On Euler systems of rank r and their Kolyvagin systems

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ABSTRACT. In this paper we set up a Kolyvagin system machinery for Euler systems of rank r (in the sense of Perrin-Riou) associated to a self-dual Galois representation T, building on our previous work on Kolyvagin systems of Rubin-Stark units and generalizing the results of Kato, Rubin and Perrin-Riou. Our machinery produces a bound on the size of the classical Selmer group attached to T in terms of a certain $r \times r$ determinant; a bound which remarkably goes hand in hand with Bloch-Kato conjectures. At the end, we present an application based on a conjecture of Perrin-Riou on p-adic L-functions, which lends further evidence to Bloch-Kato conjectures.

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INTRODUCTION

Fix once and for all an odd prime p. Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and a finite residue field $\mathbb{F}$ which has characteristic p. In [MR04], Mazur and Rubin determine the structure of a Selmer group attached to a Galois representation T (which is free of finite rank over R) in terms of a Kolyvagin system, when the *core Selmer rank* (in the sense of Definition 4.1.11 of loc.cit.) is one. In fact, when the core Selmer rank is one, they prove that the module of Kolyvagin systems is cyclic and it is therefore possible to choose 'the best' Kolyvagin system (which they call a *primitive* Kolyvagin system, it is by definition a generator for the cyclic module of Kolyvagin systems) which may be used to obtain the best possible bound on the associated Selmer group. Further, in most of the classical examples given in loc.cit., a primitive Kolyvagin system is obtained from an Euler system via Kolyvagin's descend.

When the core Selmer rank is $r > 1$, the whole picture is more complicated. In this case, one would hope that a certain *Euler system of rank* r (in the sense of [PR98]) would give rise to the sought for Kolyvagin systems via

(1) obtaining an Euler system (in the sense of [Rub00]; these correspond to *Euler systems of rank one* in the terminology of [PR98]), following the recipe of [PR98, §1.2.3],
(2) applying Kolyvagin's descend on these Euler systems of rank one.

The basic issue in this case is that the module of Kolyvagin systems is no longer cyclic[1] and the procedure above leaves us with many choices.

Fix a totally real number field k, and write G_k for its absolute Galois group. Only in this paragraph, let T denote the rank one G_k-representation $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, where χ is a totally even character $\chi : G_k \to \mathbb{Z}_p^\times$ of finite order. In this case, it turns out that the core Selmer rank[2] $\mathcal{X}(T)$ equals $[k : \mathbb{Q}]$. When $k \neq \mathbb{Q}$, the machinery of [MR04] is not sufficient as it is to treat this example. The Euler system of rank $r = [k : \mathbb{Q}]$ in this setting is obtained from (conjectural) Rubin-Stark elements [Rub96]. The author has studied this example extensively in [Büy07a, Büy07c] and has developed a Kolyvagin system machinery to make use of this most basic example of an Euler system of rank $r > 1$.

Note that one feature of the Galois representation $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$ studied in [Büy07a, Büy07c] is that it is totally odd in the sense that

$$\left(\mathrm{Ind}_{k/\mathbb{Q}} T\right)^- = \mathrm{Ind}_{k/\mathbb{Q}} T.$$

This property is essential for the treatment of [Büy07a, Büy07c, Büy08]. The aim of this article is to generalize the methods of [Büy07a, Büy07c, Büy08] in order to develop an appropriate

[1] In fact Howard [MR04, Appendix B] shows that the $\mathbb{F}$-vector space of Kolyvagin systems for the residual representation $T/\mathfrak{m}T$ is infinite dimensional.

[2] The reader who is experienced with the terminology of [MR04] might have realized that we talk about the core Selmer rank for a Galois representation without referring to a *Selmer structure*. When we say *core Selmer rank* of a Galois representation T, we implicitly mean the core Selmer rank for the *canonical Selmer structure* on T; this (generically, see [MR04, Theorem 5.2.15] for details) equals

$$\mathcal{X}(T) = \mathrm{rank}_R \left(\mathrm{Ind}_{k/\mathbb{Q}} T\right)^- \ (= d_- \text{ in the language of [PR98]})$$

when R is an integral domain: The rank of the minus eigenspace for the complex conjugation acting on the induced representation. Note that we may alternatively write

$$\mathcal{X}(T) = [k : \mathbb{Q}] \cdot \mathrm{rank}_R T - \sum_{v|\infty} \mathrm{rank}_R H^0(k_v, T)$$

where k_v is the completion of k at the infinite place v.

Kolyvagin system machinery for Galois representations T which are self-dual[3] in the sense that there is a skew-symmetric isomorphism $T \xrightarrow{\sim} \mathrm{Hom}_R(T, R(1))$. Many important Galois representations fall in this category:

(1) $T = T_p(E)$ is the p-adic Tate module of an elliptic curve E/k; $k \neq \mathbb{Q}$,
(2) $A/\mathbb{Q}$ is an abelian variety of dimension $g > 1$ and $T = T_p(A)$.

Before we state the main results of this paper, we fix our notation and set the hypotheses which we will refer to in the main body of our article.

Notation and Hypotheses. For any field K, let G_K be the Galois group of a fixed separable closure of K. Throughout, k is a fixed totally real number field and k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k. We set $\Gamma = \mathrm{Gal}(k_\infty/k)$ and $\Lambda = \mathbb{Z}_p[[\Gamma]]$ as usual. We write k_n for the unique sub-extension of k_∞/k of degree p^n, and set $\Gamma_n = \mathrm{Gal}(k_n/k)$. Our first hypothesis which we will assume for our Iwasawa theoretical results is the following:

($\mathbf{H}.\mathbf{Iw}$.) Every prime $\wp \subset k$ above p totally ramifies in k_∞/k.

For any prime $\lambda \subset k$, we fix a decomposition group $\mathcal{D}_\lambda \subset G_k$. We will occasionally identify $\mathcal{D}_\lambda$ by the absolute Galois group of the completion k_λ. We denote the inertia subgroup inside $\mathcal{D}_\lambda$ by $\mathcal{I}_\lambda$. We write $\mathrm{Fr}_\lambda \in \mathcal{D}_\lambda/\mathcal{I}_\lambda$ for the Frobenius element.

Let $\mathcal{O}$ be the ring of integers of a finite extension Φ of $\mathbb{Q}_p$ with $\mathfrak{m}$ being its maximal ideal and $\mathbb{F} = \mathcal{O}/\mathfrak{m}$ its residue field. Write $\boldsymbol{\mu}_{p^n}$ for the (Galois module of) p^n-th roots of unity, and set $\mathbb{Z}_p(1) = \varprojlim \boldsymbol{\mu}_{p^n}$ and $\boldsymbol{\mu}_{p^\infty} = \varinjlim \boldsymbol{\mu}_{p^n}$. We define $\mathcal{O}(1) := \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$, and for any $\mathcal{O}[[G_k]]$-module M, we write $M(1) := M \otimes_{\mathcal{O}} \mathcal{O}(1)$ (allowing G_k act both on M and $\mathcal{O}(1)$). We also define $M^* = \mathrm{Hom}(M, \Phi/\mathcal{O})(1)$, the Cartier dual of M; and $M^\vee = \mathrm{Hom}(M, \Phi/\mathcal{O})$, the Pontryagin dual of M.

For any field K and a topological abelian group A which is endowed with a continuous action of G_K, we write $H^i(K, A)$ for the i-th group cohomology $H^i(G_K, A)$ computed with continuous cochains. We also define

$$A^\wedge := \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

to be the p-adic completion of A.

For any commutative ring R, an ideal $I \subset R$ and an R-module A, we write $A[I]$ for the submodule of A consisting of elements that are killed by all I. For $x \in R$, we write $A[x]$ for $A[Rx]$.

Let T be a free $\mathcal{O}$-module of finite rank, endowed with a continuous action of G_k. Suppose further that T is self-dual, i.e., there exists a skew-hermitian isomorphism

$$T \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})(1). \tag{0.1}$$

In particular, the $\mathcal{O}$-rank of T is even; write $2d$ for this rank. We define

$$r := d \cdot [k : \mathbb{Q}] = \mathrm{rank}_{\mathcal{O}}(\mathrm{Ind}_{k/\mathbb{Q}}T)^- = \mathrm{rank}_{\mathcal{O}}(\mathrm{Ind}_{k/\mathbb{Q}}T)^+ = \sum_{v|\infty} \mathrm{rank}_{\mathcal{O}}\, H^0(k_v, T),$$

where k_v stands for the completion of k at the infinite place v. Note that the third equality above follows from our assumption that T is self-dual, and the second equality follows from the third. We also remark that r defined above is exactly what Perrin-Riou [PR98] calls d_-.

Write $\mathbb{T} = T \otimes \Lambda$, where we allow G_k act on both T and Λ. (The action of G_k on Λ is induced from the canonical surjection $G_k \twoheadrightarrow \Gamma$.) Define $V = T \otimes_{\mathcal{O}} \Phi$, and $V^* = \mathrm{Hom}(V, \Phi)(1)$.

[3] In fact, in slightly greater generality than this; see Remark 3.10 below.

Fix a set $\mathcal{P}$ of (non-archimedean) primes of k which does *not* contain any prime above p and any prime at which T is ramified. Following [MR04, Definition 3.1.6], we define $\mathcal{P}_s$ ($s \in \mathbb{Z}^+$) as the set of primes $\lambda \subset k$ at which T is unramified, which do not lie above p and which satisfy:

(1) $T/(\mathfrak{m}^s T + (\mathrm{Fr}_\lambda - 1)T)$ is a free $\mathcal{O}/\mathfrak{m}^s$-module of rank one,
(2) $I_\lambda := \mathrm{span}_{\mathcal{O}}\{\mathbf{N}\lambda - 1, \det(1 - \mathrm{Fr}_\lambda|T)\} \subset \mathfrak{m}^s$.

For any group Δ, and a $\mathcal{O}[\Delta]$-module M, we write $\wedge^s M$ for the sth exterior power of M computed in the category of $\mathcal{O}[\Delta]$-modules. For example, we will be dealing below with exterior powers of the sort $\wedge^s H^i(K, M)$, where K is a finite extension of k with Galois group Δ, and M is an $\mathcal{O}[[G_k]]$-module. This naturally makes $H^i(K, M)$ an $\mathcal{O}[\Delta]$-module and $\wedge^s H^i(K, M)$ is calculated in the category of $\mathcal{O}[\Delta]$-modules.

Below we record a list of properties which will play a role in what follows:

(H.1) $T/\mathfrak{m}T$ is an absolutely irreducible $\mathbb{F}[[G_k]]$-representation.
(H.2) There is a $\tau \in G_k$ such that $\tau = 1$ on $\boldsymbol{\mu}_{p^\infty}$ and the $\mathcal{O}$-module $T/(\tau - 1)T$ is free of rank one.
(H.3) $H^1(k(T, \boldsymbol{\mu}_{p^\infty}), T/\mathfrak{m}T) = H^1(k(T, \boldsymbol{\mu}_{p^\infty}), T^*[\mathfrak{m}]) = 0$, where $k(T, \boldsymbol{\mu}_{p^\infty}) = k(T)(\boldsymbol{\mu}_{p^\infty}) \subset \overline{k}$, and $k(T)$ is the smallest extension of k such that the G_k-action on T factors through $\mathrm{Gal}(k(T)/k)$.
(H.4) $p > 4$.
(H.5) The set of primes $\mathcal{P}$ satisfies $\mathcal{P}_t \subset \mathcal{P} \subset \mathcal{P}_1$ for some $t \in \mathbb{Z}^+$.
(H.T) (*Tamagawa Condition*) $(T \otimes \Phi/\mathcal{O})^{\mathcal{I}_\lambda}$ is $\mathcal{O}$-divisible for any prime $\lambda \subset k$ prime to p.
(H.nE) (*Non-exceptionality*) $H^0(k_p, T^*) := \oplus_{\wp|p} H^0(k_\wp, T^*) = 0$.
(H.D) (*A condition on 'denominators'*) $H^0(k_{\wp,\infty}, T) = 0$ for every $\wp|p$.
(H.pS) The representation V is potentially semistable (in the sense of [FPR94, §I.2]) at any place $\wp$ dividing p.
(H.O) (*Ordinarity*) The Galois representation T is ordinary at all primes $\wp \subset k$ above p in the following sense: There exists a $\mathcal{O}[G_{k_\wp}]$-stable submodule $\mathrm{F}^+T \subset T$ (depending on $\wp$) such that

$$\mathrm{F}^\pm T \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{F}^\mp T, \mathcal{O})(1)$$

as $G_{k_\wp}$-modules (under the isomorphism induced from (0.1)) and $\mathrm{F}^-T := T/\mathrm{F}^+T$ is free as a $\mathcal{O}$-module. We also set

$$\mathrm{F}^+\mathbb{T} := \mathrm{F}^+T \otimes \Lambda \ ; \ \mathrm{F}^-\mathbb{T} := \mathbb{T}/\mathrm{F}^+\mathbb{T} = \mathrm{F}^-T \otimes \Lambda.$$

(H.TZ) (*Trivial zero condition*) Under **(H.O)**,

$$H^0(k_p, \mathrm{F}^-T \otimes \Phi/\mathcal{O}) = 0.$$

We will not need the truth of *all* of these hypotheses for all of our results, and we will carefully state which of these hypotheses are in effect before stating each claim. Finally, we remark that the hypotheses **H.1-5** are already present in [MR04, §3.5]. A variant of the hypotheses **H.6** of loc.cit. will appear shortly (in fact, we will show that it holds for the cases of interest in this paper).

Statements of the Main Results. For a Galois representation T as above, assume that the hypotheses **H.1-H.5**, **H.nE** and **H.D** hold true. Suppose that $\mathbf{c}^{(r)} = \{c_K^{(r)}\}$ is an Euler system of rank r, in the sense of Definition 2.1 below. For any number field F, define

$$\mathrm{loc}_p^s : H^1(F, T) \longrightarrow H^1_s(F_p, T),$$

where $H^1_s(F_p, T)$ is the *singular quotient* (see §1.2.1). We write loc_p^s also for the induced map $\wedge^r H^1(F,T) \to \wedge^r H^1_s(F_p,T)$. Let $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)$ denote the Bloch-Kato Selmer group attached to T^* (see §1.3.1).

Theorem A (Corollary 3.6)**.** *In addition to the hypotheses above, suppose that* $\mathbf{H.pS}$ *holds for* T. *Then*

$$\#H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*) \leq [\wedge^r H^1(k,T) : \mathcal{O}\cdot \mathrm{loc}_p^s(c_k^{(r)})].$$

See Theorem 3.9 below for our Iwasawa theoretic main result, which proves that the characteristic ideal of an appropriately defined Greenberg Selmer group divides the characteristic ideal of a certain Λ-module determined by the Euler system $\mathbf{c}^{(r)}$.

We illustrate one concrete application of our technical results, which relies on Perrin-Riou's conjectures [PR95] on p-adic L-functions (see Conjectures 1 and 2 below). Suppose that $V = T \otimes \Phi$ is the p-adic realization of a pure, self-dual motive $\mathcal{M}$ defined over k. Assume in addition that V is crystalline at p, and that 1 is not an eigenvalue for the Frobenius acting on $D_{\mathrm{cris}}(V)$. Let $L(\mathcal{M}, s)$ denote the complex L-function associated to $\mathcal{M}$.

Theorem B (Theorem 3.14)**.** *Assume Conjectures 1 and 2 of Perrin-Riou, as stated in §3.3.2 and §3.3.3 below. If* $L(\mathcal{M},0) \neq 0$*, then the Bloch-Kato Selmer group* $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)$ *is finite.*

Furthermore, the proof of Theorem 3.14 gives a bound on the size $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)$ that is explicitly related to the L-value, which goes hand in hand with the Bloch-Kato conjectures.

Although the existence of the Euler system of rank r which is used to prove Corollary 3.6, Theorems 3.9 and 3.14 is conjectural, the existence of the derived classes (which play an essential role in the proofs) is *not*, thanks to the results of [MR04] and [Büy07b].

1. Preliminaries: Local conditions and Selmer groups

1.1. **Selmer structures on T.** The notation that we have set above is in effect.

We first recall Mazur and Rubin's definition of a *Selmer structure*, in particular the *canonical Selmer structure* on T and $\mathbb{T}$.

1.1.1. *Local conditions.* Let R be a complete local noetherian ring, and let M be a $R[[G_k]]$-module which is free of finite rank over R. In this paper, we will be interested in the case when $R = \Lambda$ or its certain quotients, and M is $\mathbb{T}$ or its relevant quotients by an ideal of Λ. (For example, taking the quotient by the augmentation ideal of Λ will give us $\mathcal{O}$ and the representation T.)

For each place λ of k, a *local condition* $\mathcal{F}$ (at λ) on M is a choice of an R-submodule $H^1_{\mathcal{F}}(k_\lambda, M)$ of $H^1(k_\lambda, M)$. For the prime p, a local condition $\mathcal{F}$ at p will be a choice of an R-submodule $H^1_{\mathcal{F}}(k_p, M)$ of the semi-local cohomology group $H^1(k_p, M) := \oplus_{\wp | p} H^1(k_\wp, M)$, where the direct sum is over all the primes $\wp$ of k which lie above p.

For examples of local conditions see [MR04] Definitions 1.1.6 and 3.2.1.

Suppose that $\mathcal{F}$ is a local condition (at λ) on M. If M' is a submodule of M (resp., M'' is a quotient module), then $\mathcal{F}$ induces local conditions, which we still denote by $\mathcal{F}$, on M' (resp., on M''), by taking $H^1_{\mathcal{F}}(k_\lambda, M')$ (resp., $H^1_{\mathcal{F}}(k_\lambda, M'')$) to be the inverse image (resp., the image) of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the natural maps induced by

$$M' \hookrightarrow M, \qquad M \twoheadrightarrow M''.$$

Definition 1.1. *Propagation* of a local condition $\mathcal{F}$ on M to a submodule M' (and a quotient M'') of M is the local condition $\mathcal{F}$ on M' (and on M'') obtained following the procedure above.

For example, if I is an ideal of R, then a local condition on M induces local conditions on M/IM and $M[I]$, by *propagation*.

Definition 1.2. Define the *Cartier dual* of M to be the $R[[G_k]]$-module

$$M^* := \mathrm{Hom}(M, \mu_{p^\infty})$$

where μ_{p^∞} stands for the p-power roots of unity.

Let λ be a prime of k. There is the perfect local Tate pairing

$$<\,,\,>_\lambda : H^1(k_\lambda, M) \times H^1(k_\lambda, M^*) \longrightarrow H^2(k_\lambda, \mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

Definition 1.3. The *dual local condition* $\mathcal{F}^*$ on M^* of a local condition $\mathcal{F}$ on M is defined so that $H^1_{\mathcal{F}^*}(k_\lambda, M^*)$ is the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the local Tate pairing $<\,,\,>_\lambda$.

1.1.2. *Selmer structures and Selmer groups.* Notation from §1.1.1 is in effect throughout this section.

Definition 1.4. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

- a finite set $\Sigma(\mathcal{F})$ of places of k, including all infinite places and primes above p, and all primes where M is ramified.
- for every $\lambda \in \Sigma(\mathcal{F})$ a local condition (in the sense of §1.1.1) on M (which we view now as a $R[[\mathcal{D}_\lambda]]$-module), i.e., a choice of R-submodule

$$H^1_{\mathcal{F}}(k_\lambda, M) \subset H^1(k_\lambda, M)$$

If $\lambda \notin \Sigma(\mathcal{F})$ we will also write $H^1_{\mathcal{F}}(k_\lambda, M) = H^1_{\mathrm{f}}(k_\lambda, M)$, where the module $H^1_{\mathrm{f}}(k_\lambda, M)$ is the *finite* part of $H^1(k_\lambda, M)$, defined as in [MR04] Definition 1.1.6.

For a Selmer structure $\mathcal{F}$ on M and for each prime λ of k, define $H^1_{\mathcal{F}^*}(k_\lambda, M^*) := H^1_{\mathcal{F}}(k_\lambda, M)^\perp$ as the orthogonal complement of $H^1_{\mathcal{F}}(k_\lambda, M)$ under the local Tate pairing. The Selmer structure $\mathcal{F}^*$ on M^* (with $\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^*)$) defined in this way will be called the *dual Selmer structure*.

