ON CERTAIN QUESTIONS OF THE FREE GROUP AUTOMORPHISMS THEORY

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Abstract. Certain subgroups of the groups $Aut(F_n)$ of automorphisms of a free group F_n are considered. Comparing Alexander polynomials of two poly-free groups Cb_4^+ and P_4 we prove that these groups are not isomorphic, despite the fact that they have a lot of common properties. This answers the question of Cohen-Pakianathan-Vershinin-Wu from [9]. The questions of linearity of subgroups of $Aut(F_n)$ are considered. As an application of the properties of poison groups in the sense of Formanek and Procesi, we show that the groups of the type $Aut(G * \mathbb{Z})$ for certain groups G and the subgroup of IA-automorphisms $IA(F_n) \subset Aut(F_n)$ are not linear for $n \geq 3$. This generalizes the recent result of Pettet that $IA(F_n)$ are not linear for $n \geq 5$.

1. Introduction

1.1. **The group of basis conjugating automorphisms.** Let F_n be a free group of rank $n \geq 2$ with a free generator set $\{x_1, x_2, ..., x_n\}$ and $Aut(F_n)$ the group of automorphisms of F_n. Taking the quotient of F_n by its commutator subgroup F_n', we get a natural homomorphism

$$\xi : \mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(F_n/F_n') = \mathrm{GL}_n(\mathbb{Z}),$$

where $\mathrm{GL}_n(\mathbb{Z})$ is the general linear group over the ring of integers. The kernel of this homomorphism consists of automorphisms acting trivially modulo the commutator subgroup F_n'. It is called *the group of IA-automorphisms* and denoted by $\mathrm{IA}(F_n)$ (see [2, Chapter 1, § 4]). The group $\mathrm{IA}(F_2)$ is isomorphic to the group of inner automorphisms $\mathrm{Inn}(F_2)$, which is isomorphic to the free group F_2.

D. Nilsen (for the case $n \leq 3$) and W. Magnus (for all n) have shown that (see [2, Chapter 1, § 4]) the group $\mathrm{IA}(F_n)$ is generated by the following automorphisms

$$\varepsilon_{ijk} : \begin{cases} x_i \longmapsto x_i[x_j, x_k] & k \neq i, j, \\ x_l \longmapsto x_l & l \neq i, \end{cases} \qquad \varepsilon_{ij} : \begin{cases} x_i \longmapsto x_j^{-1} x_i x_j & i \neq j, \\ x_l \longmapsto x_l & l \neq i. \end{cases}$$

The subgroup of the group $\mathrm{IA}(F_n)$ is generated by the automorphisms ε_{ij}, $1 \leq i \neq j \leq n$ is called *the group of basis conjugating automorphisms*. We will denote this group by Cb_n. The group Cb_n is a subgroup of *the group of conjugating automorphisms* C_n. (Recall that any automorphism of C_n sends a generator x_i to an element of the type $f_i^{-1} x_{\pi(i)} f_i$, where $f_i \in F_n$, and π is a permutation from the symmetric group S_n.) Clearly, if π is the identity permutation, then the described element lies in the group Cb_n.

It is shown by D. McCool [3] that the group of basis conjugating automorphisms Cb_n is generated by the automorphisms ε_{ij}, $1 \leq i \neq j \leq n$, defined above has the following relations

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(we denote different indexes by different symbols):

$$\varepsilon_{ij}\varepsilon_{kl} = \varepsilon_{kl}\varepsilon_{ij},$$

$$\varepsilon_{ij}\varepsilon_{kj} = \varepsilon_{kj}\varepsilon_{ij},$$

$$(\varepsilon_{ij}\varepsilon_{kj})\varepsilon_{ik} = \varepsilon_{ik}(\varepsilon_{ij}\varepsilon_{kj}).$$

It is shown in [7] that the group of conjugating automorphisms Cb_n, $n \geq 2$ decomposes into a semi-direct product:

$$Cb_n = D_{n-1} \leftthreetimes (D_{n-2} \leftthreetimes (\ldots \leftthreetimes (D_2 \leftthreetimes D_1))\ldots), \qquad (1)$$

where the group D_i, $i = 1, 2, \ldots, n-1$, is generated by the elements $\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}$, $\varepsilon_{1,i+1}$, $\varepsilon_{2,i+1}$, $\ldots, \varepsilon_{i,i+1}$. The elements $\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}$ generate a free group of rank i, but the elements $\varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}$ generate a free abelian group of rank i.

1.2. **Braid group as a subgroup of $Aut(F_n)$.** Braid group B_n, $n \geq 2$, on n strands is defined by the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with defining relators

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad i = 1, 2, \ldots, n-2,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2.$$

There exists a natural homomorphism from the braid group B_n onto the permutation group S_n, sending the generator σ_i to the transposition $(i, i+1)$, $i = 1, 2, \ldots, n-1$. The kernel of this homomorphism is called *the pure braid group* and denoted by P_n. The group P_n is generated by the elements a_{ij}, $1 \leq i < j \leq n$ which can be expressed in terms of generators of B_n as follows:

$$a_{i,i+1} = \sigma_i^2,$$

$$a_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \ i+1 < j \leq n.$$

The pure braid group P_n is a semi-direct product of the normal subgroup U_n, which is the free group with free generators $a_{1n}, a_{2n}, \ldots, a_{n-1,n}$ with the group P_{n-1}. Analogously, P_{n-1} is a semi-direct product of the free subgroup U_{n-1} with the free generators $a_{1,n-1}, a_{2,n-1}, \ldots, a_{n-2,n-1}$ with the subgroup P_{n-2} etc. Hence the group P_n has the following decomposition

$$P_n = U_n \leftthreetimes (U_{n-1} \leftthreetimes (\ldots \leftthreetimes (U_3 \leftthreetimes U_2))\ldots), \ U_i \simeq F_{i-1}, i = 2, 3, \ldots, n. \qquad (2)$$

