REGULAR REPRESENTATION ON THE BIG CELL AND BIG PROJECTIVE MODULES IN THE CATEGORY $\mathcal{O}$.

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Abstract. The regular representation $\mathfrak{R}(G_0)$ consists of the regular functions on the big cell $G_0 \subset G$, arising from the Gauss decomposition of a simple complex Lie group G. The algebra $\mathfrak{R}(G_0)$ is isomorphic to the subalgebra of $\mathcal{U}(\mathfrak{g})^*$, spanned by the matrix elements of integral weight modules in the category $\mathcal{O}$.

We study the blocks $\mathfrak{R}(G_0)_\lambda$, corresponding to central characters, associated with anti-dominant integral weights λ. We show that $\mathfrak{R}(G_0)_\lambda$ is spanned by the matrix elements of the big projective module P_λ, and prove that $\mathfrak{R}(G_0)_\lambda \cong P_\lambda^* \otimes_{Z(\mathfrak{g})} P_\lambda$, which gives a version of the Peter-Weyl theorem for the projective modules.

The Whittaker functor associates to any $\mathfrak{g}$-module the subspace of Whittaker vectors, transforming under $\mathfrak{n}_+$ according to a one-dimensional representation $\boldsymbol{\eta}$. We apply the Whittaker functor to the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathfrak{R}(G_0)_\lambda$, and prove that for nonsingular $\boldsymbol{\eta}$ we have $\overline{\mathrm{Wh}}_\eta(\mathfrak{R}(G_0)_\lambda) \cong P_\lambda$, which yields the big projective module analogue of the Borel-Weil realization of the simple finite-dimensional $\mathfrak{g}$-modules.

Our results admit an invariant reformulation in terms of the algebras of endomorphisms of projective generators of $\mathcal{O}_\lambda$. This allows for immediate generalizations to the quantum group case, which also hold for the small quantum groups when q is a root of unity.

0. Introduction.

A fundamental object in the representation theory of a simple complex Lie group G is the regular representation $\mathfrak{R}(G)$, realized as the algebra of regular functions on G. The space $\mathfrak{R}(G)$ carries compatible structures of a commutative associative algebra and of a $G \times G$-module, which encode important information about the category of finite-dimensional representations of G.

The algebro-geometric version of the classical Peter-Weyl theorem asserts that $\mathfrak{R}(G)$ decomposes into a direct sum of subspaces $\mathbb{M}(L_\lambda)$, spanned by matrix elements of finite-dimensional G-modules L_λ, indexed by the set $\mathbf{P}^+$ of dominant integral weights. In other words, we have a decomposition

$$\mathfrak{R}(G) = \bigoplus_{\lambda \in \mathbf{P}^+} \mathbb{M}(L_\lambda) \cong \bigoplus_{\lambda \in \mathbf{P}^+} L_\lambda^* \otimes L_\lambda, \tag{0.1}$$

where L_λ^* are the modules dual to L_λ, and the second isomorphism of $G \times G$-modules is a reformulation of the fact that the matrix elements, corresponding to any fixed basis of L_λ, are linearly independent and form a basis of $\mathbb{M}(L_\lambda)$.

Another classical result provides realizations of simple G-modules inside the regular representation $\mathfrak{R}(G)$. Let B_- be a Borel subgroup of G, and let T be the corresponding maximal torus. For any $\lambda \in \mathbf{P}^+$ we consider the character $\xi_\lambda : B_- \to \mathbb{C}$, determined by λ on T and trivially extended to the unipotent part N_- of B_-. The Borel-Weil theorem [Bo] asserts that the simple module L_λ can be realized as the subspace of $\mathfrak{R}(G)$,

$$L_\lambda \;\cong\; \left\{ \varphi \in \mathfrak{R}(G) \;\middle|\; \varphi(b\,x) = \xi_\lambda(b)\,\varphi(x) \text{ for any } b \in B_-,\, x \in G \right\}, \tag{0.2}$$

on which G acts by the right shifts, $(g\,\varphi)(x) = \varphi(x\,g)$ for any $x, g \in G$. It also follows from the Borel-Weil theorem that the subspace of N_--invariant elements in $\mathfrak{R}(G)$ gives a model for finite-dimensional G-modules, or equivalently

$$\mathfrak{R}(N_-\backslash G) \cong \left\{ \varphi \in \mathfrak{R}(G) \;\middle|\; \varphi(n\,g) = \varphi(g) \text{ for any } n \in N_-,\, g \in G \right\} \cong \bigoplus_{\lambda \in \mathbf{P}^+} L_\lambda. \tag{0.3}$$

Every G-module has a natural infinitesimal action of the Lie algebra $\mathfrak{g}$ of the group G. Elements of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be identified with the differential operators acting on germs of functions at the unit element $e \in G$, and evaluation at e yields a linear map $\vartheta : \mathfrak{R}(G) \to \mathcal{U}(\mathfrak{g})^*$. The image of $\mathfrak{R}(G)$ is the so-called Hopf dual subalgebra of $\mathcal{U}(\mathfrak{g})^*$, and the Lie algebraic version of the Peter-Weyl theorem identifies it with the subspace spanned by the matrix elements of finite-dimensional $\mathfrak{g}$-modules L_λ for all $\lambda \in \mathbf{P}^+$.

In this paper we consider the extension $\mathfrak{R}(G_0)$ of the regular representation $\mathfrak{R}(G)$, which consists of regular functions on the big cell $G_0 \subset G$ of the Gauss decomposition of the group G. We call the algebra $\mathfrak{R}(G_0)$ the regular representation on the big cell. The group G does not act on $\mathfrak{R}(G_0)$ because G_0 is not invariant under left and right shifts, but the infinitesimal regular actions of $\mathfrak{g}$ are still well-defined. The $\mathfrak{g} \oplus \mathfrak{g}$-module structure of $\mathfrak{R}(G_0)$ cannot be described in terms of finite-dimensional $\mathfrak{g}$-modules, so we are naturally led to a more general class of representations of $\mathfrak{g}$.

The Gauss decomposition $G_0 = N_-\,T\,N_+$ of the group G induces the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of the Lie algebra $\mathfrak{g}$. The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ consists of finitely generated, $\mathfrak{h}$-diagonalizable and locally $\mathfrak{n}_+$-finite $\mathfrak{g}$-modules, and contains simple highest weight modules L_λ associated with λ not necessarily dominant. The category $\mathcal{O}$ is not semisimple, and each simple modules L_λ has the indecomposable projective cover P_λ. The projective modules P_λ corresponding to strictly anti-dominant λ play a particularly important role; we call them the big projective modules.

Similarly to $\mathfrak{R}(G)$, the algebra $\mathfrak{R}(G_0)$ can be identified with a subalgebra of $\mathcal{U}(\mathfrak{g})^*$. We prove that this subalgebra is spanned by the matrix elements of all modules with integral weights in $\mathcal{O}$, and decomposes into a direct sum of $\mathfrak{g} \oplus \mathfrak{g}$-submodules $\mathbb{M}_\lambda \subset \mathcal{U}(\mathfrak{g})^*$,

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda \in -\mathbf{P}^{++}} \mathbb{M}_\lambda, \qquad \mathbb{M}_\lambda = \sum_{V \in \mathcal{O}_\lambda} \mathbb{M}(V) \tag{0.4}$$

where the subspaces $\mathbb{M}_\lambda$ are spanned by matrix elements of the modules in the block $\mathcal{O}_\lambda$, corresponding to the strictly anti-dominant integral weights $\lambda \in -\mathbf{P}^{++}$. Furthermore, we prove that it suffices to consider the matrix elements of the big projective modules, i.e. for every $\lambda \in -\mathbf{P}^{++}$ we have $\mathbb{M}_\lambda = \mathbb{M}(P_\lambda)$.

In contrast to the finite-dimensional G-modules, the matrix elements, corresponding to a fixed basis of the big projective module P_λ, are not linearly independent. Therefore, the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathbb{M}(P_\lambda)$ is not isomorphic to $P_\lambda^* \otimes P_\lambda$, but rather to its quotient by the ideal, corresponding to identically vanishing matrix elements. We establish a $\mathfrak{g} \oplus \mathfrak{g}$-module isomorphism $\mathbb{M}(P_\lambda) \cong P_\lambda^* \otimes_{Z(\mathfrak{g})} P_\lambda$, where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Unlike the finite-dimensional case, the action of $Z(\mathfrak{g})$ on P_λ does not reduce to scalars, and generates the entire endomorphism ring $\mathrm{End}_{\mathcal{O}}(P_\lambda)$ of the big projective module [So]. Thus, we obtain $\mathfrak{g}\oplus\mathfrak{g}$-module isomorphisms

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}(P_\lambda) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} P_\lambda^* \otimes_{Z(\mathfrak{g})} P_\lambda, \tag{0.5}$$

which gives an analogue of the Peter-Weyl theorem for $\mathfrak{R}(G_0)$ and the big projective modules. Another useful description of $\mathfrak{R}(G_0)$ is given by

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathcal{P}_\lambda^* \otimes_{\mathcal{A}_\lambda} \mathcal{P}_\lambda, \tag{0.6}$$

where $\mathcal{A}_\lambda$ is the algebra of endomorphisms of the projective generator $\mathcal{P}_\lambda$ of the block $\mathcal{O}_\lambda$.

The big projective modules P_λ for $\lambda\in-\mathbf{P}^{++}$ admit a realization, similar to the one provided by the Borel-Weil theorem for finite-dimensional G-modules. However, instead of the N_--invariant elements of $\mathfrak{R}(G)$, we need to consider the more general notion of Whittaker vectors, studied by Kostant [K1]. For any character $\boldsymbol{\eta}^- : N_- \to \mathbb{C}$, we consider

$$(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)) = \left\{ \varphi\in\overline{\mathfrak{R}(G_0)} \;\middle|\; \varphi(n\,x) = \boldsymbol{\eta}^-(n)\,\varphi(x) \text{ for any } n\in N_-,\ x\in G_0 \right\}, \tag{0.7}$$

where $\overline{\mathfrak{R}(G_0)}$ is the power series completion of the polynomial algebra $\mathfrak{R}(G_0)$. The notation $(\overline{\mathrm{Wh}}_\eta^- \otimes 1)$ indicates that we consider the subspace of functions with special transformation properties under the left shifts by N_-, which remains a $\mathfrak{g}$-module with respect to the right regular action. We prove that for nonsingular $\boldsymbol{\eta}^-$ the Whittaker vectors in $\mathfrak{R}(G_0)$ give a model for big projective $\mathfrak{g}$ modules, corresponding to strictly anti-dominant weights:

$$(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} P_\lambda, \tag{0.8}$$

Imposing further transformation properties with respect to the center $Z(\mathfrak{g})$, one extracts the individual big projective modules P_λ for each $\lambda\in-\mathbf{P}^{++}$.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be deformed into the quantum group $\mathcal{U}_q(\mathfrak{g})$, and the Lie algebraic realization of $\mathfrak{R}(G_0)$ as a subspace of $\mathcal{U}(\mathfrak{g})^*$ yields the "quantum coordinate algebra" $\mathfrak{R}_q(G_0)$, defined as the appropriate subspace in $\mathcal{U}_q(\mathfrak{g})^*$. When q is generic, the representation theories of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ are completely parallel, and the structural results obtained for $\mathfrak{R}(G_0)$ also hold for $\mathfrak{R}_q(G_0)$. In particular, we get the quantum versions of the Peter-Weyl and Borel-Weil theorems for quantized big projective modules.

When q is a root of unity, one can define a finite-dimensional version $\mathfrak{U}$ of the quantum group $\mathcal{U}_q(\mathfrak{g})$. The dual algebra $\mathfrak{U}^*$ exhibits properties similar to those of $\mathfrak{R}(G_0)$, and its structure can be described as in (0.6) in terms of projective generators in the appropriate category of representations, and their endomorphism algebras.

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1. Notation and preliminaries.

1.1. **Representations of $\mathfrak{g}$ and category $\mathcal{O}$.** Let $\mathfrak{g}$ be a simple complex Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We denote by Δ the root system of $\mathfrak{g}$, and make a choice of positive roots Δ_+ and simple roots Π.

Denote $r = \dim \mathfrak{h}$, and let $\{\mathbf{e}_i, \mathbf{h}_i, \mathbf{f}_i\}_{i=1}^r$ be the Chevalley generators of the Lie algebra $\mathfrak{g}$. The nilpotent subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are generated by $\{\mathbf{f}_i\}$ and $\{\mathbf{e}_i\}$ respectively. We choose root vectors $\{\mathbf{e}_\beta, \mathbf{f}_\beta\}_{\beta\in\Delta_+}$ such that for any simple root α_i we have $\mathbf{e}_{\alpha_i} \equiv \mathbf{e}_i$ and $\mathbf{f}_{\alpha_i} \equiv \mathbf{f}_i$.

We refer to elements of $\mathfrak{h}^*$ as weights. A weight λ is called integral (resp. dominant and strictly dominant) if for all $i = 1, \dots, r$ we have $\langle \lambda, \mathbf{h}_i\rangle \in \mathbb{Z}$ (resp. $\langle \lambda, \mathbf{h}_i\rangle \geq 0$ and $\langle \lambda, \mathbf{h}_i\rangle > 0$). The sets of integral (resp. dominant integral and strictly dominant integral) weights are denoted $\mathbf{P}$ (resp. $\mathbf{P}^+$ and $\mathbf{P}^{++}$). Elements of $-\mathbf{P}^+$ (resp. $-\mathbf{P}^{++}$) are called antidominant (resp. strictly antidominant) weights.

For any Weyl group element $w \in W$, the length $l(w)$ is defined as the smallest number l such that w can be represented as $w = s_{j_1} \dots s_{j_l}$, where $\{s_i\}_{i=1}^r$ are the simple root reflections. We denote $w_\circ$ the unique longest element of W.