Definition 1.5. If $\mathcal{F}$ is a Selmer structure on M, we define the *Selmer module* $H^1_{\mathcal{F}}(k, M)$ to be the kernel of the sum of the restriction maps

$$H^1(\mathrm{Gal}(k_{\Sigma(\mathcal{F})}/k), M) \longrightarrow \bigoplus_{\lambda \in \Sigma(\mathcal{F})} H^1(k_\lambda, M)/H^1_{\mathcal{F}}(k_\lambda, M)$$

where $k_{\Sigma(\mathcal{F})}$ is the maximal extension of k which is unramified outside $\Sigma(\mathcal{F})$. We also define the dual Selmer structure in a similar fashion; just replace M by M^* and $\mathcal{F}$ by $\mathcal{F}^*$ above.

Example 1.6. In this example we recall [MR04, Definitions 3.2.1 and 5.3.2].

(i) Let $R = \mathcal{O}$ and let M be a free R-module endowed with a continuous action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_{\text{can}}$ on M by setting

$$\Sigma(\mathcal{F}_{\text{can}}) = \{\lambda : M \text{ is ramified at } \lambda\} \cup \{\wp \subset k : \wp | p\} \cup \{v | \infty\}$$

and

– if $\lambda \in \Sigma(\mathcal{F}_{\text{can}})$, $\lambda \nmid p\infty$, we set
$$H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, M) = \ker[H^1(k_\lambda, M) \longrightarrow H^1(k_\lambda^{\text{unr}}, M \otimes \Phi)],$$
where k_λ^{unr} is the maximal unramified extension of k_λ,

– if $\wp | p$, we set
$$H^1_{\mathcal{F}_{\text{can}}}(k_\wp, M) = H^1(k_\wp, M).$$

The Selmer structure $\mathcal{F}_{\text{can}}$ is called the *canonical Selmer structure* on M.

(ii) Let now $R = \Lambda$ be the cyclotomic Iwasawa algebra, and let $\mathbb{M}$ be a free R-module endowed with a continuous action of G_k, which is unramified outside a finite set of places of k. We define a Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{M}$ by setting
$$\Sigma(\mathcal{F}_\Lambda) = \{\lambda : M \text{ is ramified at } \lambda\} \cup \{\wp \subset k : \wp | p\} \cup \{v | \infty\},$$
and $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}) = H^1(k_\lambda, \mathbb{M})$ for $\lambda \in \Sigma(\mathcal{F}_\Lambda)$. The Selmer structure $\mathcal{F}_\Lambda$ is called the *canonical Selmer structure* on $\mathbb{M}$.

As in Definition 1.1, induced Selmer structure on the quotients $\mathbb{M}/I\mathbb{M}$ is still denoted by $\mathcal{F}_\Lambda$. Note that $H^1_{\mathcal{F}_\Lambda}(k_\lambda, \mathbb{M}/I\mathbb{M})$ will not usually be the same as $H^1(k_\lambda, \mathbb{M}/I\mathbb{M})$. In particular, when I is the augmentation ideal inside Λ, the Selmer structure $\mathcal{F}_\Lambda$ on $\mathbb{M}$ will not always propagate to $\mathcal{F}_{\text{can}}$ on $M := \mathbb{M} \otimes \Lambda/I$.

However, when $M = T$ and $\mathbb{T} = T \otimes \Lambda$ as in the Introduction, $\mathcal{F}_\Lambda$ on $\mathbb{T}$ *does* propagate to $\mathcal{F}_{\text{can}}$ on T, under the hypotheses $\mathbf{H}.\mathbf{T}$ and $\mathbf{H}.\mathbf{nE}$.

Remark 1.7. When $R = \Lambda$ and $\mathbb{T} = T \otimes \Lambda$ (which is one of the cases of interest), the Selmer structure $\mathcal{F}_{\text{can}}$ defined in [Büy07b, §2.1] on the quotients $T \otimes \Lambda/(f)$ may be identified, under the hypotheses $\mathbf{H}.\mathbf{T}$ and $\mathbf{H}.\mathbf{nE}$, by the propagation of $\mathcal{F}_\Lambda$ to the quotients $T \otimes \Lambda/(f)$, for every distinguished polynomial $f \in \Lambda$. Indeed, for every prime $\lambda \subset k$, the submodule
$$H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, T \otimes \Lambda/(f)) \subset H^1(k_\lambda, T \otimes \Lambda/(f))$$
is the image of the canonical map $H^1(k_\lambda, T \otimes \Lambda) \to H^1(k_\lambda, T \otimes \Lambda/(f))$, by the proofs of [Büy07b, Proposition 2.10 and 2.12]. By definition, $H^1_{\mathcal{F}_\Lambda}(k_\lambda, T \otimes \Lambda/(f))$ is exactly the same thing.

Definition 1.8. A *Selmer triple* is a triple $(M, \mathcal{F}, \mathcal{P})$, where $\mathcal{F}$ is a Selmer structure on M and $\mathcal{P}$ is a set of primes as in the Introduction, namely a set of non-archimedean primes of k disjoint from $\Sigma(\mathcal{F})$.

1.2. **Modifying local conditions at p.** When the core Selmer rank of a Selmer structure (in the sense of [MR04], see also §1.4 below) is greater than one, it produces a Selmer group which is difficult to control using the Kolyvagin system machinery of Mazur and Rubin. We will see in §1.4 that $\mathcal{F}_{\text{can}}$ on T (resp., $\mathcal{F}_\Lambda$ on $\mathbb{T} = T \otimes \Lambda$) has core Selmer rank $r := d \cdot [k : \mathbb{Q}]$ where $d = \frac{1}{2}\text{rank}_{\mathcal{O}} T$ (under the hypotheses $\mathbf{H}.\mathbf{nE}$). Hence, to be able to utilize the Kolyvagin system machinery, we need to modify $\mathcal{F}_{\text{can}}$ and $\mathcal{F}_\Lambda$ appropriately. This is what we do in this section.

1.2.1. *Local conditions at* p *over* k.

Lemma 1.9. *Under the hypotheses* $\mathbf{H}.\mathbf{nE}$ *and* $\mathbf{H}.\mathbf{D}$*,*
$$H^1(k_p, T) := \bigoplus_{\wp | p} H^1(k_\wp, T)$$
is a free $\mathcal{O}$*-module of rank* $2r$.

Proof. All the references here are to [Büy07c, Appendix] and the results quoted here are originally due to Perrin-Riou.

By Theorem A.8(i), Λ-torsion submodule $H^1(k_p, \mathbb{T})_{\text{tors}}$ is isomorphic to $\oplus_{\wp|p} T^{H_{k_\wp}}$, where $H_{k_\wp} = \text{Gal}(\overline{k_\wp}/k_{\wp,\infty})$, and this module is trivial thanks to $\mathbf{H.D}$. Theorem A.8(ii) now concludes that the Λ-module $H^1(k_p, \mathbb{T})$ is free rank $2r$. Furthermore,

$$\text{coker}[H^1(k_p, \mathbb{T}) \longrightarrow H^1(k_p, T)] = H^2(k_p, \mathbb{T})[\gamma - 1],$$

where γ is any topological generator of Γ. Since we assumed $\mathbf{H.nE}$ holds, it follows from [Büy07b, Lemma 2.11] that $H^2(k_p, \mathbb{T}) = 0$, hence the map

$$H^1(k_p, \mathbb{T}) \longrightarrow H^1(k_p, T)$$

is surjective. Lemma now follows. □

Bloch and Kato [BK90, §3] define a subspace $H^1_f(k_\wp, V) \subset H^1(k_\wp, V)$ by letting

$$H^1_f(k_\wp, V) := \ker\left(H^1(k_\wp, V) \longrightarrow H^1(k_\wp, V \otimes B_{\text{cris}})\right),$$

where B_{cris} is Fontaine's crystalline period ring. We propagate the *Bloch-Kato local condition* $H^1_f(k_\wp, V)$ on V to T:

$$\begin{aligned} H^1_f(k_\wp, T) &:= \ker\left(H^1(k_\wp, T) \longrightarrow \frac{H^1(k_\wp, V)}{H^1_f(k_\wp, V)}\right) \\ &= \ker\left(H^1(k_\wp, T) \longrightarrow H^1(k_\wp, V \otimes B_{\text{cris}})\right) \end{aligned}$$

We define the singular quotient as $H^1_s(k_\wp, T) := H^1(k_\wp, T)/H^1_f(k_\wp, T)$. Note that $H^1_s(k_\wp, T)$ is a free $\mathcal{O}$-module as it injects, by definition, into $H^1(k_\wp, V)/H^1_f(k_\wp, V)$.

Assume until the end of §1.2.1 that V satisfies $\mathbf{H.pS}$. In this case, it is well known that $H^1_f(k_\wp, V)$ and $H^1_f(k_\wp, V^*)$ are orthogonal complements under the local Tate pairing (see [FPR94, Proposition I.3.3.9(iii)]). Since we assumed that T is self-dual, we conclude from Lemma 1.9 the following:

Proposition 1.10. *Both $\mathcal{O}$-modules*

$$H^1_f(k_p, T) := \bigoplus_{\wp|p} H^1_f(k_\wp, T), \text{ and } H^1_s(k_p, T) := \bigoplus_{\wp|p} H^1_s(k_\wp, T)$$

are free of rank r.

Fix an $\mathcal{O}$-rank one direct summand $\mathcal{L} \subset H^1(k_p, T)$ such that

$$\mathcal{L} \cap H^1_f(k_p, T) = \{0\}.$$

We will also write $\mathcal{L}$ for the (isomorphic) image of $\mathcal{L}$ inside $H^1_s(k_p, T)$ under the surjection

$$H^1(k_p, T) \longrightarrow H^1_s(k_p, T).$$

Definition 1.11. Define the *$\mathcal{L}$-modified Selmer structure $\mathcal{F}_\mathcal{L}$* on T as follows:

- $\Sigma(\mathcal{F}_\mathcal{L}) = \Sigma(\mathcal{F}_{\text{can}})$,
- if $\lambda \nmid p$, $H^1_{\mathcal{F}_\mathcal{L}}(k_\lambda, T) = H^1_{\mathcal{F}_{\text{can}}}(k_\lambda, T)$,
- $H^1_{\mathcal{F}_\mathcal{L}}(k_p, T) := H^1_f(k_p, T) \oplus \mathcal{L} \subset H^1(k_p, T) = H^1_{\mathcal{F}_{\text{can}}}(k_p, T)$.

1.2.2. *Local conditions at p over k_∞.* Recall that k_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of k, and $\Gamma = \mathrm{Gal}(k_\infty/k)$. Assume that the hypothesis $\mathbf{H.Iw}$. holds in this section. Let $k_\wp$ denote the completion of k at $\wp$, and let $k_{\wp,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $k_\wp$. We may therefore identify $\mathrm{Gal}(k_{\wp,\infty}/k_\wp)$ by Γ for all $\wp|p$ and henceforth Γ will stand for any of these Galois groups. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra, as usual. We also fix a topological generator γ of Γ, and set $\mathbf{X} = \gamma - 1$ (and we occasionally identify Λ by the power series ring $\mathbb{Z}_p[[\mathbf{X}]]$).

Lemma 1.12. *Suppose* $\mathbf{H.nE}$ *and* $\mathbf{H.D}$ *holds. Then*

$$H^1(k_p, \mathbb{T}) := \oplus_{\wp|p} H^1(k_\wp, \mathbb{T})$$

is a free Λ-module of rank $2r$.

Proof. This is already proved in the first part of the proof of Lemma 1.9. □

Assume $\mathbf{H.O}$ and $\mathbf{H.TZ}$ until the end of §1.2.2. We define the *Greenberg local conditions* at p by setting

$$H^1_{\mathrm{Gr}}(k_\wp, \mathbb{T}) := \ker\left(H^1(k_\wp, \mathbb{T}) \longrightarrow H^1(k_\wp, \mathrm{F}^-\mathbb{T})\right).$$

By definition, there is an exact sequence of Λ-modules

$$0 \longrightarrow \mathrm{F}^-\mathbb{T} \xrightarrow{\gamma-1} \mathrm{F}^-\mathbb{T} \longrightarrow \mathrm{F}^-T \longrightarrow 0. \tag{1.1}$$

Taking $G_{k_\wp}$-invariance of the sequence (1.1) and using $\mathbf{H.TZ}$ and Nakayama's lemma, we conclude that $H^0(k_\wp, \mathrm{F}^-\mathbb{T}) = 0$. This in return implies that the map

$$H^1(k_\wp, \mathrm{F}^+\mathbb{T}) \longrightarrow H^1(k_\wp, \mathbb{T})$$

(induced from the $G_{k_\wp}$-cohomology of the sequence $0 \to \mathrm{F}^+\mathbb{T} \to \mathbb{T} \to \mathrm{F}^-\mathbb{T} \to 0$) is injective and the image of $H^1(k_\wp, \mathrm{F}^+\mathbb{T})$ is exactly $H^1_{\mathrm{Gr}}(k_\wp, \mathbb{T})$.

Proposition 1.13. *Let $r_\wp := [k_\wp : \mathbb{Q}_p] \cdot \mathrm{rank}_{\mathcal{O}}\, \mathrm{F}^+T$.*

(i) *$H^1(k_\wp, \mathrm{F}^+\mathbb{T})$ is a free Λ-module of rank $r_\wp$.*

(ii) *The natural map*

$$H^1(k_\wp, \mathrm{F}^+\mathbb{T}) \longrightarrow H^1(k_\wp, \mathrm{F}^+T)$$

is surjective.

(iii) *$H^1(k_\wp, \mathrm{F}^+T)$ is a free $\mathcal{O}$-module of rank $r_\wp$.*

Proof. The long exact sequence of the $G_{k_\wp}$-cohomology yields an exact sequence

$$H^0(k_\wp, \mathrm{F}^-\mathbb{T}) \longrightarrow H^1(k_\wp, \mathrm{F}^+\mathbb{T}) \longrightarrow H^1(k_\wp, \mathbb{T}).$$

As explained above, one may deduce from $\mathbf{H.TZ}$ that $H^0(k_\wp, \mathrm{F}^-\mathbb{T}) = 0$, so it follows from Lemma 1.12 that $H^1(k_\wp, \mathrm{F}^+\mathbb{T})$ is Λ-torsion free. (i) now follows from [Büy07c, Theorem A.8(ii)].

Long exact sequence of the $G_{k_\wp}$-cohomology of the sequence

$$0 \longrightarrow \mathrm{F}^+\mathbb{T} \xrightarrow{\gamma-1} \mathrm{F}^+\mathbb{T} \longrightarrow \mathrm{F}^+T \longrightarrow 0$$

gives

$$\mathrm{coker}\left(H^1(k_\wp, \mathrm{F}^+\mathbb{T}) \longrightarrow H^1(k_\wp, \mathrm{F}^+T)\right) = H^2(k_\wp, \mathrm{F}^+\mathbb{T})[\gamma - 1].$$

As in the proof of Lemma 1.9,

$$H^2(k_\wp, \mathrm{F}^+\mathbb{T})[\gamma - 1] = 0 \iff H^0(k_\wp, (\mathrm{F}^+T)^*) = 0,$$

and the latter vanishing follows from the condition

$$\mathrm{F}^{-}T \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{F}^{+}T, \mathcal{O})(1) \tag{1.2}$$

of **H.O**, and the hypotheses **H.TZ**. This completes the proof of (ii). (iii) follows at once from (i) and (ii). ☐

Corollary 1.14. *The* Λ*-module*

$$H^1_{\mathrm{Gr}}(k_p, \mathbb{T}) := \bigoplus_{\wp|p} H^1_{\mathrm{Gr}}(k_\wp, \mathbb{T})$$

is free of rank r.

Proof. As a consequence of (1.2), $\mathrm{rank}_{\mathcal{O}}\,\mathrm{F}^{+}T = \frac{1}{2}\mathrm{rank}_{\mathcal{O}}\,T$, hence $\sum_{\wp|p} r_\wp = r$. ☐

Definition 1.15. Fix a Λ-rank one direct summand $\mathbb{L} \subset H^1(k_p, \mathbb{T})$ such that $\mathbb{L} \cap H^1_{\mathrm{Gr}}(k_p, \mathbb{T}) = \{0\}$. Define the $\mathbb{L}$*-modified Selmer structure* $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$ as follows:

- $\Sigma(\mathcal{F}_{\mathbb{L}}) = \Sigma(\mathcal{F}_{\Lambda})$,
- if $\lambda \nmid p$, $H^1_{\mathcal{F}_{\mathbb{L}}}(k_\lambda, \mathbb{T}) = H^1_{\mathcal{F}_{\Lambda}}(k_\lambda, \mathbb{T})$,
- $H^1_{\mathcal{F}_{\mathbb{L}}}(k_p, \mathbb{T}) := H^1_{\mathrm{Gr}}(k_p, \mathbb{T}) \oplus \mathbb{L} \subset H^1(k_p, \mathbb{T}) = H^1_{\mathcal{F}_{\Lambda}}(k_p, \mathbb{T})$.

Remark 1.16. Note that we used two different approaches to choose local conditions in §1.2.1 (over k) and in §1.2.2 (over k_∞). Starting from $H^1_{\mathrm{Gr}}(k_p, \mathbb{T})$, we may consider the image of $H^1_{\mathrm{Gr}}(k_p, \mathbb{T})$ under the canonical map

$$H^1(k_p, \mathbb{T}) \longrightarrow H^1(k_p, T)$$

and denote this image by $H^1_{\mathrm{Gr}}(k_p, T) \subset H^1(k_p, T)$. The choice $H^1_{\mathrm{Gr}}(k_p, T) \subset H^1(k_p, T)$ will be called the *Greenberg local condition* on T. It is easy to see (thanks to Proposition 1.13(ii) and (iii)) that $H^1_{\mathrm{Gr}}(k_p, T)$ coincides with the image of $H^1(k_p, \mathrm{F}^{+}T) \hookrightarrow H^1(k_p, T)$. In several cases of interest, the Selmer group determined by the Bloch-Kato definition agrees with the Selmer group determined by the Greenberg definition.

1.3. **Global duality and a comparison of Selmer groups.** In this section, we compare classical Selmer groups (which we wish to relate to the L-values) to modified Selmer groups (for which we are able to apply the Kolyvagin system machinery and compute in terms of L-values). The necessary tool to accomplish this comparison is the Poitou-Tate global duality.

1.3.1. *Comparison over* k. We first define the classical (Bloch-Kato) Selmer structure and Selmer group for T (resp., for T^*). Let $\mathcal{F}_{\mathrm{BK}}$ denote the Selmer structure on T given by

- $\Sigma(\mathcal{F}_{\mathrm{BK}}) = \Sigma(\mathcal{F}_{\mathrm{can}}) = \Sigma(\mathcal{F}_{\mathcal{L}})$,
- For $\lambda \nmid p$, $H^1_{\mathcal{F}_{\mathrm{BK}}}(k_\lambda, T) = H^1_{\mathcal{F}_{\mathrm{can}}}(k_\lambda, T) = H^1_{\mathcal{F}_{\mathcal{L}}}(k_\lambda, T)$,
- $H^1_{\mathcal{F}_{\mathrm{BK}}}(k_p, T) = H^1_f(k_p, T) \subset H^1(k_p, T) = H^1_{\mathcal{F}_{\mathrm{can}}}(k_p, T)$.

Following the procedure of Definition 1.3, define also $\mathcal{F}^*_{\mathrm{BK}}$ on T^*. Then, by definition, we have the following exact sequences:

$$0 \longrightarrow H^1_{\mathcal{F}_{\mathrm{BK}}}(k, T) \longrightarrow H^1_{\mathcal{F}_{\mathcal{L}}}(k, T) \xrightarrow{\mathrm{loc}_p^s} \mathcal{L}$$

$$0 \longrightarrow H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*) \longrightarrow H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k, T^*) \xrightarrow{\mathrm{loc}_p^*} \frac{H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k_p, T^*)}{H^1_{\mathcal{F}^*_{\mathcal{L}}}(k_p, T^*)}$$

where loc_p^s is the compositum $\mathrm{loc}_p^s : H^1(k,T) \to H^1(k_p,T) \to H^1_s(k_p,T)$. The Poitou-Tate global duality theorem (c.f., [Rub00, Theorem I.7.3], [Mil86, Theorem I.4.10], [MR04, Theorem 2.3.4]) allows us to compare the image of loc_p^s to the image of loc_p^*:

Proposition 1.17. *There is an exact sequence*

$$0 \to \frac{H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)}{H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T)} \xrightarrow{\mathrm{loc}_p^s} \mathcal{L} \xrightarrow{(\mathrm{loc}_p^*)^\vee} \left(H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)\right)^\vee \to \left(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)\right)^\vee \to 0,$$

where the map $(\mathrm{loc}_p^*)^\vee$ is induced from localization at p and the local Tate pairing between $H^1(k_p,T)$ and $H^1(k_p,T^*)$.

