Braiding of Lie algebra $sl(2)$

J. Donin *
D. Gurevich **

*
Department of Mathematics
Bar-Ilan University
52900 Ramat-Gan

Israel

**
Fac. de Math.
Université des Sc. et Tech. de Lille
59655 Villeneuve d'ASCQ Cedex

France

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Abstract

We construct a (flat) braided deformation of the enveloping algebra $U(sl(2))$. This means that the deformed algebra lies in the category of $U_q(sl(2))$-modules. We consider the space generating this deformed enveloping algebra as a braided version of the Lie algebra $sl(2)$. Quantum counterpart of Feigin algebras $gl(\lambda)$ are defined and a braided pairing and involution are introduced there. Poisson brackets generating this deformation are investigated.

0 Introduction

It is not a big exaggeration to say that the most popular object connected with the quantum Yang-Baxter equation (QYBE) is the so called quantum group $U_q(\mathfrak{g})$. It is well-known that this object has a Hopf algebra structure which is a deformation of the usual one of $U(\mathfrak{g})$. Nevetheless there exists another type of deformation arising from the YBE, namely *braiding*. Let us give some examples of braided or-in more general context- *twisted* objects.[1]

In [G1] one of the authors has introduced a notion of generalized Lie algebras (called in some papers S-Lie algebras) assuming S to be an involutive ($S^2 = \mathrm{id}$) solution of the QYBE. The enveloping algebra of an S-Lie algebra is a twisted object i.e. it has a twisted Hopf structure. It means that the compatibility of the multiplication μ and the comultiplication Δ can be expressed by means of the standard relation

$$\Delta\mu(a \otimes b) = \mu(\Delta a \otimes \Delta b)$$

[1]According to a tradition we use the term *braided* to denote the objects connected with a non-involutive (quasi-triangular) solution of the QYBE. The term *S-Lie algebras* is used only for involutive S and the term *twisted* is used in the both cases.

but the multiplication in r.h.s. of this relation is defined via the operator S (like in the definition of a super-Hopf algebras).

Remark that as far as we know the twisted Hopf algebras (assuming S to be involutive) were first introduced by S.MacLane.

The dual object of the enveloping algebra mentioned above can be considered as a twisted analogue of a formal (co)group. Global group-like twisted objects of GL and SL type have been considered in the paper [G3].

Sh.Majid [M1] has introduced a notion of a braided group and discovered a process of transmutation converting the quantum group $U_q(\mathfrak{g})$ into a braided group. The braided groups constructed by Sh.Majid have a braided Hopf structure.

Note that there exist solutions of the QYBE which can not be obtained by means of deformation from the usual permutation $S=\sigma$ $(\sigma(x\otimes y) = y\otimes x)$, for example a super-permutation. A new classes of non-deformational solutions of the QYBE have been constructed in [G2], [G3] (some of them have been independently discovered by M.Dubois-Violette and G.Launer [DL]).

All above objects represent examples of twisted (in particular, braided) structures.

However in the present paper we use the term *braided* in a slightly different sense assuming a *braided* object to lie in the category $U_q(\mathfrak{g}) - Mod$ of $U_q(\mathfrak{g})$-modules. If this object is an algebra A this means that the multiplication $\mu : A^{\otimes 2} \rightarrow A$ satisfies the condition

$$X\mu = \mu\Delta(X), X \in U_q(\mathfrak{g}) \tag{1}$$

where Δ is the coproduct in $U_q(\mathfrak{g})$ i.e. $U_q(\mathfrak{g})$ plays the role of the group of symmetries of the algebra A.

Remark that compared to $U_q(\mathfrak{g})$, that is defined by means of some relations including analytic ones, the braided counterpart of the algebra $U(sl(2))$ considered in the paper is the so called *quadratic-linear* algebra.

Since all such objects of this category have a deformational nature it is reasonable to put a question about the *flatness* of that *braiding* deformation. Roughly speaking the *flatness* means that the "quantity" of elements of the deformed object is stable under deformation. In particular, we call a space $V \in U_q(\mathfrak{g}) - Mod$ equipped with a bracket $[,] : V^{\otimes 2} \rightarrow V$ a braided Lie algebra if the bracket $[,]$ is a morhpism in the category $U_q(\mathfrak{g}) - Mod$ and if its enveloping algebra is a flat deformation of the initial object.

Emphasize that the problem of definition of a braided counterpart of Lie algebras (compared to S-Lie algebras) has been a subject of numerous papers. The main difficulty is to give a proper "braided" version of the Jacobi identity. In some paper they reproduce the form given in [G1] for the involutive S. However the simplest example considered in the present paper shows that this definition is not reasonable.

We give another version of the Jacobi axiom ensuring the flatness of the deformation of the enveloping algebra (this version is a specialization of the Jacobi identity from [PP] for the braided case)[2].