The pure braid group P_n is defined by relations (for $\nu = \pm 1$)

$$a_{ik}^{-\nu}a_{kj}a_{ik}^{\nu} = (a_{ij}a_{kj})^{\nu}\, a_{kj}\, (a_{ij}a_{kj})^{-\nu},$$

$$a_{km}^{-\nu}a_{kj}a_{km}^{\nu} = (a_{kj}a_{mj})^{\nu}\, a_{kj}\, (a_{kj}a_{mj})^{-\nu}, \qquad m < j,$$

$$a_{im}^{-\nu}a_{kj}a_{im}^{\nu} = \left[a_{ij}^{-\nu}, a_{mj}^{-\nu}\right]^{\nu} a_{kj} \left[a_{ij}^{-\nu}, a_{mj}^{-\nu}\right]^{-\nu}, \quad i < k < m,$$

$$a_{im}^{-\nu}a_{kj}a_{im}^{\nu} = a_{kj}, \qquad k < i;\ m < j \quad m < k,$$

where $[a, b] = a^{-1}b^{-1}ab$ is the commutator of elements a and b.

The braid group B_n can be embedded into the automorphism group $\mathrm{Aut}(F_n)$. In this embedding the generator σ_i, $i = 1, 2, \ldots, n-1$, defines the following automorphism

$$\sigma_i : \begin{cases} x_i \longmapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \longmapsto x_i, \\ x_l \longmapsto x_l & l \neq i, i+1. \end{cases}$$

The generator a_{rs} of the pure braid group P_n defines the following automorphism

$$a_{rs}: \begin{cases} x_i \longmapsto x_i & s < i \quad i < r, \\ x_r \longmapsto x_r x_s x_r x_s^{-1} x_r^{-1}, & \\ x_i \longmapsto [x_r^{-1}, x_s^{-1}] x_i [x_r^{-1}, x_s^{-1}]^{-1} & r < i < s, \\ x_s \longmapsto x_r x_s x_r^{-1}. & \end{cases}$$

As was shown by E. Artin, the automorphism β from $\mathrm{Aut}(F_n)$ belongs to the braid group B_n if and only if β satisfies the following two conditions:

$$1)\ \beta(x_i) = a_i^{-1} x_{\pi(i)} a_i,\ 1 \leq i \leq n,$$
$$2)\ \beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n,$$

where π is a permutation from S_n, and $a_i \in F_n$.

The set of automorphisms from C_n, which act trivially on the product $x_1 x_2 ... x_n$ is exactly the braid group B_n. It was shown by A. Savushkina [4], that the braid group B_n intersects with the subgroup Cb_n by the pure braid group P_n. Thus the group P_n is a subgroup of the basis conjugating automorphism group Cb_n for all $n \geq 2$. Furthermore, the concordance of decompositions (1) and (2) of groups Cb_n and P_n respectively, takes a place: there are correspondent embeddings $U_{i+1} \leq D_i$, $i = 1, 2, \ldots, n-1$.

1.3. **The welded braid group.** The welded braid group WB_n is one of natural generalizations of the braid group. It was introduced in [6]. This group contains the braid group B_n and the permutation group S_n. The group WB_n is generated by elements

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \alpha_1, \alpha_2, \ldots, \alpha_{n-1},$$

and has the following relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ i = 1, 2, \ldots, n-2, \tag{3}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i,\ |i - j| \geq 2, \tag{4}$$

$$\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1},\ i = 1, 2, \ldots, n-2, \tag{5}$$
$$\alpha_i \alpha_j = \alpha_j \alpha_i,\ |i - j| \geq 2, \tag{6}$$
$$\alpha_i^2 = 1\ i = 1, 2, \ldots, n-1, \tag{7}$$

$$\alpha_i \sigma_j = \sigma_j \alpha_i,\ |i - j| \geq 2, \tag{8}$$
$$\sigma_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \sigma_{i+1},\ i = 1, 2, \ldots, n-2, \tag{9}$$
$$\sigma_{i+1} \sigma_i \alpha_{i+1} = \alpha_i \sigma_{i+1} \sigma_i,\ i = 1, 2, \ldots, n-2. \tag{10}$$

The relations (3)–(4) are the defining relations of the braid group B_n, the relations (5)–(7) are the defining relations of the permutation group S_n. Observe that the following relation takes a place in the group WB_n:

$$\sigma_{i+1} \alpha_i \alpha_{i+1} = \alpha_i \alpha_{i+1} \sigma_i,\ i = 1, 2, \ldots, n-2, \tag{11}$$

which is symmetric to the relation (9). Observe also that we don't have in WB_n the relation

$$\sigma_i \sigma_{i+1} \alpha_i = \alpha_{i+1} \sigma_i \sigma_{i+1},\ i = 1, 2, \ldots, n-2, \tag{12}$$

which is symmetric to the relation (10).

There exists a natural homomorpism from the group WB_n onto the group S_n, sending the generators σ_i to the generators α_i and leaving the element α_i as it is. The kernel of this homomorphism if called the *pure welded braid group* and is denoted by WP_n. It is shown in [6] that the welded braid group WB_n is isomorphic to the conjugating automorphism group C_n, but the pure welded braid group WP_n is isomorphic to the basis conjugating automorphism group Cb_n.

