Simple characters for principal orders and their matching

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Introduction

In this paper we want to explain how one can obtain a simple character from a simple stratum, which is founded on a principal order, for an arbitrary local central simple algebra. Moreover, we want to match these characters among various algebras. Our way to construct the simple characters is a mixture of the approach of [Z7] and [BK3]. From [Z7] we use the results about finite nilpotent rings and by using the methods of [Br3] we combine it with the results of [BK3]. We use the results of [BG] to obtain so called 'special approximation sequences', which can be used for the final matching of simple characters. Finally, we are able to extend the definition of a ps-character due to [BH] to the context of arbitrary simple algebras. I am grateful to E.-W. Zink for proofreading and valuable discussions.

0 Notation and preparations

Let F be a non-archimedean local field and A a finite-dimensional central simple algebra over F. Let V be a left simple A-module. V is a right $D = \mathrm{End}_A(V)$-vector space and

$$A \simeq \mathrm{End}_D(V) \simeq M_m(D)$$

(where $M_m(D)$ denotes the ring of $m \times m$ matrices in D). Denote by d the index of the division algebra D (that is, $\dim_F(D) = d^2$), fix an isomorphism $A \simeq M_m(D)$, and identify A with $M_m(D)$. Denote by $\mathfrak{o}_F$, $\mathfrak{o}_D$, ... the rings of integers of F, D, ..., by $\mathfrak{p}_F$, $\mathfrak{p}_D$, ... the respective unique prime ideals and by $\mathfrak{k}_F$, $\mathfrak{k}_D$, ... the associated residual fields.

(0.1) *An $\mathfrak{o}_F$-order $\mathfrak{A}$ in A is a subring with the same unit element as A, which is a complete $\mathfrak{o}_F$-lattice in A. Here complete $\mathfrak{o}_F$-lattice means that $\mathfrak{A}$ is a finitely generated free $\mathfrak{o}_F$-module and contains an F-basis of the F-vectorspace A. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A, i. e., an $\mathfrak{o}_F$-order which is (left- or right-) hereditary as a ring. We denote by $\mathfrak{P}$ its Jacobson radical and by $\mathcal{K}_{\mathfrak{A}}$ its normalizer. It is well known (see* [R](39.12)*) that $\mathfrak{A}$ is hereditary iff $\mathfrak{P}$ is a projective left or right $\mathfrak{A}$-module, which is also the case iff $\mathfrak{P}$ is a fractional invertible ideal of $\mathfrak{A}$. Hence, there exists an ($\mathfrak{A}$,$\mathfrak{A}$)-bisubmodule $\mathfrak{P}^{-1}$ of A which is a complete $\mathfrak{o}_F$-lattice in A such that $\mathfrak{P}\mathfrak{P}^{-1} = \mathfrak{P}^{-1}\mathfrak{P} = \mathfrak{A}$.*

For an hereditary order $\mathfrak{A}$ we set $U^j(\mathfrak{A}) := 1 + \mathfrak{P}^j$ for all $j \in \mathbb{N}$. These groups lead to a filtration $\mathcal{K}_{\mathfrak{A}} \supset \mathfrak{A}^\times \supset U^1(\mathfrak{A}) \supset U^2(\mathfrak{A}) \supset ...$, where $\mathcal{K}_{\mathfrak{A}}$ denotes the $A^\times$-normalizer of $\mathfrak{A}$.

An $\mathfrak{o}_D$-lattice chain in V is a family $\{X_j\}_{j\in\mathbb{Z}}$ of complete $\mathfrak{o}_F$-lattices in V that are right $\mathfrak{o}_D$-modules such that

a) $\forall j \in \mathbb{Z}, X_j \supsetneq X_{j+1}$,

b) $\exists r \in \mathbb{N}, \forall j \in \mathbb{Z}, X_j\mathfrak{p}_D = X_{j+r}$.

(0.2) *One knows from* [BF] *that $\mathfrak{A}$ is also hereditary iff there exists a (unique up to a translation of the index) $\mathfrak{o}_D$-lattice chain $\{X_k\}_{k\in\mathbb{Z}}$ in V such that*

$$\mathfrak{A} = \mathrm{End}_{\mathfrak{o}_D}\{X_k\}_{k\in\mathbb{Z}} = \{x \in A \mid \forall k \in \mathbb{Z}, xX_k \subset X_k\}.$$

(0.3) *We have*

$$\mathfrak{A}/\mathfrak{P} \simeq \bigoplus_{i=1}^{r} \mathrm{End}_{\mathfrak{k}_D}(X_{i-1}/X_i) \simeq M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$$

such that the ordered numbers $n_1,\ldots,n_r$ provide numerical invariants, up to a cyclic permutation, of the conjugacy class of $\mathfrak{A}$. These numbers also come from a normal form (see [R] 39.14)

$$g\mathfrak{A}g^{-1} = \begin{pmatrix} (\mathfrak{o}_D)\,, (\mathfrak{p}_D) & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,, (\mathfrak{o}_D) & \cdots, \ \cdots \\ \cdots\ ,\ \cdots & \cdots, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,,\ \cdots & \cdots, (\mathfrak{o}_D) \end{pmatrix}^{\{n_1,\ldots,n_r\}},$$

where $(\mathfrak{o}_D)$ (and $(\mathfrak{p}_D)$, respectively) mean $n_i \times n_j$-matrices, where the lower-diagonal and diagonal blocks have entries in $\mathfrak{o}_D$ and the upper-diagonal blocks have entries in $\mathfrak{p}_D$ (note that $n_1+\ldots+n_r = m$) and g is a suitable element of $A^\times$. Here the number r is uniquely determined and called the period of $\mathfrak{A}$ and $\{X_j\}_{j\in\mathbb{Z}}$, respectively. The numbers $n_1,\ldots,n_r$ are unique up to a cyclic permutation.

The modules $X_0,\ldots,X_{r-1}$ of the lattice chain of $\mathfrak{A}$ in V (as above) are a system of representatives of the isomorphy classes of direct indecomposable right $\mathfrak{A}$-modules (1)

(see [BF](1.2.8)(ii) *and* [R](39.23)*).*

We also have

$$g\mathfrak{P}g^{-1} = \begin{pmatrix} (\mathfrak{p}_D)\,, (\mathfrak{p}_D)\,, \cdots, (\mathfrak{p}_D)\,, (\mathfrak{p}_D) \\ (\mathfrak{o}_D)\,, (\mathfrak{p}_D)\,, \cdots,\ \cdots\ ,\ \cdots \\ \cdots\ , (\mathfrak{o}_D)\,, \cdots,\ \cdots\ ,\ \cdots \\ (\mathfrak{o}_D)\,,\ \cdots\ , \cdots, (\mathfrak{p}_D)\,,\ \cdots \\ (\mathfrak{o}_D)\,,\ \cdots\ , \cdots, (\mathfrak{o}_D)\,, (\mathfrak{p}_D) \end{pmatrix}^{\{n_1,\ldots,n_r\}}.$$

If $\mathfrak{A}$ is given in normal form 0.3, then, for the simple A-module $V := D^m$ (column vectors), we can choose the numbering of the chain $\{X_j\}_{j\in\mathbb{Z}}$ such that $\forall j \in \mathbb{Z}$, ${}^tX_j = \mathfrak{p}_D^l((\bigoplus_{i=1}^{k}(\bigoplus_{\mu=1}^{n_i}\mathfrak{p}_D))\bigoplus(\bigoplus_{i=k+1}^{r}(\bigoplus_{\mu=1}^{n_j}\mathfrak{o}_D)))$, where $k,l \in \mathbb{Z}$ such that $j = lr+k$, $0 \le k < r$, and ${}^t(.)$ means the set of transposed vectors.

An order $\mathfrak{A}$ is called principal if its Jacobson radical $\mathfrak{P}$ is a principal ideal. By 0.1 such an $\mathfrak{A}$ is hereditary, too.

(0.4) *From* [BF](1.3.2) *we know that a hereditary order $\mathfrak{A}$ with invariants $n_1,\ldots,n_r$ is a principal order if and only if $n_1 = \ldots = n_r = s$. In this situation we call s the invariant of $\mathfrak{A}$.*

1 Strata and duality

Let ψ_F be a character of F^+ with conductor $\mathfrak{p}_F$ and let $\psi_A := \psi_F \circ Trd_{A/F}$, where Trd denotes the reduced trace. From [BF] we know that the pairing $A \times A \to \mathbb{C}^\times$, $(x,y) \mapsto \psi_A(xy)$ allows us to identify A with its Pontrjagin dual. Let us define the dual of any subset S of A by

(1.1) $S^* := \{a \in A \mid \psi_A(sa) = 1,$ *for all* $s \in S\}$.

If $u \in \mathbb{Z}$, we have

(1.2) $(\mathfrak{P}^u)^* = \mathfrak{P}^{1-u}$

(see [BF] and take into account that ψ_F has conductor $\mathfrak{p}_F$). For all integers $n \geq 1$ and $n > q > [\frac{n}{2}]$, where [.] denotes the least integer function, we have a canonical isomorphism of groups:

(1.3) $U^{q+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A}) \to \mathfrak{P}^{q+1}/\mathfrak{P}^{n+1}$,

induced by the map $1+x \mapsto x$. In particular, the quotient $U^{q+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A})$ is finite abelian. Its dual is isomorphic to $\mathfrak{P}^{-n}/\mathfrak{P}^{-q}$ via the map:

(1.4) $b + \mathfrak{P}^{-q} \mapsto \psi_b^\times \in U^{q+1}\widehat{(\mathfrak{A})/U}^{n+1}(\mathfrak{A}), \psi_b^\times(1+x) := \psi_A(bx)$.

Moreover, for $n > q \geq 0$, we have a canonical isomorphism of groups:

(1.5) $\mathfrak{P}^{-n}/\mathfrak{P}^{-q} \to \mathfrak{P}^{\widehat{q+1}/\mathfrak{P}}{}^{n+1}$

via the map

(1.6) $b + \mathfrak{P}^{-q} \mapsto \psi_b^+, \psi_b^+(x) := \psi_A(bx)$.

(1.7) **Notation** *For later purposes we set* $S^{\perp\psi_b^+} := \{x \in \mathfrak{P} \mid \forall s \in S : \psi_b^+(xs - sx) = 1\}$ *for any subset S of $\mathfrak{P}$.*

As in [Br2](1.1.3) we define:

(1.8) Definitions

(i) A stratum in A is a quadruple $[\mathfrak{A}, n, q, b]$, where

(a) $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A;

(b) n and q are integers with $q \leq n$; hence, $\mathfrak{P}^{-q} \subset \mathfrak{P}^{-n}$;

(c) $b \in \mathfrak{P}^{-n}$.

(ii) Two strata $[\mathfrak{A}, n, q, b_i]$, $i = 1, 2$, are said to be equivalent if $b_1 - b_2 \in \mathfrak{P}^{-q}$.

From the isomorphism (1.4), it is clear that the equivalence classes of strata of the form $[\mathfrak{A}, n, q, b]$ parametrize the dual of $U^{q+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A})$ if $n \geq 1$ and $n > q > [\frac{n}{2}]$. For $G := A^\times$, we define the formal intertwining of the stratum $[\mathfrak{A}, n, q, b]$ as the set $\mathcal{I}_G[\mathfrak{A}, n, q, b] := \{x \in G \mid x^{-1}(b + \mathfrak{P}^{-q})x \cap (b + \mathfrak{P}^{-q}) \neq \emptyset\}$, which in the case of $n > q > [\frac{n}{2}]$ is the same as the intertwinig $I_G(\psi_b^\times)$ of our character $\psi_b^\times$. We need the following important lemma, which is a generalization of [BK3](2.4.11):

(1.9) Lemma

Consider a stratum $[\mathfrak{A}, n, n-1, b]$ and suppose that $\mathcal{I}_G[\mathfrak{A}, n, n-1, b] = G$, then $(b + \mathfrak{P}^{1-n}) \cap F \neq \emptyset$.

Moreover, if χ is a continuous character of $U^{\nu+1}(\mathfrak{A})$, where $\nu \geq 0$ and where χ is intertwined by each element of G, then χ factors through the reduced norm $Nrd_{A|F} : A^\times \to F^\times$.

Proof:

As our first assertion is invariant under conjugation, we can assume that our hereditary order $\mathfrak{A}$ is given in normal form (0.3), where we identify A with $M_m(D)$. Taking into account the explicit descriptions of the powers of $\mathfrak{P}$, we may assume that there exists an integer t with $0 \leq t \leq r-1$ and

(1.10)

$$b \equiv \begin{pmatrix} 0 & , & \cdots & , 0 , & 0 , & b_1 , & 0 , & \cdots & , & 0 \\ \cdots & , & \cdots & , \cdots , & \cdots , & \cdots , & \cdots , & \cdots & , & \cdots \\ 0 & , & \cdots & , \cdots , & \cdots , & 0 , & 0 , & b_{r-t-1} & , & 0 \\ 0 & , & \cdots & , \cdots , & \cdots , & \cdots , & \cdots , & 0 & , & b_{r-t} \\ b_{r-t+1} & , & 0 & , \cdots , & \cdots , & \cdots , & \cdots , & \cdots & , & 0 \\ 0 & , & b_{r-t+2} & , 0 , & \cdots , & \cdots , & \cdots , & \cdots & , & 0 \\ \cdots & , & \cdots & , \cdots , & \cdots , & \cdots , & \cdots , & \cdots & , & \cdots \\ 0 & , & \cdots & , 0 , & b_r , & 0 , & 0 , & \cdots & , & 0 \end{pmatrix} \quad mod\ \mathfrak{P}^{1-n},$$

where r is the period of $\mathfrak{A}$. We use the block description as in (0.3). To be more precise, t is the unique integer with $-n = kr + t$ and $0 \leq t \leq r-1$, where k is also an appropriate integer. Here b_i is an $n_i \times n_{\tau_r^t(i)}$-matrix, where τ_r is the usual cyclic permutation of the numbers $\{1, \ldots, r\}$ and b_i can be chosen with entries in a system of representatives of $\mathfrak{p}_D^k/\mathfrak{p}_D^{k+1}$ if it is on or above the block diagonal, or in a system of representatives of $\mathfrak{p}_D^{k+1}/\mathfrak{p}_D^{k+2}$ if it is below the block diagonal.

If b is in $\mathfrak{P}^{1-n}$, we have $0 \in (b + \mathfrak{P}^{1-n}) \cap F$ and there is nothing to show for the first assertion. We may assume that one of the b_i is nonzero. So let i be such that $b_i \neq 0$ and let $x := \mathrm{diag}(I_{n_1}, \ldots, I_{n_{i-1}}, \pi_F^2 I_{n_i}, I_{n_{i+1}}, \ldots, I_{n_r})$, where $\mathrm{diag}(., \ldots, .)$ means a block diagonal matrix and I_n the $n \times n$ identity matrix. Then $x(b + \mathfrak{P}^{1-n}) \cap (b + \mathfrak{P}^{1-n})x \neq \emptyset$ by assumption. The elements in the second set have an entry in the $(i, \tau_r^t(i))$'s block which is in $\mathfrak{p}_D^k \backslash \mathfrak{p}_D^{k+2}$, but if we have $t \neq 0$, then, for the elements in the first set, all entries in the same block lie in $\mathfrak{p}_D^{k+2d}$, which is impossible.

Therefore, we can assume that

(1.11) $b \equiv \mathrm{diag}(b_1, \ldots, b_r)$ *mod* $\mathfrak{P}^{1-n}$,

and, in particular, $-n = kr$. Let $b' := \mathrm{diag}(b'_1, \ldots, b'_r)$ be a matrix such that $b \equiv \pi_D^k b' \bmod \mathfrak{P}^{1-n}$, where the matrices $b'_1, \ldots, b'_r$ have entries in a system of representatives of $\mathfrak{o}_D/\mathfrak{p}_D \simeq \mathfrak{k}_D$.

Consider $x \in \mathfrak{o}_D^\times I_m \subset \mathfrak{A}^\times$, then, by assumption, $x(b + \mathfrak{P}^{1-n})x^{-1} \cap (b + \mathfrak{P}^{1-n}) \neq \emptyset$, which is the same as

(1.12) $\pi_D^{-k} x \pi_D^k b' x^{-1} \equiv b'$ *mod* $\mathfrak{P}$,

because $\mathfrak{A}$ is normalized by π_D (by (0.3)), $\pi_F = \pi_D^d$ (for an appropriate choice of π_D) and $\pi_F \mathfrak{A} = \mathfrak{P}^{rd}$.

Let $\bar{x} := (x(mod\ \mathfrak{P}))$ and $\bar{b} := (b'(mod\ \mathfrak{P}))$, then we can view $\bar{x}$ as an element of $\mathfrak{k}_D^\times$ and $\bar{b}$ as an element of $M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$, where $\mathfrak{k}_D$ is viewed as being embedded diagonally into the center of the latter algebra. The equation (1.12) now says that $\sigma^k(\bar{x})\bar{b}\bar{x}^{-1} = \bar{b}$, where σ is a generator of $\mathrm{Gal}(\mathfrak{k}_D/\mathfrak{k}_F)$ induced by the adjoint action of π_D on a suitable maximal unramified extension field L/F in D (note that for suitable L we have $\mathfrak{o}_D = \mathfrak{o}_L + \mathfrak{o}_L\pi_D + \ldots + \mathfrak{o}_L\pi_D^{d-1}$ and $\mathfrak{p}_D = \mathfrak{p}_L + \mathfrak{o}_L\pi_D + \ldots + \mathfrak{o}_L\pi_D^{d-1}$); hence, because $\bar{b} \neq 0$ and $\bar{x}$ centralizes $\bar{b}$, we obtain $\sigma^k(\bar{x}) = \bar{x}$. As $\bar{x} \in \mathfrak{k}_D^\times$ was arbitrarily chosen, we conclude that d must be a divisor of k. As $\pi_D^k = \pi_F^{\frac{k}{d}} \in F^\times$, replacing b by a multiple with a central element in A (namely $\pi_F^{-\frac{k}{d}}$), it is sufficient to prove our first assertion under the assumption that $n = 0$ and $b = b'$.