Note that deforming the enveloping algebra $U(\mathfrak{g})$ we deforme simultaneously its graded adjoint algebra, i.e. the commutative algebra of polynomials on $\mathfrak{g}^*$. Therefore it is natural to consider a quasiclassical limit of this deformation. It is a so called R-matrix Poisson bracket., i.e. the Poisson brackets generated by an R-matrix.

In the case under consideration the corresponding R-matrix is of the form $R = \frac{1}{2} X \wedge Y$. It is a particular case of the following (modified) R-matrix defined for any simple Lie algebra $\mathfrak{g}$

$$R = \frac{1}{2} \sum_{\alpha \in \Omega_+} X_\alpha \wedge X_{-\alpha} \in \wedge^2 \mathfrak{g}, \tag{2}$$

where $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ is the Cartan–Weyl basis in the Chevalley normalization and Ω_+ is the set of positive roots of $\mathfrak{g}$.

There exists a natural way to assign to any R-matrix a bracket $f \otimes g \to \{f, g\}_R$ defined on any homogeneous space M and in particular on any orbit in $\mathfrak{g}^*$. However the bracket defined by means of the R-matrix (2) satisfies the Jacobi identity and therefore it is Poisson one only under some conditions on M(we call such homogeneous space to be of an R-matrix type).

All R-matrix type orbits in $\mathfrak{g}^*$ for any simple Lie algebra $\mathfrak{g}$ over the field $k = \mathbf{C}$ have been classified (over the field $k = \mathbf{C}$) in [GP]. Though the problem of a quantization of all R-matrix brackets on R-matrix type orbits is open it seems very plausible that the result of quantization can be represented as a braided deformation of the quotient-algebras of $U(\mathfrak{g})$ corresponding to these orbits.

[2]When the paper was almost completed we have received a preprint [M2] where the author gives a definition of a braided Lie algebra based on the notion of braided groups but does not investigate the flatness of the braiding deformation. Remark that the braided group is not any flat deformation of the initial group structure.

From this point of view the algebra Lie $sl(2)$ has an exceptional position since the R-matrix bracket is Poisson on the whole $sl(2)^*$ (i.e. all orbits in $sl(2)^*$ are of the R-matrix type). Therefore only for this algebra a braided flat deformation of the enveloping algebra can exist. We construct this deformation in Section 3.

In Section 4 we consider a braided counterpart of the algebra $gl(\lambda)$ introduced by B.Feigin. These algebras have been defined in [Fei] as some quotient-algebras of $U(sl(2))$. We introduce a braided pairing in the deformed algebras that is a morphism in the category of $U_q(sl(2))$-modules.

Emphasize that a deformation of the pairing arising from the classical R-matrices has been studied in [GRZ]. There some enlarged scheme of quantum mechanics has been suggested. The present paper can be considered as a first step aiming to include the deformational algebras arising from quantization of some modified R-matrices in this scheme.

1 Family of Poisson brackets

First of all we recall the definition of a flat deformation.

Let A_h be an associative algebra over the field $k[[h]]$ where k is the ground field $k = \mathbf{R}$ or $k = \mathbf{C}$ and h is a formal parametre. This algebra is called a flat deformation of the algebra $A = A_0$ if A_h and $A[[h]]$ are isomorphic to each other as $k[[h]]$-modules and the multiplication μ_h in A_h is equal to

$$\mu_h = \mu_0 + h\mu_1 + h^2\mu_2 + \dots$$

with some maps $\mu_i : A^{\otimes 2} \to A$ and where $\mu_0 = \mu$ is the multiplication in the initial algebra A (extended to $A[[h]]$). In a similar way a deformation of a coassociative structure can be defined.

It is well-known that if A_h is a flat deformation of the algebra A then μ_1 is a Hochschild cocycle on A. Assuming A to be commutative we get more information about μ_1. Namely the antisymmetric part of μ_1 denoted usually by

$$\{a, b\} = \mu_1(a, b) - \mu_1(b, a)$$

and called a Poisson bracket satisfies the Leibnitz and Jacobi identities .

Any Poisson bracket defined in the algebra of (smooth) functions on a (smooth) manifold M can be described by means of some skew bi-vector field on M. If such a Poisson bracket $\{,\}$ is given, then a procedure of constructing an algebra A_h with properties described above is called a deformational quantization.

If a Poisson bracket is nowhere degenerated (this means that the manifold has a symplectic structure) then a deformational quantization always exists (cf., for example, [Fed] and [DWL]). As far as we know there does not exist any example of a non-quantizable (degenerated) bracket.

