\nabla_W J)X, (\nabla_Y J)JZ). \quad (3.7)$$

Proof. Combine the first formula in Lemma 3.2 and identity (3.3):

$$\begin{aligned} \nabla^2\sigma(W, X, Y, Z) - \nabla^2\sigma(X, W, Y, Z) &= g(JR(X, W)Y, Z) + g(JY, R(X, W)Z) \\ &= g(JR(X, W)Y, Z) - g(R(X, W)JY, Z) \\ &= g((R(X, W)J)Y, J^2Z) \\ &= R(X, W, JY, J^2Z) - R(X, W, Y, JZ) \\ &= g((\nabla_X J)W, (\nabla_Y J)JZ). \end{aligned} \quad (3.8)$$

On the other hand

$$\begin{aligned} \nabla^2\sigma(W, W, Y, Z) &= -\nabla^2\sigma(W, Y, W, Z) \\ &= \nabla^2\sigma(Y, W, W, Z) - \nabla^2\sigma(W, Y, W, Z) \\ &= g((\nabla_W J)Y, (\nabla_W J)JZ). \end{aligned}$$

Polarising the latter, one obtains

$$\begin{aligned} \nabla^2\sigma(W + X, W + X, Y, Z) &= \nabla^2\sigma(W, W, Y, Z) + \nabla^2\sigma(W, X, Y, Z) \\ &\quad + \nabla^2\sigma(X, W, Y, Z) + \nabla^2\sigma(X, X, Y, Z) \\ &= g((\nabla_W J)Y, (\nabla_W J)JZ) + g((\nabla_X J)Y, (\nabla_X J)JZ) \\ &\quad + \nabla^2\sigma(W, X, Y, Z) + \nabla^2\sigma(X, W, Y, Z), \end{aligned}$$

whence

$$\begin{aligned} \nabla^2\sigma(W, X, Y, Z) + \nabla^2\sigma(X, W, Y, Z) &= -g((\nabla_W J)Y, (\nabla_W J)JZ) - g((\nabla_X J)Y, (\nabla_X J)JZ) + g((\nabla_{W+X} J)Y, (\nabla_{W+X} J)JZ) \\ &= g((\nabla_W J)Y, (\nabla_X J)JZ) + g((\nabla_X J)Y, (\nabla_W J)JZ). \end{aligned} \quad (3.9)$$

Adding (3.8) to (3.9) and using usual symmetries of ∇J the claim follows. \square

Now we define the Ricci and the Ricci-* endomorphisms. We still work in dimension $2n$, switching to dimension six in Proposition 3.8.

Definition 3.6. Given any local, orthonormal frame E_1, \dots, E_{2n} , the Ricci and the Ricci-* endomorphisms $\text{Ric}, \text{Ric}^* \in \Lambda^1 \otimes \Lambda^1$ are given by

$$g(\text{Ric}X, Y) := \sum_{i=1}^{2n} R(X, E_i, E_i, Y), \quad g(\text{Ric}^*X, Y) := \sum_{i=1}^{2n} R(X, E_i, JE_i, JY).$$

Because of (3.3) we can write their difference as

$$g((\text{Ric} - \text{Ric}^*)X, Y) = \sum_{i=1}^{2n} g((\nabla_X J)E_i, (\nabla_Y J)E_i). \quad (3.10)$$

Obviously $\text{Ric} - \text{Ric}^*$ is self-adjoint, and so is its covariant derivative. Moreover, $\text{Ric} - \text{Ric}^*$ and J commute: set $A := \text{Ric} - \text{Ric}^*$, so that

$$\begin{aligned} g(JAX, Y) &= -g(AX, JY) \\ &= -\sum_i g((\nabla_X J)E_i, (\nabla_{JY} J)E_i) = -\sum_i g(J(\nabla_X J)E_i, (\nabla_Y J)E_i) \\ &= \sum_i g((\nabla_{JX} J)E_i, (\nabla_Y J)E_i) = g(AJX, Y). \end{aligned}$$

We can then prove a last useful result.

Lemma 3.7. For $X, Y, Z \in \mathfrak{X}(M)$ we have the following formula:

$$\begin{aligned} 2g((\nabla_Z(\text{Ric} - \text{Ric}^*))X, Y) &= g((\text{Ric} - \text{Ric}^*)JX, (\nabla_Z J)Y) \\ &\quad + g((\text{Ric} - \text{Ric}^*)JY, (\nabla_Z J)X). \end{aligned} \quad (3.11)$$

Proof. Start differentiating (3.10) with $X = Y$, still with $A := \text{Ric} - \text{Ric}^*$:

$$\begin{aligned} g((\nabla_Z A)X, X) + 2g(AX, \nabla_Z X) &= Z(g(AX, X)) \\ &= 2 \sum_{i=1}^{2n} g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i). \end{aligned}$$

Rearranging the terms

$$g((\nabla_Z A)X, X) = 2 \sum_{i=1}^{2n} g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i). \quad (3.12)$$

Note that $\sum_{i=1}^{2n} g((\nabla_X J)\nabla_Z E_i, (\nabla_X J)E_i) = 0$: setting $\nabla_Z E_i = \sum_{j=1}^{2n} B_i^j E_j$ we have

$$0 = Z(g(E_i, E_j)) = g(\nabla_Z E_i, E_j) + g(E_i, \nabla_Z E_j) = \sum_k B_i^k \delta_{kj} + \sum_r B_j^r \delta_{ir} = B_i^j + B_j^i.$$

Thus

$$\begin{aligned} \sum_i g((\nabla_X J)\nabla_Z E_i, (\nabla_X J)E_i) &= \sum_{i,j} g((\nabla_X J)B_i^j E_j, (\nabla_X J)E_i) \\ &= -\sum_{i,j} g((\nabla_X J)E_j, (\nabla_X J)B_j^i E_i) \\ &= -\sum_j g((\nabla_X J)E_j, (\nabla_X J)\nabla_Z E_j) = 0. \end{aligned}$$

This last term appears in the expansion of $\nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i)$ as well. Simplifying we get

$$\begin{aligned}
\nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i) &= -Z(g((\nabla_X J)E_i, (\nabla_X J)E_i)) - g((\nabla_{\nabla_Z X} J)(\nabla_X J)E_i, E_i) \\
&\quad - g((\nabla_X J)\nabla_Z((\nabla_X J)E_i), E_i) - g((\nabla_X J)(\nabla_X J)E_i, \nabla_Z E_i) \\
&= -2g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) + g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i) \\
&\quad + g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)(\nabla_X J)E_i, \nabla_Z E_i) \\
&= g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i) - g((\nabla_Z(\nabla_X J))E_i, (\nabla_X J)E_i) \\
&\quad + g((\nabla_X J)E_i, (\nabla_X J)\nabla_Z E_i).
\end{aligned}$$

Therefore, by formula (3.7), identity (3.12) becomes (all sums are over $i = 1, \dots, 2n$)

$$\begin{aligned}
&g((\nabla_Z(\text{Ric} - \text{Ric}^*))X, X) \\
&= 2 \sum g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i) \\
&= -2 \sum \nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i) \\
&= \sum g((\nabla_Z J)X, (\nabla_{(\nabla_X J)E_i} J)E_i) + g((\nabla_Z J)(\nabla_X J)E_i, (\nabla_{E_i} J)JX) \\
&\quad + g((\nabla_Z J)E_i, (\nabla_X J)J(\nabla_X J)E_i) \\
&= \sum g((\nabla_{E_i} J)(\nabla_Z J)X, J(\nabla_X J)E_i) + g((\nabla_Z J)(\nabla_X J)E_i, J(\nabla_X J)E_i) \\
&\quad + g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)J(\nabla_X J)E_i).
\end{aligned}$$