Consider $x = \pi_D^\nu I_m$ for an arbitrary integer ν. In this case our assumption implies that $\sigma^\nu(\bar{b}) = \bar{b}$, where σ acts on each entry of the matrix $\bar{b} \in M_{n_1}(\mathfrak{k}_D) \times \ldots \times M_{n_r}(\mathfrak{k}_D)$. Therefore $b_1, \ldots, b_r$ can be chosen with entries in a system of representatives of $\mathfrak{o}_F/\mathfrak{p}_F$.

Consider an arbitrary invertible x of the same kind as b, then our assumption leads to $\bar{x}^{-1}\bar{b}\bar{x} = \bar{b}$; hence, we can assume that $b_i = a_i I_{n_i}$, where a_i is chosen from a system of representatives of $\mathfrak{o}_F/\mathfrak{p}_F$. Finally, consider x as an arbitrary permutation matrix of $A = M_m(D)$, then we conclude that $a_1 = a_2 = \ldots = a_r =: a$ and we obtain $(b + \mathfrak{P}) = (a + \mathfrak{P})$.

This leads to $(b + \mathfrak{P}^{1-n}) \cap F \neq \emptyset$ as required.

To prove now the second assertion, we can assume the first assertion in full generality. So let χ be a continuous character of $U^{\nu+1}(\mathfrak{A})$ that is intertwined by the whole of G, then also $\chi_{|U^n(\mathfrak{A})}$ is intertwined by the whole of G for all $n \geq \nu + 1$. Let n be the least integer such that $\chi_{|U^{n+1}(\mathfrak{A})} \equiv 1$, then there exists a stratum $[\mathfrak{A}, n, n-1, b]$ such that $\chi_{|U^n(\mathfrak{A})} = \psi_b^\times$, and by the first part, we can choose $b \in F$. Moreover, if we follow the arguments of the proof, we conclude that dr is a divisor of n.

Let L/F be an unramified field extension which splits A. Then, clearly, the stratum $[\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L, n, n-1, b\otimes 1]$ is intertwined by the whole of $(A\otimes_F L)^\times$ and we conclude from the split case of [BK3](2.4.11) and its proof that $\psi^\times_{b\otimes 1} = \psi^\times_{L,b}\circ \det_{A\otimes_F L/L}$, where $\psi^\times_{L,b}(x) := \psi_L(b(x-1))$ for $x\in U^{\frac{n}{rd}}(\mathfrak{o}_L)$ and ψ_L is an additive character of L with conductor $\mathfrak{p}_L$ and restriction ψ_F to F (also $\psi^\times_{b\otimes 1}$ is defined by ψ_L, similarly as $\psi^\times_b$ by ψ_F). Hence, $\psi^\times_b = \psi^\times_{F,b}\circ Nrd_{A/F}$ by the definition of the reduced norm, where $\psi_{F,b}$ is the restriction of $\psi_{L,b\otimes 1}$ to $U^{\frac{n}{rd}}(\mathfrak{o}_F)$. Let $\chi_0 := \psi^\times_{F,b}\circ Nrd_{A/F}$, then $\chi' := \chi_0^{-1}\chi$ is a character of $U^{\nu+1}(\mathfrak{A})$ which is intertwined by the whole of G and which is trivial on $U^n(\mathfrak{A})$. Moreover, we have $\chi = \chi'\chi_0$.

By induction on n we can realize χ as a product of characters that factor through the reduced norm; hence, so does χ.

□

2 Sound elements and their canonical mappings

We want to give a brief description of the connection between the general theory of pure elements explained in [BG] and sound embeddings as introduced in [F]. For this purpose we study the embedding data as defined in [BG](2.3.1) for sound embeddings.

To abbreviate the assertions we introduce the following notation as given in [BG] above [BG](2.1.2):

(2.1) Notation *Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A and E/F an extension in A such that $E^\times\subset\mathcal{K}_{\mathfrak{A}}$, then the pair $(E,\mathfrak{A})$ is called an embedding (with respect to $\mathfrak{A}$).*

(2.2) Proposition *If $\mathfrak{A}$ is a principal order with invariant s and period r in $A = M_m(D)$ given in normal form (0.3) and $(L,\mathfrak{A})$ an embedding, where L/F is an unramified extension of degree f dividing d, then the following assertions are equivalent:*

(i) $\mathfrak{A}\cap C_A(L)$ is a principal order of $C_A(L)$ continued by $\mathfrak{A}$ in the sense of [G](0.6)(i.e., $\mathcal{K}_{\mathfrak{A}}\cap C_A(L) = \mathcal{K}_{\mathfrak{A}\cap C_A(L)}$).

(ii) There exists a divisor d_1 of f that is prime to r such that $\lambda[d_1] := ((\lambda^1_1,\ldots,\lambda^1_f),\ldots,(\lambda^r_1,\ldots,\lambda^r_f))$, with

$$\lambda^i_j := \begin{cases} \frac{sd_1}{f} & \text{if there exists an } n\in\mathbb{Z} \text{ such that } i\equiv 1+nd_1 \pmod r \text{ and } j\equiv 1+[\frac{nd_1}{r}] \pmod f \\ 0 & \text{else,}\end{cases}$$

is an embedding datum in the sense of [BG](2.3.1) (in particular $\frac{f}{d_1}$ is a divisor of s) such that the embedding $(L,\mathfrak{A})$ is equivalent in the sense of [BG](2.2.1) to the embedding $(\lambda[d_1])$ defined in [BG](2.3.1).

(iii) The embedding $(L,\mathfrak{A})$ is sound in the sense of [F](2.11a)(2.11b).

If the equivalent conditions (i)-(iii) are fulfilled, the sound-embedding $L^\times\subset\mathcal{K}_{\mathfrak{A}}$ in the sense of [F](2.11a)(2.11b) has invariant $d_1 = d^{(1)}$ as defined in [F](2.12) and $\mathfrak{A}\cap C_A(L)$ has invariant $\frac{sd_1}{f}$ and period $\frac{rf}{d_1}$. The invariants d_1 and f determine the embedding $(L,\mathfrak{A})$ up to the equivalence relation in

[BG](2.2.1). Finally, we want to note that if f is a divisor of d and d_1 a divisior of f that is prime to r and such that $\frac{f}{d_1}$ divides s, then λ as defined in (ii) is always an embedding datum in the sense of [BG](2.3.1).

Proof:

$(i) \Rightarrow (ii)$: Clearly, there exist an embedding datum λ of degree f such that $(L, \mathfrak{A})$ is equivalent to $(\lambda) = ((\lambda_1^1, \ldots, \lambda_f^1), \ldots, (\lambda_1^r, \ldots, \lambda_f^r))$. If $\mathfrak{A} \cap C_A(L)$ is a principal order continued by $\mathfrak{A}$, then all λ_j^i must either be 0 or equal to the invariant s' of $\mathfrak{A} \cap C_A(L)$. Therefore, by

$$\sum_{j=1}^{f} \lambda_j^i = s, \tag{2}$$

we conclude that $s' \mid s$. By $sr = m = s'r'$ we have $\frac{s}{s'} = \frac{r'}{r}$ and, as $\mathfrak{A}$ is a continuation of $\mathfrak{A} \cap C_A(L)$, [BG](2.4.4) applies and λ is 1-zero-periodical in the sense of [BG](2.4.1) (i.e., in the infinite sequence of numbers $\ldots, \lambda_f^r, \lambda_1^1, \ldots, \lambda_1^r, \ldots, \lambda_f^1, \ldots, \lambda_f^r, \lambda_1^1, \ldots$ the number of zeros between two non-zero terms is always the same). Since we have r' non-zero numbers in the sequence

$$\lambda_1^1, \ldots, \lambda_1^r, \ldots, \lambda_f^1, \ldots, \lambda_f^r \tag{3}$$

of rf numbers, we conclude that r' must be a divisor of rf; hence, $\frac{r'}{r}$ is a divisor of f.

Let $d_1 := \frac{f}{(\frac{r'}{r})} = \frac{f}{(\frac{s}{s'})}$ and conclude that $s' = \frac{sd_1}{f}$ and $r' = \frac{rf}{d_1}$, then up to equivalence we can assume that $\lambda_1^1 = \frac{sd_1}{f}$ and the number between two non-zero numbers in the sequence (3) is $d_1 - 1$; hence, they are the $\lambda^{\tau_r^{nd_1 - [\frac{nd_1}{r}]r}(1)}_{\tau_f^{[\frac{nd_1}{r}]}(1)} = \lambda^{\tau_r^{nd_1}(1)}_{\tau_f^{[\frac{nd_1}{r}]}(1)}$ for $n \in \mathbb{Z}$, where $\tau_r \in S_r$ is the cyclic permutation of $\{1, \ldots, r\}$ and $\tau_f \in S_f$ is the cyclic permutation of $\{1, \ldots, f\}$. Finally, again by (2), we conclude that exactly $\frac{f}{d_1}$ non-zero terms must appear in a sequence $\lambda_1^i, \ldots, \lambda_f^i$. For $i = 1$ we obtain that $\lambda_j^1 \neq 0$ iff $\frac{r}{\langle d_1, r\rangle} \mid n$ and $j \equiv [\frac{nd_1}{r}] \pmod{f}$; hence, $j = k\frac{d_1}{\langle d_1, r\rangle}$ for $k \in \{1, \ldots, \frac{f\langle d_1, r\rangle}{d_1}\}$. We deduce that $\langle d_1, r\rangle = 1$ and we are done with the first direction.

$(ii) \Rightarrow (i)$: This direction follows immediately by [BG](2.4.4) and it also implies that $\mathfrak{A} \cap C_A(L)$ has invariant $\frac{sd_1}{f}$ and period $\frac{rf}{d_1}$.

By [F](Theorem 2) we also have $(i) \Leftrightarrow (iii)$ and we have finished the proof of the first assertion.

If (i)-(iii) are fulfilled, then again the first of the latter assertions follows by [F](Theorem 2). For the very last assertion we have to show that $\sum_{j=1}^{f} \lambda_j^i = s$ for $i = 1, \ldots, r$, under the asserted assumptions i.e., there are exactly $\frac{f}{d_1}$ different numbers $j \in \{1, \ldots, f\}$ such that there exists an integer n with $j \equiv 1 + [\frac{nd_1}{r}] \pmod{f}$ and $i \equiv 1 + nd_1 \pmod{r}$ for fixed i. As d_1 is prime to r, the latter congruence has a solution n' and clearly all numbers translated by a multiple of r fulfil the same congruence. Therefore, the numbers $j \in \{1, \ldots, f\}$ are the representatives of the classes $1 + [\frac{n'd_1}{r}] + kd_1 \pmod{f}$ for $k \in \mathbb{Z}$ and, obviously, there are exactly $\frac{f}{d_1}$ different ones of them.

□

Remark: It is clear that all sound embeddings $M^\times \subset \mathcal{K}_\mathfrak{A}$, where M/L is an unramified extension of degree prime to $\frac{d}{f}$, are in the same equivalence class as $(L, \mathfrak{A})$ and have invariant $d^{(1)} = d_1$ in the sense of [F](2.12), and the latter assertion provides the connection between all sound embeddings in the sense of [F](2.11a)(2.11b) and sound pearl-embeddings. □

(2.3) Proposition *Let $\mathfrak{A}$ be a principal $\mathfrak{o}_F$-order in A, $(E, \mathfrak{A})$ an embedding and $B := C_A(E)$, then the following assertions are equivalent:*

(i) $\mathcal{K}_{\mathfrak{A}} \cap B = \mathcal{K}_{\mathfrak{A} \cap B}$ and $\mathfrak{A} \cap B$ is a principal order;

(ii) for the maximally unramified subextesion K/F in E/F the embedding $(K, \mathfrak{A})$ is sound in the sense of [F](2.11a)(2.11b);

(iii) for the maximally unramified subextension L/F of E/F with a degree dividing d (i.e., $[L : F] = \langle f(E/F), d\rangle$) the embedding $(L, \mathfrak{A})$ is sound in the sense of [F](2.11a)(2.11b);

(iv) there exists an embedding $(M, \mathfrak{A})$ such that M/F is a maximal extension in A and M/E an extension in B (i.e., the embedding $(E, \mathfrak{A})$ extends to an embedding of a maximal field).

Proof:

$(i) \Rightarrow (ii)$: If $\mathfrak{A} \cap B$ is a principal order, then, by [F](Corollary 1) there exists an embedding $(M, \mathfrak{A} \cap B)$ such that the extension M/E is maximal in B; hence, by definition and (i) $(M, \mathfrak{A})$ is a maximal embedding which extends the embedding $(K, \mathfrak{A})$. By [F](Theorem 2(i)) the embedding $(K, \mathfrak{A})$ is sound.

$(ii) \Rightarrow (iii)$: For the embedding $(L, \mathfrak{A})$ the extension L/F is a subextension of $(K, \mathfrak{A})$; hence, again by [F](Theorem 2(i)) the embedding $(L, \mathfrak{A})$ is sound.

$(iii) \Rightarrow (iv)$: Again, by [F](Theorem 2(i)) or (2.2) $\mathfrak{A} \cap C_A(L)$ is principal and continued by $\mathfrak{A}$. As $C_A(L)$ has index $\frac{d}{f}$ and $\langle f(E/L), \frac{d}{f}\rangle = 1$, we deduce by [G](Theorem 2.2(ii)) that the order $\mathfrak{A} \cap B$ is principal and continued by $\mathfrak{A}$. Now again by [F](Corollary 1) there exists an embedding $(M, \mathfrak{A} \cap B)$ such that the extension M/E is maximal in B; hence, by definition and because $\mathfrak{A}$ continues $\mathfrak{A} \cap B$ the embedding $(M, \mathfrak{A})$ is well defined and a maximal embedding which extends the embedding $(E, \mathfrak{A})$.

$(iv) \Rightarrow (i)$: As before we observe that the field extension M/E is maximal in B; hence, our conclusion is given by [Z1](Theorem 2).

□

Remark: In the split case $d = 1$ every embedding $(E, \mathfrak{A})$ is sound because condition (iii) is allways fulfilled. □

Now assume again that $\mathfrak{A}$ is a principal order with invariant s and period r $(sr = m)$. Then the last Proposition enables us to extend the definition of a sound embedding of [F] from unramified extensions to arbitrary extensions.

(2.4) Definition *(i) An embedding $(E, \mathfrak{A})$ is called sound if it fulfils the equivalent conditions of (2.3).*

(ii) Similarly as in [BG] $\beta \in A$ is called $\mathfrak{A}$-pure if $E := F[\beta]$ is a field with $E^\times \subset \mathcal{K}_{\mathfrak{A}}$.

(iii) Moreover, β as in (ii) is called $\mathfrak{A}$-sound iff the embedding $(F[\beta], \mathfrak{A})$ is sound in the sense of (i).

Let β and $E := F[\beta]$ be as in Definition (2.4)(iii), then there exists a field extension M/E such that the extension M/F is maximal in A/F and $M^\times \subset \mathcal{K}_{\mathfrak{A}}$. Let $B := C_A(E)$ be the centralizer of E in A and $\mathfrak{B} := \mathfrak{A} \cap B$. By [Z1](Theorem 2) $\mathfrak{B}$ is a principal order and β is $(e(M/F), f(M/F))$-pure in the sense of [Z1](Definition 4). Let $\mathfrak{Q}$ be the Jacobson radical of $\mathfrak{B}$, then by [Z1](Theorem 2) for all $j \in \mathbb{Z}$ we have $\mathfrak{P}^j \cap B = \mathfrak{Q}^{[\frac{j+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}]}$, where $e_{\mathfrak{A}/\mathfrak{B}} := \langle f(E/F), \frac{f(M/F)}{s}\rangle$. Clearly, $e_{\mathfrak{A}/\mathfrak{B}}$ just depends on the embedding of E and not on the special choice of M.

(2.5) Notation

We call $e_{\mathfrak{A}/\mathfrak{B}}$ the ramification exponent of $\mathfrak{A}$ over $\mathfrak{B}$ and define a function $\sigma_{\mathfrak{A}/\mathfrak{B}} : \mathbb{Z} \to \mathbb{Z}$ by $\sigma_{\mathfrak{A}/\mathfrak{B}}(j) := [\frac{j+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}]$ for all $j \in \mathbb{Z}$.

(2.6) Lemma *(i) For all $j \in \mathbb{Z}$ we have $\sigma_{\mathfrak{A}/\mathfrak{B}}(1-j) = 1 - \sigma_{\mathfrak{A}/\mathfrak{B}}(j)$.*

(ii) For all $j \in \mathbb{Z}$ such that $e_{\mathfrak{A}/\mathfrak{B}} \mid j$, we have $\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{j}{2}]+1) = [\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(j)}{2}]+1$.

(iii) If r' is the period of $\mathfrak{B}$, r the period of $\mathfrak{A}$ and d' is the index of B, then we have $e_{\mathfrak{A}/\mathfrak{B}} = \frac{rd}{r'd'e(E/F)}$.

Proof:

(i) Let us write $j = e_{\mathfrak{A}/\mathfrak{B}}k - t$, where $0 \le t < e_{\mathfrak{A}/\mathfrak{B}}$, then $\sigma_{\mathfrak{A}/\mathfrak{B}}(1-j) = [\frac{1-e_{\mathfrak{A}/\mathfrak{B}}k+t+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}] = [\frac{e_{\mathfrak{A}/\mathfrak{B}}(1-k)+t}{e_{\mathfrak{A}/\mathfrak{B}}}] = 1-k = 1-[\frac{e_{\mathfrak{A}/\mathfrak{B}}k-t+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}] = 1-\sigma_{\mathfrak{A}/\mathfrak{B}}(j)$.

(ii) Let us write $j = e_{\mathfrak{A}/\mathfrak{B}}k$, then we have two cases:

(a) We have $2 \mid k$; hence, we can write $k = 2l$ and obtain $\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{j}{2}]+1) = [\frac{e_{\mathfrak{A}/\mathfrak{B}}l+1+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}] = l+1 = [\frac{(\frac{j}{e_{\mathfrak{A}/\mathfrak{B}}})}{2}]+1 = [\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(j)}{2}]+1$.