Let $\mathfrak{g}$ be a Lie algebra with structure constants c_{ij}^k in a fixed base $\{X_i\}$ i.e. $[X_i, X_j] = c_{ij}^k X_k$. Consider the linear Poisson-Lie[3] bracket $\{,\}_{PL}$ defined in the space $Pol(\mathfrak{g}^*)$ of polynomials on $M = \mathfrak{g}^*$ as follows

$$\{f,g\}(\xi) =< [df, dg], \xi >, \xi \in \mathfrak{g}^*.$$

Then the enveloping algebra $U(\mathfrak{g})$ with the parameter h introduced in structure constants ($c_{ij}^k \rightarrow hc_{ij}^k$) is a quantization of the Poisson-Lie bracket (in what follows we use notation $U^h(\mathfrak{g})$ for $U(\mathfrak{g})$ equipped with the parameter h as above). That follows from the PBW theorem.

The restriction of the Poisson-Lie bracket to any of its symplectic leaves defines the so called Kirillov-Kostant-Souriau bracket (we denote it $\{,\}_{KKS}$).

Let $R = r^{ij} X_i \otimes X_j \in \wedge^{\otimes 2}\mathfrak{g}$ be a classical or a modified R-matrix. Let us consider a new bracket

$$\{f,g\}_R = r^{ij}\{x_i, f\}\{x_j, g\} \tag{3}$$

where $x =< X, \xi >, \xi \in \mathfrak{g}^*$ (we call this bracket the R-matrix one).

It is obvious that the bracket (3) can be restricted to any symplectic leaf of the Poisson-Lie bracket (the orbits of the corresponding group G). It is also clear that the bracket (3) is a Poisson if R is a classical R-matrix. If R is a modified R-matrix the bracket (3) is a Poisson only being restricted to a some orbits in $\mathfrak{g}^*$ (we call them the R-matrix type ones, cf. Introduction).

We reproduce (partially) the result from [GP] where all R-matrix type orbits have been classified (over the field $k = \mathbf{C}$).

Proposition 1.1 *Let $\mathcal{O}$ be a non-zero orbit in $\mathfrak{g}^*$ of an element $x \in \mathfrak{g}^*$.*
1. Suppose the orbit $\mathcal{O}$ to be of R-matrix type. Then x is either semisimple or nilpotent.

[3]Nowadays this name is more often used to denote some quadratic Poisson bracket defined on the corresponding Lie group. We prefer to call it the Sklyanin bracket.

2. If $\mathcal{O}$ is semisimple, then $\mathcal{O}$ is of R-matrix type iff $\mathcal{O}$ is a symmetric space (it is true for $k = \mathbf{R}$ as well).

We do not reproduce any description of R-matrix type nilpotent orbits since we do not need it. Let us only remark that the orbit of the heighest weight vector is an orbit of such type (cf. also [DG1], [DGM], [DG2])

As it follows from this Proposition for any Lie algebra $\mathfrak{g} \neq sl(2)$ there exist orbits which are not of R-matrix type, for example non-symmetric orbits of semisimple elements.

Thus the Lie algebra $\mathfrak{g} = sl(2)$ is only simple Lie algebra such that all orbits in $\mathfrak{g}^*$ are of R-matrix type and therefore the bracket (3) is Poisson one on the whole $\mathfrak{g}^*$.

It is easy to see that R-matrix bracket (assuming R to be a classical R-matrix) and Poisson-Lie bracket are compatible. The brackets $\{,\}_{KKS}$ and $\{,\}_R$ have the same property for any modified R-matrix and any R-matrix type orbit. This means that all brackets of the family

$$\{,\}_{a,b} = \{,\}_{KKS} + \{,\}_R \tag{4}$$

are Poisson.

Note that compatibility of the bracket $\{,\}_{KKS}$ and reduced Sklyanin bracket has been investigated in [KRR] for the orbits equipped with an hermitien structure.

2 Quantization and quadratic-linear algebras

There exists a problem of great interest: to quantize the whole family (4) simultaneously. The scheme of simultaneous quantization assuming R to be a non-modified classical R-matrix was suggested in [GRZ]. If R is a modified R-matrix (2) then the quantization is done only for some brackets from the family (4) (cf. [DG1], [DGM], [DG2]).

Note that the brackets (4) are in general degenerated and therefore they can not be quantized by methods of the papers [Fed] and [DWL].

Let us reproduce a scheme of quantization from [GRZ] in a short form assuming R to be a classical R-matrix. Consider the element

$$F_\nu \in U(\mathfrak{g})^{\otimes 2}[[\nu]]$$

constructed by Drinfeld [D1] and satisfying the following relations

$$F_\nu = 1 \bmod \nu,\ F_\nu - \sigma F_\nu = R \bmod \nu^2,\ (\varepsilon \otimes \mathrm{id})F_\nu = (\mathrm{id} \otimes \varepsilon)F_\nu = 1,$$

$$\Delta^{12} F_\nu F_\nu{}^{12} = \Delta^{23} F_\nu F_\nu{}^{23}$$

where σ is the usual permutation. Deform now the initial multiplication

$$\mu_h : U^h(\mathfrak{g})^{\otimes 2} \to U^h(\mathfrak{g})$$

as follows

$$\mu_{h,\nu} = \mu_h(\rho \otimes \rho)F_\nu$$

where ρ is the extension of the representation $\rho_X(Y) = [X, Y]$ to $U^h(\mathfrak{g})$ and $(\rho \otimes \rho)F_\nu$ is the corresponding map from $U^h(\mathfrak{g})^{\otimes 2}[[\nu]]$ to itself.