Corollary 1.18. *The quotient $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)/H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)$ is finite if and only if $\mathrm{loc}_p^s(H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)) \neq 0$.*

Proof. Since $\mathcal{L}$ is a free $\mathcal{O}$-module of rank one, this is immediate from Proposition 1.17. □

Corollary 1.19. *Suppose $H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T)=0$. Let $c \in H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)$ be any class. Then the following sequence is exact:*

$$0 \to \frac{H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)}{\mathcal{O}\cdot c} \xrightarrow{\mathrm{loc}_p^s} \frac{\mathcal{L}}{\mathcal{O}\cdot \mathrm{loc}_p^s(c)} \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)\right)^\vee \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k,T^*)\right)^\vee \to 0.$$

Proof. Note that the assumption $H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T)=0$ forces the map loc_p^s to be injective. Corollary follows from Proposition 1.17. □

Remark 1.20. The assumption that $H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T)=0$ may seem like an unreasonably strong assumption at the moment, however, we will be able to rephrase this assumption in terms of an Euler system of rank r later on.

1.3.2. *Comparison over k_∞.* For a fixed topological generator γ of Γ, set $\gamma_n := \gamma^{p^n}$, and let $\mathcal{L}_n \subset H^1(k_p,\mathbb{T}/(\gamma_n-1)\mathbb{T})$ be the image of $\mathbb{L}$ under the map $H^1(k_p,\mathbb{T}) \to H^1(k_p,\mathbb{T}/(\gamma_n-1)\mathbb{T})$. Let $\mathcal{F}_{\mathcal{L}_n}$ denote the Selmer structure on $\mathbb{T}/(\gamma_n-1)\mathbb{T}$, which is obtained by propagating the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$ to its quotient $\mathbb{T}/(\gamma_n-1)\mathbb{T}$. The propagated Selmer structure from $\mathcal{F}_{\mathrm{Gr}}$ on $\mathbb{T}$ to the quotient $\mathbb{T}/(\gamma_n-1)\mathbb{T}$ will still be denoted by $\mathcal{F}_{\mathrm{Gr}}$.

By Shapiro's lemma, one may canonically identify $H^1(k,\mathbb{T}/(\gamma_n-1)\mathbb{T})$ by $H^1(k_n,T)$; and for every prime $\lambda \subset k$, one may identify $H^1(k_\lambda,\mathbb{T}/(\gamma_n-1)\mathbb{T})$ by $H^1((k_n)_\lambda,T)$; c.f., [Rub00, Appendix B.4 and B.5]. This way, we may view $\mathcal{F}_{\mathcal{L}_n}$ and $\mathcal{F}_{\mathrm{Gr}}$ as Selmer structures on the G_{k_n}-representation T.

Repeating the argument of Proposition 1.17 for each field k_n (instead of k) with Selmer structures $\mathcal{F}_{\mathrm{Gr}}$ and $\mathcal{F}_{\mathcal{L}_n}$ and passing to inverse limit we obtain the following:

Proposition 1.21. *The following sequences of Λ-modules are exact:*

(i) $0 \to \frac{H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})}{H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,\mathbb{T})} \xrightarrow{\mathrm{loc}_p^s} \mathbb{L} \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(k,\mathbb{T}^*)\right)^\vee \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)\right)^\vee \to 0.$

If further $H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,T)$ defined in Remark 1.16 vanishes, then,

(ii) *for any class $c \in H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})$,*

$$0 \longrightarrow \frac{H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})}{\Lambda\cdot c} \xrightarrow{\mathrm{loc}_p^s} \frac{\mathbb{L}}{\Lambda\cdot \mathrm{loc}_p^s(c)} \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(k,\mathbb{T}^*)\right)^\vee \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)\right)^\vee \longrightarrow 0.$$

Proof. We give a sketch of the proof. As in Proposition 1.17, we have an exact sequence

$$0 \longrightarrow \frac{H^1_{\mathcal{F}_{\mathcal{L}_n}}(k_n,T)}{H^1_{\mathcal{F}_{\mathrm{Gr}}}(k_n,T)} \xrightarrow{\mathrm{loc}_p^s} \mathcal{L}_n \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(k_n,T^*)\right)^\vee \longrightarrow \left(H^1_{\mathcal{F}^*_{\mathcal{L}_n}}(k_n,T^*)\right)^\vee \longrightarrow 0$$

for each n. Passing to inverse limit (and making use of [Rub00, Proposition B.1.1]) we obtain the exact sequence of (i).

For (ii), note that there is an injection $H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,\mathbb{T})/(\gamma-1) \hookrightarrow H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,T)$ induced from the exact sequence

$$H^1(k,\mathbb{T}) \xrightarrow{\gamma-1} H^1(k,\mathbb{T}) \longrightarrow H^1(k,T).$$

Therefore, our assumption that $H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,T)=0$ implies, by Nakayama's lemma, that $H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,\mathbb{T})=0$. (ii) now follows from (i). □

1.4. **Kolyvagin systems for modified Selmer structures.** Throughout §1.4 we assume that the hypotheses $\mathbf{H}.\mathbf{1}$-$\mathbf{5}$ hold for T. Assume in addition that $\mathbf{H}.\mathbf{nE}$ and $\mathbf{H}.\mathbf{T}$ (also $\mathbf{H}.\mathbf{O}$ whenever we refer to Greenberg's local conditions) hold.

One may apply [MR04, Lemma 3.7.1] to verify that all the three Selmer triples $(T,\mathcal{F}_{\mathrm{BK}},\mathcal{P})$, $(T,\mathcal{F}_{\mathrm{Gr}},\mathcal{P})$ and $(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ satisfy the hypothesis $\mathbf{H}.\mathbf{6}$ of [MR04, §3.5] (with base field $\mathbb{Q}$ in their treatment replaced by k). Therefore, the existence of Kolyvagin systems for these Selmer structures will be decided by their *core Selmer ranks* (c.f., [MR04, Definition 4.1.8 and 4.1.11]). Let $\mathcal{X}(T,\mathcal{F})$ denote the core Selmer rank of the Selmer structure $\mathcal{F}$ on T, for $\mathcal{F}=\mathcal{F}_{\mathrm{BK}},\mathcal{F}_{\mathrm{Gr}}$ or $\mathcal{F}_{\mathcal{L}}$.

Proposition 1.22. $\mathcal{X}(T,\mathcal{F}_{\mathrm{BK}})=\mathcal{X}(T,\mathcal{F}_{\mathrm{Gr}})=0$.

Proof. It follows from our assumption that T is self-dual that

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T/\mathfrak{m}T)\cong H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T^*[\mathfrak{m}]),\ \text{and}\ H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,T/\mathfrak{m}T)\cong H^1_{\mathcal{F}_{\mathrm{Gr}}}(k,T^*[\mathfrak{m}]).$$

Proposition now follows from the definition of the core Selmer rank (see [MR04, Definition 5.2.4 and Proposition 5.2.5]) .

□

Proposition 1.23. $\mathcal{X}(T,\mathcal{F}_{\mathcal{L}})=1$.

Proof. By [MR04, Definition 5.2.4 and Proposition 5.2.5] and [Wil95, Proposition 1.6]

$$\mathcal{X}(T,\mathcal{F})=\dim_{\mathbb{F}} H^1_{\mathcal{F}}(k,T/\mathfrak{m}T)-\dim_{\mathbb{F}} H^1_{\mathcal{F}^*}(k,T^*[\mathfrak{m}])=$$

$$\dim_{\mathbb{F}} H^0(k,T/\mathfrak{m}T)-\dim_{\mathbb{F}} H^0(k,T^*[\mathfrak{m}])-\sum_{\lambda\in\Sigma(\mathcal{F})}\left\{\dim_{\mathbb{F}} H^0(k_\lambda,T/\mathfrak{m}T)-\dim_{\mathbb{F}} H^1_{\mathcal{F}}(k_\lambda,T/\mathfrak{m}T)\right\}$$

Applying this formula with $\mathcal{F}=\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}=\mathcal{F}_{\mathrm{BK}}$ we see that

$$\mathcal{X}(T,\mathcal{F}_{\mathcal{L}})-\mathcal{X}(T,\mathcal{F}_{\mathrm{BK}})=\dim_{\mathbb{F}} H^1_{\mathcal{F}_{\mathcal{L}}}(k_p,T/\mathfrak{m}T)-\dim_{\mathbb{F}} H^1_{\mathcal{F}_{\mathrm{BK}}}(k_p,T/\mathfrak{m}T)$$

and this equals one by the very definition of the $\mathcal{L}$-modified Selmer structure. We already know by Proposition 1.22 that $\mathcal{X}(T,\mathcal{F}_{\mathrm{BK}})=0$ and the proof follows.

Note that if we assumed $\mathbf{H}.\mathbf{O}$ (instead of assuming $\mathbf{H}.\mathbf{pS}$), and used $\mathcal{F}_{\mathrm{Gr}}$ of Remark 1.16 (instead of $\mathcal{F}_{\mathrm{BK}}$) in order to define $H^1_{\mathcal{F}_{\mathcal{L}}}(k_p,T)=H^1_{\mathrm{Gr}}(k_p,T)\oplus\mathcal{L}$, the same proof would lead us to the identical result about $\mathcal{X}(T,\mathcal{F}_{\mathcal{L}})$. □

Remark 1.24. When $\mathbf{H}.\mathbf{nE}$ holds, one may also check using [MR04, Theorem 5.2.15] that $\mathcal{X}(T,\mathcal{F}_{\mathrm{can}})=r$.

1.4.1. *Kolyvagin systems over k.* We write $\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ for the $\mathcal{O}$-module of Kolyvagin systems for the Selmer triple $(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$. We refer the reader to [MR04, Definition 3.1.3] for a definition of this module. Assume that the hypotheses $\mathbf{H.1\text{-}5}$ and $\mathbf{H.nE}$ (also $\mathbf{H.O}$ whenever we refer to Greenberg's local conditions) hold.

Proposition 1.25. *The $\mathcal{O}$-module $\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ is free of rank* one*, generated by a Kolyvagin system $\kappa \in \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ whose image (under the canonical map induced from reduction mod $\mathfrak{m}$) in $\mathbf{KS}(T/\mathfrak{m}T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ is nonzero.*

Proof. This is immediate after Proposition 1.23 and [MR04, Theorem 5.2.10]. □

Remark 1.26. Note that the choice of a rank one direct summand $\mathcal{L} \subset H^1(k_p,T)$ makes our approach somewhat unnatural. This issue is addressed in this paragraph. Put

$$H^1(k_p,T)=\bigoplus_{i=1}^{r}\mathcal{L}_i \oplus H^1_f(k_p,T) \tag{1.3}$$

(where each $\mathcal{L}_i$ is a free $\mathcal{O}$-submodule of $H^1(k_p,T)$ of rank one) and consider

$$\sum_{i=1}^{r}\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i},\mathcal{P}) \subset \mathbf{KS}(T,\mathcal{F}_{\text{can}},\mathcal{P}). \tag{1.4}$$

Claim. *The sum in (1.4) is in fact a direct sum.*

Proof. Assume contrary: Suppose there exist $\boldsymbol{\kappa}^i \in \mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i},\mathcal{P})$ such that $\sum_{i=1}^{r}\boldsymbol{\kappa}^i=0$, and not all $\boldsymbol{\kappa}^i=0$; say without loss of generality $\boldsymbol{\kappa}^1\neq 0$. Then

$$\boldsymbol{\kappa}^1=-\sum_{i\neq 1}\kappa^i \in \sum_{i\neq 1}\mathbf{KS}(T,\mathcal{F}_{\mathcal{L}_i},\mathcal{P}).$$

This means, for every $\eta \in \mathcal{N}(\mathcal{P})$ (:= square free products of primes in $\mathcal{P}$)

$$\mathcal{L}_1/I_\eta\mathcal{L}_1 \ni \mathrm{loc}_p^s(\kappa_\eta^1)=-\sum_{i\neq 1}\mathrm{loc}_p^s(\kappa_\eta^i)\in\bigoplus_{i\neq 1}\mathcal{L}_i/I_\eta\mathcal{L}_i \tag{1.5}$$

Here $I_\eta := \prod_{\lambda|\eta} I_\lambda \subset \mathcal{O}$, and for $\lambda \in \mathcal{P}$, the ideal $I_\lambda \subset \mathcal{O}$ is as defined in the introduction. The equality of (1.5) takes place in

$$H^1_s(k_p,T/I_\eta T):=\frac{H^1(k_p,T/I_\eta T)}{H^1_f(k_p,T/I_\eta T)},$$

where $H^1_f(k_p,T/I_\eta T)$ is the image of $H^1_f(k_p,T)$ under the surjective (thanks to $\mathbf{H.nE}$) map

$$H^1(k_p,T)\twoheadrightarrow H^1(k_p,T/I_\eta T),$$

which is induced from the surjection $T \twoheadrightarrow T/I_\eta T$. We therefore have a decomposition

$$H^1(k_p,T/I_\eta T)\cong H^1_f(k_p,T/I_\eta T)\oplus\bigoplus_{i=1}^{r}\mathcal{L}_i/I_\eta\mathcal{L}_i.$$

Since $\left(\bigoplus_{i\neq 1}\mathcal{L}_i/I_\eta\mathcal{L}_i\right)\cap\mathcal{L}_1/I_\eta\mathcal{L}_1=\{0\}$, it follows from (1.5) that $\mathrm{loc}_p^s(\kappa_\eta^1)=0$, i.e.,

$$\mathrm{loc}_p(\kappa_\eta^1)\in H^1_f(k_p,T/I_\eta T)$$

for every $\eta \in \mathcal{N}(\mathcal{P})$. This means $\boldsymbol{\kappa}^1 \in \mathbf{KS}(T,\mathcal{F}_{\text{BK}},\mathcal{P})$. On the other hand $\mathbf{KS}(T,\mathcal{F}_{\text{BK}},\mathcal{P})=0$ by Proposition 1.22 and [MR04, Theorem 5.2.10(i)], hence we proved $\boldsymbol{\kappa}^1=0$, a contradiction. □

As in [Büy08, Remark 1.27], we pose the following:
Question: Is the direct sum

$$\bigoplus_{i=1}^{r} \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}_i}, \mathcal{P}) \subset \mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$$

independent of the choice of the decomposition (1.3)?

When the answer to this question is affirmative, we would have a *canonical* rank r submodule of $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$. It would be even more interesting to see if this rank r submodule descends from Euler systems (via the Euler systems to Kolyvagin systems map of Mazur and Rubin [MR04, Theorem 3.2.4]). Below, we construct such a (rank r) submodule of $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ which descends from an Euler system of rank r (in case it exists); however, this module *does* depend on the decomposition (1.3).

1.4.2. *Kolyvagin systems over k_∞.* For every $s, m \in \mathbb{Z}^+$ and for a fixed topological generator of γ of Γ, write $T_{s,m} = \mathbb{T}/(p^s, (\gamma-1)^m)$.

Definition 1.27.(Compare to [MR04, Definition 3.1.6].) Define the module of Λ-*adic Kolyvagin systems* as

$$\overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\mathbb{L}}, \mathcal{P}) := \varprojlim_{s,m} \varinjlim_{j} \mathbf{KS}(T_{s,m}, \mathcal{F}_{\mathbb{L}}, \mathcal{P}_j),$$

where $\mathbf{KS}(T_{s,m}, \mathcal{F}_{\mathbb{L}}, \mathcal{P}_j)$ is the module of Kolyvagin systems for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on the representation $T_{s,m}$, as in [MR04, Definition 3.1.3].

The analogue of [MR04, Theorem 5.2.10], which we used to prove Proposition 1.25, for the big Galois representation $\mathbb{T}$ has been proved by the author in [Büy07b, Theorem 3.23]. Under the hypotheses $\mathbf{H}.\mathbf{1}$-$\mathbf{5}$, $\mathbf{H}.\mathbf{nE}$, $\mathbf{H}.\mathbf{T}$ and $\mathbf{H}.\mathbf{O}$, this result together with Proposition 1.25 (now using $\mathcal{F}_{\text{Gr}}$ on $\mathbb{T}$ (instead of $\mathcal{F}_{\text{BK}}$) to define $\mathcal{F}_{\mathcal{L}}$ on T) can be used to show:

Proposition 1.28. *The Λ-module of Kolyvagin Systems $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P})$ for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$ is free of rank one. Furthermore, the canonical map*

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P}) \longrightarrow \mathbf{KS}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$$

is surjective.

Proof. Theorem 3.23 of [Büy07b] is proved for the canonical Selmer structure $\mathcal{F}_\Lambda = \mathcal{F}_{\text{can}}$ on $\mathbb{T}$, under the condition that $\mathcal{X}(T, \mathcal{F}_{\text{can}}) = 1$. Under the running hypotheses, which in particular imply (Proposition 1.23) that $\mathcal{X}(T, \mathcal{F}_{\mathcal{L}}) = 1$, the proof of [Büy07b, Theorem 3.23] applies *verbatim* for the Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$. $\square$

Remark 1.29. In this remark, we do not assume any longer that T is self-dual. Let $d_+ = \text{rank}_{\mathcal{O}}(\text{Ind}_{k/\mathbb{Q}} T)^+$, the rank of the $(+1)$-eigenspace of the action of a complex conjugation on $\text{Ind}_{k/\mathbb{Q}} T$. Let $H^1_f(k_p, T)$ (resp., $H^1_{\text{Gr}}(k_p, \mathbb{T})$) be as in §1.2.1 (resp., as in §1.2.2). Remark 1.24 still holds, and if we assume

$$d_+ = \text{rank}_{\mathcal{O}} H^1_f(k_p, T) \ \ (\text{resp., } d_+ = \text{rank}_\Lambda H^1_{\text{Gr}}(k_p, \mathbb{T})), \tag{1.6}$$

it is not hard to see that Propositions 1.22, 1.23 and 1.25 (resp., Proposition 1.28) hold as well under their running hypotheses (with $\mathcal{F}_{\mathcal{L}}$ (resp., with $\mathcal{F}_{\mathbb{L}}$) still defined as in Definition 1.11 (resp., Definition 1.15)). We note that if T is self-dual and $\mathbf{H}.\mathbf{pS}$ (resp., $\mathbf{H}.\mathbf{O}$) holds, then (1.6) is satisfied.

2. EULER SYSTEMS OF RANK r AND THE EULER SYSTEMS TO KOLYVAGIN SYSTEMS MAP

Suppose k, T, r and $\mathcal{P}$ are as above. We write $\mathcal{N} = \mathcal{N}(\mathcal{P})$ for the collection of integral ideals $\tau \subset k$ which are square free products of primes in $\mathcal{P}$. As before, we write $k(\mathfrak{q})$ for the maximal p-extension of k inside the ray class field of k modulo $\mathfrak{q}$ and let $\mathrm{Fr}_{\mathfrak{q}}$ denote an arithmetic Frobenius at $\mathfrak{q}$ in G_k. If $\tau = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ is an ideal in $\mathcal{N}$, we let $k(\tau)$ denote the compositum

$$k(\tau) := k(\mathfrak{q}_1) \cdots k(\mathfrak{q}_s),$$

and set $k_n(\tau) := k_n \cdot k(\tau)$. We write $\mathfrak{C} = \{k_n(\tau) : \tau \in \mathcal{N}, n \in \mathbb{Z}_{\geq 0}\}$, and $\mathcal{K} = \bigcup_{F \in \mathfrak{C}} F$. We set $\Delta^\tau = \mathrm{Gal}(k(\tau)/k)$ and $\Delta_n^\tau = \mathrm{Gal}(k_n(\tau/k)) = \Delta^\tau \times \Gamma_n$. Finally, define[4] $T^{\mathcal{D}} = \mathrm{Hom}(T, \mathcal{O}(1))$ and let

$$P_{\mathfrak{q}}(x) := \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} \cdot x | T^{\mathcal{D}}) \in \mathcal{O}[x]$$

be the Euler factor at the prime $\mathfrak{q} \in \mathcal{P}$ associated with the Galois representation $T^{\mathcal{D}}$.

For any finite group G and a finitely generated $\mathbb{Z}_p[G]$-module M we define (following [Rub96, §1.2])

$$\begin{aligned} \wedge_0^r M := \{m \in \mathbb{Q}_p \otimes \wedge^r M : (\psi_1 \wedge \cdots \wedge \psi_r)(m) \in \mathbb{Z}_p[G] \\ \text{for every } \psi_1, \ldots, \psi_r \in \mathrm{Hom}_{\mathbb{Z}_p[G]}(M, \mathbb{Z}_p[G])\}. \end{aligned}$$

where the exterior power is calculated in the category of $\mathbb{Z}_p[G]$-modules.