(b) We have $2 \nmid k$; hence, we can write $k = 2l+1$ and obtain $\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{j}{2}]+1) = [\frac{e_{\mathfrak{A}/\mathfrak{B}}l+[\frac{e_{\mathfrak{A}/\mathfrak{B}}}{2}]+1+e_{\mathfrak{A}/\mathfrak{B}}-1}{e_{\mathfrak{A}/\mathfrak{B}}}] = l+1 = [\frac{(\frac{j}{e_{\mathfrak{A}/\mathfrak{B}}})}{2}]+1 = [\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(j)}{2}]+1$.

(iii) The proof is essentially the same as for [BG](4.3.4), where we just resign the assumption that E/F is unramified and note that then $\nu_{\mathfrak{B}}(\pi_F) = r'd'e(E/F)$.

□

The second main interest of this section is to introduce two canonical mappings associated to an element which generates a field extension in A/F. As in [BK3](1.3)(1.4) we define a tame co-restriction and the adjoint map for β:

(2.7) Definition *Let β be an element which generates a field extension E/F in A and let $B := C_A(E)$.*

(i) For all $x \in A$ let $a_\beta(x) := \beta x - x\beta$ and we call the map $a_\beta : A \to A$ the adjoint map associated to β.

(ii) A tame co-restriction $\mathfrak{s}_\beta$ on A with respect to β is a (B,B)-bimodule homomorphism $\mathfrak{s}_\beta : A \to B$ such that for all principal orders $\mathfrak{A}$ for which β is $\mathfrak{A}$-sound we have $\mathfrak{s}_\beta(\mathfrak{A}) = \mathfrak{A} \cap B$.

As in [BK3](1.3.4) one can prove (using Lemma (2.6)(i) and (1.2) above):

(2.8) Proposition *(i) Let ψ_E, ψ_F be continuous additive characters of E, F with conductors $\mathfrak{p}_E$ and $\mathfrak{p}_F$, respectively ($E := F[\beta]$). Define additive characters ψ_B, ψ_A of B, A as above in section (1) for A. There exists a unique map $\mathfrak{s}_\beta : A \to B$ such that for all $a \in A$, $b \in B$ we have $\psi_A(ab) = \psi_B(\mathfrak{s}_\beta(a)b)$ and this map is a tame co-restriction with respect to β.*

(ii) For $\mathfrak{A}$ as in this section (i.e., $\mathfrak{A}$ is a principal order), β $\mathfrak{A}$-sound and all $j \in \mathbb{Z}$ we have: $\mathfrak{s}_\beta(\mathfrak{P}^j) = \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(j)}$.

□

Moreover, as in [BK3] we obtain an infinite exact sequence:

(2.9) $\ldots \stackrel{\mathfrak{s}_\beta}{\to} A \stackrel{a_\beta}{\to} A \stackrel{\mathfrak{s}_\beta}{\to} A \stackrel{a_\beta}{\to} \ldots$

3 Simple strata and embeddings

Let us fix a simple left A-module V and identify A with $End_D(V)$, where V is a right $D \simeq End_A(V)$-vectorspace. Again let $\mathfrak{A}$ be a principal $\mathfrak{o}_F$-order in A as in the last section. Let $\{X_j\}_{j\in\mathbb{Z}}$ be the lattice chain of $\mathfrak{A}$ in V as in (0.2), which means that $\mathfrak{A} = End_{\mathfrak{o}_D}(\{X_j\}_{j\in\mathbb{Z}})$. Let $\tilde{A} := End_F(V)$ and $\tilde{\mathfrak{A}} := End_{\mathfrak{o}_F}(\{X_j\}_{j\in\mathbb{Z}})$, then we have canonical embeddings $A \hookrightarrow \tilde{A}$ and $\mathfrak{A} \hookrightarrow \tilde{\mathfrak{A}}$. Similarly as in [Br2](1.1.7) we define:

(3.1) Definitions

(i) A stratum $[\mathfrak{A}, n, q, \beta]$ is said to be pure if β is $\mathfrak{A}$-pure and $n = -\nu_{\mathfrak{A}}(\beta)$.

(ii) A pure stratum as in (i) is said to be sound if β is $\mathfrak{A}$-sound.

(iii) A stratum as in (ii) is said to be simple if the associated stratum $[\tilde{\mathfrak{A}}, n, q, \beta]$ is simple in the sense of [BK3](1.5.5).

In fact, one can define a simple stratum in the same way as in [BK](1.5.5) if one generalizes the notion of the critical exponent $k_0(\beta, \mathfrak{A})$:

(3.2) Definition

(i) For all $j \in \mathbb{Z}$ we set (as in [BK3](1.4.3)) $\mathfrak{N}_j(\beta, \mathfrak{A}) := \{x \in \mathfrak{A} \mid a_\beta(x) \in \mathfrak{P}^j\}$;

(ii) for an $\mathfrak{A}$-sound element β we set $k_0(\beta, \mathfrak{A}) := max\{k \in \mathbb{Z} \mid \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subseteq \mathfrak{B} + \mathfrak{P}\}$, where $\mathfrak{B} := \mathfrak{A} \cap C_A(E)$.

To prove that the latter generalization makes sense we have to make clear more of the details with respect to the connection of the orders $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$. For this purpose we set $\tilde{B} := C_{\tilde{A}}(E)$ and $\tilde{\mathfrak{B}} := \tilde{\mathfrak{A}} \cap \tilde{B}$. Moreover, let $\tilde{\mathfrak{Q}}$ be the Jacobson radical of $\tilde{\mathfrak{B}}$. The following lemma is a slight modification of [Br4](I,4.2):

(3.3) Lemma *Let $A = M_m(D)$, and $V = D^m$ (column vectors) be the standard simple A-module, then the isomorphism $M_m(D) \simeq End_D(V)$ enables us to identify A via the canonical embedding* *into $\tilde{A} = End_F(V)$ and the canonical isomorphism $\tilde{A} = End_F(V) \simeq End_D(V) \otimes_F D^{op}$ (where $(.)^{op}$ denotes the opposite algebra structure of a given algebra) with the subset $A \otimes 1$ of $A \otimes_F D^{op}$. In the same manner we have $\tilde{B} = End_E(V) \simeq B \otimes D^{op}$ and we can identify B with the subset $B \otimes 1$. Moreover, these canonical isomorphisms restrict for all $j \in \mathbb{Z}$ to isomorphisms*

(i) $\tilde{\mathfrak{P}}^j \simeq \oplus_{k=0}^{d-1} \mathfrak{P}^{j+kr} \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L$ *and*

(ii) $\tilde{\mathfrak{Q}}^j \simeq \oplus_{k=0}^{d-1} \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(j+kr)} \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L$,

where L/F is an unramified field extension in D of degree d (i.e., L splits D).

(3.4) Lemma *In the notation of (3.3) we have $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) = \oplus_{k=0}^{d-1} \mathfrak{P}^{kr} \cap \mathfrak{N}_{j+kr}(\beta, \mathfrak{A}) \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L$. In particular we have $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) \cap A = \mathfrak{N}_j(\beta, \mathfrak{A})$.*

Proof:

Let $x \in \tilde{\mathfrak{A}}$, then by (3.3) we have $x = \oplus_{k=0}^{d-1} x_k \otimes \pi_D^{-k} l_k$, where $x_k \in \mathfrak{P}^{kr}$ and $l_k \in \mathfrak{o}_L$ for $k = 0, \ldots, d-1$ and $\tilde{\mathfrak{P}}^j = \oplus_{k=0} \mathfrak{P}^{j+kr} \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L$. Therefore $a_{\beta \otimes 1}(x) \in \tilde{\mathfrak{P}}^j$ if and only if $a_\beta(x_k) \in \mathfrak{P}^{j+kr}$ for $k = 0, \ldots, d-1$. Hence, $x \in \mathfrak{N}_j(\beta, \tilde{\mathfrak{A}})$ if and only if $x_k \in \mathfrak{P}^{kr} \cap \mathfrak{N}_{j+kr}(\beta, \mathfrak{A})$ as required.

□

We need a lemma which generalizes [BK3](2.1.1):

(3.5) Lemma *If $\nu, j \in \mathbb{Z}$, then $\mathfrak{P}^\nu \cap \mathfrak{N}_{\nu+j}(\beta, \mathfrak{A}) = \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)} \mathfrak{N}_j(\beta, \mathfrak{A})$.*

Proof:

It is easy to see that the right-hand side is contained in the left-hand side. By [BK](2.1.1) we have $\tilde{\mathfrak{P}}^\nu \cap \mathfrak{N}_{\nu+j}(\beta, \tilde{\mathfrak{A}}) = \tilde{\mathfrak{Q}}^\nu \mathfrak{N}_j(\beta, \tilde{\mathfrak{A}})$. If we use the notions of (3.3) and identify A with its image $A \otimes 1$, we deduce that $\mathfrak{P}^\nu \cap \mathfrak{N}_{\nu+j}(\beta, \mathfrak{A}) = \tilde{\mathfrak{P}}^\nu \cap \mathfrak{N}_{\nu+j}(\beta, \tilde{\mathfrak{A}}) \cap A = (\tilde{\mathfrak{Q}}^\nu \mathfrak{N}_j(\beta, \tilde{\mathfrak{A}})) \cap A =$ (3.3(ii)+3.4) $\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)} \mathfrak{N}_j(\beta, \mathfrak{A}) + \pi_F^{-1} \sum_{k=1}^{d-1} \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu+kr)} \mathfrak{N}_{j+(d-k)r}(\beta, \mathfrak{A}) = \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu)} \mathfrak{N}_j(\beta, \mathfrak{A})$, where the latter equality comes from the fact that the last summands are contained in the first one.

□

Now we see that the use of the two different kinds of critical exponents actually leads to the same result:

(3.6) Proposition

We have $k_0(\beta, \mathfrak{A}) = k_0(\beta, \tilde{\mathfrak{A}})$.

Proof:

In the notations of (3.3) and the definition of critical exponents we have to show that $\mathfrak{N}_j(\beta, \mathfrak{A}) \subset (\mathfrak{B} + \mathfrak{P})$ if and only if $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) \subset (\tilde{\mathfrak{B}} + \tilde{\mathfrak{P}})$.

First assume that $\mathfrak{N}_j(\beta, \mathfrak{A}) \subset (\mathfrak{B} + \mathfrak{P})$, then $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) =$ (3.4) $\oplus_{k=0}^{d-1} \mathfrak{P}^{kr} \cap \mathfrak{N}_{j+kr}(\beta, \mathfrak{A}) \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L =$ (Lemma 3.5) $\oplus_{k=0}^{d-1} \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(kr)} \mathfrak{N}_j(\beta, \mathfrak{A}) \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L \subset$ (by assumption) $\oplus_{k=0}^{d-1} \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(kr)} (\mathfrak{B} + \mathfrak{P}) \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L \subset \oplus_{k=0}^{d-1} \mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(kr)} \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L + \oplus_{k=0}^{d-1} \mathfrak{P}^{1+kr} \otimes_{\mathfrak{o}_F} \pi_D^{-k} \mathfrak{o}_L =$ (3.3((ii)+(i))) $\tilde{\mathfrak{B}} + \tilde{\mathfrak{P}}$.

Conversely, assume that $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) \subset (\tilde{\mathfrak{B}} + \tilde{\mathfrak{P}})$, then it follows that $\mathfrak{N}_j(\beta, \mathfrak{A}) =$ (3.4) $\mathfrak{N}_j(\beta, \tilde{\mathfrak{A}}) \cap A \subset$ (by assumption) $(\tilde{\mathfrak{B}} + \tilde{\mathfrak{P}}) \cap A =$ (3.3((ii)+(i))) $\mathfrak{B} + \mathfrak{P}$.

So we are done.

□

As in the split case one can introduce the notion of a simple pair due to [BH](1.5):

(3.7) Definition *A simple pair is a pair* $[q^0, \beta']$ *of an integer* q^0 *and an element* β' *from an algebraic closure of* F *such that* $q^0 < \min\{-\nu_E(\beta'), -k_0(\beta', \mathfrak{A}(E))\}$, *where* $E := F[\beta']$ *and* $\mathfrak{A}(E)$ *is the principal order in* $A(E) := End_F(E)$ *with* $\mathfrak{o}_F$*-lattice chain* $\{\mathfrak{p}_E^j\}_{j\in\mathbb{Z}}$ *in the simple* $A(E)$*-module* E.

As in the split case the stratum $[\mathfrak{A}, n, q, \beta]$ is simple for all $\mathfrak{A}$-sound embeddings β of β' to A (where we must assume now that $[E : F] \mid md)$ and all $q \in \mathbb{Z}$ with $q^0 = [\frac{e(E/F)q}{rd}]$, where $n = -\nu_{\mathfrak{A}}(\beta)$ and r is the period of $\mathfrak{A}$. Conversely, a simple stratum $[\mathfrak{A}, n, q, \beta]$ leads to a simple pair $[[\frac{e(E/F)q}{rd}], \beta]$. We also note that $k_0(\beta, \mathfrak{A}) = k_0(\beta, \mathfrak{A}(E))\frac{rd}{e(E/F)}$ and $\nu_{\mathfrak{A}}(\beta) = \nu_E(\beta')\frac{rd}{e(E/F)}$.

From [BG](5.1) and its proof it is straightforward that each sound stratum $[\mathfrak{A}, n, q, \beta]$ is equivalent to a simple stratum $[\mathfrak{A}, n, q, \gamma]$ in the sense of (3.1). In fact, one can see from the proof of [BG](5.1)(5.4) that it could have been shown that:

(3.8) Proposition *Let* $[\mathfrak{A}, n, q, \beta]$ *be a sound stratum,* K/F *the maximally unramified subextension of the extension* E/F. *Let* M/K *be an unramified extension in* A *such that* $M[\beta]$ *is a field, and* $M^\times \subset \mathcal{K}_{\mathfrak{A}}$ *is sound embedded in the sense of (2.4). Let* $\mathfrak{C} := \mathfrak{A} \cap C_A(M)$, *then there exists a* $\mathfrak{C}$*-sound element* γ *such that the stratum* $[\mathfrak{A}, n, q, \gamma]$ *is simple and equivalent to* $[\mathfrak{A}, n, q, \beta]$. *Moreover, we have* $F[\gamma]^{ur} \subset K$ *and finally the stratum* $[\mathfrak{C}, n', q', \gamma]$ *is simple as well as equivalent to* $[\mathfrak{C}, n', q', \beta]$, *where* $n' := \sigma_{\mathfrak{A}/\mathfrak{C}}(n)$ *and* $q' := \sigma_{\mathfrak{A}/\mathfrak{C}}(q)$.

Proof:

Because of $\mathfrak{o}_E \subset \mathfrak{A}$, $\mathfrak{o}_M \subset \mathfrak{A}$ and because there is an integral basis of $\mathfrak{o}_{M[\beta]} \mid \mathfrak{o}_M$ which is contained in $\mathfrak{o}_{F[\beta]}$ we also have $\mathfrak{o}_{M[\beta]} \subset \mathfrak{A}$. Since the extension $M[\beta]/F[\beta]$ is unramified, we conclude that $M[\beta]^\times \subset \mathcal{K}_{\mathfrak{A}}$. Of course, we also have $M[\beta]^\times \subset \mathcal{K}_{\mathfrak{C}}$ and the stratum $[\mathfrak{C}, n', q', \beta]$ is sound.

As in the beginning of the proof of [BG](5.1) one reduces to the case of $\frac{rd}{e(E/F)} \mid q$ and then, by (2.6)(iii), we have $\sigma_{\mathfrak{A}/\mathfrak{C}}(q) = q\frac{\bar{r}\bar{d}}{rd}$, where $\bar{r}$ is the period of $\mathfrak{C}$ and $\bar{d}$ is the index of $C_A(M)$.

If we now pay attention to the changed notation and if we follow the proof of [BG](5.1) where we replace K by M with the above assumptions, then we find a $\mathfrak{C}$-sound element γ such that the stratum $[\mathfrak{C}, n', q', \gamma]$ (in [BG] denoted by $[\mathfrak{B}, n', q', \gamma_1]$) is simple and equivalent to $[\mathfrak{C}, n', q', \beta]$ (in [BG] denoted by $[\mathfrak{B}, n', q', \beta]$). The argument $F[\beta] = K[\beta]$ used following [BG](5.5) is not necessary for proving $F[\gamma]^\times \subset \mathcal{K}_{\mathfrak{A}}$ (in the notation of [BG] $F[\gamma_1]^\times \subset \mathcal{K}$), because, as we still remarked above, $M[\beta]^\times \subset \mathcal{K}_{\mathfrak{A}}$. Therefore, we can follow the proof of [BG](5.1) as before to show that the stratum $[\mathfrak{A}, n, q, \gamma]$ (in the notation of [BG] $[\mathfrak{A}, n, q, \gamma_1]$) is pure (in the sense of [BG]) and equivalent to $[\mathfrak{A}, n, q, \beta]$. Clearly in our situation the $\mathfrak{C}$-soundness of β implies its $\mathfrak{A}$-soundness and we have finished the main part of our proposition. The last assertion is straightforward.

□

Remark: (i) From [BG](5.1)(ii) we deduce that all pure strata $[\mathfrak{A}, n, q, \gamma]$ that are equivalent to $[\mathfrak{A}, n, q, \beta]$ such that $[F[\gamma] : F]$ has minimal degree among all strata fulfilling the same property are necessarily simple. Moreover, in this situation we have $f(F[\gamma]/F) \mid f(F[\beta]/F)$ and $e(F[\gamma]/F) \mid e(F[\beta]/F)$.