We set $\rho = \frac{1}{2}\sum_{\beta\in\Delta_+}\beta$, and define the "dot" action of the Weyl group on $\mathfrak{h}^*$ by

$$s_i \cdot \lambda = \lambda - \langle \lambda + \rho, \mathbf{h}_i\rangle \alpha_i. \tag{1.1}$$

For any $\lambda \in \mathfrak{h}^*$, denote W_λ the subgroup of W stabilizing λ, and pick a set W^λ of representatives of the cosets W/W_λ. We call λ regular if $W_\lambda = \{e\}$. We also denote by l_λ the smallest length of an element $w \in W$ such that $w \cdot \lambda = w_\circ \cdot \lambda$; if λ is regular, then $l_\lambda = l(w_\circ) = |\Delta_+|$.

The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ consists of finitely generated, locally $\mathfrak{n}_+$-finite $\mathfrak{g}$-modules V, decomposing into the direct sum of finite-dimensional weight subspaces

$$V = \bigoplus_{\mu\in\mathfrak{h}^*} V[\mu], \qquad V[\mu] = \{v \in V | \mathbf{h}\, v = \langle \mu, \mathbf{h}\rangle\, v \text{ for any } \mathbf{h} \in \mathfrak{h}\}. \tag{1.2}$$

The formal character of V is defined by

$$\operatorname{ch} V = \sum_\mu \dim V[\mu]\; e^\mu. \tag{1.3}$$

Simple objects in $\mathcal{O}$ are the irreducible highest weight modules L_λ, parameterized by $\lambda \in \mathfrak{h}^*$. Each simple module L_λ has a projective cover P_λ with simple top, isomorphic to L_λ. The category $\mathcal{O}$ also contains the "standard" Verma modules M_λ and the "costandard" contragredient Verma modules M^c_λ, parameterized by $\lambda \in \mathfrak{h}^*$.

A central character of $\mathfrak{g}$ is a homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$, where $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. For any central character χ, we define

$$\mathcal{O}(\chi) = \left\{V \in \mathcal{O} \,\middle|\, \forall z \in Z(\mathfrak{g}) \text{ the operator } z - \chi(z) \text{ acts nilpotently in } V\right\} \tag{1.4}$$

Each $\mathcal{O}(\chi)$ is a full subcategory of $\mathcal{O}$, and we have the decomposition $\mathcal{O} = \bigoplus_\chi \mathcal{O}(\chi)$, such that $\operatorname{Ext}^\bullet_{\mathcal{O}}(M, N) = 0$ if $M \in \mathcal{O}(\chi), N \in \mathcal{O}(\chi')$ and $\chi \neq \chi'$.

For any $\lambda \in \mathfrak{h}^*$ elements $z \in Z(\mathfrak{g})$ act in the simple module L_λ as scalars. We denote χ_λ the corresponding central character; then $\chi_\lambda = \chi_\mu$ if and only if $\lambda = w \cdot \mu$ for some $w \in W$.

We denote $\mathcal{O}_\lambda = \mathcal{O}(\chi_\lambda)$. For generic $\lambda \in \mathfrak{h}^*$ the simple module L_λ is projective, and the category $\mathcal{O}_\lambda$ consists of finite direct sums of L_λ. In this paper we study the blocks $\mathcal{O}_\lambda$ associated with integral weights λ, which have much richer structure.

All projective modules in $\mathcal{O}_\lambda$ have Verma flags, i.e. filtrations with quotients isomorphic to Verma modules. Denote $(P_{x\cdot\lambda} : M_{y\cdot\lambda})$ the number of times $M_{y\cdot\lambda}$ appears as a quotient in a Verma flag of $P_{x\cdot\lambda}$; it does not depend on the choice of the filtration. The Bernstein-Gelfand-Gelfand reciprocity states that for any $x, y \in W^\lambda$

$$(P_{x\cdot\lambda} : M_{y\cdot\lambda}) = [M_{y\cdot\lambda} : L_{x\cdot\lambda}], \tag{1.5}$$

where $[M_{y\cdot\lambda} : L_{x\cdot\lambda}]$ denotes the multiplicity of the simple modules $L_{x\cdot\lambda}$ in the composition series for $M_{y\cdot\lambda}$, computed by the appropriate Kazhdan-Lusztig polynomial. As a consequence, one obtains the symmetry of the so-called Cartan matrix of $\mathcal{O}_\lambda$:

$$[P_{x\cdot\lambda} : L_{y\cdot\lambda}] = \sum_{w\in W^\lambda} (P_{x\cdot\lambda} : M_{w\cdot\lambda})\,[M_{w\cdot\lambda} : L_{y\cdot\lambda}] = \sum_{w\in W^\lambda} [M_{w\cdot\lambda} : L_{x\cdot\lambda}]\,(P_{y\cdot\lambda} : M_{w\cdot\lambda}) = [P_{y\cdot\lambda} : L_{x\cdot\lambda}].$$

Let λ be a dominant integral weight. There are two important modules in the block $\mathcal{O}_\lambda$: the unique finite-dimensional simple module L_λ, and the big projective module $P_{w_\circ\cdot\lambda}$. One of the remarkable similarities between them is exhibited by via their characters. The Weyl character formula for the finite-dimensional module L_λ can be written as

$$\operatorname{ch} L_\lambda = \sum_{w\in W^\lambda} (-1)^{|w|} \operatorname{ch} M_{w\cdot\lambda}, \tag{1.6}$$

and we have a similar expression for the character of the big projective module:

$$\operatorname{ch} P_{w_\circ\cdot\lambda} = \sum_{w\in W^\lambda} \operatorname{ch} M_{w\cdot\lambda}. \tag{1.7}$$

For any $V \in \mathcal{O}$ we define the restricted dual $V^* = \bigoplus_\mu V[\mu]^*$. Dualizing the $\mathfrak{g}$-action on V, we get a natural *right* action of $\mathfrak{g}$ on V^*, such that $\langle v^*, x\,v\rangle = \langle v^*\,x, v\rangle$ for any $x \in \mathfrak{g}$ and $v \in V, v^* \in V^*$. We transform it into the more traditional left $\mathfrak{g}$-action by means of the Lie algebra anti-involution $x \mapsto x^* = -x$. In other words, we have

$$\langle x\,v^*, v\rangle = \langle v^* x^*, v\rangle = -\langle v^*\,x, v\rangle = -\langle v^*, x\,v\rangle, \qquad x \in \mathfrak{g}. \tag{1.8}$$

The dual module V^* belongs to the "mirror" category $\mathcal{O}^*$, associated with the lowest weight $\mathfrak{g}$-modules. More precisely, $\mathcal{O}^*$ consists of finitely generated, $\mathfrak{h}$-diagonalizable and locally $\mathfrak{n}_-$-nilpotent $\mathfrak{g}$-modules.

1.2. **Loewy length and radical, socle series.** Let V be a finite length module over some algebra. The Loewy length $ll(V)$ is an invariant of V, defined as the length of the shortest filtration of V with semisimple quotients. Such shortest filtration is not necessarily unique; in fact, there are two canonical choices.

The radical of V is defined as the intersection of all maximal proper submodules of V. The radical filtration, also called the radical series or the upper Loewy series,

$$0 = \operatorname{rad}^l V \subset \operatorname{rad}^{l-1} V \subset \cdots \subset \operatorname{rad}^1 V \subset \operatorname{rad}^0 V = V \tag{1.9}$$

is defined inductively by $\operatorname{rad}^{i+1} V = \operatorname{rad}(\operatorname{rad}^i V)$ for $i \geq 0$. One can show that the length l of the radical filtration of V is equal to $ll(V)$.

Similarly, the socle of V is defined as the sum of all semisimple submodules of V. The socle filtration, also called the socle series or the lower Loewy series,

$$0 = \operatorname{soc}^0 V \subset \operatorname{soc}^1 V \subset \cdots \subset \operatorname{soc}^{l-1} V \subset \operatorname{soc}^l V = V \tag{1.10}$$

is defined inductively by $\operatorname{soc}^{i+1} V/\operatorname{soc}^i V = \operatorname{soc}(V/\operatorname{soc}^i V)$ for $i \geq 0$. Again, the length l of the socle filtration of V is equal to $ll(V)$.

The semisimple modules $\overline{\operatorname{soc}}^i V = \operatorname{soc}^i V/\operatorname{soc}^{i-1} V$ and $\overline{\operatorname{rad}}^i V = \operatorname{rad}^{i-1} V/\operatorname{rad}^i V$ are called the layers of the corresponding series. Moreover, any other filtration $0 \subset V_0 \subset \cdots \subset V_m = V$ with semisimple quotients satisfies

$$\operatorname{rad}^{m-i} V \subset V_i \subset \operatorname{soc}^i V. \tag{1.11}$$

A module is called rigid, if its socle and radical series coincide; in that case they give the unique shortest filtration with semisimple quotients, which we simply call the Loewy series.

Examples of rigid modules in the category $\mathcal{O}$ include Verma modules M_λ for all λ, and the big projective modules P_λ for anti-dominant λ. For more information we refer the reader to [Ir] and references therein.

2. The regular representation on the big cell

2.1. **The regular representations.** Let G be the simple complex Lie group, corresponding to $\mathfrak{g}$. Denote by $\mathfrak{R}(G)$ the algebra of regular functions on G; it has the structure of a $G\times G$-module, given by

$$(\varrho_1(g)\psi)(x) = \psi(g^{-1}x), \qquad (\varrho_2(g)\psi)(x) = \psi(x\,g), \qquad g, x \in G. \tag{2.1}$$

The Lie algebra $\mathfrak{g}$ can be defined as the tangent space to the group at the unit element $e \in G$. To any $\xi \in \mathfrak{g}$, we can associate two vector fields on G: the left-invariant vectors field $\mathcal{L}_\xi$ such that $\mathcal{L}_\xi(e) = \xi$, and the right-invariant vector field $\mathcal{R}_{-\xi}$ such that $\mathcal{R}_{-\xi}(e) = -\xi$. These maps yield two commuting embeddings of $\mathfrak{g}$ into the Lie algebra $\mathrm{Vect}(G)$, and define the two regular $\mathfrak{g}$-actions on $\mathfrak{R}(G)$ by first order differential operators. Equivalently, they can be defined by the infinitesimal versions of (2.1):

$$(\varrho_1(\xi)\psi)(x) = \frac{d}{dt}\psi(e^{-t\xi}g)\Big|_{t=0}, \qquad (\varrho_2(\xi)\psi)(x) = \frac{d}{dt}\psi(ge^{t\xi})\Big|_{t=0}, \qquad \xi\in\mathfrak{g},\ x\in G. \tag{2.2}$$

Under both regular actions, the elements of the center $Z(\mathfrak{g})$ act on $\mathfrak{R}(G)$ by $G\times G$-invariant differential operators; for example, the quadratic Casimir operator in $Z(\mathfrak{g})$ corresponds to the Laplace operator on G. The left and right actions of $Z(\mathfrak{g})$ on $\mathfrak{R}(G)$ differ by an involution $z \mapsto z^*$, induced by the antipode $x \mapsto x^* = -x$ for $x \in \mathfrak{g}$, cf. (1.8):

$$\varrho_1(z)\varphi = \varrho_2(z^*)\varphi, \qquad z \in Z(\mathfrak{g}),\ \varphi \in \mathfrak{R}(G), \tag{2.3}$$

Equivalently, in terms of the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$, it corresponds to the involution of $S(\mathfrak{h})^W$ induced by the map $\mathbf{h} \mapsto -\mathbf{h}$ for $\mathbf{h} \in \mathfrak{h}$.

Similarly to (1.4), we define the $\mathfrak{g}\oplus\mathfrak{g}$-submodules $\mathfrak{R}(G)_\lambda$ for any $\lambda \in \mathfrak{h}^*$ by

$$\mathfrak{R}(G)_\lambda = \left\{ \varphi \in \mathfrak{R}(G) \,\middle|\, \forall z \in Z(\mathfrak{g}) \exists n \in \mathbb{N}:\ (\varrho_1(z^*) - \chi_\lambda)^n\varphi = (\varrho_2(z) - \chi_\lambda)^n\varphi = 0 \right\}. \tag{2.4}$$

Since $\mathfrak{R}(G)$ is locally $G \times G$-finite, it decomposes into the direct sum

$$\mathfrak{R}(G) = \bigoplus_{\lambda \in \mathbf{P}^+} \mathfrak{R}(G)_\lambda \tag{2.5}$$

of submodules, corresponding to dominant integral highest weights $\lambda \in \mathbf{P}^+$.

Let T and $N_\pm$ denote the maximal torus and the unipotent subgroups of G, corresponding to $\mathfrak{h}$ and $\mathfrak{n}_\pm$. The Gauss decomposition determines the dense open subset $G_0 = N_-\,T\,N_+$ of the group G. We refer to G_0 as the big cell of the Gauss decomposition of G, and denote by $\mathfrak{R}(G_0)$ the algebra of regular functions on G_0.

The group G does not act on the algebra $\mathfrak{R}(G_0)$, because G_0 is not invariant under left and right shifts by G. Nevertheless, the infinitesimal regular actions (2.2) of $\mathfrak{g}$ on $\mathfrak{R}(G_0)$

are still well-defined. We define the $\mathfrak{g}\oplus\mathfrak{g}$-submodules $\mathfrak{R}(G_0)_\lambda$ the same way as in (2.4), and obtain the direct sum decomposition analogous to (2.5):

$$\mathfrak{R}(G_0) = \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathfrak{R}(G_0)_\lambda \tag{2.6}$$

The fact that the summation above is over the strictly anti-dominant $\lambda\in-\mathbf{P}^{++}$ immediately follows from the explicit polynomial realization of $\mathfrak{R}(G_0)$, which we consider in the next subsection.

2.2. **The polynomial realization of $\mathfrak{R}(G_0)$.** The algebra $\mathfrak{R}(G_0)$ can be realized as a certain polynomial algebra, with the action of $\mathfrak{g}$ given by first order differential operators.

