A note on Zariski pairs

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§0. Introduction

In [1], Artal Bartolo defined the notion of *Zariski pairs* as follows:

Definition. A couple of complex reduced projective plane curves C_1 and C_2 of a same degree is said to make a Zariski pair, if there exist tubular neighborhoods $T(C_i) \subset \mathbb{P}^2$ of C_i for $i = 1, 2$ such that $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are diffeomorphic, while the pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic; that is, the singularities of C_1 and C_2 are topologically equivalent, but the embeddings of C_1 and C_2 into $\mathbb{P}^2$ are not topologically equivalent.

The first example of Zariski pair was discovered and studied by Zariski in [9] and [11]. He showed that there exist projective plane curves C_1 and C_2 of degree 6 with 6 cusps and no other singularities such that $\pi_1(\mathbb{P}^2 \setminus C_1)$ and $\pi_1(\mathbb{P}^2 \setminus C_2)$ are not isomorphic. Indeed, the placement of the 6 cusps on the sextic curve has a crucial effect on the fundamental group of the complement. Let C_1 be a sextic curve defined by an equation $f^2 + g^3 = 0$, where f and g are general homogeneous polynomials of degree 3 and 2, respectively. Then C_1 has 6 cusps lying on a conic defined by $g = 0$. In [9], it was shown that $\pi_1(\mathbb{P}^2 \setminus C_1)$ is isomorphic to the free product $\mathbb{Z}/(2) * \mathbb{Z}/(3)$ of cyclic groups of order 2 and 3. On the other hand, in [11], it was proved that there exists a sextic curve C_2 with 6 cusps which are not lying on any conic, and that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C_2)$ is cyclic of order 6. In [4], Oka gave an explicit defining equation of C_2. In [1], Artal Bartolo presented a simple way to construct (C_1, C_2) from a cubic curve C by means of a Kummer covering of $\mathbb{P}^2$ of exponent 2 branched along three lines tangent to C at its points of inflection.

Except for this example, very few Zariski pairs are known ([1], [8]). In [5], and independently in [7], infinite series of Zariski pairs have been constructed from the above example of Zariski by means of covering tricks of the plane.

In this paper, we present a method to construct Zariski pairs, which yields two infinite series of new examples of Zariski pairs as special cases.

A germ of curve singularity is called of type (p, q) if it is locally defined by $x^p + y^q = 0$.

Series I. This series consists of pairs $(C_1(q), C_2(q))$ of curves of degree $3q$, where q runs through the set of integers ≥ 2 prime to 3. Each of $C_1(q)$ and $C_2(q)$ has $3q$ singular points of type $(3, q)$ and no other singularities. The fundamental group $\pi_1(\mathbb{P}^2 \setminus C_1(q))$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus C_2(q))$ is abelian. When $q = 2$, this example is nothing but the classical one of the sextic curves due to Zariski.

Series II. This series consists of pairs $(D_1(q), D_2(q))$ of curves of degree $4q$, where q runs through the set of odd integers > 2. Each of $D_1(q)$ and $D_2(q)$ has $8q$ singular points of type $(2, q)$ – that is, rational double points of type A_{q-1} – and no other singularities. The fundamental group $\pi_1(\mathbb{P}^2 \setminus D_1(q))$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus D_2(q))$ is abelian.

Our method is a generalization of Artal Bartolo's method for re-constructiong the classical example of Zariski to higher dimensions and arbitrary exponents of the Kummer covering. Indeed, when $q = 2$ in Series I, our construction coincides with his.

Instead of the computation of the first Betti number of the cyclic branched covering of $\mathbb{P}^2$, which was employed in [1], we use the fundamental groups of the complements in order to distinguish two embeddings of curves in $\mathbb{P}^2$. For the calculation of the fundamental groups, we use [6; Theorem 1] and a result of [3] and [7].

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§1. A method of constructing Zariski pairs

1.1. Non-abelian members. Let p and q be integers ≥ 2 prime to each other. We choose homogeneous polynomials $f \in H^0(\mathbb{P}^2, \mathcal{O}(pk))$ and $g \in H^0(\mathbb{P}^2, \mathcal{O}(qk))$, where k is an integer ≥ 1. Suppose that f and g are generally chosen. Consider the projective plane curve

$$C_{p,q,k} \quad : \quad f^q + g^p = 0$$

of degree pqk (cf. [2]). It is easy to see that the singular locus of this curve consists of pqk^2 points of type (p, q). In [7; Example (3) in §0], the following is shown.