The second term in the latter sum vanishes by (2.2). The sum $\sum g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)J(\nabla_X J)E_i)$ vanishes as well. To see this, we set $C := J(\nabla_Z J)$. In the first place C lies in $\mathfrak{so}(2n)$, because

$$g(J(\nabla_Z J)E_i, E_j) = g((\nabla_Z J)J(\nabla_X J)E_i, E_j) = -g(J(\nabla_Z J)E_j, E_i).$$

Consequently, the following chain of identities leads to our claim (indices i, j vary from 1 to $2n$):

$$\begin{aligned}
\sum g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)J(\nabla_X J)E_i) &= - \sum g((\nabla_X J)J(\nabla_Z J)E_i, (\nabla_X J)E_i) \\
&= - \sum g((\nabla_X J)C_i^j E_j, (\nabla_X J)E_i) \\
&= \sum g((\nabla_X J)E_j, (\nabla_X J)C_j^i E_i) \\
&= \sum g((\nabla_X J)J(\nabla_Z J)E_j, (\nabla_X J)E_j) = 0.
\end{aligned}$$

We then go back to our first expansion recalling that $\text{Ric} - \text{Ric}^*$ commutes with J .

$$\begin{aligned}
-2 \sum \nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i) &= \sum g((\nabla_{E_i} J)(\nabla_Z J)X, J(\nabla_X J)E_i) \\
&= - \sum g((\nabla_{J(\nabla_Z J)X} J)E_i, (\nabla_X J)E_i) \\
&= -g((\text{Ric} - \text{Ric}^*)J(\nabla_Z J)X, X) \\
&= g((\text{Ric} - \text{Ric}^*)JX, (\nabla_Z J)X).
\end{aligned}$$

Thus $g((\nabla_Z(\text{Ric} - \text{Ric}^*))X, X) = g((\text{Ric} - \text{Ric}^*)JX, (\nabla_Z J)X)$. By polarisation and the symmetry of $\nabla_Z(\text{Ric} - \text{Ric}^*)$ the result follows. \square

Let us restrict to the six-dimensional case now, so $n = 3$. Recall that in Lemma 3.1 we proved the existence of a special function μ on M satisfying (3.1).

Proposition 3.8. *If M is a nearly Kähler six-manifold, the function μ is constant.*

Proof. We only prove μ is locally constant, then the claim follows from the connectedness of M . Mapping X into $A + B$ in (3.1) one has

$$g((\nabla_{A+B}J)Y, (\nabla_{A+B}J)Y) = \mu^2(\|A + B\|^2\|Y\|^2 - g(A + B, Y)^2 - \sigma(A + B, Y)^2),$$

which can be simplified as

$$g((\nabla_A J)Y, (\nabla_B J)Y) = \mu^2(g(A, B)\|Y\|^2 - g(A, Y)g(B, Y) - g(JA, Y)g(JB, Y)).$$

On the other hand

$$\begin{aligned} g((\text{Ric} - \text{Ric}^*)A, B) &= \sum_{i=1}^3 g((\nabla_A J)E_i, (\nabla_B J)E_i) + g((\nabla_A J)JE_i, (\nabla_B J)JE_i) \\ &= \mu^2(6g(A, B) - g(A, B) - g(JA, JB)) = 4\mu^2g(A, B). \end{aligned}$$

Thus $\text{Ric} - \text{Ric}^* = 4\mu^2 \text{Id}$, but now formula (3.11) implies

$$\begin{aligned} 2g((\nabla_Z(\text{Ric} - \text{Ric}^*))X, Y) &= g((\text{Ric} - \text{Ric}^*)JX, (\nabla_Z J)Y) + g((\text{Ric} - \text{Ric}^*)JY, (\nabla_Z J)X) \\ &= 4\mu^2(g(JX, (\nabla_Z J)Y) + g(JY, (\nabla_Z J)X)) = 0. \end{aligned}$$

This proves $\nabla_Z(\text{Ric} - \text{Ric}^*) = 0 = 4Z(\mu^2) \text{Id}$ for every Z , hence μ is locally constant. \square

We have thus proved that on connected nearly Kähler six-manifolds there exists a constant μ such that

$$\|(\nabla_X J)Y\|^2 = \mu^2(\|X\|^2\|Y\|^2 - g(X, Y)^2 - \sigma(X, Y)^2), \quad X, Y \in \mathfrak{X}(M).$$

Observe μ cannot vanish because of the nearly Kähler condition, so we assume it to be positive according to Lemma 3.1. Using the terminology introduced by Gray [10, Proposition 3.5] we say that connected nearly Kähler six-manifolds have *global constant type*.

4 The Einstein condition

The aim of this section is to push our calculations further in order to prove that nearly Kähler six-manifolds are Einstein. We follow [11] to do this. We first introduce a connection adapted to the $U(3)$ -structure (g, J) . A quick computation of the torsion of J will help us go smoothly towards it. We then work out some relevant symmetries satisfied by the curvature tensor of the new connection. We conclude proving that $\text{Ric}_g = 5\mu^2g$, where Ric_g is the Ricci curvature $(2, 0)$ -tensor of the Levi-Civita connection and μ is the function defined in (3.1).

Let us now compute the Nijenhuis tensor of J , i.e. the type $(2, 1)$ -tensor field N on M defined by

$$4N(X, Y) := [X, Y] - [JX, JY] + J[JX, Y] + J[X, JY], \quad X, Y \in \mathfrak{X}(M).$$

Proposition 4.1. *If M is nearly Kähler then $N(X, Y) = J(\nabla_X J)Y$, where $X, Y \in \mathfrak{X}(M)$.*

Proof. The key property we use here is that the Levi-Civita connection ∇ is torsion-free. Expanding the commutators one gets

$$\begin{aligned} 4N(X, Y) &= \nabla_X Y - \nabla_Y X + 2J(\nabla_X J)Y + J\nabla_X JY - J\nabla_Y JX \\ &= 2J(\nabla_X J)Y + J(\nabla_X J)Y - J(\nabla_Y J)X \\ &= 4J(\nabla_X J)Y, \end{aligned}$$

and we are done. \square

The difference $\nabla - \frac{1}{2}N$ defines a covariant derivative $\widehat{\nabla}$:

$$\widehat{\nabla}_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y, \quad X, Y \in \mathfrak{X}(M).$$

Proposition 4.2. $\widehat{\nabla}$ is a $U(n)$ -connection.

Remark 4.3. In Proposition 5.7 below we prove that on nearly Kähler six-manifolds $\widehat{\nabla}$ is actually an $SU(3)$ -connection, first exhibiting a complex volume form $\psi_{\mathbb{C}}$ on M and then proving it is $\widehat{\nabla}$ -parallel.