1.4. **The group Cb_n^+.** Denote by Cb_n^+ the subgroup of the group Cb_n, $n \geq 2$, generated by the elements ε_{ij}, $i > j$. Groups Cb_n^+ are poly-free groups: it is shown in [15] that there exists the splitting exact sequence:

$$1 \to F_{n-1} \to Cb_n^+ \to Cb_{n-1}^+ \to 1,$$

where F_{n-1} is the free subgroup in Cb_n^+, generated by the elements

$$\varepsilon_{n1}, \varepsilon_{n2}, \dots, \varepsilon_{n,n-1}.$$

Description of the Lie algebra constructed from lower central filtration of the basis conjugating automorphism group, is a non-trivial problem; the methods used in the description of Lie algebras for pure braid groups, do not work in this case. However, in the case of the group Cb_n^+, the situation is different. Lie algebras and cohomology rings of groups Cb_n^+ were described in [9]. It was also shown in [9] that the group Cb_n^+ is isomorphic to the pure braid group P_n for $n = 2, 3$. Groups Cb_n^+ and P_n are quite similar. For instance, the groups Cb_n^+ and P_n are *stably isomorphic*, namely, suspensions over their classifying spaces $\Sigma K(Cb_n^+, 1)$ and $\Sigma K(P_n, 1)$ are homotopically equivalent for all $n \geq 1$ (see [9]). The conjecture that Cb_n^+ and P_n are isomorphic for all $n \geq 1$ comes naturally (question 1, [9]). One of the main results of this paper is the proof that the groups Cb_4^+ and P_4 are *non-isomorphic* (Theorem 1).

1.5. **The questions of linearity.** As it was shown above, for any $n \geq 2$, there is the following chain of subgroups of the automorphism group $Aut(F_n)$:

$$P_n \subset Cb_n \subset IA_n(F_n) \subset Aut(F_n).$$

It was shown in [16], [17] that the braid groups B_n are linear. Hence the pure braid groups P_n are also are linear. The situation with groups $Aut(F_n)$ is as follows. The group $\mathrm{Aut}(F_n)$ is not linear for all $n \geq 3$ (see [11]), however the group $\mathrm{Aut}(F_2)$ is linear (such a presentation was constructed, for example, in [12]). A certain non-linear subgroup (called *poison group*) of $\mathrm{Aut}(F_n)$ was constructed in [11]. The following question rises naturally (see question 15.17 [13]): are the groups of IA-automorphisms $\mathrm{IA}(F_n)$ and the groups of basis conjugating automorphisms Cb_n linear for $n \geq 2$? Recently A. Pettet showed that the group $\mathrm{IA}(F_n)$ is not linear for $n \geq 5$. In this paper we show that the groups $\mathrm{IA}(F_n)$ are not linear for all $n \geq 3$ (Theorem 5). Therefore, the complete answer to the question 15.17 from [13] is given.

2. The group Cb_4^+ is not isomorphic to the pure braid group P_4

2.1. **Fox differential calculus.** Recall the definition and main properties of Fox derivatives [5, chapter 3], [8, chapter 7].

Let F_n be a free group of rank n with free generators $x_1, x_2, \dots, x_n$. Let φ be an endomorphism of the group F_n. Denote by F_n^φ the image of F_n under the endomorphism φ. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Z}G$ the integral group ring of G.

For every $j = 1, 2, \dots, n$ define a map

$$\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$$

by setting

$$1)\ \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

$$2)\ \frac{\partial x_i^{-1}}{\partial x_j} = \begin{cases} -x_i^{-1} & \text{for } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

$$3)\ \frac{\partial (wv)}{\partial x_j} = \frac{\partial w}{\partial x_j}(v)^{\tau} + w\frac{\partial v}{\partial x_j},\ w, v \in \mathbb{Z}F_n,$$

where $\tau : \mathbb{Z}F_n \longrightarrow \mathbb{Z}$ is the trivialization operation, sending all elements of F_n to the identity,

$$4)\ \frac{\partial}{\partial x_j}\left(\sum a_g g\right) = \sum a_g \frac{\partial g}{\partial x_j},\ g \in F_n,\ a_g \in \mathbb{Z}.$$

Denote by Δ_n the augmentation ideal of the ring $\mathbb{Z}F_n$, i.e. the kernel of the homomorphism τ. It is easy to see that for every $v \in \mathbb{Z}F_n$, the element $v - v^{\tau}$ belongs to Δ_n. Furthermore, there is the following formula

$$v - v^{\tau} = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}(x_j - 1), \tag{13}$$

which is called *the main formula of Fox calculus.* Formula (13) implies that the elements $\{x_1 - 1, x_2 - 1, \ldots, x_n - 1\}$ determine the basis of the augmentation ideal Δ_n.

2.2. **Groups Cb_4^+ and P_4.** As it was noted in the introduction, the poly-free groups Cb_n^+ and P_n have similar properties. Clearly, one has

$$Cb_2^+ \simeq P_2 \simeq \mathbb{Z};$$
$$Cb_3^+ \simeq P_3 \simeq F_2 \rtimes \mathbb{Z}.$$

However, the next result shows that the groups Cb_4^+ and P_4 being semidirect products of F_3 with $F_2 \rtimes \mathbb{Z}$, are not isomorphic.

Theorem 1. *Groups Cb_4^+ and P_4 are not isomorphic.*

Proof. It is well-known that (see [10]) that the group P_4 decomposes as a direct product of its center generated by a single element $a_{12}a_{13}a_{23}a_{14}a_{24}a_{34}$ and a subgroup $H = U_4 \leftthreetimes U_3$, $U_4 = \langle a_{14}, a_{24}, a_{34}\rangle$, $U_3 = \langle a_{13}, a_{23}\rangle$. It is easy to check that the center of the group Cb_4^+ is the infinite cyclic group generated by the element $\varepsilon_{21}\varepsilon_{31}\varepsilon_{41}$ and the whole group Cb_4^+ is a direct product of its center and a subgroup $G = D_3^+ \leftthreetimes D_2^+$, where $D_3^+ = \langle \varepsilon_{41}, \varepsilon_{42}, \varepsilon_{43}\rangle$, $D_2^+ = \langle \varepsilon_{31}, \varepsilon_{32}\rangle$. Since a center is a characteristic subgroup for every group, it is enough to show that the group G is not isomorphic to the group H. We will compare Alexander polynomials for the groups G and H.