Proposition 1. *The fundamental group $\pi_1(\mathbb{P}^2 \setminus C_{p,q,k})$ is isomorphic to the group $\langle\ a, b, c \mid a^p = b^q = c,\ \ c^k = 1\ \rangle$. In particular, it is non-abelian.*

See also [3], in which the fundamental groups of the complements of curves of this type are calculated. There the groups are presented in a different way.

This curve $C_{p,q,k}$ will be a member C_1 of a Zariski pair.

1.2. Abelian partners. We shall construct the other member C_2 of the Zariski pair such that $\pi_1(\mathbb{P}^2 \setminus C_2)$ is abelian.

Let p, q and k be integers as above. We put $n = pk$. Interchanging p and q if necessary, we may assume that $n \geq 3$. Let $S_0 \subset \mathbb{P}^{n-1}$ be a hypersurface of degree n defined by $F_0(X_1, \ldots, X_n) = 0$. We consider a linear pencil of hypersurfaces

$$S_t \quad : \quad F_0(X_1, \ldots, X_n) + t \cdot X_1 \cdots X_n = 0,$$

which is spanned by S_0 and $S_\infty := \{X_1 \cdots X_n = 0\}$. We put $H_i = \{X_i = 0\}$ $(i = 1, \ldots, n)$. We consider the morphism $\phi_q : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ given by

$$(Y_1 : \ldots : Y_n) \quad \longmapsto \quad (X_1 : \ldots : X_n) \ = \ (Y_1^q : \ldots : Y_n^q),$$

which is a covering of degree q^{n-1} branched along S_∞.

Proposition 2. *Suppose that (1) every member S_t is reduced, and that (2) S_0 contains none of the hyperplanes H_i. Then $\pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t))$ is abelian for a general member S_t.*

Proof. Let $\mathbb{P}^1$ be the t-line, and we put $\mathbb{A}^1 := \mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{W} \subset \mathbb{P}^{n-1} \times \mathbb{A}^1$ be the divisor defined by

$$X_1 \cdots X_n \cdot (\ F_0(X_1, \dots, X_n) + t \cdot X_1 \cdots X_n\ ) = 0,$$

which is the union of $S_\infty \times \mathbb{A}^1$ and the universal family of the affine part $\{\ S_t\ ;\ t \in \mathbb{A}^1\}$ of the pencil. For $t \in \mathbb{A}^1$, we denote by $W_t \subset \mathbb{P}^{n-1}$ the divisor obtained from the scheme theoretic intersection $(\{t\} \times \mathbb{P}^{n-1}) \cap \mathcal{W}$, which is equal with the divisor $S_t + S_\infty$.

First, we shall show that $\pi_1(\mathbb{P}^{n-1} \setminus W_t)$ is abelian for a general t. Remark that the assumption (2) implies that S_t contains none of H_i unless $t = \infty$. Combining this with the assumption (1), we see that W_t is reduced for all $t \in \mathbb{A}^1$. Hence, by [6; Theorem 1], the inclusion $\mathbb{P}^{n-1} \setminus W_t \hookrightarrow (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}$ induces an isomorphism on the fundamental groups for a general t. Therefore, it is enough to show that $\pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W})$ is abelian. In order to prove this, we consider the first projection

$$p \quad : \quad (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W} \quad \longrightarrow \quad \mathbb{P}^{n-1} \setminus S_\infty.$$

Since $\{\ S_t\ ;\ t \in \mathbb{P}^1\}$ is a pencil whose base locus is contained in S_∞, there is a unique member $S_{t(P)}$ $(t(P) \neq \infty)$ containing P for each point $P \in \mathbb{P}^{n-1} \setminus S_\infty$. Therefore $p^{-1}(P)$ is a punctured affine line $\mathbb{A}^1 \setminus \{t(P)\}$ for every $P \in \mathbb{P}^{n-1} \setminus S_\infty$. Consequently, p is a locally trivial fiber space. Moreover, p has a section

$$s \quad : \quad \mathbb{P}^{n-1} \setminus S_\infty \quad \longrightarrow \quad (\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W},$$

which is given by, for example, $s(P) = (P,\ t(P) + 1)$. Hence the homotopy exact sequence of p splits. Combining this with the fact that the image of the injection $\pi_1(\mathbb{A}^1 \setminus \{t(P)\}) \to \pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W})$ is contained in the center, we see that

$$\pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W}) \quad \cong \quad \pi_1(\mathbb{P}^{n-1} \setminus S_\infty) \ \times\ \pi_1(\mathbb{A}^1 \setminus \{\text{ a point }\}).$$

This shows that $\pi_1((\mathbb{P}^{n-1} \times \mathbb{A}^1) \setminus \mathcal{W})$ is abelian.