(ii) From (2.6)(iii) and in the notation of (2.5),(3.8) one deduces that if $\frac{rd}{e(E/F)} \mid q$, then we have $q' = \frac{q}{e_{\mathfrak{A}/\mathfrak{C}}}$. □

The last proposition makes it possible to define the notion of an approximation sequence:

(3.9) Definition

(A) Let $[\mathfrak{A}, n, l, \beta]$ be a sound stratum, then the last jump of the stratum is the smallest integer q that is greater than or equal to l and smaller than or equal to n such that there exists a sound stratum $[\mathfrak{A}, n, q, \gamma]$ equivalent to $[\mathfrak{A}, n, q, \beta]$ with the property that $[F[\gamma] : F] < [F[\beta] : F]$.

(B) Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, then the last proposition enables us to find for each integer q, where $0 \le q \le n$, a simple stratum $[\mathfrak{A}, n, q, \gamma^{(q)}]$ that is equivalent to $[\mathfrak{A}, n, q, \beta]$ and, of course, $[\mathfrak{A}, n, q_2, \gamma^{(q_2)}]$ is equivalent to $[\mathfrak{A}, n, q_2, \gamma^{(q_1)}]$ if $q_2 \ge q_1$. An approximation sequence for $[\mathfrak{A}, n, 0, \beta]$ is a sequence of simple strata $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$, $\mu \in \mathbb{N}$, where

(i) $q_1 = 0$ and $\gamma_1 = \beta$,

(ii) $q_\mu = n$ and $\gamma_\mu = 0$,

(iii) $q_{i+1} > q_i$ is the last jump for the stratum $[\mathfrak{A}, n, q_i, \gamma_i]$ in the sense of (A) and, finally,

(iv) $[\mathfrak{A}, n, q_{i+1}, \gamma_{i+1}]$ is a simple stratum equivalent to $[\mathfrak{A}, n, q_{i+1}, \gamma_i]$ for $i = 1, \ldots, \mu - 1$.

The numbers $q_1, \ldots, q_\mu$ called the jumps of the approximation sequence.

(C) A sequence $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$, where we have $q_1 < q_2 < \ldots < q_\mu$ and $\mu \in \mathbb{N}$ and which fulfils (A)(i)(ii)(iv) and contains an approximation sequence for the simple stratum $[\mathfrak{A}, n, 0, \beta]$ in the sense of (B) is called a weak approximation sequence for the simple stratum $[\mathfrak{A}, n, 0, \beta]$ if we have $\frac{rd}{e(F[\gamma_{i-1}]/F)} \mid q_i$ for all $i = 2, \ldots, \mu$. Then we also call the numbers $q_1, \ldots, q_\mu$ the jumps of the weak approximation sequence.

Remark: From [BG](Theorem (5.1)(ii)) and the construction it is clear that (if $n > 0$) such an approximation sequence for $[\mathfrak{A}, n, 0, \beta]$ always exists and that the sequence of numbers $q_1, \ldots, q_\mu$ is the same for all approximation sequences. In fact, we have $q_{i+1} = -k_0(\gamma_i, \mathfrak{A})$ for $i = 1, \ldots, \mu - 2$. In particular, all approximation sequences are weak approximation sequences.

(3.10) Definition *A simple stratum $[\mathfrak{A}, n, 0, \beta]$ with $n > 0$ that has an approximation sequence of length $\mu = 2$ is said to be minimal over F.*

If $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$, where $n > 0$, is an approximation sequence to a simple stratum, then it is easy to see that the stratum $[\mathfrak{A}, n, 0, \gamma_j]$ is simple and has an approximation sequence $[\mathfrak{A}, n, q_k, \gamma_k]$, $k = j, \ldots, \mu$ for $j = 1, \ldots, \mu - 1$. In particular, the stratum $[\mathfrak{A}, n, 0, \gamma_{\mu-1}]$ is minimal over F. These facts enable us to introduce inductive arguments along an approximation sequence as in [BK3].

The main statement of this section is a consequence of the stronger version (3.8) of [BG](5.1):

(3.11) Theorem

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and K be the maximal unramified part of the extension E/F, where $E := F[\beta]$. Let $C := C_A(K)$ be the centralizer of K in A and let $\mathfrak{C} := \mathfrak{A} \cap C$. By assumption the embedding $K^\times \subset \mathcal{K}_\mathfrak{A}$ is sound and, therefore, $\mathfrak{C}$ is a principal order. Moreover, the stratum $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), 0, \beta]$ is simple and in this situation, there exists an approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$, for $[\mathfrak{A}, n, 0, \beta]$ with the following properties:

(i) $K = F[\gamma_1]^{ur} \supset F[\gamma_2]^{ur} \supset \ldots \supset F[\gamma_\mu]^{ur}$, where $F[\gamma_i]^{ur}$ is the maximal unramified part of the extension $F[\gamma_i]/F$ for $i = 1, \ldots, \mu$ and

(ii) $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(q_i), \gamma_i]$, $i = 1, \ldots, \mu$, is a weak approximation sequence for $[\mathfrak{C}, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), 0, \beta]$.

Proof:

First choose an arbitrary approximation sequence with jumps $q_1 < \ldots < q_\mu$ and apply (3.8) to the stratum $[\mathfrak{A}, n, q_2, \beta]$ with $M = K$ to obtain the first two special approximations $[\mathfrak{A}, n, q_1, \gamma_1] = [\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}, n, q_2, \gamma_2] = [\mathfrak{A}, n, q_2, \gamma]$. Clearly, $K[\gamma_2]$ is a field and we have performed the beginning of an induction over μ.

So let us assume that we have approximations $[\mathfrak{A}, n, q_1, \gamma_1], \ldots, [\mathfrak{A}, n, q_{i-1}, \gamma_{i-1}]$ such that $K[\gamma_1], \ldots, K[\gamma_{i-1}]$ are fields, then (3.8) applies again and the obtained approximation $[\mathfrak{A}, n, q_i, \gamma_i]$ has also the property that $K[\gamma_i]$ is a field.

Now we obtain the approximation sequence inductively with the above properties as required.

□

This special kind of approximations is not necessary for introducing simple characters, but one can use such approximations for the later matching of simple characters among various simple algebras.

(3.12) Definition

An approximation sequence as in (3.11) is said to be special.

4 Orbitfilters of additive characters and their stabilizers

Let $\mathfrak{A}$ be a principal order and $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum. Moreover, use the conventions for E, B and $\mathfrak{B}$ from section (2). We define ψ_β^+ as in section (1). In the next section (5) we want to construct certain admissible pairs to representation filters of the desending series $U^1(\mathfrak{A}) \supset U^2(\mathfrak{A}) \supset U^3(\mathfrak{A}) \supset \ldots$ (see [Z2]) and in section (6) we want to match them with similar admissible pairs in other simple algebras with the later aim of generalizing the results of [Z3] and [Z4]. The admissible pairs generate stable representation filters, and as in [Z5] (or [Z7]) we want to obtain the associated groups as stabilizers of certain orbitfilters associated with ψ_β^+. In [BK3] the underlying Heisenberg characters are called 'simple characters' and we want to use the same conventions. As in [Br3] we restrict ourselves to the split case by using similar constructions for the canonical splittings, where we use the conventions of section (3) above for $\tilde{A}$ and $\tilde{\mathfrak{A}}$.

Let us consider the family of orbits $\mathcal{F}(\psi_\beta^+) := \{(U^j(\mathfrak{A})) \circ \psi^+_{\beta|\mathfrak{P}^j}\}_{j\in\mathbb{N}}$, where $U^j(\mathfrak{A}) \circ \psi^+_{\beta|\mathfrak{P}^j}$ means the character orbit of $\psi^+_{\beta|\mathfrak{P}^j}$ under the adjoint action of $U^j(\mathfrak{A})$ and $\psi^+_{\beta|\mathfrak{P}^j}$ is the restriction of ψ_β^+ to $\mathfrak{P}^j$.

Let us fix a weak approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$ for $[\mathfrak{A}, n, 0, \beta]$.

(4.1) Definition *(i) Set* $J^1(\psi_\beta^+) := Stab_{U^1(\mathfrak{A})}(\mathcal{F}(\psi_\beta^+))$, *where* $U^1(\mathfrak{A})$ *acts again by conjugation,*
(ii) let $1 + \mathfrak{J}^1(\psi_\beta^+) := J^1(\psi_\beta^+)$,
(iii) let $\mathfrak{H}^1(\psi_\beta^+) := \mathfrak{J}^1(\psi_\beta^+)^{\perp\psi_\beta^+}$,
(iv) let $H^1(\beta, \mathfrak{A}) := 1 + \mathfrak{H}^1(\psi_\beta^+)$,
(v) let $\mathfrak{J}^1(\beta, \mathfrak{A}) := \sum_{i=1}^{\mu} \mathfrak{P}^{[\frac{q_i+1}{2}]} \cap C_{\mathfrak{P}}(\gamma_i)$,
(vi) let $J^1(\beta, \mathfrak{A}) := 1 + \mathfrak{J}^1(\beta, \mathfrak{A})$ *and, finally,*
(vii) let $\mathfrak{H}^1(\beta, \mathfrak{A}) := \sum_{i=1}^{\mu} \mathfrak{P}^{[\frac{q_i}{2}]+1} \cap C_{\mathfrak{P}}(\gamma_i)$.

Clearly, by definition, $\mathfrak{J}^1(\psi_\beta^+)$ and $\mathfrak{H}^1(\psi_\beta^+)$ are rings that are normalized by $\mathcal{K}_{\mathfrak{B}}$. From (1.2) we conclude that we have $(\mathfrak{P}^i)^{\perp\psi_\beta^+} = \{x \in \mathfrak{P} \mid \forall y \in \mathfrak{P}^i : \psi_\beta^+(xy - yx) = \psi^+_{a_\beta(x)}(y) = 1\} = \{x \in \mathfrak{P} \mid a_\beta(x) \in \mathfrak{P}^{-i+1}\} = \mathfrak{P} \cap \mathfrak{N}_{-i+1}(\beta, \mathfrak{A})$ for all $i \in \mathbb{Z}$; hence, from [Z7](1.8) we obtain:

(4.2) Lemma *(i)* $\mathfrak{J}^1(\psi_\beta^+) = \sum_{i=1}^{\infty} \mathfrak{P}^i \cap \mathfrak{N}_{-i}(\beta, \mathfrak{A})$ *and*
(ii) $\mathfrak{H}^1(\psi_\beta^+) = \sum_{i=1}^{\infty} \mathfrak{P}^i \cap \mathfrak{N}_{-i+1}(\beta, \mathfrak{A})$.

□

From these descriptions of the rings $\mathfrak{J}^1(\psi_\beta^+)$, $\mathfrak{H}^1(\psi_\beta^+)$ and the definition of a weak approximation sequence ($(\beta - \gamma_i) \in \mathfrak{P}^{-q_i}$ and $-a_{\gamma_i}(C_A(\gamma_i)) = \{0\}$ for $i = 1, \ldots, \mu$) one easily deduces the inclusions $\mathfrak{J}^1(\beta, \mathfrak{A}) \subset \mathfrak{J}^1(\psi_\beta^+)$ and $\mathfrak{H}^1(\beta, \mathfrak{A}) \subset \mathfrak{H}^1(\psi_\beta^+)$.

To see the opposite inclusion, let us consider the same situation in the split algebra $\tilde{A}$. For that purpose let $\tilde{\psi}_\beta^+$ be the additive character of the Jacobson radical $\tilde{\mathfrak{P}}$ of $\tilde{\mathfrak{A}}$ defined similarly as ψ_β^+ of $\mathfrak{P}$. By Lemma (4.2)(ii) and [BK3]((3.1.9)(i) with $t = 2i - 1$ for $i = 1, 2, \ldots$, its proof, induction over the weak approximation sequence, and (3.1.2)) we have $\mathfrak{H}^1(\beta, \tilde{\mathfrak{A}}) \supset \mathfrak{H}^1(\tilde{\psi}_\beta^+)$. By Lemma (4.2)(i) and [BK3]((3.1.10)(i) with $t = 2i$ for $i = 1, 2, \ldots$, its proof, induction over the weak approximation sequence and (3.1.2)) we have $\mathfrak{J}^1(\beta, \tilde{\mathfrak{A}}) \supset \mathfrak{J}^1(\tilde{\psi}_\beta^+)$. As the opposite inclusions hold by the same argument as above, we deduce that $\mathfrak{H}^1(\beta, \tilde{\mathfrak{A}}) \cap A = \mathfrak{H}^1(\tilde{\psi}_\beta^+) \cap A$ and $\mathfrak{J}^1(\beta, \tilde{\mathfrak{A}}) \cap A = \mathfrak{J}^1(\tilde{\psi}_\beta^+) \cap A$. Similarly as in [Br3] we conclude:

(4.3) Proposition *We have* $\mathfrak{H}^1(\beta, \mathfrak{A}) = \mathfrak{H}^1(\psi_\beta^+)$ *and* $\mathfrak{J}^1(\beta, \mathfrak{A}) = \mathfrak{J}^1(\psi_\beta^+)$. *In particular,* $H^1(\beta, \mathfrak{A})$ *and* $J^1(\beta, \mathfrak{A})$ *are groups that are normalized by* $\mathcal{K}_{\mathfrak{B}}$. *Moreover, the definitions of the latter groups are, in fact, independent of the special choice of the fixed weak approximation sequence* $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$ *for* $[\mathfrak{A}, n, 0, \beta]$.

Another important property follows easily by the explicit description of Lemma (4.2):

(4.4) Lemma *(i)* $\mathfrak{H}^1(\beta,\mathfrak{A})$ *is a two-sided ideal in* $\mathfrak{J}^1(\beta,\mathfrak{A})$ *and*

(ii) $\mathfrak{H}^1(\beta,\mathfrak{A})$, $\mathfrak{J}^1(\beta,\mathfrak{A})$ *are* $(\mathfrak{B},\mathfrak{B})$*-bimodules.*

□

As an example of an inductive argument along the weak approximation sequence for $[\mathfrak{A},n,0,\beta]$, we can generalize the Propositions [BK3](3.1.16) and [BK3](3.1.22). Following the proofs of [BK3] argument by argument without great change and using arguments as in [Br3] to obtain group inclusions for subgroups of A from the split case in $\tilde{A}$, we obtain:

(4.5) Proposition *Let* $[\mathfrak{A},n,0,\beta]$ *be a simple stratum with weak approximation sequence* $[\mathfrak{A},n,q_i,\gamma_i]$, $i=1,\dots,\mu$, *where* $\mu>1$, *then we have:*

(i) For $0\le\nu\le q_2-1$ *we have an exact sequence*

$$\mathfrak{P}^{q_2-\nu}\cap\mathfrak{N}_{-\nu}(\beta,\mathfrak{A})+\mathfrak{P}^{[\frac{q_2+1}{2}]}\cap\mathfrak{J}^1(\beta,\mathfrak{A})\xrightarrow{a_\beta}(\mathfrak{P}^{\nu+1}\cap\mathfrak{H}^1(\beta,\mathfrak{A}))^*\xrightarrow{\mathfrak{s}_\beta}\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(-\nu)}\to 0.$$

(ii) For $1\le\nu\le q_2$ *we have an exact sequence*

$$\mathfrak{P}^{1+q_2-\nu}\cap\mathfrak{N}_{1-\nu}(\beta,\mathfrak{A})+\mathfrak{P}^{[\frac{q_2}{2}]+1}\cap\mathfrak{H}^1(\beta,\mathfrak{A})\xrightarrow{a_\beta}(\mathfrak{J}^{\nu}(\beta,\mathfrak{A}))^*\xrightarrow{\mathfrak{s}_\beta}\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(1-\nu)}\to 0.$$

(iii) In particular, for $1\le\nu\le[\frac{q_2+1}{2}]$, *we have an exact sequence*

$$0\to\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{q_2}{2}]+1)}\to\mathfrak{P}^{[\frac{q_2}{2}]+1}\cap\mathfrak{H}^1(\beta,\mathfrak{A})\xrightarrow{a_\beta}(\mathfrak{J}^{\nu}(\beta,\mathfrak{A}))^*\xrightarrow{\mathfrak{s}_\beta}\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(1-\nu)}\to 0.$$

□

5 Simple characters for principal orders

In this section we want to construct a $\mathcal{K}_{\mathfrak{B}}J^1(\beta,\mathfrak{A})$ stable character θ_β of $H^1(\beta,\mathfrak{A})$ with the property that we have $\theta_\beta((1+x)(1+y)(1+x)^{-1}(1+y)^{-1})=\psi_\beta^+(xy-yx)$ for all $1+x,1+y\in J^1(\beta,\mathfrak{A})$. By construction it is clear that the right-hand side of the equation defines a non-degenerate alternating form $\mathfrak{J}^1(\beta,\mathfrak{A})/\mathfrak{H}^1(\beta,\mathfrak{A})\times\mathfrak{J}^1(\beta,\mathfrak{A})/\mathfrak{H}^1(\beta,\mathfrak{A})\to\mathbb{C}^\times$; hence, the left-hand side defines a non-degenerate alternating form $J^1(\beta,\mathfrak{A})/H^1(\beta,\mathfrak{A})\times J^1(\beta,\mathfrak{A})/H^1(\beta,\mathfrak{A})\to\mathbb{C}^\times$, too. Consequently, θ_β determines a $\mathcal{K}_{\mathfrak{B}}J^1(\beta,\mathfrak{A})$ stable Heisenberg representation η_β of $J^1(\beta,\mathfrak{A})$, which is the admissible pair, and which we referred to at the beginning of the previous section (4).

Let us be more definite now. Again, let $[\mathfrak{A},n,0,\beta]$ be a simple stratum and let us fix a weak approximation sequence $[\mathfrak{A},n,q_i,\gamma_i]$, $i=1,\dots,\mu$. Further, let us use the notation for $\mathfrak{P}$, E, B, $\mathfrak{B}$ and $\mathfrak{Q}$ as above. If $n=0$, we set $\eta_\beta=\theta_\beta\equiv 1$. Otherwise we have $\mu\ge 2$ and can construct our simple character inductively along the weak approximation sequence.