The presentation of the group Cb_4 implies that the group G is defined by the following set of relations:

$$\varepsilon_{31}^{-1}\varepsilon_{41}\varepsilon_{31} = \varepsilon_{41},\ \varepsilon_{31}^{-1}\varepsilon_{42}\varepsilon_{31} = \varepsilon_{42},\ \varepsilon_{31}^{-1}\varepsilon_{43}\varepsilon_{31} = \varepsilon_{41}\varepsilon_{43}\varepsilon_{41}^{-1},$$
$$\varepsilon_{32}^{-1}\varepsilon_{41}\varepsilon_{32} = \varepsilon_{41},\ \varepsilon_{32}^{-1}\varepsilon_{42}\varepsilon_{32} = \varepsilon_{42},\ \varepsilon_{32}^{-1}\varepsilon_{43}\varepsilon_{32} = \varepsilon_{42}\varepsilon_{43}\varepsilon_{42}^{-1}.$$

We present these relations in the following form:

$$r_{11} = \varepsilon_{41}^{-1}\varepsilon_{31}^{-1}\varepsilon_{41}\varepsilon_{31},\ r_{21} = \varepsilon_{42}^{-1}\varepsilon_{31}^{-1}\varepsilon_{42}\varepsilon_{31},\ r_{31} = \varepsilon_{43}^{-1}\varepsilon_{41}^{-1}\varepsilon_{31}^{-1}\varepsilon_{43}\varepsilon_{31}\varepsilon_{41},$$
$$r_{12} = \varepsilon_{41}^{-1}\varepsilon_{32}^{-1}\varepsilon_{41}\varepsilon_{32},\ r_{22} = \varepsilon_{42}^{-1}\varepsilon_{32}^{-1}\varepsilon_{42}\varepsilon_{32},\ r_{32} = \varepsilon_{43}^{-1}\varepsilon_{42}^{-1}\varepsilon_{32}^{-1}\varepsilon_{43}\varepsilon_{32}\varepsilon_{42}.$$

We define the homomorphism $\varphi : G \to \langle t\rangle$, by setting

$$\varphi(\varepsilon_{31}) = \varphi(\varepsilon_{32}) = \varphi(\varepsilon_{41}) = \varphi(\varepsilon_{42}) = \varphi(\varepsilon_{43}) = t$$

and extend it to the homomorphism of integral group rings by linearity

$$\varphi : \mathbb{Z}G \longrightarrow \mathbb{Z}\langle t\rangle.$$

Let us find Fox derivatives of the relators r_{ij}, with respect to the symbols ε_{31}, ε_{32}, ε_{41}, ε_{42}, ε_{43} and their φ-images. We have the following

$$\left(\frac{\partial r_{11}}{\partial \varepsilon_{31}}\right)^{\varphi} = t^{-2}(t-1), \left(\frac{\partial r_{11}}{\partial \varepsilon_{41}}\right)^{\varphi} = t^{-2}(1-t), \left(\frac{\partial r_{12}}{\partial \varepsilon_{32}}\right)^{\varphi} = t^{-2}(t-1),$$
$$\left(\frac{\partial r_{12}}{\partial \varepsilon_{41}}\right)^{\varphi} = t^{-2}(1-t), \left(\frac{\partial r_{21}}{\partial \varepsilon_{31}}\right)^{\varphi} = t^{-2}(t-1), \left(\frac{\partial r_{21}}{\partial \varepsilon_{42}}\right)^{\varphi} = t^{-2}(1-t),$$
$$\left(\frac{\partial r_{22}}{\partial \varepsilon_{32}}\right)^{\varphi} = t^{-2}(t-1), \left(\frac{\partial r_{22}}{\partial \varepsilon_{42}}\right)^{\varphi} = t^{-2}(1-t), \left(\frac{\partial r_{31}}{\partial \varepsilon_{31}}\right)^{\varphi} = t^{-1}(t-1),$$
$$\left(\frac{\partial r_{31}}{\partial \varepsilon_{41}}\right)^{\varphi} = t^{-2}(t-1), \left(\frac{\partial r_{31}}{\partial \varepsilon_{43}}\right)^{\varphi} = t^{-3}(1-t^2), \left(\frac{\partial r_{32}}{\partial \varepsilon_{32}}\right)^{\varphi} = t^{-1}(t-1),$$
$$\left(\frac{\partial r_{32}}{\partial \varepsilon_{42}}\right)^{\varphi} = t^{-2}(t-1), \left(\frac{\partial r_{32}}{\partial \varepsilon_{43}}\right)^{\varphi} = t^{-3}(1-t^2).$$

Derivatives with respect to other generators are zero.

With the obtained values of derivatives we form the Alexander matrix. After elementary transformations of rows and columns and deleting zero rows and columns, we get the diagonal matrix

$$(1-t)\ \mathrm{diag}(-\mathrm{t}^{-2}, -\mathrm{t}^{-2}, -\mathrm{t}^{-2}, -\mathrm{t}^{-2}(1+\mathrm{t})).$$

Since the Alexander polynomial is defined up to the multiplication with a unit of the ring $\mathbb{Z}\langle t\rangle$, we get

$$\Delta_G(t) = (1-t)^4(1+t).$$

We now consider the group H. Defining relations of the group P_4 imply that the group H can be defined by the following relations

$$a_{13}a_{14}a_{13}^{-1} = a_{34}^{-1}a_{14}a_{34}, a_{13}^{-1}a_{24}a_{13} = [a_{14}^{-1}, a_{34}^{-1}]a_{24}[a_{34}^{-1}, a_{14}^{-1}],$$
$$a_{23}^{-1}a_{14}a_{23} = a_{14}, a_{23}a_{24}a_{23}^{-1} = a_{34}^{-1}a_{24}a_{34},$$
$$a_{13}^{-1}a_{34}a_{13} = a_{14}a_{34}a_{14}^{-1}, a_{23}^{-1}a_{34}a_{23} = a_{24}a_{34}a_{24}^{-1}.$$

