WEIL LINEAR SYSTEMS ON SINGULAR K3 SURFACES

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§ 0. Introduction.

We recall that K3 surface is a smooth projective algebraic surface X over an algebraically closed field k with $K_X=0$ and $H^1(X,\mathcal{O}_X)=0$.

A normal projective algebraic surface Y is a singular K3 surface if for the minimal resolution of singularities $\sigma:X\longrightarrow Y$ the nonsingular surface X is a K3 one. In this case, all singularities of Y are Du Val singularities $\mathbf{A}_m$, $\mathbf{D}_m$, $\mathbf{E}_m$, and we get Y if we blow down trees of nonsingular rational curves of the type $\mathbf{A}_m$, $\mathbf{D}_m$, $\mathbf{E}_m$ on X.

V.A.Alekseev asked me: what one can say about a complete ample linear system $|\bar{D}|$ of integral Weil divisors $\bar{D}$ on singular K3 surface Y. For example, what one can say about the fixed part of the linear system, multiplicities of fixed components with respect to $\bar{D}^2$, $\dim|\bar{D}|$?

This problem is very important maybe from the viewpoint of a classification of Fano threefolds F with $\mathbb{Q}$-factorial terminal singularities. If the linear system $|-K_F|$ has a good member $Y\in|-K_F|$ then, by the adjunction formula, Y is a singular K3 surface and the restriction of the linear system $|-K_F|$ on Y is a complete ample linear system of Y. Thus, we can reduce a description of the $|-K_F|$ to a linear system on the surface Y. And a classification of Fano threefolds is very closely related with a description of linear systems on K3 surfaces.

Unfortunately, it is not proved yet that this good member does exist. Recently, V.A.Alekseev got some results in this direction, and it was the reason why he asked me about. On the other hand, as I think, one can consider results about linear systems on singular K3 surfaces as a good model for the system $|-K_F|$ on Fano threefolds with terminal singularities and can try to generalize these results for Fano threefolds with terminal singularities.

It was very strange to me that I did not see in literature some results devoted to linear systems on singular K3 surfaces. Except, of course, Saint-Donat's paper [S-D] devoted to nonsingular ones. It is required to construct some theory devoted to this problem.

At first, me and a little later Alekseev considered the case when rk Pic Y=1 (see § 3, 3.3 here). It was solved by different methods. I worked with nonsingular K3 surface X, and Alekseev used Riemann-Roch for singular K3 surface Y. Later, I considered the general case when rk Pic Y is arbitrary. The last case is much more complicated, and we will consider this case here. On the other hand, for the case rk Pic Y=1 we have a very precise answer. For an arbitrary rk Pic Y, we have a theory only. Using this theory, one can get the full description of all cases in principle.

At last, we recall some results about linear systems on nonsingular K3 surfaces. Here we have the

Proposition 0.1. *Let* $H\in\mathrm{Pic}\ X$ *is* nef. *Then one of the cases* (i)-(iv) *below holds:*

(i) $H^2>0$, $|H|$ *contains an irreducible curve and has not fixed points,* $\dim|H|=H^2/2+1>0$;

(ii) $H^2=0$, $|H|=m|E|$, $m>0$, *where* $|E|$ *is an elliptic pencil* ($|H|$ *contains an irreducible curve for* $m=1$ *only*).

(iii) $H=0$, $|H|=\emptyset$.

(iv) $H^2>0$ *and* $|H|=m|E|+\Gamma$, $m>1$, *where* $|E|$ *is an elliptic pencil,* Γ *is an irreducible curve with* $\Gamma^2=-2$, *and* $E\cdot\Gamma=1$. *Here* $m=\dim|H|=H^2/2+1$, Γ *is the fixed part of* $|H|$.

Proof. It is well known to specialists and follows very easy from [S-D]. We will give a proof.

Let $H\neq 0$. Since H is nef, $H^2\geq 0$. Then, by Riemann-Roch theorem, $\dim|H|>0$. Let $|C|$ be the moving part of $|H|$ and Δ the fixed part. By [S-D], (i), or (ii) holds for $|C|$.

At first, let $|C|$ contains an irreducible curve C. By Riemann-Roch theorem, $(C+\Delta)^2\leq C^2$. Thus, $\Delta\cdot(2C+\Delta)\leq 0$. It follows $\Delta\cdot(C+\Delta)+\Delta\cdot C\leq 0$. Since $C+\Delta$ and C are nef, $\Delta\cdot C=\Delta\cdot(C+\Delta)=0$. Then $\Delta^2=0$. If $\Delta=0$, we get the case (i). If $\Delta\neq 0$, by Riemann-Roch theorem, $\dim|\Delta|\geq 1$, and we get the contradiction.

Let $|C|=m|E|$ where $|E|$ is an elliptic pencil. By Riemann-Roch theorem, $(mE+\Delta)^2/2+1\leq m$. Thus, $(mE+\Delta)\cdot\Delta+mE\cdot\Delta\leq 2m-2$. Since $mE+\Delta$ is nef, either $E\cdot\Delta=0$ or $E\cdot\Delta=1$ and $\Delta^2\leq -2$. We consider these possibilities.

Let $E\cdot\Delta=0$. By Hodge index theorem, $\Delta^2\leq 0$. Since $E+\Delta$ is nef, $\Delta^2=0$. If $\Delta=0$, we get the case (ii). If $\Delta\neq 0$, we get the contradiction since $\dim|\Delta|\geq 1$.

Let $E\cdot\Delta=1$ and $\Delta^2\leq -2$. Then $\Delta=\Gamma+\Delta'$ where Γ is an irreducible curve with $\Gamma^2=-2$, and $E\cdot\Gamma=1$, and $E\cdot\Delta'=0$, and Γ is not a component of the divisor Δ'. If $\Delta'=0$, we get (iv). Let $\Delta'\neq 0$. By Hodge index theorem and Riemann-Roch theorem, $(\Delta')^2<0$ and $(\Gamma+\Delta')^2=-2+2\Gamma\cdot\Delta'+(\Delta')^2<0$. Since Picard lattice of K3 surface is even, $2\Gamma\cdot\Delta'+(\Delta')^2=(\Gamma+\Delta')\cdot\Delta'+\Gamma\cdot\Delta'\le 0$. Since $C+\Gamma+\Delta'$ is nef and $C\cdot\Delta'=0$, $(\Gamma+\Delta')\cdot\Delta'\ge 0$. Since Γ is not a component of Δ', $\Gamma\cdot\Delta'\ge 0$. It follows $(\Gamma+\Delta')\cdot\Delta'=\Gamma\cdot\Delta'=0$. Thus, $(\Delta')^2=0$. We get the contradiction. ∎

We want to get something similar for singular K3 surfaces. On the other hand, the Proposition 0.1 will be very important for us in the case of singular K3 surfaces also.

§ 1 Fixed part of linear system on nonsingular K3 surfaces.

Let X be a nonsingular K3 surface, H an effective divisor on X and $|H|$ the corresponding complete linear system. Let $|H|=|C|+\Delta$, where $|C|$ is the moving part and Δ is the fixed part of $|H|$. What one can say about the $|C|$ and Δ?

From the Proposition 0.1, it follows the following statement:

(*) $|C|$ satisfies the condition (i), (ii), or (iii) of the Proposition 0.1, and $\Delta=\sum k_i\Gamma_i$, where any Γ_i is an irreducible -2 curve and $k_i\in\mathbb{N}$. If $|C|=m|E|$ where E is an elliptic curve and $m\ge 2$ then there does not exist more than one irreducible component Γ_i of Δ such that $E\cdot\Gamma_i\ge 1$; if here $m\ge 4$, then the multiplicity k_i of the Γ_i is $k_i=1$.

Our question is: If (*) holds, when

$$|C+\Delta|=|C|+\Delta? \qquad (1.1)$$

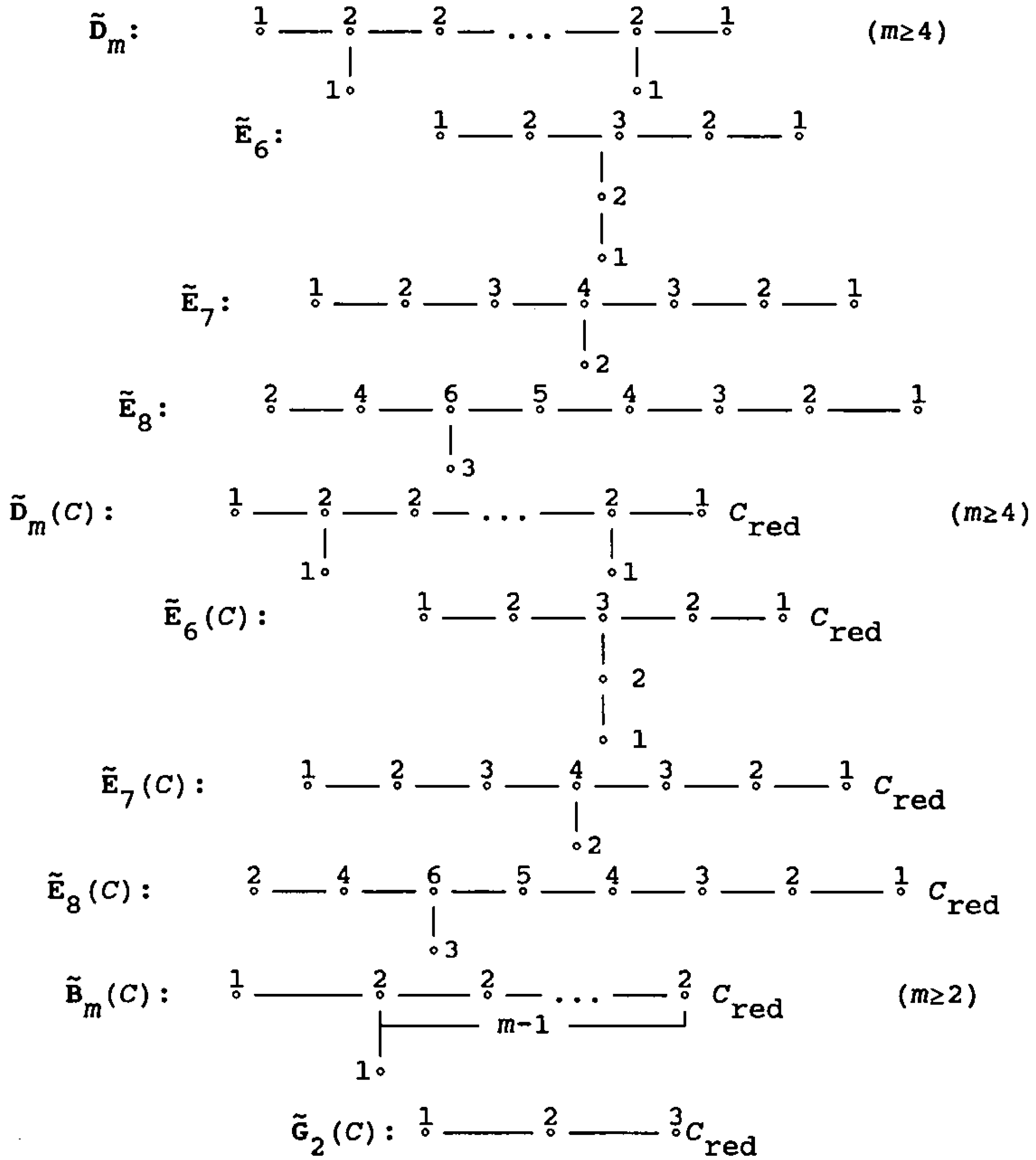
We correspond to this situation a graph $G(C,\Delta)$ (and $G(\Delta)$) by the obvious way. The $G(C,\Delta)$ is the dual graph of intersections of the irreducible components C and Γ_i of $C+\Delta$. Here C is a general member of $|C|$ if $C^2>0$, and $C=mE$ where E is a general member of the pencil $|E|$ if $C^2=0$ and $|C|=m|E|$. The weight of the vertex C is equal to C^2, the weight of the vertex Γ_i is equal to -2. The multiplicity of C is equal to 1 if $C^2>0$, and is equal to m if $|C|=m|E|$ where $|E|$ is an elliptic pencil; the multiplicity of Γ_i is equal to k_i. For the case (i) let $C_{red}=C$, and for the case (ii) $C_{red}=E$ where $|C|=m|E|$. We denote by $\circ$ a vertex of the weight -2, and by $C\circ$ (or $C^2\circ$) a vertex of the weight C^2.