It is convenient to order the positive roots $\beta_1,\ldots,\beta_m$ according to their heights,

$$i<j \;\Rightarrow\; \mathrm{ht}\,\beta_i \le \mathrm{ht}\,\beta_j, \qquad i,j=1,\ldots,m=|\Delta_+|. \tag{2.7}$$

and choose the coordinate system $G_0 = N_-\,T\,N_+$ by using the parameterization

$$g(\vec{x},\vec{y},\vec{z}) = \exp(x_{\beta_m}\,\mathbf{f}_{\beta_m})\ldots\exp(x_{\beta_1}\,\mathbf{f}_{\beta_1})\,z_1^{\mathbf{h}_1}\ldots z_r^{\mathbf{h}_r}\,\exp(y_{\beta_1}\,\mathbf{e}_{\beta_1})\ldots\exp(y_{\beta_m}\,\mathbf{e}_{\beta_m}), \tag{2.8}$$

where $\vec{z}=\{z_i\}\in(\mathbb{C}^\times)^r$ are coordinates on T, and $\vec{x}=\{x_\beta\}\in\mathbb{C}^m$ (resp. $\vec{y}=\{y_\beta\}\in\mathbb{C}^m$) are coordinates on N_- (resp. N_+). We also abbreviate $x_i\equiv x_{\alpha_i}$ and $y_i\equiv y_{\alpha_i}$ for $i=1,\ldots,r$. The Gauss decomposition yields the algebra isomorphisms

$$\mathfrak{R}(G_0)\cong\mathfrak{R}(N_-)\otimes\mathfrak{R}(T)\otimes\mathfrak{R}(N_+)\cong\mathbb{C}[\vec{x}]\otimes\mathbb{C}[\mathbf{P}]\otimes\mathbb{C}[\vec{y}], \tag{2.9}$$

where $\mathfrak{R}(T)=\mathbb{C}[z_1^{\pm1},\ldots,z_r^{\pm1}]$ is identified with the group algebra $\mathbb{C}[\mathbf{P}]$ by

$$z^\lambda \;\equiv\; z_1^{\langle\lambda,\mathbf{h}_1\rangle}\ldots z_r^{\langle\lambda,\mathbf{h}_r\rangle} \qquad \text{for any } \lambda\in\mathbf{P}. \tag{2.10}$$

In general the explicit formulas for the differential operators describing the action of $\mathfrak{g}$ are quite complicated. For our purposes we only need a modest amount of information about the structure of this realization.

Proposition 2.1. *There exist polynomials $p_{i,\beta},q_{i,\beta},r_{i,\beta},s_{i,\beta}$ for $i=1,\ldots,r$ and $\beta\in\Delta_+$, such that the $\mathfrak{g}\oplus\mathfrak{g}$-action on $\mathfrak{R}(G_0)$ is given by*

$$\begin{aligned}
\varrho_1(\mathbf{e}_i) &= x_i^2\,\partial_{x_i} - x_i\,z_i\,\partial_{z_i} - z_i^{-2}\,\partial_{y_i} - \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} r_{i,\beta}(\vec{x})\,\partial_{x_\beta} - z_i^{-2}\sum_{\beta\in\Delta_+\setminus\Pi} s_{i,\beta}(\vec{y})\,\partial_{y_\beta},\\
\varrho_1(\mathbf{h}_i) &= -z_i\,\partial_{z_i} + 2x_i\,\partial_{x_i} - \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} q_{i,\beta}(\vec{x})\,\partial_{x_\beta},\\
\varrho_1(\mathbf{f}_i) &= -\partial_{x_i} - \sum_{\beta\in\Delta_+\setminus\Pi} p_{i,\beta}(\vec{x})\,\partial_{y_\beta},
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\varrho_2(\mathbf{e}_i) &= \partial_{y_i} + \sum_{\beta\in\Delta_+\setminus\Pi} p_{i,\beta}(\vec{y})\,\partial_{y_\beta},\\
\varrho_2(\mathbf{h}_i) &= z_i\,\partial_{z_i} - 2y_i\,\partial_{y_i} + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} q_{i,\beta}(\vec{y})\,\partial_{y_\beta},\\
\varrho_2(\mathbf{f}_i) &= -y_i^2\,\partial_{y_i} + y_i\,z_i\,\partial_{z_i} + z_i^{-2}\,\partial_{x_i} + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} r_{i,\beta}(\vec{y})\,\partial_{y_\beta} + z_i^{-2}\sum_{\beta\in\Delta_+\setminus\Pi} s_{i,\beta}(\vec{x})\,\partial_{x_\beta}
\end{aligned} \tag{2.12}$$

Proof. The algorithm for computing these explicit actions is to push the infinitesimal exponentials in (2.2) inside until they are absorbed by the corresponding factor in (2.8). To handle the adjustment factors appearing in this commutation process, it suffices to do the computations modulo t^2 using the identity

$$\exp(B)\exp(tA) \equiv \exp\left(t \sum_{j=0}^{\infty} \frac{1}{j!} \underbrace{[B, \dots, [B, [B, A]] \dots]}_{j \text{ commutators}} \right) \exp(B) \mod t^2,$$

and similar special cases of the Campbell-Hausdorff formula. Using this approach, one can verify that the right regular action has the form (2.12) for some $p_{i,\beta}, q_{i,\beta}, r_{i,\beta}, s_{i,\beta}$; we skip the technical details.

It is easy to see that the transposition anti-automorphism $g \mapsto g^\top$ interchanges the $\vec{x}$ and $\vec{y}$ variables, i.e. $g(\vec{x}, \vec{y}, \vec{z})^\top = g(\vec{y}, \vec{x}, \vec{z})$, and transforms the right regular action ϱ_2 into the negative left regular action $-\varrho_1$. This implies that the left regular action (2.11) is described in terms of the same polynomials $p_{i,\beta}, q_{i,\beta}, r_{i,\beta}, s_{i,\beta}$. □

In other words, Proposition 2.1 states that for each simple root α_i the action on $\mathfrak{R}(G_0)$ of the subalgebra $\mathfrak{sl}_{\alpha_i}(2, \mathbb{C}) = \mathbb{C}\mathbf{e}_i \oplus \mathbb{C}\mathbf{h}_i \oplus \mathbb{C}\mathbf{f}_i$ is given by the standard $\mathfrak{sl}(2, \mathbb{C})$ action (see e.g. [FS]), plus additional terms determined by $p_{i,\beta}, q_{i,\beta}, r_{i,\beta}, s_{i,\beta}$. Other properties of these polynomials - for example, they are homogeneous in the appropriate sense - are not crucial for our further analysis.

Proposition 2.2. *For any $\lambda \in -\mathbf{P}^{++}$ we have*

$$\operatorname{ch} \mathfrak{R}(G_0)_\lambda = \sum_{w \in W^\lambda} \operatorname{ch}\left(M^*_{w\cdot\lambda} \otimes M_{w\cdot\lambda} \right). \tag{2.13}$$

Equivalently, for any $x, y \in W^\lambda$ we have

$$[\mathfrak{R}(G_0)_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = [P_{y\cdot\lambda} : L_{x\cdot\lambda}]. \tag{2.14}$$

Proof. Consider the collection $\{\mathfrak{R}^{\leq\lambda}\}_{\lambda\in\mathbf{P}}$ of subspaces of $\mathfrak{R}(G_0)$, defined by

$$\mathfrak{R}^{\leq\lambda} = \bigoplus_{\mu\leq\lambda} \mathbb{C}[\vec{x}] \otimes \mathbb{C}z^\mu \otimes \mathbb{C}[\vec{y}] \tag{2.15}$$

where $\mu \leq \lambda$ is equivalent to $\lambda - \mu \in \mathbf{P}^+$. It is clear from (2.11), (2.12) that $\mathfrak{R}^{\leq\lambda}$ is a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathfrak{R}(G_0)$ for any $\lambda \in \mathbf{P}$. Similarly, one defines the $\mathfrak{g} \oplus \mathfrak{g}$-submodules $\{\mathfrak{R}^{<\lambda}\}_{\lambda\in\mathbf{P}}$ of $\mathfrak{R}(G_0)$.

The induced $\mathfrak{g} \oplus \mathfrak{g}$-actions in the quotients $\mathfrak{R}^{\leq\lambda}/\mathfrak{R}^{<\lambda}$ are given by

$$\begin{aligned}
\varrho_1^{(\lambda)}(\mathbf{e}_i) &= x_i^2\, \partial_{x_i} - x_i\, z_i\, \partial_{z_i} - \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} r_{i,\beta}(\vec{x})\, \partial_{x_\beta}, \\
\varrho_1^{(\lambda)}(\mathbf{h}_i) &= -z_i\, \partial_{z_i} + 2x_i\, \partial_{x_i} - \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} q_{i,\beta}(\vec{x})\, \partial_{x_\beta}, \\
\varrho_1^{(\lambda)}(\mathbf{f}_i) &= -\partial_{x_i} - \sum_{\beta\in\Delta_+\setminus\Pi} p_{i,\beta}(\vec{x})\, \partial_{y_\beta},
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\varrho_2^{(\lambda)}(\mathbf{e}_i) &= \partial_{y_i} + \sum_{\beta\in\Delta_+\setminus\Pi} p_{i,\beta}(\vec{y})\,\partial_{y_\beta}, \\
\varrho_2^{(\lambda)}(\mathbf{h}_i) &= z_i\,\partial_{z_i} - 2y_i\,\partial_{y_i} + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} q_{i,\beta}(\vec{y})\,\partial_{y_\beta}, \\
\varrho_2^{(\lambda)}(\mathbf{f}_i) &= -y_i^2\,\partial_{y_i} + y_i\,z_i\,\partial_{z_i} + \sum_{\beta\in\Delta_+\setminus\{\alpha_i\}} r_{i,\beta}(\vec{y})\,\partial_{y_\beta}
\end{aligned} \tag{2.17}$$

The $\mathfrak{g}$-action (2.16) is homogeneous in $\vec{z}$ and independent of $\vec{x}$, and therefore defines an action of $\mathfrak{g}$ on the space $\mathbb{C}z^\lambda\otimes\mathbb{C}[\vec{x}]$, which is identified with the functions (or sections of line bundles) on the big cell of the flag variety. The corresponding $\mathfrak{g}$-module can be shown to be is isomorphic to M_λ^*. Similarly, the action (2.17) in $\mathbb{C}z^\lambda\otimes\mathbb{C}[\vec{y}]$ yields the contragredient Verma module M_λ^c. Therefore, we have a $\mathfrak{g}\oplus\mathfrak{g}$-module isomorphism

$$\mathfrak{R}^{\le\lambda}/\mathfrak{R}^{<\lambda} = \mathbb{C}[\vec{y}]\otimes\mathbb{C}z^\lambda\otimes\mathbb{C}[\vec{x}] \cong M_\lambda^*\otimes M_\lambda^c. \tag{2.18}$$

Collecting the constituents associated with the central character χ_λ, i.e. arising from the quotients $\mathfrak{R}^{\le\mu}/\mathfrak{R}^{<\mu}$ with $\mu\in W^\lambda\cdot\lambda$, we obtain

$$\operatorname{ch}\mathfrak{R}(G_0)_\lambda = \sum_{w\in W^\lambda}\operatorname{ch}\left(M_{w\cdot\lambda}^*\otimes M_{w\cdot\lambda}^c\right) = \sum_{w\in W^\lambda}\operatorname{ch}\left(M_{w\cdot\lambda}^*\otimes M_{w\cdot\lambda}\right), \tag{2.19}$$

establishing (2.13). Similarly, we have

$$\begin{aligned}
[\mathfrak{R}(G_0)_\lambda : L_{x\cdot\lambda}\otimes L_{y\cdot\lambda}^*] &= \sum_{w\in W^\lambda}[M_{w\cdot\lambda}^*\otimes M_{w\cdot\lambda}^c : L_{x\cdot\lambda}^*\otimes L_{y\cdot\lambda}] = \\
= \sum_{w\in W^\lambda}[M_{w\cdot\lambda}^* : L_{x\cdot\lambda}^*][M_{w\cdot\lambda}^c : L_{y\cdot\lambda}] &= \sum_{w\in W^\lambda}[M_{w\cdot\lambda} : L_{x\cdot\lambda}]\;(P_{y\cdot\lambda} : M_{w\cdot\lambda}) = [P_{y\cdot\lambda} : L_{x\cdot\lambda}].
\end{aligned}$$

□

3. Whittaker vectors in $\mathfrak{R}(G_0)$ and the big projective modules

3.1. **Whittaker vectors and Whittaker functor.** Let $\eta=(\eta_1,\ldots,\eta_r)\in\mathbb{C}^r$. There exists a unique character $\boldsymbol{\eta}^+ : \mathcal{U}(\mathfrak{n}_+)\to\mathbb{C}$, such that $\boldsymbol{\eta}^+(\mathbf{e}_i)=\eta_i$ for $i=1,\ldots,r$. In order for $\boldsymbol{\eta}^+$ to be an algebra homomorphism we must put $\boldsymbol{\eta}^+(\mathbf{e}_\beta)=0$ for $\beta\notin\Pi$.

The Whittaker functor Wh_η^+ associates to any $\mathfrak{g}$-module V the subspace

$$\mathrm{Wh}_\eta^+(V) = \left\{v\in V \,\middle|\, \ker\boldsymbol{\eta}^+\cdot v = 0\right\}. \tag{3.1}$$

Elements $v\in\mathrm{Wh}_\eta^+(V)$ are called the Whittaker vectors of V with respect to $\boldsymbol{\eta}^+$.

One can show that two Whittaker functors Wh_η^+ and $\mathrm{Wh}_{\eta'}^+$ are isomorphic if and only if the corresponding characters vanish on the same Chevalley generators: $\eta_i=0\Leftrightarrow\eta_i'=0$. Thus, there are 2^r nonequivalent Whittaker functors. The two extreme cases correspond to the trivial character with all $\eta_i=0$, and to the nonsingular characters with all $\eta_i\ne0$.

In the case of trivial $\eta=0$, the functor Wh_0^+ gives the subspace of singular vectors in V:

$$\mathrm{Wh}_0^+(V) = \mathrm{Sing}^+(V) = \left\{v\in V \,\middle|\, \mathfrak{n}_+\cdot v = 0\right\}, \tag{3.2}$$

which are important in the study of modules in category $\mathcal{O}$. However, for nontrivial η we have $\mathrm{Wh}_\eta^+ V=0$ for all $V\in\mathcal{O}$, which is easily proved using the weight decomposition of V.