These relations can be presented in the following form

$$q_{11} = a_{13}a_{14}a_{13}^{-1}a_{34}^{-1}a_{14}^{-1}a_{34},\ \ q_{21} = a_{13}^{-1}a_{24}a_{13}a_{14}a_{34}a_{14}^{-1}a_{34}^{-1}a_{24}^{-1}a_{34}a_{14}a_{34}^{-1}a_{14}^{-1},$$
$$q_{12} = a_{14}^{-1}a_{23}^{-1}a_{14}a_{23},\ \ q_{22} = a_{23}a_{24}a_{23}^{-1}a_{34}^{-1}a_{24}^{-1}a_{34},$$
$$q_{31} = a_{13}^{-1}a_{34}a_{13}a_{14}a_{34}^{-1}a_{14}^{-1},\ \ q_{32} = a_{23}^{-1}a_{34}a_{23}a_{24}a_{34}^{-1}a_{24}^{-1}.$$

As above, we define the homomorphism φ from the group H to the infinite cyclic group $\langle t\rangle$, by setting

$$\varphi(a_{13}) = \varphi(a_{23}) = \varphi(a_{14}) = \varphi(a_{24}) = \varphi(a_{34}) = t,$$

and extend it by linearity to the homomorphism of group rings

$$\varphi : \mathbb{Z}H \longrightarrow \mathbb{Z}\langle t\rangle.$$

We compute the Fox derivatives of the relators q_{ij}, written above, with respect to the variables a_{13}, a_{23}, a_{14}, a_{24}, a_{34} and find their images under the homomorphism φ. We get the following identities (the derivatives of all other relators can be written analogously)

$$\left(\frac{\partial q_{11}}{\partial a_{13}}\right)^{\varphi} = 1-t, \left(\frac{\partial q_{11}}{\partial a_{14}}\right)^{\varphi} = t^{-1}(t^2-1), \left(\frac{\partial q_{11}}{\partial a_{34}}\right)^{\varphi} = t^{-1}(1-t),$$
$$\left(\frac{\partial q_{21}}{\partial a_{13}}\right)^{\varphi} = t^{-1}(t-1), \left(\frac{\partial q_{21}}{\partial a_{14}}\right)^{\varphi} = -(1-t)^2, \left(\frac{\partial q_{21}}{\partial a_{24}}\right)^{\varphi} = t^{-1}(1-t),$$
$$\left(\frac{\partial q_{21}}{\partial a_{34}}\right)^{\varphi} = (t-1)^2, \left(\frac{\partial q_{32}}{\partial a_{23}}\right)^{\varphi} = t^{-1}(t-1), \left(\frac{\partial q_{32}}{\partial a_{24}}\right)^{\varphi} = t-1,$$
$$\left(\frac{\partial q_{32}}{\partial a_{34}}\right)^{\varphi} = t^{-1}(1-t^2).$$

We form the matrix from the values of the calculated Fox derivatives. After elementary transformations over rows and columns, we get the diagonal matrix

$$(1-t)\ \mathrm{diag}(1, -\mathrm{t}^{-2}, 1, \mathrm{t}^{-2}+\mathrm{t}^{-1}+1).$$

The Alexander polynomial of the group H is equal to

$$\Delta_H(t) = (1-t)^4(t^2+t+1).$$

Thus the Alexander polynomials of the groups G and H are different, and hence, the groups G and H are not isomorphic (see, for example [8] for the proof that Alexander invariants, in particular the Alexander polynomial, are invariants of a group and do not depend on a given group presentation). □

3. Isomorphism problem for a certain class of Lie algebras

3.1. **A class of Lie algebras.** The last section gives a motivation for finding methods for proving that two given subgroup of automorphism groups are not isomorphic. In this section we will consider a similar question.

Consider a homomorphism

$$\phi : F_n \to IA(F_k),$$

where $n, k \geq 2$. We can form a natural semi-direct product

$$G_\phi := F_k \rtimes F_n$$

and ask whether G_{ϕ_1} is isomorphic to G_{ϕ_2} for two different homomorphisms $\phi_1, \phi_2 : F_n \to IA(F_k)$.

For a given group G, denote by $L(G)$ the Lie algebra constructed from the lower central series filtration:

$$L(G) := \bigoplus_{i\geq 1} \gamma_i(G)/\gamma_{i+1}(G),$$
$$\gamma_1(G) = G,\ \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Here we will consider the class of Lie algebras $L(G_\phi)$ for different homomorphisms ϕ and prove that some Lie algebras from this class are not isomorphic.

Since F_n acts trivially on $H_1(F_k)$, there is the following exact splitting sequence of Lie algebras:

$$0 \to L(F_k) \to L(G_\phi) \to L(F_n) \to 0$$

by [18].

3.2. **Scheuneman's invariants.** Let k be a field and L a finitely-generated Lie algebra of nilpotence class two. Let us choice a basis of L: $\{x_1, \dots, x_n, y_1, \dots, y_r\}$ such that $\{y_1, \dots, y_r\}$ is a basis of $[L, L]$. Let $U(L)$ be a universal enveloping algebra of L. Then $U(L)$ can be identified with a polynomial algebra over k with non-commutative variables $x_1, \dots, x_n, y_1, \dots, y_r$ and relations

$$x_i y_j = y_j x_i, \text{ for all } i, j;$$
$$y_i y_j = y_j y_i, \text{ for all } i, j;$$
$$x_i x_j - x_j x_i = \text{linear combination of } y_k\text{'s};$$

Consider an alternative sum

$$I(x_1, \dots, x_n) = \sum_{\sigma=(i_1,\dots,i_n)\in S_n} (-1)^{|\sigma|} x_{i_1} \dots x_{i_n}, \tag{14}$$

where $|\sigma|$ denotes the sign of the permutation σ.