Proof. It is enough to show $\widehat{\nabla}g = 0$ and $\widehat{\nabla}J = 0$. Since $\nabla g = 0$ and by the usual symmetries one has

$$\begin{aligned} \widehat{\nabla}g(X, Y, Z) &= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &\quad + \frac{1}{2}(g(J(\nabla_X J)Y, Z) + g(Y, J(\nabla_X J)Z)) \\ &= \frac{1}{2}(g(Y, (\nabla_X J)JZ)) - g(Y, (\nabla_X J)JZ) = 0. \end{aligned}$$

The second claim follows easily by expanding $(\widehat{\nabla}_X J)Y = \widehat{\nabla}_X JY - J\widehat{\nabla}_X Y$ and simplifying. \square

Let us call \widehat{R} the curvature tensor of $\widehat{\nabla}$: $\widehat{R}(W, X)Y := \widehat{\nabla}_W \widehat{\nabla}_X Y - \widehat{\nabla}_X \widehat{\nabla}_W Y - \widehat{\nabla}_{[W, X]}Y$. Standard computations give

$$\begin{aligned} \widehat{R}(W, X)Y &= R(W, X)Y + \frac{1}{4}((\nabla_X J)(\nabla_W J)Y - (\nabla_W J)(\nabla_X J)Y) \\ &\quad - \frac{1}{2}J(R(W, X)JY - JR(W, X)Y). \end{aligned}$$

A contraction with the metric and identity (3.3) applied to the last term yield a type $(4, 0)$ -tensor field, which we still denote by \widehat{R} . Its expression is

$$\begin{aligned} \widehat{R}(W, X, Y, Z) &= R(W, X, Y, Z) + \frac{1}{2}g((\nabla_W J)X, (\nabla_Y J)Z) \\ &\quad + \frac{1}{4}(g((\nabla_X J)Y, (\nabla_W J)Z) - g((\nabla_W J)Y, (\nabla_X J)Z)). \end{aligned} \quad (4.1)$$

We can go a bit further rewriting every summand in terms of the curvature tensor R : by formula (3.3) and the first Bianchi identity, (4.1) becomes

$$\begin{aligned} \widehat{R}(W, X, Y, Z) &= R(W, X, Y, Z) + \frac{1}{2}(R(W, X, JY, JZ) - R(W, X, Y, Z)) \\ &\quad + \frac{1}{4}(R(X, Y, JW, JZ) - R(X, Y, W, Z)) \\ &\quad - R(W, Y, JX, JZ) + R(W, Y, X, Z) \\ &= \frac{1}{4}(3R(W, X, Y, Z) + 2R(W, X, JY, JZ) \\ &\quad + R(X, Y, JW, JZ) - R(W, Y, JX, JZ)). \end{aligned}$$

Recalling that R is J -invariant and lies in $\text{Sym}^2(\Lambda^2)$ we obtain the final expression

$$\begin{aligned} \widehat{R}(W, X, Y, Z) &= \frac{1}{4}(3R(W, X, Y, Z) + 2R(W, X, JY, JZ) \\ &\quad + R(W, Z, JX, JY) + R(W, Y, JZ, JX)). \end{aligned} \quad (4.2)$$

Lemma 4.4. The tensor \widehat{R} lies in $\Lambda^2 \otimes [\Lambda^{1,1}]$.

Proof. Skew-symmetry in the first two arguments is straightforward by definition of \widehat{R} . That $\widehat{R}(W, X)$ sits in $[\Lambda^{1,1}]$ is a simple consequence of Proposition 4.2. \square

Lemma 4.5. The tensor \widehat{R} sits inside $\text{Sym}^2(\Lambda^2)$.

Proof. Lemma 4.4 implies that we only need to check $\widehat{R}(W, X, Y, Z) = \widehat{R}(Y, Z, W, X)$. This can be done using (4.2) and applying J -invariance of R . \square

We now want more information about the exact expression of $\nabla \widehat{R}$. We keep working on a nearly Kähler manifold of generic dimension $2n$, focussing on the six-dimensional case only after Proposition 4.7. Incidentally, in the course of the proof of that result we will need an explicit formula for the cyclic sum $\nabla_V \widehat{R}(W, X, Y, Z) + \nabla_W \widehat{R}(X, V, Y, Z) + \nabla_X \widehat{R}(V, W, Y, Z)$, specifically the case where V, W, X are elements of a local unitary frame. The goal now is to work out this expression.

Let us start computing $\nabla_V \widehat{R}(W, X, Y, Z)$. Differentiating (4.1) one gets

$$\begin{aligned} V(\widehat{R}(W, X, Y, Z)) &= V(R(W, X, Y, Z)) + \frac{1}{4}g(\nabla_V((\nabla_X J)Y), (\nabla_W J)Z) \\ &\quad + \frac{1}{4}g((\nabla_X J)Y, \nabla_V((\nabla_W J)Z)) - \frac{1}{4}g(\nabla_V((\nabla_W J)Y), (\nabla_X J)Z) \\ &\quad - \frac{1}{4}g((\nabla_W J)Y, \nabla_V((\nabla_X J)Z)) + \frac{1}{2}g(\nabla_V((\nabla_W J)X), (\nabla_Y J)Z) \\ &\quad + \frac{1}{2}g((\nabla_W J)X, \nabla_V((\nabla_Y J)Z)). \end{aligned}$$

Expanding both sides and isolating $\nabla_V \widehat{R}(W, X, Y, Z)$ on the left we have

$$\begin{aligned} \nabla_V \widehat{R}(W, X, Y, Z) &= -\widehat{R}(\nabla_V W, X, Y, Z) - \widehat{R}(W, \nabla_V X, Y, Z) - \widehat{R}(W, X, \nabla_V Y, Z) - \widehat{R}(W, X, Y, \nabla_V Z) \\ &\quad + R(\nabla_V W, X, Y, Z) + R(W, \nabla_V X, Y, Z) + R(W, X, \nabla_V Y, Z) + R(W, X, Y, \nabla_V Z) \\ &\quad + \frac{1}{4}(g((\nabla_V(\nabla_X J))Y + (\nabla_X J)\nabla_V Y, (\nabla_W J)Z) \\ &\quad \quad + g((\nabla_X J)Y, (\nabla_V(\nabla_W J))Z + (\nabla_W J)\nabla_V Z)) \\ &\quad - \frac{1}{4}(g((\nabla_V(\nabla_W J))Y + (\nabla_W J)\nabla_V Y, (\nabla_X J)Z) \\ &\quad \quad + g((\nabla_W J)Y, (\nabla_V(\nabla_X J))Z + (\nabla_X J)\nabla_V Z)) \\ &\quad + \frac{1}{2}(g((\nabla_V(\nabla_W J))X + (\nabla_W J)\nabla_V X, (\nabla_Y J)Z) \\ &\quad \quad + g((\nabla_W J)X, (\nabla_V(\nabla_Y J))Z + (\nabla_Y J)\nabla_V Z)) + \nabla_V R(W, X, Y, Z). \end{aligned}$$

One can expand the first four summands on the right hand side making use of (4.1). Recall that $(\nabla_{A,B}^2 J)C = (\nabla_A(\nabla_B J))C - (\nabla_{\nabla_A B} J)C$, then simplifying we are left with

$$\begin{aligned} \nabla_V \widehat{R}(W, X, Y, Z) &= \nabla_V R(W, X, Y, Z) \\ &\quad + \frac{1}{2}(g((\nabla_{V,W}^2 J)X, (\nabla_Y J)Z) + g((\nabla_{V,Y}^2 J)Z, (\nabla_W J)X)) \\ &\quad + \frac{1}{4}(g((\nabla_{V,W}^2 J)Z, (\nabla_X J)Y) + g((\nabla_{V,X}^2 J)Y, (\nabla_W J)Z)) \\ &\quad - \frac{1}{4}(g((\nabla_{V,W}^2 J)Y, (\nabla_X J)Z) + g((\nabla_{V,X}^2 J)Z, (\nabla_W J)Y)). \end{aligned}$$