For this purpose we first assume that $[\mathfrak{A},n,0,\beta]$ is minimal over F. In this case $H^1(\beta,\mathfrak{A})=1+\mathfrak{Q}+\mathfrak{P}^{[\frac{n}{2}]+1}=U^1(\mathfrak{B})U^{[\frac{n}{2}]+1}(\mathfrak{A})$ and $\psi_\beta^\times$ is a character of $U^{[\frac{n}{2}]+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A})$. Clearly, we can restrict $\psi_\beta^\times$ to $U^{[\frac{n}{2}]+1}(\mathfrak{A})\cap B=U^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{n}{2}]+1)}(\mathfrak{B})$ and using the duality (1.4) and Lemma (1.9) we see that this restriction of $\psi_\beta^\times$ factors through the reduced norm $Nrd_{B/E}$. Therefore it makes sense to define the following:

(5.1) Definition *If the stratum* $[\mathfrak{A}, n, 0, \beta]$ *is minimal over* F*, then a simple character* θ_β *is a character chosen as the unique character, which is defined by* $\psi_\beta^\times$ *on* $U^{[\frac{n}{2}]+1}(\mathfrak{A})$ *and by* $\lambda_\beta \circ Nrd_{B/E}$ *on* $U^1(\mathfrak{B})$*, where* λ_β *is a character of* $U^1(\mathfrak{o}_E)$ *such that* $(\lambda_\beta \circ Nrd_{B/E})_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{n}{2}]+1)}(\mathfrak{B})} = (\psi_\beta^\times)_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{n}{2}]+1)}(\mathfrak{B})}$.

From the latter definition it is clear that, in the minimal case, θ_β is normalized by $\mathcal{K}_\mathfrak{B}$.

Now let $[\mathfrak{A}, n, 0, \beta]$ be an arbitrary simple stratum with $n > 0$ again. We can choose a simple character $\theta_{\gamma_{\mu-1}}$ for the minimal stratum $[\mathfrak{A}, n, 0, \gamma_{\mu-1}]$ and define a simple character θ_β by an inductive procedure along the weak approximation sequence for $[\mathfrak{A}, n, 0, \beta]$.

Clearly we do not lose anything of the generality of our arguments if we assume that a simple character is already defined for $\gamma_2, \ldots \gamma_{\mu-1}$ and define θ_β under the assumption of the knowledge of a simple character θ_γ, where $\gamma := \gamma_2$, because this is the inductionstep we are looking for. Throughout this section let $q := q_2$. Note that since $\mathfrak{P}^{[\frac{q}{2}]+1} \cap \mathfrak{H}^1(\gamma, \mathfrak{A})$ is a $(\mathfrak{B}, \mathfrak{B})$-bimodule normalized by $\mathcal{K}_\mathfrak{B}$ and because of Lemma (4.2), we have $H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{B})(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A}))$ and each factor of the right-hand side is normalized by $\mathcal{K}_\mathfrak{B}$. What we need for the inductive definition is the proof of the following Lemma:

(5.2) Lemma *Assume that we have a simple character* θ_γ *of* $H^1(\gamma, \mathfrak{A})$*, then the character* $(\psi_{\beta-\gamma}^\times \theta_\gamma)_{|U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A})}$ *is normalized by* $\mathcal{K}_\mathfrak{B}$ *and, if restricted to* $U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A}) \cap B$*, it factors through the reduced norm* $Nrd_{B/E}$.

Note that by (1.9) this Lemma holds true if the stratum $[\mathfrak{A}, n, 0, \beta]$ is minimal over F, if we set $\theta_{\gamma_\mu} \equiv 1$ and $H^1(\gamma_\mu, \mathfrak{A}) := U^1(\mathfrak{A})$, where we already know that $q_\mu = n$ and $\gamma_\mu = 0$. Thus, under the assumption that Lemma (5.2) holds true in general, the following definition (similar to [BK3](3.2.3)) makes sense:

(5.3) Definition *(i) Suppose that a simple character* θ_γ *for the simple stratum* $[\mathfrak{A}, n, 0, \gamma]$ *is already chosen, then a simple character* θ_β *for* $[\mathfrak{A}, n, 0, \beta]$ *is a character chosen as the unique character, which is defined by* $(\psi_{\beta-\gamma}^\times \theta_\gamma)_{|U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A})}$ *on* $U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A})$ *and by* $\lambda_\beta \circ Nrd_{B/E}$ *on* $U^1(\mathfrak{B})$*, where* λ_β *is a character of* $U^1(\mathfrak{o}_E)$ *such that* $(\lambda_\beta \circ Nrd_{B/E})_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{q}{2}]+1)}(\mathfrak{B})} = (\psi_{\beta-\gamma}^\times \theta_\gamma)_{|U^{\sigma_{\mathfrak{A}/\mathfrak{B}}([\frac{q}{2}]+1)}(\mathfrak{B})}$. *We also demand from a simple character that it is normalized by the whole of* $\mathcal{K}_\mathfrak{B}$.

(ii) Denote the set of all simple characters for $[\mathfrak{A}, n, 0, \beta]$ *by* $\mathcal{C}(\beta, \mathfrak{A})$

From each step of the inductive definition of a simple character θ_β along a weak approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$ we obtain a uniquely defined character λ_{γ_i}, $i = 1, \ldots, \mu$, and conversely from the sequence $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=1,\ldots,\mu}$ we can recover simple characters $\theta_{\gamma_{\mu-1}}, \ldots, \theta_{\gamma_2}$ and finally $\theta_{\gamma_1} = \theta_\beta$.

(5.4) Definition *(i) A sequence* $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=1\ldots,\mu}$ *as we obtain from the inductive definition of a simple character* θ_β *is called a defining sequence for* θ_β.

(ii) We denote by $\mathcal{D}(\beta,\mathfrak{A})$ the set of all defining sequences for simple characters for the stratum $[\mathfrak{A},n,0,\beta]$, where we also vary over all possible weak approximation sequences.

Clearly, a defining sequence for a simple character is not unique, as is a weak approximation sequence $[\mathfrak{A},n,0,\gamma_i]$, $i=1,\dots,\mu$ for a simple stratum $[\mathfrak{A},n,0,\beta]$. Until proving Corollary (5.14)(5.15) we have to fix our weak approximation sequence.

To prove our main Lemma (5.2), we first need a result about the intertwining of the simple characters that already exist. Then the property that θ_β is normalized by the whole of $\mathcal{K}_{\mathfrak{B}}$ will also follow, because $\mathcal{K}_{\mathfrak{B}}$ normalizes $J^1(\beta,\mathfrak{A})$. To simplify notation, we sometimes assume that there already exists a simple character for $[\mathfrak{A},n,0,\beta]$, if it is allowed according to our induction argument.

(5.5) Notation *For $x,y\in\mathfrak{P}$ set $[1+x,1+y]:=(1+x)(1+y)(1+x)^{-1}(1+y)^{-1}$.*

First we prove the main property of simple characters which is a generalization of [Z7](4.3 miii):

(5.6) Proposition *Let θ_β be a simple character, then for all $x,y\in\mathfrak{J}^1(\beta,\mathfrak{A})$ we have*

$$\theta_\beta([1+x,1+y])=\psi^+_\beta(xy-yx).$$

Proof:

First let us assume that $[\mathfrak{A},n,0,\beta]$ is minimal over F. We write $1+x,1+y\in J^1(\beta,\mathfrak{A})$ in the form $(1+x)=(1+x_1)(1+x_2)$ and $(1+y)=(1+y_1)(1+y_2)$, where $1+x_1,1+y_1\in U^1(\mathfrak{B})$ and $1+x_2,1+y_2\in U^{[\frac{n+1}{2}]}(\mathfrak{A})$. With these conventions we obtain $\theta_\beta([1+x,1+y])=\theta_\beta([(1+x_1)(1+x_2),(1+y_1)(1+y_2)])=\theta_\beta({}^{(1+x_1)}[1+x_2,1+y])\theta_\beta([1+x_1,1+y])=\theta_\beta([1+x_2,1+y])\theta_\beta([1+x_1,1+y])$, because, by construction, θ_β is invariant under the adjoint action of $\mathcal{K}_{\mathfrak{B}}$. Moreover, $[1+x_1,1+y]=1+(x_1y-yx_1)(1+x_1)^{-1}(1+y)^{-1}\in U^{[\frac{n}{2}]+1}(\mathfrak{A})$; hence, $\theta_\beta([1+x_1,1+y])=\psi^\times_\beta([1+x_1,1+y])=\psi^+_\beta((x_1y-yx_1)(1+x_1)^{-1}(1+y)^{-1})=\psi^+_\beta(x_1y-yx_1)=1$, because ψ^+_β is trivial on $\mathfrak{P}^{n+1}$ and β commutes with x_1. With the same kind of arguments we have $\theta_\beta([1+x_2,1+y])=\theta_\beta([1+x_2,1+y_1])\theta({}^{(1+y_1)}[1+x_2,1+y_2])=\theta_\beta([1+x_2,1+y_2])=\psi^\times_\beta(1+(x_2y_2-y_2x_2)(1+x_2)^{-1}(1+y_2)^{-1})=\psi^+_\beta(x_2y_2-y_2x_2)=\psi^+_\beta(xy-yx)$, where the latter equation again holds true because x_1,y_1 commute with β and $2[\frac{n+1}{2}]+1\geq n+1$.

So let us turn to the general case and assume that our assertion is true for θ_γ and, as above, we can set $1+x=(1+x_1)(1+x_2)$ and $1+y=(1+y_1)(1+y_2)$, where $x_1,y_1\in\mathfrak{Q}$ and $x_2,y_2\in\mathfrak{P}^{[\frac{q+1}{2}]}\cap\mathfrak{J}^1(\beta,\mathfrak{A})$. As above we obtain $\theta_\beta([1+x,1+y])=\theta_\beta({}^{(1+x_1)}[1+x_2,1+y])\theta_\beta([1+x_1,1+y])=\theta_\beta([1+x_2,1+y])\theta_\beta([1+x_1,1+y])=\theta_\beta([1+x_2,1+y_1])\theta({}^{(1+y_1)}[1+x_2,1+y_2])\theta_\beta([1+x_1,1+y])=\theta_\beta([1+x_2,1+y_1])\theta_\beta([1+x_2,1+y_2])\theta_\beta([1+x_1,1+y])$. Again one has to use similar arguments to prove the triviality of $\theta_\beta([1+x_2,1+y_1])$ and $\theta_\beta([1+x_1,1+y])$. As an example, we show that $\theta_\beta([1+x_2,1+y_1])=1$. By Lemma (4.4) $[1+x_2,1+y_1]=1+(x_2y_1-y_1x_2)(1+x_2)^{-1}(1+y_1)^{-1}\in(\mathfrak{P}^{[\frac{q}{2}]+1}\cap\mathfrak{H}^1(\beta,\mathfrak{A}))=(\mathfrak{P}^{[\frac{q}{2}]+1}\cap\mathfrak{H}^1(\gamma,\mathfrak{A}))$; hence, by induction we have $\theta_\beta([1+x_2,1+y_1])=\psi^\times_{\beta-\gamma}(1+(x_2y_1-y_1x_2)(1+x_2)^{-1}(1+y_1)^{-1})\theta_\gamma([1+x_2,1+y_1])=$

$\psi^+_{\beta-\gamma}(x_2y_1-y_1x_2)\psi^+_\gamma(x_2y_1-y_1x_2) = \psi^+_\beta(x_2y_1-y_1x_2) = 1$. Here we also have to note that $\psi^+_{\beta-\gamma}$ is trivial on $\mathfrak{P}^{q+1}$. Finally, we obtain $\theta_\beta([1+x_2,1+y_2]) = \psi^+_{\beta-\gamma}(x_2y_2-y_2x_2)\psi^+_\gamma(x_2y_2-y_2x_2) = \psi^+_\beta(xy-yx)$, where the last equation again holds because $x_1+x_2+x_1x_2 \equiv x_2 \pmod{\mathfrak{H}^1(\beta,\mathfrak{A})}$ and $y_1+y_2+y_1y_2 \equiv y_2 \pmod{\mathfrak{H}^1(\beta,\mathfrak{A})}$ and Definition (4.1)(i)(ii)(iii).

□

As another example of an inductive argument along the weak approximation sequence for $[\mathfrak{A},n,0,\beta]$, we can generalize the Propositions [BK3](3.2.6) and [BK3](3.2.8)(3.2.11). Following the proofs of [BK3] argument by argument without great change and using arguments as in [Br3] to obtain group inclusions for subgroups of A from the split case in $\tilde{A}$, we obtain:

(5.7) Proposition *Let θ_β be a simple character and $[\frac{q}{2}] \le \nu \le q-1$. Let $k,l \ge 1$ be integers such that $k+l \ge \nu+1$ and $k+2l \ge q+1$. If, moreover, $x \in \mathfrak{P}^k \cap \mathfrak{N}_{-q+k}(\beta,\mathfrak{A})$ and $y \in \mathfrak{P}^l \cap \mathfrak{N}_{-q+l}(\beta,\mathfrak{A})$, then the commutator $[1+x,1+y]$ is in $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})$ and $\theta_\beta([1+x,1+y]) = \psi^\times_{(1+x)^{-1}\beta(1+x)-\beta}(1+y)$.*

□

(5.8) Proposition *Let $[\frac{q}{2}] \le \nu \le q-1$ and θ_β be a simple character. Let $k,l \ge 1$ be integers such that $k+l \ge \nu+1$ and $k+2l \ge q+1$. If, moreover, $x \in \mathfrak{P}^k \cap \mathfrak{N}_{-r+k}(\beta,\mathfrak{A})$ and $y \in \mathfrak{P}^l \cap \mathfrak{J}^1(\beta,\mathfrak{A})$, then $[1+x,1+y]$ is in $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})$ and $\theta_\beta([1+x,1+y]) = \psi^\times_{(1+x)^{-1}\beta(1+x)-\beta}(1+y)$.*

□

(5.9) Corollary *Let $[\frac{q}{2}] \le \nu \le q-1$ and θ_β be a simple character. Let $k,l \ge 1$ be integers such that $k+l \ge \nu+1$ and $k+2l \ge q+1$. If, moreover, $x \in \mathfrak{P}^k \cap \mathfrak{N}_{-r+k}(\beta,\mathfrak{A})$ and $y \in \mathfrak{P}^l \cap \mathfrak{H}^1(\beta,\mathfrak{A})$, then $[1+x,1+y]$ is in $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})$ and $\theta_\beta([1+x,1+y]) = \psi^\times_{(1+x)^{-1}\beta(1+x)-\beta}(1+y)$.*

□

(5.10) Proposition *If θ_β is a simple character for $[\mathfrak{A},n,0,\beta]$ and $0 \le \nu \le q-1$, then $(\theta_\beta)_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})}$ is normalized by $J^1(\beta,\mathfrak{A})$.*

Proof:

Follows from the fact that $J^1(\beta,\mathfrak{A})$ normalizes $U^{\nu+1}(\mathfrak{A}) \cap H^1(\beta,\mathfrak{A})$, Proposition (5.6) and the definition of $\mathfrak{J}^1(\beta,\mathfrak{A})$, $\mathfrak{H}^1(\beta,\mathfrak{A})$, respectively.

□

The result on the intertwining of simple characters, which we need to prove our main Lemma (5.2), is a generalization of [BK3](3.3.2) and [Z7](4.5):

(5.11) Theorem *Let θ_β be a simple character and $0 \le \nu \le q-1$, then the intertwining of $(\theta_\beta)_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})}$ consists of the union of all double classes*

$$B^\times//(1+\mathfrak{P}^{q-\nu}\cap\mathfrak{N}_{-\nu}(\beta,\mathfrak{A})+\mathfrak{P}^{[\frac{q+1}{2}]}\cap\mathfrak{J}^1(\beta,\mathfrak{A})).$$

Proof:

The proof is in fact the same as in [BK3](3.3.2). One has to use [Br2](4.1.1) instead of [BK3](1.5.8), [Br2](4.3.3) instead of [BK3](1.4.16), Proposition (4.5)(i) instead of [BK3](3.1.16), Proposition (5.7) instead of [BK3](3.2.6), Proposition (5.8) instead of [BK3](3.2.8) and Proposition (5.10) instead of [BK3](3.3.1). Moreover, one has to use arguments as in [Br3] to obtain inclusions of subrings of A from the split case in $\tilde{A}$. For example, to generalize Lemma [BK3](3.3.16) one has to use [BK3](1.3.16) in the split case and will obtain for $x \in B^\times$: $(\mathfrak{P}^{\nu+1}+x^{-1}\mathfrak{P}^{\nu+1}x)\cap B = (\tilde{\mathfrak{P}}^{\nu+1}+x^{-1}\tilde{\mathfrak{P}}^{\nu+1}x)\cap\tilde{B}\cap A =$ [BK3](1.3.16) $(\tilde{\mathfrak{Q}}^{\nu+1}+x^{-1}\tilde{\mathfrak{Q}}^{\nu+1}x)\cap A =$ (3.3(ii)) $\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu+1)}+x^{-1}\mathfrak{Q}^{\sigma_{\mathfrak{A}/\mathfrak{B}}(\nu+1)}x$, where $\tilde{B} = C_{\tilde{A}}(E)$ and $\tilde{\mathfrak{Q}} = \tilde{\mathfrak{P}}\cap\tilde{B}$.