Therefore, we should consider a modification $\overline{\mathrm{Wh}}_\eta^+$ of the Whittaker functor Wh_η^+. For any $V = \bigoplus_\mu V[\mu]$ introduce its completion $\bar V = \prod_\mu V[\mu]$. We define

$$\overline{\mathrm{Wh}}_\eta^+(V) = \mathrm{Wh}_\eta^+(\bar V) = \left\{ v \in \bar V \,\middle|\, \ker \boldsymbol{\eta}^+ \cdot v = 0 \right\}. \tag{3.3}$$

The center $Z(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ acts on the space of Whittaker vectors in V. Therefore, for each η we get a functor

$$\overline{\mathrm{Wh}}_\eta^+ : \mathfrak{g}\text{-mod} \to Z(\mathfrak{g})\text{-mod}, \qquad V \mapsto \overline{\mathrm{Wh}}_\eta^+(V). \tag{3.4}$$

We will be particularly interested in the restriction of the functor $\overline{\mathrm{Wh}}_\eta^+$ to the category $\mathcal{O}$.

Similarly, one defines versions $\mathrm{Wh}_\eta^-, \overline{\mathrm{Wh}}_\eta^- : \mathfrak{g}\text{-mod} \to Z(\mathfrak{g})\text{-mod}$ of the Whittaker functors, corresponding to the character $\boldsymbol{\eta}^- : \mathcal{U}(\mathfrak{n}_-) \to \mathbb{C}$ defined by $\boldsymbol{\eta}^-(\mathbf{f}_i) = \eta_i$. The functor $\overline{\mathrm{Wh}}_\eta^-$ plays the same role for the mirror category $\mathcal{O}^*$ as $\overline{\mathrm{Wh}}_\eta^+$ plays for $\mathcal{O}$.

If V is equipped with two commuting $\mathfrak{g}$-actions, then the space of Whittaker vectors associated with one copy of $\mathfrak{g}$ retains the $\mathfrak{g}$-module structure with respect to the other copy of $\mathfrak{g}$. In the next section we study the functors

$$\begin{aligned} \overline{\mathrm{Wh}}_\eta^- \otimes 1 &: \mathfrak{g} \oplus \mathfrak{g}\text{-mod} \to Z(\mathfrak{g}) \oplus \mathfrak{g}\text{-mod}, \\ 1 \otimes \overline{\mathrm{Wh}}_\eta^+ &: \mathfrak{g} \oplus \mathfrak{g}\text{-mod} \to \mathfrak{g} \oplus Z(\mathfrak{g})\text{-mod}, \\ \overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+ &: \mathfrak{g} \oplus \mathfrak{g}\text{-mod} \to Z(\mathfrak{g}) \oplus Z(\mathfrak{g})\text{-mod}, \end{aligned}$$

applied to the regular representation on the big cell.

3.2. Polynomial realization of $(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0))$.

We show that the space of Whittaker vectors with respect to the left regular action admits a polynomial realization, such that the right regular action of $\mathfrak{g}$ is very similar to the $\mathfrak{g}$-actions on functions on the flag variety.

Proposition 3.1. *The $\mathfrak{g}$-module $(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0))$ can be realized in the polynomial algebra $\mathbb{C}[\mathbf{P}] \otimes \mathbb{C}[\vec y]$, with the $\mathfrak{g}$-action given by*

$$\begin{aligned} \varrho(\mathbf{e}_i) &= \partial_{y_i} + \sum_{\beta \in \Delta_+ \setminus \Pi} p_{i,\beta}(\vec y)\,\partial_{y_\beta}, \\ \varrho(\mathbf{h}_i) &= z_i\,\partial_{z_i} - 2y_i \partial_{y_i} + \sum_{\beta \in \Delta_+ \setminus \{\alpha_i\}} q_{i,\beta}(\vec y)\,\partial_{y_\beta}, \\ \varrho(\mathbf{f}_i) &= -y_i^2 \partial_{y_i} + y_i\, z_i \partial_{z_i} + \sum_{\beta \in \Delta_+ \setminus \{\alpha_i\}} r_{i,\beta}(\vec y)\,\partial_{y_\beta} + \eta_i\, z_i^{-2} \end{aligned} \quad , \tag{3.5}$$

where $p_{i,\beta}, q_{i,\beta}, r_{i,\beta}$ are the same as in Proposition 2.1.

Proof. The desired Whittaker vectors in $\mathfrak{R}(G_0)$ are represented by functions satisfying

$$\left(\partial_{x_i} + \sum_{\beta \in \Delta^+} p_{i,\beta}(\vec x) \partial_{x_\beta} + \eta_i \right) \psi(\vec x, \vec y, \vec z) = 0, \qquad i = 1, \dots, r \tag{3.6}$$

These differential equations are independent of $\vec y, \vec z$, and thus the space of solutions is spanned by functions of the form $\psi(\vec x, \vec y, \vec z) = \phi(\vec y, \vec z)\, \tau(\vec x)$ where $\tau(\vec x)$ satisfies the equations (3.6), and represents a Whittaker vector in $\mathbb{C}[\vec x] \cong M_0^*$. It is known that for any $\mu \in \mathfrak{h}^*$ and $\eta \in \mathbb{C}^r$ we have $\dim \overline{\mathrm{Wh}}_\eta^-(M_\mu^*) = 1$, see e.g. [Ba]. Therefore, there exists a unique up to proportionality solution $\tau(\vec{x}) \in \mathbb{C}[[x]]$ of (3.6), and a direct verification shows that $\tau(\vec{x}) = \exp\left(\sum_{i=1}^r \eta_i\, x_i\right)$.

We conclude that the Whittaker vectors in $\mathfrak{R}(G_0)$ are identified with functions

$$\psi(\vec{x}, \vec{y}, \vec{z}) = \phi(\vec{y}, \vec{z}) \ \exp\left(\sum_{i=1}^r \eta_i\, x_i\right), \tag{3.7}$$

and it is straightforward to check that the specialization of the formulas (2.12) to the subspace of functions (3.7) results in the action (3.5) on the space $\mathbb{C}[\mathbf{P}] \otimes \mathbb{C}[\vec{y}]$. □

The polynomial realization gives us the following important information on the size of the Whittaker vector spaces.

Proposition 3.2. *For any $\eta \in \mathbb{C}^r$ we have*

$$\mathrm{ch}(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) = \sum_{w \in W^\lambda} \mathrm{ch}\, M_{w \cdot \lambda}. \tag{3.8}$$

Equivalently, for any $w \in W^\lambda$ we have

$$[(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) : L_{x \cdot \lambda}] = [P_\lambda : L_{x \cdot \lambda}]. \tag{3.9}$$

Proof. Consider the collection of submodules $\mathfrak{P}^{\leq \lambda}$, defined by

$$\mathfrak{P}^{\leq \lambda} = \bigoplus_{\mu \leq \lambda} \mathbb{C} z^\mu \otimes \mathbb{C}[\vec{y}]. \tag{3.10}$$

The corresponding quotients $\mathfrak{P}^{\leq \lambda}/\mathfrak{P}^{< \lambda}$ are given by (2.16), and therefore

$$\mathfrak{P}^{\leq \lambda}/\mathfrak{P}^{< \lambda} \cong M_\lambda^c. \tag{3.11}$$

Collecting the constituents from $\mathcal{O}_\lambda$, we conclude that

$$\mathrm{ch}(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) = \sum_{w \in W^\lambda} \mathrm{ch}\, M_{w \cdot \lambda}^c = \sum_{w \in W^\lambda} \mathrm{ch}\, M_{w \cdot \lambda}. \tag{3.12}$$

Similarly, using the fact that for every $w \in W^\lambda$ we have $(P_\lambda : M_{w \cdot \lambda}) = 1$, we obtain

$$[(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) : L_{x \cdot \lambda}] = \sum_{w \in W^\lambda} [M_{w \cdot \lambda}^c : L_{x \cdot \lambda}] = [P_\lambda : L_{x \cdot \lambda}]. \tag{3.13}$$

□

3.3. **Big projective modules and the Borel-Weil realization.** The Borel-Weil realization of finite-dimensional simple $\mathfrak{g}$-modules L_λ with dominant integral highest weights $\lambda \in \mathbf{P}^+$, given by (0.2), can be reformulated in the following form.

Theorem 3.3. *For any $\lambda \in \mathbf{P}^+$, there is an isomorphism*

$$L_\lambda \cong (\mathrm{Sing}^- \otimes 1)(\mathfrak{R}(G)_\lambda) = \left\{ \varphi \in \mathfrak{R}(G)_\lambda \,\middle|\, \varrho_1(\mathfrak{n}_-)\varphi = 0 \right\}. \tag{3.14}$$

Replacing the functor Sing^- by its generalization $\overline{\mathrm{Wh}}_\eta^-$ and applying it to the regular representation on the big cell, we obtain a similar realization of the big projective modules in the category $\mathcal{O}$.

Theorem 3.4. *Let $\eta \in \mathbb{C}^r$ be nonsingular. Then for any $\lambda \in -\mathbf{P}^{++}$ there is an isomorphism*

$$(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) \cong P_\lambda. \tag{3.15}$$

Proof. Set for brevity $V = (\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda)$. The proof is based on two properties of V.

First, the module V has a quotient, isomorphic to the contragredient Verma module $M^c_{w_\circ\cdot\lambda}$, which immediately follows from (3.11).

Second, V has a submodule, isomorphic to the Verma module $M_{w_\circ\cdot\lambda}$. Indeed, consider the vector $v \in V$ represented by the polynomial $z^{w_\circ\cdot\lambda}$; it is annihilated by $\mathfrak{n}_+$ and has dominant highest weight $w_\circ \cdot \lambda$. Therefore, the submodule generated by v must be isomorphic to some quotient of $M_{w_\circ\cdot\lambda}$; we show that in fact it coincides with $M_{w_\circ\cdot\lambda}$.

The socle of the Verma module $M_{w_\circ\cdot\lambda}$ is generated by a singular vector $v_{sing} = q(\lambda)v_0$, where v_0 is the highest weight vector generating $M_{w_\circ\cdot\lambda}$, and $q(\lambda)$ is some element in $\mathcal{U}(\mathfrak{n}_-)$ of weight $w_\circ \cdot \lambda - \lambda$. Let $\{n_i\}_{i=1}^r$ be the nonnegative integers such that $\sum_{i=1}^r n_i\alpha_i = w_\circ \cdot \lambda - \lambda$. Consider the PBW basis of $\mathcal{U}(\mathfrak{n}_-)$, associated with the ordering $\{\mathbf{f}_{\beta_1}, \dots, \mathbf{f}_{\beta_m}\}$. Then with respect to this basis we have an expansion

$$q(\lambda) = c\, \mathbf{f}_1^{n_1} \dots \mathbf{f}_r^{n_r} + \text{ terms with nonsimple root vectors,}$$

with some nonzero coefficient c, see e.g. [Ba]. It is clear that in the polynomial realization of V the action of $\mathbf{f}_\beta$ contains the multiplication by $z^{-\beta}$ if and only if β is a simple root. Therefore, the coefficient before z^λ in the expansion of $q(\lambda)v$ is equal to $c\,\eta_1^{m_1} \dots \eta_r^{m_r} \neq 0$, and thus $q(\lambda)v \neq 0$. Hence v generates a submodule of V isomorphic to $M_{w_\circ\cdot\lambda}$.

We now return to the main proof. It follows from (3.9) that the composition series for V contains exactly one simple module $L_{w_\circ\cdot\lambda}$ with the dominant highest weight $w_\circ \cdot \lambda$. This unique constituent appears in the socle of M^c_λ and in the top layer of M_λ. It is known that both M_λ and M^c_λ are rigid and have Loewy length $l_\lambda + 1$, and therefore V has Loewy length at least $2l_\lambda + 1$.

On the other hand, modules in the category $\mathcal{O}_\lambda$ have Loewy Length at most $2l_\lambda + 1$, so we have $ll(V) = 2l_\lambda + 1$. Moreover, the only indecomposable module of Loewy length $2l_\lambda + 1$ is the big projective module P_λ. Therefore, V must contain P_λ as a direct summand, and comparing the characters (1.7) and (3.8) we see that in fact $V \cong P_\lambda$. ☐

Similar arguments show that for any $\lambda \in -\mathbf{P}^{++}$ there is an isomorphism

$$(1 \otimes \overline{\mathrm{Wh}}_\eta^+)(\mathfrak{R}(G_0)_\lambda) \cong P^*_\lambda. \tag{3.16}$$

3.4. **Whittaker functions on G_0.** Let $\eta, \eta' \in \mathbb{C}^r$. The associated Whittaker functions on the group G are defined as elements of $(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0))$. When $G = SL(2, \mathbb{C})$, they are directly related to the Whittaker functions $W_{k,m}(z)$, satisfying the Whittaker differential equation [WW]. For Lie groups of other type the Whittaker functions were used by Kazhdan and Kostant to prove the integrability of the quantum Toda system: the restriction of the Laplacian to the subspace of Whittaker functions coincides with the Toda Hamiltonian, and higher Casimir operators yield the quantum integrals of motion.

The decomposition (2.6) induces the $Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$-module decomposition

$$(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)) = \bigoplus_{\lambda \in -\mathbf{P}^{++}} (\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)_\lambda), \tag{3.17}$$

The subspaces $(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)_\lambda)$ are spanned by the generalized eigenfunctions for the quantum Toda system, corresponding to the central character χ_λ.

For any $\lambda \in -\mathbf{P}^{++}$ denote $\mathcal{C}_\lambda = \mathrm{End}_{\mathcal{O}}(P_\lambda)$. The algebra $\mathcal{C}_\lambda$ was studied in [So, Be], and can be interpreted as the cohomology ring of a suitable complex flag variety. In particular, $\mathcal{C}_\lambda$ is commutative.