Two polynomials $f(z_1, \dots, z_m)$ and $g(z_1, \dots, z_m)$ are *k-equivalent* if

$$f(z_1, \dots, z_m) = ag(z'_1, \dots, z'_m),$$

where $a \in k, a \neq 0$,

$$z'_j = \sum b_{ij} z_j, \; b_{ij} \in k, \; det(b_{ij}) \neq 0.$$

The following theorem is due to Scheuneman:

Theorem 2. [14] *The k-equivalence class of $I(x_1, \dots, x_n)$ is an invariant of k-isomorphism of the Lie algebra L.*

For k-equivalent polynomials $f(z_1, \dots, z_m)$ and $g(z_1, \dots, z_m)$ and $z'_i = \sum b_{ij} z_j, \; b_{ij} \in k$, $det(b_{ij}) \neq 0$, such that $f(z_1, \dots, z_m) = ag(z'_1, \dots, z'_m)$,

$$Hes(f)(z_1, \dots, z_m) = a^m (det(b_{ij}))^2 Hes(g)(z'_i, \dots, z'_m),$$

where $Hes(f)(z_1, \dots, z_m)$ is the Hessian $det(\frac{\partial^2 f}{\partial z_i \partial z_j})$ (see, for example, Lemma 8 [14]).

3.3. **Non-isomorphic Lie algebras.** Consider the following homomorphism ϕ_α of a free group F_3 with basis $\{u_1, u_2, u_3\}$ to the automorphism group of a free group F_3 with basis $\{t_1, t_2, t_3\}$:

$$\phi_\alpha : u_1 \mapsto \begin{cases} t_1 \longmapsto t_1\alpha_1 & \\ t_i \longmapsto t_i & \text{for } t = 2, 3, \end{cases}$$

$$\phi_\alpha : u_2 \mapsto \begin{cases} t_2 \longmapsto t_2\alpha_2 & \\ t_i \longmapsto t_i & \text{for } t = 1, 3, \end{cases}$$

$$\phi_\alpha : u_3 \mapsto \begin{cases} t_3 \longmapsto t_3\alpha_3 & \\ t_i \longmapsto t_i & \text{for } t = 1, 2, \end{cases}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_1 \in \{[t_1, t_2], [t_2, t_3], [t_1, t_3]\},$$
$$\alpha_2 \in \{[t_2, t_1], [t_2, t_3], [t_1, t_3]\},$$
$$\alpha_3 \in \{[t_1, t_2], [t_3, t_2], [t_3, t_1]\}.$$

Lie algebras $L(G_\alpha)$ which correspond to the groups G_α can be described as Lie algebras with generators $t_1, t_2, t_3, u_1, u_2, u_3$ and defining relations:

$$[t_1, u_1] = \alpha_1, \; [t_2, u_2] = \alpha_2, \; [t_3, u_3] = \alpha_3,$$
$$[t_2, u_1] = [t_3, u_1] = [t_1, u_2] = [t_3, u_2] = [t_1, u_3] = [t_2, u_3] = 0.$$

Consider the quotient

$$L_\alpha := L(G_{\phi_\alpha})/[L(G_{\phi_\alpha}), L(G_{\phi_\alpha}), L(G_{\phi_\alpha})].$$

Clearly, L_α is 12-dimensional algebra with basis

$$\{t_1, t_2, t_3, u_1, u_2, u_3, y_1, y_2, y_3, y_4, y_5, y_6\},$$

where $y_1 = [t_1, t_2], y_2 = [t_1, t_3], y_3 = [t_2, t_3], y_4 = [u_1, u_2], y_5 = [u_1, u_3], y_6 = [u_2, u_3]$. Consider the polynomial $I(t_1, t_2, t_3, u_1, u_2, u_3)$. The direct computation gives the following expression

$$\begin{aligned} I(t_1, t_2, t_3, u_1, u_2, u_3) = \\ [t_1, t_2][t_3, u_3][u_1, u_2] - [t_1, t_3][t_2, u_2][u_1, u_3] + [t_1, u_1][t_2, t_3][u_2, u_3] - [t_1, u_1][t_2, u_2][t_3, u_3] \\ = y_1 y_4 \alpha_3 - y_2 y_5 \alpha_2 + y_3 y_6 \alpha_1 - \alpha_1 \alpha_2 \alpha_3 = P_{(\alpha_1, \alpha_2, \alpha_3)}(y_1, \dots, y_6). \end{aligned}$$

By Theorem 2, we have

$$\{k - \text{equivalence classes of cubic forms } P_{(\alpha_1, \alpha_2, \alpha_3)}\} \subseteq \{k - \text{isomorphism classes of } G_{\phi_\alpha}\}$$

Consider the following cases:

1) Let $a_1 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3],\ \alpha_2 = [t_1, t_3],\ \alpha_3 = [t_1, t_2]$. Then

$$P_{a_1}(y_1, \dots, y_6) = y_1^2 y_4 - y_2^2 y_5 + y_3^2 y_6 - y_1 y_2 y_3,$$
$$Hes(P_{a_1})(y_1, \dots, y_6) = 64 y_1^2 y_2^2 y_3^2.$$

2) Let $a_2 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3],\ \alpha_2 = [t_2, t_3],\ \alpha_3 = [t_1, t_2]$. Then

$$P_{a_2}(y_1, \dots, y_6) = y_1^2 y_4 - y_2 y_5 y_3 + y_3^2 y_6 - y_1 y_3^2,$$
$$Hes(P_{a_2}) = 16 y_1^2 y_3^4.$$

3) Let $a_3 = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = [t_2, t_3],\ \alpha_2 = [t_2, t_3],\ \alpha_3 = [t_1, t_2]$. Then

$$P_{a_3}(y_1, \dots, y_6) = y_1 y_4 y_3 - y_2 y_5 y_3 + y_3^2 y_5 - y_3^3,$$
$$Hes(P_{a_3}) = 4 y_3^6.$$

Clearly, all these three cases define non-k-equivalent polynomials. Hence, the correspondent groups and Lie algebras are not isomorphic.