Definition 2.1. An *Euler system of rank r* is a collection $\mathbf{c} = \{c_{k_n(\tau)}\}$ such that

(i) $c_{k_n(\tau)} \in \wedge_0^r H^1(k_n(\tau), T)$,
(ii) for $\tau' | \tau$ and $n \leq n'$

$$\mathrm{Cor}^r_{k_n(\tau)/k_{n'}(\tau')}\left(c_{k_n(\tau)}\right) = \left(\prod_{\substack{\mathfrak{q} | \tau \\ \mathfrak{q} \nmid \tau'}} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\right) c_{k_{n'}(\tau')}$$

where $\mathrm{Cor}^r_{k_n(\tau)/k_{n'}(\tau')}$ is the map induced from the corestriction

$$\mathrm{Cor}_{k_n(\tau)/k_{n'}(\tau')} : H^1(k_n(\tau), T) \longrightarrow H^1(k_{n'}(\tau'), T).$$

We note that the $\wedge^r H^1(k_n(\tau), T)$ is the r-th exterior power of the $\mathbb{Z}_p[\Delta_n^\tau]$-module $H^1(k_n(\tau), T)$ in the category of $\mathbb{Z}_p[\Delta_n^\tau]$-modules.

Remark 2.2. Note that we demand the collection $\mathbf{c}$ to be integral in a weaker sense than [PR98, §1.2.2]. This, of course, is inspired from [Rub96], and this weaker version is sufficient for our purposes.

Remark 2.3. For any number field K, let

$$\mathrm{loc}_p : \wedge_0^r H^1(K, T) \longrightarrow \wedge_0^r H^1(K_p, T)$$

(resp.,

$$\mathrm{loc}_p^s : \wedge_0^r H^1(K, T) \longrightarrow \wedge_0^r H^1_s(K_p, T))$$

be the map induced from

$$H^1(K, T) \longrightarrow H^1(K_p, T)$$

[4]Note that $T^{\mathcal{D}} \cong T$ since we assume T is self-dual. This definition may therefore seem unnecessary, yet we still introduce it for a good comparison with the notation of [PR98, Rub00, MR04].

(resp., from the compositum

$$H^1(K,T) \longrightarrow H^1(K_p,T) \longrightarrow H^1_s(K_p,T)).$$

Suppose $\mathbf{c} = \{c_{k_n(\tau)}\}$ is an Euler system of rank r. Then our results regarding the freeness of the semi-local cohomology from §1.2.1 and §1.2.2 above, together with [Rub96, Example 1 on page 38] show that

$$\wedge_0^r H^1(K_p,T) = \wedge^r H^1(K_p,T) \text{ and } \wedge_0^r H^1_s(K_p,T) = \wedge^r H^1_s(K_p,T). \tag{2.1}$$

Here the equalities are induced from the canonical inclusion $\wedge^r M \hookrightarrow \mathbb{Q}_p \otimes \wedge^r M$. It follows from 2.1 that

$$\mathrm{loc}_p(c_{k_n}) \in \wedge^r H^1((k_n)_p,T) \text{ and } \mathrm{loc}_p^s(c_{k_n}) \in \wedge^r H^1_s((k_n)_p,T).$$

Remark 2.4. The 'Euler factors' $P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})$ which appear in the distribution relation (ii) above matches with the Euler factors in [PR98, Rub00] but differ from the Euler factors chosen in [MR04, Definition 3.2.3]. However, thanks to [Rub00, §IX.6], it is possible to go back and forth between these two choices and [MR04, Theorem 3.2.4] still applies.

Remark 2.5. Suppose $r = 1$. In this case

$$\wedge_0^r H^1(K,T) = \wedge^r H^1(K,T) = H^1(K,T)$$

for any number field $K \subset \mathcal{K}$ (where the first equality is [Rub96, Proposition 1.2(ii)]) and our definition agrees with Perrin-Riou's definition [PR98, §1.2.1] of an *Euler system of rank one*; and these both agree with Rubin's [Rub00, Definition II.1.1 and Remark II.1.4] definition of an *Euler system*. We also will henceforth call an Euler system of rank one simply an 'Euler system'.

2.1. **Euler systems to Kolyvagin systems map.** We first recall what Mazur and Rubin call the *Euler systems to Kolyvagin systems map*. Suppose $T, \mathcal{P}$ and $\mathcal{K}$ are as above. Let $\mathbf{ES}(T) = \mathbf{ES}(T,\mathcal{K})$ denote the collection of Euler systems (i.e., *Euler systems of rank one* in the sense of Definition 2.1) for $(T,\mathcal{K})$. Fix a topological generator γ of Γ and set $\gamma_n = \gamma^{p^n}$, and let $\mathfrak{m}_\Lambda$ be the maximal ideal of $\Lambda = \mathcal{O}[[\Gamma]]$.

Definition 2.6. For $\mathbb{F} = \mathcal{F}_\Lambda$ or $\mathcal{F}_{\mathbb{L}}$, we set

$$\overline{\mathbf{KS}}'(\mathbb{T},\mathbb{F},\mathcal{P}) := \varprojlim_{m,n} \varinjlim_{j} \mathbf{KS}(\mathbb{T}/(p^m,\gamma_n - 1)\mathbb{T},\mathbb{F},\mathcal{P}_j),$$

where $\mathbf{KS}(\mathbb{T}/(p^m,\gamma_n - 1)\mathbb{T},\mathbb{F},\mathcal{P}_j)$ is the $\Lambda/(p^m,\gamma_n-1)$-module of Kolyvagin systems (in the sense of [MR04, Definition 3.1.3]) for the Selmer structure $\mathbb{F}$ propagated to the quotient $\mathbb{T}/(p^m,\gamma_n - 1)\mathbb{T}$.

Remark 2.7. We introduced the module $\overline{\mathbf{KS}}'(\mathbb{T},\mathbb{F},\mathcal{P})$ above because, after applying Kolyvagin's descent procedure [Rub00, §IV] (modified appropriately in [MR04, Appendix A]) on an Euler system, one obtains elements of $\overline{\mathbf{KS}}'(\mathbb{T},\mathcal{F}_\Lambda,\mathcal{P})$. On the other hand, it is not hard to see that the module $\overline{\mathbf{KS}}'(\mathbb{T},\mathbb{F},\mathcal{P})$ defined above is naturally isomorphic to the module $\overline{\mathbf{KS}}(\mathbb{T},\mathbb{F},\mathcal{P})$ of Definition 1.27, using the fact that each of the collections $\{p^m,\gamma_n - 1\}_{m,n}$ and $\{p^m,(\gamma-1)^n\}_{m,n}$ forms a base of neighborhoods at *zero*. Furthermore, using the fact that the collection $\{\mathfrak{m}_\Lambda^\alpha\}_{\alpha\in\mathbb{Z}^+}$ also forms a base of neighborhoods at *zero*, one may identify these two modules Kolyvagin systems by the generalized module of Kolyvagin systems defined in [MR04, Definition 3.1.6]. By slight abuse, we will write $\overline{\mathbf{KS}}(\mathbb{T},\mathbb{F},\mathcal{P})$ for any of the three modules of Kolyvagin systems given by three different definitions (i.e., by Definitions 1.27 and 2.6 here; and by [MR04, Definition 3.1.6]). For our purposes in this section, we will use Definition 2.6 to define this module.

Consider the following hypotheses:

KS1. $T/(\mathrm{Fr}_{\mathfrak{q}}-1)T$ is a cyclic $\mathcal{O}$-module for every $\mathfrak{q}\in\mathcal{P}$.

KS2. $\mathrm{Fr}_{\mathfrak{q}}^{p^k}-1$ is injective on T for every $\mathfrak{q}\in\mathcal{P}$ and $k\geq 0$.

Theorem 2.8. ([MR04, Theorem 3.2.4 & 5.3.3]) *Suppose the hypotheses* **KS1-2** *hold. Then there are canonical maps*

- $\mathbf{ES}(T)\longrightarrow\overline{\mathbf{KS}}(T,\mathcal{F}_{\mathrm{can}},\mathcal{P})$,
- $\mathbf{ES}(T)\longrightarrow\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\Lambda},\mathcal{P})$

with the properties that

- *if* $\mathbf{c}$ *maps to* $\boldsymbol{\kappa}\in\overline{\mathbf{KS}}(T,\mathcal{F}_{\mathrm{can}},\mathcal{P})$ *then* $\kappa_1=c_k$,
- *if* $\mathbf{c}$ *maps to* $\boldsymbol{\kappa}^{\mathrm{Iw}}\in\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\Lambda},\mathcal{P})$ *then*

$$\kappa_1^{\mathrm{Iw}}=\{c_{k_n}\}\in\varprojlim_n H^1(k_n,T)=H^1(k,\mathbb{T}).$$

Starting from an Euler system of rank r, one first applies Perrin-Riou's procedure [PR98, §1.2.3] (based on an idea due to Rubin [Rub96, §6]) to obtain an Euler system. After this, we would like to apply the Euler systems to Kolyvagin systems map (Theorem 2.8) on these Euler systems. Note however that Theorem 2.8 will only give rise to Kolyvagin systems for the coarser Selmer structures $\mathcal{F}_\Lambda$ and $\mathcal{F}_{\mathrm{can}}$ (rather than the finer Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$).

Let $\mathbf{ES}^{(r)}(T)=\mathbf{ES}^{(r)}(T,\mathcal{K})$ denote the collection of Euler systems of rank r. The previous paragraph is summarized in the diagram below:

$$\begin{tikzcd}
\mathbf{ES}^{(r)}(T) \arrow[r, "{[PR98]}"] \arrow[dr, dotted, "\mathcal{R}"'] & \mathbf{ES}(T) \arrow[r, "{[MR02]}"] & \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_\Lambda,\mathcal{P}) \arrow[r] & \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathrm{can}},\mathcal{P}) \\
& (?) \arrow[u, hook] \arrow[r, dotted, "\mathcal{D}_\Lambda"] \arrow[rr, dotted, bend right, "\mathcal{D}"'] & \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}},\mathcal{P}) \arrow[u, hook] \arrow[r] & \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P}) \arrow[u, hook]
\end{tikzcd}$$

To be able to obtain Kolyvagin systems for the modified Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$, we need to analyze the structure of semi-local cohomology groups for $\mathbb{T}$ and T at p, over various ray class fields of k. This is carried out in §2.2. We then apply the results of §2.2 in §2.3 to choose carefully a map $\mathcal{R}$ such that the image of the map $\mathcal{R}$ determines the correct submodule **(?)** $\subset\mathbf{ES}(T)$, on which the Euler systems to Kolyvagin systems map restricts to $\mathcal{D}_\Lambda$ and $\mathcal{D}$; and gives (see §2.4) Kolyvagin systems for the modified Selmer structures $\mathcal{F}_{\mathbb{L}}$ and $\mathcal{F}_{\mathcal{L}}$.

2.2. **Semi-local preparation.** Throughout §2.2 we will assume **H.nE** and **H.D** hold true.

Lemma 2.9. *For every* $k_n(\tau)\in\mathfrak{C}$*, the corestriction maps*

(i) $H^1(k_n(\tau)_p,T)\longrightarrow H^1(k(\tau)_p,T)$,
(ii) $H^1(k(\tau)_p,T)\longrightarrow H^1(k_p,T)$,
(iii) $H^1(k_n(\tau)_p,T)\longrightarrow H^1(k_p,T)$

on the semi-local cohomology at p *are all surjective.*

Proof. The cokernel of the map
$$H^1(k(\tau),\mathbb{T})=\varprojlim_n H^1(k_n(\tau)_p,T)\longrightarrow H^1(k(\tau)_p,T)$$
is given by $H^2(k(\tau)_p,\mathbb{T})[\gamma-1]$, where γ is any topological generator of $\Gamma=\mathrm{Gal}(k_\infty/k)$. Since it is known that (c.f., [PR94]) $H^2(k(\tau)_p,\mathbb{T})$ is a finitely generated $\mathcal{O}$-module, it follows that
$$H^2(k(\tau)_p,\mathbb{T})[\gamma-1]=0 \iff H^2(k(\tau)_p,\mathbb{T})/(\gamma-1)=0.$$
Since the cohomological dimension of the absolute Galois group of any local field is 2,
$$H^2(k(\tau)_p,\mathbb{T})/(\gamma-1)\cong H^2(k(\tau)_p,\mathbb{T}/(\gamma-1))=H^2(k(\tau)_p,T).$$
It therefore suffices to check that
$$H^2(k(\tau)_p,T):=\bigoplus_{v|p}H^2(k(\tau)_v,T)=0,$$
which, via local duality is equivalent to checking that $H^0(k(\tau)_v,T^*)=0$ for each $v|p$.

Write $\mathcal{D}_v^\tau$ for the decomposition group of v inside $\mathrm{Gal}(k(\tau)/k):=\Delta^\tau$. We may identify $\mathcal{D}_v^\tau\subset\Delta^\tau$ by the local Galois group $\mathrm{Gal}(k(\tau)_v/k_\wp)$ where $\wp\subset k$ is the prime below v. Since Δ^τ is generated by the inertia groups at the primes dividing τ, all of which act trivially on T^* (since $\tau\in\mathcal{N}(\mathcal{P})$, by definition), it follows that $H^0(k(\tau)_v,T^*)=H^0(k_\wp,T^*)$, and $H^0(k_\wp,T^*)=0$ since we assumed $\mathbf{H.nE}$, and thus (i) is proved.

Now set $T_\tau:=\mathrm{Ind}_{k(\tau)/k}T$. The semi-local version of Shapiro's lemma (which is explained in [Rub00, §A.5]) gives an isomorphism
$$H^1(k(\tau)_p,T)\cong H^1(k_p,T_\tau)$$
and the map
$$\mathrm{Cor}_\tau:\ H^1(k_p,T_\tau)\cong H^1(k(\tau)_p,T)\longrightarrow H^1(k_p,T)$$
is induced from the augmentation sequence
$$0\longrightarrow\mathcal{A}_\tau\cdot T_\tau\longrightarrow T_\tau\longrightarrow T\longrightarrow 0,$$
where $\mathcal{A}_\tau$ is the augmentation ideal of the local ring $\mathcal{O}[\Delta^\tau]$. The argument above shows that the cokernel of Cor_τ is dual to
$$H^0(k_p,(\mathcal{A}_\tau\cdot T_\tau)^*):=\oplus_{\wp|p}H^0(k_\wp,(\mathcal{A}_\tau\cdot T_\tau)^*).$$
Furthermore,
$$(\mathcal{A}_\tau\cdot T_\tau)^*:=\mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau,\Phi/\mathcal{O}(1))=\mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau,\Phi/\mathcal{O})\otimes\mathcal{O}(1),$$
and $\mathrm{Hom}(\mathcal{A}_\tau\cdot T_\tau,\Phi/\mathcal{O})=\mathcal{A}_\tau\cdot\mathrm{Hom}(T_\tau,\Phi/\mathcal{O})$, hence
$$H^0(k_p,(\mathcal{A}_\tau\cdot T_\tau)^*)\hookrightarrow H^0(k_p,T_\tau^*).$$
It therefore suffices to show that $H^0(k_p,T_\tau^*)=0$. By local duality this is equivalent to proving $H^2(k_p,T_\tau)=0$, which by the semi-local Shapiro's Lemma equivalent to show $H^2(k(\tau)_p,T)=0$, which again by local duality equivalent to the statement $H^0(k(\tau)_p,T^*)=0$; and this we have already verified in the first two paragraphs of this Proof. This completes the proof of (ii).

(iii) clearly follows from (i) and (ii). □

Proposition 2.10. *For every $\tau\in\mathcal{N}(\mathcal{P})$:*
(i) *The semi-local cohomology group $H^1(k(\tau)_p,T)$ is a free $\mathcal{O}[\Delta^\tau]$-module of rank $2r$.*
(ii) *For every $n\in\mathbb{Z}_{\geq 0}$, the semi-local cohomology group $H^1(k_n(\tau)_p,T)$ is a free $\mathcal{O}[\Delta_n^\tau]$-module of rank $2r$.*

Proof. We start with the remark that $H^1(k(\tau)_p, T)$ is a free $\mathcal{O}$-module of rank $2r \cdot |\Delta^\tau|$. Indeed, this may be proved following the proof of Lemma 1.9 (again relying on the hypotheses $\mathbf{H.nE}$ and $\mathbf{H.D}$). Further, we know thanks to Lemma 2.9 that the map $H^1(k(\tau)_p, T) \to H^1(k_p, T)$ (which could be thought of as reduction modulo the augmentation ideal $\mathcal{A}_\tau \subset \mathcal{O}[\Delta^\tau]$) is surjective. Nakayama's Lemma and Lemma 1.9 therefore imply that $H^1(k(\tau)_p, T)$ is generated by (at most) $2r$ elements over the ring $\mathcal{O}[\Delta^\tau]$. Let $\mathfrak{B} = \{x_1, x_2, \ldots, x_{2r}\}$ be any set of such generators. To prove (i), it suffices to check that the x_i's do not admit any non-trivial $\mathcal{O}[\Delta^\tau]$-linear relation. Assume contrary, and suppose there is a non-trivial relation

$$\sum_{i=1}^{2r} \alpha_i x_i = 0, \quad \alpha_i \in \mathcal{O}[\Delta^\tau]. \tag{2.2}$$

Write $S = \{\delta x_j : \delta \in \Delta^\tau, 1 \leq j \leq 2r\}$, note that by our assumption on the set $\mathfrak{B}$, the set S generates $H^1(k(\tau)_p, T)$ as an $\mathcal{O}$-module, and $|S| = 2r \cdot |\Delta^\tau| = \mathrm{rank}_{\mathcal{O}} H^1(k(\tau)_p, T)$. Equation (2.2) can be rewritten as

$$\sum_{\delta, j} a_{\delta,j} \cdot \delta x_j = 0$$

with $a_{\delta,j} \in \mathcal{O}$. Since we already know that $H^1(k(\tau)_p, T)$ is $\mathcal{O}$-torsion free, we may assume without loss of generality that $a_{\delta_0, j_0} \in \mathcal{O}^\times$ for some δ_0, j_0. This in return implies that

$$\delta_0 x_{j_0} \in \mathrm{span}_{\mathcal{O}}(S - \{\delta_0 x_{j_0}\}),$$

hence $H^1(k(\tau)_p, T)$ is generated by $S - \{\delta_0 x_{j_0}\}$. This, however, is a contradiction since we already know that the $\mathcal{O}$-rank of $H^1(k(\tau)_p, T)$ is $2r \cdot |\Delta^\tau| = |S|$, hence it cannot be generated by $|S| - 1$ elements over $\mathcal{O}$. The proof of (i) is now complete.

(ii) is proved in an identical fashion, now considering the *augmentation map*

$$H^1(k_n(\tau)_p, T) \longrightarrow H^1(k(\tau)_p, T),$$

which is surjective thanks to Lemma 2.9. □

Let $\mathcal{K}_0 \subset \mathcal{K}$ be the composite of all fields $k(\tau)$, where τ runs through the set $\mathcal{N} = \mathcal{N}(\mathcal{P})$. Set $\mathbf{\Delta} := \mathrm{Gal}(\mathcal{K}_0/k)$.

Corollary 2.11. *$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T)$ is a free $\mathcal{O}[[\Gamma \times \mathbf{\Delta}]]$-module of rank $2r$ and the natural projection maps*

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_m(\eta)_p, T)$$

are surjective for all $m \in \mathbb{Z}_{\geq 0}$ and $\eta \in \mathcal{N}$.

Proof. Immediate after Proposition 2.10. □

2.3. **Choosing the correct homomorphisms.** In this section we use the results from §2.2 to choose useful homomorphisms which will be utilized in §2.4 to construct Kolyvagin systems for the modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ (resp., $\mathcal{F}_{\mathbb{L}}$) on T (resp., on $\mathbb{T}$). This will be carried out in two steps: Under the hypotheses $\mathbf{H.pS}$ on T, we will make our choice of homomorphisms in §2.3.1 and use the results of this section in §2.4.1 to construct an element of $\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ out of an Euler system of rank r. For the Iwasawa theoretic results, we will assume $\mathbf{H.O}$, and we will show how to choose the useful homomorphisms in §2.3.2. This choice will be utilized in §2.4.2 to construct an element of $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, T)$ starting from an Euler system of rank r.