The algebra $U^h(\mathfrak{g})$ equipped with the multiplication $\mu_{h,\nu}$ will be denoted $U^{h,q}$ (we put $q = e^\nu$). It is a two-parameter quantization of the family (4). More precisely this algebra equipped with the multiplication $\mu_{h,\nu}$ where $h = ah_1, \nu = bh_1$ is a quantization of the bracket $\{,\}_{a,b}$.

Unfortunately we can not use similar methods to quantize the family (4) when R is the R-matrix (2) since there do not exist elements F_ν with the above properties. (the last property holds in a weaker form). Nevetheless we will quantize the family (4) in the frames of theory of quadratic and quadratic-linear algebras.

We will reproduce some aspects of this theory basing mainly on the paper [PP].

Let V be a fixed linear space and I be a subspace in $V^{\otimes 2}$. Then the algebra $\wedge_+(V, I) = T(V)/\{I\}$ where $\{I\}$ is the ideal in $T(V)$ generated by elements belonging to I is called the *quadratic algebra* corresponding to I. Let $\wedge_+^l(V, I)$ denotes its homogenous component of degree l and introduce the linear spaces

$$\wedge_-^0 = k,\ \wedge_-^1 = V,\ \wedge_-^l(V, I) = \wedge_-^{l-1}(V, I) \otimes V \cap V^{\otimes(l-2)} \otimes I, l = 2; 3; \ldots$$

Note that in general case the direct sum $\oplus \wedge_-^l(V, I)$ does not have any associtive structure. The quadratic algebra is called a *Koszul algebra* if the complex

$$\ldots \xrightarrow{d} \wedge_+^k(V, I) \otimes \wedge_-^l(V, I) \xrightarrow{d} \wedge_+^{k+1}(V, I) \otimes \wedge_-^{l-1}(V, I) \xrightarrow{d} \ldots$$

is exact, here d is a natural differential:

$$d(a_1 \otimes \ldots \otimes a_k) \otimes (b_1 \otimes \ldots \otimes b_l) = (a_1 \otimes \ldots \otimes a_k \otimes b_1) \otimes (b_2 \otimes \ldots \otimes b_l).$$

The following statement is well-known.

Proposition 2.1 *If $\wedge_+(V, I)$ is a Koszul algebra then the usual relation*

$$\mathcal{P}_-(t)\mathcal{P}_-(-t) = 1$$

for the Poincaré series $\mathcal{P}_\pm(t) = \sum \dim \wedge_\pm^k (V, I)t^k$ holds.

Given a map $[,] : I \to V$ define *a quadratic-linear* algebra (an analogue of the enveloping algebra) in a natural way: $U(\mathfrak{g}) = T(V)/\{J\}$ where $\{J\} \subset T(V)$ is the ideal generated by elements $I - [,]I$. Since there exists a natural filtration in this algebra we consider the graded algebra $GrU(\mathfrak{g})$.

The following proposition proved in [PP] is a very useful generalization of the PBW theorem.

Proposition 2.2 *Let us assume that the algebra $\wedge_+(V, I)$ is a Koszul algebra and that the following conditions*

$$([,]^{12} - [,]^{23}) \wedge_-^3 (V, I) \subset I$$

and

$$[,]([,]^{12} - [,]^{23}) \wedge_-^3 (V, I) = 0$$

are fulfilled. Then $GrU(\mathfrak{g})$ and $\wedge_+(V, I)$ are isomorphic as graded algebras.

In the next Section we will use this proposition to construct a braiding of $sl(2)$.

3 Braided Lie algebras: $sl(2)$ case

First let us give a definition of a braided Lie algebra.

Definition 3.1 *Let $U_q(\mathfrak{g})$ be a quantum group and let V be a finite-dimensional object of the category $U(sl(2)) - Mod$ of all $U(sl(2))$-modules. Assume that $V^{\otimes 2}$ can be decomposed into a direct sum of two subspaces $V^{\otimes 2} = I \oplus \overline{I}$. We say that the space V is equipped with the structure of a braided Lie algebra*

if there exists an operator $[,]: V^{\otimes 2} \to V$ *satisfying the axioms*
0. the algebra $\wedge_+(V, I)$ *is a Koszul algebra;*
1.$[,]\overline{I} = 0$*;*
2. The relations of Proposition 2.2 are satisfied;
3. The spaces $I, \overline{I}$ *are objects of the category* $U(sl(2)) - Mod$ *and the operator* $[,]$ *is a morphism in this category.*

Remark that this definition can be extended up to constant-linear brackets $[,]: V^{\otimes 2} \to V \oplus k$. It is possible to make this bracket purely linear introducing a new central generator s and assuming $[,]V^{\otimes 2} \to V \oplus ks$ (we let $Xs = 0$ for any $X \in U_q(\mathfrak{g})$).