Therefore, the second Bianchi identity implies

$$\begin{aligned} \nabla_V \widehat{R}(W, X, Y, Z) + \nabla_W \widehat{R}(X, V, Y, Z) + \nabla_X \widehat{R}(V, W, Y, Z) &= \mathfrak{S}_{V,W,X} \left(\frac{1}{2}g((\nabla_{V,Y}^2 J)Z, (\nabla_W J)X) + \frac{1}{2}g((\nabla_{V,W}^2 J)X, (\nabla_Y J)Z) \right. \\ &\quad \left. + \frac{1}{4}g((\nabla_{V,W}^2 J)Z - (\nabla_{W,V}^2 J)Z, (\nabla_X J)Y) \right. \\ &\quad \left. + \frac{1}{4}g((\nabla_{V,X}^2 J)Y - (\nabla_{X,V}^2 J)Y, (\nabla_W J)Z) \right). \end{aligned} \quad (4.3)$$

Besides formula (4.3), in the proof of Proposition 4.7 we will need a last technical result.

Lemma 4.6. *Let Y be a vector field on M and $\{E_i, JE_i\}_{i=1,\dots,n}$, with $JE_i = E_{n+i}$, be a local orthonormal frame. Then the following formula holds:*

$$\sum_{j=1}^{2n} (\nabla_{E_j, E_j}^2 J)Y = -(\text{Ric} - \text{Ric}^*)JY. \quad (4.4)$$

Proof. This is a consequence of formula (3.7):

$$\begin{aligned}
g((\nabla_{E_j, E_j}^2 J)Y, X) &= \nabla^2 \sigma(E_j, E_j, Y, X) \\
&= -\frac{1}{2}(g((\nabla_{E_j} J)Y, (\nabla_X J)JE_j) + g((\nabla_{E_j} J)X, (\nabla_{E_j} J)JY)) \\
&= \frac{1}{2}(g((\nabla_{E_j} J)Y, J(\nabla_X J)E_j) - g(J(\nabla_{E_j} J)X, (\nabla_{E_j} J)Y)) \\
&= \frac{1}{2}(g((\nabla_{E_j} J)Y, J(\nabla_X J)E_j) + g(J(\nabla_X J)E_j, (\nabla_{E_j} J)Y)) \\
&= g((\nabla_{E_j} J)Y, (\nabla_{E_j} J)JX).
\end{aligned}$$

Then summing over j and identity (3.10) give

$$\begin{aligned}
\sum_{j=1}^{2n} g((\nabla_{E_j, E_j}^2 J)Y, X) &= \sum_{j=1}^{2n} g((\nabla_{E_j} J)Y, (\nabla_{E_j} J)JX) \\
&= g((\text{Ric} - \text{Ric}^*)Y, JX) = -g((\text{Ric} - \text{Ric}^*)JY, X),
\end{aligned}$$

because $\text{Ric} - \text{Ric}^*$ commutes with J . \square

Proposition 4.7. *Let W, X be two vector fields on M and $\{E_i, JE_i\}_{i=1, \dots, n}$, be a local orthonormal frame as above. Then*

$$\sum_{i,j=1}^{2n} g((\text{Ric} - \text{Ric}^*)E_i, E_j)(R(W, E_i, E_j, X) - 5R(W, E_i, JE_j, JX)) = 0. \quad (4.5)$$

Proof. Since $\widehat{R} \in \Lambda^2 \otimes [\Lambda^{1,1}]$ by Lemma 4.4 and $JE_i = E_{n+i}$ for $i = 1, \dots, n$, we have

$$\begin{aligned}
\sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) &= \frac{1}{2} \sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) + \widehat{R}(W, X, JE_i, J(\nabla_V J)E_i) \\
&= \frac{1}{2} \sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) - \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) \\
&= \frac{1}{2} \sum_{i=1}^n \widehat{R}(W, X, E_i, (\nabla_V J)E_i) - \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) - \widehat{R}(W, X, E_i, (\nabla_V J)E_i) = 0.
\end{aligned}$$

We can thus differentiate the identity obtained with respect to a vector field U viewing each summand on the left hand side as a function $p \mapsto \widehat{R}_p(\cdot, \cdot, \cdot, (\nabla_V J)_p \cdot)$:

$$\sum_i \nabla_U \widehat{R}(W, X, E_i, (\nabla_V J)E_i) + \widehat{R}(W, X, E_i, (\nabla_{U,V}^2 J)E_i) = 0. \quad (4.6)$$

Set $U = V = E_j$ and sum over $j = 1, \dots, 2n$. The second term in the latter sum becomes

$$\sum_{i,j} \widehat{R}(W, X, E_i, (\nabla_{E_j, E_j}^2 J)E_i). \quad (4.7)$$

By (4.4), sum (4.7) becomes

$$\begin{aligned}
\sum_{i,j} \widehat{R}(W, X, E_i, (\nabla_{E_j, E_j}^2 J)E_i) &= - \sum_i \widehat{R}(W, X, E_i, (\text{Ric} - \text{Ric}^*)JE_i) \\
&= - \sum_{i,j} \widehat{R}(W, X, E_i, g((\text{Ric} - \text{Ric}^*)JE_i, JE_j)JE_j) \\
&= - \sum_{i,j} g((\text{Ric} - \text{Ric}^*)E_i, E_j) \widehat{R}(W, X, E_i, JE_j).
\end{aligned}$$

Set $X = JW$. Then J -invariance of R and the first Bianchi identity give

$$\begin{aligned}
\widehat{R}(W, JW, E_i, JE_j) &= \frac{1}{4}(3R(W, JW, E_i, JE_j) - 2R(W, JW, JE_i, E_j) \\
&\quad - R(JW, E_i, JW, E_j) - R(W, E_i, W, E_j)) \\
&= \frac{1}{4}(5R(W, JW, E_i, JE_j) - R(W, E_i, W, E_j) - R(W, JE_i, W, JE_j)) \\
&= \frac{1}{4}(5R(W, E_i, JW, JE_j) - 5R(W, JE_j, JW, E_i) \\
&\quad - R(W, E_i, W, E_j) - R(W, JE_i, W, JE_j)).
\end{aligned}$$

Using (4.2) and (4.4) we have (sums over i and j)

$$\begin{aligned}
&\sum \widehat{R}(W, JW, E_i, (\nabla_{E_j, E_j}^2 J)E_i) \\
&= - \sum g((\text{Ric} - \text{Ric}^*)E_i, E_j) \widehat{R}(W, JW, E_i, JE_j) \\
&= \frac{1}{4} \sum g((\text{Ric} - \text{Ric}^*)E_i, E_j) (-5R(W, E_i, JW, JE_j) + 5R(W, JE_j, JW, E_i) \\
&\quad + R(W, E_i, W, E_j) + R(W, JE_i, W, JE_j)).
\end{aligned}$$