Note that $\phi_q : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is étale over $\mathbb{P}^{n-1} \setminus W_t$ for every t. Hence the natural homomorphism

$$\phi_{q*} \quad : \quad \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \quad \longrightarrow \quad \pi_1(\mathbb{P}^{n-1} \setminus W_t)$$

is injective. This implies that $\pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t))$ is abelian for a general t. On the other hand, since $\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)$ is a Zariski open dense subset of $\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t)$, the inclusion induces a surjective homomorphism

$$\pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(W_t)) \quad \twoheadrightarrow \quad \pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t)).$$

Thus $\pi_1(\mathbb{P}^{n-1} \setminus \phi_q^{-1}(S_t))$ is also abelian for a general t. $\square$

Proposition 3. *Suppose the following; (3) $S_0 \cap H_i$ is a non-reduced divisor pD_i of H_i of multiplicity p, where D_i is a reduced divisor of H_i, none of whose irreducible components* *is contained in* $H_i \cap (\cup_{j\neq i} H_j)$*, and (4) the singular locus of* S_t *is of codimension* ≥ 2 *in* S_t *for a general* t*. Then the general plane section* $\mathbb{P}^2 \cap \phi_q^{-1}(S_t)$ *of* $\phi_q^{-1}(S_t)$ *is a curve of degree* pqk*, and its singular locus consists of* pqk^2 *points of type* (p, q)*.*

Proof. Note that the assumption (3) implies that $S_t \cap H_i$ is also equal with pD_i for $t \neq \infty$. Let P be a general point of any irreducible component of D_i, and let Q be a point such that $\phi_q(Q) = P$, which lies on the hyperplane defined by $Y_i = 0$. By the assumption (3), Q is not contained in any of the other hyperplanes defined by $Y_j = 0$ $(j \neq i)$. Hence there exist analytic local coordinate systems $(w_1, \dots, w_{n-1})$ and $(z_1, \dots, z_{n-1})$ of $\mathbb{P}^{n-1}$ with the origins P and Q, respectively, such that H_i is given by $w_1 = 0$, $\phi_q^{-1}(H_i)$ is given by $z_1 = 0$, and ϕ_q is given by

$$(z_1, \dots, z_{n-1}) \quad \longmapsto \quad (w_1, \dots, w_{n-1}) \;=\; (z_1^q, z_2, \dots, z_{n-1}).$$

Let $t \in \mathbb{A}^1$ be general. By the assumption (3), the defining equation of S_t at P is of the form

$$u(w) \cdot w_1 + v(w_2, \dots, w_{n-1})^p = 0.$$

By the assumption (4), S_t is non-singular at P, because P is a general point of an irreducible component of D_i. This implies that $u(P) \neq 0$. On the other hand, the divisor D_i, which is defined by $v(w_2, \dots, w_{n-1}) = 0$ on the hyperplane $H_i = \{w_1 = 0\}$, is non-singular at P, because D_i is reduced by the assumption (3) and P is general. Hence we have

$$\frac{\partial v}{\partial w_j}(P) \neq 0 \quad \text{at least for one } j \geq 2.$$

The defining equation of $\phi_q^{-1}(S_t)$ is then of the form

$$\widetilde{u}(z) \cdot z_1^q + v(z_2, \dots, z_{n-1})^p = 0, \quad \text{where} \quad \widetilde{u}(Q) \neq 0.$$

Then, it is easy to see that, in terms of suitable analytic coordinates $(\widetilde{z}_1, \dots, \widetilde{z}_{n-1})$ with the origin Q, this equation can be written as follows;

$$\widetilde{z}_1^{\,q} \;+\; \widetilde{z}_2^{\,p} \;=\; 0.$$

Thus, when we cut $\phi_q^{-1}(S_t)$ by a general 2-dimensional plane passing through Q, a germ of curve singularity of type (p, q) appears at Q.