2.3.1. *Choice of Homomorphisms: Potentially semi-stable case.* In this section, we assume that the hypothesis $\mathbf{H.pS}$ holds along with $\mathbf{H.nE}$ and $\mathbf{H.D}$. Let $\mathcal{L} \subset H^1(k_p, T)$ be as in §1.2.1. As before, we denote the (isomorphic) image of $\mathcal{L}$ under $H^1(k_p,T) \to H^1_s(k_p,T)$ also by $\mathcal{L}$.

Proposition 2.12. *There exists a decomposition of the $\mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]]$-module of rank-$2r$*

$$\mathbb{V}_p := \varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) = \boldsymbol{\mathcal{L}^{(r)}} \oplus \boldsymbol{\mathcal{L}_s}$$

with a distinguished rank one direct summand $\boldsymbol{\mathcal{L}} \subset \boldsymbol{\mathcal{L}_s}$ with the following properties:

Under the maps induced from the corestriction map

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow H^1(k_p, T),$$

(1) *$\boldsymbol{\mathcal{L}^{(r)}}$ (and therefore also $\boldsymbol{\mathcal{L}_s}$) is a free $\mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]]$-module of rank r,*
(2) *$\boldsymbol{\mathcal{L}^{(r)}}$ projects onto $H^1_f(k_p,T)$, and $\boldsymbol{\mathcal{L}_s}$ onto $H^1_s(k_p,T)$,*
(3) *$\boldsymbol{\mathcal{L}}$ projects onto $\mathcal{L}$.*

The proof of Proposition 2.12 is elementary linear algebra and will be left out not to digress from our main course.

Definition 2.13. For $k_n(\tau) = K \in \mathfrak{C}$, let $\mathcal{L}_K$ (resp., $\mathcal{L}_K^{(r)}$; resp., $\mathcal{L}_K^s$) be the image of $\boldsymbol{\mathcal{L}}$ (resp., $\boldsymbol{\mathcal{L}^{(r)}}$; resp., $\boldsymbol{\mathcal{L}_s}$) under the (surjective) projection map $\mathbb{V}_p \to H^1(K_p, T)$.

Note that $\mathcal{L}_K$ (resp., $\mathcal{L}_K^{(r)}$ and $\mathcal{L}_K^s$) is a free $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$-module of rank *one* (resp., of rank r) for all $K \in \mathfrak{C}$, and that

$$(X_{K'})^{\mathrm{Gal}(K'/K)} = X_K$$

for $X = \mathcal{L}, \mathcal{L}^{(r)}$ and $\mathcal{L}^s$; for all $K \subset K'$. When $K = k$, note that $\mathcal{L}_K = \mathcal{L}$ and $\mathcal{L}_K^{(r)} = H^1_f(k_p, T)$ by definition (Proposition 2.12).

We write

$$\bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]]) := \varprojlim_{K \in \mathfrak{C}} \bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s, \mathcal{O}[\Delta_K]).$$

Here $\Delta_K = \mathrm{Gal}(K/k)$ and the inverse limit is with respect to the natural maps induced from

$$\mathcal{L}_K^s \longrightarrow (\mathcal{L}_{K'}^s)^{\mathrm{Gal}(K'/K)}$$

and the isomorphism

$$\mathcal{O}[\Delta_{K'}]^{\mathrm{Gal}(K'/K)} \xrightarrow{\sim} \mathcal{O}[\Delta_K]$$
$$\mathbf{N}_K^{K'} \longmapsto 1$$

for $K \subset K'$.

Localization at p followed by the projection onto the "singular quotient" $\mathcal{L}_K^s$ gives rise to a map

$$\mathrm{loc}_p^s : H^1(K,T) \xrightarrow{\mathrm{loc}_p} H^1(K_p,T) \longrightarrow \mathcal{L}_K^s,$$

which induces a canonical map

$$\bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]]) \longrightarrow \varprojlim_{K \in \mathfrak{C}} \bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}(H^1(K,T), \mathcal{O}[\Delta_K]).$$

The image of $\Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$ under this map will still be denoted by Ψ.

Proposition 2.14. *Suppose $\mathbf{c}^{(r)} = \{c_K^{(r)}\}_{K\in\mathfrak{C}}$ is an Euler system of rank r. For any*

$$\{\psi_K\}_{K\in\mathfrak{C}} = \Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]]),$$

define

$$H^1(K,T) \ni c_{K,\Psi} := \psi_K(c_K^{(r)}).$$

Then the collection $\{c_{K,\Psi}\}_{K\in\mathfrak{C}}$ is an Euler system for the G_k-representation T.

We will sometimes denote the Euler system $\{c_{K,\Psi}\}_{K\in\mathfrak{C}}$ by $\{c_{k_n(\tau),\Psi}\}_{n,\tau}$.

Proof. This is proved in [PR98, §1.2.3]. See also [Rub96, Proposition 6.6] for the treatment in the particular case $T=\mathbb{Z}_p(1)$. □

Proposition 2.15. *For any $K\in\mathfrak{C}$, the projection map*

$$\bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]]) \longrightarrow \bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s, \mathcal{O}[\Delta_K])$$

is surjective.

Proof. Obvious since all $\mathcal{L}_K^s$, for $K\in\mathfrak{C}$, are free $\mathcal{O}[\Delta_K]$-modules. □

If one applies the *Euler systems to Kolyvagin systems map* of Mazur and Rubin (c.f., [MR04, Theorem 5.3.3]) on the Euler system $\{c_{K,\Psi}\}_{K\in\mathfrak{C}}$ above, all one gets a priori is a Kolyvagin system for the (coarser) Selmer structure $\mathcal{F}_{\text{can}}$, and in general *not* for the (finer) Selmer structure $\mathcal{F}_{\mathcal{L}}$. Below, we will choose these homomorphisms Ψ carefully so that the resulting Kolyvagin system is indeed a Kolyvagin system for the modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ (resp., $\mathcal{F}_{\mathbb{L}}$) on T (resp., on $\mathbb{T}$).

Definition 2.16. We say that an element

$$\{\psi_K\}_{K\in\mathfrak{C}} = \Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]])$$

satisfies $\mathrm{H}_{\mathcal{L}}$ if for any $K\in\mathfrak{C}$ one has $\psi_K(\wedge^r\mathcal{L}_K^s)\subset\mathcal{L}_K$.

We now construct a specific element

$$\Psi_0 \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]])$$

that satisfies $\mathrm{H}_{\mathcal{L}}$ (and which, in a certain sense, is the best possible choice).

Fix an $\mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]]$-basis

$$\{\Psi_{\boldsymbol{\mathcal{L}}}^{(1)},\dots,\Psi_{\boldsymbol{\mathcal{L}}}^{(r-1)}\}$$

of the free $\mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]]$-module $\mathrm{Hom}_{\mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]]}(\boldsymbol{\mathcal{L}_s}/\boldsymbol{\mathcal{L}}, \mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]])$ of rank $r-1$. This in return fixes a basis $\{\psi_{\mathcal{L}_K}^{(i)}\}_{i=1}^{r-1}$ for the free $\mathcal{O}[\Delta_K]$-module $\mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s/\mathcal{L}_K, \mathcal{O}[\Delta_K])$ for all $K\in\mathfrak{C}$; such that $\{\psi_{\mathcal{L}_K}^{(i)}\}_{K\in\mathfrak{C}}$ are compatible with respect to the surjections

$$\mathrm{Hom}_{\mathcal{O}[\Delta_{K'}]}(\mathcal{L}_{K'}^s/\mathcal{L}_{K'}, \mathcal{O}[\Delta_{K'}]) \longrightarrow \mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s/\mathcal{L}_K, \mathcal{O}[\Delta_K])$$

for all $K\subset K'$. Note that the homomorphism

$$\bigoplus_{i=1}^{r-1}\psi_{\mathcal{L}_K}^{(i)} : \mathcal{L}_K^s/\mathcal{L}_K \longrightarrow \mathcal{O}[\Delta_K]^{r-1}$$

is an isomorphism of $\mathcal{O}[\Delta_K]$-modules, for all $K \in \mathfrak{C}$. Let $\psi_K^{(i)}$ denote the image of $\psi_K^{(i)}$ under the canonical injection

$$\mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s/\mathcal{L}_K, \mathcal{O}[\Delta_K]) \hookrightarrow \mathrm{Hom}_{\mathcal{O}[\Delta_K]}(\mathcal{L}_K^s, \mathcal{O}[\Delta_K]).$$

Note then that the map

$$\Psi_K := \bigoplus_{i=1}^{r-1} \psi_K^{(i)} : \mathcal{L}_K^s \longrightarrow \mathcal{O}[\Delta_K]^{r-1}$$

is surjective and $\ker(\Psi_K) = \mathcal{L}_K$.

Define

$$\varphi_K := \psi_K^{(1)} \wedge \psi_K^{(2)} \wedge \cdots \wedge \psi_K^{(r-1)} \in \bigwedge^{r-1} \mathrm{Hom}(\mathcal{L}_K^s, \mathcal{O}[\Delta_K]).$$

For $K \subset K'$, note that $\varphi_{K'}$ maps to φ_K under the homomorphism

$$\bigwedge^{r-1} \mathrm{Hom}(\mathcal{L}_{K'}^s, \mathcal{O}[\Delta_{K'}]) \twoheadrightarrow \bigwedge^{r-1} \mathrm{Hom}(\mathcal{L}_K^s, \mathcal{O}[\Delta_K]).$$

We may therefore regard $\Psi_0 := \{\varphi_K\}_{K\in\mathfrak{C}}$ as an element of $\bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \mathbf{\Delta}]])$. Composing with $\mathrm{loc}_p^s : H^1(K,T) \to \mathcal{L}_K^s$, we may further regard Ψ_0 as an element of

$$\varprojlim_{K\in\mathfrak{C}} \bigwedge^{r-1} \mathrm{Hom}(H^1(K,T), \mathcal{O}[\Delta_K]).$$

Proposition 2.17. *Suppose $\{\varphi_K\}_K = \Psi_0$ is as above. Then φ_K maps $\wedge^r \mathcal{L}_K^s$ isomorphically onto $\ker(\Psi_K) = \mathcal{L}_K$, for all $K \in \mathfrak{C}$. In particular, Ψ_0 satisfies $\mathrm{H}_{\mathcal{L}}$.*

Proof. The proof is identical to the proof of (the easy half of) [Büy07b, Lemma 3.1], which also follows the proof of [MR04, Lemma B.1] almost line by line. □

2.3.2. *Choice of Homomorphisms: The ordinary case.* Throughout §2.3.2 we assume the hypotheses $\mathbf{H.O}$, $\mathbf{H.nE}$ and $\mathbf{H.D}$ hold true. Let $H^1_{\mathrm{Gr}}(k_p, \mathbb{T})$ and $\mathbb{L}$ be the submodules of $H^1(k_p, \mathbb{T})$ defined in §1.2.2.

We start with the following Proposition whose proof is identical to the proof of Proposition 2.12:

Proposition 2.18. *There exists a decomposition of the $\mathcal{O}[[\Gamma \times \mathbf{\Delta}]]$-module of rank-$2r$*

$$\mathbb{V}_p := \varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) = \boldsymbol{\mathcal{L}^{(r)}} \oplus \boldsymbol{\mathcal{L}_s}$$

with a distinguished rank one direct summand $\boldsymbol{\mathcal{L}} \subset \boldsymbol{\mathcal{L}_s}$ with the following properties:

(1) *$\boldsymbol{\mathcal{L}^{(r)}}$ and $\boldsymbol{\mathcal{L}_s}$) are both free $\mathcal{O}[[\Gamma \times \mathbf{\Delta}]]$-modules of rank r,*

Under the maps induced from the corestriction

$$\varprojlim_{n,\tau} H^1(k_n(\tau)_p, T) \longrightarrow \varprojlim_{n} H^1((k_n)_p, T) = H^1(k_p, \mathbb{T}),$$

(2) *$\boldsymbol{\mathcal{L}^{(r)}}$ projects onto $H^1_{\mathrm{Gr}}(k_p, \mathbb{T})$, and $\boldsymbol{\mathcal{L}_s}$ onto $H^1_s(k_p, \mathbb{T}) := H^1(k_p, \mathbb{T})/H^1_{\mathrm{Gr}}(k_p, \mathbb{T})$,*

(3) *$\boldsymbol{\mathcal{L}}$ projects onto $\mathbb{L}$.*

Having defined $\boldsymbol{\mathcal{L}^{(r)}}$, $\boldsymbol{\mathcal{L}^{s}}$ and $\boldsymbol{\mathcal{L}}$, we may proceed as in §2.3.1 and define $\mathcal{L}_K^{(r)}$, $\mathcal{L}_K^s$ and $\mathcal{L}_K$ as above; and use these to define a particular element

$$\Psi_0 \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$$

in an identical fashion. We also note that

$$H^1(k_p, \mathbb{T}) \supset \mathbb{L} = \{\mathcal{L}_{k_n}\} \subset \varprojlim_n H^1((k_n)_p, T).$$

Definition 2.19. We say that an element

$$\{\psi_K\}_{K \in \mathfrak{C}} = \Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$$

satisfies $\mathrm{H}_{\mathbb{L}}$ if for any $K \in \mathfrak{C}$ one has $\psi_K(\wedge^r \mathcal{L}_K^s) \subset \mathcal{L}_K$.

Although the definition of the property $\mathrm{H}_{\mathbb{L}}$ is identical to the definition of $\mathrm{H}_{\mathcal{L}}$ (Definition 2.16), we wish to distinguish between these two in order to remind us that we used Greenberg's local conditions as a start for one, and Bloch-Kato local conditions for the other. Finally, we note that the following (almost identical) version of Proposition 2.17 holds:

Proposition 2.20. *Let $\Psi_0 = \{\varphi_K\}_K$ be as above. Then φ_K maps $\wedge^r \mathcal{L}_K^s$ isomorphically onto $\ker(\Psi_K) = \mathcal{L}_K$, for all $K \in \mathfrak{C}$. In particular, Ψ_0 satisfies $\mathrm{H}_{\mathbb{L}}$.*

Remark 2.21. The diagram in §2.1 now looks as follows:

$$\begin{tikzcd}
\mathbf{ES}^{(r)}(T) \arrow[r, "{[PR98]}"] \arrow[dr, dotted, "\Psi_0"'] & \mathbf{ES}(T) \arrow[r, "{[MR02]}"] & \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_\Lambda, \mathcal{P}) \arrow[r] & \overline{\mathbf{KS}}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \\
 & \Psi_0(\mathbf{ES}^{(r)}(T)) \arrow[u, hook] \arrow[r, dotted, "\mathcal{D}_\Lambda"] \arrow[rr, dotted, bend right, "\mathcal{D}"'] & \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathbb{L}}, \mathcal{P}) \arrow[u, hook] \arrow[r] & \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P}) \arrow[u, hook]
\end{tikzcd}$$

where $\Psi_0(\mathbf{ES}^{(r)}(T))$ stands for the collection of Euler systems (of rank one) obtained from Euler systems of rank r, following the procedure of Perrin-Riou and Rubin (Proposition 2.14) with the choice $\Psi_0 \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$. In the following section, we verify that the restriction of the Euler systems to Kolyvagin systems map of [MR04] on $\Psi_0(\mathbf{ES}^{(r)}(T))$ really restricts to the maps $\mathcal{D}_\Lambda$ and $\mathcal{D}$.

Remark 2.22. Since the maps $H^1_s(k_p.\mathbb{T}) \longrightarrow \mathcal{L}^s_{k_n}$, for $n \in \mathbb{Z}^+$, are all surjective (by our choices made in Proposition 2.18) and $H^1_s(k_p, \mathbb{T})$ (resp., $\mathcal{L}^s_{k_n}$) is a free Λ-module (resp., $\mathcal{O}[\Gamma_n]$-module) of rank r, it follows that there is a canonical isomorphism

$$\varprojlim_n \wedge^r_{\mathcal{O}[\Gamma_n]} \mathcal{L}^s_{k_n} \cong \wedge^r_\Lambda \varprojlim_n \mathcal{L}^s_{k_n} = \wedge^r_\Lambda H^1_s(k_p, \mathbb{T}).$$

This and Proposition 2.20 show that $\varphi_\infty = \{\varphi_{k_n}\}_n$ maps $\bigwedge^r H^1_s(k_p, \mathbb{T})$ isomorphically onto $\mathbb{L} = \varprojlim_n \mathcal{L}_n$.

2.4. **Kolyvagin systems for modified Selmer structures (bis).** We are now ready to construct Kolyvagin systems[5] for the $\mathcal{L}$-modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T (resp., $\mathbb{L}$-modified Selmer structure $\mathcal{F}_{\mathbb{L}}$ on $\mathbb{T}$) starting from an Euler system of rank r, for each choice of a compatible homomorphisms $\Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$ that satisfies $\mathrm{H}_{\mathcal{L}}$ (resp., $\mathrm{H}_{\mathbb{L}}$). These classes will be utilized in the following section to demonstrate the main outcomes of this machinery.

2.4.1. *Kolyvagin systems over k* (bis). Theorem 2.8 gives a map
$$\mathbf{ES}(T) \longrightarrow \overline{\mathbf{KS}}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$$
where
$$\overline{\mathbf{KS}}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) := \varprojlim_{\alpha}(\varinjlim_{j} \mathbf{KS}(T/\mathfrak{m}^\alpha T,\, \mathcal{F}_{\text{can}}, \mathcal{P} \cap \mathcal{P}_j))$$
is the (generalized) module of Kolyvagin systems for the Selmer triple $(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ and $\mathcal{F}_{\text{can}}$ is the canonical Selmer structure on T as in Example 1.6 (and its *propagation* to the quotients of T). One of the main attributes of this map is that if an Euler system $\left\{c_{k_n(\tau)}\right\}_{n,\tau} = \mathbf{c} \in \mathbf{ES}(T)$ maps to the Kolyvagin system $\boldsymbol{\kappa} = \left\{\{\kappa_\tau(\alpha)\}_{\tau \in \mathcal{N}_j}\right\}_{\alpha}$ under this map, then

$$\begin{array}{lcl} \kappa_1 \overset{\text{def}}{=} \varprojlim_{\alpha} \kappa_1(\alpha) & \in & \varprojlim_{\alpha} H^1(k, T/\mathfrak{m}^\alpha T) = H^1(k, T) \\ \| & & \\ c_k \quad\quad \in \; H^1(k,T). & & \end{array} \tag{2.3}$$

Let $\boldsymbol{\kappa}^{\Psi_0} = \left\{\left\{\kappa_\tau^{\Psi_0}(\alpha)\right\}_{\tau \in \mathcal{N}_j}\right\}_{\alpha}$ be the image of the Euler system $\mathbf{c}^{(r)}_{\Psi_0} = \left\{c^{(r)}_{k_n(\tau),\Psi_0}\right\}_{n,\tau}$, which itself is obtained from an Euler system $\mathbf{c}^{(r)} = \{c_K^{(r)}\}_{K \in \mathfrak{E}}$ of rank r via Proposition 2.14 applied with $\Psi_0 = \{\varphi_K\}_{K \in \mathfrak{E}}$ above. Thus the equation (2.3) reads
$$\kappa_1^{\Psi_0} = c^{(r)}_{k,\Psi_0} = \varphi_k(c_k^{(r)}). \tag{2.4}$$

Theorem 2.23. $\boldsymbol{\kappa}^{\Psi_0} := \left\{\left\{\kappa_\tau^{\Psi_0}(\alpha)\right\}_{\tau \in \mathcal{N}}\right\}_{\alpha} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$.

Here
$$\overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P}) = \varprojlim_{\alpha}(\varprojlim_{j} \mathbf{KS}(T/\mathfrak{m}^\alpha T, \mathcal{F}_{\mathcal{L}}, \mathcal{P} \cap \mathcal{P}_j))$$
is the (generalized) module of Kolyvagin systems for the $\mathcal{L}$-modified Selmer structure $\mathcal{F}_{\mathcal{L}}$ on T.

Remark 2.24. We could have used *any* element $\Psi \in \bigwedge^{r-1} \mathbf{Hom}(\boldsymbol{\mathcal{L}_s}, \mathcal{O}[[\Gamma \times \boldsymbol{\Delta}]])$ in Theorem 2.23 that satisfies $\mathrm{H}_{\mathcal{L}}$ (rather then the particular element Ψ_0) and still obtain Kolyvagin systems for the $\mathcal{L}$-modified Selmer structure.