We will compare the definition above with the one from [G4]. Let S be a solution of the quantum Yang-Baxter equation (QYBE). Let $\{e_i\}$ be a fixed base in V and $\{e^i\}$ be the dual base in V^*. Consider the linear space $L = V \otimes V^*$ with base $\{e_i^j = e_i \otimes e^j\}$ equipped with the operator

$$S_Q : L^{\otimes 2} \to L^{\otimes 2},\ S_Q(e_i^j \otimes e_k^l) = S_{ik}^{ab}\,(S^{-1})_{mn}^{jl}\, e_a^m \otimes e_b^n.$$

Consider also the algebra A_{RTF} of Reshetikhin-Takhtadzhyan-Faddeev type $A(S) = T(L)/\{J\}$ where $\{J\}$ is the ideal generated by the image of the operator $\mathrm{id} - S_Q$. In some cases this algebra with an additional condition on the quantum determinant (assuming it to be well-defined) has a Hopf structure.

In [RTF] it is shown that if S is a solution of the QYBE corresponding to a vector representation of the quantum group $U_q(\mathfrak{g})$ then there exists a pairing of Hopf algebras

$$U_q(\mathfrak{g}) \otimes A_{RTF} \to k.$$

In the case under consideration the category of finite-dimensional $U(\mathfrak{g})$-modules and the category of finite-dimensional A_{RTF}-comodules coincide. In the general case (including non-deformational solutions of the QYBE from [G3]) the category $U(sl(2)) - Mod$ in the axiom 3 of the Definition above should be replaced by the last category as it was done in [G4].

Let restrict ourselves now to the case $\mathfrak{g} = sl(2)$. In this case the Hopf algebra structure in $U_q(sl(2))$ can be described as follows. This algebra is generated by the elements $\{H, X, Y\}$ satisfying the relations

$$[H, X] = 2X,\ [H, Y] = -2Y,\ [X, Y] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

The coproduct is defined by the following formulae

$$\Delta(X) = X \otimes 1 + q^{-H} \otimes X,\ \Delta(Y) = 1 \otimes Y + Y \otimes q^{H},\ \Delta(H) = H \otimes 1 + 1 \otimes H$$

(we do not need the antipode).

Let V be a $U_q(sl(2))$-module of spin 1 with the base $\{u, v, w\}$ and of the following module structure:

$$Hu = 2u,\ Hv = 0,\ Hw = -2w,\ Xu = 0,\ Xv = -(q + q^{-1})u,\ Xw = v,$$

$$Yu = -v,\ Yv = (q + q^{-1})w,\ Yw = 0.$$

The space $V^{\otimes 2}$ can be decomposed into a direct sum of three irreducible $U_q(sl(2))$-modules: V_0, V_1, and V_2 of spin 0; 1 and 2 respectively. Let us describe them explicitly:

$$V_0 = \mathrm{span}((q^3 + q)uw + vv + (q + q^{-1})wu),$$

$$V_1 = \mathrm{span}(q^2uv - vu,\ (q^3 + q)(uw - wu) + (1 - q^2)vv,\ -q^2vw + wv),$$

$$V_2 = \mathrm{span}(uu,\ uv + q^2vu,\ uw - qvv + q^4wu,\ vw + q^2wv,\ ww)$$

(we omit the sign $\otimes$ if it does not lead to a misunderstanding). Note that the element $C_q = (q^3 + q)uw + vv + (q + q^{-1})wu$ is a q-analogue of the Casimir element, i.e., it is $U_q(sl(2))$-invariant.

We define the subspaces $I = I_q$ and $\overline{I} = \overline{I}_q$ and the bracket [,] as follows $I = V_1$, $\overline{I} = V_0 \oplus V_2$,

$$[,]\overline{I} = 0,\ [,](q^2uv - vu) = -2hu,$$

$$[,]((q^3 + q)(uw - wu) + (1 - q^2)vv) = 2hv, [,](-q^2vw + wv) = 2hw.$$

We leave to the reader to verify that the bracket [,] is a morphism of the category $U_q(sl(2)) - Mod$. Indeed it is possible to prove these relations assuming the highest weight vectors of V and V_1 to be equal to each other and applying the decreasing operator Y to this equality. This implies that the multiplication $\mu^{h,q}$ in the algebra $U^{h,q}(sl(2))$ satisfies the condition (1).