Proposition 3.5. *Let $\eta, \eta' \in \mathbb{C}^r$ be nonsingular. Then*

$$(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)_\lambda) \cong \mathcal{C}_\lambda. \tag{3.18}$$

Proof. Using Theorem 3.4, we get

$$(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)_\lambda) = (1 \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\overline{\mathrm{Wh}}_\eta^- \otimes 1)(\mathfrak{R}(G_0)_\lambda) \cong \overline{\mathrm{Wh}}_{\eta'}^+(P_\lambda). \tag{3.19}$$

It was shown in [Ba] that for nonsingular η the restriction of the Whittaker functor $\overline{\mathrm{Wh}}_\eta^+$ is isomorphic to Soergel's functor

$$\mathbb{V} : \mathcal{O}_\lambda \to \mathcal{C}_\lambda\text{-mod}, \qquad \mathbb{V}(M) = \mathrm{Hom}_{\mathcal{O}}(P_\lambda, M), \tag{3.20}$$

when we regard $\mathbb{V}$ as a functor from $\mathcal{O}_\lambda$ to $Z(\mathfrak{g})$-modules via the surjection $Z(\mathfrak{g}) \to \mathcal{C}_\lambda$, arising from the action of $\mathcal{U}(\mathfrak{g})$ on P_λ (see [So]). Therefore, we have

$$\overline{\mathrm{Wh}}_{\eta'}^+(P_\lambda) \cong \mathrm{Hom}_{\mathcal{O}}(P_\lambda, P_\lambda) = \mathcal{C}_\lambda. \tag{3.21}$$

□

Using the equivalence of Whittaker and Soergel's functors, we reformulate Theorem 3.4.

Corollary 3.6. *For any $\lambda \in \mathbf{P}^{++}$, there are $\mathfrak{g}$-module isomorphisms*

$$\mathrm{Hom}_{\varrho_1}(P_\lambda^*, \mathfrak{R}(G_0)) \cong P_\lambda, \qquad \mathrm{Hom}_{\varrho_2}(P_\lambda, \mathfrak{R}(G_0)) \cong P_\lambda^*, \tag{3.22}$$

where we regard $\mathfrak{R}(G_0)$ as a $\mathfrak{g}$-module with respect to the indicated regular action ϱ_1 or ϱ_2, and $\mathfrak{g}$ acts on the space of morphisms via the other regular action.

Another consequence of Proposition 3.5 is that $\dim(\overline{\mathrm{Wh}}_\eta^- \otimes \overline{\mathrm{Wh}}_{\eta'}^+)(\mathfrak{R}(G_0)_\lambda) = |W^\lambda|$. Thus, the space of generalized eigenfunctions of the quantum Toda system has dimension $|W^\lambda|$, and has a natural grading, inherited from the corresponding cohomology ring. It would be interesting to interpret this grading in the context of quantum completely integrable systems.

4. Matrix elements of modules in the category $\mathcal{O}$.

4.1. **The algebra $\mathcal{U}(\mathfrak{g})^*$ and subspaces of matrix elements.** Let $\mathcal{U}(\mathfrak{g})^*$ denote the dual space of the enveloping algebra $\mathcal{U}(\mathfrak{g})$; it has a natural $\mathfrak{g} \oplus \mathfrak{g}$-module structure, defined by

$$(\varrho_1(\xi)\psi)(x) = -\psi(\xi\, x), \qquad (\varrho_2(\xi)\psi)(x) = \psi(x\, \xi), \qquad \xi \in \mathfrak{g},\ x \in \mathcal{U}(\mathfrak{g}). \tag{4.1}$$

Let V be a $\mathfrak{g}$-module. For any $v \in V$, $v^* \in V^*$ we define the "matrix element" functional

$$\Phi_{v^* \otimes v}(x) = \langle v^*, x\, v\rangle, \qquad x \in \mathcal{U}(\mathfrak{g}), \tag{4.2}$$

and extend it by linearity to the map

$$\Phi : V^* \otimes V \to \mathcal{U}(\mathfrak{g})^*. \tag{4.3}$$

We denote by $\mathbb{M}(V)$ the subspace $\Phi(V^* \otimes V) \subset \mathcal{U}(\mathfrak{g})^*$, spanned by the matrix elements of V. It is easy to check that the map Φ is a $\mathfrak{g} \oplus \mathfrak{g}$-homomorphism, and thus $\mathbb{M}(V)$ is a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})^*$. We use the same notation Φ for all $\mathfrak{g}$-modules V, i.e. regard Φ as the "universal" matrix elements map.

Dualizing the comultiplication in the Hopf algebra $\mathcal{U}(\mathfrak{g})$, we obtain a commutative associative multiplication in $\mathcal{U}(\mathfrak{g})^*$. The commutative algebra $\mathcal{U}(\mathfrak{g})^*$ is isomorphic to the algebra of the formal power series in $n = \dim \mathfrak{g}$ variables, see e.g. [Di].

We consider two smaller subalgebras of $\mathcal{U}(\mathfrak{g})^*$. The Hopf dual $\mathcal{U}(\mathfrak{g})^*_{Hopf}$ is defined by

$$\mathcal{U}(\mathfrak{g})^*_{Hopf} = \left\{ \varphi \in \mathcal{U}(\mathfrak{g})^* \;\middle|\; \ker \varphi \text{ contains a two-sided ideal } J \subset \mathcal{U}(\mathfrak{g}) \text{ of finite codimension} \right\}.$$

One immediately checks that $\mathcal{U}(\mathfrak{g})^*_{Hopf}$ is a subalgebra and a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})^*$. Moreover, the $\mathfrak{g} \oplus \mathfrak{g}$ action on $\mathcal{U}(\mathfrak{g})^*_{Hopf}$ is locally finite, since for any $\varphi \in \mathcal{U}(\mathfrak{g})^*_{Hopf}$ all elements in $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))(\varphi)$ also annihilate the corresponding ideal J, and $\dim \mathrm{Ann}(J) < \infty$. This is the exact analogue of the locally finite $G \times G$ action on $\mathfrak{R}(G)$; in fact one has the algebra and $\mathfrak{g} \oplus \mathfrak{g}$-module isomorphism $\mathcal{U}(\mathfrak{g})^*_{Hopf} \cong \mathfrak{R}(G)$. In other words, $\mathcal{U}(\mathfrak{g})^*_{Hopf}$ gives the Lie algebraic model of the regular representation $\mathfrak{R}(G)$, and in particular is spanned by the matrix elements of simple finite-dimensional $\mathfrak{g}$-modules L_λ for $\lambda \in \mathbf{P}^+$.

The matrix elements of modules in the category $\mathcal{O}$ span a larger subspace of $\mathcal{U}(\mathfrak{g})^*$. Consider the triangular decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+)$, and let $\mathcal{U}(\mathfrak{n}_\pm)^*$ be the restricted duals of the corresponding enveloping algebras with respect to the principal gradation. We also consider the subalgebra of $\mathcal{U}(\mathfrak{h})^*$, spanned by functionals e^μ, defined by

$$\langle e^\mu, \mathbf{h}_1^{k_1} \dots \mathbf{h}_r^{k_r} \rangle = \langle \mu, \mathbf{h}_1 \rangle^{k_1} \dots \langle \mu, \mathbf{h}_r \rangle^{k_r}, \qquad \mu \in \mathfrak{h}^*,\ k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}.$$

We note that under the identification of $\mathcal{U}(\mathfrak{h})^*$ with the algebra of formal power series, the functionals e^μ correspond to the exponential functions, and hence naturally correspond to the elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$. We define

$$\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}} = \mathcal{U}(\mathfrak{n}_-)^* \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{U}(\mathfrak{n}_+)^*. \tag{4.4}$$

The subspace $\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$ is also a subalgebra and a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathcal{U}(\mathfrak{g})^*$. It follows from the definition that the $\mathfrak{g} \oplus \mathfrak{g}$-action on it is locally $\mathfrak{n}_- \oplus \mathfrak{n}_+$-finite, and that $\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$ is $\mathfrak{h}$-diagonalizable with respect to both left and right actions.

Proposition 4.1. *The subalgebra $\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$ coincides with the subspace $\mathbb{M}_{\mathcal{O}}$, defined by*

$$\mathbb{M}_{\mathcal{O}} = \sum_{V \in \mathcal{O}} \mathbb{M}(V) \subset \mathcal{U}(\mathfrak{g})^*. \tag{4.5}$$

Equivalently, $\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$ is spanned by the matrix elements of modules in category $\mathcal{O}$.

Proof. Suppose $V \in \mathcal{O}$, and let $v \in V$, $v^* \in V^*$ be $\mathfrak{h}$-homogeneous. For any $\mathfrak{h}$-homogeneous $x_\pm \in \mathcal{U}(\mathfrak{n}_\pm)$ and any $x_0 = \mathbf{h}_1^{k_1} \dots \mathbf{h}_r^{k_r} \in \mathcal{U}(\mathfrak{h})$ we have

$$\Phi_{v^* \otimes v}(x_- x_0 x_+) = \langle v^*, x_- x_0 x_+ v \rangle = \langle \mu, \mathbf{h}_1 \rangle^{k_1} \dots \langle \mu, \mathbf{h}_r \rangle^{k_r} \ \langle x_-^* v^*, x_+ v \rangle, \tag{4.6}$$

where $\mu \in \mathfrak{h}^*$ is the weight of the homogeneous vector $x_+ v$. This implies that $\Phi_{v^* \otimes v} \in \mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$, and therefore $\mathbb{M}(V) \subset \mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$.

We now prove the opposite inclusion. Let $\varphi \in \mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$, and consider the subspace

$$V = (1 \otimes \mathcal{U}(\mathfrak{g}))\, \varphi \subset \mathcal{U}(\mathfrak{g})^* \tag{4.7}$$

With respect to the right regular action V is an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module, generated by a single $\mathfrak{n}_+$-nilpotent element φ, and hence $V \in \mathcal{O}$. Let φ^* denote the restriction of $1 \in \mathcal{U}(\mathfrak{g})$ to the subspace V, i.e. $\langle \varphi^*, \psi \rangle = \psi(1)$; then $\varphi^* \in V^*$ and

$$\Phi_{\varphi^* \otimes \varphi}(x) = \langle \varphi^*, x\, \varphi \rangle = (x\varphi)(1) = \varphi(x), \tag{4.8}$$

which means that $\varphi \in \mathbb{M}(V)$, and completes the proof. $\square$

The subalgebra $\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}}$ decomposes into a direct sum of $\mathfrak{g}\oplus\mathfrak{g}$-submodules, corresponding to the central characters of $\mathfrak{g}$. Let $\mathbb{M}_\lambda$ be subspace of matrix elements of modules in $\mathcal{O}_\lambda$,

$$\mathbb{M}_\lambda = \sum_{V\in\mathcal{O}_\lambda} \mathbb{M}(V), \qquad \lambda\in\mathfrak{h}^*. \tag{4.9}$$

Another important subalgebra of $\mathcal{U}(\mathfrak{g})^*$ is the subspace

$$\mathcal{U}(\mathfrak{g})^*_{\mathcal{O}_{int}} = \mathcal{U}(\mathfrak{n}_-)^*\otimes\mathbb{C}[\mathbf{P}]\otimes\mathcal{U}(\mathfrak{n}_+)^* \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}_\lambda, \tag{4.10}$$

corresponding to the matrix elements of module $V\in O$ with integral weights. This subspace gives the Lie algebraic model of (2.6), and in fact is isomorphic to $\mathfrak{R}(G_0)$ as an algebra and as a $\mathfrak{g}\oplus\mathfrak{g}$-module, see Theorem 5.4.

4.2. **Rigidity and Loewy series of $\mathbb{M}_\lambda$.** For any $\lambda\in-\mathbf{P}^{++}$ the $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathbb{M}_\lambda$ admits an increasing filtration $0=\mathbb{M}_\lambda^{(0)}\subset\mathbb{M}_\lambda^{(1)}\subset\cdots\subset\mathbb{M}_\lambda^{(2l_\lambda)}\subset\mathbb{M}_\lambda^{(2l_\lambda+1)}=\mathbb{M}_\lambda$, defined by

$$\mathbb{M}_\lambda^{(k)} = \sum_{\substack{V\in\mathcal{O}_\lambda,\\ ll(V)\le k}} \mathbb{M}(V), \qquad k=1,\ldots,2l(w_\lambda)+1. \tag{4.11}$$

Since the Loewy length of any $V\in\mathcal{O}_\lambda$ is at most $2l_\lambda+1$, we have $\mathbb{M}_\lambda^{(2l_\lambda+1)}=\mathbb{M}_\lambda$.

Proposition 4.2. *For any $\lambda\in-\mathbf{P}^{++}$ the socle filtration of $\mathbb{M}_\lambda$ coincides with* (4.11)*.*

Proof. Consider the layers $\overline{\mathbb{M}}_\lambda^{(k)} = \mathbb{M}_\lambda^{(k)}/\mathbb{M}_\lambda^{(k-1)}$ of this filtration. Let $\varphi\in\mathbb{M}_\lambda^{(k)}$, and let $V\in\mathcal{O}_\lambda$ be such that $ll(V)\le k$ and $\varphi\in\mathbb{M}(V)$. Since $ll(\operatorname{rad}V)=ll(V)-1\le k-1$, we have

$$\Phi(V^*\otimes\operatorname{rad}V)=\Phi((\operatorname{rad}V)^*\otimes\operatorname{rad}V)\subset\mathbb{M}_\lambda^{(k-1)},$$

and similarly $\Phi(\operatorname{rad}V^*\otimes V)\subset\mathbb{M}_\lambda^{(k-1)}$. Therefore, we can complete the commutative diagram

$$\begin{tikzcd} V^*\otimes V \arrow[r, "\Phi"] \arrow[d, "\pi\otimes\pi"'] & \mathbb{M}_\lambda^{(k)} \arrow[d, "\pi"] \\ (V^*/\operatorname{rad}V^*)\otimes(V/\operatorname{rad}V) \arrow[r, dotted] & \mathbb{M}_\lambda^{(k)}/\mathbb{M}_\lambda^{(k-1)} \end{tikzcd} \tag{4.12}$$

and $\pi(\varphi)\in\overline{\mathbb{M}}_\lambda^{(k)}$ belongs to the image of $(V^*/\operatorname{rad}V^*)\otimes(V/\operatorname{rad}V)$, which is semisimple. Therefore the $\mathfrak{g}\oplus\mathfrak{g}$-module $\overline{\mathbb{M}}_\lambda^{(k)}$ is semisimple, and (1.11) implies that $\mathbb{M}_\lambda^{(k)}\subset\operatorname{soc}^k\mathbb{M}_\lambda$.