For the rest of this section the integer α will be fixed, and we denote the element $\kappa_\tau^{\Psi_0}(\alpha) \in H^1(k, T/\mathfrak{m}^\alpha T)$ by $\kappa_\tau^{\Psi_0}$. Note that the statement of Theorem 2.23 claims for each $\tau \in \mathcal{N}_\alpha$ that
$$\kappa_\tau^{\Psi_0} \in H^1_{\mathcal{F}_{\mathcal{L}}(\tau)}(k, T/\mathfrak{m}^\alpha T),$$
where $\mathcal{F}_{\mathcal{L}}(\tau)$ is defined as in [MR04, Example 2.1.8]. However, [MR04, Theorem 5.3.3] already concludes that
$$\kappa_\tau^{\Psi_0} \in H^1_{\mathcal{F}_{\text{can}}(\tau)}(k, T/\mathfrak{m}^\alpha T).$$

[5]which we proved to exist in §1.4.

Since $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\text{can}}$ determine the same local conditions outside p, it suffices to prove the following in order to prove Theorem 2.23:

Proposition 2.25. *Let*

$$\mathrm{loc}_p^s : H^1(k, T/\mathfrak{m}^\alpha T) \longrightarrow H^1_s(k_p, T/\mathfrak{m}^\alpha T) := \frac{H^1(k_p, T/\mathfrak{m}^\alpha T)}{H^1_f(k_p, T/\mathfrak{m}^\alpha T)}$$

be the localization map into the semi-local cohomology at p, *followed by the projection onto the singular quotient. Then*

$$\mathrm{loc}_p^s(\kappa_\tau^{\Psi_0}) \in \mathcal{L}/\mathfrak{m}^\alpha\mathcal{L} \subset H^1_s(k_p, T/\mathfrak{m}^\alpha T).$$

Proposition 2.25 will be proved below. We first note that $H^1_f(k_p, T/\mathfrak{m}^\alpha T)$ is by definition the propagation of $H^1_f(k_p, T)$. Similarly,

$$H^1_f(k_p, T/\mathfrak{m}^\alpha T) \oplus \mathcal{L}/\mathfrak{m}^\alpha\mathcal{L} = H^1_{\mathcal{F}_{\mathcal{L}}(\tau)}(k_p, T/\mathfrak{m}^\alpha T)$$

is the propagation of the $\mathcal{L}$-modifed condition $H^1_{\mathcal{F}_{\mathcal{L}}(\tau)}(k_p, T) := H^1_f(k_p, T) \oplus \mathcal{L}$ at p. Let

$$\left\{\tilde{\kappa}_\tau^{\Psi_0}(\alpha) \in H^1(k_p, T/\mathfrak{m}^\alpha T)\right\}_{\tau\in\mathcal{N}_\alpha}$$

be the collection that [Rub00, Definition 4.4.10] associates to the Euler system $\left\{c^{(r)}_{k_n(\tau),\Psi_0}\right\}_{n,\tau}$. Here we write $\tilde{\kappa}_\tau^{\Psi_0}(\alpha)$ for the class denoted by $\kappa_{[k,\tau,\alpha]}$ in loc.cit. Since we have fixed α until the end of this section, we will safely drop α from the notation and denote $\tilde{\kappa}_\tau^{\Psi_0}(\alpha)$ by $\tilde{\kappa}_\tau^{\Psi_0}$. Note that Equation (33) in [MR04, Appendix A] relates this class to $\kappa_\tau^{\Psi_0}$.

Lemma 2.26. *If* $\mathrm{loc}_p^s(\tilde{\kappa}_\tau^{\Psi_0}) \in \mathcal{L}/\mathfrak{m}^\alpha\mathcal{L}$ *then* $\mathrm{loc}_p^s(\kappa_\tau^{\Psi_0}) \in \mathcal{L}/\mathfrak{m}^\alpha\mathcal{L}$ *as well.*

Proof. Obvious using Equation (33) in [MR04, Appendix A]. □

Let D_τ denote the derivative operator of Kolyvagin, defined as in [Rub00, Definition 4.4.1]. Rubin [Rub00, Definition 4.4.10] defines $\tilde{\kappa}_\tau^{\Psi_0}$ as a canonical inverse image of $D_\tau c^{(r)}_{k(\tau),\Psi_0}$ (mod $\mathfrak{m}^\alpha$) under the restriction map[6]

$$H^1(k, T/\mathfrak{m}^\alpha T) \longrightarrow H^1(k(\tau), T/\mathfrak{m}^\alpha T)^{\Delta^\tau}.$$

Therefore, $\mathrm{loc}_p^s(\tilde{\kappa}_\tau^{\Psi_0})$ maps to $\mathrm{loc}_p^s\left(D_\tau c^{(r)}_{k(\tau),\Psi_0}\right)$ (mod $\mathfrak{m}^\alpha$) under the map[7]

$$H^1_s(k_p, T/\mathfrak{m}^\alpha T) \longrightarrow H^1_s(k(\tau)_p, T/\mathfrak{m}^\alpha T)^{\Delta^\tau} (:= \left(\mathcal{L}^s_{k(\tau)}/\mathfrak{m}^\alpha\mathcal{L}^s_{k(\tau)}\right)^{\Delta^\tau}).$$

Under this isomorphism, $\mathcal{L}/\mathfrak{m}^\alpha\mathcal{L} \subset H^1(k_p, T/\mathfrak{m}^\alpha T)$ is mapped isomorphically onto the rank one $\mathcal{O}/\mathfrak{m}^\alpha\mathcal{O}$-module $\left(\mathcal{L}_{k(\tau)}/\mathfrak{m}^\alpha\mathcal{L}_{k(\tau)}\right)^{\Delta_\tau}$, by the definition of $\mathcal{L}_{k(\tau)}$ and by the fact that $\mathcal{L}_{k(\tau)}$ is a free $\mathcal{O}[\Delta^\tau]$-module. The diagram below summarizes the discussion in this paragraph:

$$\begin{array}{ccc}
H^1(k_p, T/\mathfrak{m}^\alpha T) & \xrightarrow{\sim} & H^1(k(\tau)_p, T/\mathfrak{m}^\alpha T)^{\Delta_\tau} \\
\uparrow & & \uparrow \\
\mathcal{L}/\mathfrak{m}^\alpha\mathcal{L} & \xrightarrow{\quad\sim\quad} & \left(\mathcal{L}_{k(\tau)}/\mathfrak{m}^\alpha\mathcal{L}_{k(\tau)}\right)^{\Delta^\tau}
\end{array}$$

[6]This restriction map is an isomorphism if we assume (H.3): The kernel and cokernel of this map are both annihilated by $\#(T/\mathfrak{m}^\alpha T)^{G_{k_\tau}}$. Furthermore, $(T/\mathfrak{m}^\alpha T)^{G_{k(\tau)}} = (T/\mathfrak{m}^\alpha T)^{G_k}$ since $\Delta^\tau = \mathrm{Gal}(k(\tau)/k)$ is generated by the inertia groups at the primes of k dividing τ; and all of these act trivially on $T/\mathfrak{m}^\alpha T$. On the other hand it follows from hypotheses (H.3) (c.f., [MR04, Lemma 3.5.2]) that $(T/\mathfrak{m}^\alpha T)^{G_k} = 0$.

[7]This map is also an isomorphism thanks to $\mathbf{H.nE}$.

Proposition 2.27. $\mathrm{loc}_p(\tilde{\kappa}_\tau^{\Psi_0}) \in \mathcal{L}/\mathfrak{m}^\alpha\mathcal{L}$.

Proof. Since loc_p is Galois equivariant $\mathrm{loc}_p(D_\tau c^{(r)}_{k(\tau),\Psi_0}) = D_\tau \mathrm{loc}_p(c^{(r)}_{k(\tau),\Psi_0})$. Furthermore,

$$\mathrm{loc}_p\left(c^{(r)}_{k(\tau),\Psi_0}\right) \in \mathcal{L}_{k(\tau)},$$

since Ψ_0 satisfies $\mathrm{H}_{\mathcal{L}}$. On the other hand, by [Rub00, Lemma 4.4.2], the class $D_\tau c^{(r)}_{k(\tau),\Psi_0}$ (mod $\mathfrak{m}^\alpha$) is fixed by Δ^τ, which in return implies that

$$\mathrm{loc}_p\left(c^{(r)}_{k(\tau),\Psi_0}\right) \pmod{\mathfrak{m}^\alpha} \in \left(\mathcal{L}_{k(\tau)}/\mathfrak{m}^\alpha\mathcal{L}_{k(\tau)}\right)^{\Delta^\tau}.$$

This shows that $\mathrm{loc}_p(\tilde{\kappa}_\tau^{\Psi_0})$ maps into $\mathcal{L}/\mathfrak{m}^\alpha\mathcal{L}$ by the discussion in the paragraph preceding the statement of this Proposition. $\square$

Proof of Proposition 2.25. Immediate after Lemma 2.26 and Proposition 2.27. $\square$

By the discussion following the statement of Theorem 2.23, this also completes the proof of Theorem 2.23.

2.4.2. *Kolyvagin systems over k_∞ (bis).* For $\mathbb{F} = \mathcal{F}_\Lambda$ or $\mathcal{F}_{\mathbb{L}}$, recall

$$\overline{\mathbf{KS}}(\mathbb{T},\mathbb{F},\mathcal{P}) := \varprojlim_{\alpha,n}(\varinjlim_{j} \mathbf{KS}(\mathbb{T}/(\mathfrak{m}^\alpha,\gamma_n-1)\mathbb{T},\mathbb{F},\mathcal{P}\cap\mathcal{P}_j)),$$

the (generalized) module of Λ-adic Kolyvagin systems for the Selmer structure[8] $\mathbb{F}$ on $\mathbb{T}$. Our definition slightly differs from that of Mazur an Rubin [MR04, Definition 3.1.6], however, as noted in Remark 2.7, it is possible to identify their generalized module Kolyvagin systems with ours using the fact that both $\{(\mathfrak{m}^\alpha,\gamma_n-1)\}_{\alpha,n}$ and $\{\mathfrak{m}_\Lambda^\beta\}_\beta$ (where $\mathfrak{m}_\Lambda$ is the maximal ideal of Λ) forms a base of neighborhoods at 0.

Suppose that $\Psi_0 \in \bigwedge^{r-1}\mathbf{Hom}(\boldsymbol{\mathcal{L}_s},\mathcal{O}[[\Gamma\times\boldsymbol{\Delta}]])$ is as in §2.3.2; in particular Ψ_0 satisfies $\mathrm{H}_{\mathbb{L}}$. Let $\mathbf{c}^{(r)} = \{c_K^{(r)}\}_{K\in\mathfrak{C}}$ be any Euler system of rank r and let $\mathbf{c}_{\Psi_0} = \{c_{k_n(\tau),\Psi_0}\}$ be the Euler system of rank one obtained from $\mathbf{c}^{(r)}$ via Proposition 2.14 applied with Ψ_0. As before, let

$$\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}} \in \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_\Lambda,\mathcal{P})$$

be the image of $\mathbf{c}_{\Psi_0}$ under the Euler system to Kolyvagin system map of Theorem 2.8. The proof of the following Theorem is very similar to the proof of Theorem 2.23 above and will be skipped; see also the proofs of [Büy07c, Theorem 3.23] and [Büy07a, Theorem 2.19].

Theorem 2.28. $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}} \in \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}},\mathcal{P})$.

Remark 2.29. We know (by the definition of the Euler systems to Kolyvagin systems map) that

$$\begin{array}{l} \kappa_1^{\Psi_0,\mathrm{Iw}} \stackrel{\mathrm{def}}{=\!=} \varprojlim_{\alpha,n} \kappa_1^{\Psi_0,\mathrm{Iw}}(\alpha,n) \in \varprojlim_{\alpha,n} H^1(k,\mathbb{T}/(\mathfrak{m}^\alpha,\gamma_n-1)\mathbb{T}) = H^1(k,\mathbb{T}) \\ \quad \Vert \\ \{c_{k_n,\Psi_0}\}_n \xlongequal{\mathrm{def}} \left\{\varphi_{k_n}(c^{(r)}_{k_n})\right\}_n \in \varprojlim_n H^1(k_n,T) = H^1(k,\mathbb{T}). \end{array}$$

Remark 2.30. In this paragraph, T is no longer assumed to be self-dual. Suppose (1.6) holds for T. As in Remark 1.29, we note that the results of §2.4 (under their running hypotheses) apply verbatim for T with this property.

[8]As usual, we write $\mathbb{F}$ also for the *propagation* of $\mathbb{T}$ to the quotients $\mathbb{T}/(\mathfrak{m}^\alpha,\gamma_n-1)\mathbb{T}$.

3. Applications

Throughout this section, the hypotheses $\mathbf{H.1}$-$\mathbf{H.5}$, $\mathbf{H.nE}, \mathbf{H.D}$ are in effect.

3.1. **Applications over k.** Aside from the hypotheses we assumed above, suppose in §3.1 that the hypothesis $\mathbf{H.pS}$ holds as well.

We start with proving a bound on the size of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)$, which we will use, together with the comparison result from §1.3.1, to obtain a bound on the classical Selmer group.

Theorem 3.1. *Under the running hypotheses,*

$$|H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)| \leq |H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)/\mathcal{O} \cdot \kappa_1^{\Psi_0}|,$$

with equality if and only if the Kolyvagin system $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ is primitive (in the sense of [MR04, Definition 4.5.5]).

Proof. This is the standard application of $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$, see [MR04, Corollary 5.2.13]. $\square$

Consider the following condition on the Euler system $\mathbf{c}^{(r)}$ of rank r:

(H.nV) $\mathrm{loc}_p^s\left(c_k^{(r)}\right) \neq 0$.

Lemma 3.2. *Suppose* **(H.nV)** *holds. Then* $\mathrm{loc}_p^s(\kappa_1^{\Psi_0}) \neq 0$*, in particular,* $\kappa_1^{\Psi_0} \neq 0$.

Proof. The following equalities follow from the definitions:

$$\mathrm{loc}_p^s(\kappa_1^{\Psi_0}) = \mathrm{loc}_p^s\left(c_{k,\Psi_0}\right) = \mathrm{loc}_p^s\left(\varphi_k\left(c_k^{(r)}\right)\right) = \varphi_k\left(\mathrm{loc}_p^s\left(c_k^{(r)}\right)\right). \tag{3.1}$$

Since $\varphi_k : \wedge^r H^1_s(k_p, T) \to \mathcal{L}$ is an isomorphism and since we assumed **(H.nV)**, Lemma follows. $\square$

Corollary 3.3. *If* **(H.nV)** *holds, then*

(i) $H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)$ *is finite,*
(ii) $H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)$ *is a free $\mathcal{O}$-module of rank one.*

Proof. By Lemma 3.2 and [MR04, Corollary 5.2.13(i)] applied with $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$, it follows that $H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)$ is finite.

We have $H^1(k, T)_{\mathrm{tors}} \cong H^0(k, T \otimes \Phi/\mathcal{O})$ for the $\mathcal{O}$-torsion submodule $H^1(k, T)_{\mathrm{tors}}$. As explained in [MR04, Lemma 3.5.2], it follows from our hypothesis $\mathbf{H.3}$ that $H^0(k, T \otimes \Phi/\mathcal{O}) = 0$. We therefore conclude that $H^1_{\mathcal{F}_{\mathcal{L}}}(k, T) \subset H^1(k, T)$ is $\mathcal{O}$-torsion free, hence it is a free $\mathcal{O}$-module. Using [MR04, Corollary 5.2.6], we conclude that

$$\begin{aligned} \mathrm{rank}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)\right) &= \mathrm{rank}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\mathcal{L}}}(k, T)\right) - \mathrm{corank}_{\mathcal{O}}\left(H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)\right) \\ &= \mathcal{X}(T, \mathcal{F}_{\mathcal{L}}) - \mathcal{X}(T^*, \mathcal{F}^*_{\mathcal{L}}) = 1, \end{aligned}$$

where $\mathcal{X}(T, \mathcal{F}_{\mathcal{L}})$ and $\mathcal{X}(T^*, \mathcal{F}^*_{\mathcal{L}})$ denote the core Selmer rank, see §1.4. $\square$

Corollary 3.4. *If* **(H.nV)** *holds, then* $H^1_{\mathcal{F}_{\mathrm{BK}}}(k, T) = 0$.

Proof. By Lemma 3.2, we have $\mathrm{loc}_p^s(\kappa_1^{\Psi_0}) \neq 0$, in particular, the map $\mathrm{loc}_p^s : H^1_{\mathcal{F}_{\mathcal{L}}}(k,T) \to \mathcal{L}$ is non-trivial. Since both $H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)$ and $\mathcal{L}$ are free $\mathcal{O}$-modules of rank one, it follows that loc_p^s is injective, i.e.,

$$H^1_{\mathcal{F}_{\mathrm{BK}}}(k,T) = \ker\left(H^1_{\mathcal{F}_{\mathcal{L}}}(k,T) \xrightarrow{\mathrm{loc}_p^s} \mathcal{L}\right) = 0.$$

□

Theorem 3.5. *Under the hypothesis* **(H.nV)***,*

$$|H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)| \leq |\mathcal{L}/\mathcal{O}\cdot \mathrm{loc}_p^s(\kappa_1^{\Psi_0})|,$$

and we have equality if and only if $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ *is primitive.*

Proof. This follows from Theorem 3.1 and Corollary 1.19 applied with the class $c = \kappa_1^{\Psi_0} \in H^1_{\mathcal{F}_{\mathcal{L}}}(k,T)$. Note that Corollary 1.19 applies thanks to Corollary 3.4. □

Corollary 3.6. (i) $|H^1_{\mathcal{F}^*_{\mathrm{BK}}}(k,T^*)| \leq |\wedge^r H^1_s(k,T)/\mathcal{O}\cdot \mathrm{loc}_p^s(c_k^{(r)})|$.

(ii) *Suppose* **(H.nV)** *holds. We then have equality in* (i) *if and only if the inequality of Theorem 3.5 is an equality.*

Proof. By construction,

$$\begin{aligned} \varphi_k : \wedge^r H^1_s(k_p,T) &\xrightarrow{\sim} \mathcal{L} \\ \mathrm{loc}_p^s(c_k^{(r)}) &\longmapsto \mathrm{loc}_p^s(\kappa_1^{\Psi_0}) \end{aligned}$$

If **(H.nV)** fails, then there is nothing to prove, hence we may assume without loss of generality that **(H.nV)** holds. In this case, Corollary follows from Theorem 3.5 and the diagram above.

□

3.2. **Applications over k_∞.** Along with the hypotheses we set at the beginning of §3.3.4, suppose also that $\mathbf{H.T}$, $\mathbf{H.O}$ and $\mathbf{H.TZ}$ hold. Recall that we write char(M) for the characteristic ideal of a finitely generated Λ-module M, with the convention that char$(M)=0$ unless M is Λ-torsion.

We proceed as in the previous section: We first prove a bound for the characteristic ideal of the dual Selmer group $H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)^\vee$, which we use, together with Proposition 1.21, to obtain a bound on the characteristic ideal of the (Pontryagin dual of the) classical Selmer group.

Let $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}} \in \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}},\mathcal{P})$ be the Λ-adic Kolyvagin system obtained from an Euler system of rank r as in §2.4.2. Note that $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}}$ maps to $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ under the map

$$\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathbb{L}},\mathcal{P}) \longrightarrow \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P}).$$

We note that $\mathcal{F}_{\mathcal{L}}$ in this section is defined using the Greenberg local condition (see Remark 1.16), whereas $\mathcal{F}_{\mathcal{L}}$ that we used in the previous section is defined by relaxing Bloch-Kato local conditions (see §1.2.1).

Theorem 3.7. *Under the running hypotheses:*

(i) $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathbb{L}}}(k,\mathbb{T}^*)^\vee\right) \Big| \mathrm{char}\left(H^1_{\mathcal{F}_{\mathbb{L}}}(k,\mathbb{T})/\Lambda\cdot\kappa_1^{\Psi_0,\mathrm{Iw}}\right)$.

(ii) *The divisibility in* (i) *is an equality if* $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T,\mathcal{F}_{\mathcal{L}},\mathcal{P})$ *is primitive.*

Proof. (i) is [MR04, Theorem 5.3.10(i)], and the assertion (ii) follows from [MR04, Theorem 5.3.10(iii)], once we check that $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}}$ is Λ-primitive (in the sense of [MR04, Definition 5.3.9]), provided that $\boldsymbol{\kappa}^{\Psi_0} \in \overline{\mathbf{KS}}(T, \mathcal{F}_{\mathcal{L}}, \mathcal{P})$ is primitive. This is what we verify now.