The question is: What are graphs of this kind possible? It is obvious that if $G(C,\Delta)$ is possible (has the property (1.1)) then an every subgraph of $G(C,\Delta)$ is possible. Here a subgraph corresponds to a divisor D such that $0\le D\le C+\Delta$.

We prove the following basic theorem.

Theorem 1.1. *Let C and Δ are divisors on the nonsingular K3 surface X which satisfy the condition (*) above.*

Then $|C+\Delta|=|C|+\Delta$ if and only if $G(C,\Delta)$ is a tree (particularly, all components of $C+\Delta$ are intersected transversely in no more than one point) and $G(C,\Delta)$ has no subtrees $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_m(C)$ or $\tilde{G}_2(C)$ below:



<u>Proof</u>. By the Proposition 0.1, these conditions are necessary: The divisors corresponding to subgraphs $\tilde{A}_m$, $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{A}_m(C)$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_m(C)$, $\tilde{G}_2(C)$ are nef.

Let us prove the inverse statement which is much more difficult.

If $\Delta=0$, the statement is trivial. If $|C|=m|E|$, where $|E|$ is an elliptic pencil, $m\geq 2$ and $\Delta=\Gamma$ is an irreducible curve, then the statement holds by the Proposition 0.1 and the condition (*).

Let $\Delta\neq\varnothing$ and Δ is not an irreducible curve if $|C|=m|E|$, $m\geq 2$ and $|E|$ is an elliptic pencil. Let $G(C,\Delta)$ be a tree and it has no subtrees $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_m(C)$ or $\tilde{G}_2(C)$. We will show that then there exists an irreducible component Γ_i of Δ such that $\Gamma_i\cdot(C+\Delta)<0$. It follows the Theorem. Indeed, then Γ_i is a fixed component of $|C+\Delta|$, and the conditions of the Theorem hold for $C+(\Delta-\Gamma_i)$. Thus, we shall obtain the Theorem by the induction and the Proposition 0.1.

In such a way, we must prove that there exists an irreducible component Γ_i of Δ such that $\Gamma_i\cdot(C+\Delta)<0$. If it is not true, then the divisor $C+\Delta$ is nef. In this case we call the tree $G(C,\Delta)$ nef also. To prove the Theorem, we have to show that, if the tree $G(C,\Delta)$ is nef, then the tree $G(C,\Delta)$ contains one of subtrees $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_m(C)$ or $\tilde{G}_2(C)$. We can reformulate this by the following way. We say that the nontrivial nef tree $G(C,\Delta)$ is minimal if it has no nontrivial nef subtrees (Here, the nef tree is called trivial if it corresponds to the divisors C, or kE, or $kE+\Gamma$ where $k\geq 2$, or 0.) We must show that an every nontrivial minimal nef tree is one of the trees $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, $\tilde{B}_m(C)$ or $\tilde{G}_2(C)$. In such a way, we have to obtain the classification of nef minimal trees.

Let $G(C,\Delta)$ be a nontrivial minimal nef tree. Evidently, then trees $G(C,\Delta)$ and $G(\Delta)$ are connected.

Since $G(C,\Delta)$ is a tree, it has at least two ends. Thus, there exists a terminal vertex v_1 of $G(C,\Delta)$ with the weight -2.

Let $G(C,\Delta)$ be a chain of vertices $v_1,v_2,\dots,v_m$ and $k_1,k_2,\dots,k_m$ are their multiplicities. Then the chain of multiplicities $0,k_1,k_2,\dots,k_m$ is convex below, and, if the vertex v_m has the weight -2, the chain $0,k_1,k_2,\dots,k_m,0$ is convex below also. Here, the chain $0,k_1,k_2,\dots,k_m$ is convex below if $k_i-k_{i-1}+k_i-k_{i+1}\leq 0$ for $1\leq i\leq m-1$. It follows that the vertex v_m has the weight ≥ 0 (thus, we have a case (i) or (ii)) and the chain of multiplicities $0,k_1,k_2,\dots,k_m$ is strongly increased. It follows very easy that $m=3$, $k_1=1$, $k_2=2$, $k_3=3$ and the vertex $v_3=E$ where $|E|$ is an elliptic pencil ($G(C,\Delta)$ is nontrivial minimal nef!). Thus, $G(C,\Delta)$ is the tree $\tilde{G}_2(C)$.

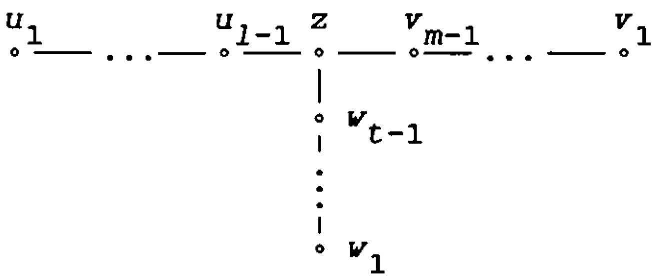
We recall that the valence of a vertex v of a tree is the number of edges of the tree which come out from v. Suppose that $G(C,\Delta)$ is not a chain. Then we can suppose that the chain $v_1, v_2, \ldots, v_m$ consists of vertices $v_2, \ldots, v_{m-1}$ of the valence 2, and the v_m has a valence ≥ 3. For the cases (i) and (ii), the vertex C is a terminal vertex of the tree $G(C,\Delta)$ since the tree $G(\Delta)$ is connected. Thus, the vertex v_m has the weight -2. The multiplicity k_m of the v_m is ≥ 2 since the chain of multiplicities $0, k_1, \ldots k_m$ is increased.

If the vertex v_m has the valence ≥ 4, then $G(C,\Delta)$ contains a subtree of type $\tilde{D}_4$ or $\tilde{D}_4(C)$ with the vertex v_m of the subtree of the valence 4. It follows that $G(C,\Delta)$ is this subtree, since $G(C,\Delta)$ is a minimal nef tree.

Thus, further, we can suppose that v_m has the valence 3. Let $\alpha_1, \alpha_2, \ldots \alpha_n = v_m$ and $\beta_1, \beta_2, \ldots, \beta_p = v_m$ be two other chains of vertices of $G(C,\Delta)$ which are different from the chain $v_1, v_2, \ldots, v_m$ and come out from v_m. Here we suppose that the valence of $\alpha_2, \ldots \alpha_{n-1}$ and $\beta_2, \ldots, \beta_{p-1}$ is 2 and the vertices a_1 and β_1 have valence 1 or ≥ 3.

Suppose that the vertex α_1 has the valence ≥ 3. Let $t_1, \ldots, t_n = k_m \geq 2$ are multiplicities of $\alpha_1, \alpha_2, \ldots, \alpha_n$. In this case, if all multiplicities $t_1, t_2, \ldots, t_n$ are strongly greater than 1, the tree $G(C,\Delta)$ contains a subtree $\tilde{D}_{n+2}$ or $\tilde{D}_{n+2}(C)$ with the vertices α_1 and v_m of the valence 3 in this subtree. Then $G(C,\Delta)$ is coincided with this subtree. Thus, we can suppose that there exists $i \geq 1$ such that $t_i = 1$ and all $t_{i+1}, \ldots, t_n = k_m$ are strongly greater than 1.

It follows that we can find nef subtree T of $G(C,\Delta)$ with vertices z, $u_1, \ldots, u_{l-1}$, $v_1, \ldots, v_{m-1}$, $w_1, \ldots, w_{t-1}$ and with the form

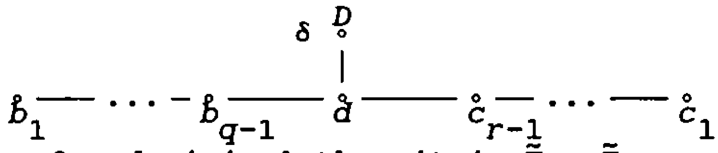


where $l \geq 2$, $m \geq 2$, $t \geq 2$.

To get this tree, one should set up $(u_1, \ldots u_{l-1}) = (\alpha_1, \ldots, \alpha_{n-1})$ if α_1 has the valence 1, and $(u_1, \ldots u_{l-1}) = (\alpha_i, \ldots, \alpha_{n-1})$ if α_1 has the valence ≥ 3. By the same way, one gets the chain $w_1, \ldots, w_{m-1}$ using the chain $\beta_1, \ldots, \beta_p$. Since $G(C,\Delta)$ is minimal, $G(C,\Delta) = T$. We should prove that $G(C,\Delta) = \tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$, or $\tilde{B}_m(C)$. We prove it in the Lemmas below. We denote by D the curve C in the case (i), the curve E in the case (ii), and one of the terminal vertices of $G(C,\Delta) = G(\Delta)$ in the case (iii). We denote by δ the multiplicity of D.

Thus, $\delta=1$ in the case (i), $|C|=\delta|E|$ in the case (ii) where $|E|$ is an elliptic pencil, δ is equal to the multiplicity of D in the case (iii). Indexes near vertices on pictures are multiplicities of the vertices.

Lemma 1. *If a tree* T *of the form*



where $2\le q\le r$, *is* nef *and* minimal *then it is* $\tilde{E}_7$, $\tilde{E}_8$, *or* $\tilde{B}_3(C)$.

Proof. The chains $0,b_1,\dots,b_{q-1},d$ and $0,c_1,\dots,c_{r-1},d$ are convex below and $\delta\ge d-b_{q-1}+d-c_{r-1}$ where $d-b_{q-1}\ge1$ and $d-c_{r-1}\ge1$. It follows that $\delta\ge2$ and $d\ge2$. Thus, we have the case (ii) or (iii).

Let us consider the case (ii). Since $\delta\ge2$, $d\ge2$ and $2\le q\le r$, the tree T contains the subtree $\tilde{B}_3(C)$. It follows $T=\tilde{B}_3(C)$, since T is minimal.

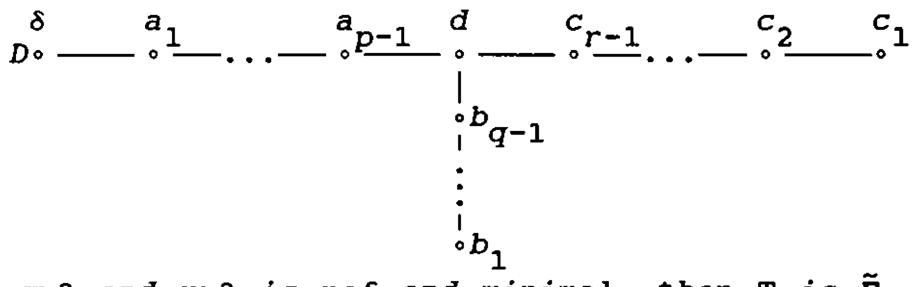
Let us consider the case (iii). Then the chain $0,\delta,d$ is also convex below. It follows that we have an inequality

$$d/2\ge\delta\ge d/q+d/r. \qquad (1.2)$$

It follows that $3\le q\le r$. Let $q=3$. From (1.2), $1/2\ge1/3+1/r$ and $r\ge6$. It follows $c_i\ge i$ and $d\ge r\ge6$ since the chain $0,c_1,\dots,c_{r-1},d$ is convex below. From (1.2), then $\delta\ge3$. If $b_1=1$, then $d/2\ge\delta\ge(d-1)/2+d/r$. It follows, $d/r\le1/2$. We obtain the contradiction, since $d\ge r$. Thus, $b_1\ge2$. Since the chain $0,b_1,b_2,d$ is convex below, $b_2\ge4$. As a result, we prove that T contains a subtree $\tilde{E}_8$. Then $T=\tilde{E}_8$.

Suppose $4\le q\le r$. Since $\delta\ge2$ and the chains $0,b_1,\dots,b_{q-1},d$ and $0,c_1,\dots,c_{r-1},d$ are convex below, T contains a subtree $\tilde{E}_7$. Then $T=\tilde{E}_7$. ■

Lemma 2. *If a tree* T *of the form*



where $p\ge1$, $q\ge3$ *and* $r\ge3$ *is* nef *and* minimal, *then* T *is* $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ *or* $\tilde{E}_6(C)$.

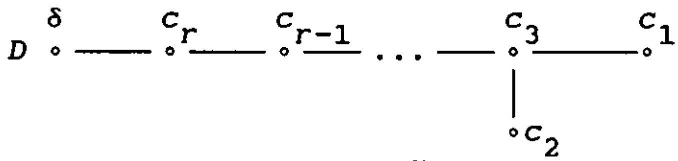
Proof. Let us use an induction by p. For $p=1$ it was proved in the Lemma 1 that T is $\tilde{E}_7$ or $\tilde{E}_8$.