Next, let $\varphi\in\operatorname{soc}^k\mathbb{M}_\lambda$, and let V be as in (4.7). Then $ll(V)\le ll(\operatorname{soc}^k\mathbb{M}_\lambda)=k$ and $\varphi\in\mathbb{M}(V)$, hence $\varphi\in\mathbb{M}_\lambda^{(k)}$. Therefore $\operatorname{soc}^k\mathbb{M}_\lambda\subset\mathbb{M}_\lambda^{(k)}$, and the statement follows. $\square$

Proposition 4.3. *Let $\lambda\in-\mathbf{P}^{++}$. Then*

$$\operatorname{ch}\mathbb{M}_\lambda = \sum_{w\in W^\lambda}\operatorname{ch}\left(M^*_{w\cdot\lambda}\otimes M_{w\cdot\lambda}\right). \tag{4.13}$$

Equivalently, for any $x,y\in W^\lambda$ we have

$$[\mathbb{M}_\lambda : L^*_{x\cdot\lambda}\otimes L_{y\cdot\lambda}] = [P_{y\cdot\lambda} : L_{x\cdot\lambda}]. \tag{4.14}$$

Proof. Let $x, y \in W^\lambda$, and let $L = L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}$ be a simple constituent of the composition series of the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathbb{M}_\lambda$, belonging to some layer $\overline{\mathbb{M}}^{(k)}_\lambda$ of the filtration (4.11). It follows from (4.12) that L can be generated by some $V \in \mathcal{O}_\lambda$ such that $ll(V) = k$. We can assume without loss of generality that $V/\operatorname{rad} V \cong L_x$ and $V^*/\operatorname{rad} V^* \cong L^*_y$.

Let $P = P_{\mu_1} \oplus \cdots \oplus P_{\mu_j}$ be a projective cover of V. Then $V^* \subset \operatorname{soc}^k P^*$ and (4.12) implies that for $\mu_i \neq x \cdot \lambda$ the matrix elements of P_{μ_i} do not contribute to $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \overline{\mathbb{M}}^{(k)}_\lambda$. Therefore, L can be generated by matrix elements of $P_{x\cdot\lambda}$, and moreover L belongs to $\Phi(\operatorname{soc}^k P^*_{x\cdot\lambda} \otimes P_{x\cdot\lambda})$.

In other words, any simple constituent $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \overline{\mathbb{M}}^{(k)}_\lambda$ corresponds to a pair of submodules $L_{y\cdot\lambda} \subset \overline{\operatorname{rad}}^1 P_{x\cdot\lambda}$ and $L^*_{x\cdot\lambda} \subset \overline{\operatorname{soc}}^k P^*_{y\cdot\lambda} \cong (\overline{\operatorname{rad}}^k P_{y\cdot\lambda})^*$. Since the projective module $P_{y\cdot\lambda}$ has simple top isomorphic to $L_{y\cdot\lambda}$, we see that the occurrences of $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}$ in $\overline{\mathbb{M}}^{(k)}_\lambda$ are in bijective correspondence with submodules $L^*_{x\cdot\lambda} \subset (\overline{\operatorname{rad}}^k P_{y\cdot\lambda})^*$. Therefore,

$$[\mathbb{M}_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = \sum_k [\overline{\mathbb{M}}^{(k)}_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = \sum_k [(\overline{\operatorname{rad}}^k P_{y\cdot\lambda})^* : L^*_{x\cdot\lambda}] = [P_{y\cdot\lambda} : L_{x\cdot\lambda}], \quad (4.15)$$

which implies the desired statement. □

The simple constituents $L_{x\cdot\lambda}$ of the projective module $P_{y\cdot\lambda}$ are in bijective correspondence with the morphisms from $\operatorname{Hom}_{\mathcal{O}}(P_{x\cdot\lambda}, P_{y\cdot\lambda})$ - to any such morphism we associate the constituent in $P_{y\cdot\lambda}$, determined by the image of the simple top of $P_{x\cdot\lambda}$. Therefore, we get

Corollary 4.4. *For any $\lambda \in -\mathbf{P}^{++}$ and $x, y \in W^\lambda$ we have*

$$[\mathbb{M}_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = \dim \operatorname{Hom}_{\mathcal{O}}(P_{x\cdot\lambda}, P_{y\cdot\lambda}). \quad (4.16)$$

Thus, the structure of the $\mathfrak{g} \oplus \mathfrak{g}$-module $\mathbb{M}_\lambda$ is governed by the morphisms between projective modules in the category $\mathcal{O}_\lambda$.

Proposition 4.5. *For any $\lambda \in -\mathbf{P}^{++}$ the radical filtration of $\mathbb{M}_\lambda$ coincides with* (4.11).

Proof. To verify that $\mathbb{M}^{(k)}_\lambda \subset \operatorname{rad} \mathbb{M}^{(k+1)}_\lambda$, it suffices to show that for $k = 1 \ldots, 2l_\lambda$, every simple constituent of the layer $\overline{\mathbb{M}}^{(k)}_\lambda$ is non-trivially linked with the above layer $\overline{\mathbb{M}}^{(k+1)}_\lambda$.

Consider a simple component $L = L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \overline{\mathbb{M}}^{(k)}_\lambda$, where $x, y \in W^\lambda$. It follows from the proof of Proposition 4.3 that L corresponds to a pair of simple constituents $L_{y\cdot\lambda} \in \overline{\operatorname{rad}}^1 P_{y\cdot\lambda}$ and $L^*_{x\cdot\lambda} \in (\overline{\operatorname{rad}}^k P_{y\cdot\lambda})^*$. Moreover, one can independently prove that L can be represented by matrix elements of the big projective module P_λ, see Corollary 5.2 below. Thus we may also assume that L is represented by a pair $L_{y\cdot\lambda} \in \overline{\operatorname{rad}}^j P_\lambda$ and $L^*_{x\cdot\lambda} \in (\overline{\operatorname{rad}}^{j+k-1} P_\lambda)^*$. Since $k \leq 2l_\lambda$, we either have $j > 1$ or $j + k - 1 < 2l_\lambda + 1$, or possibly both.

If $j > 1$, then $\overline{\operatorname{rad}}^{j-1} P_\lambda \neq 0$, and due to rigidity of P_λ the constituent $L_{y\cdot\lambda}$ is non-trivially linked with some $L_{w\cdot\lambda} \subset \overline{\operatorname{rad}}^{j-1} P_\lambda$. It follows that $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}$ is non-trivially linked with $L^*_{x\cdot\lambda} \otimes L_{w\cdot\lambda} \subset \overline{\mathbb{M}}^{(k+1)}_\lambda$, and hence $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \operatorname{rad} \mathbb{M}^{(k+1)}_\lambda$.

If $j + k - 1 < 2l_\lambda + 1$, then $\overline{\operatorname{rad}}^{j+k} P_\lambda \neq 0$, and as above we see that $L^*_{x\cdot\lambda}$ is non-trivially linked with some $L^*_{w\cdot\lambda} \subset (\overline{\operatorname{rad}}^{j+k} P_\lambda)^*$. Again, it follows that $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}$ is non-trivially linked with $L^*_{w\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \overline{\mathbb{M}}^{(k+1)}_\lambda$, and thus $L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \subset \operatorname{rad} \mathbb{M}^{(k+1)}_\lambda$.

We have established that for every k we have $\mathbb{M}^{(k)}_\lambda \subset \operatorname{rad} \mathbb{M}^{(k+1)}_\lambda$; the opposite inclusion follows from (1.11). Therefore, (4.11) is the radical filtration of $\mathbb{M}_\lambda$. □

Combining Proposition 4.2 and Proposition 4.5, we get

Corollary 4.6. *For any $\lambda \in -\mathbf{P}^{++}$ the module $\mathbb{M}_\lambda$ is rigid and has Loewy length $2l_\lambda + 1$.*

5. Big projective modules and the structure of $\mathfrak{R}(G_0)$.

5.1. **Matrix elements of big projectives modules.**

Proposition 5.1. *For any $\lambda \in -\mathbf{P}^{++}$ there is a $\mathfrak{g} \oplus \mathfrak{g}$-module isomorphism*

$$\mathbb{M}(P_\lambda) \cong P_\lambda^* \otimes_{Z(\mathfrak{g})} P_\lambda \tag{5.1}$$

Proof. By construction, we have $\mathbb{M}(P_\lambda) = P_\lambda^* \otimes P_\lambda / \ker \Phi$. We need to check that $\ker \Phi$ coincides with the submodule $J \subset P_\lambda^* \otimes P_\lambda$,

$$J = \left\langle z^* v^* \otimes v - v^* \otimes z\, v \right\rangle, \qquad v \in P_\lambda,\ v^* \in P_\lambda^*,\ z \in Z(\mathfrak{g}) \tag{5.2}$$

It is easy to see that $J \subset \ker \Phi$; indeed, for every $v \in P_\lambda$, $v^* \in P_\lambda^*$ and $z \in Z(\mathfrak{g})$ we have

$$\Phi_{z^* v^* \otimes v}(x) = \langle z^* v^*, x\, v \rangle = \langle v^*, z\, x v \rangle = \langle v^*, x\, z v \rangle = \Phi_{v^* \otimes z v}(x), \qquad x \in \mathcal{U}(\mathfrak{g}). \tag{5.3}$$

To prove the opposite inclusion, we use Soergel's deformation of the projective modules. We can think of it as a family of $\mathfrak{g}$-modules $P_{\lambda;\varepsilon}$, which are identical as vector spaces, but the action of $\mathfrak{g}$ depends on the deformation parameter $\varepsilon \in \mathfrak{h}^*$. The specialization $\varepsilon = 0$ yields $P_{\lambda,\varepsilon} \cong P_\lambda$; see [So] for complete details.

When λ is regular, we consider ε generic in a small neighborhood of $0 \in \mathfrak{h}^*$. If λ lies on some wall(s) of the Weyl chamber (i.e. $W_\lambda \neq \{e\}$), then we consider ε generic such that $W_\varepsilon = W_\lambda$, i.e. from the same wall(s) as λ. For such ε, the specialization $P_{\lambda;\varepsilon}$ is a direct sum of the Verma modules,

$$P_{\lambda;\varepsilon} \cong \bigoplus_{w \in W^\lambda} M_{w\cdot\lambda+\varepsilon}. \tag{5.4}$$

Moreover, the central characters corresponding to $w \cdot \lambda + \varepsilon$ are all distinct, and therefore

$$P_{\lambda;\varepsilon}^* \otimes_{Z(\mathfrak{g})} P_{\lambda;\varepsilon} \cong \bigoplus_{w \in W^\lambda} M_{w\cdot\lambda+\varepsilon}^* \otimes M_{w\cdot\lambda+\varepsilon} \cong \mathbb{M}(P_{\lambda;\varepsilon}). \tag{5.5}$$

Thus, generically we have $\ker \Phi_\varepsilon = J_\varepsilon$; the discrepancies between J_ε and $\ker \Phi_\varepsilon$ may occur when ε satisfies $\langle \beta, \varepsilon \rangle = 0$ for one or more $\beta \in \Delta^+$. We are most interested, of course, in the extreme degenerate case $\varepsilon = 0$.

As in [So], it suffices to check that the linear equations determining submodules $\ker \Phi_\varepsilon$ and J_ε still have the same rank in the *subgeneric* case, when ε satisfies $\langle \beta, \varepsilon \rangle = 0$ for a unique positive root β such that $\langle \beta, \lambda + \rho \rangle \neq 0$. This reduces our problem to the verification of the Proposition for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, which we check directly in the remaining part of the proof.

Let μ be a regular anti-dominant weight for $\mathfrak{sl}(2, \mathbb{C})$; then $W^\mu = \{e, s\}$. Under the standard identification of the weight lattice for $\mathfrak{sl}(2, \mathbb{C})$ with $\mathbb{Z}$, we have $\mu \in \{-2, -3, \dots\}$.

We represent the structures of the big projective module P_μ and its dual P_μ^* pictorially, with the top blocks corresponding to the tops of the modules, and the bottom blocks representing the socles:

$$P_\mu \sim \begin{array}{c} \boxed{L_\mu} \\ \boxed{L_{s\cdot\mu}} \\ \boxed{L_\mu} \end{array}, \qquad\qquad P^*_\mu \sim \begin{array}{c} \boxed{L^*_\mu} \\ \boxed{L^*_{s\cdot\mu}} \\ \boxed{L^*_\mu} \end{array}.$$

Tensoring these two series, we get a filtration for $P^*_\mu \otimes P_\mu$, with layers depicted by

$$P^*_\mu \otimes P_\mu \sim \begin{array}{c} \boxed{L^*_\mu \otimes L_\mu} \\ \boxed{L^*_{s\cdot\mu} \otimes L_\mu} \oplus \boxed{L^*_\mu \otimes L_{s\cdot\mu}} \\ \boxed{L^*_\mu \otimes L_\mu} \oplus \boxed{L^*_{s\cdot\mu} \otimes L_{s\cdot\mu}} \oplus \boxed{L^*_\mu \otimes L_\mu} \\ \boxed{L^*_\mu \otimes L_{s\cdot\mu}} \oplus \boxed{L^*_{s\cdot\mu} \otimes L_\mu} \\ \boxed{L^*_\mu \otimes L_\mu} \end{array}$$

The natural action of $Z(\mathfrak{g})$ on P_μ gives a surjection $Z(\mathfrak{g}) \to \mathrm{End}_{\mathcal{O}}(P_\mu) \cong \mathbb{C}[Q]/\langle Q^2\rangle$. In other words, the endomorphism algebra $\mathrm{End}_{\mathcal{O}}(P_\mu)$ is linearly spanned by the identity and an endomorphism Q, satisfying $Q^2 = 0$, which can be constructed as the composition map

$$Q: \quad \begin{array}{c} \boxed{L_\mu} \\ \boxed{L_{s\cdot\mu}} \\ \boxed{L_\mu} \end{array} \quad \boxed{L_\mu} \quad \begin{array}{c} \boxed{L_\mu} \\ \boxed{L_{s\cdot\mu}} \\ \boxed{L_\mu} \end{array}.$$

The action of $Z(\mathfrak{g})$ on P^*_μ is completely analogous.