Since the degree of D_i is $k = n/p$, the inverse image $\phi_q^{-1}(D_i)$ is a reduced hypersurface of degree qk in the hyperplane defined by $Y_i = 0$. Moreover $\phi_q^{-1}(D_i)$ and $\phi_q^{-1}(D_j)$ have no common irreducible components when $i \neq j$ because of the assumption (3). Hence the intersection points of $\phi_q^{-1}(\sum_{i=1}^n D_i)$ with a general plane $\mathbb{P}^2 \subset \mathbb{P}^n$ is pqk^2 in number. Moreover, $\mathbb{P}^2 \cap \phi_q^{-1}(S_t)$ is non-singular outside of these intersection points, because of the assumption (4). □

1.3. Summary. Suppose that we have constructed a hypersurface $S_0 \subset \mathbb{P}^{n-1}$ of degree $n \geq 3$ which satisfies the assumptions (1)-(4) in Propositions 2 and 3. Let C_2 be a general plane section of $\phi_q^{-1}(S_t)$, where t is general. Because of Zariski's hyperplane section theorem [10] and Propositions 1, 2 and 3, we see that the curve C_2 has the same type of singularities as that of $C_{p,q,k}$, but the fundamental group $\pi_1(\mathbb{P}^2 \setminus C_2)$ is abelian. Hence (C_1, C_2) is a Zariski pair, with $C_1 = C_{p,q,k}$.

§2. Construction of Series I

We carry out the construction of the previous section with $p = 3$, $k = 1$, $n = 3$ and q an arbitrary integer ≥ 2 prime to 3.

We fix a homogeneous coordinate system $(X : Y : Z)$ of $\mathbb{P}^2$, and put

$$L_1 = \{X = 0\}, \quad L_2 = \{Y = 0\}, \quad L_3 = \{Z = 0\}, \quad \text{and}$$

$$R_1 = (0 : 1 : -1) \ \in \ L_1, \quad R_2 = (1 : 0 : -1) \ \in \ L_2.$$

Let $\mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathcal{O}(3)))$ be the space of all cubic curves on $\mathbb{P}^2$, which is isomorphic to the projective space of dimension 9, and let $\mathcal{F} \subset \mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathcal{O}(3)))$ be the family of cubic curves C which satisfy the following conditions;

(a) C intersects L_1 at R_1 with multiplicity ≥ 3,
(b) C intersects L_2 at R_2 with multiplicity ≥ 3, and
(c) C intersects L_3 at a point with multiplicity ≥ 3.

(We consider that C intersects a line L_i with multiplicity ∞, if L_i is contained in C.)

Proposition 4. *The family $\mathcal{F}$ consists of 3 projective lines. They meet at one point corresponding to $C_\infty := \{XYZ = 0\}$.*

Proof. Let $F(X, Y, Z) = 0$ be the defining equation of a member C of this family $\mathcal{F}$. By the condition (a), F is of the form

$$F(X, Y, Z) = A(Y + Z)^3 + X \cdot G(X, Y, Z),$$

where A is a constant, and $G(X, Y, Z)$ is a homogeneous polynomial of degree 2. By the condition (b), we have $F(X, 0, Z) = A(Z + X)^3$, and hence we get

$$G(X, Y, Z) = A(3Z^2 + 3ZX + X^2) + Y \cdot H(X, Y, Z),$$

where $H(X, Y, Z)$ is a homogeneous polynomial of degree 1. By the condition (c), we have $F(X, Y, 0) = A(Y + \alpha X)^3$ for some α. Then α must be a cubic root of unity, and we get

$$H(X, Y, Z) = 3A\alpha^2 X + 3A\alpha Y + BZ,$$

where B is a constant. Combining all of these, we get

$$\begin{aligned}
&F(X, Y, Z) \\
=&A(X^3 + Y^3 + Z^3) + 3A(\alpha^2 X^2 Y + \alpha XY^2 + Y^2 Z + YZ^2 + Z^2 X + ZX^2) + BXYZ \\
=&A(X + Y + Z)^3 + 3A(\alpha^2 - 1)X^2 Y + 3A(\alpha - 1)XY^2 + (B - 6A)XYZ.
\end{aligned}$$

This curve $C = \{F = 0\}$ intersects L_3 at

$$R_3 = R_3(\alpha) := (1 : -\alpha : 0) \in L_3$$

with multiplicity ≥ 3. This means that the family $\mathcal{F}$ consists of three lines $\mathcal{L}(1)$, $\mathcal{L}(\omega)$ and $\mathcal{L}(\omega^2)$ in the projective space $\mathbb{P}_*(\Gamma(\mathbb{P}^2, \mathcal{O}(3)))$, where $\omega = \exp(2\pi i/3)$, such that a general cubic C in $\mathcal{L}(\alpha)$ intersects L_3 at $R_3(\alpha)$ with multiplicity 3. The ratio of the coefficients $t := B/A$ gives an affine coordinate on each line $\mathcal{L}(\alpha)$. The three lines $\mathcal{L}(1)$, $\mathcal{L}(\omega)$, $\mathcal{L}(\omega^2)$ intersect at one point $t = \infty$ corresponding to the cubic $C_\infty = L_1 + L_2 + L_3$. □