We now split this expression in four different sums where the indices i, j always run from 1 to n . Set $A := \text{Ric} - \text{Ric}^*$ and

$$\begin{aligned}
L(E_i, E_j) &:= -5R(W, E_i, JW, JE_j) + 5R(W, JE_j, JW, E_i) \\
H(E_i, E_j) &:= R(W, E_i, W, E_j) + R(W, JE_i, W, JE_j),
\end{aligned}$$

so we can write $\sum_{i,j} \widehat{R}(W, JW, E_i, (\nabla_{E_j, E_j}^2 J)E_i)$ as

$$\begin{aligned}
&\frac{1}{4} \sum_{i,j=1}^n (g(AE_i, E_j)(L + H)(E_i, E_j) + g(AE_i, JE_j)(L + H)(E_i, JE_j) \\
&\quad + g(AJE_i, E_j)(L + H)(JE_i, E_j) + g(AJE_i, JE_j)(L + H)(JE_i, JE_j)).
\end{aligned}$$

The symmetries of R , its J -invariance and the identity $AJ = JA$ yield

$$\begin{aligned}
&\sum \widehat{R}(W, JW, E_i, (\nabla_{E_j, E_j}^2 J)E_i) \\
&= \frac{1}{2} \sum_{i,j=1}^n (g(AE_i, E_j)(L(E_j, E_i) + H(E_i, E_j)) + g(AE_i, JE_j)(L(E_i, JE_j) + H(E_i, JE_j))).
\end{aligned}$$

Going back to our usual notation we find

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \widehat{R}(W, JW, E_i, (\nabla_{E_j, E_j}^2 J)E_i) \\
&= \frac{1}{2} \sum_{i,j=1}^{2n} g(AE_i, E_j) (R(W, E_i, W, E_j) - 5R(W, E_i, JW, JE_j)) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^n g(AE_i, JE_j) (R(W, E_i, W, JE_j) + 5R(W, E_i, JW, E_j)) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^n g(AJE_i, E_j) (R(W, JE_i, W, E_j) - 5R(W, JE_i, JW, JE_j)) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^n g(AJE_i, JE_j) (R(W, JE_i, W, JE_j) + 5R(W, JE_i, JW, E_j)) \\
&= \frac{1}{2} \sum_{i,j=1}^{2n} g(AE_i, E_j) (R(W, E_i, W, E_j) - 5R(W, E_i, JW, JE_j)).
\end{aligned}$$

Let us go back to (4.6) and focus on the first term now. Setting again $U = V = E_j, X = JW$, applying Lemma 4.5, and summing over j (and k) from 1 to $2n$ we have:

$$\begin{aligned} \sum_{i,j} \nabla_{E_j} \widehat{R}(W, JW, E_i, (\nabla_{E_j} J)E_i) &= \sum_{i,j,k} \nabla_{E_j} \widehat{R}(W, JW, E_i, g((\nabla_{E_j} J)E_i, E_k)E_k) \\ &= \sum_{i,j,k} \nabla \sigma(E_j, E_i, E_k) \nabla_{E_j} \widehat{R}(E_i, E_k, W, JW) \\ &= \frac{1}{2} \sum_{i < j < k} \nabla \sigma(E_i, E_j, E_k) \mathfrak{S}_{i,j,k} \nabla_{E_i} \widehat{R}(E_j, E_k, W, JW). \end{aligned}$$

The sum $\mathfrak{S}_{i,j,k} \nabla_{E_i} \widehat{R}(E_j, E_k, W, JW)$ actually vanishes: by formula (4.3)

$$\begin{aligned} &\nabla_{E_i} \widehat{R}(E_j, E_k, W, JW) + \nabla_{E_j} \widehat{R}(E_k, E_i, W, JW) + \nabla_{E_k} \widehat{R}(E_i, E_j, W, JW) \\ &= \frac{1}{2} \mathfrak{S}_{i,j,k} (g((\nabla_{E_k, W}^2 J)JW, (\nabla_{E_i} J)E_j) + g((\nabla_{E_i, E_j}^2 J)E_k, (\nabla_W J)JW)) \\ &\quad + \frac{1}{4} \mathfrak{S}_{i,j,k} (g((\nabla_{E_i, E_j}^2 J)JW - (\nabla_{E_j, E_i}^2 J)JW, (\nabla_{E_k} J)W) \\ &\quad + \frac{1}{4} \mathfrak{S}_{i,j,k} (g((\nabla_{E_j, E_i}^2 J)W - (\nabla_{E_i, E_j}^2 J)W, (\nabla_{E_k} J)JW)). \end{aligned}$$

Recall that $\nabla^2 \sigma(W, X, Y, Z) = g((\nabla_{W, X}^2 J)Y, Z)$. Applying (3.7) and simplifying we have

$$\begin{aligned} &\nabla_{E_i} \widehat{R}(E_j, E_k, W, JW) + \nabla_{E_j} \widehat{R}(E_k, E_i, W, JW) + \nabla_{E_k} \widehat{R}(E_i, E_j, W, JW) \\ &= \frac{1}{2} \mathfrak{S}_{i,j,k} \nabla^2 \sigma(E_k, W, JW, (\nabla_{E_i} J)E_j) \\ &\quad + \frac{1}{4} \mathfrak{S}_{i,j,k} (\nabla^2 \sigma(E_i, E_j, JW, (\nabla_{E_k} J)W) - \frac{1}{4} \nabla^2 \sigma(E_j, E_i, JW, (\nabla_{E_k} J)W)) \\ &\quad - \frac{1}{4} \mathfrak{S}_{i,j,k} (\nabla^2 \sigma(E_i, E_j, W, (\nabla_{E_k} J)JW) + \frac{1}{4} \nabla^2 \sigma(E_j, E_i, W, (\nabla_{E_k} J)JW)) \\ &= \frac{1}{2} g((\nabla_W J)(\nabla_{E_i} J)E_j, (\nabla_{E_k} J)W) + \frac{1}{2} g((\nabla_{E_k} J)E_i, (\nabla_W J)(\nabla_{E_j} J)W) \\ &\quad + \frac{1}{2} g((\nabla_W J)(\nabla_{E_j} J)E_k, (\nabla_{E_i} J)W) + \frac{1}{2} g((\nabla_{E_i} J)E_j, (\nabla_W J)(\nabla_{E_k} J)W) \\ &\quad + \frac{1}{2} g((\nabla_{E_j} J)E_k, (\nabla_W J)(\nabla_{E_i} J)W) + \frac{1}{2} g((\nabla_W J)(\nabla_{E_k} J)E_i, (\nabla_{E_j} J)W) = 0. \end{aligned}$$

Then $\sum_{i,j=1}^{2n} \nabla_{E_j} \widehat{R}(W, JW, E_i, (\nabla_{E_i} J)E_j) = 0$. Polarisation of (4.6) with $X = JW$ concludes the proof. \square

Let now Ric_g be the Ricci curvature (2,0)-tensor field of g .