Now suppose that $p\ge2$.

At first, suppose that $a_1=1$. Then, evidently, $D^2\ge0$ and we have the case (i) or (ii). Let us set up $C'=C+\Gamma_1$ where Γ_1 is the component corresponding to the vertex with the multiplicity a_1 and the weight of C' is 1. Then we get the statement by induction: The case $p=2$ is impossible; if $p\geq 3$ then $p=3$ and T is $\tilde{E}_6(C')$. It follows that T contains the subtree $\tilde{E}_6$ and $T=\tilde{E}_6$. We get the contradiction.

Let $a_1\geq 2$. The chains $\delta,a_1,\dots,a_{p-1},d$, and $0,b_1,\dots,b_{q-1},d$, and $0,c_1,\dots,c_{r-1},d$ are convex below. Since $q\geq 3$ and $r\geq 3$, we get $d\geq 3$, $b_{q-1}\geq 2$, and $c_{r-1}\geq 2$. If $\delta=1$, then also $a_{p-1}\geq 2$, since $a_1\geq 2$. It follows $T=\tilde{E}_6$ or $\tilde{E}_6(C)$. If $\delta=2$, then the chain $\delta=2,a_1,\dots,a_{p-1},d$ is increased (may be not strongly), since $a_1\geq 2$. It follows T contains the subtree $\tilde{B}_{p+2}(C)$. Then $T=\tilde{B}_{p+2}(C)$, and we get the contradiction, since $q\geq 3$ and $r\geq 3$. If $\delta\geq 3$, then T contains the subtree $\tilde{G}_2(C)$ since $a_1\geq 2$ and $p\geq 2$. Then $T=\tilde{G}_2(C)$, and we get the contradiction, since $q\geq 3$ and $r\geq 3$. ∎

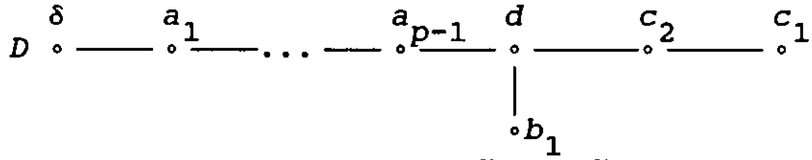
Lemma 3. *If a tree* T *of the form*



where $r\geq 4$, is nef *and minimal, then* $T=\tilde{B}_r(C)$.

Proof. The chains $0,c_1,c_3$, and $0,c_2,c_3$, and $\delta,c_r,\dots,c_3$ are convex below and $c_4\geq c_3-c_1+c_3-c_2$. It follows $c_4\geq c_3\geq 2$, and the chain $\delta,c_r,\dots,c_3$ is decreased. It follows, $D^2=0$ and we have the case (ii). Then T contains the subtree $\tilde{B}_r(C)$, hence $T=\tilde{B}_r(C)$. ∎

Lemma 4. *If a tree* T *of the form*



where $p\geq 1$, is nef *and minimal, then* $T=\tilde{E}_8$ *or* $\tilde{E}_8(C)$.

Proof. The case (i). Then $\delta=1$. The chains $1,a_1,\dots,a_{p-1},d$, and $0,b_1,d$, and $0,c_1,c_2,d$ are convex below, and $b_1+c_2+a_{p-1}\geq 2d$. It follows that $d/2+2d/3+1+(d-1)(p-1)/p\geq 2d$. Thus, $d(2-1/2-2/3-(p-1)/p)\leq$ $\leq 1-(p-1)/p$. Or $d(1/p-1/6)\leq 1/p$. Evidently, $d\geq 3$. It follows, $p\geq 4$. If $p=4$, we get $d/12\leq 1/4$. It follows $d=3$. One can see very easy that this case is impossible. If $p=5$, we get $d/30\leq 1/5$. If follows that $d\leq 6$. One can see very easy, that then $d=6$ and $T=\tilde{E}_8(C)$.

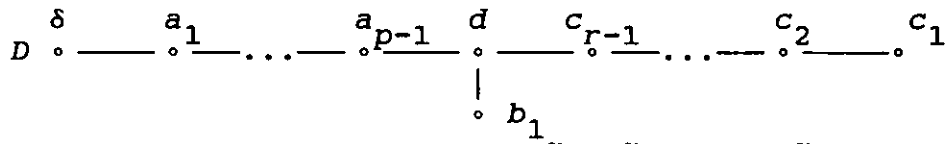
Let us suppose that $p\geq 6$. If $a_1=1$, we set up $C_1=C+\Gamma_1$ where Γ_1 corresponds to the vertex with the multiplicity a_1, and this case is reduced to the case $p-1$: we obtain that $p=6$ and $T=\tilde{E}_8(C_1)$. Then T contains $\tilde{E}_8$ and $T=\tilde{E}_8$. We get the contradiction. If $a_1\geq 2$, then $d\geq p+1\geq 7$

and $pd/(p+1)+2d/3+b_1\geq 2d$. Or $d(1/3+1/(p+1))\leq b_1$. Since $d\geq 7$, we get $b_1\geq 3$. If $c_1=1$, we get $pd/(p+1)+d/2+1+(d-1)/2\geq 2d$. It follows $d/(p+1)\leq 1/2$. This is impossible since $d\geq p+1$. It proves that T contains the subtree $\tilde{E}_8$ and $T=\tilde{E}_8$. We get the contradiction.

The case (ii). If $a_1=1$, we set up $C_1=C+\Gamma_1$ (like above). It reduces the case to the previous one, and we get that T contains the subtree $\tilde{E}_8$. Particularly, it holds if $\delta\geq 4$. Let $a_1\geq 2$. If $\delta=3$, then T contains the subtree $\tilde{G}_2(C)$ and $T=\tilde{G}_2(C)$. We get the contradiction. If $\delta=2$, we get that the chain $\delta,a_1,\ldots,a_{p-1},d$ is increased since $a_1\geq 2$. It follows that T contains the subtree $\tilde{D}_{p+1}(C)$, and $T=\tilde{D}_{p+1}(C)$. We get the contradiction. If $\delta=1$, the proof is the same as for the case (i).

The case (iii). Then the chain $0,a_1,\ldots,a_{p-1},d$ is convex below, and the proof is similar to the case (i). ∎

Lemma 5. *If a tree* T *of the form*



where $r\geq 4$, *is* nef *and minimal, then* $T=\tilde{E}_7$, $\tilde{E}_7(C)$ *or* $\tilde{E}_8$.

Proof. The case (i).

If $p=1$, we get the statement from the Lemma 1.

Let $p=2$. Then $d-b_1\geq d/2$, and $d-a_1\geq (d-1)/2$, and we get $c_{r-1}\geq d-b_1+d-a_1\geq d-1/2$. It follows that $c_{r-1}\geq d$. We get the contradiction since the chain $0,c_1,\ldots,c_{r-1},d$ is convex below.

Let $p\geq 3$. If $a_1=1$, then we reduce the case to the case $p-1$ like above. Let $a_1\geq 2$. Then $d\geq p+1$, $a_i\geq 1+i$, and $d\geq r$, $c_i\geq i$. Let $b_1=1$. Then $dp/(p+1)+d(r-1)/r+1\geq 2d$. It follows, $d(2-p/(p+1)-(r-1)/r)\leq 1$. Or $d(1/(p+1)+1/r)\leq 1$. But $d\geq p+1$ and $d\geq r$. We get the contradiction. Thus, $b_1\geq 2$. It follows, T contains the subtree $\tilde{E}_7(C)$.

The case (ii). The proof is the same as for the Lemma 4.

The case (iii). We have the inequality $2d\leq d(p/(p+1)+1/2+(r-1)/r)$. Thus, $1/(p+1)+1/r\leq 1/2$ and $p\geq 2$. The case $p=2$ follows from the Lemma 4. Let $p\geq 3$. Then $\delta\geq 1$, $a_i\geq i+1$, $d\geq 4$, $c_j\geq j$.

Let $b_1=1$. Then $d/(p+1)+d/r\leq 1$. But $d\geq p+1$ and $d\geq r$. We get the contradiction, and $b_1\geq 2$.

As a result, we proved that T contains a subtree $\tilde{E}_7$. Then $T=\tilde{E}_7$. It finishes the proof of the Lemma and the Theorem 1.1. ∎

The basic Theorem 1.1 reduces a description of all possible graphs $G(C,\Delta)$ with the condition (1.1) to a description of nonsingular curves trees $\mathcal{T}$ on K3 surfaces which satisfy the condition

(*)′ $\mathcal{T}$ does not contain more than one curve C with a square $C^2 \geq 0$ (if $\mathcal{T}$ has not such a curve, we set up $C=0$); all other curves Γ_i, $i \in I$, of the $\mathcal{T}$ are nonsingular rational.

To obtain all possible graphs $G(C,\Delta)$, one should prescribe to the curves C and Γ_i of the trees $\mathcal{T}$ multiplicities m and k_i such that the condition (*) holds, and prove the condition of the Theorem 1.1. Here any tree $\mathcal{T}$ is possible if these multiplicities are equal to one:

Corollary 1.2. *If $\mathcal{T}$ is a tree satisfying to the condition* (*)′, *then for the divisor* $\Delta = \sum_{i \in I} \Gamma_i$ *holds that* $|C+\Delta| = |C| + \Delta$.

Proof. This follows from the theorem 1.1, or one can prove it independently (consider a terminal vertex with a weight -2 of $G(C)$).∎ △

§ 2. Trees of nonsingular curves on a nonsingular K3 surface.

2.1. General remarks. We consider here results on a classification of nonsingular curves trees $\mathcal{T}$ on K3 surfaces which satisfy to the condition (*)′. $G(C)$ is the graph of intersections of curves of $\mathcal{T}$ and G the graph of intersections of the curves Γ_i, $i \in I$.

To obtain this classification, we use the following reasons (I), (II), (III), (IV) below, which are purely algebraic.

(I). *Hodge index theorem:* A *tree* $G(C)$ *should not be more than hyperbolic - the corresponding intersection matrix has not more than one positive square.*

By (I), connected component G_i of G may be elliptic (with negative definite intersection matrix), parabolic (with semidefinite intersection matrix), and hyperbolic (with hyperbolic intersection matrix).

Proposition 2.1.1. (1) *An elliptic connected component of* G *is a tree* $\mathbf{A}_m, \mathbf{D}_m, \mathbf{E}_6, \mathbf{E}_7$ *or* $\mathbf{E}_8$.

(2) *A parabolic connected component of* G *is a tree* $\tilde{\mathbf{D}}_m, \tilde{\mathbf{E}}_6, \tilde{\mathbf{E}}_7$ *or* $\tilde{\mathbf{E}}_8$.

(3) *A hyperbolic connected component* G_{hyp} *of* G *is unique.*

(4) *If* $G_{\mathrm{hyp}} \neq \emptyset$, *then all other components of* G *are elliptic.*

(5) *If* $C^2 \geq 0$ *and* $C \neq 0$ *and* $G_{\mathrm{hyp}} \neq \emptyset$, *then* C *is joined to a vertex* Γ_{hyp} *of* G_{hyp}. *If* $C^2 > 0$ *and* G_i *is a parabolic component of* G, *then* C *is joined to a vertex* Γ_i *of* G_i.

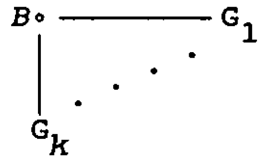
Proof. It is obvious. ∎

For the matrix M we denote by $D(M)$ the determinant of M, and $\bar{D}(M) = D(-M)$. For the subgraph T of $G(C)$ we denote by the same letter the corresponding intersection matrix. It is obvious that

$$\bar{D}(T) \quad \text{is} \begin{cases} >0 \text{ if T is elliptic,} \\ =0 \text{ if T is parabolic,} \\ \le 0 \text{ if T is hyperbolic,} \\ <0 \text{ if T is hypebolic and linearly independent.} \end{cases} \qquad (2.1)$$

We use the following simple formula: Let a tree G has a form:

G:



Let v_i be the vertex of G_i joined to B. Then

$$\bar{D}(G)=\bar{D}(G_1)\bar{D}(G_2)\cdots\bar{D}(G_k)(-B^2)-\bar{D}(G_1-v_1)\bar{D}(G_2)\cdots\bar{D}(G_k)-$$
$$-\bar{D}(G_1)\bar{D}(G_2-v_2)\bar{D}(G_3)\cdots\bar{D}(G_k)-\ldots-\bar{D}(G_1)\bar{D}(G_2)\cdots\bar{D}(G_{k-1})\bar{D}(G_k-v_k)=$$
$$=\bar{D}(G_1)\bar{D}(G_2)\cdots\bar{D}(G_k)(-B^2-\bar{D}(G_1-\{v_1\})/\bar{D}(G_1)-\ \ldots-\bar{D}(G_k-\{v_k\})/\bar{D}(G_k)). \qquad (2.2)$$

(II). *On a K3 surface, if E is an effective curve with $E^2=0$, then $C\cdot E\ge 2$ for any irreducible curve C with $C^2\ge 0$.*

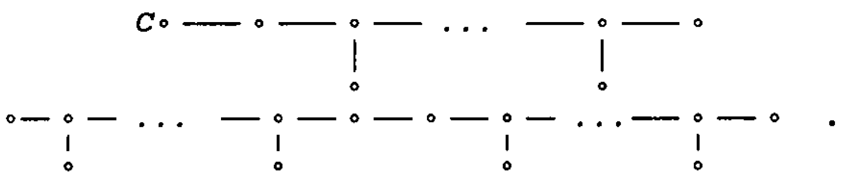
We can use (II) by the following way.