Proposition 1. *The Lie algebras $L(G_{a_1}), L(G_{a_2}), L(G_{a_3})$ are pairwise non-isomorphic.*

Remark. The invariant polynomials for the two-step nilpotent quotients of Lie algebras of the groups $Cb_4^+/Z(Cb_4^+)$ and $P_4/Z(P_4)$ define cubic forms which lie on Hilbert's null-cone, hence they do not differ from each other by an invariant of a Hessian type[1].

4. On non-linearity of certain automorphism groups

4.1. **Poison group.** Let G be a group. Consider the group $\mathcal{H}(G)$, defined as an HNN-extension:

$$\mathcal{H}(G) = \langle G \times G, t \mid t(g, g)t^{-1} = (g, 1),\ g \in G\rangle.$$

It is easy to see that the group $\mathcal{H}(G)$ can be presented as a semi-direct product

$$\mathcal{H}(G) \simeq (G * \mathbb{Z}) \rtimes G,$$

where the action of G on $G * \mathbb{Z} = G * \langle t \rangle$ is defined by

$$g : g_1 t^{e_1} g_2 t^{e_2} \dots g_k t^{e_k} \mapsto g_1 (tg^{-1})^{e_1} g_2 (tg^{-1})^{e_2} \dots g_k (tg^{-1})^{e_n},\ g, g_i \in G, e_i \in \mathbb{Z}.$$

[1]Authors thank V.L. Popov for helping analyzing these cubic forms and for this remark

For example, if $G = \mathbb{Z}$, we get the following group

$$\mathcal{H}(G) = \langle a, b \mid [a, [b, a]] = 1 \rangle.$$

It follows from elementary Titse moves that

$$H(\mathbb{Z}) = \langle a, a', t \mid [a, a'] = 1,\ taa't^{-1} = a \rangle = \langle a, a', t \mid [a, a'] = 1,\ a' = a^{-1}t^{-1}at = [a, t] \rangle =$$

$$= \langle a, t \mid [a, [a, t]] = 1 \rangle,$$

where $a = (a, 1), a' = (1, a) \in G \times G$.

Denote by $\mathcal{NAF}$ the class of nilpotent-by-abelian-by-finite groups. The following result due to Formanek and Procesi presents a way of constructing non-linear groups.

Theorem 3. [11] *If $G \notin \mathcal{NAF}$, then the group $\mathcal{H}(G)$ is non- linear.*

The simplest example of a group which does not lie in the class $\mathcal{NAF}$, clearly, is a free non-cyclic group. Thus, for $G = F_2$, the group $\mathcal{H}(F_2)$, called a *poison group*, is non-linear. This fact plays an important role in the proof of non-linearity of groups $Aut(F_n)$ for ≥ 3; the poison group is a subgroup in $Aut(F_n),\ n \geq 3$. This statement can be generalized.

Theorem 4. *Let $G \notin \mathcal{NAF}$, then $Aut(G * \mathbb{Z})$ is non-linear.*

Proof. We will realize the group $\mathcal{H}(G)$ as a subgroup in $Aut(G * \mathbb{Z})$ and the statement will follow from Theorem 3. Elements of the subgroup $G \times G$ in $\mathcal{H}(G)$ we will denote as (g, g'), i.e. we put dash for elements from the second copy of G. The group $G * \mathbb{Z}$ we will describe in terms of generators $g \in G$ and a free generator t. Consider the homomorphism

$$f : \mathcal{H}(G) \to Aut(G * \mathbb{Z}),$$

given by setting

$$f : g \mapsto i_g,\ g \in G,\ t \mapsto i_t,\ g' \mapsto s_{g'},\ g' \in G,$$

where i_g is the conjugation by g, i_t is the conjugation by t, $s_{g'}$ ($g' \in G$) is the automorphism of $G * \mathbb{Z}$ acting trivially on G and sending the element t to the element tg'^{-1}. It can be checked that f is a group homomorphism. Every element of the group $\mathcal{H}(G)$ can be written without 'dash' elements, since $tgg't^{-1} = g$ and, therefore, $g' = [g, t]$. Hence, in the case of the existence of a non-trivial kernel of f, there is an element of $G * \mathbb{Z}$, acting trivially by conjugation, i.e. lying in the center of $G*\mathbb{Z}$. However, any non-trivial free product has a trivial center, therefore, f is a monomorphism. Thus $\mathcal{H}(G)$ is a subgroup of $Aut(G * \mathbb{Z})$. Theorem 3 implies that the group $Aut(G * \mathbb{Z})$ is non-linear. □

Clearly, one can consider different embeddings of groups $\mathcal{H}(G)$ in correspondent automorphism groups. Consider the case $G = F_2$.