Theorem 4.8. *Nearly Kähler six-manifolds are Einstein with positive scalar curvature.*

Proof. Consider the six-dimensional case in Proposition 4.7, i.e. $n = 3$. In the course of the proof of Proposition 3.8 we got $\text{Ric} - \text{Ric}^* = 4\mu^2 \text{Id}$, with $\mu > 0$ constant. Thus, since $g(E_i, E_j) = \delta_{ij}$, (4.5) reduces to

$$\sum R(W, E_i, E_i, X) - 5R(W, E_i, JE_i, JX) = 0,$$

which is equivalent to saying $\text{Ric} = 5\text{Ric}^*$. Therefore, $\text{Ric} - \text{Ric}^* = \text{Ric} - \frac{1}{5}\text{Ric} = 4\mu^2 \text{Id}$, namely $\text{Ric}_g = 5\mu^2 g$, and M is Einstein with positive scalar curvature. \square

5 Formulation in terms of PDEs

We now go through the details behind Theorem 1.3, following [4, Section 4.3] for this last part. It will be convenient to work on the complexified tangent bundle $T \otimes \mathbb{C}$ of M . We use the standard notations

$T^{1,0}$ and $T^{0,1}$ for the eigenspaces of J corresponding to the eigenvalues i and $-i$ respectively, so that $T \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$. All linear operations are extended by \mathbb{C} -linearity.

A first step in the direction we want to take was Proposition 2.3, where we proved that having a nearly Kähler structure on (M, g, J) is equivalent to saying $\nabla\sigma$ is a type $(3, 0) + (0, 3)$ form or that $d\sigma = 3\nabla\sigma$, for $\sigma = g(J \cdot, \cdot)$. We now give further characterisations.

Lemma 5.1. *The following assertions hold:*

1. M is nearly Kähler if and only if $\nabla_X Y + \nabla_Y X \in T^{1,0}$ for $X, Y \in T^{1,0}$.
2. If M is nearly Kähler then $\nabla_{\bar{X}} Y \in T^{1,0}$, for $X, Y \in T^{1,0}$.

Proof. For $X, Y \in T^{1,0}$ we have

$$\begin{aligned} J(\nabla_X Y + \nabla_Y X) &= \nabla_X JY + \nabla_Y JX - (\nabla_X J)Y - (\nabla_Y J)X \\ &= i(\nabla_X Y + \nabla_Y X) - (\nabla_X J)Y - (\nabla_Y J)X, \end{aligned}$$

from which our first claim follows. The second is a plain check that $(\nabla_{\bar{X}} J)Y = 0$, for $J\nabla_{\bar{X}} Y = \nabla_{\bar{X}} JY - (\nabla_{\bar{X}} J)Y = i\nabla_{\bar{X}} Y - (\nabla_{\bar{X}} J)Y$. \square

Lemma 5.2. *Let us consider $\{F_i\}_{i=1,2,3}$, a local orthonormal basis of $T^{1,0}$ on M . Denote by $\{f^i\}_{i=1,2,3}$ its dual in $\Lambda^{1,0}$. The following facts are equivalent:*

1. $\nabla_X Y + \nabla_Y X \in T^{1,0}$ for $X, Y \in T^{1,0}$.
2. There exists a constant, complex-valued function λ such that $[F_i, F_j]^{0,1} = -\lambda \bar{F}_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.
3. There exists a constant, complex-valued function λ such that $(df^i)^{0,2} = \lambda \bar{f}^j \wedge \bar{f}^k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Proof. Let us prove that 2 and 3 are equivalent first. Suppose $(df^i)^{0,2} = \lambda \bar{f}^j \wedge \bar{f}^k$ for some constant $\lambda \in \mathbb{C}$. Since type $(0, 1)$ forms vanish on $(1, 0)$ vectors and $(d\bar{f}^k)^{2,0} = \overline{(df^k)^{0,2}} = \bar{\lambda} f^i \wedge f^j$, we get

$$\begin{aligned} [F_i, F_j]^{0,1} &= \sum_{k=1}^3 \bar{f}^k([F_i, F_j])\bar{F}_k = \sum_{k=1}^3 (F_i(\bar{f}^k(F_j)) - F_j(\bar{f}^k(F_i)) - d\bar{f}^k(F_i, F_j))\bar{F}_k \\ &= - \sum_{k=1}^3 (d\bar{f}^k)^{2,0}(F_i, F_j)\bar{F}_k = -\lambda \bar{F}_k. \end{aligned}$$

Conversely, assume $[F_i, F_j]^{0,1} = -\lambda \bar{F}_k$ holds for some complex constant λ . If we set $\bar{X} = \sum_{\ell=1}^3 a_\ell \bar{F}_\ell, \bar{Y} = \sum_{\ell=1}^3 b_\ell \bar{F}_\ell$, using that $[\bar{F}_j, \bar{F}_k]^{1,0} = [F_j, F_k]^{0,1} = -\lambda F_i$ we have

$$\begin{aligned} \lambda \bar{f}^j \wedge \bar{f}^k(\bar{X}, \bar{Y}) &= \lambda(a_j b_k - b_j a_k) \\ &= -f^i \left(\sum_{i,j,k} (a_j b_k - b_j a_k) (-\lambda F_i) \right) \\ &= -f^i \left(\sum_{j < k} (a_j b_k - b_j a_k) ([\bar{F}_j, \bar{F}_k]^{1,0}) \right) \\ &= -f^i([\bar{X}, \bar{Y}]^{1,0}) \\ &= -\bar{X}(f^i(\bar{Y})) + \bar{Y}(f^i(\bar{X})) + df^i(\bar{X}, \bar{Y}) \\ &= df^i(\bar{X}, \bar{Y}). \end{aligned}$$

This yields our first equivalence.

Let us assume now that $[F_i, F_j]^{0,1} = -\bar{\lambda}\bar{F}_k$ for $\lambda \in \mathbb{C}$. We use that $g(\nabla_{F_j}F_k, F_k) = 0$ to compute $g(\nabla_{F_1}F_2 + \nabla_{F_2}F_1, F_i)$ for all $i = 1, 2, 3$. We have

$$\begin{aligned} g(\nabla_{F_1}F_2 + \nabla_{F_2}F_1, F_1) &= g(\nabla_{F_1}F_2 - \nabla_{F_2}F_1, F_1) \\ &= g([F_1, F_2], F_1) = -\bar{\lambda}g(\bar{F}_3, F_1) = 0. \\ g(\nabla_{F_1}F_2 + \nabla_{F_2}F_1, F_2) &= g(-\nabla_{F_1}F_2 + \nabla_{F_2}F_1, F_2) \\ &= g([F_2, F_1], F_2) = \bar{\lambda}g(\bar{F}_3, F_2) = 0. \\ g(\nabla_{F_1}F_2 + \nabla_{F_2}F_1, F_3) &= g(\nabla_{F_1}F_2, F_3) + g(\nabla_{F_2}F_1, F_3) \\ &= -g(F_2, \nabla_{F_1}F_3) - g(F_1, \nabla_{F_2}F_3). \end{aligned}$$

Now note that $g(\nabla_{F_2}F_3 - \nabla_{F_3}F_2, F_1) = g(\nabla_{F_3}F_1 - \nabla_{F_1}F_3, F_2) = -\bar{\lambda}$. This yields

$$\begin{aligned} -g(F_2, \nabla_{F_1}F_3) - g(F_1, \nabla_{F_2}F_3) &= -g(F_2, \nabla_{F_1}F_3) + \bar{\lambda} - g(F_1, \nabla_{F_3}F_2) \\ &= g(F_2, \nabla_{F_3}F_1 - \nabla_{F_1}F_3) + \bar{\lambda} \\ &= -\bar{\lambda} + \bar{\lambda} = 0. \end{aligned}$$

The other cases are analogous and **1** follows.