Let we have a connected parabolic subtree $\mathcal{P}$ of $G(C)$: (C), where $C^2=0$ and $C\ne 0$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. This tree corresponds to all components of an elliptic pencil fiber on a K3 surfaces. Thus, an every vertex v of $\mathcal{P}$ has the invariant $m(\mathcal{P},v)$ which is equal to the multiplicity of the corresponding to v irreducible component of the fiber. (This invariants are shown as the multiplicities of the vertices of the trees $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ of the Theorem 1.1.) By (II), we have the

Proposition 2.1.2. (1) *Let $C^2\ge 0$ and $C\ne 0$. Let $\mathcal{P}$ be a connected parabolic subtree of $G(C)$, let p be a vertex of $\mathcal{P}$ joined to C. Then* $m(\mathcal{P},p)>1$.

(2) *Let $\mathcal{P}$ and Q be two connected parabolic subtrees of $G(C)$ which have not common vertices. Let p be a vertex of $\mathcal{P}$ and q of Q and pq an edge of $G(C)$. Then either $m(\mathcal{P},p)>1$ or $m(Q,q)>1$.* ∎

For example, it follows that $G(C)$ has not subtrees (where $C^2\ge 0$):



(III). *An elliptic pencil on a K3 surface has not multiple fibers.*

It follows the

Proposition 2.1.3. *Let a tree $G(C)$ has two disjoint connected parabolic subtrees $\mathcal{P}$ and Q, and a vertex w of $G(C)-\mathcal{P}$ is joined to a* *vertex p of the $\mathcal{P}$. Then w is joined to some vertex q of the Q and*

$$m(\mathcal{P},p)=m(Q,q). \blacksquare$$

Let $\mathcal{E}=\mathbf{A}_n$, $\mathbf{D}_n$ or $\mathbf{E}_n$ be an elliptic subtree of $G(C)$ and e be a vertex of $\mathcal{E}$. We can introduce the invariant $m(\mathcal{E},e)$ which is equal to the set of multiplicities of the vertex e under all possible embeddings of $\mathcal{E}$ into all parabolic connected graphs $\tilde{\mathbf{A}}_m$, $\tilde{\mathbf{D}}_m$, $\tilde{\mathbf{E}}_m$.

Proposition 2.1.4. *Let $G(C)$ has two disjoint connected subtrees $\mathcal{P}$ and $\mathcal{E}$ where $\mathcal{P}$ is parabolic and $\mathcal{E}$ is elliptic. Let a vertex w of $G(C)-\mathcal{P}$ is joined to a a vertex p of $\mathcal{P}$ and to a vertex e of $\mathcal{E}$.*

Then $m(\mathcal{P},p)\geq\min m(\mathcal{E},e)$.

Proof. Vertices of $\mathcal{P}$ correspond to all components of a degenerate fiber of an elliptic pencil E on X. Vertices of $\mathcal{E}$ correspond to some components of an other degenerate fiber of E and also have multiplicities. By (III), we get the statement. $\blacksquare$

(IV). *The rank of Picard lattice of* K3 *surface* ≤ 22, *and it is* ≤ 20 *if a basic field has the characteristic* 0.

It follows the

Proposition 2.1.5. rk $G(C)\leq 22$, *and* rk $G(C)\leq 20$ *if* char=0.

We describe all possible trees $G(C)$ which satisfy the condition (I), the Proposition 2.1.2 of (II), the Propositions 2.1.3 and 2.1.4 of (III) and the Proposition 2.1.5 of (IV). It is a purely algebraic problem about sets of vectors in a linear space with a symmetric pairing.

2.2. The case $C^2>0$. For K3 surface $C^2\geq 2$, and we have the

Theorem 2.2.1. 1. *Let $C^2\geq 2$ and the $G_{\text{hyp}}\neq\emptyset$, let Γ_{hyp} be the vertex of the G_{hyp} joined to C. Then $G_{\text{hyp}}-\Gamma_{\text{hyp}}$ is elliptic.*

2. *Let $H_1,\dots,H_t$ be all connected components of $G_{\text{hyp}}-\Gamma_{\text{hyp}}$ and Γ_i be a vertex of H_i joined to Γ_{hyp}. Then*

$$\bar{D}(G_{\text{hyp}})=\bar{D}(H_1)\cdots\bar{D}(H_t)(2-\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)-\dots-\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t))<0$$

where

$$\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\dots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)>2.$$

3. *For $G_{\text{hyp}}(C)$ we have*

$$\bar{D}(G_{\text{hyp}}(C))=\bar{D}(H_1)\cdot\cdot\bar{D}(H_t)(-C^2(2-\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)-\dots-\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t))-1)\leq 0$$

$$2\leq C^2\leq\bar{D}(G_{\text{hyp}}-\Gamma_{\text{hyp}})/(-\bar{D}(G_{\text{hyp}}))=1/(\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\dots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)-2).$$

4. *It follows:* $2<\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\dots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)\leq 5/2.$

5.

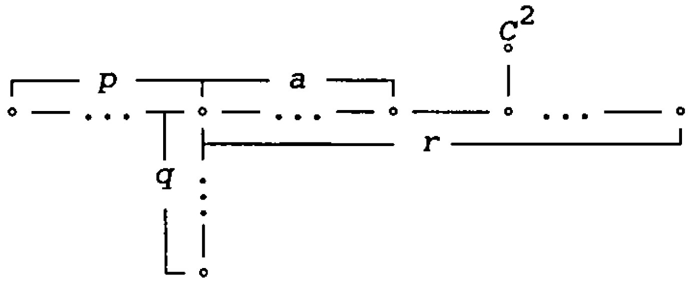
$$\operatorname{rk} G_{hyp}(C) = \begin{cases} \#G(C) \ \text{if} \ C^2<1/(\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\ldots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)-2) \\ \#G(C)-1 \ \text{if} \ C^2=1/(\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\ldots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)-2) \end{cases}.$$

6. *From 1-4 and the Proposition 2.1.2, it follows:*

If $C^2>42$ *then* $G_{hyp}=\emptyset$. *If* $G_{hyp}\neq\emptyset$, *then the tree* G_{hyp} *has three (only if* $C^2\leq 42$*), four (only if* $C^2\leq 6$*) or five (only if* $C^2=2$*) ends, and* $G_{hyp}(C)$ *is one of the following trees:*

G_{hyp} *has three ends:*

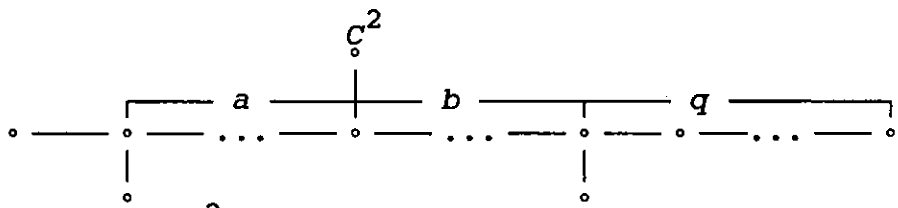
$G_1(C^2;p,q,r;a)$



where $a\geq 0$, $2\leq p\leq q\leq r$, *if* $p\geq 3$ *then* $a\leq 1$, *if* $q\geq 3$ *then* $a\leq 4$, *if* $q\geq 4$ *then* $a\leq 2$; $1/p+1/q+1/r<1$, $1/p+1/q+1/a>1$,
$2\leq C^2\leq (r-a)(-pqa+pq+qa+ap)/(pqr-pq-qr-rp)\leq 42$ *(the case* $C^2=42$ *corresponds to the case* $a=0$, $p=2$, $q=3$, $r=7$ *only).*
$\bar{D}(G_1(C^2;p,q,r;a))=C^2(pqr-pq-qr-rp)-(r-a)(pqa-pq-qa-ap)$,
$\bar{D}(G_{hyp})=-pqr+pq+qr+rp$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=(r-a)(-pqa+pq+qa+ap)$.

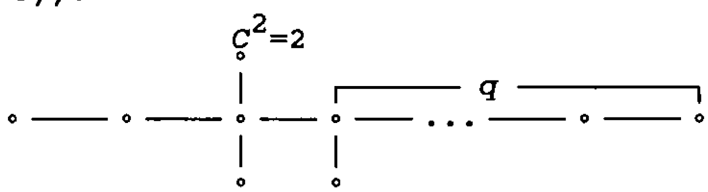
G_{hyp} *has four ends:*

$G_2(C^2;2,q;a,b,)$:



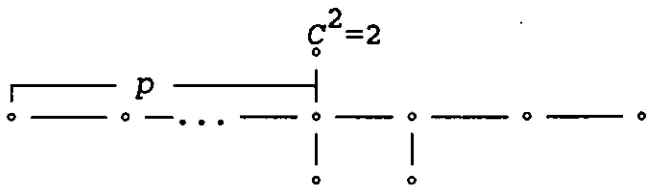
where $2\leq C^2\leq 6$, $a\geq 1$, $b\geq 1$, $q\geq 3$, $C^2+b\leq 3+4/(q-2)$.
$\bar{D}(G_2(C^2;2,q;a,b))=4((q-2)C^2+((q-2)(b-3)-4))$, $\bar{D}(G_{hyp})=8-4q$,
$\bar{D}(G_{hyp}-\Gamma_{hyp})=4(4-(q-2)(b-3))$.

$G_2(2;3,q;1,2)$



where $q=3,4$. $\bar{D}(G_2(2;3,q;1,2))=8(q-4)$, $\bar{D}(G_{hyp})=-7q+10$,
$\bar{D}(G_{hyp}-\Gamma_{hyp})=6(q+2)$.

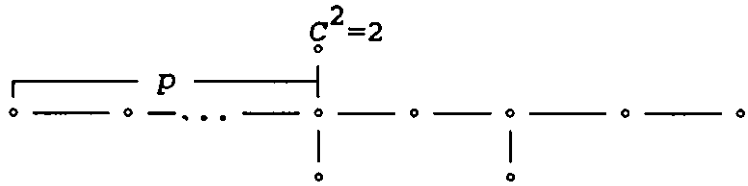
$G_2(2;p,3;1,2)$



where $p=4,5$. $\bar{D}(G_2(2;p,3;1,2))=4(p-5)$, $\bar{D}(G_{hyp})=-7p+10$.

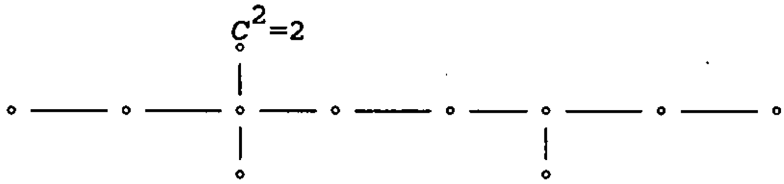
$\bar{D}(G_{hyp}-\Gamma_{hyp})=10p$.

$G_2(2;p,3;1,3)$



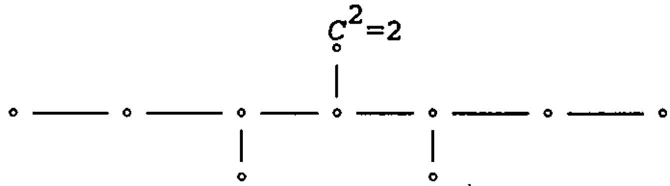
where $p=3,4$. $\bar{D}(G_2(2;p,3;1,3)=4p-16$, $\bar{D}(G_{hyp})=8-6p$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=8p$.