Let $\overline{T}$ be the residual representation $\mathbb{T}/\mathfrak{m}_\Lambda\mathbb{T} = T/pT$. For a Kolyvagin system $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T})$ (resp., $\kappa \in \overline{\mathbf{KS}}(T)$), let $\overline{\boldsymbol{\kappa}}$ (resp., $\overline{\kappa}$) denote the image of $\boldsymbol{\kappa}$ (resp., κ) under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \mathbf{KS}(\overline{T})$ (resp., under the map $\overline{\mathbf{KS}}(T) \to \mathbf{KS}(\overline{T})$). Since $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}}$ maps to the element $\boldsymbol{\kappa}^{\Psi_0}$ under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \overline{\mathbf{KS}}(T)$, it is clear that $\overline{\boldsymbol{\kappa}}^{\Psi_0,\mathrm{Iw}} = \overline{\boldsymbol{\kappa}}^{\Psi_0}$, and we henceforth write $\overline{\kappa}$ for both. By our assumption that $\boldsymbol{\kappa}^{\Psi_0}$ is primitive, it follows that $\overline{\kappa} \neq 0$. This proves that the image of $\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}}$ under the map $\overline{\mathbf{KS}}(\mathbb{T}) \to \overline{\mathbf{KS}}(\mathbb{T}/\mathfrak{p}\mathbb{T})$ is non-zero for any height-one prime $\mathfrak{p} \subset \Lambda$; since we have a commutative diagram

$$\begin{tikzcd}
\boldsymbol{\kappa}^{\Psi_0,\mathrm{Iw}} \arrow[d, mapsto] & \in & \overline{\mathbf{KS}}(\mathbb{T}) \arrow[d] \arrow[dr] & \\
 & & & \overline{\mathbf{KS}}(\mathbb{T}/\mathfrak{p}\mathbb{T}) \arrow[dl] \\
\overline{\kappa} & \in & \mathbf{KS}(\overline{T}) &
\end{tikzcd}$$

and $\overline{\kappa} \neq 0$. $\square$

Corollary 3.8. *Suppose the hypothesis* **(H.nV)** *holds.*

(i) $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(k, \mathbb{T}^*)^\vee\right) \Big|\, \mathrm{char}\left(\mathbb{L}/\Lambda \cdot \mathrm{loc}^s_p(\kappa_1^{\Psi_0,\mathrm{Iw}})\right)$.

(ii) *The inequality of* (i) *is an equality if and only if* $\boldsymbol{\kappa}^{\Psi_0}$ *is primitive.*

Proof. As in Corollary 3.4, **(H.nV)** implies that $H^1_{\mathcal{F}_{\mathrm{Gr}}}(k, T)$ vanishes. (i) now follows from Theorem 3.7(i) and Proposition 1.21(ii) applied with the class $c = \kappa_1^{\Psi_0,\infty} \in H^1_{\mathcal{F}_{\mathbb{L}}}(k, \mathbb{T})$. The assertion (ii) is immediate from Theorem 3.7(ii). $\square$

Define $c^{(r)}_{k_\infty} := \{c^{(r)}_{k_n}\}_n \in \varprojlim_n \wedge^r_0 H^1(k_n, T)$. Recall that the subscript '0' here is to remind us that the elements $\{c^{(r)}_{k_n}\}$ are allowed to have *denominators*. As explained in Remark 2.2, the singular projections of these elements have no *denominators*: $\mathrm{loc}^s_p(c^{(r)}_{k_n}) \in \wedge^r H^1_s((k_n)_p, T)$. Hence,

$$\mathrm{loc}^s_p(c^{(r)}_{k_\infty}) := \{c^{(r)}_{k_n}\} \in \varprojlim_n \wedge^r H^1_s((k_n)_p, T) = \wedge^r H^1_s(k_p, \mathbb{T}),$$

where the last equality is because each $H^1_s((k_n)_p, T)$ is a free $\mathcal{O}[\Gamma_n]$-module of rank r and the maps $H^1_s(k_p, \mathbb{T}) \to H^1_s((k_n)_p, T)$ are all surjective.

Theorem 3.9. *Under the hypotheses of Corollary 3.8,*

(i) $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{Gr}}}(k, \mathbb{T}^*)^\vee\right) \Big|\, \mathrm{char}\left(\wedge^r H^1_s(k_p, \mathbb{T})/\Lambda \cdot \mathrm{loc}^s_p(c^{(r)}_{k_\infty})\right)$,

(ii) *the divisibility in (i) is an equality if and only if* $\boldsymbol{\kappa}^{\Psi_0}$ *is primitive.*

Proof. Recall $\varphi_\infty = \{\varphi_{k_n}\}_n$, which we defined in Remark 2.22. By definition, we have the following diagram:

$$\begin{aligned}
\varphi_\infty : \wedge^r H^1_s(k_p, \mathbb{T}) &\xrightarrow{\ \sim\ } \mathbb{L} \\
\mathrm{loc}^s_p(c^{(r)}_{k_\infty}) &\longmapsto \mathrm{loc}^s_p(\kappa_1^{\Psi_0,\infty})
\end{aligned}$$

(i) now follows from Corollary 3.8(i) and the diagram above, and (ii) is immediate after Corollary 3.8(ii). $\square$

Remark 3.10. In this remark, we no longer assume that T is self-dual. As in Remarks 1.29 and 2.30 above, we note that the results of §3.1 and §3.2 apply (under their running hypotheses) for T which satisfies (1.6).

3.3. **Perrin-Riou's (conjectural) p-adic L-functions.** Rubin [Rub00, §VIII] sets up a connection between Euler systems of rank r and p-adic L-functions via the work of Perrin-Riou [PR94, PR95]. We will apply the results of §3.1 and §3.2 with the (conjectural) Euler system of Perrin-Riou and Rubin. Since these Euler systems arise from p-adic L-functions, Corollary 3.6 and Theorem 3.9 will relate Selmer groups to L-values.

3.3.1. *The setting.* For notational convenience, we restrict ourselves to the case $\Phi = \mathbb{Q}_p$ and $\mathcal{O} = \mathbb{Z}_p$. Let $\mathbb{Q}_p(1) = \mathbb{Q}_p \otimes \mathbb{Z}_p(1)$ and $\mathbb{Q}(j) = \mathbb{Q}_p(1)^{\otimes j}$ for every $j \in \mathbb{Z}$. We also write $V(j) = V \otimes \mathbb{Q}_p(j)$ for a Galois representation V, and $V^* = \mathrm{Hom}(V, \mathbb{Q}_p(1))$. Throughout this section, we assume that the G_k-representation $V = T \otimes \mathbb{Q}_p$ is the p-adic realization M_p of a (pure) motive $M_{/k}$ in the sense of [FPR94, §III.2.1.1]. Write $w = w(M)$ for the weight of M and let $L(M, s)$ denote the L-function of M. This is defined as an Euler product

$$L(M, s) = \prod_{\ell} L_{\ell}(M, s)$$

which is absolutely convergent in the half-plane $\Re(s) > 1 + \frac{w}{2}$. We will assume without loss that $k = \mathbb{Q}$; as in general one could consider the induced representation $\mathrm{Ind}_{k/\mathbb{Q}}T$ in place of T. We will suppose further that the representation $V = M_p$ is crystalline at p.

Write $\check{M}$ for the dual motive. We shall be interested in the case of a *self-dual motive* $M \xrightarrow{\sim} \check{M}(1)$. In this case, we have $w = -1$, and $s = 0$ is the center of symmetry of the conjectural functional equation that the associated complex L-function $L(M, s)$ satisfies. Serre's [Ser86, §3] general recipe implies that the Archimedean factor $L_\infty(M, s)$ at infinity is non-vanishing at $s = 0$, hence the central point $s = 0$ is critical in the sense of Deligne [Del79].

Example 3.11. In the examples below, suppose k is an arbitrary totally real field.

1. Let A be an abelian variety over k. Set $M = h^1(A)(1)$. The p-adic realization of M is given by $M_p = \mathbb{Q}_p \otimes T_p(A)$. Falting's [Fal83] proof of the Tate conjecture implies that the motive M determines the abelian variety A up to an isogeny over k. Let $A^\vee$ denote the dual abelian variety, and fix a polarization $f : A \to A^\vee$. This isogeny induces an isomorphism of motives $h^1(A) \xrightarrow{\sim} h^1(A^\vee)$ and the Weil pairing shows that $M \xrightarrow{\sim} \check{M}(1)$, i.e., M is self-dual. One has $L(M, s) = L(A_{/k}, s+1)$, where $L(A_{/k}, s)$ is the Hasse-Weil L-function attached to A. The study of $L(M, s)$ at the central critical point $s = 0$ therefore amounts to the study of $L(A_{/k}, s)$ at $s = 1$. The representation V is crystalline at p if and only if A has good reduction at p (by the work of Fontaine [Fon79] for the "if" part of this statement; and the "only if" part by Coleman and Iovita [CI99], see also [Mok93] for the case when $A_{/\mathbb{Q}_p}$ is potentially a product of Jacobians).

2. Suppose that f is a cuspidal Hilbert eigenform of even parallel weight $(w, w, \dots, w)$ (for brevity, we say of weight $w \in 2\mathbb{Z}^+$), of level $\mathfrak{n} \subset \mathcal{O}_k$ and central character φ. Thanks to [Shi78, Proposition 1.3], there exists a number field L_f such that its ring of integers $\mathcal{O}_f := \mathcal{O}_{L_f}$ contains the values of φ and all Hecke eigenvalues $\theta_f(\mathfrak{a})$ for $(\mathfrak{a}, \mathfrak{n}) = 1$. Let $\mathfrak{p}$ be any prime of L_f above p. The work of Carayol [Car86], Wiles [Wil88], Taylor [Tay89] and Blasius and Rogawski [BR93] attaches f a motive

M such that the $\mathfrak{p}$-adic realization $M_{\mathfrak{p}} = V_{\mathfrak{p}}(f)$ is an irreducible [Tay95] two dimensional representation of G_k over $L_{f,\mathfrak{p}}$. (When $k = \mathbb{Q}$, this construction is due to Eichler, Shimura, Deligne [Del71] and Scholl [Sch90].) At least when p is large enough, Blasius and Rogawski show that $V_{\mathfrak{p}}(f)$ is crystalline at $\mathfrak{p}$ if $(\mathfrak{p}, \mathfrak{n}) = 1$. Let $\mathbb{A}_k$ denote the idéles of k, and suppose $\chi : \mathbb{A}_k/k^\times \to L_f^\times$ is a character such that $\varphi = \chi^{-2}$. As Nekovář explains in [Nek06, §12.5.5], the G_k-representation $V = V_{\mathfrak{p}}(f)(w/2) \otimes \chi$ is self-dual in the sense that $V \xrightarrow{\sim} \mathrm{Hom}(V, \mathbb{Q}_p(1))$.

Let B_{dR} denote Fontaine's [Fon82] field of p-adic periods; it is a discretely valued field whose valuation ring contains $\overline{\mathbb{Q}}_p$. There is a natural descending filtration $\cdots \supset B^i_{\mathrm{dR}} \supset B^{i+1}_{\mathrm{dR}} \supset \ldots$, which is obtained by letting $B^i_{\mathrm{dR}} \subset B_{\mathrm{dR}}$ to be the set of elements whose valuation is at least i. For an arbitrary Galois representation W (which is finite dimensional over $\mathbb{Q}_p$) and a finite extension $\mathfrak{L}$ of $\mathbb{Q}_p$, write $D_{\mathrm{dR}}(\mathfrak{L}, W) = H^0(\mathfrak{L}, B_{\mathrm{dR}} \otimes W)$, and $D_{\mathrm{dR}}(\mathbb{Q}_p, W) = D_{\mathrm{dR}}(W)$. The filtration on B_{dR} induces a decreasing filtration $\{D^i_{\mathrm{dR}}(W)\}_{i\in\mathbb{Z}}$ on $D_{\mathrm{dR}}(W)$. One always has

$$\dim_{\mathfrak{L}} D_{\mathrm{dR}}(\mathfrak{L}, W) \leq \dim_{\mathbb{Q}_p} W$$

by [Fon82, §5.1] and the $G_{\mathfrak{L}}$-representation W is called de Rham if $\dim_{\mathfrak{L}}(D_{\mathrm{dR}}(\mathfrak{L}, W)) = \dim_{\mathbb{Q}_p}(W)$. A $G_{\mathbb{Q}_p}$-representation W is de Rham if and only if it is de Rham as a $G_{\mathfrak{L}}$-representation; and one has

$$\mathfrak{L} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(W) \xrightarrow{\sim} D_{\mathrm{dR}}(\mathfrak{L}, W),$$

if W is de Rham.

For any de Rham representation W of $G_{\mathfrak{L}}$ as above, Bloch and Kato [BK90] construct a canonical homomorphism

$$\exp^* : H^1(\mathfrak{L}, W) \longrightarrow D^0_{\mathrm{dR}}(\mathfrak{L}, W)$$

called the dual exponential map. By its construction, it factors through the singular quotient $H^1_s(\mathfrak{L}, W)$. In Section 3.3.2 below, we will explain Perrin-Riou's [PR94] interpolation of the dual exponential maps for *crystalline*[9] representations (which we define next), as one climbs up the cyclotomic tower.

Let B_{cris} be Fontaine's crystalline period ring, see [Fon94] for its construction and other properties we note here. For a $G_{\mathbb{Q}_p}$-representation W as above, let $D_{\mathrm{cris}}(W) = H^0(\mathbb{Q}_p, B_{\mathrm{cris}} \otimes W)$ be Fontaine's filtered vector space associated to W which is endowed by a Frobenius action. If W is also a $G_{\mathbb{Q}}$-representation, we set

$$D_{\mathrm{cris}}(F, W) = D_{\mathrm{cris}}(\mathrm{Ind}_{F/\mathbb{Q}} W)$$

for a finite abelian extension F of $\mathbb{Q}$ which is unramified above p.

For any $G_{\mathbb{Q}_p}$-representation W, it is known that $D_{\mathrm{cris}}(W) \subset D_{\mathrm{dR}}(W)$, and hence

$$\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}(W) \leq \dim_{\mathbb{Q}_p} D_{\mathrm{dR}}(W) \leq \dim_{\mathbb{Q}_p} W,$$

and we say that W is crystalline if $\dim_{\mathbb{Q}_p} D_{\mathrm{cris}}(W) = \dim_{\mathbb{Q}_p} W$. Hence, if W is crystalline, then W is de Rham as well, and one has $D_{\mathrm{cris}}(W) = D_{\mathrm{dR}}(W)$.

We define final more ring which plays an important role in what follows. Define $G_\infty := \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}) = \Delta \times \Gamma$ where $\Delta = \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q})$ is a finite group of order prime to p, and Γ is defined as before. Set $G_n = \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^n})/\mathbb{Q})$. Fixing a topological generator γ of Γ, we

[9] Kato claims in [Kat04, Remark 16.5] that this assumption is not necessary and refers to his preprint with Kurihara and Tsuji [KKT96].

may identify $\mathbb{Z}_p[[G_\infty]]$ by the power series ring $\mathbb{Z}_p[\Delta][[\gamma-1]]$ over the group ring $\mathbb{Z}_p[\Delta]$. For any integer $h \geq 1$, set

$$\mathcal{H}_h = \left\{ \sum_{\substack{n\geq 0,\\ \delta\in\Delta}} a_{n,\delta}\cdot\delta\cdot(\gamma-1)^n \in \mathbb{Q}_p[\Delta][[\gamma-1]] : \right.$$
$$\left. \lim_{n\to\infty} |a_{n,\delta}|_p \cdot n^{-h} = 0, \text{ for every } \delta\in\Delta \right\},$$

where $|\cdot|_p$ is the p-adic norm on $\mathbb{Q}_p$, normalized by setting $|p|_p = \frac{1}{p}$. Define $\mathcal{H}_\infty = \varinjlim_h \mathcal{H}_h$. Any continuous character $\chi : G_\infty \to \overline{\mathbb{Q}}_p^\times$ induces a homomorphism $\mathcal{H}_\infty \to \overline{\mathbb{Q}}_p^\times$, which we still denote by χ. We write ρ_{cyc} for the cyclotomic character

$$\rho_{\text{cyc}} : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$$

and following [PR94, §4.1.5], we say that ρ is a geometric character of G_∞ if there is an integer $j = j_\rho$ such that $\rho_{\text{cyc}}^{-j}\cdot\rho = \chi_\rho$ is a character of finite order.

Finally, for every field F and a G_F-module T which is free of finite rank over $\mathbb{Z}_p$, write

$$H^1_\infty(F,T) = \varprojlim_n H^1(F(\boldsymbol{\mu}_{p^n}),T),$$

and if $W = T\otimes\mathbb{Q}_p$, write $H^1_\infty(F,W) = \mathbb{Q}_p\otimes H^1_\infty(F,T)$.

3.3.2. *Perrin-Riou's extended logarithm and conjectures.* We are now ready to state Perrin-Riou's theorem [PR95, Theorem 1.2.5], following [Kat04, Theorem §16.4].

Theorem 3.12 (Perrin-Riou)**.** *Suppose W is a $G_\mathbb{Q}$-representation which is finite dimensional as a $\mathbb{Q}_p$-vector space. Assume W is de Rham at p and $D_{\text{cris}}(W^*) \subset D^0_{\text{dR}}(W^*)$. Then for every finite extension F of $\mathbb{Q}$ which is unramified above p, there is a unique homomorphism*

$$\mathfrak{Log}^F : H^1_\infty(F,W) \longrightarrow \mathcal{H}_\infty \otimes_{\mathbb{Q}_p} D_{\text{cris}}(F,W)$$

which satisfies the following properties (i)-(ii)*, for every $\eta\in D_{\text{cris}}(W^*)$ and for every integer $j\geq 1$:*

(i) *Let $\mathfrak{Log}^F_\eta$ be the composite map*

$$\mathfrak{Log}^F_\eta : H^1_\infty(F,W) \xrightarrow{\mathfrak{Log}^F} \mathcal{H}_\infty\otimes_{\mathbb{Q}_p} D_{\text{cris}}(F,W) \xrightarrow{\eta} \mathcal{H}_\infty\otimes_{\mathbb{Q}_p} F,$$

where the second map is induced from the canonical pairing

$$D_{\text{dR}}(W)\times D_{\text{cris}}(W^*) \longrightarrow \mathbb{Q}_p$$

and from

$$D_{\text{cris}}(F,W)\subset D_{\text{dR}}(F,W)\cong F\otimes D_{\text{dR}}(W).$$

Then for $n\geq 1$, for every character $\chi : G_n \to \overline{\mathbb{Q}}_p^\times$ which does not factor through G_{n-1} and for any $x\in H^1_\infty(F,W)$, we have

$$\rho^j_{\text{cyc}}\chi^{-1}\left(\mathfrak{Log}^F_\eta\right) \approx \sum_{\sigma\in G_n}\chi(\sigma)\langle\sigma(\exp^*(x_{-j,n})),(p^{-j}\varphi)^{-n}(\eta)\rangle.$$

Here:

- *'$\approx$' means equality up to simple non-zero factors which are omitted for brevity,*
- *φ is the (geometric) Frobenius at p,*

– $x_{-j,n}$ *is the image of* x *under the composite*
$$H^1_\infty(F_p, W) \xrightarrow{\sim} H^1_\infty(F_p, W(-j)) \xrightarrow{\text{proj}} H^1(F(\boldsymbol{\mu}_{p^n})_p, W(-j)),$$
– $\exp^*$ *is the semi-local Bloch-Kato dual exponential*
$$\begin{aligned}\exp^* : H^1(F(\boldsymbol{\mu}_{p^n}), W(-j)) &\longrightarrow D^0_{\mathrm{dR}}(F(\boldsymbol{\mu}_{p^n}), W(-j))\\ &\subset D_{\mathrm{dR}}(F(\boldsymbol{\mu}_{p^n}), W(-j))\\ &= D_{\mathrm{dR}}(F(\boldsymbol{\mu}_{p^n}), W)\\ &= F(\boldsymbol{\mu}_{p^n}) \otimes D_{\mathrm{dR}}(W)\end{aligned}$$
– $\langle\, ,\, \rangle$ *is the pairing*
$$F(\boldsymbol{\mu}_{p^n}) \otimes D_{\mathrm{dR}}(W) \times D_{\mathrm{cris}}(W^*) \longrightarrow F(\boldsymbol{\mu}_{p^n}) \otimes \mathbb{Q}_p$$
induced from the pairing $D_{\mathrm{dR}}(W) \times D_{\mathrm{cris}}(W^*) \longrightarrow \mathbb{Q}_p$.