The relations similar to the last three ones have been constructed in [E] but the author of that paper does not define any bracket and does not investigate the flatness of the deformation of the enveloping algebra.

Let us express the bracket [,] in the basis form:

$$[u,u]=0,\ [u,v]=-2q^2Mu,\ [u,w]=2((q+q^{-1}))^{-1}Mv,$$

$$[v,u]=2Mu,\ [v,v]=2(1-q^2)Mv,\ [v,w]=-2q^2Mw,$$

$$[w,u]=-2((q+q^{-1}))^{-1}Mv,\ [w,v]=2Mw,\ [w,w]=0,$$

where $M=h(1+q^4)^{-1}$.

Let $U^{h,q}(sl(2))$ denote the enveloping algebra $T(V)/\{J\}$ where J is a linear space in $V^{\otimes 2}\oplus V$ generated by the elements

$$q^2uv-vu+2hu,\ (q^3+q)(uw-wu)+(1-q^2)vv-2hv,-q^2vw+wv-2hw,$$

and $\{J\}$ is the corresponding ideal. Note that $U^{0,q}(sl(2))=\wedge_+(V,I_q)$.

Proposition 3.1 *The algebra $\wedge_+(V,I_q)$ is a Koszul algebra.*

Proof. Fix a base $D_3=\{u^av^bw^c, a+b+c=3\}$ in the homogeneous component $\wedge^3_+(V,I_0)$. Let us show now that this set is still the base in the space $\wedge^3_+(V,I_q)$, i.e. the elements of this family are linearly independent.

Note first that the space $\wedge^3_-(V,I_q)$ is a one-dimensional space. It is easy to find its generator:

$$Z=-(q+q^{-1})w(q^2uv-vu)+v((q^3+q)(uw-wu)+(1-q^2)vv)+$$

$$(q^3+q)u(-q^2vw+wv)=-(q^3+q)(q^2uv-vu)w+((q^3+q)(uw-wu)+$$

$$(1-q^2)vv)v+(q+q^{-1})(-q^2vw+wv)u.$$

Thus the space V_1 generates a 17-dimensional subspace in $V^{\otimes 3}$. It is obvious that any element of $V^{\otimes 3}$ can be expressed via the elements from the set D_3. Therefore the elements from D_3 are independent.

By virtue of the "diamond lemma" (its useful version is given in a [PP]) we can state that the set $\{u^av^bw^c$ for all integer $a,b,c\geq 0\}$ is PBW base in $\wedge_+(V,I_q)$. Then by virtue of the Priddy theorem (cf. [PP]) the algebra $\wedge_+(V,I_q)$ is a Koszul one. That completes the proof.

Since $\dim\wedge^2_-(V,I_q)=3$ and $\dim\wedge^3_-(V,I_q)=1$ for any q, then by Proposition 2.1 $\mathcal{P}_+(t)$ does not change in the process of deformation i.e. $\wedge_+(V,I_q)$ is a flat deformation of $\wedge_+(V,I_0)$.

It is easy to see that $([,]^{12} - [,]^{23})Z = 0$ and therefore the following data $(V,\ I,\ \overline{I},\ [,])$ defines a braided Lie algebra. By virtue of Proposition 2.2 $GrU^{h,q}(sl(2)) = \wedge_+(V, I_q)$ and therefore the algebra $GrU^{h,q}(sl(2))$ is a flat two-parameter deformation of the commutative algebra $\wedge_+(V, I_0)$.

We put $q = e^{\nu}$ and computing the quasiclassical limit we find that the algebra $U^{0,q}(sl(2)) = \wedge_+(V, I_q)$ is a quantization of the R-matrix bracket

$$\{u, v\}_R = -uv;\ \{u, w\}_R = \frac{v^2}{2},\ \{v, w\}_R = vw$$

where $R = \frac{1}{2}(u \otimes w - w \otimes u)$. Since the algebra $U^{h,1}(sl(2))$ is a quantization of the Poisson-Lie bracket it is clear that the algebra $U^{h,q}(sl(2))$ is a quantization of the family (4).

Thus we have quantized the family (4) simutaneously. In the next Section we investigate some quotient algebras of the algebras $U^{h,q}(sl(2))$.

4 Braided algebra $gl(\lambda)$

The main goal of this Section is to treat the algebra $A_c^{h,q}$ as a braided analogue of the Feigin algebra $gl(\lambda)$ and to introduce a pairing such that it is a morphism in the category $U_q(sl(2)) - Mod$ (for the sake of brevity we call it a $U_q(sl(2))$-pairing).

The algebra $gl(\lambda)$ was introduced by B.Feigin [Fei] as a quotient algebra of $U(sl(2))$ over the ideal generated by the element $C - \lambda(\lambda - 1)/2$ where C is the Casimir element in $sl(2)$. This algebra is associative (and it is a Lie algebra with respect to the natural bracket). As a $sl(2)$-module $gl(\lambda)$ is the direct sum of irreducible $sl(2)$-modules $V_i, 0 \le i < \infty$.