Let F_3 be a free group with basis x_1, x_2, x_3 and a_1, a_2, a_3 some elements of F_3, such that the subgroup $\langle a_1, a_2, a_3 \rangle$ is free of rank 3. Define automorphisms $\alpha_i,\ i = 1, 2, 3$ of the group F_3 as conjugation with a_i. The following statement can be checked straightforwardly:

Proposition 2. *Let $\phi_1, \phi_2 \in Aut(F_3)$ be automorphisms which satisfy the following conditions:*

$$\begin{aligned} \phi_1(a_1) &= \phi_2(a_1) = a_1, \\ \phi_1(a_2) &= \phi_2(a_2) = a_2, \\ \phi_1(a_3) &= a_3a_1,\ \phi_2(a_3) = a_3a_2. \end{aligned}$$

Then the subgroup of $Aut(F_3)$, generated by elements $\alpha_1, \alpha_2, \alpha_3, \phi_1, \phi_2$ is isomorphic to the poison group $\mathcal{H}(F_2)$.

As an example lets take $a_3 = x_3$, with a_1 and a_2 arbitrary elements of $\langle x_1, x_2\rangle$, which does not lie in one cyclic subgroup. Define

$$\phi_1 : \begin{cases} x_i \mapsto x_i, & i = 1, 2 \\ x_3 \mapsto x_3 a_1, \end{cases}$$

$$\phi_2 : \begin{cases} x_i \mapsto x_i, & i = 1, 2 \\ x_3 \mapsto x_3 a_2. \end{cases}$$

The conditions 2 can be checked straightforwardly. Then the subgroup of $Aut(F_3)$, generated by the elements $\alpha_i, \phi_j,\ i = 1, 2, 3, j = 1, 2$ is isomorphic to $\mathcal{H}(F_2)$. In particular, in the case $a_1, a_2 \in \gamma_2(F_3)$, we have the following

Theorem 5. *The group $IA(F_3)$ contains a subgroup isomorphic to $\mathcal{H}(F_2)$, and hence, $IA(F_3)$ is not linear.*

Observe also that the poison group $\mathcal{H}(F_2)$ is residually finite. It follows from Baumslag's theorem which states that every finitely generated subgroup of an automorphism group of a residually finite group is itself residually finite. Also observe that the non-linearity of the poison group can be used for construction of other non-linear groups given by commutator relations. For example, the group $\mathcal{H}(F_2)$ contains the following normal subgroup of index 2:

$$H = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_3] = [x_2, x_4] = [x_1^{x_5}, x_3] = [x_2^{x_5}, x_4] = 1,$$
$$[x_1 x_3, x_2] = [x_2 x_4, x_1] = [x_1^{x_5} x_3, x_4] = [x_2^{x_5} x_4, x_3] = 1\rangle,$$

which is of cause non-linear.

5. Questions

(1) Describe the Lie algebra of the group $Cb_n,\ n \geq 2$.
(2) Let L_n be a free Lie algebra with n generators, $n \geq 3$. Does the group $Aut(L_n)$ contain the poison group as a subgroup?
(3) Are the groups Cb_n^+ linear for $n \geq 3$?
(4) Define the chain of subgroups

$$\mathrm{Aut}(F_n) = \mathrm{IA}_n^1 \geq \mathrm{IA}_n^2 \geq \mathrm{IA}_n^3 \geq \dots,$$

where IA_n^k, $k \geq 1$ is the subgroup of $\mathrm{Aut}(F_n)$, which consists of automorphisms acting trivially modulo the k-th term of the lower central series of F_n. This chain was introduced in [1]. For which $k \geq 3$, $n \geq 3$ the groups IA_n^k are non-linear?
(5) Do the groups Cb_n contain the poison group as a subgroup for $n \geq 3$?
(6) Is the group $\mathcal{H}(\mathbb{Z}) = \langle a, b \mid [a, [b, a]] = 1\rangle$ linear?

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References

[1] Andreadakis, S.: On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc., 15 (1965), 239–268.
[2] Lyndon R. and Schupp P.: Combinatorial group theory, *Springer-Verlag* (1977).
[3] McCool,J.: On basis–conjugating automorphisms of free groups, Can. J. Math., 38 (1986), 1525–1529.
[4] Savushkina, A.: On group of conjugating automorphisms of a free group, Mat. Zametki, 60 (1996), 92–108.

[5] Birman, J.: Braids, links and mapping class group, Princeton–Tokyo: Univ. press, 1974.
[6] Fenn, R., Rimányi, R. and Rourke, C.: The braid–permutation group, Topology, 36 (1997), 123–135.
[7] Bardakov, V.: The structure of the group of conjugating automorphisms, Algebra i Logik, 42 (2003), 515–541.
[8] Crowell, R. and Fox, R.: Introduction to knot theory, Ginn and company, 1963.
[9] Cohen, F., Pakianathan J., Vershinin, V. and Wu, J.: Basis-conjugating automorphisms of a free group and associated Lie algebras, Preprint: arxiv math.GR/0610946.
[10] Neschadim, M.: Normal automorphisms of braid groups, Preprint RAS, Siberian branch, Novosibirsk, 1993, 19 p.
[11] Formanek, E. and Procesi, C.: The automorphism groups of a free group is not linear, J. Algebra, 149 (1992), 494–499.
[12] Bardakov, V.: Linear presentations of the conjugating automorphism groups and braid groups of some manifolds, Sib. Mat. J., 46, 1 (2005), 17–31.
[13] Kourovka Notebook: Open problems of group theory, 15-th issue, Novosibirsk, 2002.
[14] Scheuneman, J.: Two-step nilpotent Lie algebras, J. Algebra 7, (1967), 152-159.
[15] Cohen, D., Cohen F. and Pakianathan J.: Centralizers of Lie algebras associated to the descending central series of certain poly-free groups, preprint arxiv: 0603470.
[16] Bigelow, S.: Braid groups are linear, J. Amer. Math. Soc. 14, (2001), 471-486.
[17] Krammer, D.: Braid groups are linear, Ann. Math. 151, (2002), 131-156.
[18] Falk M. and Randell, R.: The lower central series of a fiber-type arrangements, *Inv. Math.* 82, (1985), 77-88.