Hence we get three pencils of cubic curves $\{\ C(1)_t\ ;\ t \in \mathcal{L}(1)\}$, $\{\ C(\omega)_t\ ;\ t \in \mathcal{L}(\omega)\}$, and $\{\ C(\omega^2)_t\ ;\ t \in \mathcal{L}(\omega^2)\}$. It is easy to check that these pencils satisfy the assumptions (2), (3) and (4) in the previous section. Note that the pencil $\mathcal{L}(1)$ does not satisfy the assumption (1) because $C(1)_6$ is a triple line. However, the other two satisfy (1). Indeed, if a cubic curve C in the family $\mathcal{F}$ is non-reduced, then the conditions (a)-(c) imply that it must be a triple line. Therefore the three points R_1, R_2 and $R_3(\alpha)$ are co-linear, which is equivalent to $\alpha = 1$. Consequently, C must be a member of $\mathcal{L}(1)$.

Now, by using the pencil $\mathcal{L}(\omega)$ or $\mathcal{L}(\omega^2)$, we complete the construction of Series I.

Note that, if $C(1)_a$ is a non-singular member of $\mathcal{L}(1)$, then $\pi_1(\mathbb{P}^2 \setminus \phi_q^{-1}(C(1)_a))$ is isomorphic to the free product $\mathbb{Z}/(3) * \mathbb{Z}/(q)$. Indeed, since $C(1)_a$ is defined by

$$(X + Y + Z)^3 + (a - 6)XYZ = 0,$$

the pull-back $\phi_q^{-1}(C(1)_a)$ is defined by

$$(U^q + V^q + W^q)^3 + (a - 6)(UVW)^q = 0,$$

which is of the form $\widetilde{f}^{\,3} + \widetilde{g}^{\,q} = 0$. The polynomials $\widetilde{f}$ and $\widetilde{g}$ are not general by any means. However, since the type of singularities of $\phi_q^{-1}(C(1)_a)$ is the same as that of $C_{3,q,1}$, we have an isomorphism $\pi_1(\mathbb{P}^2 \setminus \phi_q^{-1}(C(1)_a)) \cong \pi_1(\mathbb{P}^2 \setminus C_{3,q,1})$.

§3. Construction of Series II

It is enough to show the following:

Proposition 5. *The quartic surface*

$$S_0 \quad : \quad F_0(x_1, x_2, x_3, x_4) \ := \ (x_1^2 + x_2^2)^2 + 2x_3x_4(x_1^2 - x_2^2) + x_3^2x_4^2 \ = \ 0$$

in $\mathbb{P}^3$ satisfies the assumptions (1)-(4) with $p = 2$ and $k = 2$.

Proof. The assumptions (2) and (3) can be trivially checked. To check the assumptions (1) and (4), we put

$$F_t \ := \ F_0 + t \cdot x_1x_2x_3x_4,$$

and calculate the partial derivatives $\partial F_t/\partial x_i$ for $i = 1, \ldots, 4$. Let $Q_t \subset \mathbb{P}^3$ be the quadric surface defined by

$$2x_1^2 - 2x_2^2 + 2x_3x_4 + tx_1x_2 \ = \ 0.$$

It is easy to see that Q_t is irreducible for all $t \neq \infty$. It is also easy to see that Q_t is the unique common irreducible component of the two cubic surfaces

$$\frac{\partial F_t}{\partial x_3} = 0, \qquad \text{and} \qquad \frac{\partial F_t}{\partial x_4} = 0.$$

Suppose that a surface $S_a = \{F_a = 0\}$ in this pencil contains a non-reduced irreducible component mT ($m \geq 2$). Then, both of $\partial F_a / \partial x_3$ and $\partial F_a / \partial x_4$ must vanish on T. Hence T must coincide with Q_a, and we get $S_a = 2Q_a$. Comparing the defining equations of S_a and $2Q_a$, we see that there are no such a. Thus the assumption (1) is satisfied. To check the assumption (4), we remark that the condition $\dim \operatorname{Sing} S_t \leq 0$ is an open condition for t. Hence it is enough to prove, for example, $\dim \operatorname{Sing} S_2 = 0$. It is easy to show that $\operatorname{Sing} S_2$ consists of four points $(1 : \pm\sqrt{-1} : 0 : 0)$, $(0 : 0 : 0 : 1)$ and $(0 : 0 : 1 : 0)$. $\square$

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