Consider now the associative algebra $A_c^{h,q} = U^{h,q}(sl(2))/\{C_q - c\}$ and let $\mu_c^{h,q}$ denotes the multiplication in it. Here $\{C_q - c\}$ denotes as usually the ideal generated by the element $C_q - c$.

Assume now that q is not a root of unity. It is well-known that for a generic q the q-analogue of the representation theory is very close to the classical one. Let two $U_q(sl(2))$-modules $\rho_k : U_q(\mathfrak{g}) \to End(V_k), k = i; j$ (where k is the spin) be given. Then $V_i \otimes V_j$ can be decomposed into the direct sum

$$V_i \otimes V_j = \oplus_{|i-j| \le m \le k+l} V_m$$

of $U_q(sl(2))$-modules.

As an example we describe explicitly this decomposition for the product $V_1 \otimes V_1$

Consider an operator $S_{i,j} = \sigma(\rho_i \otimes \rho_j)R$ where R is the universal R-matrix for $U_q(sl(2))$ (c.f. [KT])

$$R = \exp_{q^{-2}}((q-q^{-1})X \otimes Y)q^{\frac{H \otimes H}{2}} =$$

$$(1+(q-q^{-1})X \otimes Y + (q-q^{-1})^2(1+q^{-2})^{-1}X^2 \otimes Y^2 + \ldots)q^{\frac{H \otimes H}{2}}.$$

The operator $S_{i,j}$ being restricted to an irreducible component of the product $V_i \otimes V_j$ is a scalar one. Let λ_{ij}^k denote the eigenvalue of $S_{i,j}$ restricted on V_k. Let Pr_{ij}^k denote the projection operator $V_i \otimes V_j \to V_k$ in the category $U_q(\mathfrak{g}) - Mod$. Then $V_1 \otimes V_1 = V_0 \oplus V_1 \oplus V_2$ and

$$\lambda_{11}^0 = q^{-4},\ \lambda_{11}^1 = -q^{-2},\ \lambda_{11}^2 = q^2, Pr_{11}^2 = \frac{(S-q^{-4})(S+q^{-2})}{(q^2-q^{-4})(q^2+q^{-2})}.$$

Using the standart method of the deformation theory we show for generic q that as a $U_q(\mathfrak{g})$-module the algebra $A_c^{h,q}$ is the direct sum of the irreducible $U_q(\mathfrak{g})$-modules V_i.

Now we introduce a braided Lie bracket in the algebra $A_c^{h,q}$. To do this we introduce an operator $\tilde{S} = \{\tilde{S}_{i,j}\}$ such that $\tilde{S}_{i,j} = \sum(-1)^{a_k} Pr_{ij}^k$, where $a_k = 0$ if $\lambda_{ij}^k \geq 0$ and $a_k = 1$ if $\lambda_{ij}^k < 0$ (assuming $q \in \mathbf{R}$). This operator is an involutive morphism in the category $U_q(sl(2)) - Mod$ because all operators Pr_{ij}^k are morphisms in it.[4]

Define the braided Lie bracket as follows

$$[,] = \mu^{h,q}(id - \tilde{S}).$$

It is obvious that this bracket is a morphism in the category $U_q(sl(2)) - Mod$ but the problem whether the Jacobi identity holds is still open.

We illustrate now the notion of a braided Lie algebra once more treating the algebra $A_c^{h,q}$ itself as the enveloping algebra of a braided Lie algebra. To do this let us introduce a new decomposition $V^{\otimes 2} = I \oplus \overline{I}$ as follows $I = V_0 \oplus V_1$, $\overline{I} = V_2$ and define the bracket $[,]$ as above with the only

[4]Note that according to Drinfeld's results [D2] $S = F\sigma q^{\frac{t}{2}} F^{-1}$ where t is the split Casimir element and F is an intertwining operator. Then $\tilde{S} = F\sigma F^{-1}$ (c.f.[DGM],[DG1]).

difference $[,](t) = c$ where t is the split q-Casimir, i.e., C_q which is regarded as an element of $V^{\otimes 2}$.