Within the proof we also obtain a generalization of [BK3](3.3.9):

(5.12) Proposition *Let $[\mathfrak{A},n,0,\beta]$ be a simple non-minimal stratum and for a fixed l, $0\le l\le q=q_2$ (q the last jump of our weak approximation sequence as above) let $[\mathfrak{A},n,l,\gamma]$ be simple and equivalent to $[\mathfrak{A},n,l,\beta]$. Further, let t be the last jump of a weak approximation sequence for $[\mathfrak{A},n,0,\gamma]$ and let $[\frac{l}{2}]\le\nu\le q-1$. If θ_β is a simple character for $[\mathfrak{A},n,0,\beta]$, which is defined depending on a simple character θ_γ for $[\mathfrak{A},n,0,\gamma]$ as in (5.3), and if x is an element of a double class $\bar{x}\in C_A(\gamma)^\times//(1+\mathfrak{P}^{t-l}\cap\mathfrak{N}_{-t}(\beta,\mathfrak{A})+\mathfrak{P}^{[\frac{t+1}{2}]}\cap\mathfrak{J}^1(\gamma,\mathfrak{A}))$, then we have for all $h\in(x^{-1}(U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A}))x\cap(U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})))$*

$$\theta_\beta^x(h) = \theta_\beta(h)\psi^\times_{x^{-1}\beta x-\beta}(h),$$

$$\theta_\gamma^x(h) = \theta_\gamma(h)\psi^\times_{x^{-1}\gamma x-\gamma}(h).$$

Proof:

In the split case (see [BK](3.3.9)) the proposition was formulated only for $l=q$, where the notation $q=r$ was used. However, for the purposes of [BK3](3.3.20) the case of $l<q$ has to be dealt with too, which is in fact implicitly given in the proofs of [BK](3.3.5) and [BK](3.3.9). Analogously, to prove (5.14) below, we need to show (5.12) for any $l\le q$. Proceeding as in [BK3] one has to replace [BK3](3.1.16) by our Proposition (4.5)(i).

□

□

As a corollary we can now prove our main lemma:

Proof of Lemma (5.2):

Using [Br2](4.1.1) instead of [BK3](1.5.8) and Proposition (5.12) instead of [BK3](3.3.9) we can use the same arguments as in [BK3] to generalize [BK3](3.3.18) and obtain:

(5.13) Corollary *Let $[\frac{q}{2}]\le\nu\le q-1$, and assume that $q<n$. Let $[\mathfrak{A},n,q,\gamma]$ be simple and equivalent to $[\mathfrak{A},n,q,\beta]$, then the map*

$$\mathcal{C}(\beta,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})}\to\mathcal{C}(\gamma,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\gamma,\mathfrak{A})},$$

$$(\theta_\beta)_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})} \mapsto (\theta_\beta)_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\gamma,\mathfrak{A})}(\psi^\times_{\beta-\gamma})^{-1}$$

is bijective, where $\mathcal{C}(\beta,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\beta,\mathfrak{A})}$ *and* $\mathcal{C}(\gamma,\mathfrak{A})_{|U^{\nu+1}(\mathfrak{A})\cap H^1(\gamma,\mathfrak{A})}$ *denote the sets of restricted characters.*

In fact, our lemma (5.2) is just a reformulation of (5.13).

□

Following the arguments of [BK3], using Proposition (5.12) instead of [BK3](3.3.9) and [Br2](4.1.1) instead of [BK3](1.5.8), we can generalize [BK3](Corollary (3.3.20)) without great changes:

(5.14) Corollary *(i) The definition of the set* $\mathcal{C}(\beta,\mathfrak{A})$ *is, in fact, independent of the special choice of a weak approximation sequence for* $[\mathfrak{A},n,0,\beta]$.

(ii) Let $0 \leq l \leq q-1$, *and let* $[\mathfrak{A},n,l,\beta']$ *be a simple stratum equivalent to* $[\mathfrak{A},n,l,\beta]$, *then* $\mathcal{C}(\beta',\mathfrak{A})_{|U^{[\frac{l}{2}]+1}(\mathfrak{A})\cap H^{[\frac{l}{2}]+1}(\beta',\mathfrak{A})} = \mathcal{C}(\beta,\mathfrak{A})_{|U^{[\frac{l}{2}]+1}(\mathfrak{A})\cap H^{[\frac{l}{2}]+1}(\beta,\mathfrak{A})}\psi^\times_{\beta'-\beta}$, *where* $\mathcal{C}(\beta',\mathfrak{A})_{|U^{[\frac{l}{2}]+1}(\mathfrak{A})\cap H^{[\frac{l}{2}]+1}(\beta',\mathfrak{A})}$, $\mathcal{C}(\beta,\mathfrak{A})_{|U^{[\frac{l}{2}]+1}(\mathfrak{A})\cap H^{[\frac{l}{2}]+1}(\beta,\mathfrak{A})}$ *are the sets of restricted characters.*

□

Note that in the case $l=0$, i.e. $\beta-\beta'\in\mathfrak{A}$, we have $\psi^\times_{\beta'-\beta}\equiv 1$. In particular, the set $\mathcal{C}(\beta,\mathfrak{A})$ depends only on the equivalence class of the simple stratum $[\mathfrak{A},n,0,\beta]$. As a reformulation of the last corollary we obtain:

(5.15) Corollary *(i) Let* θ_β *be a fixed simple character for the simple stratum* $[\mathfrak{A},n,0,\beta]$, *then, for each weak approximation sequence* $[\mathfrak{A},n,q_i,\gamma_i]$, $i=1,\ldots,\mu$, *we can find a defining sequence* $\{(\gamma_i,\lambda_{\gamma_i})\}_{i=1,\ldots,\mu}$ *for* θ_β.

(ii) The above inductive procedure of defining a simple character as in (5.1)(5.3) leads to a well defined surjective map $\mathcal{D}(\beta,\mathfrak{A})\rightarrow\mathcal{C}(\beta,\mathfrak{A})$ *from defining sequences (see (5.4)) to simple characters.*

For later purposes we need a proposition of [BF] which was published with a misprint. The version of [BF](2.8.3) that we use here is:

(5.16) Lemma *Let* $\mathfrak{A}$ *be a principal order with period* r *in a simple algebra* A *of index* d. *Let* $e_{\mathfrak{A}/\mathfrak{o}_F} := rd$ *and let* $\sigma_{\mathfrak{A}/\mathfrak{o}_F}:\mathbb{Z}\rightarrow\mathbb{Z}$ *be given by* $\sigma_{\mathfrak{A}/\mathfrak{o}_F}(j) := [\frac{j+e_{\mathfrak{A}/\mathfrak{o}_F}-1}{e_{\mathfrak{A}/\mathfrak{o}_F}}]$ *for* $j\in\mathbb{Z}$, *then the restrictions of the reduced norm mapping* $Nrd_{A/F}$ *induce surjective mappings* $U^j(\mathfrak{A})\rightarrow U^{\sigma_{\mathfrak{A}/\mathfrak{o}_F}(j)}(\mathfrak{o}_F)$ *for all* $j\in\mathbb{N}$.

□

Remark: Because the definition of $\sigma_{\mathfrak{A}/\mathfrak{o}_F}$ and $\sigma_{\mathfrak{A}/\mathfrak{B}}$ is analogous, also the analogue of Lemma (2.6)(i)(ii) holds true for $e_{\mathfrak{A}/\mathfrak{o}_F}$ and $\sigma_{\mathfrak{A}/\mathfrak{o}_F}$. □

The following proposition makes the relationship between defining sequences and simple characters more obvious:

(5.17) Proposition *If* $\{\lambda_{\gamma_i}\}_{i=1,\dots,\mu}$ *is a defining sequence for a simple character* $\theta_\beta \in \mathcal{C}(\beta,\mathfrak{A})$*, then* θ_β *determines the characters* λ_{γ_i} *on the groups* $U^{s_i+1}(\mathfrak{o}_{F[\gamma_i]})$ *with* $s_i = [\frac{\sigma_{\mathfrak{A}\cap C_A(\gamma_i)/\mathfrak{o}_{F[\gamma_i]}}(\sigma_{\mathfrak{A}/\mathfrak{A}\cap C_A(\gamma_i)}(q_i))}{2}]$ *for* $i = 1,\dots,\mu$ *and, conversely, the restrictions of the* λ_{γ_i} *to the latter groups for* $i = 1,\dots,\mu$ *determine* θ_β*. Moreover, each sequence of continuations* $\{\lambda'_{\gamma_i}\}_{i=1,\dots,\mu}$ *of these restrictions to the whole groups* $U^1(\mathfrak{o}_{F[\gamma_i]})$ *for* $i = 1,\dots,\mu$ *is a defining sequence for* θ_β*.*

Proof:

In the minimal case there is nothing to show because $q_1 = 0$ and $\lambda_{\gamma_2} \equiv 1$; hence, θ_β determines its defining sequence uniquely. Suppose now that the assertion holds true for all simple characters $\theta_\gamma \in \mathcal{C}(\gamma,\mathfrak{A})$, where $\gamma = \gamma_2$ is the last approximation of the simple stratum $[\mathfrak{A},n,0,\beta]$ of our given weak approximation sequence.

By (5.13) the defining character θ_γ of the inductive definition of θ_β is uniquely determined on $U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma,\mathfrak{A})$, where we have $q = q_2$. As $H^1(\gamma,\mathfrak{A}) = U^1(\mathfrak{A}\cap C_A(\gamma))(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma,\mathfrak{A}))$ and θ_γ is given on $U^1(\mathfrak{A}\cap C_A(\gamma))$ by $\lambda_\gamma \circ Nrd_{A\cap C_A(\gamma)/F[\gamma]}$, we deduce from (5.16) that λ_γ is determined by θ_β on $U^{[\frac{\sigma_{\mathfrak{A}\cap C_A(\gamma)/\mathfrak{o}_{F[\gamma]}}(\sigma_{\mathfrak{A}/\mathfrak{A}\cap C_A(\gamma)}(q))}{2}]+1}(\mathfrak{o}_{F[\gamma]})$, whereas a twist of λ_γ by an arbitrary character of $U^1(\mathfrak{o}_{F[\gamma]})/U^{[\frac{\sigma_{\mathfrak{A}\cap C_A(\gamma)/\mathfrak{o}_{F[\gamma]}}(\sigma_{\mathfrak{A}/\mathfrak{A}\cap C_A(\gamma)}(q))}{2}]+1}(\mathfrak{o}_{F[\gamma]})$ shifts θ_γ within $\mathcal{C}(\gamma,\mathfrak{A})$ but has no influence on θ_β. Now our assertion follows by induction along the approximation sequence.

□

6 Matching of simple characters among various algebras

In this section let $n > 0$ again. To prove the main result of this section, we need two propositions:

(6.1) Proposition *Let* $[\mathfrak{A},n,0,\beta]$ *be a simple stratum in a central simple algebra* A/F *and* $E := F[\beta]$*. Let* K/F *be the maximal unramified part of the extension* E/F*,* $C := C_A(K)$ *and* $\mathfrak{C} := \mathfrak{A}\cap C$*, then the stratum* $[\mathfrak{C},\sigma_{\mathfrak{A}/\mathfrak{C}}(n),0,\beta]$ *is simple and* $H^1(\beta,\mathfrak{A})\cap C \supset H^1(\beta,\mathfrak{C})$*. Moreover, we assume that the simple characters* $\mathcal{C}(\beta,\mathfrak{C})$ *are defined with respect to the additive character* $\psi_K := \psi_F \circ Tr_{K/F}$ *(which has conductor* $\mathfrak{p}_K$*). The map* $\mathcal{C}(\beta,\mathfrak{A}) \to \mathcal{C}(\beta,\mathfrak{C})$ *induced by restriction, is then well defined and injective. If* $[\mathfrak{A},n,q_i,\gamma_i]$*,* $i = 1,\dots,\mu$*, is a* <u>special</u> *approximation sequence for* $[\mathfrak{A},n,0,\beta]$ *and* $\theta_{\beta,\mathfrak{A}} \in \mathcal{C}(\beta,\mathfrak{A})$ *has a defining sequence* $\{(\gamma_i,\lambda_{\gamma_i})\}_{i=1,\dots,\mu}$*, then* $\{(\gamma_i,\lambda_{\gamma_i}\circ Nrd_{K[\gamma_i]/F[\gamma_i]})\}_{i=1,\dots,\mu}$ *is a defining sequence for the restriction* $\theta_{\beta,\mathfrak{C}} \in \mathcal{C}(\beta,\mathfrak{C})$ *of* $\theta_{\beta,\mathfrak{A}}$*. Finally, the image of the map is characterized by the property that for a defining sequence* $\{(\gamma_i,\lambda'_{\gamma_i})\}_{i=1,\dots,\mu}$ *of an element of the image, the characters* λ'_{γ_i} *factor through the norm* $Nrd_{K[\gamma_i]/F[\gamma_i]}$ *for* $i = 1,\dots,\mu$*.*

Proof:

By the remark following (3.8) we know that $e_{\mathfrak{A}/\mathfrak{C}} \mid q_i$ for $i = 1,\dots,\mu$; hence, from (4.1)(vii) and (2.6)(ii) we obtain $\mathfrak{H}^1(\beta,\mathfrak{A})\cap C = (\sum_{i=1}^{\mu}\mathfrak{P}^{[\frac{q_i}{2}]+1}\cap C_{\mathfrak{P}}(\gamma_i))\cap C \supset \sum_{i=1}^{\mu}(\mathfrak{P}^{[\frac{q_i}{2}]+1}\cap C_{\mathfrak{P}}(\gamma_i)\cap C) = \sum_{i=1}^{\mu}\mathfrak{R}^{\sigma_{\mathfrak{A}/\mathfrak{C}}([\frac{q_i}{2}]+1)}\cap C_{\mathfrak{R}}(\gamma_i) = \sum_{i=1}^{\mu}\mathfrak{R}^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{C}}(q_i)}{2}]+1}\cap C_{\mathfrak{R}}(\gamma_i) = \mathfrak{H}^1(\beta,\mathfrak{C})$, where $\mathfrak{R}$ is the Jacobson radical of $\mathfrak{C}$, as required. Let $\theta_{\beta,\mathfrak{A}}$ be a simple character of $H^1(\beta,\mathfrak{A})$ and $\theta_{\beta,\mathfrak{C}}$ its restriction to $H^1(\beta,\mathfrak{C})$.

To prove the assertion about simple characters, let us use the method of induction along the weak approximation sequence:

To start the induction correctly, we have to assume the same situation for β as for a first approximation $\gamma_{\mu-1}$; hence, that $K \supset F[\beta]^{ur}$, where $F[\beta]^{ur}$ is the maximal unramified subextension of $F[\beta]/F$, that $K[\beta]$ is a field not necessarily equal to E and, finally, that the embedding $K[\beta]^\times \subset \mathcal{K}_\mathfrak{A}$ is sound (i.e. β is $\mathfrak{C}$-sound). If $[\mathfrak{A},n,0,\beta]$ is minimal over F, we have $H^1(\beta,\mathfrak{A}) = U^1(\mathfrak{B})U^{[\frac{n}{2}]+1}(\mathfrak{A})$ and $H^1(\beta,\mathfrak{C}) = U^1(\mathfrak{B}\cap C_B(K[\beta]))U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{C}}(n)}{2}]+1}(\mathfrak{C})$. If $\theta_{\beta,\mathfrak{A}}$ is given on $U^1(\mathfrak{B})$ by $\lambda_\beta \circ Nrd_{B/E}$, then $\theta_{\beta,\mathfrak{C}}$ on $U^1(\mathfrak{B}\cap C_B(K[\beta]))$ by $(\lambda_\beta \circ Nrd_{B/E})_{|C_B(K[\beta])} = \lambda_\beta \circ Nr_{K[\beta]/E} \circ Nrd_{C_B(K[\beta])/K[\beta]}$. Moreover, because of $(Trd_{A/F})_{|C} = Tr_{K/F} \circ Trd_{C/K}$ and $\psi_K = \psi_F \circ Tr_{K/F}$, we have that $\theta_{\beta,\mathfrak{C}}$ is given on $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{C}}(n)}{2}]+1}(\mathfrak{C})$ by $\psi^\times_{\beta,K}$, where $\psi^\times_{\beta,K}$ is defined similarly with respect to ψ_K as $\psi^\times_\beta$ with respect to ψ_F. It follows that $\theta_{\beta,\mathfrak{C}}$ is a simple character. As the norm mapping $Nr_{K[\beta]/E} : U^1(\mathfrak{o}_{K[\beta]}) \to U^1(\mathfrak{o}_E)$ is surjective (see for example [Se](V,3,Prop. 3)), as the reduced norm mapping $Nrd_{B/E} : U^1(\mathfrak{B}) \to U^1(\mathfrak{o}_E)$ is surjective by (5.16) and as, again by (5.16), the reduced norm mapping $Nrd_{C_B(K[\beta])/K[\beta]} : U^1(\mathfrak{B}\cap C_B(K[\beta])) \to U^1(\mathfrak{o}_{K[\beta]})$ is surjective too, we can recover $\theta_{\beta,\mathfrak{A}}$ from $\theta_{\beta,\mathfrak{C}}$.

If, conversely, we start from a simple character $\theta_{\beta,\mathfrak{C}}$ with defining sequence $\{(\beta,\lambda'_\beta)\}$ and assume that it has the form $\lambda'_\beta = \lambda_\beta \circ Nr_{K[\beta]/F[\beta]}$, then, by Lemma 1.9, $\psi^\times_\beta$ is a character that factors on $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(n)}{2}]+1}(\mathfrak{B})$ through the reduced norm $Nrd_{B/E}$ and its restriction to $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n)}{2}]+1}(\mathfrak{C}\cap C_A(K[\beta]))$ agrees with $\psi^\times_{\beta,K}$; hence, with the restriction of $\lambda_\beta \circ Nr_{K[\beta]/F[\beta]} \circ Nrd_{C_A(K[\beta])/K[\beta]}$.