$G_2(2;3,3;1,4)$



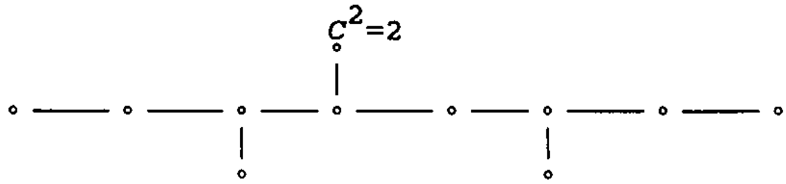
$\bar{D}(G_2(2;3,3;1,4)=0$, $\bar{D}(G_{hyp})=-9$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=18$.

$G_2(2;3,3;2,2)$



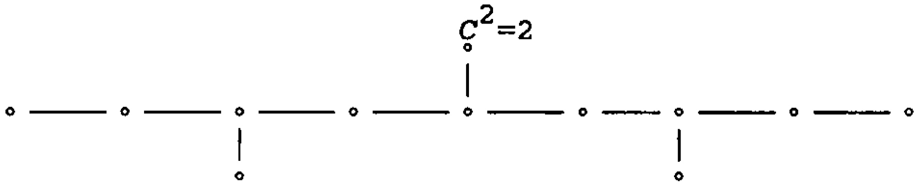
$\bar{D}(G_2(2;3,3;2,2)=-5$, $\bar{D}(G_{hyp})=-10$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=25$.

$G_2(2;3,3;2,3)$



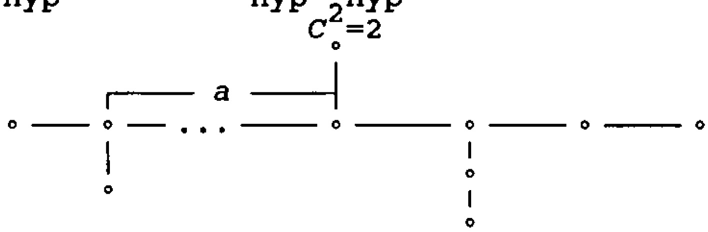
$\bar{D}(G_2(2;3,3;2,3)=-2$, $\bar{D}(G_{hyp})=-9$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=20$.

$G_2(2;3,3;3,3)$



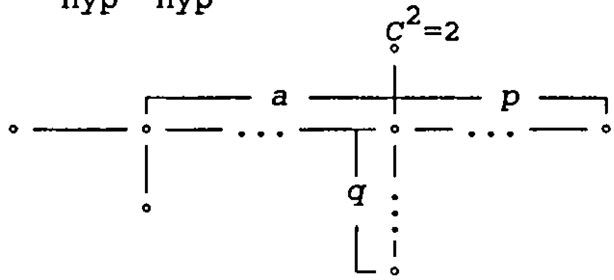
$\bar{D}(G_2(2;3,3;3,3)=0$, $\bar{D}(G_{hyp})=-8$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=16$.

$G_3(2;a)$:



$\bar{D}(G_3(2;a))=0$, $\bar{D}(G_{hyp})=-12$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=24$;

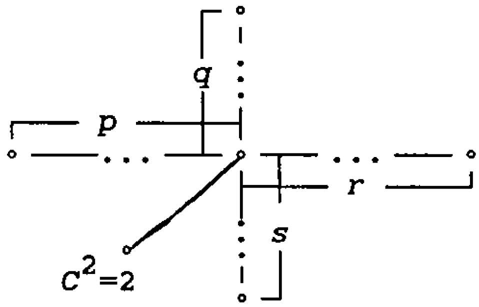
$G_4(2;a;p,q)$



where $(p,q)=(3,q)$, $3\leq q\leq 6$, *and* $(p,q)=(4,4)$.

$\bar{D}(G_4(2;a;p,q))=4(pq-2p-2q)$, $\bar{D}(G_{hyp})=4(-pq+p+q)$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=4pq$.

$G_5(2;p,q,r,s)$



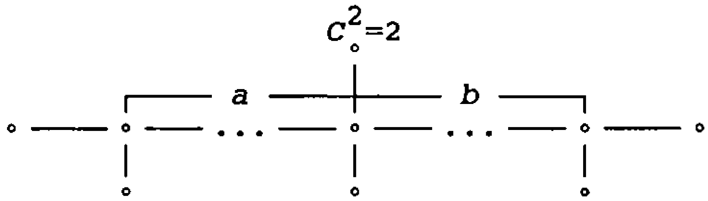
where $p\geq q\geq r\geq s\geq 2$, *and* $2>1/p+1/q+1/r+1/s\geq 3/2$;

$\bar{D}(G_5(2;p,q,r,s)=3pqrs-2pqr-2qrs-2rsp-2spq\leq 0$,

$\bar{D}(G_{hyp})=-2pqrs+pqr+qrs+rsp+spq$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=pqrs$.

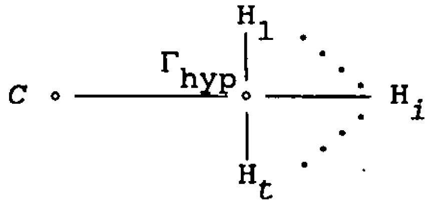
G_{hyp} *has five ends*:

$G_6(2;a,b)$



$\bar{D}(G_6(2;a,b))=0$, $\bar{D}(G_{hyp})=-16$, $\bar{D}(G_{hyp}-\Gamma_{hyp})=32$.

<u>Proof.</u> $G_{hyp}-\Gamma_{hyp}$ is orthogonal to C with $C^2>0$. Since $G(C)$ is hyperbolic, it follows that $G-\Gamma_{hyp}$ is elliptic. Thus, $\bar{D}(G-\Gamma_{hyp})>0$ and $\bar{D}(H_i)>0$. $G(C)$ has the form



It follows that G is linearly independent and $\bar{D}(G)<0$ since G is hyperbolic. Applying (2.2) to $D=\Gamma_{hyp}$ and then to C, we get

$$\bar{D}(G_{hyp})=\bar{D}(H_1)\cdots\bar{D}(H_t)(2-\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)-\ldots-\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t))<0$$

and $\bar{D}(G_{hyp}(C))=$

$$=\bar{D}(H_1)\cdots\bar{D}(H_t)(-C^2(2-\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)-\ldots-\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t))-1)\leq 0.$$

It follow the statements 1 - 5.

To get the last statement 6, we should find all possible sets $(H_1,\Gamma_1;\ldots;H_t,\Gamma_t)$ such that

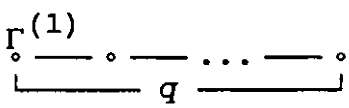
$$2<\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\ldots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)\leq 5/2.$$

For an elliptic tree $H=\mathbf{A}_n$, $\mathbf{D}_m$, $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and its vertex Γ let

$$I(H,\Gamma)=\bar{D}(H-\Gamma)/\bar{D}(H).$$

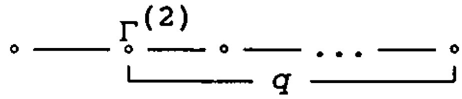
One can find very easy the full list of such pairs with $I(H,\Gamma)\leq 5/2$:

$(\mathbf{A}_q,\Gamma^{(1)})$:



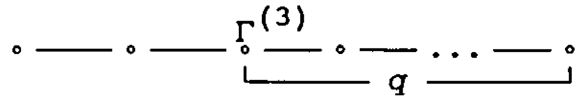
with $I(\mathbf{A}_q,\Gamma^{(1)})=1-1/(1+q)\geq 1/2$, where $q\geq 1$;

$(\mathbf{A}_{1+q},\Gamma^{(2)})$:



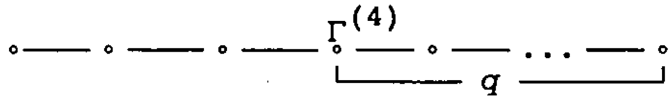
with $I(\mathbf{A}_{1+q},\Gamma^{(2)})=2-4/(2+q)\geq 1$, where $q\geq 2$;

$(\mathbf{A}_{2+q},\Gamma^{(3)})$:



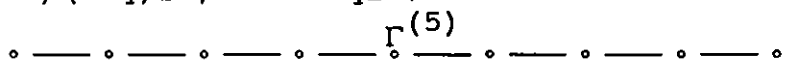
with $I(\mathbf{A}_{2+q},\Gamma^{(3)})=3-9/(3+q)\geq 3/2$, where $q\geq 3$;

$(\mathbf{A}_{3+q},\Gamma^{(4)})$:



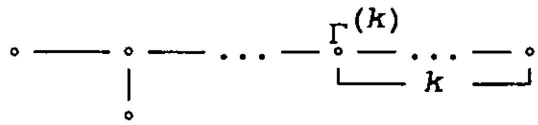
with $I(\mathbf{A}_{3+q},\Gamma^{(4)})=4-16/(4+q)\geq 2$, where $q\geq 4$;

$(\mathbf{A}_9,\Gamma^{(5)})$:



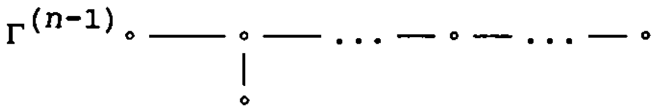
with $I(\mathbf{A}_9,\Gamma^{(5)})=5/2$.

$(\mathbf{D}_n,\Gamma^{(k)})$:



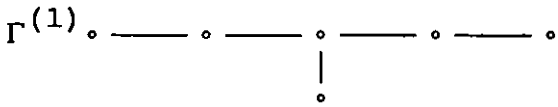
with $I(\mathbf{D}_n,\Gamma^{(k)})=k$, where $n-2\geq k\geq 1$;

$(\mathbf{D}_n,\Gamma^{(n-1)})$:



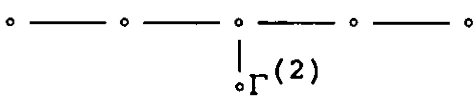
with $I(\mathbf{D}_n,\Gamma^{(n-1)})=n/4\geq 5/4$, where $n\geq 5$;

$(\mathbf{E}_6,\Gamma^{(1)})$:



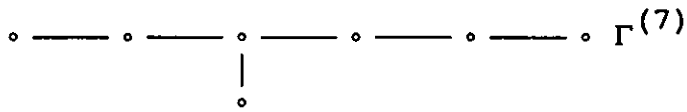
with $I(\mathbf{E}_6,\Gamma^{(1)})=4/3$;

$(\mathbf{E}_6,\Gamma^{(2)})$:



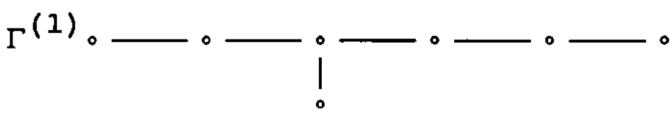
with $I(\mathbf{E}_6,\Gamma^{(2)})=2$;

$(\mathbf{E}_7,\Gamma^{(7)})$:



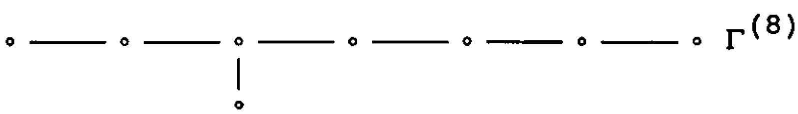
with $I(\mathbf{E}_7,\Gamma^{(7)})=3/2$;

$(\mathbf{E}_7,\Gamma^{(1)})$:



with $I(\mathbf{E}_7,\Gamma^{(1)})=2$;

$(\mathbf{E}_8,\Gamma^{(8)})$:



with $I(\mathbf{E}_8,\Gamma^{(8)})=2$.