In the case under consideration we have $\dim\wedge^3_-(V, I_q) = 4$. It is obvious that $Z \in \wedge^3_-(V, I_q)$. One can find other three generators of this space using the relations

$$u \otimes t - t \otimes u \in \{V_1\},\ v \otimes t - t \otimes v \in \{V_1\},\ w \otimes t - t \otimes w \in \{V_1\}.$$

Thus the following elements (together with Z) generate the space $\wedge^3_-(V, I_q)$

$$Z_1 = ut - u((q^3+q)(uw - wu) + (1-q^2)vv) - q^{-2}v(q^2uv - vu) =$$

$$tu + q^{-2}((q^3+q)(uw - wu) + (1-q^2)vv)u + (q^2uv - vu)v,$$

$$Z_2 = vt + (q^3+q)u(-q^2vw + wv) + w(q^{-1} + q^{-3})(q^2uv - vu) =$$

$$tv - (q^3+q)(q^2uv - vu) - (q^{-1} + q^{-3})(-q^2vw + wv)u,$$

$$Z_3 = wt + q^{-2}w((q^3+q)(uw - wu) + (1-q^2)vv) - v(-q^2vw + wv) =$$

$$tw - ((q^3+q)(uw - wu) + (1-q^2)vv)w + q^{-2}(-q^2vw + wv)v.$$

Applying now the operator $[,]^{12} - [,]^{23}$ to the elements above we obtain 0. Thus the axioms 1-3 of the Definition of a braided Lie algebra are satisfied. As for the axiom 0 we can only conjecture it.

Thus we have represented the algebra $A^{h,q}$ (up to the conjecture) as the enveloping algebra of a braided Lie algebra.

Introduce now a $U_q(sl(2))$-pairing in the algebra $A^{h,q}$ as follows. First remark that $\mathrm{Im}Pr^0_{ij} \neq 0$ iff $i = j$ and $\dim\mathrm{Im}Pr^0_{ii} = 1$. Let f_i denote a generator of the space $\mathrm{Im}Pr^0_{ii}$ and put $<1, 1>= 1$,

$$<,>|_{V_i \otimes V_j} = 0 \text{ if } i \neq j,\ <,>|_{V_i \otimes V_i} = b_i$$

where b_i is defined via the relation $Pr^0_{ij} = b_i f_i$ (we take the split Casimir as f_1).

In the case of the algebra $A^{h,q}_0$ there exists a lot of pairings since we can choose f_i in arbitrarily way. As for the algebras $A^{h,q}_c$, $c \neq 0$ we should choose the elements f_i more carefully. Namely we put $f_i = c^i f^i_1 \bmod \{V_1\}$. It is natural to do so because of the equations $< f^k_1, f^l_1 >= c^{k+l}$.

It is clear that this pairing in algebra the $A^{h,q}_0$ is a $U_q(sl(2))$-one and it is $\tilde{S}$-commutative i.e. $<,>=<,>\tilde{S}$.

Thus a $U_q(sl(2))$-pairing in the algebra $A_c^{h,q}$ is constructed. This approach enables us to treat the braided algebra $A_c^{h,q}$ as an operator algebra (with respect to right or left multiplication) equipped with a braided pairing. It would be very interesting to calculate this pairing in a more explicit form.

It is not difficult to construct an involution which is a morphism in the category $U_q(sl(2)) - Mod$ but its role is not yet clear. Finishing this Section we want to remark that for the algebras lying in the category of $U_q(\mathfrak{g})$-modules it is natural to require the pairing and involution operators to be morphisms of this category. From our point of view it is more natural to introduce the so called "quantum sphere" (c.f. [P]) equipped with pairing and involution of such a type. It will be done elsewhere.

5 Discussion

At the end of the paper we would like to compare the braided Lie algebras and the S-Lie algebras introduced in [G1] for an involutive solution of the QYBE S. Let us recall that the last object was defined by the followng axioms:

1. $[,]S = -[,]$,
2. $[,][,]^{12}(\mathrm{id} + S^{12}S^{23} + S^{23}S^{12}) = 0$,
3. $S[,]^{12} = [,]^{23}S^{12}S^{23}$.

Introducing the subspaces I and $\overline{I} \in V^{\otimes 2}$ as follows

$$I = \mathrm{Im}(\mathrm{id} - S) = \mathrm{Ker}(\mathrm{id} + S),\ \overline{I} = \mathrm{Im}(\mathrm{id} + S) = \mathrm{Ker}(\mathrm{id} - S)$$

one can see that the first axioms coincide. The axiom 0 for the S-Lie algebra is fulfilled since in the involutive case the algebra $\wedge(V, I)$ is always Koszul one (it is proved in [G3] in a more general context). The last axiom from the last definition means that the bracket $[,]$ is a morphism of the category generated by the space V.

However the axioms 2 have different forms. And it is easy to check that the Jacobi identity in the last form is not true for the braided Lie algebra $sl(2)$. In particular that the "adjoint" operator $X \to [X, Y]$ is not a representation of the braided Lie algebra $sl(2)$ (but it is a representation for any S-Lie algebra).

Remark that one often introduces the Jacobi identity for non-involutive S in the same form as for S-Lie algebras (cf. for example [W], [O]). An example of the bracket in the braided $sl(2)$ shows that it is reasonable to claim the Jacobi identity for non-involutive S in a weaker form of Proposition 2.2.

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