(ii) *Suppose* $\eta = (1 - p^{-j}\varphi)\eta'$,*with* $\eta' \in D_{\mathrm{cris}}(W^*)$*, and let* $\mathfrak{Log}^F_\eta$ *be as in* (i)*. Then for any* $x \in H^1_\infty(F_p, W)$ *we have*
$$\rho^j_{cyc}(\mathfrak{Log}^F_\eta(x)) = (j-1)! \cdot \langle \exp^*(x_{-j,0}), (1 - p^{j-1}\varphi^{-1})\eta' \rangle.$$

Let $M_{/\mathbb{Q}}$ be a pure motive. For a geometric character ρ of G_∞, set $M(\rho) = M(j_\rho)(\chi_\rho)$. For every positive integer $\mathfrak{f}$, one can then attach $M(\rho)$ a complex L-function with Euler factors at primes dividing $\mathfrak{f}$ removed:
$$L_{\mathfrak{f}}(M(\rho), s) = \prod_{\ell \nmid \mathfrak{f}} L_\ell(M(\rho), s)^{-1}.$$
Here, for a prime $\ell \neq p$ at which the p-adic realization $M(\rho)_p$ is unramified, the Euler factor at ℓ is given by
$$L_\ell(M, s) = \det\left(1 - \mathrm{Fr}_\ell^{-1} x \mid M(\rho)_p\right)\Big|_{s=\ell^{-s}}.$$
Let $\mathfrak{K} = \mathrm{Frac}(\mathcal{H}_\infty)$, the fraction field of $\mathcal{H}_\infty$. Write $d_- = \dim M_p^-$ for the dimension of the (-1)-eigenspace of a complex conjugation acting on the p-adic realization M_p which we henceforth assume to be crystalline.

Conjecture 1 (Perrin-Riou [PR95] §4.2.2)**.** *For every positive integer* $\mathfrak{f}$ *which is prime to* p *and to every prime at which* M_p *is ramified, there exists an element* $\mathfrak{l}_{\mathfrak{f}}(M) \in \mathfrak{K} \otimes \wedge^{d_-} D_{\mathrm{cris}}(M_p)$ *and* $\overline{\eta} = \eta_1 \wedge \cdots \wedge \eta_{d_-} \in \wedge^{d_-} D_{\mathrm{cris}}(M_p^*)$ *such that*
$$\mathbf{L}^{(p)}_{\mathfrak{f}}(M) = \overline{\eta}(\mathfrak{l}_{\mathfrak{f}}(M)) \in \mathfrak{K}$$
is the 'p*-adic* L*-function' attached to* M*, which interpolates the special values of the complex* L*-functions attached to twists of* M *by geometric characters, with their Euler factors at primes dividing* $\mathfrak{f}$ *removed.*

See [PR95, §4.2] for a detailed description of the properties which characterize this p-adic L-function. The statement above is Rubin's extrapolation [Rub00, Conjecture VIII.2.1] of Perrin-Riou's conjecture by introducing the level $\mathfrak{f}$. The interpolation property alluded to above (roughly) reads as follows:

For every geometric character ρ of G_∞ such that $\chi_\rho(p) \cdot p^{j_\rho}$ and $\overline{\chi}_\rho(p) \cdot p^{-j_\rho - 1}$ are not eigenvalues of φ on $D_{\mathrm{cris}}(M_p)$,

$$\rho^{-1}(\mathbf{L}^{(p)}_{\mathfrak{f}}(M)) = \mathcal{E}_p(M(\rho)) \times \frac{L_{\mathfrak{f}}(M(\rho), 0)}{\mathrm{Per}_\infty(M(\rho))} \times \mathrm{Per}_p(M(\rho)) \tag{3.2}$$

where $\mathcal{E}_p(M(\rho))$ is the Euler factor at p and $\mathrm{Per}_\infty(M(\rho))$ (resp., $\mathrm{Per}_p(M(\rho))$) is the archimedean (resp., p-adic) period attached to $M(\rho)$, see [PR95, §3.1 and 4.1.4].

3.3.3. *Connection with Euler systems of rank r.* Let $M_{/\mathbb{Q}}$ be a pure motive as above, and let M_p be its p-adic realization which is crystalline. Fix a $G_{\mathbb{Q}}$-stable lattice $\mathcal{T} \subset M_p$ and an integer $\mathcal{B} = \mathcal{B}(\mathcal{T})$ which is divisible by p and all bad primes for M_p. For any integer $\mathfrak{f}$, write $R_{\mathfrak{f}} = \mathbb{Q}(\boldsymbol{\mu}_{\mathfrak{f}})^+$ for the maximal real field of $\mathbb{Q}(\boldsymbol{\mu}_{\mathfrak{f}})$ and define

$$\mathcal{C} = \bigcup_{\substack{(\mathfrak{f},\mathcal{B})=1\\ n\geq 1}} R_{\mathfrak{f}}(\boldsymbol{\mu}_{p^n}).$$

For notational consistency, we write $r = d_- = d_-(M_p)$. Recall that an Euler system of rank r for the pair $(\mathcal{T},\mathcal{C})$ is a collection $\mathbf{c}^{(r)} = \left\{c_K^{(r)}\right\}_{K\subset\mathcal{C}}$ with the properties that

- $c_K^{(r)} \in \wedge_0^r H^1(K,\mathcal{T})$,
- for $K \subset K' \subset \mathcal{C}$ such that $K'/\mathbb{Q}$ is a finite extension,

$$\mathrm{Cor}^r_{K'/K}(c_{K'}) = \left(\prod_{\mathfrak{q}} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\right) c_K,$$

  where the product is over all rational primes $\mathfrak{q} \nmid \mathcal{B}$ which does not ramify in $K/\mathbb{Q}$, but does ramify in $K'/\mathbb{Q}$.

See §2 above for further details.

Until the end of this section we assume the following conditions hold for $\mathcal{T}$:

(A) $H^0(\mathbb{Q}_p(\boldsymbol{\mu}_p),\mathcal{T}^*) = 0$,
(B) $H^0(\mathbb{Q}_p(\boldsymbol{\mu}_{p^\infty}),\mathcal{T}) = 0$.

where $\mathcal{T}^* = \mathrm{Hom}(\mathcal{T},\boldsymbol{\mu}_{p^\infty})$ is as before. The conditions above are the hypotheses $\mathbf{H.nE}$ and $\mathbf{H.D}$ with $k = \mathbb{Q}(\boldsymbol{\mu}_p)$, and as in §1.2.1, one may prove under these conditions that:

(i) $H^1_\infty(\mathbb{Q}_p,\mathcal{T})$ is a free $\mathbb{Z}_p[[G_\infty]]$-module of rank $d = \dim M_p$,
(ii) the canonical projection $H^1_\infty(\mathbb{Q}_p,\mathcal{T}) \to H^1(\mathbb{Q}_p(\boldsymbol{\mu}_{p^n}),\mathcal{T})$ is surjective,
(iii) $H^1(\mathbb{Q}_p(\boldsymbol{\mu}_{p^n}),\mathcal{T})$ is a free $\mathbb{Z}_p[G_n]$-module of rank d.

Furthermore, as noted in Remark 2.3, these together with [Rub96, Example (1), page 38] show that

(1) $\wedge_0^r H^1(\mathbb{Q}_p(\boldsymbol{\mu}_{p^n}),\mathcal{T}) = \wedge^r H^1(\mathbb{Q}_p(\boldsymbol{\mu}_{p^n}),\mathcal{T})$,
(2) $\wedge^r H^1_\infty(\mathbb{Q}_p,\mathcal{T}) = \varprojlim_n \wedge^r H^1(\mathbb{Q}_p(\boldsymbol{\mu}_{p^n}),\mathcal{T})$,

where the exterior products in (1) is taken in the category of $\mathbb{Z}_p[G_n]$-modules, whereas in (2), the exterior products are taken in the category of $\mathbb{Z}_p[[G_\infty]]$-modules.

For any number field K, write as usual

$$\mathrm{loc}_p : H^1(K,\mathcal{T}) \longrightarrow H^1(K_p,\mathcal{T})$$

for the localization map at p. If $\mathbf{c}^{(r)} = \left\{c_K^{(r)}\right\}_{K\in\mathcal{K}}$ is an Euler system of rank r for $(\mathcal{T},\mathcal{C})$, we may regard $\mathrm{loc}_p\left(c_\infty^{(r)}\right) := \left\{\mathrm{loc}_p\left(c^{(r)}_{\mathbb{Q}(\boldsymbol{\mu}_{p^n})}\right)\right\}_n$ as an element of $\wedge^r H^1_\infty(\mathbb{Q}_p,\mathcal{T})$, and apply Perrin-Riou's extended logarithm

$$\mathfrak{Log}^{\otimes r} : \wedge^r H^1_\infty(\mathbb{Q}_p,\mathcal{T}) \longrightarrow \mathfrak{K} \otimes \wedge^r D_{\mathrm{cris}}(\mathcal{M}_p)$$

on it. Here we write $\mathfrak{Log}$ for $\mathfrak{Log}^{\mathbb{Q}_p}$ above.

Finally, assume that the weak Leopoldt conjecture (see [PR95] §1.3) holds for the representation $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{T}, \mathbb{Z}_p(1))$.

Conjecture 2 ([PR95] §4.4 and [Rub00] Lemma VIII.5.1)**.** *Assuming the hypotheses above, there exists an Euler system $\mathbf{c}^{(r)} = \left\{c_K^{(r)}\right\}_{K\in\mathcal{C}}$ of rank r for $(\mathcal{T}, \mathcal{C})$ so that*

$$\overline{\eta}\left(\mathfrak{Log}^{\otimes r}\left(\mathrm{loc}_p\left(c_\infty^{(r)}\right)\right)\right) = \mathbf{L}^{(p)}(M),$$

where $\overline{\eta} = \eta_1 \wedge \cdots \wedge \eta_r \in \wedge^r D_{\mathrm{cris}}(M_p^*)$, and $\mathbf{L}^{(p)}(M) = \mathbf{L}_1^{(p)}(M)$ is as in Conjecture 1.

We will write $\mathfrak{Log}_{\overline{\eta}}^{\otimes r}$ as a short-cut for the composite

$$\overline{\eta} \circ \mathfrak{Log}^{\otimes r} : \wedge^r H^1_\infty(\mathbb{Q}_p, \mathcal{T}) \to \mathcal{H}_\infty.$$

3.3.4. *Applications.* We apply the results of §3.3.4 together with the (conjectural) Euler system of rank r given in Conjecture 2.

Suppose V is the p-adic realization of a fixed self-dual pure motive $\mathcal{M} \xrightarrow{\sim} \check{\mathcal{M}}(1)$ defined over $k = \mathbb{Q}$ and with coefficients in $L = \mathbb{Q}$. As remarked before, taking $k = \mathbb{Q}$ is not too serious as one may always consider $\mathrm{Ind}_{k/\mathbb{Q}}\mathcal{M}$ in place of $\mathcal{M}$; and the assumption that $L = \mathbb{Q}$ is only made for notational convenience. The p-adic realization V is then a finite dimensional $\mathbb{Q}_p$-vector space endowed with a $G_\mathbb{Q}$-action, which is unramified outside a finite set of places. We will also assume that V is crystalline at p. Fix a $G_\mathbb{Q}$-stable lattice $T \subset V$. We assume until the end of this paper that T satisfies the hypotheses (A) and (B) from the previous section, as well as $\mathbf{H}.\mathbf{1}$-$\mathbf{H}.\mathbf{5}$ from the introduction.

Along with the motive $\mathcal{M}_{/\mathbb{Q}}$, we will consider its Tate-twists $\mathcal{M}(j)$ for very large integers j; the p-adic realization $\mathcal{M}(j)_p$ of $\mathcal{M}(j)$ is $V(j) = V \otimes \mathbb{Q}_p(j)$. The $G_\mathbb{Q}$-representation $V(j)$ is also unramified outside a finite set of places and is crystalline at p. We write $T(j) = T \otimes \mathbb{Z}_p(j)$ which is naturally a lattice inside $V(j)$.

Lemma 3.13. *For any j, the hypotheses* (A) *and* (B) *hold for $T(j)$.*

Proof. (B) obviously holds for $T(j)$ if it holds for T. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ with $\Gamma = \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_{p^\infty})/\mathbb{Q}(\boldsymbol{\mu}_p))$ as usual. The statement of (A) for T is equivalent to the vanishing of $H^2(\mathbb{Q}_p(\boldsymbol{\mu}_p), T \otimes \Lambda) = 0$ (see the proof of Lemma 1.9), and the proof of Lemma follows using the natural isomorphism

$$H^2(\mathbb{Q}_p(\boldsymbol{\mu}_p), T \otimes \Lambda) \longrightarrow H^2(\mathbb{Q}_p(\boldsymbol{\mu}_p), T(j) \otimes \Lambda).$$

□

Fix a large enough $j \in 2\mathbb{Z}$ so that $D^0_{\mathrm{dR}}(V(j)^*) = D_{\mathrm{dR}}(V(j)^*)$. Such an integer j exists because

$$D^0_{\mathrm{dR}}(V(j)^*) = D^0_{\mathrm{dR}}(V^*(-j)) = D^{-j}_{\mathrm{dR}}(V^*).$$

Since we insist that j is even, it follows that $r = \dim(V^-) = \dim(V(j)^-)$.

Assume that the weak Leopoldt conjecture is true for the representation $\mathrm{Hom}(T(j), \mathbb{Z}_p(1)) \cong T(-j)$, and suppose that the Conjecture 2 holds for $M = \mathcal{M}(j)$.

Theorem 3.14. *Suppose 1 is not an eigenvalue for the action of φ on $D_{\mathrm{cris}}(V)$, and assume that $L(\mathcal{M}, 0) \neq 0$. Then the Bloch-Kato Selmer group $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(\mathbb{Q}, T^*)$ is finite.*

Remark 3.15. Since V is self-dual, it follows that 1 is an eigenvalue of φ acting on $D_{\mathrm{cris}}(V)$ if and only if p^{-1} is an eigenvalue. The assumption that 1 is not an eigenvalue (therefore neither p^{-1}) rules out the possibility that the p-adic L-function $\mathbf{L}^{(p)}(\mathcal{M})$ may have an *exceptional zero* at the trivial character of G_∞.

Note also that 1 (resp., p^{-1}) is an eigenvalue of φ acting on $D_{\mathrm{cris}}(V^*) = D_{\mathrm{cris}}(V)$ if and only if p^j (resp., p^{j-1}) is an eigenvalue of φ acting on $D_{\mathrm{cris}}(V(j)^*)$. In particular, under the assumption that 1 is not an eigenvalue for $\varphi\big|_{D_{\mathrm{cris}}(V)}$, the operators $1-p^{-j}\varphi$ and $1-p^{j-1}\varphi^{-1}$ acting on $D_{\mathrm{cris}}(V(j)^*)$ (which appear in the statement of Theorem 3.12(ii)) are both invertible.

Proof of Theorem 3.14. Let $\mathbf{c}^{(r)}(j)$ denote the Euler system of rank r for the pair $(T(j),\mathcal{C})$ predicted by Conjecture 2, where $\mathcal{C}$ is as in the previous section. Applying Rubin's twisting formalism [Rub00, §VI], we obtain an Euler system $\mathbf{c}^{(r)} = \{c_K^{(r)}\}_{K\in\mathcal{C}}$ of rank r for $(T,\mathcal{C})$. Corollary 3.6 gives an inequality

$$|H^1_{\mathcal{F}^*_{\mathrm{BK}}}(\mathbb{Q},T^*)| \le |\wedge^r H^1_s(\mathbb{Q},T)/\mathbb{Z}_p\cdot \mathrm{loc}_p^s(c_{\mathbb{Q}}^{(r)})\,|,$$

and the theorem is proved once we verify that $\mathrm{loc}_p^s(c_{\mathbb{Q}}^{(r)})\neq 0$.

Let $c_\infty^{(r)}(j)=\{c^{(r)}_{\mathbb{Q}(\boldsymbol{\mu}_{p^n})}(j)\}_n\in H^1_\infty(\mathbb{Q},T(j))$, and consider

$$\rho^j_{\mathrm{cyc}}\mathbf{L}^{(p)}(\mathcal{M}(j)) = \rho^j_{\mathrm{cyc}}\mathfrak{Log}_{\overline{\eta}}^{\otimes r}\left(\mathrm{loc}_p\left(c_\infty^{(r)}(j)\right)\right) \tag{3.3}$$

where the equality follows from the defining property of $\mathbf{c}^{(r)}(j)$. If we take j large enough and assume that 1 is not an eigenvalue for $\varphi\big|_{D_{\mathrm{cris}}(V)}$, one may calculate $\rho^j_{\mathrm{cyc}}\mathbf{L}^{(p)}(\mathcal{M}(j))$ using the interpolation property of the (conjectural) p-adic L-function $\mathbf{L}^{(p)}(\mathcal{M}(j))$ and conclude that

$$\rho^j_{\mathrm{cyc}}\mathbf{L}^{(p)}(\mathcal{M}(j))\neq 0 \tag{3.4}$$

by our assumption that $L(\mathcal{M},0)\neq 0$. On the other hand, the interpolation property of Perrin-Riou's extended logarithm (see Theorem 3.12(ii)) shows that the image of $\mathrm{loc}_p\left(c_\infty^{(r)}(j)\right)$ under

$$\wedge^r H^1_\infty(\mathbb{Q}_p,T(j)) \xrightarrow{\mathfrak{Log}_{\overline{\eta}}^{\otimes r}} \mathcal{H}_\infty \xrightarrow{\rho^j_{\mathrm{cyc}}} \mathbb{Q}_p$$

coincides with the image of $\mathrm{loc}_p\left(c_{\mathbb{Q}}^{(r)}\right)$ under

$$\wedge^r H^1(\mathbb{Q}_p,T) \xrightarrow{(\exp^*)^{\otimes r}} \wedge^r D_{\mathrm{dR}}(V) \xrightarrow{\alpha^{-1}\beta\cdot\overline{\eta}} \mathbb{Q}_p\ ,$$

and since the Bloch-Kato dual exponential $\exp^*$ factors through the singular quotient $H^1_{\mathrm{s}}(\mathbb{Q}_p,T) := H^1(\mathbb{Q}_p,T)/H^1_f(\mathbb{Q}_p,T)$, this agrees with the image of $\mathrm{loc}_p^{\mathrm{s}}\left(c_{\mathbb{Q}}^{(r)}\right)$ under the composite

$$\wedge^r H^1_{\mathrm{s}}(\mathbb{Q}_p,T) \xrightarrow{(\exp^*)^{\otimes r}} \wedge^r D_{\mathrm{dR}}(V) \xrightarrow{\alpha^{-1}\beta\cdot\overline{\eta}} \mathbb{Q}_p\ . \tag{3.5}$$

Here

$$\alpha=\det(1-p^{-j}\varphi|D_{\mathrm{cris}}(V(j)^*)) \ \text{ and } \ \beta=\det(1-p^{j-1}\varphi^{-1}|D_{\mathrm{cris}}(V(j)^*)).$$

Both α and β are non-zero thanks to our assumption that 1 is not an eigenvalue for $\varphi\big|_{D_{\mathrm{cris}}(V)}$ (see Remark 3.15).

It then follows from (3.3) and (3.4) that the image of $\mathrm{loc}_p^{\mathrm{s}}\left(c_{\mathbb{Q}}^{(r)}\right)$ under the map (3.5) is non-zero, which in return implies that $\mathrm{loc}_p^{\mathrm{s}}\left(c_{\mathbb{Q}}^{(r)}\right)\neq 0$ and the Theorem is proved. □

Remark 3.16. The proof of Theorem 3.14 gives a bound on the Bloch-Kato Selmer group $H^1_{\mathcal{F}^*_{\mathrm{BK}}}(\mathbb{Q}, T^*)$ which is closely related to L-values. This lends evidence to Bloch-Kato conjectures.

Remark 3.17. One may possibly prove an Iwasawa theoretic version of Theorem 3.14. However, the author is unable to state this application of the conjectural Euler System of rank r (Conjecture 2) and Theorem 3.9 in a satisfactory level of generality because he does not know how to compare the Bloch-Kato local condition with the Greenberg local condition for a general Galois representation, besides the comparison for the Galois representations attached to elliptic modular forms [Kat04, Lemma 17.9] and for Hilbert modular forms [Nek06, Proposition 12.5.8].

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