Again we have $(Nrd_{B/E})_{|C_B(K[\beta])} = Nr_{K[\beta]/E} \circ Nrd_{C_B(K[\beta])/K[\beta]}$, $(Trd_{A/F})_{|C} = Tr_{K/F} \circ Trd_{C/K}$ and the maps $Nr_{K[\beta]/E} : U^i(\mathfrak{o}_{K[\beta]}) \to U^i(\mathfrak{o}_E)$ are surjective for all $i \geq 1$ (see [Se](V, 3, Prop. 3)). By (5.16), its concluding remark, $e_{\mathfrak{B}\cap C_B(K[\beta])/\mathfrak{o}_{K[\beta]}} \mid \sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n)$ (by (2.6)(ii)) and $e_{\mathfrak{B}/\mathfrak{o}_E} \mid \sigma_{\mathfrak{A}/\mathfrak{B}}(n)$ (again by (2.6)(ii)), we have surjective maps $Nrd_{C_B(K[\beta])/K[\beta]} : U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n)}{2}]+1}(\mathfrak{B}\cap C_A(K[\beta])) \to U^{[\frac{\sigma_{\mathfrak{B}\cap C_B(K[\beta])/\mathfrak{o}_{K[\beta]}}(\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n))}{2}]+1}(\mathfrak{o}_{K[\beta]})$ and $Nrd_{B/E} : U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(n)}{2}]+1}(\mathfrak{B}) \to U^{[\frac{\sigma_{\mathfrak{B}/\mathfrak{o}_E}(\sigma_{\mathfrak{A}/\mathfrak{B}}(n))}{2}]+1}(\mathfrak{o}_E)$. Because of $\sigma_{\mathfrak{B}\cap C_B(K[\beta])/\mathfrak{o}_{K[\beta]}}(\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n)) = -\nu_{K[\beta]}(\beta) = -\nu_{F[\beta]}(\beta) = \sigma_{\mathfrak{B}/\mathfrak{o}_E}(\sigma_{\mathfrak{A}/\mathfrak{B}}(n))$ and $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(n)}{2}]+1}(\mathfrak{B}) \cap C_B(K[\beta]) = U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(n)}{2}]+1}(\mathfrak{B}\cap C_B(K[\beta]))$, we conclude that $\psi^\times_\beta$ can be given on $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{B}}(n)}{2}]+1}(\mathfrak{B})$ only by the restriction of $\lambda_\beta \circ Nrd_{B/E}$ and $\theta_{\beta,\mathfrak{C}}$ lifts to a simple character $\theta_{\beta,\mathfrak{A}}$ of $H^1(\beta,\mathfrak{A})$ with defining sequence $\{(\beta,\lambda_\beta)\}$ as required. Therefore the assertion is true in the minimal case.

For the induction step let $\gamma := \gamma_2$ be the last approximation of β and $q := q_2$ be the last jump. Furthermore, assume that $K \supset F[\beta]^{ur}$ and that $K[\beta]$ is a field, which is not nessecarily equal to E. By the induction hypothesis we assume that the restriction $\theta_{\gamma,\mathfrak{C}}$ of a simple character $\theta_{\gamma,\mathfrak{A}}$ of $H^1(\gamma,\mathfrak{A})$ to $H^1(\gamma,\mathfrak{C})$ is again simple and that one can recover $\theta_{\gamma,\mathfrak{A}}$ from $\theta_{\gamma,\mathfrak{C}}$. We have $H^1(\beta,\mathfrak{A}) =$

$U^1(\mathfrak{B})(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma,\mathfrak{A}))$ and $H^1(\beta,\mathfrak{C}) = U^1(\mathfrak{B} \cap C_B(K[\beta]))(U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{C}}(q)}{2}]+1}(\mathfrak{C}) \cap H^1(\gamma,\mathfrak{C}))$. Again, as in the minimal case, if $\theta_{\beta,\mathfrak{A}}$ is given on $U^1(\mathfrak{B})$ by $\lambda_\beta \circ Nrd_{B/E}$, then $\theta_{\beta,\mathfrak{C}}$ on $U^1(\mathfrak{B} \cap C_B(K[\beta]))$ by $\lambda_\beta \circ Nrd_{K[\beta]/E} \circ Nrd_{C_B(K[\beta])/K[\beta]}$. Also $\theta_{\beta,\mathfrak{C}}$ is given on $U^{[\frac{\sigma_{\mathfrak{A}/\mathfrak{C}}(q)}{2}]+1}(\mathfrak{C}) \cap H^1(\gamma,\mathfrak{C})$ by $\psi^\times_{\beta-\gamma,K}\theta_{\gamma,\mathfrak{C}}$; hence, $\theta_{\beta,\mathfrak{C}}$ is a simple character. By induction and the same argument as in the minimal case, we can recover $\theta_{\beta,\mathfrak{A}}$ from $\theta_{\beta,\mathfrak{C}}$ and obtain a well defined injective map. Assuming the equation $\sigma_{\mathfrak{B}\cap C_B(K[\beta])/\mathfrak{o}_{K[\beta]}}(\sigma_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])}(q)) = \sigma_{\mathfrak{B}/\mathfrak{o}_E}(\sigma_{\mathfrak{A}/\mathfrak{B}}(q))$, also the characterization of the image of the map holds true by the same argument as in the minimal case if we use (5.2) and the induction assumption that a simple character $\theta_{\gamma,\mathfrak{C}}$ with defining sequence $\{(\gamma_i, \lambda_{\gamma_i} \circ Nr_{K[\gamma_i]/F[\gamma_i]})\}_{i=2,\dots,\mu}$ lifts to a simple character $\theta_{\gamma,\mathfrak{A}}$ with defining sequence $\{(\gamma_i,\lambda_{\gamma_i})\}_{i=2,\dots,\mu}$.

For the required equation we just have to note that by (2.6)(iii) and $e(K[\beta]/F) = e(E/F)$, we have $e_{\mathfrak{B}\cap C_B(K[\beta])/\mathfrak{o}_{K[\beta]}} e_{\mathfrak{A}/\mathfrak{B}\cap C_B(K[\beta])} = \frac{rd}{e(E/F)} = e_{\mathfrak{B}/\mathfrak{o}_E} e_{\mathfrak{A}/\mathfrak{B}}$; hence, we are done.

□

(6.2) Lemma *Let $[\mathfrak{A},n,q,\beta]$ be a sound stratum in a central simple algebra A/F and $E := F[\beta]$. Let L/F be an unramified extension such that $E \otimes_F L$ is again a field, then the stratum $[\mathfrak{A},n,q,\beta]$ in A is simple if and only if the stratum $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, q, \beta \otimes 1]$ in $A \otimes_F L$ is simple.*

Proof:

By definition (3.1)(iii) and [BK3](1.5.5) the stratum $[\mathfrak{A},n,q,\beta]$ is simple if and only if $-k_0(\beta,\tilde{\mathfrak{A}}) > q$, where $\tilde{\mathfrak{A}}$ is the principal order attached to the same lattice chain in a simple A-module that is also a simple $\tilde{A}$-module and $\tilde{A}$ the canonical splitting of A as described at the beginning of section (3). By Proposition (3.6) we have $k_0(\beta,\mathfrak{A}) = k_0(\beta,\tilde{\mathfrak{A}})$ and with similar notations we have $k_0(\beta\otimes 1, \mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L) = k_0(\beta\otimes 1, \widetilde{\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L})$. Now, clearly, our assertion follows from $k_0(\beta,\mathfrak{A}) = k_0(\beta\otimes 1, \mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L)$, which can be deduced from straightforward computations using the fact that if $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$, then $\mathfrak{P}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$ is the Jacobson radical of $\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L$.

□

(6.3) Lemma *Let ψ_F be an additive character of F with conductor $\mathfrak{p}_F$ and L/F an unramified extension, then there exists an extension ψ_L of ψ_F to L^+ with conductor $\mathfrak{p}_L$.*

Proof:

The character ψ_F can be identified with the sequence $\{\psi_{F,i}\}_{i=0,1,2,\dots}$ in the projective limit $\varprojlim_{i=0,1,2,\dots} \widehat{\mathfrak{p}_F^{-i}/\mathfrak{p}_F}$, where $\psi_{F,i} := (\psi_F)_{|\mathfrak{p}_F^{-i}/\mathfrak{p}_F}$ for $i = 0,1,2,\dots$. Here the limit is to be understood in such a way that a sequence $\{\chi_i\}_{i=0,1,2,\dots} \in \prod_{i=0,1,2,\dots} \widehat{\mathfrak{p}_F^{-i}/\mathfrak{p}_F}$ is in $\varprojlim_{i=0,1,2,\dots} \widehat{\mathfrak{p}_F^{-i}/\mathfrak{p}_F}$ if and only if we have $(\chi_k)_{|\mathfrak{p}_F^{-l}/\mathfrak{p}_F} = \chi_l$ for all $l < k$.

In the same way we define $\varprojlim_{i=0,1,2,\dots} \widehat{\mathfrak{p}_L^{-i}/\mathfrak{p}_L}$, and finding an extension ψ_L of ψ_F with conductor $\mathfrak{p}_L$ is the same as finding a sequence $\{\psi_{L,i}\}_{i=0,1,2,\dots} \in \varprojlim_{i=0,1,2,\dots} \widehat{\mathfrak{p}_L^{-i}/\mathfrak{p}_L}$ such that $(\psi_{L,i})_{|\mathfrak{p}_F^{-i}/\mathfrak{p}_F} = \psi_{F,i}$ for $i = 0,1,2,\dots$, where we regard $\mathfrak{p}_F^{-i}/\mathfrak{p}_F$ as a subgroup of $\mathfrak{p}_L^{-i}/\mathfrak{p}_L$ via the natural embedding induced by the inclusions $\mathfrak{p}_F^{-i} \subset \mathfrak{p}_L^{-i}$ for all $i \geq 0$ and $\mathfrak{p}_F \subset \mathfrak{p}_L$ (here we have to use the assumption that the extension L/F is unramified).

We construct ψ_L inductively. First we extend $\psi_{F,0}$ to $\mathfrak{o}_L/\mathfrak{p}_L$ to obtain $\psi_{L,0}$, which provides the beginning of our induction. Now assume that for $i \geq 0$ we have constructed extensions $\psi_{L,k}$ of the characters $\psi_{F,k}$ for $0 \leq k \leq i$ such that we have $(\psi_{L,k})_{|\mathfrak{p}_L^{-l}/\mathfrak{p}_L} = \psi_{L,l}$ for $l \leq k$ and $l, k \in \{0, \ldots, i\}$.

Now let ψ' be an extension of $\psi_{L,i}$ to $\mathfrak{p}_L^{-i-1}/\mathfrak{p}_L$, then $\psi'_{|\mathfrak{p}_F^{-i-1}/\mathfrak{p}_F}$ differs from $\psi_{F,i+1}$ by a character χ of $\mathfrak{p}_F^{-i-1}/\mathfrak{p}_F^{-i}$, which can be extended to a character χ' of $\mathfrak{p}_L^{-i-1}/\mathfrak{p}_L^{-i}$. Now $\psi_{L,i+1} = \psi'\chi'$ extends $\psi_{L,i}$ as well as $\psi_{F,i+1}$ as required.

□

(6.4) Proposition *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in a central simple algebra A/F and $E := F[\beta]$. Assume that E/F is totally ramified. Let L/F be an unramified splitting field of A, then the stratum $[\mathfrak{A} \otimes_{o_F} o_L, n, 0, \beta \otimes 1]$ is simple, $\mathfrak{H}^1(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L) = \mathfrak{H}^1(\beta, \mathfrak{A}) \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$. Moreover, we assume that the simple characters $\mathcal{C}(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)$ are defined with respect to an additive character ψ_L with conductor $\mathfrak{p}_L$ of L, which extends ψ_F (which exists by the above Lemma (6.3)). The map $\mathcal{C}(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L) \to \mathcal{C}(\beta, \mathfrak{A})$, induced by restricting a character of $H^1(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)$ to the subset $1 \otimes 1 + \mathfrak{H}^1(\beta, \mathfrak{A}) \otimes 1 \simeq H^1(\beta, \mathfrak{A})$, is then well defined and surjective. If $[\mathfrak{A}, n, q_i, \gamma_i]$, $i = 1, \ldots, \mu$, is a weak approximation sequence for $[\mathfrak{A}, n, 0, \beta]$, then $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, q_i, \gamma_i \otimes 1]$, $i = 1, \ldots, \mu$, is a weak approximation sequence for $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, 0, \beta \otimes 1]$, and if $\theta_{\beta \otimes 1} \in \mathcal{C}(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L)$ has a defining sequence $\{(\gamma_i \otimes 1, \lambda_{\gamma_i \otimes 1})\}_{i=1,\ldots,\mu}$, then $\{(\gamma_i, (\lambda_{\gamma_i \otimes 1})_{|U^1(\mathfrak{o}_{F[\gamma_i]})}\}_{i=1,\ldots,\mu}$ is a defining sequence for the restriction $\theta_\beta \in \mathcal{C}(\beta, \mathfrak{A})$ of $\theta_{\beta \otimes 1}$. Moreover, to each fixed simple character θ_β, each fixed defining sequence $\{\lambda_{\gamma_i}\}_{i=1,\ldots,\mu}$ to θ_β and each fixed preimage $\theta_{\beta \otimes 1}$, we can find a defining sequence $\{\lambda_{\gamma_i \otimes 1}\}_{i=1,\ldots,\mu}$ for $\theta_{\beta \otimes 1}$ such that $\{(\lambda_{\gamma_i \otimes 1})_{|U^1(\mathfrak{o}_{F[\gamma_i]})}\}_{i=1,\ldots,\mu} = \{\lambda_{\gamma_i}\}_{i=1,\ldots,\mu}$.*

Proof:

Since E/F is totally ramified, $E \otimes_F L$ is a field and Lemma (6.2) applies to our weak approximation sequences for $[\mathfrak{A}, n, 0, \beta]$ and for $[\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, n, 0, \beta \otimes 1]$. By (4.1)(vii) and as $\mathfrak{P} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ is the Jacobson radical of $\mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, the equation $\mathfrak{H}^1(\beta \otimes 1, \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L) = \mathfrak{H}^1(\beta, \mathfrak{A}) \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$ is trivial. Let $\bar{A} := A \otimes_F L$, $\bar{\mathfrak{A}} := \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$, $\bar{B} := C_{\bar{A}}(E \otimes_F L)$ and $\bar{\mathfrak{B}} := \bar{\mathfrak{A}} \cap \bar{B}$.

The proof again argues by induction. So first, we assume that $[\mathfrak{A}, n, 0, \beta]$ is minimal over F. We have $H^1(\beta \otimes 1, \bar{\mathfrak{A}}) = U^1(\bar{\mathfrak{B}})U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}})$ and $H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{B})U^{[\frac{n}{2}]+1}(\mathfrak{A})$. If $\theta_{\beta \otimes 1}$ is a simple character of $H^1(\beta \otimes 1, \bar{\mathfrak{A}})$, then it is given by a character $\lambda_{\beta \otimes 1} \circ det_{\bar{B}/E \otimes_F L}$ on $U^1(\bar{\mathfrak{B}})$ and by $\psi^\times_{\beta \otimes 1}$ on $U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}})$, where for all $x \in U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}})$ the character $\psi^\times_{\beta \otimes 1}$ is given by $\bar{\psi}^\times_{\beta \otimes 1}(x) := \psi_L \circ Tr_{\bar{A}/L}(\beta \otimes 1(x-1))$. By the definition of a reduced norm and a reduced trace, the restriction of $\theta_{\beta \otimes 1}$ to $H^1(\beta, \mathfrak{A})$ is given on $U^1(\mathfrak{B})$ by $(\lambda_{\beta \otimes 1})_{|U^1(\mathfrak{o}_E)} \circ Nrd_{B/E}$ and on $U^{[\frac{n}{2}]+1}(\mathfrak{A})$ by $(\bar{\psi}^\times_{\beta \otimes 1})_{|U^{[\frac{n}{2}]+1}(\mathfrak{A})} = \psi^\times_\beta$. Hence, the restriction of $\theta_{\beta \otimes 1}$ is a simple character as asserted above.

Conversely, if θ_β is a simple character of $H^1(\beta, \mathfrak{A})$ given by $\lambda_\beta \circ Nrd_{B/E}$ on $U^1(\mathfrak{B})$ and by $\psi^\times_\beta$ on $U^{[\frac{n}{2}]+1}(\mathfrak{A})$, then it can clearly be extended to $U^1(\mathfrak{B})U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}})$ by $\bar{\psi}^\times_{\beta \otimes 1}$ on $U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}})$. Since, by Lemma 1.9, $(\bar{\psi}^\times_{\beta \otimes 1})_{|U^{[\frac{n}{2}]+1}(\bar{\mathfrak{A}}) \cap \bar{B}}$ factors through the determinant map $det_{\bar{B}|E \otimes_F L}$ and, by the definition of the reduced norm, λ_β extends to a character $\bar{\lambda}_\beta$ of $U^1(\mathfrak{o}_{E\otimes_F L})$ such that $(\bar{\lambda}_\beta \circ det_{\bar{B}/(E\otimes_F L)})_{|U^{[\frac{n}{2}]+1}(\bar{\mathfrak{B}})} = (\bar{\psi}^{\times}_{\beta\otimes 1})_{|U^{[\frac{n}{2}]+1}(\bar{\mathfrak{B}})}$. Therefore θ_β extends to a simple character of $H^1(\beta\otimes 1, \bar{\mathfrak{A}})$ and we have shown that our restriction map is surjective in the minimal case.

So let us assume now that $[\mathfrak{A}, n, 0, \beta]$ has a weak approximation sequence with the final approximation $[\mathfrak{A}, n, q, \gamma]$. It is well known from [BG] that $f(F[\gamma]/F) \mid f(E/F)$ and $e(F[\gamma]/F) \mid e(E/F)$ in this situation; hence, the extension $F[\gamma]/F$ is again totally ramified. We have $H^1(\beta\otimes 1, \bar{\mathfrak{A}}) = U^1(\bar{\mathfrak{B}})(U^{[\frac{q}{2}]+1}(\bar{\mathfrak{A}}) \cap H^1(\gamma\otimes 1, \bar{\mathfrak{A}}))$ and $H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{B})(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A}))$. If $\theta_{\beta\otimes 1}$ is a simple character of $H^1(\beta\otimes 1, \bar{\mathfrak{A}})$, then it is given by a character $\lambda_{\beta\otimes 1} \circ det_{\bar{B}/E\otimes_F L}$ on $U^1(\bar{\mathfrak{B}})$ and by $\psi^{\times}_{(\beta-\gamma)\otimes 1}\theta_{\gamma\otimes 1}$ on $(U^{[\frac{q}{2}]+1}(\bar{\mathfrak{A}}) \cap H^1(\gamma\otimes 1, \bar{\mathfrak{A}}))$, where $\theta_{\gamma\otimes 1}$ is a simple character of $H^1(\gamma\otimes 1, \bar{\mathfrak{A}})$ and $\bar{\psi}^{\times}_{(\beta-\gamma)\otimes 1}$ is defined as in the minimal case. By induction hypothesis, we can assume that the restriction θ_γ of $\theta_{\gamma\otimes 1}$ to $H^1(\gamma, \mathfrak{A})$ is again a simple character and fulfils the assertion about the defining sequences. The restriction of $\theta_{\beta\otimes 1}$ to $H^1(\beta, \mathfrak{A})$ is given on $U^1(\mathfrak{B})$ by $(\lambda_{\beta\otimes 1})_{|U^1(\mathfrak{o}_E)} \circ Nrd_{B/E}$ and on $(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A}))$ by $(\bar{\psi}^{\times}_{(\beta-\gamma)\otimes 1}\theta_{\gamma\otimes 1})_{|(U^{[\frac{q}{2}]+1}(\mathfrak{A})\cap H^1(\gamma,\mathfrak{A}))} = (\psi^{\times}_{\beta-\gamma}\theta_\gamma)_{|(U^{[\frac{q}{2}]+1}(\mathfrak{A})\cap H^1(\gamma,\mathfrak{A}))}$. Therefore, the restriction of $\theta_{\beta\otimes 1}$ is a simple character as asserted above.