It is now straightforward (cf. [FS]) to see that both J and $\ker\Phi$ are described by

$$J = \ker\Phi \sim \begin{array}{c} \boxed{L^*_\mu \otimes L_\mu} \\ \boxed{L^*_\mu \otimes L_{s\cdot\mu}} \oplus \boxed{L^*_{s\cdot\mu} \otimes L_\mu} \\ \boxed{L^*_\mu \otimes L_\mu} \end{array},$$

with the top constituent $L^*_\mu \otimes L_\mu$ corresponding to the "skew-symmetric" part of the middle layer of $P^*_\mu \otimes P_\mu$, which contained two copies of $L^*_\mu \otimes L_\mu$.

Finally, there is only one singular antidominant weight $\mu = -1$, in which case the statement is obvious because $P_{-1} = M_{-1} = L_{-1}$, and therefore $J = \ker\Phi = 0$.

This concludes the analysis of the $\mathfrak{sl}(2,\mathbb{C})$ case, and the proof of the Proposition. □

In other words, we only need matrix elements of the big projective module P_λ to span the entire block $\mathbb{M}_\lambda$.

Corollary 5.2. *For any $\lambda \in -\mathbf{P}^{++}$ we have $\mathbb{M}_\lambda = \mathbb{M}(P_\lambda)$.*

Proof. From the definitions it follows that $\mathbb{M}(P_\lambda) \subset \mathbb{M}_\lambda$. On the other hand, (5.5) implies,

$$\mathrm{ch}\,\mathbb{M}(P_\lambda) = \sum_{w\in W^\lambda} \mathrm{ch}\left(M^*_{w\cdot\lambda} \otimes M_{w\cdot\lambda}\right),$$

or equivalently for any $x, y \in W^\lambda$ we have

$$[\mathbb{M}(P_\lambda) : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = [\bigoplus_{w\in W^\lambda} M^*_{w\cdot\lambda} \otimes M_{w\cdot\lambda} : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] =$$

$$= \sum_{w\in W^\lambda} [M^*_{w\cdot\lambda} : L^*_{x\cdot\lambda}][M_{w\cdot\lambda} : L_{y\cdot\lambda}] = \sum_{w\in W^\lambda} [M_{w\cdot\lambda} : L_{x\cdot\lambda}][P_{y\cdot\lambda} : M_{w\cdot\lambda}] = [P_{y\cdot\lambda} : L_{x\cdot\lambda}]. \quad (5.6)$$

Comparing it with (4.14), we conclude that $\mathbb{M}(P_\lambda) = \mathbb{M}_\lambda$. □

5.2. **The $\mathfrak{g} \oplus \mathfrak{g}$-module structure of $\mathfrak{R}(G_0)$ and the Peter-Weyl theorem.** Recall that the classical Peter-Weyl theorem asserts that for a compact Lie group $\bar{G}$ the matrix elements of finite-dimensional $\bar{G}$-modules produce an L^2-basis of the space $L^2(\bar{G})$. Its algebraic version can be formulated as follows:

Theorem 5.3. *Let G be a simple complex Lie group, and let $\mathfrak{R}(G)$ denote the algebra of regular functions on G. Then there are isomorphisms of $G \times G$-modules*

$$\mathfrak{R}(G) \cong \bigoplus_{\lambda\in\mathbf{P}^+} \mathbb{M}(L_\lambda) \cong \bigoplus_{\lambda\in\mathbf{P}^+} L^*_\lambda \otimes L_\lambda \quad (5.7)$$

The first isomorphism $\mathfrak{R}(G) \cong \bigoplus_{\lambda\in\mathbf{P}^+} \mathbb{M}(L_\lambda)$ reflects the spanning property, and the second isomorphism $\mathfrak{R}(G) \cong \bigoplus_{\lambda\in\mathbf{P}^+} L^*_\lambda \otimes L_\lambda$ corresponds to the linear independence of matrix elements functions, corresponding to fixed bases of L_λ.

We can now formulate the projective analogue of the Peter-Weyl theorem.

Theorem 5.4. *Let G be a simple complex group, and let G_0 be the big cell of G associated with the Gauss decomposition. Then there are $\mathfrak{g} \oplus \mathfrak{g}$-module isomorphisms*

$$\mathfrak{R}(G_0) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} \mathbb{M}(P_\lambda) \cong \bigoplus_{\lambda\in-\mathbf{P}^{++}} P^*_\lambda \otimes_{Z(\mathfrak{g})} P_\lambda. \quad (5.8)$$

Proof. The map $\mathcal{D} : \mathcal{U}(\mathfrak{g}) \to \mathrm{Diff}(G)$, induced by the identification of $\mathfrak{g}$ with left-invariant vector fields on G, yields a homomorphism of $\mathfrak{g} \oplus \mathfrak{g}$-modules,

$$\vartheta : \mathfrak{R}(G_0) \to \mathcal{U}(\mathfrak{g})^*, \qquad (\vartheta(\psi))(x) = (\mathcal{D}_x\psi)(e), \quad (5.9)$$

where e is the unit of the group G. We claim that ϑ is an injection. Indeed, if $\psi \in \ker\vartheta$, then the function ψ and all of its derivatives vanish at the unit of the group. Since ψ is regular, this means that ψ is identically zero, and thus $\ker\vartheta = 0$.

It is obvious from the polynomial realization that $\mathfrak{R}(G_0)$ is locally $\mathfrak{n}_- \otimes \mathfrak{n}_+$-nilpotent, and is $\mathfrak{h}$-diagonalizable with respect to both regular actions. The same argument as in Proposition 4.1 implies that for any $\lambda \in -\mathbf{P}^{++}$ the submodule $\mathfrak{R}(G_0)_\lambda$ is isomorphic to a $\mathfrak{g} \oplus \mathfrak{g}$-submodule of $\mathbb{M}_\lambda$. On the other hand, combining (2.14) and (4.14), we get

$$[\mathfrak{R}(G_0)_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] = [P_{x\cdot\lambda} : L_{y\cdot\lambda}] = [\mathbb{M}_\lambda : L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda}] \quad (5.10)$$

for every $x, y \in W^\lambda$, which means that $\vartheta(\mathfrak{R}(G_0)_\lambda) = \mathbb{M}_\lambda$. Taking the direct sum over λ and using Propositions 5.1 and 5.2, we get the desired statement. □

5.3. **Projective generators and their endomorphisms.** For any $\lambda \in -\mathbf{P}^{++}$, define

$$\mathcal{P}_\lambda = \bigoplus_{w \in W^\lambda} P_{w\cdot\lambda} \tag{5.11}$$

to be the projective generator of $\mathcal{O}_\lambda$, and let $\mathcal{A}_\lambda$ denote its endomorphism algebra,

$$\mathcal{A}_\lambda = \mathrm{End}_{\mathcal{O}_\lambda}(\mathcal{P}_\lambda) = \bigoplus_{x,y \in W^\lambda} \mathrm{Hom}_{\mathcal{O}}(P_{x\cdot\lambda}, P_{y\cdot\lambda}). \tag{5.12}$$

The fundamental role of the algebra $\mathcal{A}_\lambda$ in the study of the category $\mathcal{O}_\lambda$ is reflected by the equivalence of categories [BGG]:

$$\mathcal{O}_\lambda \;\cong\; \text{finite dimensional } \mathcal{A}_\lambda\text{-mod}. \tag{5.13}$$

Theorem 5.5. *For any $\lambda \in -\mathbf{P}^{++}$ there is an isomorphism of $\mathfrak{g}\oplus\mathfrak{g}$-modules*

$$\mathbb{M}_\lambda \cong \mathcal{P}_\lambda^* \otimes_{\mathcal{A}_\lambda} \mathcal{P}_\lambda. \tag{5.14}$$

Proof. It is clear that $\mathbb{M}_\lambda \cong (\mathcal{P}_\lambda^* \otimes \mathcal{P}_\lambda)/\ker\Phi$, because matrix elements of projectives in $\mathcal{O}_\lambda$ span $\mathbb{M}_\lambda$. We need to show that $\ker\Phi$ coincides with the $\mathfrak{g}\oplus\mathfrak{g}$-submodule $\mathcal{J}_\lambda \subset \mathcal{P}_\lambda^* \otimes \mathcal{P}_\lambda$,

$$\mathcal{J}_\lambda = \Big\langle \phi^*(v^*)\otimes v - v^*\otimes\phi(v) \Big\rangle, \qquad v\in\mathcal{P}_\lambda,\ v^*\in\mathcal{P}_\lambda^*,\ \phi\in\mathcal{A}_\lambda. \tag{5.15}$$

Using the fact that $\phi \in \mathcal{A}_\lambda$ are $\mathfrak{g}$-intertwining operators, we compute

$$\Phi_{\phi^*(v^*)\otimes v}(x) = \langle \phi^*(v^*), xv\rangle = \langle v^*, \phi(xv)\rangle = \langle v^*, x\,\phi(v)\rangle = \Phi_{v^*\otimes\phi(v)}(x) \tag{5.16}$$

for any $x \in \mathcal{U}(\mathfrak{g})$, which shows that $\mathcal{J}_\lambda \subset \ker\Phi$, and that $\mathbb{M}_\lambda$ is a quotient of $\mathcal{P}_\lambda^* \otimes_{\mathcal{A}_\lambda} \mathcal{P}_\lambda$. To complete the proof, we use Soergel's deformations of projective modules $P_{w\cdot\lambda}$ for $w \in W^\lambda$.

We use the same notion of generic ε as in the proof of Proposition 5.1. Then for any $x \in W$ the specialization $P_{x\cdot\lambda;\varepsilon}$ is a direct sum of the Verma modules,

$$P_{x\cdot\lambda;\varepsilon} \cong \bigoplus_{y\in W^\lambda} M_{y\cdot\lambda+\varepsilon}\otimes V_{x,y}, \tag{5.17}$$

where $V_{x,y}$ are the multiplicity spaces such that $\dim V_{x,y} = (P_{x\cdot\lambda} : M_{y\cdot\lambda}) = [M_{y\cdot\lambda} : L_{x\cdot\lambda}]$. It follows that $\mathcal{P}_{\lambda;\varepsilon}$ is isomorphic to the sum of Verma modules $M_{w\cdot\lambda+\varepsilon}$ with some multiplicities. For generic ε the modules $M_{w\cdot\lambda+\varepsilon}$ are irreducible, and the central characters corresponding to $w\cdot\lambda+\varepsilon$ are all distinct. Therefore, elements of $\mathcal{A}_{\lambda;\varepsilon} = \mathrm{End}(\mathcal{P}_{\lambda;\varepsilon})$ just reshuffle the isotypic components $M_{w\cdot\lambda+\varepsilon}$, and we have

$$\mathcal{P}_{\lambda;\varepsilon}^* \otimes_{\mathcal{A}_{\lambda;\varepsilon}} \mathcal{P}_{\lambda;\varepsilon} \cong \bigoplus_{w\in W} M_{w\cdot\lambda+\varepsilon}^* \otimes M_{w\cdot\lambda+\varepsilon}. \tag{5.18}$$

Thus, generically we have $\ker\Phi_\varepsilon = \mathcal{J}_\varepsilon$. As in the proof of Proposition 5.1, it suffices to check that no discrepancies occur when ε is subgeneric, which reduces the problem to the rank one verification. The rest of the proof establishes the required statement for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.

As in the proof of Proposition 5.1, let $\mu \in \{-2,-3,\dots\}$ be a regular antidominant weight for $\mathfrak{g}$. The two indecomposable projective modules in $\mathcal{O}_\mu$ are the big projective module P_μ and the Verma module $M_{s\cdot\mu}$. The algebra $\mathcal{A}_\mu$ has dimension 5. It has two obvious idempotents $\mathbf{1}_\mu$ and $\mathbf{1}_{s\cdot\mu}$, and contains the inclusion $\iota : M_{s\cdot\mu} \to P_\mu$ and the map $\tau : P_\mu \to M_{s\cdot\mu}$, which factors through L_μ; we set for convenience $\iota(P_\mu) = \tau(M_{s\cdot\mu}) = 0$. The remaining element $Q \in \mathrm{End}_{\mathcal{O}}(P_\mu)$ is then equal to the composition $Q = \tau\circ\iota$.

Using the idempotents $\mathbf{1}_\mu, \mathbf{1}_{s\cdot\mu}$, we immediately see that

$$M^*_{s\cdot\mu} \otimes_{\mathcal{A}_\mu} P_\mu = P^*_\mu \otimes_{\mathcal{A}_\mu} M_{s\cdot\mu} = 0. \tag{5.19}$$

Let now $v^* \otimes v \in M^*_{s\cdot\mu} \otimes M_{s\cdot\mu}$. We can find $u^* \in P^*_\mu$ such that $v^* = \iota^*(u^*)$, and therefore

$$v^* \otimes v = \iota^*(u^*) \otimes v \equiv u^* \otimes \iota(v) \in P^*_\mu \otimes P_\mu. \tag{5.20}$$

Thus, any element in $\mathcal{P}^*_\mu \otimes_{\mathcal{A}_\mu} \mathcal{P}_\mu$ can be represented by an element from $P^*_\mu \otimes P_\mu$, and Proposition 5.1 implies

$$\mathcal{P}^*_\mu \otimes_{\mathcal{A}_\mu} \mathcal{P}_\mu \cong P^*_\mu \otimes_{\mathcal{C}_\mu} P_\mu \cong \mathbb{M}_\mu. \tag{5.21}$$

This concludes the analysis of the $\mathfrak{sl}(2,\mathbb{C})$ case, and the proof of the Proposition. $\square$

The algebra $\mathcal{A}_\lambda$ is a graded algebra, and is the Koszul dual of the algebra $\mathrm{Ext}^\bullet(\mathcal{L}_\lambda, \mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda = \bigoplus_{w\in W^\lambda} L_{w\cdot\lambda}$, see [So, BGS]. The algebra $\mathcal{A}^\bullet_\lambda$ is generated over $\mathcal{A}^0_\lambda$ by elements of $\mathcal{A}^1_\lambda$ with relations of degree 2. It is easy to see that the degree 1 generators can be represented by elements of $\mathrm{Hom}_{\mathcal{O}}(P_{x\cdot\lambda}, P_{y\cdot\lambda})$, such that the simple top of $P_{x\cdot\lambda}$ is sent into $\overline{\mathrm{rad}}^2 P_{y\cdot\lambda}$. More generally, elements of degree k can be represented by morphisms sending the top of $P_{x\cdot\lambda}$ to the layer $\overline{\mathrm{rad}}^{k+1} P_{y\cdot\lambda}$. Thus, we can refine Corollary 4.4 as follows.