It follows, we have the following and only following possibilities for $(H_1,\Gamma_1;\ldots;H_t,\Gamma_t)$ with $2<I=I(H_1)+\ldots+I(H_t)\leq 5/2$:

$\{(\mathbf{D}_n,\Gamma^{(n-1)})\}$, $n=9,10$ with $I=n/4$;

$\{(\mathbf{A}_9,\Gamma^{(5)})\}$ with $I=5/2$;

$\{(\mathbf{A}_{3+q},\Gamma^{(4)})\}$, $q=5,6$ with $I=4-16/(4+q)$;

$\{(\mathbf{A}_{2+q},\Gamma^{(3)})\}$, $7\le q\le 15$, with $I=3-9/(3+q)$;
$\{(\mathbf{E}_8,\Gamma^{(8)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{E}_7,\Gamma^{(1)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{E}_7,\Gamma^{(7)}),(\mathbf{D}_n,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{E}_7,\Gamma^{(7)}),(\mathbf{A}_3,\Gamma^{(2)})\}$ with $I=5/2$;
$\{(\mathbf{E}_7,\Gamma^{(7)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, $q\ge 2$, with $I=5/2-1/(q+1)$;
$\{(\mathbf{E}_6,\Gamma^{(2)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{E}_6,\Gamma^{(1)}),(\mathbf{D}_n,\Gamma^{(1)})\}$ with $I=7/3$;
$\{(\mathbf{E}_6,\Gamma^{(1)}),(\mathbf{A}_3,\Gamma^{(2)})\}$ with $I=7/3$;
$\{(\mathbf{E}_6,\Gamma^{(1)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, $q\ge 3$, with $I=7/3-1/(1+q)$;
$\{(\mathbf{D}_5,\Gamma^{(4)}),(\mathbf{D}_5,\Gamma^{(4)})\}$ with $I=5/2$;
$\{(\mathbf{D}_m,\Gamma^{(m-1)}),(\mathbf{D}_n,\Gamma^{(1)})\}$, $m=5,6$, with $I=m/4+1$;
$\{(\mathbf{D}_5,\Gamma^{(4)}),(\mathbf{A}_{1+q},\Gamma^{(2)})\}$, $q=2,3$, with $I=13/4-4/(2+q)$;
$\{(\mathbf{D}_6,\Gamma^{(5)}),(\mathbf{A}_3,\Gamma^{(2)})\}$ with $I=5/2$;
$\{(\mathbf{D}_5,\Gamma^{(4)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, $q\ge 4$, with $I=9/4-1/(1+q)$;
$\{(\mathbf{D}_6,\Gamma^{(5)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, $q\ge 2$, with $I=5/2-1/(1+q)$;
$\{(\mathbf{D}_7,\Gamma^{(6)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, $1\le q\le 3$, with $I=11/4-1/(1+q)$;
$\{(\mathbf{D}_8,\Gamma^{(7)}),(\mathbf{A}_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(\mathbf{D}_n,\Gamma^{(2)}),(\mathbf{A}_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(\mathbf{D}_n,\Gamma^{(1)}),(\mathbf{A}_5,\Gamma^{(3)})\}$ with $I=5/2$;
$\{(\mathbf{D}_n,\Gamma^{(1)}),(\mathbf{A}_{1+q},\Gamma^{(2)})\}$, $3\le q\le 6$, with $I=3-4/(2+q)$;
$\{(\mathbf{A}_7,\Gamma^{(4)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{A}_5,\Gamma^{(3)}),(\mathbf{A}_3,\Gamma^{(2)})\}$ with $I=5/2$;
$\{(\mathbf{A}_{2+p},\Gamma^{(3)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, where $p\ge 3$, $q\ge 1$, $2>9/(3+p)+1/(1+q)\ge 3/2$, with $I=4-9/(3+p)-1/(1+q)$;
$\{(\mathbf{A}_{1+p},\Gamma^{(2)}),(\mathbf{A}_{1+q},\Gamma^{(2)})\}$, where $2\le p\le q$, $2<q$, $1/(2+p)+1/(2+q)\ge 3/8$, with $I=4-4/(2+p)-4/(2+q)$
$\{(\mathbf{A}_{1+p},\Gamma^{(2)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, where $p>2$, $q\ge 1$, $1>4/(2+p)+1/(1+q)\ge 1/2$, with $I=3-4/(2+p)-1/(1+q)$;
$\{(\mathbf{E}_7,\Gamma^{(7)}),(\mathbf{A}_1,\Gamma^{(1)}),(\mathbf{A}_1,\Gamma^{(1)})\}$, with $I=5/2$.
$\{(\mathbf{E}_6,\Gamma^{(1)}),(\mathbf{A}_p,\Gamma^{(1)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, where $1\le p\le q$, $1/(1+p)+1/(1+q)\ge 5/6$, with $I=10/3-1/(1+p)-1/(1+q)$;
$\{(\mathbf{D}_5,\Gamma^{(4)}),(\mathbf{A}_p,\Gamma^{(1)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, where $1\le p\le q$, $1/(1+p)+1/(1+q)\ge 3/4$, with $I=13/4-1/(1+p)-1/(1+q)$;
$\{(\mathbf{D}_6,\Gamma^{(5)}),(\mathbf{A}_1,\Gamma^{(1)}),(\mathbf{A}_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(\mathbf{D}_m,\Gamma^{(1)}), (\mathbf{D}_n,\Gamma^{(1)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{D}_m,\Gamma^{(1)}), (\mathbf{A}_3,\Gamma^{(2)}),(\mathbf{A}_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(\mathbf{D}_m,\Gamma^{(1)}),(\mathbf{A}_p,\Gamma^{(1)}),(\mathbf{A}_q,\Gamma^{(1)})\}$, where $1\le p\le q$, $q>1$,

$1/(1+p)+1/(1+q)\geq 1/2$, with $I=3-1/(1+p)-1/(1+q)$;
$\{(A_5,\Gamma^{(3)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(A_3,\Gamma^{(2)}),(A_3,\Gamma^{(2)}),(A_1,\Gamma^{(1)})\}$ with $I=5/2$;
$\{(A_{1+p},\Gamma^{(2)}),(A_q,\Gamma^{(1)}),(A_r,\Gamma^{(1)})\}$ where $p\geq 2$, $1\leq q\leq r$,
$3/2\leq 4/(2+p)+1/(1+q)+1/(1+r)<2$, with $I=4-4/(2+p)-1/(1+q)-1/(1+r)$;
$\{(A_p,\Gamma^{(1)}),(A_q,\Gamma^{(1)}),(A_r,\Gamma^{(1)})\}$ where $1\leq p\leq q\leq r$,
$1/2\leq 1/(1+p)+1/(1+q)+1/(1+r)<1$, with $I=3-1/(1+p)-1/(1+q)-1/(1+r)$;
$\{(D_n,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(A_3,\Gamma^{(2)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)})\}$, with $I=5/2$;
$\{(A_p,\Gamma^{(1)}),(A_q,\Gamma^{(1)}),(A_r,\Gamma^{(1)}),(A_s,\Gamma^{(1)})\}$ where $1\leq p\leq q\leq r\leq s$, $s>1$,
$3/2\leq 1/(1+p)+1/(1+q)+1/(1+r)+1/(1+s)$,
with $I=4-1/(1+p)-1/(1+q)-1/(1+r)-1/(1+s)$;
$\{(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)}),(A_1,\Gamma^{(1)})\}$, with $I=5/2$.

If we draw the trees corresponding to all this possibilities, we get all trees of the Theorem and one additional tree corresponding to the case $\{(D_n,\Gamma^{(2)}),(A_1,\Gamma^{(1)})\}$. The last tree is impossible by the Proposition 2.1.2. The same Proposition gives the additional inequalities: if $p\geq 3$ then $a\leq 1$, if $q\geq 3$ then $a\leq 4$, if $q\geq 4$ then $a\leq 2$ for the tree $G_1(C^2;p,q,r;a)$. ∎

Proposition 2.2.2. 1. *Let $C^2\geq 2$ and the hyperbolic connected component $G_1=G_{hyp}\neq\emptyset$. Let G_i, $1\leq i\leq k$, are all connected components of G which are connected by the edge Cv_i, $v_i\in G_i$, with C, and G_j, $k<i\leq l$ are all other connected components of G (disconnected with C).*

Then all connected components G_i, $2\leq i\leq l$ are elliptic and

1. $\bar{D}(G(C))=\bar{D}(G_1)\bar{D}(G_2)\cdots\bar{D}(G_l)(-C^2-$
$-\bar{D}(G_1-v_1)/\bar{D}(G_1)-\bar{D}(G_2-v_2)/\bar{D}(G_2)-\ldots\ldots-\bar{D}(G_k-v_k)/\bar{D}(G_k))\leq 0$,

where

$$2\leq C^2\leq\bar{D}(G_1-v_1)/(-\bar{D}(G_1))-\bar{D}(G_2-v_2)/\bar{D}(G_2)-\ldots\ldots-\bar{D}(G_k-v_k)/\bar{D}(G_k)).$$

2. rk $G(C)=\#G(C)$ *if the right inequality above is strong, and* rk $G(C)=\#G(C)-1$ *if this inequality is an equality.*

3. *If $\mathcal{P}$ is a parabolic subtree of the tree $G_1=G_{hyp}$, then*

$$m(\mathcal{P},v_1)\geq\min m(G_i,v_i),\ 2\leq i\leq k.$$

Proof. Use the formula (2.2) for $B=C$ and the Proposition 2.1.4. ∎

Remark 2.2.3. If $G_{hyp}=\emptyset$, then all restrictions for the tree $G(C)$ we can give here follow from the Propositions 2.1.1 - 2.1.5. We would like to emphasis the difference of this case from the case $G_{hyp}\neq\emptyset$. For the case $G_{hyp}=\emptyset$ the C^2 (equivalently, the $\dim|C|=C^2/2+1$) may be arbitrary large. ∎

2.3. The case $C^2=0$ and $C\neq 0$. Here we have the

Theorem 2.3.1. *Let* $C^2=0$ *and* $C\neq 0$, *and the* $G_{\text{hyp}}\neq\emptyset$, *let* Γ_{hyp} *be the vertex of the* G_{hyp} *joined to* C.

Then all connected components $H_1,\dots,H_t$ *of the* $G_{\text{hyp}}-\Gamma_{\text{hyp}}$ *are parabolic or elliptic. Let* Γ_i *be the vertex of* H_i *joined to* Γ_{hyp}. *Then* $m(H_i,\Gamma_i)=1$ *if the component* H_i *is parabolic, and* $\min m(H_i,\Gamma_i)=1$ *if the component* H_i *is elliptic. The* $\text{rk}\, G(C)=\#G(C)-p$ *where* p *is the number of parabolic components from* $H_1,\dots,H_t$.

Proof. Use the Propositions 2.1.3 and 2.1.4 ■

Remark 2.3.2. If $G_{\text{hyp}}=\emptyset$, then all restrictions for the tree $G(C)$, we can give here, follow from the Propositions 2.1.1 - 2.1.5. ■

2.4. The case $C=0$. In this case $G(C)=G$. The problem is to classify hyperbolic trees G_{hyp}. In [M], G.Maxwell investigated the case $\text{rk}\, G_{\text{hyp}} = \#G_{\text{hyp}}$. Fortunately, it is necessary only to reformulate his results to consider the general case which we need.

Theorem 2.4.1. *Let* G *be a connected tree of nonsingular* -2 *curves on a* K3 *surface. Then one of the two cases* (a) *or* (b) *holds:*

(a) *There exists a vertex* Γ *of* G *such that all connected components* $H_1,\dots,H_t$ *of* $G_{\text{hyp}}-\Gamma$ *are parabolic or elliptic. If one of these components* $H_1,\dots,H_t$ *is parabolic, then* G *is hyperbolic. If all the components* $H_1,\dots,H_t$ *are elliptic then* G *is hyperbolic iff*

$$\bar{D}(H_1-\Gamma_1)/\bar{D}(H_1)+\dots+\bar{D}(H_t-\Gamma_t)/\bar{D}(H_t)>2$$

where Γ_i *is the vertex of* H_i *joined to* Γ. *If* G *is hyperbolic, then*

$$\text{rk}\, G=\#G-\max(\alpha-1,0)$$

where α *is the number of parabolic components from* $H_1,\dots,H_t$.

(b) *There exists an edge* $\Gamma_1\Gamma_2$ *of* G *such that all connected components of* $G-\{\Gamma_1\Gamma_2\}$ *are elliptic and, if* G_1 *is the connected component of* $G-\Gamma_2$ *containing* Γ_1 *and* G_2 *is the connected components of* $G-\Gamma_1$ *containing* Γ_2, *then both* G_1 *and* G_2 *are hyperbolic. In this situation the matrices of* G_1,G_2 and G are *hyperbolic iff* $\bar{D}(G_1)<0$, $\bar{D}(G_2)<0$ *and* $\bar{D}(G)\le 0$.