Conversely, if θ_β is a simple character of $H^1(\beta, \mathfrak{A})$ given by $\lambda_\beta \circ Nrd_{B/E}$ on $U^1(\mathfrak{B})$ and by $\psi^{\times}_{\beta-\gamma}\theta_\gamma$ on $(U^{[\frac{q}{2}]+1}(\mathfrak{A}) \cap H^1(\gamma, \mathfrak{A}))$, where θ_γ is a simple character of $H^1(\gamma, \mathfrak{A})$, then it can clearly be extended to $U^1(\mathfrak{B})(U^{[\frac{q}{2}]+1}(\bar{\mathfrak{A}}) \cap H^1(\gamma\otimes 1, \bar{\mathfrak{A}}))$ by $\bar{\psi}^{\times}_{(\beta-\gamma)\otimes 1}\theta_{\gamma\otimes 1}$ on $(U^{[\frac{q}{2}]+1}(\bar{\mathfrak{A}}) \cap H^1(\gamma\otimes 1, \bar{\mathfrak{A}}))$, where $\theta_{\gamma\otimes 1}$ is a simple extension of θ_γ, which exists by induction hypothesis. By Lemma 5.2, $(\bar{\psi}^{\times}_{(\beta-\gamma)\otimes 1}\theta_{\gamma\otimes 1})_{|U^{[\frac{q}{2}]+1}(\bar{\mathfrak{B}})}$ factors through the determinant map $det_{\bar{B}|E\otimes_F L}$ and, by the definition of the reduced norm, λ_β extends to a character $\bar{\lambda}_\beta$ of $U^1(\mathfrak{o}_{E\otimes_F L})$ such that $(\bar{\lambda}_\beta \circ det_{\bar{B}/(E\otimes_F L)})_{U^{[\frac{q}{2}]+1}(\bar{\mathfrak{B}})} = (\bar{\psi}^{\times}_{(\beta-\gamma)\otimes 1}\theta_{\gamma\otimes 1})_{U^{[\frac{q}{2}]+1}(\bar{\mathfrak{B}})}$. Hence, θ_β extends to a simple character of $H^1(\beta\otimes 1, \bar{\mathfrak{A}})$ and by induction we have shown that our restriction map is surjective in the general case.

Moreover, if $\theta_{\beta\otimes 1}$ is an arbitrary but fixed preimage of θ_β and $\{\lambda_{\gamma_i}\}_{i=1,\ldots,\mu}$ is a fixed defining sequence of θ_β, then the charcters $\lambda'_{\gamma_i\otimes 1}$ are uniquely determined on $U^{\bar{s}_i+1}(\mathfrak{o}_{F[\gamma_i]\otimes_F L})$ with $\bar{s}_i = [\frac{\sigma_{\mathfrak{A}\cap C_A(\gamma_i)\otimes_{\mathfrak{o}_F}\mathfrak{o}_L/\mathfrak{o}_{F[\gamma_i]\otimes_F L}}(q_i)}{2}]$, by (5.17) and because of $e_{\mathfrak{A}/\mathfrak{A}\cap C_A(\gamma_i)} = 1$ (see [G] Theorem 2.2(i)) for each defining sequence $\{\lambda'_{\gamma_i\otimes 1}\}_{i=1,\ldots,\mu}$ with respect to our given approximation sequence $[\mathfrak{A}\otimes_{\mathfrak{o}_F}\mathfrak{o}_L, n, q_i, \gamma_i\otimes 1]$ $i = 1, \ldots, \mu$ and, conversely, these restrictions determine $\theta_{\beta\otimes 1}$. Obviously, as $\theta_{\beta\otimes 1}$ is a preimage of θ_β, we have $(\lambda'_{\gamma_i\otimes 1})_{|U^{s_i+1}(\mathfrak{o}_{F[\gamma_i]})} = (\lambda_{\gamma_i})_{|U^{s_i+1}(\mathfrak{o}_{F[\gamma_i]})}$, where $s_i = [\frac{\sigma_{\mathfrak{A}\cap C_A(\gamma_i)/\mathfrak{o}_{F[\gamma_i]}}(q_i)}{2}]$. Therefore its possible to extend the characters $(\lambda'_{\gamma_i\otimes 1})_{|U^{\bar{s}_i+1}(\mathfrak{o}_{F[\gamma_i]\otimes_F L})}$ to $U^1(\mathfrak{o}_{F[\gamma_i]})U^{\bar{s}_i+1}(\mathfrak{o}_{F[\gamma_i]\otimes_F L})$ and, finally, to characters $\lambda_{\gamma_i\otimes 1}$ of $U^1(\mathfrak{o}_{F[\gamma_i]\otimes_F L})$ such that they coincide with the λ_{γ_i} on $U^1(\mathfrak{o}_{F[\gamma_i]})$ for $i = 1, \ldots, \mu$. Again by (5.17) the sequence $\{\lambda_{\gamma_i\otimes 1}\}_{i=1,\ldots,\mu}$ is a defining sequence for $\theta_{\beta\otimes 1}$ as required.

□

For the rest of this section, let β be an element of an algebraic closure of F which generates a field extension E/F of finite degree, where we set $E := F[\beta]$. As in [BK3] let $A(E) := End_F(E)$ and $\mathfrak{A}(E) := End_{\mathfrak{o}_F}(\{\mathfrak{p}_E^j\}_{j\in\mathbb{Z}})$. Further, assume that $[0, \beta]$ is a simple pair, i.e., $0 > \nu_E(\beta), k_0(\beta, \mathfrak{A}(E))$. Moreover, let K/F be the maximal unramified subextension of E/F, let $C(E) := End_K(E) = C_{A(E)}(K)$ and $\mathfrak{C}(E) := End_{\mathfrak{o}_K}(\{\mathfrak{p}_E^j\}_{j\in\mathbb{Z}}) = \mathfrak{A}(E) \cap C(E)$. Let $n_0 := -\nu_E(\beta)$, then the strata

$[\mathfrak{A}(E), n_0, 0, \beta]$, $[\mathfrak{C}(E), n_0, 0, \beta]$ are simple. The main result of this section is a generalization of [BK3](3.6.13):

(6.5) Theorem *Let A be a central simple algebra over F with the property that $[E:F] \mid N$, where $N := \sqrt{\dim_F(A)}$ is the reduced degree of A. Moreover, let d be the index of A and write $N = md$. Let $\mathfrak{A}$ be a principal $\mathfrak{o}_F$-order in A with invariant s and period r such that $e(E/F) \mid rd$ and $f(E/F) \mid \ddot{d}s$, where $\ddot{d}$ is the greatest divisor of d that is prime to r, i.e., E/F has an embedding in A/F which is $\mathfrak{A}$-sound (see [G](1.9)(iii) with $t = 1$).*

(i) Let $\iota : E \hookrightarrow A$ be an F-embedding of E with $E^\times \subset \mathcal{K}_{\mathfrak{A}}$ which is sound. Moreover, let $n = n_0 \frac{rd}{e(E/F)}$ and if $q_{0,i}$, $i = 1, \ldots, \mu$, are the unique numbers coming from an approximation sequence $[\mathfrak{A}(E), n_0, q_{0,i}, \gamma_i]$, $i = 1, \ldots, \mu$ for $[\mathfrak{A}(E), n_0, 0, \beta]$, then set $q_i := q_{0,i} \frac{rd}{e(E/F)}$ for $i = 1, \ldots, \mu$. There exist embeddings $\iota_1, \ldots, \iota_\mu$ such that $[\mathfrak{A}, n, q_i, \iota_i(\gamma_i)]$, $i = 1, \ldots, \mu$, is a special approximation sequence for $[\mathfrak{A}, n, 0, \iota(\beta)]$ and $\iota_1 = \iota$.

(ii) There exists a unique bijection $\tau_{\mathfrak{A}(E),\mathfrak{A},\beta,\iota} : \mathcal{C}(\beta, \mathfrak{A}(E)) \to \mathcal{C}(\iota(\beta), \mathfrak{A})$ with the following property:

If we are given a simple character θ_β of $H^1(\beta, \mathfrak{A}(E))$, fix a defining sequence $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=1,\ldots,\mu}$, which comes from a special approximation sequence $[\mathfrak{A}(E), n_0, q_{0,i}, \gamma_i]$ for $[\mathfrak{A}(E), n_0, 0, \beta]$ and fix embeddings $\iota_i : F[\gamma_i] \to A$, $i = 1, \ldots, \mu$, as in (i), then $\tau_{\mathfrak{A}(E),\mathfrak{A},\beta,\iota}(\theta_\beta)$ is a simple character with defining sequence $\{(\iota_i(\gamma_i), \lambda_{\gamma_i} \circ \iota_i^{-1})\}_{i=1,\ldots,\mu}$.

Proof:

(i) By [BG]((5.1) and its proof) and Theorem (3.11), respectively, we can find a special approximation sequence $[\mathfrak{A}, n, q_i, \gamma_i']$, $i = 1, \ldots, \mu$ for $[\mathfrak{A}, n, 0, \iota(\beta)]$. Following the arguments of the proof of [BG](5.1) between [BG](5.2) and [BG](5.3), choosing a (W, E)-decomposition for $\tilde{\mathfrak{B}}$ as in [BK3](1.2.8) and using intertwining-implies-conjugacy arguments, we can choose γ_2' in such a way that it has the same minimal polynomial over F as γ_2. Going along the approximation sequence inductively, we obtain a choice of the $\gamma_1', \ldots, \gamma_\mu'$ with the same minimal polynomials as the $\gamma_1, \ldots, \gamma_\mu$ over F; hence, the maps induced by mapping γ_i at γ_i' for $i = 1, \ldots, \mu$ lead to the embeddings $\iota_1, \ldots, \iota_\mu$ as required.

(ii) If the map exists, then, obviously, it is uniquely determined by the property. For the existence we have to note that by Proposition (6.1) there exists a map $\mathcal{C}(\beta, \mathfrak{A}(E)) \to \mathcal{C}(\beta, \mathfrak{C}(E))$ with a well known description of simple characters in terms of defining sequences and being injective. By Proposition (6.4) there exists a surjective map $\mathcal{C}(\beta \otimes 1, \mathfrak{C}(E) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L) \to \mathcal{C}(\beta, \mathfrak{C}(E))$, where L/K is an unramified splitting field of C. The latter map gives us the possibility to obtain a defining sequence of an image from the defining sequence of a preimage.

If we choose to each image of the first map an arbitrary but fixed preimage with respect to the second map, we obtain an injective map $\mathcal{C}(\beta, \mathfrak{A}(E)) \to \mathcal{C}(\beta \otimes 1, \mathfrak{C}(E) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L)$ with the property that, to a defining sequence $\{(\gamma_i, \lambda_{\gamma_i})\}_{i=1,\ldots,\mu}$ of a preimage (coming from a special approximation sequence for $[\mathfrak{A}(E), n_0, 0, \beta]$), we obtain a defining sequence $\{(\gamma_i \otimes 1, \lambda_{\gamma_i \otimes 1})\}_{i=1,\ldots,\mu}$ of the image such that $(\lambda_{\gamma_i \otimes 1})_{|U^1(\mathfrak{o}_{F[\gamma_i]})} = \lambda_{\gamma_i} \circ Nr_{K[\gamma_i]/F[\gamma_i]}$ for $i = 1, \ldots, \mu$.

Now the embeddings $\iota_1, \ldots, \iota_\mu$ induce embeddings $\iota'_1, \ldots, \iota'_\mu$ of $K[\gamma_1 \otimes 1]/K, \ldots, K[\gamma_\mu \otimes 1]/K$ such that $[\mathfrak{C} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), \sigma_{\mathfrak{A}/\mathfrak{C}}(q_i), \iota'_i(\gamma_i \otimes 1)]$, $i = 1, \ldots, \mu$, is a weak approximation sequence for $[\mathfrak{C} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, \sigma_{\mathfrak{A}/\mathfrak{C}}(n), 0, \iota'_1(\beta \otimes 1)]$.

By [BK3](3.6.13) we have a canonical bijection

$$\tau_{\mathfrak{C}(E)\otimes_{\mathfrak{o}_K}\mathfrak{o}_L, \mathfrak{C}\otimes_{\mathfrak{o}_K}\mathfrak{o}_L, \beta, 0} : \mathcal{C}(\beta \otimes 1, \mathfrak{C}(E) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L) \to \mathcal{C}(\iota(\beta) \otimes 1, \mathfrak{C} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L)$$

such that for each defining sequence $\{(\gamma_i \otimes 1, \lambda_{\gamma_i \otimes 1})\}_{i=1,\ldots,\mu}$ of a fixed preimage the image is described by the defining sequence $\{(\gamma_i \otimes 1, \lambda_{\gamma_i \otimes 1} \circ \iota_i'^{-1})\}_{i=1,\ldots,\mu}$.

Using Proposition (6.4) again, we obtain a simple character of $H^1(\beta, \mathfrak{C})$ with defining sequence $\{(\gamma_i, (\lambda_{\gamma_i \otimes 1} \circ Nr_{K[\gamma_i \otimes 1]/F[\gamma_i \otimes 1]} \circ \iota_i'^{-1})_{|U^1(\mathfrak{o}_{K[\gamma_i]})})\}_{i=1,\ldots,\mu}$ and because of $(\lambda_{\gamma_i \otimes 1} \circ Nr_{K[\gamma_i \otimes 1]/F[\gamma_i \otimes 1]} \circ \iota_i'^{-1})_{|U^1(\mathfrak{o}_{K[\gamma_i]})}) = \lambda_{\gamma_i} \circ \iota_i^{-1} \circ Nr_{K[\iota_i(\gamma_i)]/F[\iota_i(\gamma_i)]}$ for $i = 1, \ldots, \mu$ also Proposition (6.1) applies again. By taking a preimage under the map $\mathcal{C}(\iota(\beta), \mathfrak{A}) \to \mathcal{C}(\iota(\beta), \mathfrak{C})$, we finally obtain a character of $H^1(\beta, \mathfrak{A})$ with defining sequence $\{(\iota_i(\gamma_i), \lambda_{\gamma_i} \circ \iota_i^{-1})\}_{i=1,\ldots,\mu}$ as required.

Again, the last two steps do not depend on the special choice of embeddings and defining sequences; hence, following the diagram

$$\begin{array}{ccccc}
\mathcal{C}(\beta, \mathfrak{A}(E)) & \overset{\text{res}(6.1)}{\hookrightarrow} & \mathcal{C}(\beta, \mathfrak{C}(E)) & \overset{\text{res}(6.4)}{\twoheadleftarrow} & \mathcal{C}(\beta \otimes 1, \mathfrak{C}(E) \otimes_{\mathfrak{o}_K} \mathfrak{o}_L) \\
 & & & & \updownarrow [BK3] \\
\mathcal{C}(\iota(\beta), \mathfrak{A}) & \overset{\text{res}(6.1)}{\hookrightarrow} & \mathcal{C}(\iota(\beta), \mathfrak{C}) & \overset{\text{res}(6.4)}{\twoheadleftarrow} & \mathcal{C}(\iota(\beta) \otimes 1, \mathfrak{C} \otimes_{\mathfrak{o}_K} \mathfrak{o}_L),
\end{array}$$

we obtain a map $\tau_{\mathfrak{A}(E), \mathfrak{A}, \beta, \iota}$ with the asserted properties with respect to the embeddings and the defining sequences. As the situation is completely symmetric, we can show the bijectivity by constructing the inverse map. If we fix special defining sequences and embeddings, the last assertion of (6.4) makes obvious that our map cannot depend on the special choices of the preimages with respect to the horizontal surjective maps of our diagram; hence, we are done.

□

The latter result enables us to generalize Definition [BH](8.2. . .):

(6.6) Definition *To a simple pair $[0, \beta]$ a ps-character is defined as a pair (θ, β), where $\theta = \{(A, \iota, \mathfrak{A}) \mapsto \theta(A, \iota, \mathfrak{A}) \in \mathcal{C}(\iota(\beta), \mathfrak{A})\}$ is a simple character valued function as follows: to each triple $(A, \iota, \mathfrak{A})$ fulfilling the assumptions of Theorem (6.5), θ associates a simple character of $H^1(\iota(\beta), \mathfrak{A})$ such that $\tau_{\mathfrak{A}(E), \mathfrak{A}, \beta}(\theta(A(E), \kappa, \mathfrak{A}(E)) = \theta(A, \iota, \mathfrak{A})$, where κ is the canonical embedding of E to $A(E) = End_F(E)$.*

Clearly, a ps-character is uniquely determined by a value on one triple and we have:

(6.7) Corollary *Let θ_β be a simple character of $H^1(\beta, \mathfrak{A}(E))$, then there exists a unique ps-character (θ, β) such that $\theta(A(E), \kappa, \mathfrak{A}(E)) = \theta_\beta$, where κ is the canonical embedding of E to $A(E) = End_F(E)$.*

□

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