Corollary 5.6. *Let $\lambda \in -\mathbf{P}^{++}$. Then the graded $\mathfrak{g}\oplus\mathfrak{g}$-module $\mathrm{Gr}^\bullet\,\mathbb{M}_\lambda$, associated with the filtration* (4.11)*, is isomorphic to*

$$\mathrm{Gr}^\bullet\,\mathbb{M}_\lambda \cong \bigoplus_{x,y\in W^\lambda} L^*_{x\cdot\lambda} \otimes L_{y\cdot\lambda} \otimes (\mathcal{A}^\bullet_\lambda)_{x,y}\,, \tag{5.22}$$

where $(\mathcal{A}^\bullet_\lambda)_{x,y} = \mathrm{Hom}^\bullet_{\mathcal{O}_\lambda}(P_{x\cdot\lambda}, P_{y\cdot\lambda})$ are the multiplicity spaces, trivial as $\mathfrak{g}\oplus\mathfrak{g}$-modules.

Thus, $\mathbb{M}_\lambda$ is in a sense a "categorification" of the algebra $\mathcal{A}_\lambda$. It would be interesting to interpret our results in the general context of mixed geometry [BGS].

6. Generalizations to quantum groups.

6.1. **Quantum groups with generic q.** The problem of constructing a q-deformation of the regular representation $\mathfrak{R}(G)$ was one of the main motivations for the development of the theory of quantum groups. The quantum group $\mathcal{U}_q(\mathfrak{g})$ is a q-deformation of the Hopf algebra $\mathcal{U}(\mathfrak{g})$, generated by $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm1}$, subject to the standard relations, see e.g. [L2]. Using the analogy with the Lie algebraic realization of $\mathfrak{R}(G)$, we can define the quantum coordinate algebra $\mathfrak{R}_q(G)$ as the Hopf dual $\mathcal{U}_q(\mathfrak{g})^*_{Hopf}$ of the quantum group, cf. [L1, APW]. The associative algebra $\mathfrak{R}_q(G)$ is equipped with two commuting quantum regular actions ϱ_1, ϱ_2.

When $q \in \mathbb{C}^\times$ is generic (or is regarded as a formal variable), the representation theory of $\mathcal{U}_q(\mathfrak{g})$ is parallel to the classical case. In particular, one has the q-analogues of the BGG category $\mathcal{O}_q$, which contains the quantum counterparts $L_{\lambda,q}$, $M_{\lambda,q}$, $P_{\lambda,q}$ of the simple, Verma and projective modules. The category $\mathcal{O}_q$ decomposes into direct sum of blocks $\mathcal{O}_{\lambda,q}$, according to the characters of the center $Z_q(\mathfrak{g})$ of the quantum group.

We have the quantum version of the Peter-Weyl theorem, which asserts that

$$\mathfrak{R}_q(G) = \bigoplus_{\lambda\in\mathbf{P}^+} \mathbb{M}(L_{\lambda,q}) \cong \bigoplus_{\lambda\in\mathbf{P}^+} L^*_{\lambda,q} \otimes L_{\lambda,q}, \tag{6.1}$$

and it is also clear from this decomposition that the space of left $\mathcal{U}_q(\mathfrak{n}_-)$-invariant elements gives a model for finite-dimensional simple modules $L_{\lambda,q}$, yielding the quantum version of the Borel-Weil realization.

The regular representation $\mathfrak{R}(G_0)$, studied in this paper, also has a quantum analogue. Again, we define it by analogy with the Lie algebraic realization as a suitable subspace of $\mathcal{U}_q(\mathfrak{g})^*$, so that we have $\mathfrak{R}_q(G_0) = \mathcal{U}_q(\mathfrak{n}_-)^* \otimes \mathbb{C}[\mathbf{P}] \otimes \mathcal{U}_q(\mathfrak{n}_+)^*$. The space $\mathfrak{R}_q(G_0)$ is an associative, almost commutative algebra, and satisfies the quantum version of the Peter-Weyl theorem for the big projective modules.

Theorem 6.1. *The algebra $\mathfrak{R}_q(G_0)$ has a decomposition*

$$\mathfrak{R}_q(G_0) = \bigoplus_{\lambda \in -\mathbf{P}^{++}} \mathbb{M}(P_{\lambda,q}) \cong \bigoplus_{\lambda \in -\mathbf{P}^{++}} P^*_{\lambda,q} \otimes_{Z_q(\mathfrak{g})} P_{\lambda,q}. \tag{6.2}$$

The principal ingredient in the Borel-Weil realization of the big projective modules was the notion of the Whittaker vectors in $\mathfrak{R}(G_0)$. The obstacle to immediate generalizations to the quantum case is the absence of nonsingular characters $\boldsymbol{\eta}_q^+ : \mathcal{U}_q(\mathfrak{n}^+) \to \mathbb{C}$. Two possible approaches to quantum analogues of Whittaker functions were suggested in [E, Se], and were used in the study of the deformed quantum Toda system.

We take yet another approach, based on the equivalence between Whittaker and Soergel functors, which can also be generalized to the case when q is a root of unity. The analogue of Corollary 3.6 is given by

Theorem 6.2. *For any $\lambda \in -\mathbf{P}^{++}$, we have the isomorphisms*

$$\mathrm{Hom}_{\varrho_1}(P^*_{\lambda,q}, \mathfrak{R}_q(G_0)) \cong P_{\lambda,q}, \qquad \mathrm{Hom}_{\varrho_2}(P_{\lambda,q}, \mathfrak{R}_q(G_0)) \cong P^*_{\lambda,q}. \tag{6.3}$$

The proofs of these quantum theorems are analogous to their Lie algebra counterparts.

6.2. **Quantum groups at roots of unity.** For simplicity, we will assume now that $\mathfrak{g}$ is simply laced. Let $\ell \geq 3$ be an odd integer, and let q be a primitive ℓ-th root of unity. Then $\mathcal{U}_q(\mathfrak{g})$ contains a central ideal $\mathcal{I}$, generated by E_i^ℓ, F_i^ℓ and $(K_i^\ell - 1)$. The corresponding quotient $\mathfrak{U} = \mathcal{U}_q(\mathfrak{g})/\mathcal{I}$ is a finite-dimensional Hopf algebra [L1].

We consider the non-semisimple category $\mathfrak{O}$ of finite-dimensional $\mathfrak{U}$-modules. The simple modules $L_{\lambda;\ell}$ and their unique indecomposable projective covers $P_{\lambda;\ell}$ are parameterized by weights λ from the restricted weight lattice $\mathbf{P}_\ell = \mathbf{P}/(\ell\,\mathbf{P})$, conveniently realized as a set by

$$\mathbf{P}_\ell \cong \left\{ \lambda \in \mathbf{P}^+ \,\middle|\, \langle \lambda, \alpha_i \rangle < \ell \text{ for all } \alpha_i \in \Pi \right\}. \tag{6.4}$$

The Weyl group action on $\mathbf{P}$ induces an action of W on $\mathbf{P}_\ell$, so that

$$w \circ \lambda = w \cdot \lambda \mod \ell\,\mathbf{P}, \qquad w \in W. \tag{6.5}$$

For $\lambda \in \mathbf{P}_\ell$, we define its stabilizer $W_{\lambda;\ell}$ and the coset space $W^{\lambda;\ell}$ as in the Lie algebra case.

The linking principle implies that $\mathrm{Ext}^\bullet(L_{\lambda;\ell}, L_{\mu;\ell}) = 0$ unless $\mu = w \circ \lambda$ for some $w \in W$. Therefore, the category $\mathfrak{O}$ decomposes into a direct sum of blocks, indexed by a set $\mathbf{X}_\ell$ of representatives of the Weyl group orbits in $\mathbf{P}_\ell$.

The dual space $\mathfrak{U}^*$ is an associative algebra, and consists of the functionals from $\mathcal{U}_q(\mathfrak{g})^*$, vanishing on the ideal $\mathcal{I}$. It carries two commuting regular $\mathfrak{U}$-actions ϱ_1 and ϱ_2, and as a $\mathfrak{U} \times \mathfrak{U}$-module decomposes into blocks

$$\mathfrak{U}^* \cong \bigoplus_{\lambda \in \mathbf{X}_\ell} \mathfrak{U}^*_\lambda. \tag{6.6}$$

As in the Lie algebra case, the space $\mathfrak{U}^*_\lambda$ is spanned by the matrix elements of finite-dimensional $\mathfrak{U}^*$-modules. However, a block of $\mathfrak{U}^*$ cannot be generated by a single indecomposable projective, and to give an analogue of the Peter-Weyl theorem we must use the more general approach, provided by Theorem 5.5. We have

Theorem 6.3. *Let $\lambda \in \mathbf{X}_\ell$, and let $\mathcal{P}_{\lambda;\ell} = \bigoplus_{w \in W^{\lambda;\ell}} P_{w\circ\lambda;\ell}$ be the projective generator of the block $\mathfrak{U}^*_\lambda$. Then $\mathfrak{U}^*_\lambda$ is spanned by matrix elements of $\mathcal{P}_{\lambda;\ell}$, and we have*

$$\mathfrak{U}^*_\lambda = \mathcal{P}^*_{\lambda;\ell} \otimes_{\mathcal{A}_{\lambda;\ell}} \mathcal{P}_{\lambda;\ell}. \tag{6.7}$$

Proof. As in the proof of Proposition 4.1, every functional every functional from $\mathfrak{U}^*$ can be interpreted as a matrix element of a finite-dimensional $\mathfrak{U}$-module, and since $\mathfrak{O}$ has enough projectives [APW], it suffices to consider matrix elements of $\mathcal{P}_{\lambda;\ell}$.

For $\lambda \in \mathbf{P}_\ell$ the projective $\mathfrak{U}$-module, remains projective when regarded as a $\mathcal{U}(\mathfrak{g})$-module. We consider the q-version of Soergel's deformation $P_{\lambda+\varepsilon;\ell}$ of the projective $\mathcal{U}_q(\mathfrak{g})$-modules in $\mathfrak{O}$, and corresponding matrix elements. Generically, the deformed projectives split into a direct sum of simple modules, and the statement is clear. To check that the spaces still coincide when λ becomes integral, it suffices again to consider subgeneric ε, which reduces the problem to the rank one case.

Thus, we assume that $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. The regular orbits of the affine Weyl group are $\{\mu, \mu'\}$, where $\mu \in \mathbf{X}_\ell = \{0, 1, \ldots, \frac{\ell-3}{2}\}$ and $\mu' = \ell - \mu - 2$. The corresponding projective modules are depicted by

$$P_{\mu;\ell} \sim \begin{array}{ccc} & \boxed{L_{\mu;\ell}} & \\ \boxed{L_{\mu';\ell}} & \oplus & \boxed{L_{\mu';\ell}} \\ & \boxed{L_{\mu;\ell}} & \end{array}, \qquad P_{\mu';\ell} \sim \begin{array}{ccc} & \boxed{L_{\mu';\ell}} & \\ \boxed{L_{\mu;\ell}} & \oplus & \boxed{L_{\mu;\ell}} \\ & \boxed{L_{\mu';\ell}} & \end{array}.$$

Thus, $\mathcal{P}_{\mu;\ell} = P_{\mu;\ell} \oplus P_{\mu';\ell}$, and it is easy to describe the graded algebra $\mathcal{A}_{\mu;\ell} = \mathrm{End}_{\mathfrak{U}}(\mathcal{P}_{\mu;\ell})$ of dimension eight. Indeed, the space $\mathrm{End}(P_{\mu;\ell})$ contains the identity map of degree 0, and the nontrivial endomorphism of degree 2, factoring through L_μ . The space $\mathrm{Hom}(P_{\mu;\ell}, P_{\mu';\ell})$ has two generators of degree 1, arising from the "baby Verma" flags of the projective modules. Reversing the role of μ and μ', we get the remaining four generators of $\mathcal{A}_{\mu;\ell}$.

The verification of the fact that the analogue of (5.15) coincides with the kernel of the matrix elements map is a straightforward exercise, which we leave to the reader. □

The obvious modification of Corollary 5.6 describes the structure of the regular block $\mathfrak{U}^*_\mu$. For $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ it is given by

$$\mathfrak{U}^*_\mu \sim \begin{array}{ccccccc} & & \boxed{L^*_{\mu';\ell} \otimes L_{\mu';\ell}} & \oplus & \boxed{L^*_{\mu;\ell} \otimes L_{\mu;\ell}} & & \\ \boxed{L^*_{\mu';\ell} \otimes L_{\mu;\ell}} & \oplus & \boxed{L^*_{\mu;\ell} \otimes L_{\mu';\ell}} & \oplus & \boxed{L^*_{\mu';\ell} \otimes L_{\mu;\ell}} & \oplus & \boxed{L^*_{\mu;\ell} \otimes L_{\mu';\ell}} \\ & & \boxed{L^*_{\mu;\ell} \otimes L_{\mu;\ell}} & \oplus & \boxed{L^*_{\mu';\ell} \otimes L_{\mu';\ell}} & & \end{array}, \tag{6.8}$$

and was originally obtained in [AGL] by explicit computations. A representation-theoretic derivation of (6.8) using matrix elements of projective modules was done by A. Lyakhovskaya (A. Lachowska) and the author.

The analogue of Corollary 3.6 and Theorem 6.2 holds for $\mathfrak{U}^*$, and can also be proved using the self-duality of $\mathfrak{U}^*$.

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