Finally, we prove that **1** implies **2**. Assuming $\nabla_X Y + \nabla_Y X \in T^{1,0}$ with $X, Y \in T^{1,0}$, we have

$$g([F_i, F_j]^{0,1}, F_k) = g([F_i, F_j], F_k) = g(\nabla_{F_i}F_j, F_k) - g(\nabla_{F_j}F_i, F_k) = 2g(\nabla_{F_i}F_j, F_k),$$

as the metric is of type $(1, 1)$ and $\nabla_{F_i}F_j = -\nabla_{F_j}F_i + W$, $W \in T^{1,0}$ by assumption. The basis has type $(1, 0)$, so

$$\begin{aligned} g(\nabla_{F_i}F_j, F_k) &= g(J\nabla_{F_i}F_j, JF_k) \\ &= g(\nabla_{F_i}JF_j - (\nabla_{F_i}J)F_j, JF_k) \\ &= -g(\nabla_{F_i}F_j, F_k) - g((\nabla_{F_i}J)F_j, iF_k), \end{aligned}$$

which implies $2g(\nabla_{F_i}F_j, F_k) = -i\nabla\sigma(F_i, F_j, F_k)$, and $g(\nabla_{F_i}F_j, F_k)$ is totally skew-symmetric in i, j, k . So we can write it as

$$2g(\nabla_{F_i}F_j, F_k) = -\varepsilon_{ijk}\bar{\lambda}$$

for some complex valued function λ on M , where ε_{ijk} is the sign of the permutation (i, j, k) and takes value 0 when any two indices coincide. There remains to prove that λ is constant. To this aim, take any real, local orthonormal set $\{E_1, JE_1, E_2, JE_2\}$. We put $(\nabla_{E_1}J)E_2 := \mu E_3$, where E_3 is a unit vector and μ a non-negative real function satisfying **(3.1)**. Then set $F_k := (1/\sqrt{2})(E_k - iJE_k)$ in $T^{1,0}$, $k = 1, 2, 3$, and recall that $g(\bar{X}, \bar{Y}) = \overline{g(X, Y)}$ and $\nabla_{\bar{X}}\bar{Y} = \nabla_{\bar{X}}\bar{Y}$ for every $X, Y \in T \otimes \mathbb{C}$. Hence $-\lambda = 2g(\nabla_{\bar{F}_1}\bar{F}_2, \bar{F}_3)$. Here below we find the relationship between λ and μ :

$$\begin{aligned} -\lambda &= 2g(\nabla_{\bar{F}_1}\bar{F}_2, \bar{F}_3) \\ &= g(\nabla_{E_1}E_2 - \nabla_{JE_1}JE_2 + i(\nabla_{E_1}JE_2 + \nabla_{JE_1}E_2), \bar{F}_3) \\ &= g(\nabla_{E_1}E_2 + iJ\nabla_{E_1}E_2, \bar{F}_3) + ig(\nabla_{JE_1}E_2 + iJ\nabla_{JE_1}E_2, \bar{F}_3) \\ &\quad + g(J(\nabla_{E_1}J)E_2 + i(\nabla_{E_1}J)E_2, \bar{F}_3). \end{aligned}$$

Observe that $\nabla_{E_1}E_2 + iJ\nabla_{E_1}E_2$ and $\nabla_{JE_1}E_2 + iJ\nabla_{JE_1}E_2$ are of type $(0, 1)$, so the first two terms vanish, and expanding the last term we find

$$g(J(\nabla_{E_1}J)E_2 + i(\nabla_{E_1}J)E_2, \bar{F}_3) = i\mu g(E_3 - iJE_3, \bar{F}_3) = i\sqrt{2}\mu.$$

In Proposition **3.8** we proved that μ is constant, so λ is constant as well. \square

Theorem 5.3. *Let (M, g, J) be an almost Hermitian six-manifold. Then M is nearly Kähler if and only if there exist a complex three-form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$ and a constant function μ such that*

$$d\sigma = 3\mu\psi_+, \quad d\psi_- = -2\mu\sigma \wedge \sigma. \quad (5.1)$$

Proof. Assume M is nearly Kähler. Using the local orthonormal basis as in Lemma 5.2, we can write locally $\sigma = i \sum_{k=1}^3 f^k \wedge \bar{f}^k$, where $f^k = (1/\sqrt{2})(e^k + iJe^k)$. Let us define

$$\psi_{\mathbb{C}} = \psi_+ + i\psi_- := 2\sqrt{2}f^1 \wedge f^2 \wedge f^3.$$

We know by Proposition 2.3 that M nearly Kähler implies $d\sigma \in [\Lambda^{3,0}]$. We thus calculate its $(3,0) + (0,3)$ part.

$$(id\sigma)^{3,0} = -\left(\sum_{k=1}^3 df^k \wedge \bar{f}^k - f^k \wedge d\bar{f}^k\right)^{3,0} = \sum_{k=1}^3 (f^k \wedge d\bar{f}^k)^{3,0} = \sum_{k=1}^3 f^k \wedge (d\bar{f}^k)^{2,0}.$$

Lemma 5.2 implies

$$(id\sigma)^{3,0} = \sum_{k=1}^3 f^k \wedge (d\bar{f}^k)^{2,0} = \bar{\lambda} \underset{1,2,3}{\mathfrak{S}} f^1 \wedge f^2 \wedge f^3 = \frac{3}{2\sqrt{2}}\bar{\lambda}\psi_{\mathbb{C}}.$$

Similarly, $(id\sigma)^{0,3} = -(3/2\sqrt{2})\lambda\bar{\psi}_{\mathbb{C}}$. We found $\lambda = -i\sqrt{2}\mu$, so $id\sigma = (id\sigma)^{3,0} + (id\sigma)^{0,3} = 3i\mu\psi_+$. This implies $0 = d\psi_+$, hence $d\psi_{\mathbb{C}} = -d\bar{\psi}_{\mathbb{C}}$. Differentiating $\psi_{\mathbb{C}}$ we find

$$\begin{aligned} d\psi_{\mathbb{C}} &= 2\sqrt{2}(df^1 \wedge f^2 \wedge f^3 - f^1 \wedge df^2 \wedge f^3 + f^1 \wedge f^2 \wedge df^3) \\ &= 2\sqrt{2}((df^1)^{1,1} \wedge f^2 \wedge f^3 + (df^1)^{0,2} \wedge f^2 \wedge f^3 \\ &\quad - f^1 \wedge (df^2)^{1,1} \wedge f^3 - f^1 \wedge (df^2)^{0,2} \wedge f^3 \\ &\quad + f^1 \wedge f^2 \wedge (df^3)^{1,1} + f^1 \wedge f^2 \wedge (df^3)^{0,2}) \in \Lambda^{3,1} + \Lambda^{2,2}. \end{aligned}$$

With similar computations one can see that $d\bar{\psi}_{\mathbb{C}} \in \Lambda^{1,3} + \Lambda^{2,2}$. We proved that $d\psi_{\mathbb{C}} = -d\bar{\psi}_{\mathbb{C}}$, so the $(1,3)$ part of $d\psi_{\mathbb{C}}$ vanishes. We then have

$$id\psi_- = 2\sqrt{2}\lambda \sum_{j < k} \bar{f}^j \wedge \bar{f}^k \wedge f^j \wedge f^k = -2i\mu\sigma \wedge \sigma$$

and the first implication is done.