Let $H_{j1},\dots,H_{jk_j}$ *be all connected components of* $G_j-\Gamma_j$, $j=1,2$, *and* Γ_{ji} *be a vertex of the* H_{ji} *joined to the* Γ_j. *Then the last inequalities are equivalent to*

$$A_1=\bar{D}(H_{11}-\Gamma_{11})/\bar{D}(H_{11})+\dots+\bar{D}(H_{1k_1}-\Gamma_{1k_1})/\bar{D}(H_{1k_1})>2,$$

$$A_2=\bar{D}(H_{21}-\Gamma_{21})/\bar{D}(H_{21})+\dots+\bar{D}(H_{2k_2}-\Gamma_{2k_2})/\bar{D}(H_{2k_2})>2,$$

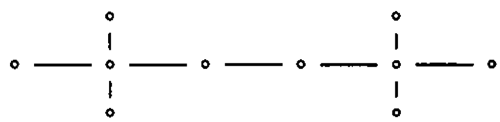
and

$$(A_1-2)(A_2-2)\le 1.$$

The rk G=#G *if the last inequality is strong, and* rk G=#G-1 *if the last inequality is equality.*

Proof. This is similar to [M]. We leave details to the reader. ∎

The Theorem 2.4.1 is sufficient to draw (in principle) all possible trees G of -2 curves on the K3 surfaces. An additional restrictions for these trees give the Propositions 2.1.1 - 2.1.5. For example, all connected trees with ≤10 vertices are either elliptic or parabolic, or hyperbolic (it is mentioned in [M]), and the full list of these trees one can find in [H]. Only the following tree with ≤10 vertices contradicts to these propositions and, hence, is impossible on K3 surfaces:



2.5. Remark. Here we only mentioned the most important and rude conditions for trees G(*C*). We hope to give other more delicate necessary and sufficient conditions in further publications. This problem is a little similar to the problem of a description of all possible singularities of quartic singular K3 surfaces. You can see the series of Urabe's articles devoted to this subject; see [U], for example. But our problem is much more complicated. It is arithmetic and is connected with the existence of an embedding of the corresponding to G(*C*) lattice into K3 cohomology lattice (it is an even unimodular lattice of the signature (3,19)). One can use here the discriminant form technique (see [N]).

§ 3. Fixed part of Weil linear systems on singular K3 surfaces.

3.1. General case. Let Y be a singular K3 surface and $\sigma:X\longrightarrow Y$ the minimal resolution of singularities of Y. Let $\Delta_s=\Sigma b_jF_j$, where $b_j\ge 0$ are integers and F_j are components of the exceptional divisor of σ. Let D be an effective divisor on X. A complete Weil linear system $\bar{D}$ on Y is the image $|\bar{D}|=\sigma_*(|D+\Delta_s|)$, where $|D+\Delta_s|$ is the complete linear system on X and we consider all possible $b_j\ge 0$. It is very easy to see that this image is stabilized if b_j are increased. Like in the § 1, we want to describe the moving part and the fixed part of the linear system $\bar{D}$. Evidently, the fixed part is the image of the fixed part Δ of the linear system $|D+\Delta_s|$. And the fixed components part of $|\bar{D}|$ is the image $\sigma_*(\Delta_r)$ of the part $\Delta_r=\Delta-\Delta_s$. It is not difficult to prove that when $b_j \gg 0$ then all components F_j of Δ_s belong to the fixed part of the linear system $|D+\Delta_s|$. We suppose that $b_j \gg 0$, or it is more convenient to suppose that the all $b_j=+\infty$.

Let $|D+\Delta_s|=|C|+\Delta$, where $|C|$ is the moving part and Δ is the fixed part. Then $\Delta=\Delta_r+\Delta_s$ where Δ_s is the part defined by the all components F_j of the multiplicity $+\infty$ and $\Delta_r=\Delta-\Delta_s$. Then the multiplicity a_i of an irreducible component Γ_i of Δ_r is defined and is a finite natural number. It defines the multiplicities of the corresponding irreducible components $\sigma_*(\Gamma_i)$ of the fixed part $\sigma_*(\Delta)$ of the complete linear system $|\bar{D}|$. As in § 1, we define the graph $G(C,\Delta)$. Its difference from the situation of the § 1 is that vertices of its subgraph $G(\Delta)$ are of the two kinds:

Black vertices of the multiplicity $+\infty$ corresponding to the components F_j of the exceptional divisor of σ;

White vertices of the finite multiplicity $a_i \in \mathbb{N}$, corresponding to the irreducible components $\sigma_*(\Gamma_i)$ of the fixed part of $|\bar{D}|$.

Thus, the problem we should solve, is the same as in the § 1: To describe all possible graphs $G(C,\Delta)$ of this kind such that $|C+\Delta|=|C|+\Delta$. It is a particular case of the problem we have solved in the § 1, and it is necessary to reformulate the results of § 1 in this situation only.

The analog of the condition (*) is the condition

(**) $|C|$ satisfies the condition (i), (ii), or (iii) of the Proposition 0.1, $\Delta=\Delta_r+\Delta_s$, where $\Delta_r=\sum a_i\Gamma_i$, $a_i \in \mathbb{N}$, and $\Delta_s=\sum b_jF_j$, $b_j=+\infty$ (or $b_j \gg 0$), and all Γ_i and F_j are irreducible -2 curve. (For the graph $G(C,\Delta)$, the vertices Γ_i are called white and the vertices F_j black.) If $|C|=m|E|$ where E is an elliptic curve and $m \geq 2$ then there does not exist more than one irreducible component R of Δ such that $E \cdot R \geq 1$; if here $m \geq 4$, then the vertex R is white and has the multiplicity $a=1$.

Our question is: If (**) holds, when

$$|C+\Delta|=|C|+\Delta \text{ ?} \qquad (3.1)$$

Theorem 3.1.1. *Let $C+\Delta$ be a divisor on a nonsingular* K3 *surface X which satisfy the condition (**) above.*

Then $|C+\Delta|=|C|+\Delta$ (equivalently, $|\sigma_*(C+\Delta)|=\sigma_*(|C|)+\sigma_*(\Delta)$ for the contraction σ of the all black curves F_j), if and only if $G(C,\Delta)$ is a tree and $G(C,\Delta)$ has not a subtree $T=\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{D}_m(C)$, $\tilde{E}_6(C)$, $\tilde{E}_7(C)$, $\tilde{E}_8(C)$ $\tilde{B}_m(C)$ or $\tilde{G}_2(C)$ of the Theorem 1.1. It means that if the tree $G(C,\Delta_{red})$ contains the subtree T_{red} (red means the reduction), *then there exists a vertex v of* T *which is a white vertex of the tree* G(C,Δ) *and its multiplicity in* G(C,Δ) *is strongly less than the multiplicity of the vertex v in the subtree* T.

Proof. This follows from the Theorem 1.1. ■

3.2. Nef case. We use notations of **3.1.** Here we want to consider the case when a linear system $|\bar{D}|$ on a singular K3 surface Y is nef or numerically ample (in the sense of Mumford intersection pairing on a normal surface [Mu]). This case is the most interesting for applications (for Fano threefolds, for example). We use the following trivial

Lemma 3.2.1. *$\bar{D}$ is* nef *iff $\sigma^*(\bar{D})$ is* nef. *In other words, if we normalize weights b_j of black vertices F_j of the Δ by the condition $F_j\cdot(C+\Delta)=0$ (here b_j are rational numbers) and not change weights a_i of the white vertices Γ_i (here a_i are natural numbers), then for any white curve Γ_i we have the inequality: $\Gamma_i\cdot(C+\Delta)\geq 0$. If $|\bar{D}|$ is ample, the last inequalities are strong: $\Gamma_i\cdot(C+\Delta)>0$.* ■

Thus, it is natural to give the

Definition 3.2.2. The graph G(C,Δ) is called convex below if for the weights (b_j) of the black vertices F_j satisfying to the condition $F_j\cdot(C+\Delta)=0$, the condition $\Gamma_i\cdot(C+\Delta)\geq 0$ holds for the white vertices Γ_i. In other words, for any component U of the Δ the inequality $U\cdot(C+\Delta)\geq 0$ holds, and, if U is black, this inequality is the equality. A diagram G(C,Δ) is called strongly convex below if it is convex below and for any white vertex Γ_i a strong inequality $\Gamma_i\cdot(C+\Delta)>0$ holds. It is sufficient to prove this conditions for connected components Δ_i of Δ only. ■

From the Theorem 3.1.1 and the Lemma 3.2.1, we get

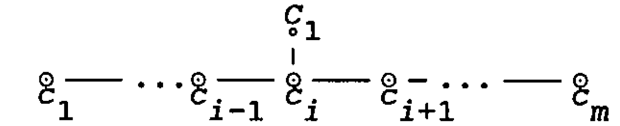
Theorem 3.2.3. *Under the conditions of the Theorem* 3.1.1,

$$|\sigma_*(C+\Delta)|=\sigma_*(|C|)+\sigma_*(\Delta) \text{ and } \sigma_*(C+\Delta) \text{ is nef}$$

if and only if G(C,Δ) *satisfies the Theorem* 3.1.1 *for the weights $b_j=+\infty$ of the black vertices F_j and the tree* G(C,Δ) *is convex below for the weights b_j of the black vertices F_j satisfying to the condition $F_j\cdot(C+\Delta)=0$ of the Definition* 3.2.2. *If $\sigma_*(C+\Delta)$ is ample, then this tree* G(C,Δ) *should be additionally strongly convex.* ■

As an example, let us consider the case when $C^2>0$ and G(Δ) has the form $\mathbf{A}_m$ or $\mathbf{D}_m$. We denote ∘ a white vertex, · a black vertex, and ⊙ a vertex which may be either white or black. Then we get the following possible trees G(C,Δ) on the K3 surfaces over a basic field of characteristic 0, where c_i is the weight of the vertex (white or black) satisfying to the conditions of the Definition 3.2.2:

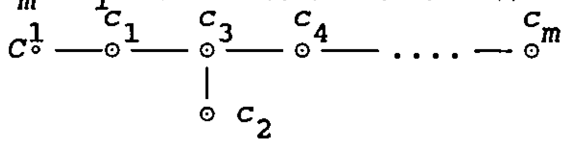
$(\mathbf{A}_m, i)$:



where the chains of weights $0, c_1, \ldots c_i$, and $c_i, \ldots, c_m, 0$ are convex below and $(c_i - c_{i-1}) + (c_i - c_{i+1}) \le 1$. Since $m \le 19$, the

$$\max\{c_1, \ldots, c_i, \ldots c_m\} = c_i \le i(m+1-i)/(m+1) \le (m+1)/4 \le 5 .$$

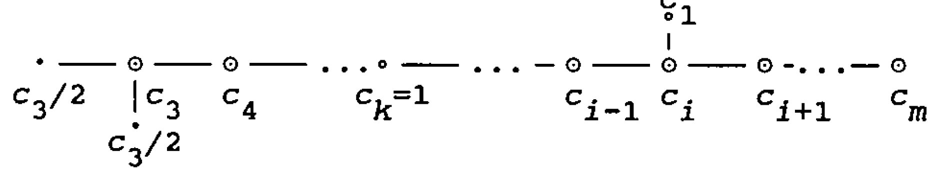
$(\mathbf{D}_m, 1)$:



where the chains $1, c_1, c_3$, and $0, c_2, c_3$, and $c_3, c_4, \ldots, c_m, 0$ are convex below and $c_3 - c_1 + c_3 - c_2 \le c_4$. Since $m \le 19$, the

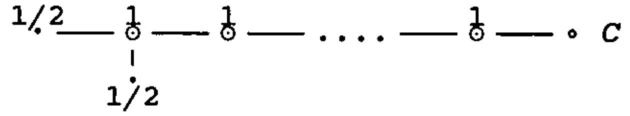
$$\max\{1, c_1, c_2, \ldots c_m\} = \max\{1, c_3\} \le (m-2)/2 \le 17/2 < 9.$$

$(\mathbf{D}_m, i)$, $3 < i < m$:



The chains of weights $c_3, c_4, \ldots, c_k = 1, \ldots c_{i-1}, c_i$ and $c_i, \ldots c_m, 0$ are convex below, and $c_4 \ge c_3$, and $(c_i - c_{i-1}) + (c_i - c_{i+1}) \le 1$. The $\max\{c_1, \ldots, c_m\} = c_i \le \dfrac{(i+1-k)(m+1-i)}{(m+1-k)} \le (m+2-k)^2/(4(m+1-k)) \le 81/17 < 5$.

$(\mathbf{D}_m, m)$:

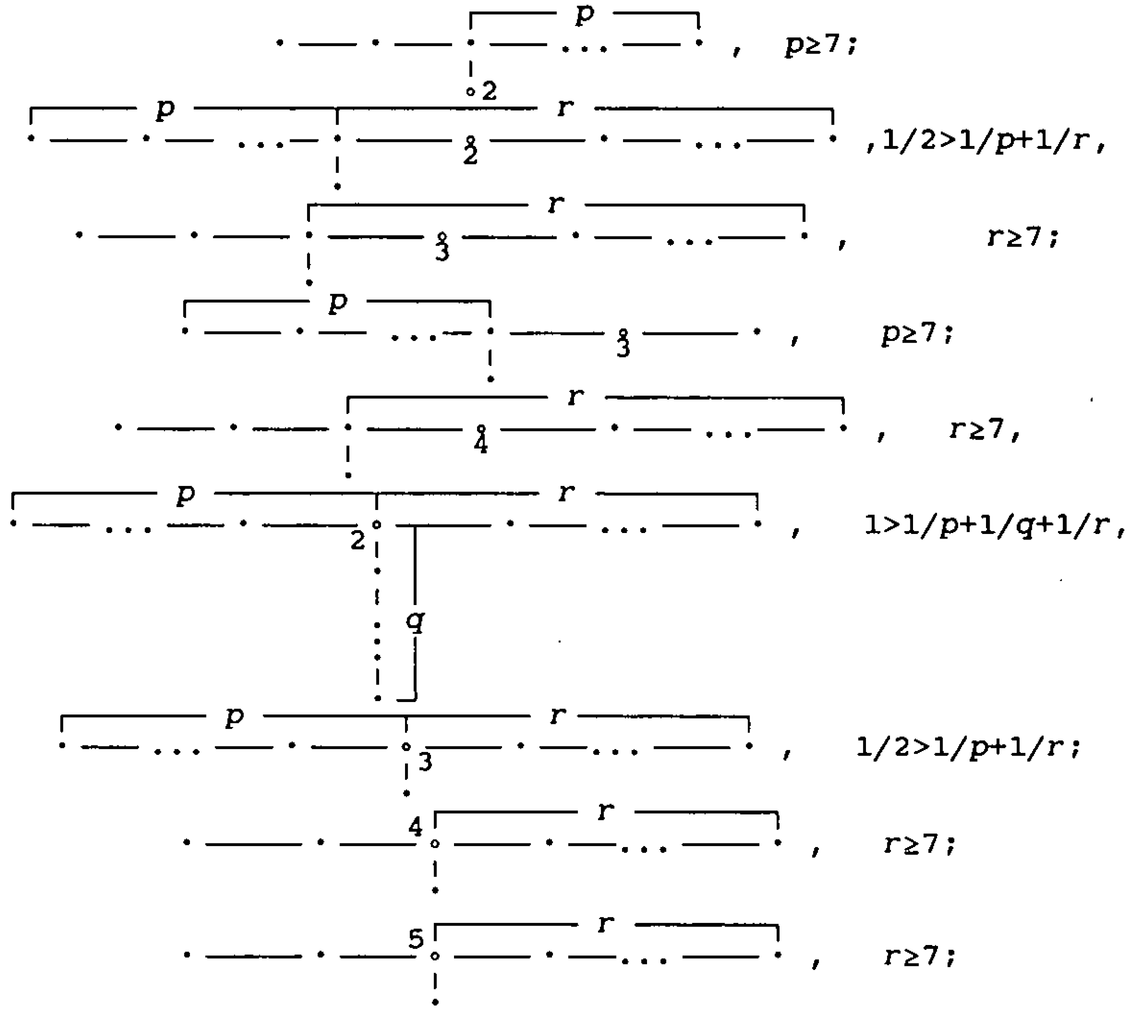


We should emphasize that this diagrams are possible for an arbitrary even $C^2 > 0$ and an arbitrary $\dim|\sigma_*(C+\Delta))| = C^2/2+1$, and here the moving part $\sigma_*|C|$ of $|\sigma_*(C+\Delta))|$ is not a pencil. A multiplicity of the fixed part components of $|\sigma_*(C+\Delta))|$ may be >1, but ≤ 8 (and the case of the maximum multiplicity 8 is possible). This shows the difference of the nef and ample linear systems on singular K3 surfaces comparing with the nonsingular case. But here the hyperbolic component $G_{\mathrm{hyp}}(\Delta) = \varnothing$. We consider an opposite example below.

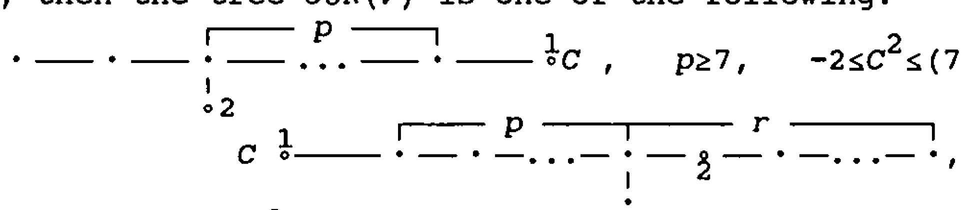
3.3. The case rk Pic Y=1. More generally, we consider the case when $\sigma_*(D_i)^2 > 0$ for any irreducible component D_i of a general member $D = \sum D_i \in |C + \Delta_r|$. This case is characterized by the following conditions: an every elliptic component of $G(\Delta)$ contains black vertices only; the tree $G(\Delta)$ does not contain parabolic components; if v is a white vertex of $G_{\mathrm{hyp}}(\Delta)$ and $N(v)$ is the maximum connected subtree of $G_{\mathrm{hyp}}(\Delta)$ which contains only the one white vertex v, then $N(v)$ is hyperbolic.

Using this conditions and the theory above, we get the following description of $G_{hyp}(C,\Delta)$ if it has a white vertex v of a multiplicity ≥ 2 (equivalently, $|\bar{D}|$ has a fixed component of a multiplicity ≥ 2): This case is the most interesting for applications.

We get that for a white vertex v of a multiplicity >1 the tree $N(v)$ is one of the following trees:

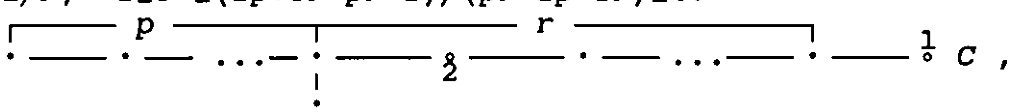


It follows very easy that this multiplicity >1 white vertex v of the $G(C,\Delta)$ is unique, and $G(C,\Delta)$ has not more then one other white vertex. We shall denote this vertex C (thus, we permit that $C^2=-2$). If there exists this additional white vertex C of the multiplicity one, then the tree $CUN(v)$ is one of the following:

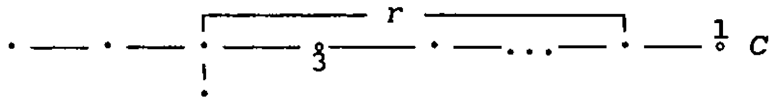


$p\geq 7$, $-2\leq C^2\leq (7-p)/(p-6)\leq 0$.

$1/2>1/p+1/r$, $-2\leq C^2\leq (2p+3r-pr-2)/(pr-2p-2r)\leq 4$.

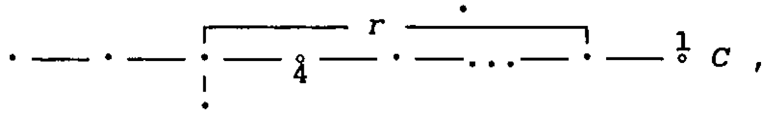


$1/2>1/p+1/r$, $-2\le C^2\le(2r+3p-pr-2)/(pr-2p-2r)\le 4$.

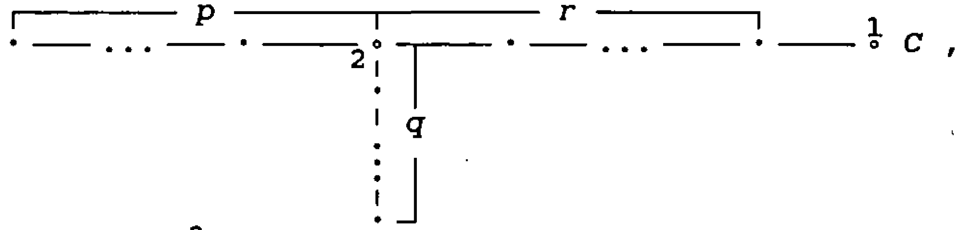


$r\ge 7$, $-2\le C^2\le(7-r)/(r-6)\le 0$.

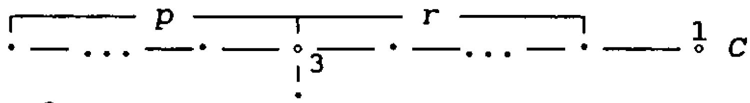
$p\ge 7$, $-2\le C^2\le(7-p)/(p-6)\le 0$.



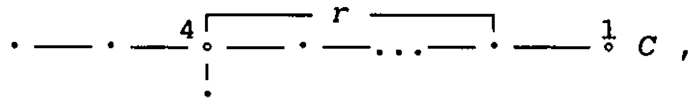
$r\ge 7$, $-2\le C^2\le(7-r)/(r-6)\le 0$.



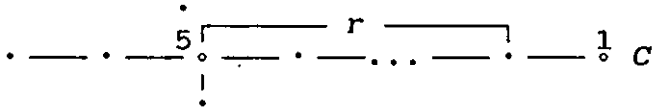
$1>1/p+1/q+1/r$, $-2\le C^2\le(2pq+pr+qr-pqr-p-q))/(pqr-pq-qr-rp)\le 10$.



$1/2>1/p+1/r$, $-2\le C^2\le(2r+3p-pr-2)/(pr-2p-2r)\le 4$.



$r\ge 7$, $-2\le C^2\le(7-r)/(r-6)\le 0$.



$r\ge 7$, $-2\le C^2\le(7-r)/(r-6)\le 0$.

It follows very easy the estimate $\bar{D}^2<20$ if $|\bar{D}|$ has a component of the multiplicity >1. We hope giving more precise description of the case rk Pic Y=1 in further publications.

We should mention that almost at the same time V.A.Alekseev got the same results for rk Pic Y=1 by other method (using Riemann-Roch theorem for singular K3 surfaces) and the more strong estimate: $\bar{D}^2<13$ if $|\bar{D}|$ has a fixed component of a multiplicity >1. Of course, the same estimate follows from the calculations above.

§ 4. Some open questions.

4.1. Fano threefolds. Let F be Fano threefold with $\mathbb{Q}$-factorial terminal singularities, and a good member $Y\in|-K_F|$, which is a singular K3 surface, exists. Let $\dim|-K_F|>0$. Then $|-K_F|\,|_Y$ is an nonempty complete ample linear system on the singular K3 surface Y. Thus, some tree $G(C,\Delta)$, we have described above, corresponds to this linear system. We can consider this tree $G(C,\Delta)$ as an invariant of the Fano threefold F. What are such invariants $G(C,\Delta)$ possible for Fano threefolds F with $\mathbb{Q}$-factorial terminal singularities?

4.2. Graded ring of a singular K3 surface. I due to participants of the conference "Algebraic and Analytic Varieties" Tokyo, August 1990, the Professors Sh.Ishii, M.Reid, M.Tomari and K.Watanabe by the following very interesting question (see their articles connected with this subject): What one can say about the graded ring

$$R(Y) = \oplus_{m\geq 0} H^0(Y, \mathcal{O}(m\bar{D}))$$

for a nef effective (or, maybe, noneffective) integral Weil divisor $\bar{D}$ on a singular K3 surface Y, its generators and relations. The nonsingular case see in [S-D]. The theory we have constructed here gives all possibilities when it is needed to investigate this ring. Moreover, this theory permits to interpret a homogeneous constituent $H^0(Y, \mathcal{O}(m\bar{D}))$ of the ring as a some precisely described complete linear system on the nonsingular K3 surface X which is the minimal resolution of singularities of Y.

References

[H] F.Harary, "Graph theory," Addison-Wesley, Reading, Mass., 1969.

[Mu] D.Mumford, *The topology of normal singularities of an algebraic* surface and a criterion for simplicity, IHES 9 (1961) 5-22.

[M] G.Maxwell, Hyperbolic trees, J. Algebra. 54 (1978) 46-49.

[N] V.V.Nikulin, Integral symmetric bilinear forms and some of their geometrical applications. Izv AN SSSR, ser. matem. 43 (1979) 111-177 (English transl: Math. USSR Izv. 14 (1980) 103-167.)

[S-D] B.Saint-Donat, Projective models of K3 surfaces, Amer. J. Math. 96 (1974) 602-639.

[U] T.Urabe, Tie transformations of Dynkin graphs and singularities on quartic surfaces, Invent. math. 100 (1990), 207-230.