Conversely, given $d\sigma = 3\mu\psi_+$ and $d\psi_- = -2\mu\sigma \wedge \sigma$, it is enough to prove that $(df^i)^{0,2} = \lambda \bar{f}^j \wedge \bar{f}^k$ for (i, j, k) cyclic permutation of $(1, 2, 3)$ and some constant $\lambda \in \mathbb{C}$. To get it, we first see that

$$\psi_{\mathbb{C}} \wedge (df^i)^{0,2} = \psi_{\mathbb{C}} \wedge df^i = d\psi_{\mathbb{C}} \wedge f^i = id\psi_- \wedge f^i = \psi_{\mathbb{C}} \wedge \lambda(\bar{f}^j \wedge \bar{f}^k).$$

Now observe that the map $\Lambda^{0,2} \rightarrow \Lambda^{3,2}$ given by the wedge product with $\psi_{\mathbb{C}}$ is injective. This implies $(df^i)^{0,2} = \lambda \bar{f}^j \wedge \bar{f}^k$. \square

Remark 5.4. We can rescale our basis so that $\sigma \mapsto \tilde{\sigma} := \mu^2\sigma$ and $\psi_{\pm} \mapsto \tilde{\psi}_{\pm} := \mu^3\psi_{\pm}$. Then

$$d\tilde{\sigma} = 3\tilde{\psi}_+, \quad d\tilde{\psi}_- = -2\tilde{\sigma} \wedge \tilde{\sigma}.$$

Theorem 5.3 provides us with a characterisation of nearly Kähler six-manifolds in terms of an $SU(3)$ -structure.

Definition 5.5. Let (M, g, J) be an almost Hermitian six-dimensional manifold with an $SU(3)$ -structure ($\sigma = g(J \cdot, \cdot)$, $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$). We say that M is *nearly Kähler* if and only if

$$d\sigma = 3\psi_+, \quad d\psi_- = -2\sigma \wedge \sigma,$$

up to homothety.

Observe that locally σ and $\psi_{\mathbb{C}}$ were expressed in terms of type $(1, 0)$ vectors f^i as $\sigma = i \sum_{k=1}^3 f^k \wedge \bar{f}^k$ and $\psi_{\mathbb{C}} = 2\sqrt{2}f^1 \wedge f^2 \wedge f^3$, thus giving the real models

$$\sigma = e^1 \wedge Je^1 + e^2 \wedge Je^2 + e^3 \wedge Je^3, \quad (5.2)$$

$$\psi_+ = e^1 \wedge e^2 \wedge e^3 - Je^1 \wedge Je^2 \wedge e^3 - e^1 \wedge Je^2 \wedge Je^3 - Je^1 \wedge e^2 \wedge Je^3, \quad (5.3)$$

$$\psi_- = e^1 \wedge e^2 \wedge Je^3 - Je^1 \wedge Je^2 \wedge Je^3 + e^1 \wedge Je^2 \wedge e^3 + Je^1 \wedge e^2 \wedge e^3, \quad (5.4)$$

which are obtained by the definition $f^k := (1/\sqrt{2})(e^k + iJe^k)$, $k = 1, 2, 3$. By $Je^i = -e^i \circ J$, expressions (5.3) and (5.4), the relation $\psi_- = -\psi_+(\cdot, \cdot, J\cdot)$ readily follows. On the other hand by equations (5.1) and Proposition 2.3 we have $d\sigma = 3\mu\psi_+ = 3\nabla\sigma$, so $\mu\psi_+ = \nabla\sigma$, but since $\nabla\sigma \in \Lambda^1 \otimes [\Lambda^{2,0}]$ we find

$$\begin{aligned} \psi_-(X, Y, Z) &= -\psi_+(X, Y, JZ) = -\mu^{-1}\nabla\sigma(X, Y, JZ) = -\mu^{-1}\nabla\sigma(JZ, X, Y) \\ &= \mu^{-1}\nabla\sigma(JZ, JX, JY) = \mu^{-1}\nabla\sigma(JX, JY, JZ) = -J\psi_+(X, Y, Z). \end{aligned}$$

Therefore $\psi_- = -J\psi_+$.

Remark 5.6. Let us set $\text{vol} := e^1 \wedge Je^1 \wedge e^2 \wedge Je^2 \wedge e^3 \wedge Je^3$. A straightforward calculation of $\psi_+ \wedge \psi_-$ and $\sigma \wedge \psi_{\pm}$ gives

$$\psi_+ \wedge \psi_- = 4\text{vol} = \frac{2}{3}\sigma^3, \quad \sigma \wedge \psi_{\pm} = 0.$$

Since $g(\psi_+, \psi_+) = 4$ the first equation tells us that $\psi_+ \wedge \psi_- = 4\text{vol} = g(\psi_+, \psi_+)\text{vol} = \psi_+ \wedge *\psi_+$, so by uniqueness of $*\psi_+$ we deduce $*\psi_+ = \psi_-$.

Recall that $\widehat{\nabla} := \nabla - \frac{1}{2}J(\nabla J)$ is a $U(3)$ -connection by Proposition 4.2.

Proposition 5.7. $\widehat{\nabla}$ is an $SU(3)$ -connection.

Proof. We calculate $\widehat{\nabla}(\nabla\sigma)$. By (3.7) we have

$$\begin{aligned} \widehat{\nabla}(\nabla\sigma)(W, X, Y, Z) &= W(\nabla\sigma(X, Y, Z)) - \nabla\sigma(\widehat{\nabla}_W X, Y, Z) \\ &\quad - \nabla\sigma(X, \widehat{\nabla}_W Y, Z) - \nabla\sigma(X, Y, \widehat{\nabla}_W Z) \\ &= \nabla^2\sigma(W, X, Y, Z) + \frac{1}{2} \mathfrak{S}_{X, Y, Z} g((\nabla_W J)X, (\nabla_Y J)JZ) = 0, \end{aligned}$$

which proves $\widehat{\nabla}(\nabla\sigma) = 0 = \widehat{\nabla}\psi_+$, thus ψ_+ is parallel. Further, by $\psi_- = -J\psi_+$ we have at once $\widehat{\nabla}\psi_- = 0$, namely $\widehat{\nabla}\psi_{\mathbb{C}} = 0$, which proves $\widehat{\nabla}$ is actually an $SU(3)$ -connection. \square

Remark 5.8. We mentioned already in Proposition 2.3 that $\nabla\sigma$ lies in $[\Lambda^{3,0}]$, so obviously it is only the $(3, 0) + (0, 3)$ part of $\nabla\sigma$ that measures the failure of M to be Kähler. Therefore, we can say that it is exactly the type of $\nabla\sigma$ that determines the class of nearly Kähler manifolds in the classification completed by Gray and Hervella. On the other hand, equation (2.1) tells us $\nabla\sigma$ may be identified with ∇J , which may in turn be identified with the Nijenhuis tensor N of J by Proposition 4.1. The latter is the *intrinsic torsion* of the $SU(3)$ -structure (σ, ψ_{\pm}) by Proposition 5.7. A detailed study of this object for $SU(3)$ - and G_2 -structures was pursued by Chiossi and Salamon (see [5], in particular Theorem 1.1 for what regards our set-up).

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